[5] der ternären definiten quadratischen Formen in Diagonalgestalt über $\boldsymbol{Z}$ mit Klassenzahl 1 kann man deshalb alle für Klassenzahl 1 „in Frage kommenden" höherdimensionalen Diagonalformen aufstellen: als einklassig kommen nur solche Formen in Frage, deren sämtliche orthogonalen Komponenten einklassig sind. Die in Frage kommenden Formen lassen sich mit der Minkowski–Siegelschen Maßformel auf Einklassigkeit testen: eine Form D ist genau dann einklassig, wenn $M(D) = 1/E(D)$, wobei $M(D)$ das Maß und $E(D)$ die Einheitenanzahl von D ist. Für die m-dimensionale Diagonalform

$$D = (\underbrace{d_1, d_1, \ldots, d_1}_{s_1\text{-mal}}, \ldots, \underbrace{d_n, \ldots, d_n}_{s_n\text{-mal}})$$

ist

$$E(D) = 2^m s_1! \ldots s_n!,$$

andererseits läßt sich der explizite Ausdruck für $M(D)$ bei Minkowski ([7], S. 171, 181) finden. Auf diese Weise erhält man die Tabelle.

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Dirichlet series with functional equations and related arithmetical identities

by

K. Chandrasekharan and H. Joris (Zürich)

To Carl L. Siegel on his completion of 75 years

§ 1. Introduction. Fifty years ago Siegel gave a short proof of Hamburger's theorem on the Riemann zeta-function $\zeta(s)$. Let G be an entire function of finite order, P a polynomial, s a complex variable, written $s = \sigma + it$, and $f(s) = G(s)/P(s)$. Let $f(s) = \sum_{m=1}^{\infty} c_m m^{-s}$, where (c_m) is a sequence of complex numbers, and the series converges absolutely for $\sigma > 1$. Let

$$(1.1) \qquad \pi^{-\frac{1}{2}s}\Gamma(\tfrac{1}{2}s)f(s) = \pi^{-\frac{1}{2}(1-s)}\Gamma(\tfrac{1}{2}-\tfrac{1}{2}s)g(1-s),$$

where $g(1-s) = \sum_{m=1}^{\infty} b_m m^{s-1}$, the series converging absolutely for $\sigma < -\varkappa < 0$. Then $f(s) = c_1\zeta(s) = g(s)$. The proof follows at once from Siegel's partial-fraction formula [9]:

$$(1.2) \qquad \sum_{m=1}^{\infty} c_m\left(\frac{1}{t+im}+\frac{1}{t-im}\right) - \pi t H(t) = 2\pi\sum_{m=1}^{\infty} b_m e^{-2\pi m t}, \qquad t > 0,$$

where $H(t)$ is a finite sum of terms of the form $t^{\alpha}\log^{\beta} t$.

Arnold Walfisz in his Göttingen dissertation, published in 1922, found an identity associated with the Dedekind zeta-function $\zeta_K(s)$ of an algebraic number field K of degree n, from which he deduced an Ω-result for the ideal function. For $\operatorname{Re} s > 1$, $\zeta_K(s) = \sum_{m=1}^{\infty} a(m)m^{-s}$, where $a(m)$ is the number of non-null integral ideals with norm m. Walfisz's identity [10] runs as follows: for $\operatorname{Re} s > 0$, we have

$$(1.3) \qquad \frac{1}{s}\sum_{m=1}^{\infty} a(m)e^{-sm^{1/n}} - \frac{n!\lambda}{s^{n+1}} - \frac{1}{s}\zeta_K(0)$$

$$= Di^{r_1+r_2}\frac{1}{s}\sum_{m=1}^{\infty}\frac{a(m)}{m}M(sm^{-1/n}).$$

Here λ is the residue of $\zeta_K(s)$ at the pole $s=1$, r_1 is the number of "real conjugates" of K, $2r_2$ the number of "imaginary conjugates" of K, $D=(2\pi)^{-n}\Delta^{1/2}$, Δ the absolute value of the discriminant of K, and

$$M(s)=\sum_{k=0}^{n}\eta_k L(Eie^{\pi ik/n}s),$$

where the (η_k) are constants depending on K, $E=(2\pi)^{-1}\Delta^{1/n}$, and the function

$$L(v)=\sum_{m=0}^{\infty}\frac{\Gamma^n\left(1+\frac{m}{n}\right)}{\Gamma(1+m)}v^m,\qquad |v|<n,$$

is continuable analytically into $C-\{v\geqslant n\}$, where C denotes the complex plane. Identity (1.3) is the basis of Walfisz's result that

$$P_K(x)=\Omega_{\pm}(x^{(n-1)/2n}),\qquad \text{as } x\to\infty,$$

where $P_K(x)=R_K(x)-\lambda x$ for $x>0$, and the 'ideal function' $R_K(x)$ is given by $R_K(x)=\sum_{m\leqslant x}a(m)$, for $x>0$. For recent work on this problem and the related literature, see Joris [8].

Although there seems to be no apparent connexion between (1.2) and (1.3), (cf. comment by S. Bochner [1], p. 353), we shall show in this article that both are special cases of an identity (2.7) which can be proved for Dirichlet series satisfying a general functional equation of the type studied by Chandrasekharan and Narasimhan [3], an identity which, in fact, is *equivalent* to the functional equation itself. Such identities have been considered by Hamburger [5] in the case of Riemann's functional equation (1.1), which has the gamma factor $\Gamma(\frac{1}{2}s)$, and by Chandrasekharan and Narasimhan [2] in the case of Hecke's functional equation, with the gamma factor $\Gamma(s)$. Here we consider equations with multiple gamma factors, which are of the form

$$\Delta(s)\varphi(s)=\Delta(\delta-s)\psi(\delta-s),\tag{1.4}$$

where

$$\Delta(s)=\prod_{k=1}^{N}\Gamma(\alpha_k s+\beta_k),\qquad N\geqslant 1,\ \alpha_k>0,\ \beta_k \text{ complex},$$

and $\varphi(s)$ and $\psi(s)$ are representable by absolutely convergent Dirichlet series of the form $\sum_{m=1}^{\infty}a_m\lambda_m^{-s}$, $\sum_{m=1}^{\infty}b_m\mu_m^{-s}$, and one of the functions, say φ, is subject to the additional restriction that it can be continued analytically all over the complex s-plane with the possible exception of a compact set, and satisfies a (mild) restriction on its growth uniformly in every vertical strip. We conclude, in particular, that if equation (1.4) is satisfied by a pair of Dirichlet series φ, ψ, and also by the pair φ, ψ_1, where $\psi_1(s)=\sum_{m=1}^{\infty}d_m\nu_m^{-s}$, then $d_n=cb_n$ and $\nu_n=c_1\mu_n$, for $n=1,2,\ldots$, where c is real, $c\neq 0$, $c_1>0$.

§ 2. Identities equivalent to the functional equation. Let $\{a_m\}$, $\{b_m\}$ be two sequences of complex numbers, not all zero, and $\{\lambda_m\}$, $\{\mu_m\}$ two sequences of real numbers, such that

$$0<\lambda_1<\lambda_2<\ldots<\lambda_m\to\infty,\qquad 0<\mu_1<\mu_2<\ldots<\mu_m\to\infty.$$

Let δ be a real number, s a complex number, $s=\sigma+it$. Let

$$\Delta(s)=\prod_{\nu=1}^{N}\Gamma(\alpha_\nu s+\beta_\nu),\qquad A=\sum_{\nu=1}^{N}\alpha_\nu,\qquad B=\sum_{\nu=1}^{N}(\beta_\nu-\tfrac{1}{2}),$$

where N is an integer, $N\geqslant 1$, β_ν complex, $\alpha_\nu>0$, for $\nu=1,2,\ldots,N$. Suppose that the Dirichlet series $\sum_{m=1}^{\infty}a_m\lambda_m^{-s}$, $\sum_{m=1}^{\infty}b_m\mu_m^{-s}$ have finite abscissae of absolute convergence denoted by σ_a^* and σ_b^* respectively, while σ_a and σ_b denote the corresponding abscissae of ordinary convergence. Suppose that the sum-function $\varphi(s)=\sum_{m=1}^{\infty}a_m\lambda_m^{-s}$, which is regular for $\mathrm{Re}\,s>\sigma_a$, can be continued analytically all over the s-plane, with the possible exception of a compact set S, and there exists an $\varepsilon>0$, such that

$$\varphi(\sigma+it)=O(e^{A\pi|t|-\varepsilon|t|})\tag{2.1}$$

as $|t|\to\infty$, uniformly in each strip $\sigma_1\leqslant\sigma\leqslant\sigma_2$, where $-\infty<\sigma_1<\sigma_2<+\infty$. Let

$$\psi(s)=\sum_{m=1}^{\infty}b_m\mu_m^{-s},\qquad \mathrm{Re}\,s>\sigma_b,\tag{2.2}$$

so that $\psi(\delta-s)$ is regular for $\sigma<\delta-\sigma_b$.

The Dirichlet series $\sum_{m=1}^{\infty}a_m\lambda_m^{-s}$, $\sum_{m=1}^{\infty}b_m\mu_m^{-s}$ are then said to satisfy the functional equation (1.4), with the gamma factor $\Delta(s)$, if

$$\Delta(s)\varphi(s)=\Delta(\delta-s)\psi(\delta-s),\qquad \text{for } \sigma<\delta-\sigma_b.\tag{2.3}$$

Conditions (2.1)–(2.3) imply, because of the Phragmén–Lindelöf principle, that there exists a function χ, which is regular outside the compact set S, with the property

$$\lim_{|t|\to\infty}\chi(\sigma+it)=0,$$

uniformly in every bounded σ-interval, and such that

$$\chi(s) = \Delta(s)\varphi(s), \quad \text{for } \sigma > c_1,$$

and

$$\chi(s) = \Delta(\delta-s)\psi(\delta-s), \quad \text{for } \sigma < c_2,$$

where c_1 and c_2 are constants.

We define

$$M(v) = \frac{1}{2\pi i}\int_{-(m_0+\frac{1}{2})} \frac{\Delta(\delta-z/2A)}{\Delta(z/2A)}\Gamma(z)v^{-z}dz, \quad \operatorname{Re} v > 0, \tag{2.4}$$

where the integration is over the line $-(m_0+\frac{1}{2})+it$, $-\infty < t < +\infty$, and m_0 is an integer, such that

$$\begin{gathered} m_0 \geqslant -1, \quad m_0+2A\delta+\tfrac{1}{2} > \big(2A\sigma_b^*, 2A\max_\nu \operatorname{Re}(-\beta_\nu/a_\nu)\big), \\ m_0+\tfrac{1}{2} > 2A\max_\nu \operatorname{Re}\left(\frac{\beta_\nu-1}{a_\nu}\right), \end{gathered} \tag{2.5}$$

and

$$R(s) = \frac{1}{2\pi i}\int_{\mathscr{C}} \Gamma(z)\varphi(z/2A)s^{-z}dz, \quad \operatorname{Re} s > 0, \tag{2.6}$$

$\mathscr{C}$ being a curve which encloses all the singularities of the integrand which lie to the right of the line $\operatorname{Re} z = -(m_0+\frac{1}{2})$.

LEMMA 1. *Functional equation* (1.4) *implies the identity*

$$\sum_{m=1}^{\infty} a_m e^{-\lambda_m^{1/2A}s} - R(s) = \sum_{m=1}^{\infty} b_m\mu_m^{-\delta}M(s\mu_m^{-1/2A}), \tag{2.7}$$

for $\operatorname{Re} s > 0$, *the series on the right converging absolutely.*

Identity (2.7) *implies, in turn, that*

$$\begin{aligned} \left(-\frac{1}{s}\frac{d}{ds}\right)^{\varrho}\left(\frac{1}{s}\sum_{m=1}^{\infty} a_m e^{-\lambda_m^{1/2A}s}\right) \\ = \left(-\frac{1}{s}\frac{d}{ds}\right)^{\varrho}\left[\frac{1}{s}R(s)+\frac{1}{s}\sum_{m=1}^{\infty} b_m\mu_m^{-\delta}M(s\mu_m^{-1/2A})\right] \end{aligned} \tag{2.8}$$

for every integer $\varrho \geqslant 0$, *and* $\operatorname{Re} s > 0$.

Conversely, given the Dirichlet series $\varphi(s) = \sum_{m=1}^{\infty} a_m\lambda_m^{-s}$ *and* $\psi(s) = \sum_{m=1}^{\infty} b_m\mu_m^{-s}$ *satisfying the conditions* (2.1) *and* (2.2), *the validity of* (2.8) *for* $s > 0$, *and for some integer* $\varrho \geqslant 0$, *implies the validity of functional equation* (1.4) (*and hence also of identity* (2.7)).

Proof. To start with, let s be real, $s > 0$, and a be such that $\sum_{m=1}^{\infty}|a_m|\lambda_m^{-a} < \infty$, with $a > 0$. Then

$$\begin{aligned} \sum_{m=1}^{\infty} a_m e^{-\lambda_m^{1/2A}s} &= \sum_{m=1}^{\infty} a_m \frac{1}{2\pi i}\int_c \Gamma(z)(\lambda_m^{1/2A}s)^{-z}dz, \quad c > 0 \\ &= \frac{1}{2\pi i}\int_c \Gamma(z)s^{-z}\sum_{m=1}^{\infty} a_m\lambda_m^{-z/2A}dz, \quad c > a\cdot 2A \\ &= \frac{1}{2\pi i}\int_c \Gamma(z)s^{-z}\varphi(z/2A)\,dz, \end{aligned}$$

by the definition of φ. The integration is along the line $c+it$ (1), $-\infty < t < \infty$.

Now let m_0 be an integer, such that $\varphi(z/2A)$ is regular for $\operatorname{Re} z \leqslant -(m_0+\frac{1}{2})$. Since

$$\varphi(z/2A) = \frac{\psi(\delta-z/2A)\Delta(\delta-z/2A)}{\Delta(z/2A)},$$

that will be the case, if the series for $\psi(\delta-z/2A)$ is absolutely convergent for $\operatorname{Re} z < -(m_0+\frac{1}{2})+\varepsilon$, $\varepsilon > 0$, and $\Delta(\delta-z/2A)$ has no singularities for $\operatorname{Re} z \leqslant -(m_0+\frac{1}{2})$. The former condition is fulfilled if $m_0+\frac{1}{2} > (\sigma_b^*-\delta)2A$, while the latter condition is fulfilled if $\operatorname{Re}\{a_\nu\delta-(a_\nu z)/2A+\beta_\nu\} > 0$, for $\nu = 1, \ldots, N$; that is, if $m_0+2A\delta+\frac{1}{2} > 2A\max_\nu \operatorname{Re}(-\beta_\nu/a_\nu)$. Hence

$$\sum_{m=1}^{\infty} a_m e^{-\lambda_m^{1/2A}s} = R(s)+\frac{1}{2\pi i}\int_{-(m_0+\frac{1}{2})} \Gamma(z)\varphi(z/2A)s^{-z}dz, \tag{2.9}$$

for $m_0+2A\delta+\frac{1}{2} > \{2A\sigma_b^*, 2A\max_\nu(-\operatorname{Re}\beta_\nu/a_\nu)\}$, where $R(s)$ is defined as in (2.6).

If m is an integer, and $m > m_0 \geqslant -1$, then clearly

$$\begin{aligned} \sum_{k=1}^{\infty} a_k e^{-\lambda_k^{1/2A}s} \\ = R(s)+\sum_{j=m_0+1}^{m}\frac{(-s)^j}{j!}\varphi(-j/2A)+\frac{1}{2\pi i}\int_{-(m+\frac{1}{2})}\Gamma(z)\varphi(z/2A)s^{-z}dz. \end{aligned} \tag{2.10}$$

We shall see that if $m \to \infty$, then the last integral tends to zero, provided that $0 < s < \varkappa$, for a certain constant $\varkappa$.

(1) The letters $c, c_1, c_2, \ldots$ denote constants which do not necessarily have the same value at all occurrences.

Since

$$\Gamma(z) = \pi/\{\Gamma(1-z)\sin\pi z\},$$

we have

$$(2.11)\qquad \frac{\Delta(\delta-z/2A)}{\Delta(z/2A)}$$
$$= \pi^{-N}\prod_{\nu=1}^{N}\{\Gamma(\alpha_\nu\delta+\beta_\nu-\alpha_\nu z/2A)\,\Gamma(1-\beta_\nu-\alpha_\nu z/2A)\sin[\pi(\beta_\nu+\alpha_\nu z/2A)]\}.$$

By Stirling's approximation for the gamma-function, we have

$$\log\Gamma(s+c) = (s+c-\tfrac{1}{2})\log s - s + \tfrac{1}{2}\log 2\pi + \sum_{\nu=1}^{m} c_\nu' s^{-\nu} + O(|s|^{-m-1}),$$

for any constant c, as $|s|\to\infty$, uniformly for $|\arg s|\leqslant \pi-\varepsilon<\pi$, where m is an arbitrary positive integer. (The c_ν' depend on c.) Hence

$$(2.12)\qquad \log\left(\frac{\Delta(\delta-z/2A)}{\Delta(z/2A)\prod_{\nu=1}^{N}\sin\{\pi(\beta_\nu+\alpha_\nu z/2A)\}}\right)$$
$$= c_1 + O(|z|^{-1}) + \sum_{\nu=1}^{N}(\alpha_\nu\delta+\beta_\nu-\tfrac{1}{2}-\alpha_\nu z/2A)\big(\log(\alpha_\nu/2A)+\log(-z)\big) +$$
$$+\sum_{\nu=1}^{N}(\tfrac{1}{2}-\beta_\nu-\alpha_\nu z/2A)\big(\log(\alpha_\nu/2A)+\log(-z)\big)+z$$
$$= c_2 + z + O(|z|^{-1}) + A\delta\log(-z) - (z/A)\sum_{\nu=1}^{N}\alpha_\nu\log(\alpha_\nu/2A) - z\log(-z),$$

where $c_1 = N\log 2$, $c_2 = c_1 + \sum_{\nu=1}^{N}(\alpha_\nu\delta)\log(\alpha_\nu/2A)$. Since

$$(2.13)\qquad \log\left(\frac{1}{\Gamma(1-z)}\right) = c_3 - z + O(|z|^{-1}) - (\tfrac{1}{2}-z)\log(-z),$$

we have, for $z = -(m+\tfrac{1}{2})+iy$, $s>0$,

$$\Gamma(z)s^{-z}\varphi(z/2A) = \Gamma(z)s^{-z}\psi(\delta-z/2A)\frac{\Delta(\delta-z/2A)}{\Delta(z/2A)}$$
$$= \frac{\pi}{\sin\pi z}s^{-z}\psi(\delta-z/2A)\frac{\Delta(\delta-z/2A)}{\Delta(z/2A)}\,\frac{1}{\Gamma(1-z)}$$
$$= O\big(s^{m+\frac{1}{2}}e^{-\pi|y|}(m+|y|)^{A\delta-\frac{1}{2}}e^{\frac{1}{2}\pi|y|}e^{cm}\big)$$
$$= O\big(s^{m+\frac{1}{2}}e^{-\frac{1}{2}\pi|y|}(m+|y|)^{A\delta-\frac{1}{2}}e^{cm}\big),$$

where

$$(2.14)\qquad c = \frac{1}{A}\sum_{\nu=1}^{N}\alpha_\nu\log(\alpha_\nu/2A).$$

Hence

$$\frac{1}{2\pi i}\int_{-(m+\frac{1}{2})}\Gamma(z)s^{-z}\varphi(z/2A)\,dz = O\Big(s^{m+\frac{1}{2}}e^{cm}\int_0^\infty (m+y)^{A\delta-\frac{1}{2}}e^{-\frac{1}{2}\pi y}\,dy\Big)$$
$$= O\Big(s^{m+\frac{1}{2}}e^{cm}\int_m^\infty y^{A\delta-\frac{1}{2}}e^{-\pi(y-m)/2}\,dy\Big)$$
$$= O\Big(s^{m+\frac{1}{2}}e^{(c+\frac{1}{2}\pi)m}\int_m^\infty y^{A\delta-\frac{1}{2}}e^{-\frac{1}{2}\pi y}\,dy\Big)$$
$$= o\big(s^{m+\frac{1}{2}}e^{(c+\frac{1}{2}\pi)m}\big),\quad \text{as } m\to\infty$$
$$= o(1),\quad \text{as } m\to\infty, \text{ if } 0<s\leqslant e^{-(c+\frac{1}{2}\pi)},$$

where c is given by (2.14). Hence (2.10) yields the identity

$$(2.15)\qquad \sum_{m=1}^{\infty}a_m e^{-\lambda_m^{1/2A}s} - R(s) = \sum_{j=m_0+1}^{\infty}\frac{(-s)^j}{j!}\varphi(-j/2A),$$
$$0<s\leqslant c_4 = e^{-(c+\frac{1}{2}\pi)}.$$

To compute $\varphi(-j/2A)$ for $j\geqslant m_0+1$ we use the functional equation and note the third restriction on m_0 in (2.5). We have

$$\varphi(-j/2A) = \psi(\delta+j/2A)\frac{\Delta(\delta+j/2A)}{\Delta(-j/2A)}$$
$$= (-1)^N\pi^{-N}\sum_{m=1}^{\infty}b_m\mu_m^{-\delta-j/2A}\;\times$$
$$\times\prod_{\nu=1}^{N}\{\Gamma(\alpha_\nu\delta+\beta_\nu+\alpha_\nu j/2A)\,\Gamma(1-\beta_\nu+\alpha_\nu j/2A)\sin[\pi(-\beta_\nu+\alpha_\nu j/2A)]\}.$$

Now

$$(2.16)\qquad \prod_{\nu=1}^{N}\sin[\pi(-\beta_\nu+\alpha_\nu j/2A)]$$
$$= (2i)^{-N}\prod_{\nu=1}^{N}\big(e^{\pi i(-\beta_\nu+\alpha_\nu j/2A)} - e^{-\pi i(-\beta_\nu+\alpha_\nu j/2A)}\big) = (2i)^{-N}\sum_{k=1}^{2^N}e^{i\gamma_k j}\eta_k,$$

where $-\frac{1}{2}\pi \leqslant \gamma_k \leqslant \frac{1}{2}\pi$, since $a_\nu > 0$; and η_k is independent of j. Hence

$$(2.17)\quad \varphi(-j/2A) = (-2\pi i)^{-N}\sum_{k=1}^{2^N}\eta_k\prod_{\nu=1}^{N}\Gamma(a_\nu\delta+\beta_\nu+a_\nu j/2A)\Gamma(1-\beta_\nu+a_\nu j/2A)\times$$
$$\times e^{i\gamma_k j}\sum_{m=1}^{\infty}b_m\mu_m^{-\delta-j/2A}$$
$$= c_5 V(j)\sum_{k=1}^{2^N}\eta_k e^{i\gamma_k j}\sum_{m=1}^{\infty}b_m\mu_m^{-\delta-j/2A},\qquad c_5 = (-2\pi i)^{-N},$$

where

$$(2.18)\qquad V(z) = \prod_{\nu=1}^{N}\Gamma(a_\nu\delta+\beta_\nu+a_\nu z/2A)\Gamma(1-\beta_\nu+a_\nu z/2A).$$

From (2.17) and (2.15) we obtain the identity

$$\sum_{m=1}^{\infty}a_m e^{-\lambda_m^{1/2A}s}-R(s) = c_5\sum_{j=m_0+1}^{\infty}\frac{V(j)}{j!}(-s)^j\sum_{k=1}^{2^N}\eta_k e^{i\gamma_k j}\sum_{m=1}^{\infty}b_m\mu_m^{-\delta-j/2A},$$

for $0 < s \leqslant e^{-(c+\frac{1}{2}\pi)}$, where c is given by (2.14).

By (2.12) and (2.13), with $-z$ in place of z, however, we have for $j \geqslant m_0+1$,

$$\frac{V(j)}{j!} = c_6 e^{cj} j^{A\delta-\frac{1}{2}} e^{o(1)},\qquad \text{as } j\to\infty,$$

where $c < 0$ (cf. (2.14)). The series $\sum_{m=1}^{\infty} b_m\mu_m^{-\delta-j/2A}$ converges absolutely for $j \geqslant m_0+1 \geqslant 0$, provided that $2A\delta+m_0+1 > 2A\sigma_b^*$, which is the case by (2.5). Hence

$$(2.19)\quad \sum_{m=1}^{\infty}a_m e^{-\lambda_m^{1/2A}s}-R(s) = c_5\sum_{m=1}^{\infty}\frac{b_m}{\mu_m^{\delta}}\sum_{k=1}^{2^N}\eta_k\sum_{j=m_0+1}^{\infty}\frac{V(j)}{j!}\left(\frac{-se^{i\gamma_k}}{\mu_m^{1/2A}}\right)^j,$$

for $0 < s \leqslant e^{-(c+\frac{1}{2}\pi)}$, and $0 < s < e^{-c}\mu_1^{1/2A}$, the latter being sufficient for the interchange in the order of summation.

Now define the functions L and M by

$$(2.20)\qquad L(v) = \sum_{j=m_0+1}^{\infty}\frac{V(j)}{j!}v^j,\qquad |v| < e^{-c},$$

and

$$(2.21)\qquad M(v) = c_5\sum_{k=1}^{2^N}\eta_k L(-e^{i\gamma_k}v),\qquad |v| < e^{-c}.$$

We see that

$$(2.22)\qquad M(v) = O(|v|^{m_0+1}),\qquad \text{as}\quad |v|\to 0.$$

From (2.19) we have

$$(2.23)\qquad \sum_{m=1}^{\infty}a_m e^{-\lambda_m^{1/2A}s}-R(s) = \sum_{m=1}^{\infty}b_m\mu_m^{-\delta}M(s\mu_m^{-1/2A}),$$

for $0 < s < c'$, say. We define the function L_1 by the relation

$$(2.24)\qquad L_1(v) = \frac{1}{2\pi i}\int_{-(m_0+\frac{1}{2})}\Gamma(z)V(-z)v^{-z}dz,\qquad v\in C-\{v\leqslant -e^{-c}\},$$

C denoting the complex plane. The integral is absolutely convergent, and L_1 is regular. Further

$$L_1(v) = L(-v),\qquad \text{for}\qquad 0 < v < e^{-(c+\frac{1}{2}\pi)},$$

if we note the third restriction on m_0 in (2.5).

Thus $L_1(-v)$ gives the analytic continuation of $L(v)$ in $C-\{v\geqslant e^{-c}\}$. Hence

$$(2.25)\quad M(v) = c_5\frac{1}{2\pi i}\int_{-(m_0+\frac{1}{2})}\Big(\sum_{k=1}^{2^N}\eta_k e^{-i\gamma_k z}\Big)\Gamma(z)V(-z)v^{-z}dz,\qquad \operatorname{Re} v > 0$$
$$= c_5\frac{1}{2\pi i}\int_{-(m_0+\frac{1}{2})}(2i)^N\prod_{\nu=1}^{N}\sin[\pi(-\beta_\nu-a_\nu z/2A)]\times$$
$$\times\Gamma(z)V(-z)v^{-z}dz\qquad \text{(by (2.16))}$$
$$= c_5\frac{1}{2\pi i}\int_{-(m_0+\frac{1}{2})}(-1)^N(2i)^N\frac{\Delta(\delta-z/2A)}{\Delta(z/2A)}\pi^N\Gamma(z)v^{-z}dz,$$
$$\text{(by (2.11) and (2.18))}$$
$$= \frac{1}{2\pi i}\int_{-(m_0+\frac{1}{2})}\frac{\Delta(\delta-z/2A)}{\Delta(z/2A)}\Gamma(z)v^{-z}dz,\qquad \operatorname{Re} v > 0,$$

since $c_5 = (-2\pi i)^{-N}$ as in (2.17).

From (2.25), (2.23), and (2.22) it follows that identity (2.23) holds for $\operatorname{Re} s > 0$, hence also (2.8), and the first part of the lemma is proved.

To prove the second part of the lemma, suppose that (2.8) holds for $s > 0$ and for some integer $\varrho > 0$. Then, since

$$\left(-\frac{1}{s}\frac{d}{ds}\right)^{\varrho}(s^{-1-z}) = \frac{2^{\varrho}\Gamma(\frac{1}{2}+\frac{1}{2}z+\varrho)s^{-1-2\varrho-z}}{\Gamma(\frac{1}{2}+\frac{1}{2}z)},$$

we have

$$(2.26)\quad \left(-\frac{1}{s}\frac{d}{ds}\right)^{\varrho}\left\{s^{-1}\sum_{m=1}^{\infty} b_m\mu_m^{-\delta} M(s\mu_m^{-1/2A})\right\}$$

$$= \frac{1}{2\pi i}\sum_{m=1}^{\infty}\frac{b_m}{\mu_m^{\delta}}2^{\varrho}\int_{-(m_0+\frac12)} \mu_m^{z/2A}\Gamma(z)\frac{\Delta(\delta-z/2A)}{\Delta(z/2A)}\frac{\Gamma(\frac12+\frac12 z+\varrho)}{\Gamma(\frac12+\frac12 z)}s^{-1-2\varrho-z}dz$$

$$= \frac{2^{\varrho}}{2\pi i}\int_{-(m_0+\frac12)}\Gamma(z)\frac{\Delta(\delta-z/2A)}{\Delta(z/2A)}\frac{\Gamma(\frac12+\frac12 z+\varrho)}{\Gamma(\frac12+\frac12 z)}\psi(\delta-z/2A)s^{-1-2\varrho-z}dz,$$

for $s>0$. On the other hand, from first principles, we have

$$(2.27)\quad \sum_{m=1}^{\infty} a_m e^{-\lambda_m^{1/2A}s}$$

$$= \frac{1}{2\pi i}\int_{d}\Gamma(z)s^{-z}\varphi(z/2A)\,dz,\qquad \operatorname{Re} s>0,\ d>0,\ d>a\cdot 2A$$

$$= R(s)+\frac{1}{2\pi i}\int_{-(m_0+\frac12)}\Gamma(z)s^{-z}\varphi(z/2A)\,dz,$$

where $\sum_{m=1}^{\infty}|a_m|\lambda_m^{-a}<\infty$, and $R(s)$ is defined as in (2.6). And (2.27) implies that

$$(2.28)\quad \left(-\frac{1}{s}\frac{d}{ds}\right)^{\varrho}\left(\frac{1}{s}\sum_{m=1}^{\infty}a_m e^{-\lambda_m^{1/2A}s}\right)$$

$$= \left(-\frac{1}{s}\frac{d}{ds}\right)^{\varrho}\left(\frac{R(s)}{s}\right)+\frac{2^{\varrho}}{2\pi i}\int_{-(m_0+\frac12)}\frac{\Gamma(z)\Gamma(\frac12+\frac12 z+\varrho)}{\Gamma(\frac12+\frac12 z)}s^{-1-2\varrho-z}\varphi(z/2A)\,dz.$$

From (2.28), (2.26), (2.8), and the conditions for the uniqueness of the Fourier transform of a function, it follows that

$$\varphi(z/2A) = \frac{\Delta(\delta-z/2A)}{\Delta(z/2A)}\psi(\delta-z/2A),$$

which is (1.4). By the first part of the lemma, this implies (2.7), and the second part of the lemma is proved.

LEMMA 2. *Given* $\varphi(s)=\sum_{m=1}^{\infty}a_m\lambda_m^{-s}$, $\psi(s)=\sum_{m=1}^{\infty}b_m\mu_m^{-s}$ *(as well as* A *and* Δ*) as in (2.1) and (2.2),* $M(s)$ *as in (2.4) with* m_0 *as in (2.5), and* $R(s)$ *as (2.6), let*

$$(2.29)\quad \sum_{m=1}^{\infty}a_m e^{-\lambda_m^{1/2A}s} = R(s)+\sum_{m=1}^{\infty}b_m\mu_m^{-\delta}M(s\mu_m^{-1/2A}),\quad \textit{for}\quad \operatorname{Re} s>0.$$

Then we have the identity

$$(2.30)\quad \frac{1}{\Gamma(\varrho+1)}\sum_{\lambda_m^{1/A}\le x}{}' (x-\lambda_m^{1/A})^{\varrho}a_m$$

$$= \frac{1}{2\pi i}\int_{\frac12\mathscr{C}}\frac{\Gamma(z)}{\Gamma(z+1+\varrho)}\varphi(z/A)x^{z+\varrho}dz+\sum_{m=1}^{\infty}b_m\mu_m^{-\delta-\varrho/A}g_{\varrho}(x\mu_m^{1/A}),$$

where $x>0$, ϱ *integral,* $\varrho\geqslant 0$, $\varrho>m_0+1+A\delta$, $\varrho>\frac12 m_0+\frac14$, *(the dash on the sum on the left-hand side indicating that when* $\varrho=0$ *and* $x=\lambda_m^{1/A}$, a_m *is to be multiplied by* $\frac12$,*)* $\mathscr{C}$ *is the curve in the definition of* $R(s)$ *in (2.6), and*

$$(2.31)\quad g_{\varrho}(y) = \frac{1}{2\pi i}\int_{A\delta+\frac12(m_0+\frac12)}\frac{\Gamma(A\delta-z)\Delta(z/A)}{\Gamma(A\delta+1+\varrho-z)\Delta(\delta-z/A)}y^{\varrho+A\delta-z}dz,$$

for $y>0$, *the integral converging absolutely for* ϱ *fulfilling the above conditions.*

Conversely, given φ, ψ *(as well as* A, Δ*) as in (2.1), and (2.2), if (2.30) holds for* $x>0$, $\varrho\geqslant 0$, ϱ *integral,* $\varrho>m_0+1+A\delta$, $\varrho>\frac12 m_0+\frac14$, *where* m_0 *is defined as in (2.5), then (2.29) holds for* $\operatorname{Re} s>0$, *with* M *and* R *defined as in (2.4) and (2.6).*

Proof. By differentiation of (2.29), we have

$$(2.29)'\quad \left(-\frac{1}{s}\frac{d}{ds}\right)^{\varrho}\left(\frac{1}{s}\sum_{m=1}^{\infty}a_m e^{-\lambda_m^{1/2A}s}\right)$$

$$= \left(-\frac{1}{s}\frac{d}{ds}\right)^{\varrho}\left(\frac{1}{s}R(s)+\frac{1}{s}\sum_{m=1}^{\infty}\frac{b_m}{\mu_m^{\delta}}M(s\mu_m^{-1/2A})\right),$$

for any integer $\varrho>0$ and $\operatorname{Re} s>0$. And by (2.26) we have

$$(2.32)\quad \left(-\frac{1}{s}\frac{d}{ds}\right)^{\varrho}\left(\frac{1}{s}\sum_{m=1}^{\infty}b_m\mu_m^{-\delta}M(s\mu_m^{-1/2A})\right)$$

$$= \frac{2^{\varrho}}{2\pi i}\sum_{m=1}^{\infty}b_m\mu_m^{-\delta}\int_{-(m_0+\frac12)}\mu_m^{z/2A}\frac{\Delta(\delta-z/2A)}{\Delta(z/2A)}\Gamma(z)\frac{\Gamma(\frac12+\frac12 z+\varrho)}{\Gamma(\frac12+\frac12 z)}s^{-1-2\varrho-z}dz.$$

Multiplying throughout by $(2\pi i)^{-1}e^{sx^{1/2A}}$, with $x>0$, and integrating along the line $\sigma+it$, with a fixed $\sigma>0$, $-\infty<t<\infty$, we get

$$(2.33)\qquad \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} e^{sx^{1/2A}}\left(-\frac{1}{s}\frac{d}{ds}\right)^{\varrho}\left(\frac{1}{s}\sum_{m=1}^{\infty} b_m\mu_m^{-\delta}M(s\mu_m^{-1/2A})\right)ds$$

$$= \frac{2^{\varrho}}{(2\pi i)^2}\int_{\sigma x^{1/2A}}\int_{-(m_0+\frac12)} e^{s}\sum_{m=1}^{\infty} b_m\mu_m^{-\delta+z/2A}\Gamma(z)\frac{\Gamma(\frac12+\frac12 z+\varrho)}{\Gamma(\frac12+\frac12 z)}\frac{\Delta(\delta-z/2A)}{\Delta(z/2A)}\times$$
$$\times s^{-1-2\varrho-z}x^{z/2A}x^{2\varrho/2A}\,ds\,dz$$

$$= \frac{2^{\varrho}}{(2\pi i)^2}\sum_{m=1}^{\infty} b_m\mu_m^{-\delta}\int_{-(m_0+\frac12)}\int_{\sigma x^{1/2A}} e^{s}\mu_m^{z/2A}\Gamma(z)\frac{\Gamma(\frac12+\frac12 z+\varrho)}{\Gamma(\frac12+\frac12 z)}\frac{\Delta(\delta-z/2A)}{\Delta(z/2A)}\times$$
$$\times s^{-1-2\varrho-z}x^{(z+2\varrho)/2A}\,dz\,ds$$

provided that the interchange of the summation and integrations is justified. We shall see that this is so, if

$$(2.34)\qquad 2A\delta+m_0+\tfrac12 > 2A\sigma_b^*,\quad\text{and}\quad m_0+A\delta+1<\varrho,\quad \tfrac12 m_0+\tfrac14<\varrho.$$

First we note that the series inside the integral sign in (2.33) converges absolutely on the line $z = -(m_0+\frac12)+iy$, since

$$\sum_{m=1}^{\infty}|b_m|\mu_m^{-\delta-(m_0+\frac12)/2A}<\infty \quad\text{for}\quad 2A\delta+m_0+\tfrac12>2A\sigma_b^*.$$

We are thus concerned with the *absolute* convergence of the integral

$$(2.35)\qquad \int_{\sigma_0}\int_{-(m_0+\frac12)}\Gamma(z)\frac{\Gamma(\frac12+\frac12 z+\varrho)}{\Gamma(\frac12+\frac12 z)}\frac{\Delta(\delta-z/2A)}{\Delta(z/2A)}s^{-1-2\varrho-z}\,ds\,dz,$$

where $\sigma_0 = \sigma x^{1/2A}>0$, $z = -(m_0+\frac12)+iy$, and $s = \sigma_0+i\tau = re^{i\theta}$, say. Since

$$|\Gamma(s)|\sim(2\pi)^{1/2}e^{-\frac12\pi|\tau|}|\tau|^{\sigma_0-\frac12},$$

as $|\tau|\to\infty$, for fixed σ_0, we have

$$(2.36)\qquad \frac{\Gamma(z)\Gamma(\frac12+\frac12 z+\varrho)\Delta(\delta-z/2A)}{\Gamma(\frac12+\frac12 z)\Delta(z/2A)}\sim \varkappa e^{-\frac12\pi|y|}|y|^{-m_0-1}|y|^{\varrho+A\delta+m_0+\frac12},$$

as $|y|\to\infty$, where $\varkappa$ is a constant; while

$$|s^{-1-2\varrho-z}| = r^{m_0-2\varrho-\frac12}e^{\theta y},\qquad \theta = \operatorname{arctg}(\tau/\sigma_0).$$

Let $B_0>0$, and be chosen sufficiently large. We consider the double integral (2.35) separately in the following cases: (i) $|y|\leqslant B_0$, $-\infty<\tau<+\infty$; (ii) $y>B_0$, $\tau>B_0$; (iii) $y>B_0$, $|\tau|\leqslant B_0$; (iv) $y>B_0$, $\tau<-B_0$; (v) $y<-B_0$, $\tau>B_0$; (vi) $y<-B_0$, $|\tau|\leqslant B_0$, and (vii) $y<-B_0$, $\tau<-B_0$.

It will be sufficient to prove the absolute convergence in cases (i), (ii), and (iii), since the other cases are similar.

In case (i), the integrand is $O(r^{m_0-2\varrho-\frac12})$, $r\geqslant\sigma_0>0$, and the integral is absolutely convergent for $\varrho>\frac12(m_0+\frac12)$.

In case (ii) we have $\theta = \frac12\pi-\operatorname{arctg}(\sigma_0/\tau) = \frac12\pi-(\sigma_0/\tau)+O(|\sigma_0/\tau|^3) = \frac12\pi-(\sigma_0/\tau)(1+\omega)$, say, where $|\omega|<\frac12$ (since B_0 is sufficiently large). In view of (2.36) it will be sufficient to consider the convergence of the integral

$$\int_{\tau=B_0}^{\infty}\int_{y=B_0}^{\infty}e^{-\frac12\pi y+\theta y}y^d\tau^c\,d\tau\,dy = \int_{\tau=B_0}^{\infty}\int_{y=B_0}^{\infty}e^{-\frac{\sigma_0}{\tau}(1+\omega)y}y^d\tau^c\,d\tau\,dy$$
$$= \int_{\tau=B_0}^{\infty}\int_{y=B_0\sigma_0(1+\omega)/\tau}^{\infty}e^{-y}y^d\tau^c\left(\frac{\tau}{\sigma_0(1+\omega)}\right)^{d+1}d\tau\,dy,$$

$d = \varrho+A\delta-\frac12$, $c = m_0-2\varrho-\frac12$. Since $1+\omega>\frac12$ and $\sigma_0>0$, this is less than a constant multiple of the integral

$$\int_{\tau=B_0}^{\infty}\tau^{c+d+1}\int_{y=(B_0\sigma_0)/2\tau}^{\infty}e^{-y}y^d\,d\tau\,dy,$$

which is convergent if $d>-1$ and $c+d+1<-1$ (or $m_0+A\delta+1<\varrho$). If $d\leqslant-1$, we need the condition $c<-1$ (or $\varrho>\frac12(m_0+\frac12)$) for the convergence. For the y-integral can be split up into two, the first going from $(B_0\sigma_0)/2\tau$ to $\frac12\sigma_0$, while the second goes from $\frac12\sigma_0$ to ∞. Thus we have to consider

$$\int_{\tau=B_0}^{\infty}\tau^{c+d+1}\left(\int_{B_0\sigma_0/2\tau}^{\frac12\sigma_0}+\int_{\frac12\sigma_0}^{\infty}\right)e^{-y}y^d\,d\tau\,dy$$
$$= \int_{\tau=B_0}^{\infty}\tau^{c+d+1}\int_{y=B_0\sigma_0/2\tau}^{\frac12\sigma_0}e^{-y}y^d\,d\tau\,dy+O(1),\qquad\text{if } c+d+1<-1$$
$$= O\left(\int_{B_0}^{\infty}(1+|\log\tau|)\tau^c\,d\tau\right) = O(1),\qquad\text{if } c<-1.$$

In case (iii) we have $|\tau|\leqslant B_0$, which implies that $|\theta|\leqslant\theta_0<\frac12\pi$, where $\theta_0 = \theta_0(B_0)$, and we are led to consider the integral

$$\int_{\tau=-B_0}^{B_0}\int_{y=B_0}^{\infty}e^{-\frac12\pi y+\theta y}y^d r^c\,d\tau\,dy<\infty,$$

since $r = |\sigma_0+i\tau|\geqslant\sigma_0>0$.

Case (vi) is similar to case (iii) while cases (iv), (v), and (vii) are similar to case (ii).

Altogether we see that if $\varrho > m_0+A\delta+1$, and $\varrho > \frac12(m_0+\frac12)$, and $2A\delta+m_0+\frac12 > 2A\sigma_b^*$, then the interchange of the integrations and the summation in (2.33) is justified, and we obtain therefore

(2.37)
$$\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} e^{sx^{1/2A}}\left(-\frac{1}{s}\frac{d}{ds}\right)^{\varrho}\left(\frac{1}{s}\sum_{m=1}^{\infty} b_m\mu_m^{-\delta}M(s\mu_m^{-1/2A})\right)ds$$
$$=\frac{1}{2\pi i}\sum_{m=1}^{\infty} b_m\mu_m^{-\delta}2^{\varrho}x^{\varrho/A}\int_{-(m_0+\frac12)}\frac{\Gamma(z)\Gamma(\frac12+\frac12 z+\varrho)}{\Gamma(\frac12+\frac12 z)}\,\frac{\Delta(\delta-z/2A)}{\Delta(z/2A)}\,\mu_m^{z/2A}x^{z/2A}\times$$
$$\times\frac{1}{2\pi i}\int_{\sigma x^{1/2A}} e^{s}s^{-1-2\varrho-z}\,ds\,dz$$
$$=\frac{1}{2\pi i}\sum_{m=1}^{\infty} b_m\mu_m^{-\delta}2^{\varrho}x^{\varrho/A}\int_{-(m_0+\frac12)}\frac{\Gamma(z)}{\Gamma(\frac12+\frac12 z)}\,\frac{\Gamma(\frac12+\frac12 z+\varrho)}{\Gamma(z+1+2\varrho)}\,\frac{\Delta(\delta-z/2A)}{\Delta(z/2A)}\times$$
$$\times\mu_m^{z/2A}x^{z/2A}\,dz,\qquad \varrho>\tfrac12(m_0+\tfrac12)$$
$$=2^{-\varrho}\sum_{m=1}^{\infty} b_m\mu_m^{-(\delta+\varrho/A)}g_{\varrho}\big((\mu_m x)^{1/A}\big),$$

since

$$\frac{1}{2\pi i}\int_{-(m_0+\frac12)}\frac{\Gamma(z)}{\Gamma(\frac12+\frac12 z)}\,\frac{\Gamma(\frac12+\frac12 z+\varrho)}{\Gamma(z+1+2\varrho)}\,\frac{\Delta(\delta-z/2A)}{\Delta(z/2A)}\,y^{\frac12 z}dz,\qquad y=(\mu_m x)^{1/A}$$
$$=\frac{2^{-1-2\varrho}}{2\pi i}\int_{-(m_0+\frac12)}\frac{\Gamma(\frac12 z)}{\Gamma(\frac12 z+1+\varrho)}\,\frac{\Delta(\delta-z/2A)}{\Delta(z/2A)}\,y^{\frac12 z}dz,$$

since $\Gamma(z)\pi^{1/2}2^{1-z}=\Gamma(\frac12 z)\Gamma(\frac12 z+\frac12)$

$$=\frac{2^{-2\varrho}y^{-\varrho}}{2\pi i}\int_{A\delta+\frac12(m_0+\frac12)}\frac{\Gamma(A\delta-z)\Delta(z/A)}{\Gamma(A\delta+1+\varrho-z)\Delta(\delta-z/A)}\,y^{A\delta+\varrho-z}dz$$
$$=2^{-2\varrho}y^{-\varrho}g_{\varrho}(y),\qquad \varrho>m_0+1+A\delta.$$

On the other hand, for any fixed $\sigma>0$, we have

(2.38)
$$\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} e^{sx^{1/2}}\left(-\frac{1}{s}\frac{d}{ds}\right)^{\varrho}\left(\frac{1}{s}R(s)\right)ds$$
$$=\frac{1}{2\pi i}\int_{\sigma} e^{sx^{1/2}}\left(-\frac{1}{s}\frac{d}{ds}\right)^{\varrho}\left(\frac{1}{s}\,\frac{1}{2\pi i}\int_{\mathscr{C}}\Gamma(z)\varphi(z/2A)s^{-z}dz\right)ds$$
$$=\frac{2^{\varrho}}{2\pi i}\int_{\sigma} e^{sx^{1/2}}\frac{1}{2\pi i}\int_{\mathscr{C}}\frac{\Gamma(z)\Gamma(\frac12+\frac12 z+\varrho)}{\Gamma(\frac12+\frac12 z)}\varphi(z/2A)s^{-1-2\varrho-z}\,dz\,ds$$

$$=\frac{\pi^{-1/2}2^{\varrho-1}x^{\varrho}}{2\pi i}\int_{\mathscr{C}}\Gamma(\tfrac12 z)2^{z}\Gamma(\tfrac12+\tfrac12 z+\varrho)x^{z/2}\varphi(z/2A)\,\frac{1}{2\pi i}\int_{\sigma} e^{s}s^{-1-2\varrho-z}dz,$$
$$\varrho>\tfrac12(m_0+\tfrac12)$$
$$=\frac{\pi^{-1/2}2^{\varrho-1}x^{\varrho}}{2\pi i}\int_{\mathscr{C}}\frac{\Gamma(\frac12 z)2^{z}\Gamma(\frac12+\frac12 z+\varrho)x^{z/2}\varphi(z/2A)}{\Gamma(z+2\varrho+1)}\,dz$$
$$=\frac{2^{-\varrho}}{2\pi i}\int_{\frac12\mathscr{C}}\frac{\Gamma(z)\varphi(z/A)}{\Gamma(z+\varrho+1)}\,x^{z+\varrho}dz.$$

Finally

(2.39)
$$\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} e^{sx^{1/2A}}\left(-\frac{1}{s}\frac{d}{ds}\right)^{\varrho}\left(\frac{1}{s}\sum_{m=1}^{\infty} a_m e^{-\lambda_m^{1/2A}s}\right)ds,\qquad x>0,\ \sigma>0,$$
$$=\sum_{m=1}^{\infty} a_m\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} e^{sx^{1/2A}}\left(-\frac{1}{s}\frac{d}{ds}\right)^{\varrho}\left(\frac{1}{s}e^{-s\lambda_m^{1/2A}}\right)ds$$
$$=\frac{2^{-\varrho}}{\Gamma(\varrho+1)}\sum_{\lambda_m^{1/A}\le x^{1/A}}{}' a_m(x^{1/A}-\lambda_m^{1/A})^{\varrho},$$

for ϱ integral, $\varrho\geq 0$, as in [2].

Now (2.39), (2.38), and (2.37) yield (2.30), and the first part of the lemma is proved.

To prove the second part, we assume (2.30) given for $x>0$ and for some integer $\varrho\geq 0$, which is such that $\varrho>m_0+1+A\delta$, $\varrho>\frac12(m_0+\frac12)$, where m_0 satisfies the restrictions of (2.5). We then multiply it throughout by $e^{s\sqrt{x}}x^{-1/2}$ where $\operatorname{Re}s>0$, and integrate relative to x from 0 to ∞. The left-hand side gives

(2.40)
$$\frac{1}{\Gamma(\varrho+1)}\int_{0}^{\infty} e^{-s\sqrt{x}}x^{-1/2}\sum_{\lambda_m^{1/A}\le x} a_m(x-\lambda_m^{1/A})^{\varrho}dx,\qquad \varrho\geq 0,$$
$$=\frac{1}{\Gamma(\varrho+1)}\sum_{m=1}^{\infty} a_m\int_{\lambda_m^{1/A}}^{\infty} e^{-s\sqrt{x}}x^{-1/2}(x-\lambda_m^{1/A})^{\varrho}dx$$
$$=\frac{2^{2}\varrho}{\Gamma(\varrho+1)}\sum_{m=1}^{\infty} a_m\frac{1}{s}\int_{\lambda_m^{1/2A}}^{\infty} xe^{-sx}(x^2-\lambda_m^{1/A})^{\varrho-1}dx,$$

by partial integration

$$=\frac{2^{2}\varrho}{\Gamma(\varrho+1)}\sum_{m=1}^{\infty} a_m\left(-\frac{1}{s}\frac{d}{ds}\right)\int_{\lambda_m^{1/2A}}^{\infty} e^{-sx}(x^2-\lambda_m^{1/A})^{\varrho-1}dx$$

$$= 2^{\varrho+1}\sum_{m=1}^{\infty} a_m\left(-\frac{1}{s}\frac{d}{ds}\right)^{\varrho}\int_{\lambda_m^{1/2A}}^{\infty} e^{-sx}dx$$

$$(2.41)\qquad = 2^{\varrho+1}\left(-\frac{1}{s}\frac{d}{ds}\right)^{\varrho}\frac{1}{s}\sum_{m=1}^{\infty} a_m e^{-\lambda_m^{1/2A}s}.$$

The first term on the right-hand side of (2.30) gives

$$(2.42)\qquad \int_0^{\infty} e^{-sx^{1/2}}x^{-1/2}\frac{1}{2\pi i}\int_{\frac{1}{2}\mathscr{C}}\frac{\Gamma(z)}{\Gamma(z+1+\varrho)}\varphi(z/A)x^{z+\varrho}\,dz\,dx$$

$$= \frac{1}{2\pi i}\int_{\frac{1}{2}\mathscr{C}}\frac{\Gamma(z)}{\Gamma(z+1+\varrho)}\varphi(z/A)\int_0^{\infty} e^{-sx^{1/2}}x^{-1/2}x^{z+\varrho}\,dx\,dz,$$

if $\varrho > \frac{1}{2}(m_0+\frac{1}{2})$,

$$= \frac{1}{2\pi i}\int_{\frac{1}{2}\mathscr{C}}\frac{\Gamma(z)\Gamma(2z+2\varrho+1)}{\Gamma(z+1+\varrho)}\,2s^{-2z-2\varrho-1}\varphi(z/A)\,dz$$

$$= \frac{1}{2\pi i}\int_{\mathscr{C}}\frac{\Gamma(z)\Gamma(\frac{1}{2}z+\varrho+\frac{1}{2})}{\Gamma(\frac{1}{2}+\frac{1}{2}z)}\,s^{-z-2\varrho-1}\varphi(z/2A)\,2^{2\varrho+1}dz$$

$$= 2^{\varrho+1}\left(-\frac{1}{s}\frac{d}{ds}\right)^{\varrho}\left(\frac{1}{s}R(s)\right),\qquad \text{because of (2.6).}$$

The second term on the right-hand of (2.30) gives

$$(2.43)\qquad \int_0^{\infty} e^{-sx^{1/2}}x^{-1/2}\Big(\sum_{m=1}^{\infty} b_m\mu_m^{-\delta-\varrho/A}g_\varrho(x\mu_m^{1/A})\Big)\,dx,\qquad \operatorname{Re}s = \sigma > 0,$$

$$= \sum_{m=1}^{\infty} b_m\mu_m^{-\delta-\varrho/A}\int_0^{\infty} 2e^{-sx}g_\varrho(x^2\mu_m^{1/A})\,dx,$$

provided that the interchange of the integration and the summation is justified.

Now

$$g_\varrho(x) = \int_{\xi-i\infty}^{\xi+i\infty} G_\varrho(z)x^{\varrho+A\delta-z}dz,\qquad z = \xi+iy,\ \ \xi = A\delta+\tfrac{1}{2}(m_0+\tfrac{1}{2}),$$

and

$$G_\varrho(z) = \frac{\Gamma(A\delta-z)\Delta(z/A)}{\Gamma(A\delta+1+\varrho-z)\Delta(\delta-z/A)}.$$

The first integral in (2.43) is absolutely convergent, provided that

$$\int_{x=0}^{\infty}\int_{y=-\infty}^{\infty} e^{-\sigma x}\Big(\sum_{m=1}^{\infty}|b_m|\mu_m^{-\xi/A}\Big)|G_\varrho(z)|x^{2(\varrho+A\delta-\xi)}dy\,dx < \infty.$$

If B_0 is sufficiently large, and $|y| > B_0 > 0$, then

$$|G_\varrho(z)| = O(|y|^{-1-\varrho-A\delta+2\xi}),$$

while $\sum_{m=1}^{\infty}|b_m|\mu_m^{-\xi/A} < \infty$, for $m_0+\frac{1}{2} > 2A\sigma_b^* - 2A\delta$, and

$$\int_{x=0}^{\infty}\int_{y=B_0}^{\infty} e^{-\sigma x}|y|^{-1-\varrho-A\delta+2\xi}x^{2(\varrho+A\delta-\xi)}dy\,dx < \infty,$$

for $\sigma > 0$, $\varrho > A\delta+m_0+\frac{1}{2}$, $\varrho > \frac{1}{2}(m_0-\frac{1}{2})$. Similarly also

$$\int_{x=0}^{\infty}\int_{y=-\infty}^{-B_0} < \infty.$$

Finally the integral

$$\int_{x=0}^{\infty}\int_{y=-B_0}^{B_0} e^{-\sigma x}|G_\varrho(z)|x^{2(\varrho+A\delta-\xi)}dy\,dx < \infty,$$

for $\sigma > 0$, provided that the line $\xi+iy$, $|y| \leqslant B_0$ is free from the poles of $G_\varrho(z)$, which is the case if $2A\delta+m_0+\frac{1}{2} > 2A\max_\nu(\operatorname{Re}(-\beta_\nu/\alpha_\nu))$. Hence (2.43) is valid.

Now, for $s > 0$,

$$\int_0^{\infty} e^{-x}g_\varrho(ax^2)dx$$

$$= \frac{1}{2\pi i}\int_0^{\infty} e^{-x}dx\int_{\xi-i\infty}^{\xi+i\infty}\frac{\Gamma(A\delta-z)\Delta(z/A)}{\Gamma(A\delta+1+\varrho-z)\Delta(\delta-z/A)}(ax^2)^{\varrho+A\delta-z}dz,$$

with $\xi = A\delta+\frac{1}{2}(m_0+\frac{1}{2})$, $a = \mu_m^{1/A}s^{-2}$. The right-hand side is

$$= \frac{1}{2\pi i}\int_{\xi-i\infty}^{\xi+i\infty}\frac{\Gamma(A\delta-z)\Delta(z/A)}{\Gamma(A\delta+1+\varrho-z)\Delta(\delta-z/A)}a^{\varrho+A\delta-z}\int_0^{\infty} e^{-x}x^{2(\varrho+A\delta-z)}dx\,dz,$$

since $\varrho > \frac{1}{2}(m_0+\frac{1}{2})$

$$= \frac{1}{2\pi i}\int_{-\frac{1}{2}(m_0+\frac{1}{2})}\frac{\Gamma(z)\Delta(\delta-z/A)}{\Gamma(1+\varrho+z)\Delta(z/A)}a^{\varrho+z}\Gamma(2z+2\varrho+1)\,dz$$

$$= \frac{2^{2\varrho+1}}{2\pi i}\int_{-(m_0+\frac{1}{2})}\frac{\Gamma(z)\Delta(\delta-z/2A)}{\Gamma(\frac{1}{2}+\frac{1}{2}z)\Delta(z/2A)}\Gamma(\varrho+\tfrac{1}{2}+\tfrac{1}{2}z)a^{\varrho+\frac{1}{2}z}\tfrac{1}{2}\,dz,$$

$$a = \mu_m^{1/A}s^{-2}.$$

This taken together with (2.43) gives, for $s>0$, hence for $\operatorname{Re} s>0$,

$$(2.44)\quad \int_0^\infty e^{-sx^{1/2}} x^{-1/2}\Big(\sum_{m=1}^\infty b_m \mu_m^{-\delta-\varrho/A} g_\varrho(x\mu_m^{1/A})\Big)\,dx$$

$$= \sum_{m=1}^\infty b_m \mu_m^{-\delta-\varrho/A}\cdot\frac{2}{s}\int_0^\infty e^{-x} g_\varrho(\mu_m^{1/A}x^2 s^{-2})\,dx$$

$$= \sum_{m=1}^\infty b_m \mu_m^{-\delta-\varrho/A}\frac{2^{2\varrho+1}}{2\pi i}\int_{-(m_0+\frac12)}\frac{\Gamma(z)\,\Delta(\delta-z/2A)\,\Gamma(\frac12+\frac12 z+\varrho)}{\Delta(z/2A)\,\Gamma(\frac12+\frac12 z)}\,\mu_m^{(\frac12 z+\varrho)/A} s^{-1-2\varrho-z}\,dz$$

$$= 2^{\varrho+1}\sum_{m=1}^\infty b_m\mu_m^{-\delta}\frac{1}{2\pi i}\int_{-(m_0+\frac12)}\frac{\Gamma(z)\,\Delta(\delta-z/2A)}{\Delta(z/2A)}\,\mu_m^{z/2A}\Big(-\frac1s\frac{d}{ds}\Big)^\varrho s^{-1-z}\,dz$$

$$= 2^{\varrho+1}\Big(-\frac1s\frac{d}{ds}\Big)^\varrho\Big\{\frac1s\sum_{m=1}^\infty b_m\mu_m^{-\delta} M(s\mu_m^{-1/2A})\Big\}.$$

Now (2.44) and (2.43), together with (2.42), (2.41), and (2.30) give (2.29)′ hence also (2.29). This completes the proof of Lemma 2.

Lemmas 1 and 2 yield

THEOREM 1. *Functional equation* (1.4), *identity* (2.7), *and identity* (2.30) *are equivalent.*

Remarks.

(i) Identity (2.30) is not new. It has been proved by Chandrasekharan and Narasimhan (see (4.6) of [3] and formula (4) of [4]), and used by them to obtain arithmetical results. It may be remarked that Theorem 4.1 and Remark (5.5) of their paper [3] yield, for example, Rankin's result on the Ramanujan function $\tau(n)$, namely $\sum_{k=1}^n \tau^2(k) = c\cdot n^{12} + O(n^{12-\frac25})$, as a consequence of Rankin's functional equation for the series $\sum_{k=1}^\infty \tau^2(k)k^{-s}$ ([6], pp. 174–182).

(ii) Theorem 1 has been proved in the case $\Delta(s)=\Gamma(s)$ by Chandrasekharan and Narasimhan [2].

(iii) Bochner (in (149) of [1]) has a 'modular relation', which is equivalent to functional equation (1.4) in case $\Delta(s)=\Gamma(\frac12 s)^{r_1}\Gamma(s)^{r_2}$, where r_1+2r_2 is the degree of an algebraic number field, and which resembles a theta-relation. It does not, however, yield Hecke's theta-relation, as Bochner himself remarks.

(iv) If $\Delta(s)=\Gamma(\frac12 s)$, $m_0=0$, $\lambda_m=\mu_m=\pi^{1/2}m$, identity (2.7) yields Siegel's partial-fraction formula (1.2). If $\Delta(s)=\Gamma(\frac12 s)^{r_1}\Gamma(s)^{r_2}$, then (2.7) yields, on taking $m_0=-1$, Walfisz's identity (1.3), for which an alternative proof has been given by Joris [7].

§ 3. Determination of A and of δ.

THEOREM 2. *Suppose that* $\varphi(s)=\sum_{m=1}^\infty a_m\lambda_m^{-s}$, *and* $\psi(s)=\sum_{m=1}^\infty b_m\mu_m^{-s}$ *satisfy the functional equation*

$$(3.1)\qquad \Delta(s)\varphi(s)=\Delta(\delta-s)\psi(\delta-s),$$

where

$$\Delta(s)=\prod_{k=1}^N\Gamma(\alpha_k s+\beta_k),\quad \alpha_k>0,\quad \beta_k\ \textit{complex},\quad \delta\ \textit{real},$$

(3.1)′

$$A=\sum_{k=1}^N\alpha_k,\qquad B=\sum_{k=1}^N(\beta_k-\tfrac12),$$

and suppose further that $\varphi(s)=\sum_{m=1}^\infty a_m\lambda_m^{-s}$ *and* $\psi_1(s)=\sum_{m=1}^\infty d_m\nu_m^{-s}$ (*where* (ν_m) *is a strictly increasing sequence of positive numbers diverging to* $+\infty$, *and the series* $\sum_{m=1}^\infty d_m\nu_m^{-s}$ *admits a finite abscissa of absolute convergence*) *satisfy the equation*

$$(3.2)\qquad \Delta_1(s)\varphi(s)=\Delta_1(\delta_1-s)\psi_1(\delta_1-s),$$

where

$$\Delta_1(s)=\prod_{k=1}^{N'}\Gamma(\alpha_k' s+\beta_k'),\quad \alpha_k'>0,\quad \beta_k'\ \textit{complex},\quad \delta_1\ \textit{real},$$

(3.2)′

$$A'=\sum_{k=1}^{N'}\alpha_k',\qquad B'=\sum_{k=1}^{N'}(\beta_k'-\tfrac12).$$

Then

$$(3.3)\qquad \delta=\delta_1\quad\textit{and}\quad A=A'.$$

Proof. If $\sigma<\delta-\sigma_b^*$, we have $\psi(\delta-s)=\sum_{m=1}^\infty b_m\mu_m^{s-\delta}$, the series converging absolutely, and the function $\psi(\delta-\sigma-it)$ is a Bohr almost periodic function of t which cannot therefore tend to zero as $|t|\to\infty$. Hence, if $\sigma<\delta-\sigma_b^*$, we have on the one hand

$$(3.4)\qquad \psi(\delta-\sigma-it)=O(1)$$

and on the other,

$$(3.5)\qquad \psi(\delta-\sigma-it)\neq o(1),\quad \text{as } |t|\to\infty.$$

Since by Stirling's formula,

$$\left|\frac{\Delta(\delta-\sigma-it)}{\Delta(\sigma+it)}\right|\sim c_\pm |t|^{A\delta-2A\sigma},\quad \text{as}\quad t\to\pm\infty,\ c_\pm>0,$$

and

$$\varphi(s) = \frac{\Delta(\delta-s)\psi(\delta-s)}{\Delta(s)},$$

(3.4) implies that for $\sigma < \delta - \sigma_b^*$,

$$\limsup_{|t|\to\infty} \frac{\log|\varphi(\sigma+it)|}{\log|t|} \leqslant A\delta - 2A\sigma,$$

while (3.5) implies the opposite inequality. Hence, for $\sigma < \delta - \sigma_b^*$,

$$\varkappa(\sigma) \equiv A\delta - 2A\sigma = \limsup_{|t|\to\infty} \frac{\log|\varphi(\sigma+it)|}{\log|t|}.$$

Since $\varkappa(\sigma)$ depends only on φ and σ, it follows that A, and $A\delta$, hence δ (since $A > 0$), depend only on φ.

§ 4. The behaviour of the function $L(v)$. By definition we have (cf. (2.24))

$$L(-v) = \frac{1}{2\pi i} \int_{-(m_0+\frac{1}{2})} \Gamma(z) V(-z) v^{-z} dz, \quad v \in C - \{v \leqslant -c'\}, c' > 0, \tag{4.1}$$

where m_0 is an integer defined as in (2.5), and

$$V(z) = \prod_{\nu=1}^{N} \Gamma\left(\beta_\nu + \alpha_\nu\delta + \frac{\alpha_\nu z}{2A}\right) \Gamma\left(1 - \beta_\nu + \frac{\alpha_\nu z}{2A}\right). \tag{4.2}$$

We may assume that $m_0 \geqslant 0$, for otherwise $m_0 = -1$, and because of the third restriction on m_0 in (2.5), we shall have

$$L(-v) = \frac{1}{2\pi i} \int_{1/2} \Gamma(z) V(-z) v^{-z} dz = \frac{1}{2\pi i} \int_{-1/2} \Gamma(z) V(-z) v^{-z} dz + \text{a constant},$$

which reduces to the case $m_0 = 0$ of (4.1).

We rewrite $V(z)$ for convenience as

$$V(z) = \prod_{\nu=1}^{N_0} \Gamma(\gamma_\nu + \varepsilon_\nu z), \tag{4.3}$$

say, where $N_0 \geqslant 1$, $0 < \varepsilon_\nu < 1$, $\sum_{\nu=1}^{N_0} \varepsilon_\nu = 1$.

If m is any positive integer, we have, by Stirling's formula,

$$\begin{aligned} \log V(-z) &= \sum_{\nu=1}^{N_0} \log \Gamma(\gamma_\nu - \varepsilon_\nu z) \\ &= \sum_{\nu=1}^{N_0} \Big\{ \log(2\pi)^{1/2} + (\gamma_\nu - \tfrac{1}{2} - \varepsilon_\nu z)\big(\log(-z) + \log\varepsilon_\nu\big) + \varepsilon_\nu z + \sum_{\mu=1}^{m} c_{\nu,\mu} z^{-\mu} + \\ &\qquad + O(|z|^{-m-1}) \Big\} \\ &= -z\log(-z) + k_1 z + k_2 \log(-z) + k_3 + \sum_{\mu=1}^{m} c_\mu z^{-\mu} + O(|z|^{-m-1}), \end{aligned} \tag{4.4}$$

where

$$\begin{aligned} k_1 &= 1 - \sum_{\nu=1}^{N_0} \varepsilon_\nu \log\varepsilon_\nu; \qquad k_2 = \sum_{\nu=1}^{N_0} (\gamma_\nu - \tfrac{1}{2}); \\ k_3 &= \tfrac{1}{2} N_0 \log 2\pi + \sum_{\nu=1}^{N_0} (\gamma_\nu - \tfrac{1}{2}) \log\varepsilon_\nu. \end{aligned} \tag{4.5}$$

If $a < 0$, and $|v| < 1$, we have

$$\begin{aligned} (1+v)^a &= \sum_{m=0}^{\infty} \binom{a}{m} v^m \\ &= \sum_{m=0}^{\infty} \frac{\Gamma(a+1)}{\Gamma(m+1)\Gamma(a-m+1)} v^m = \frac{1}{\Gamma(-a)} \sum_{m=0}^{\infty} \frac{(-1)^m \Gamma(m-a) v^m}{m!} \\ &= \frac{1}{2\pi i} \cdot \frac{1}{\Gamma(-a)} \int_{\sigma_0 - i\infty}^{\sigma_0 + i\infty} \Gamma(z)\Gamma(-a-z) v^{-z} dz, \quad \text{for } 0 < \sigma_0 < -a. \end{aligned}$$

Thus, for $a > 0$, $c' > 0$, $|v| < c'$, $0 < \sigma_0 < a$, we have

$$\begin{aligned} (c'+v)^{-a} &= (c')^{-a}(1+v/c')^{-a} = \frac{(c')^{-a}}{2\pi i\,\Gamma(a)} \int_{\sigma_0} \Gamma(z)\Gamma(a-z) v^{-z} (c')^z dz \\ &= \frac{(c')^{-a}}{2\pi i\,\Gamma(a)} \int_{-(m_0+\frac{1}{2})} \Gamma(z)\Gamma(a-z) v^{-z} (c')^z dz + P(v), \end{aligned} \tag{4.6}$$

where m_0 is an integer $\geqslant 0$, and $P(v)$ a polynomial of degree m_0.

It follows that (4.6) is valid for $v \in C - \{v \leqslant -c'\}$. Now

$$\begin{aligned} &\log\Big((c')^{-a}\big(\Gamma(a)\big)^{-1}\Gamma(a-z)(c')^z\Big) \\ &\quad = -a\log c' - \log\Gamma(a) + z\log c' + (a - \tfrac{1}{2} - z)\log(-z) + z + \\ &\qquad\qquad + \log(2\pi)^{1/2} + \sum_{n=1}^{m} a_n z^{-n} + O(|z|^{-m-1}) \\ &\quad = -z\log(-z) + z(1 + \log c') + (a - \tfrac{1}{2})\log(-z) + \log(2\pi)^{1/2} - a\log c' - \\ &\qquad\qquad - \log\Gamma(a) + \sum_{n=1}^{m} a_n z^{-n} + O(|z|^{-m-1}), \end{aligned}$$

so that, from (4.4), we have

$$\log\left(\frac{V(-z)(c')^{a}\Gamma(a)}{\Gamma(a-z)(c')^{z}}\right) \tag{4.7}$$
$$= z(k_1-1-\log c')+(k_2+\tfrac{1}{2}-a)\log(-z)+k_3-\log(2\pi)^{1/2}+$$
$$+a\log c'+\log\Gamma(a)+\sum_{n=1}^{m} b_n z^{-n}+O(|z|^{-m-1}).$$

Now choose

$$\log c' = k_1-1 = -\sum_{\nu=1}^{N_0}\varepsilon_\nu\log\varepsilon_\nu, \qquad a = k_2+\tfrac{1}{2}, \tag{4.8}$$

so that $k_2 > -\frac{1}{2}$, since $a > 0$, and set

$$k = k_3-\log(2\pi)^{1/2}+a\log c'+\log\Gamma(a). \tag{4.9}$$

Then

$$\log V(-z) \tag{4.10}$$
$$= \log\left(e^{k}(c')^{-a}\big(\Gamma(a)\big)^{-1}\Gamma(a-z)(c')^{z}\right)+\sum_{n=1}^{m} b_n z^{-n}+O(|z|^{-m-1}),$$

for $\operatorname{Re} z = -(m_0+\frac{1}{2})$, as $|z|\to\infty$. On setting

$$h = e^{k}(c')^{-a}\big(\Gamma(a)\big)^{-1} \tag{4.11}$$

we get

$$V(-z) = h\Gamma(a-z)(c')^{z}\left(1+\sum_{n=1}^{m} b_n z^{-n}+O(|z|^{-m-1})\right). \tag{4.12}$$

Using this in (4.1), we get

$$L(-v) = \frac{h}{2\pi i}\int_{-(m_0+\frac{1}{2})}\Gamma(z)\Gamma(a-z)\times$$
$$\times\left(1+\sum_{n=1}^{m}\frac{c_n}{(a-z)(a-z-1)\dots(a-z-n+1)}+O(|z|^{-m-1})\right)\left(\frac{v}{c'}\right)^{-z}dz.$$

If $l = [a]$, so that $a-1 < l \leqslant a$, then

$$L(-v) = \frac{h}{2\pi i}\sum_{n=0}^{l} c_n\int_{-(m_0+\frac{1}{2})}\Gamma(z)\Gamma(a-z-n)\left(\frac{v}{c'}\right)^{-z}dz+$$
$$+O\left(\int_{-(m_0+\frac{1}{2})} e^{\pi|\operatorname{Im} z|}\frac{|\Gamma(z)||\Gamma(a-z)|}{|z|^{l+1}}|v|^{m_0+\frac{1}{2}}|dz|\right),$$

where $c_0 = 1$. Using (4.6), with $a-n$ in place of a, we get, *if a is non-integral, $a > 0$,*

$$L(-v) = \sum_{n=0}^{l} d_n(c'+v)^{-a+n}+O(|v|^{m_0+\frac{1}{2}}), \qquad \text{for } v\in C-\{v\leqslant -c'\}, \tag{4.13}$$

the constants d_r depending of a, and $d_0 = e^{k}$.

If $a > 0$, and a is an integer, then $l = a$, and since

$$\frac{1}{2\pi i}\int_{\sigma_1}\Gamma(z)\Gamma(-z)v^{-z}dz = \sum_{n=1}^{\infty}\frac{(-1)^n v^n}{n} = -\log(1+v),$$

for $-1 < \sigma_1 < 0$, and $0 < |v| < 1$, we have

$$L(-v) = \sum_{n=0}^{l-1} d_n(c'+v)^{-a+n}+d_l\log\left(1+\frac{v}{c'}\right)+O(|v|^{m_0+\frac{1}{2}}), \tag{4.14}$$

the constants d_r depending on a. Thus we have

THEOREM 3. *If $a = A\delta+\frac{1}{2} > 0$, and $l = [a]$, and*

$$c' = \exp\left\{-\sum_{\nu=1}^{N_0}\varepsilon_\nu\log\varepsilon_\nu\right\},$$

then as $v\to c'$ in $\Omega = C-\{v\geqslant c'\}$, we have

$$L(v) = \begin{cases}\displaystyle\sum_{n=0}^{l} d_n(c'-v)^{-a+n}+O(1), & \textit{for} \quad a\neq l,\\ \displaystyle\sum_{n=0}^{l-1} d_n(c'-v)^{-a+n}+d_l\log(c'-v)+O(1), & \textit{for} \quad a = l,\end{cases}$$

where $d_0 = e^{k}$, and k is defined as in (4.9). Here we choose that determination of $(c'-v)^{-a+n}$ which is positive for $v < c'$.

We have only to note that (4.3) implies that $N_0 = 2N$, $\gamma_\nu = \beta_\nu+a_\nu\delta$, for $\nu = 1,\dots,N$; and $\gamma_\nu = 1-\beta_{\nu-N}$ for $\nu = N+1,\dots,2N$; so that

$$a = k_2+\tfrac{1}{2} = \sum_{\nu=1}^{N_0}(\gamma_\nu-\tfrac{1}{2})+\tfrac{1}{2} = A\delta+\tfrac{1}{2}$$

by (4.8) and (4.5) and (3.1)′. Again (4.3) implies that $\varepsilon_\nu = \dfrac{a_\nu}{2A}$ for $\nu = 1,\dots,N$, and $\varepsilon_\nu = \dfrac{a_{\nu-N}}{2A}$ for $\nu = N+1,\dots,2N$, so that

$$c' = \exp\left\{-\sum_{\nu=1}^{N_0}\varepsilon_\nu\log\varepsilon_\nu\right\} = \exp\left\{-\frac{1}{A}\sum_{\nu=1}^{N}a_\nu\log\frac{a_\nu}{2A}\right\} = e^{-c},$$

where c is defined as in (2.14).

Remark. The argument used to prove Theorem 3 goes through for complex a, with $\operatorname{Re} a > 0$, $l = [\operatorname{Re} a]$, $d_0 = e^k$.

§ 5. Uniqueness of the functional equation.

THEOREM 4. *Suppose that functional equations* (3.1) *and* (3.2) *hold as in Theorem* 2, *with* $A\delta + \frac{1}{2} > 0$. *Then* $d_n = \varkappa b_n$, *where* $\varkappa$ *is real*, $\varkappa \neq 0$; *and* $B' \equiv B \bmod 1$. *In particular*, $\operatorname{Im} B = \operatorname{Im} B'$. *Further* $\nu_n = c_2 \mu_n$, $c_2 > 0$, *for* $n = 1, 2, \ldots$

Proof. By Lemma 1, we have, for $\operatorname{Re} s > 0$,

$$f(s) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n^{1/2A} s} - R(s) = \sum_{n=1}^{\infty} b_n \mu_n^{-\delta} M(s\mu_n^{-1/2A}) \tag{5.1}$$

$$= \sum_{n=1}^{\infty} d_n \nu_n^{-\delta} M_1(s\nu_n^{-1/2A}), \text{ say,}$$

where $R(s)$ is analytic in $C - \{s \leqslant 0\}$. Now

$$M(s) = \varkappa_1 e^{\pi i N/2} \sum_{l=1}^{2N} \eta_l L(-e^{i\gamma_l} s), \qquad \varkappa_1 > 0,$$

where $L(v)$ is analytic for $v \in C - \{v \geqslant c'\}$, and in a neighbourhood of $v = c'$ has the form

$$\varkappa_2 (c' - v)^{-(A\delta + \frac{1}{2})} \big(1 + o(1)\big), \qquad \varkappa_2 > 0,$$

if $A\delta + \frac{1}{2} > 0$, while the (γ_l) are such that

$$\tfrac{1}{2}\pi = \gamma_1 > \gamma_2 \geqslant \ldots \geqslant \gamma_{2N-1} > \gamma_{2N} = -\tfrac{1}{2}\pi,$$

and the (η_l) are such that

$$\eta_1 = e^{-\pi i \sum_{\nu=1}^{N} \beta_\nu}, \qquad \eta_{2N} = e^{\pi i (\sum_{\nu=1}^{N} \beta_\nu - N)} \qquad \text{(cf. (2.16))}.$$

Further $L(v)$ can be analytically continued across the line $v > c'$ from above as well as from below. Hence $M(s)$ has, for $\operatorname{Re} s \geqslant 0$, the only singularities $\pm ic'$, and its behaviour in their neighbourhood is given by

$$M(s) = \varkappa_1 \varkappa_2 e^{\frac{1}{2}\pi i N} \eta_1 (c' - is)^{-(A\delta + \frac{1}{2})} \big(1 + o(1)\big), \quad \text{as} \quad s \to -ic',$$

and

$$M(s) = \varkappa_1 \varkappa_2 e^{\frac{1}{2}\pi i N} \eta_{2N} (c' + is)^{-(A\delta + \frac{1}{2})} \big(1 + o(1)\big), \quad \text{as} \quad s \to ic'.$$

If we set $B = \sum_{\nu=1}^{N} (\beta_\nu - \frac{1}{2})$, $B' = \sum_{\nu=1}^{N'} (\beta'_\nu - \frac{1}{2})$, as in (3.1)′, (3.2)′, then

$f(s)$ has for its only singularities on the line $\operatorname{Re} s = 0$ the points $s = \pm ic' \mu_n^{1/2A}$, $n = 1, 2, 3, \ldots$, and in a neighbourhood of $s = -ic'\mu_n^{1/2A}$ it has the form

$$\begin{aligned} &b_n \mu_n^{-\delta} \varkappa_3 e^{-\pi i B} (c' - is\mu_n^{-1/2A})^{-(A\delta + \frac{1}{2})} \big(1 + o(1)\big) \\ &= b_n \mu_n^{-\delta/2 + 1/4A} \varkappa_3 e^{-\pi i B} (\mu_n^{1/2A} c' - is)^{-(A\delta + \frac{1}{2})} \big(1 + o(1)\big), \qquad \varkappa_3 > 0, \end{aligned} \tag{5.2}$$

while in a neighbourhood of $s = +ic'\mu_n^{1/2A}$, it is of the form

$$b_n \mu_n^{-\delta/2 + 1/4A} \varkappa_3 e^{\pi i B} (\mu_n^{1/2A} c' + is)^{-(A\delta + \frac{1}{2})} \big(1 + o(1)\big). \tag{5.3}$$

We may assume, without loss of generality, that $b_n \neq 0$, $d_n \neq 0$ for all n. Then the representation $f(s) = \sum_{n=1}^{\infty} d_n \nu_n^{-1} M_1(s\nu_n^{-1/2A})$ in (5.1) shows that f is of the form

$$d_n \varkappa_4 \nu_n^{-\delta/2 + 1/4A} e^{-\pi i B'} (\nu_n^{1/2A} c_1' - is)^{-(A\delta + \frac{1}{2})} \big(1 + o(1)\big), \qquad c_1' > 0 \tag{5.4}$$

near $s = -ic_1' \nu_n^{1/2A}$, for $n = 1, 2, \ldots$, which are the only singularities of f on the negative imaginary axis in the s-plane. From (5.2) and (5.4) it follows that

$$\nu_n = c_2 \mu_n, \qquad n = 1, 2, \ldots; \quad c_2 = (c'/c_1')^{2A} > 0.$$

This, together with (5.4), implies that near $s = -ic_1'\nu_n^{1/2A}$, $f(s)$ is of the form

$$d_n \varkappa_4 c_2^{-\delta/2 + 1/4A} e^{-\pi i B'} \mu_n^{-\delta/2 + 1/4A} (\mu_n^{1/2A} c' - is)^{-(A\delta + \frac{1}{2})} \big(1 + o(1)\big). \tag{5.5}$$

Comparing (5.5) and (5.2), we get

$$d_n \varkappa_4 c_2^{-\delta/2 + 1/4A} e^{-\pi i B'} = b_n \varkappa_3 e^{-\pi i B},$$

that is

$$d_n = \varkappa_5 e^{\pi i (B' - B)} b_n, \qquad n = 1, 2, \ldots; \varkappa_5 > 0.$$

If one compares the singularities on the positive imaginary axis, one obtains similarly

$$d_n = \varkappa_5 e^{-\pi i (B' - B)} b_n, \qquad n = 1, 2, \ldots; \varkappa_5 > 0.$$

Hence $e^{2\pi i (B' - B)} = 1$, or $B \equiv B' \bmod 1$. In particular, $\operatorname{Im} B = \operatorname{Im} B'$. Further $d_n = \varkappa b_n$, $\varkappa$ real, $\varkappa \neq 0$. Hence $\psi_1(s) = \varkappa c_2^{-s} \psi(s)$, $c_2 > 0$, $\varkappa$ real.

Remarks.

(i) Suppose that the functional equation

$$\Delta(s)\varphi(s) = \Delta(\delta - s)\psi(\delta - s) \tag{5.6}$$

holds, but the condition $A\delta+\frac{1}{2}>0$, of Theorem 4, does not. We can then obtain another functional equation for which it does. If we define

$$\Delta_1(s) = \Delta(a+s) = \prod_{\nu=1}^{N} \Gamma(a_\nu a + a_\nu s+\beta_\nu), \qquad \delta_1 = \delta-2a,$$

$$\varphi_1(s) = \varphi(a+s), \qquad \psi_1(s) = \psi(a+s),$$

then the equation

$$\Delta_1(s)\varphi_1(s) = \Delta_1(\delta_1-s)\psi_1(\delta_1-s) \tag{5.7}$$

holds. The same A is associated with both the equations. It follows that for the new equation we have $A\delta_1+\frac{1}{2} = A\delta+\frac{1}{2}-2aA>0$, if $a<\frac{1}{2A}(A\delta+\frac{1}{2})$. Thus, with a suitably chosen a, (5.7) holds with the desired condition. So Theorem 4 holds also for $A\delta+\frac{1}{2}\leqslant 0$.

(ii) That equations of the type (3.1) and (3.2) can occur is illustrated by a simple example. Suppose that $\Delta(s)\varphi(s) = \Delta(\delta-s)\psi(\delta-s)$, with $\Delta(s) = \Gamma(s)$. If $\Delta_1(s) = \Gamma(\frac{1}{2}s)\Gamma(\frac{1}{2}+\frac{1}{2}s)$, and $\psi_1(s) = 2^{-\delta}2^{2s}\psi(s)$, then $\Delta_1(s)\varphi(s) = \Delta_1(\delta-s)\psi_1(\delta-s)$.

(iii) That $\varkappa$ in Theorem 4 can be negative is shown by an example. Let $\lambda>2$, $\mu_n = \lambda_n = n\frac{2\pi}{\lambda}$. Then it is known that there exists a function $\varphi(s) = \sum_{n=1}^{\infty} a_n\lambda_n^{-s}$, such that $\Gamma(s)\varphi(s) = \Gamma(-s)\varphi(-s)$. In our notation $\delta=0$, $A=1$, $\psi(s)=\varphi(s)$. Let $\Delta(s) = \Gamma(s)$, and $\Delta_1(s) = \Gamma(1+s)$, so that

$$\frac{\Delta_1(-s)}{\Delta_1(s)} = -\frac{\Delta(-s)}{\Delta(s)},$$

and the equation $\Delta_1(s)\varphi(s) = \Delta_1(-s)\psi_1(-s)$, with $\psi_1(s) = -\psi(s)$, holds.

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