RATIONALLY TRIVIAL HERMITIAN SPACES ARE LOCALLY TRIVIAL

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Abstract. Let R be a regular local ring, K its field of fractions and A an Azumaya algebra with involution over R. Let $\mathbf{h}$ be an ϵ-hermitian space over A. We show that if $\mathbf{h} \otimes_R K$ is hyperbolic over $A \otimes_R K$, then $\mathbf{h}$ is hyperbolic over A.

1. Introduction

Let R be a regular local ring, K its field of fractions and A an Azumaya algebra with involution over R (see §4 for a precise definition). Let $\mathbf{h}$ be an ϵ-hermitian space over A. Assume that $\mathbf{h} \otimes_R K$ is hyperbolic over $A \otimes_R K$. Is $\mathbf{h}$ hyperbolic too? We show that this is true if R is a regular local ring containing a field of characteristic different from 2.

Grothendieck [G] conjectured that, for any reductive group scheme G over R, rationally trivial G-homogeneous spaces are trivial. Our result corresponds to the case when G is the unitary group $\mathrm{U}^{\epsilon}_{2n}(A)$. If R is an essentially smooth local k-algebra and G is defined over k (we say that G *is constant*) Grothendieck's conjecture has been proved in most cases: by Colliot-Thélène and Ojanguren [4] for a perfect infinite field k and then by Raghunathan [19] for any infinite k. One notable open case is that of a finite base field. For a non-constant group G only two cases have been proved: when G is a torus, by Colliot-Thélène and Sansuc [5], and when G is the group $\mathrm{SL}_1(D)$ of norm one elements of an Azumaya R-algebra D, by Panin and Suslin [13].

Our proof has been ispired by Voevodsky's work [24]. We prove our main result in the case when the base ring is a local essentially smooth algebra over an infinite field of characteristic different from 2. The general result can be deduced from this using Popescu's theorem and some formal arguments (see §9). An essential tool is a non-degenerate trace form for finite extensions of smooth algebras, which was introduced by Euler. We recall its definition and main properties in §§ 11 and 12.

2. The specialization lemma for absolute curves

In this paper k will always denote a fixed ground field. Hence, for any smooth d-dimensional scheme X over k or any smooth d-dimensional k-algebra A we will denote by Ω_X or Ω_A, the module of Kähler differentials of X or A and by ω_x and ω_A its d-th exterior power.

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Lemma 2.1. *Let k be a field of characteristic different from 2 and X a smooth irreducible affine curve over k. Let $x \in X(k)$ be a rational point of X and $\mathbf{q}$ a quadratic space over X. If the generic fibre $\mathbf{q} \otimes_{k[X]} k(X)$ is hyperbolic, then the closed fibre $\mathbf{q} \otimes_{k[X]} k(x)$ is hyperbolic too.*

Proof. Replacing X by a suitable open neighbourhood of x we may assume that the canonical bundle ω_X is trivial. Let X_f be a principal open set over which $\mathbf{q}$ is hyperbolic. We can find a finite morphism

$$\pi : X \to \mathbb{A}^1_k$$

such that

(a) π is étale at x.
(b) $\pi^{-1}(1)$ is contained in X_f.
(c) $\pi^{-1}(0) = \{x\} \amalg D$ with D a finite subscheme of X_f.

Since ω_X and $\omega_{\mathbb{A}^1_k}$ are trivial, the Euler trace defined in §11 yields, by §12, a functor $\mathrm{Tr}^{\mathcal{E}}$ that transforms quadratic spaces over X into quadratic spaces over $\mathbb{A}^1_k$.

If V is the underlying $k[X]$-module of $\mathbf{q}$, the quadratic space $\mathrm{Tr}^{\mathcal{E}}(\mathbf{q})$ is given by the composite map

$$V \times V \xrightarrow{\mathbf{q}} k[X] \xrightarrow{\mathcal{E}} k[t] \ .$$

Clearly, for any closed point $s \in \mathbb{A}^1_k$

$$\mathrm{Tr}^{\mathcal{E}}(\mathbf{q})|_{\{s\}} \simeq \mathrm{Tr}^{\mathcal{E}(s)}(\mathbf{q}|_{\pi^{-1}(s)}) \ ,$$

where $\mathcal{E}(s) = \mathcal{E} \otimes_{k[t]} k(s)$. For any quadratic space $\mathbf{r}$ over $\mathbb{A}^1_k$ we denote by $\mathbf{r}_0$ and $\mathbf{r}_1$ its evaluations at 0 and 1. By a theorem of Karoubi ([9] or [10], VII, §4) the canonical map $\mathrm{W}(k) \to \mathrm{W}(\mathbb{A}^1_k)$ is an isomorphism, hence, for any space $\mathbf{r}$ over $\mathbb{A}^1_k$ the spaces $\mathbf{r}_0$ and $\mathbf{r}_1$ are Witt equivalent. In particular, for $\mathbf{r} = \mathrm{Tr}^{\mathcal{E}}(\mathbf{q})$ we get

$$\mathrm{Tr}^{\mathcal{E}(0)}(\mathbf{q}|_{\pi^{-1}(0)}) \sim \mathrm{Tr}^{\mathcal{E}}(\mathbf{q})_0 \sim \mathrm{Tr}^{\mathcal{E}}(\mathbf{q})_1 \sim \mathrm{Tr}^{\mathcal{E}(1)}(\mathbf{q}|_{\pi^{-1}(1)}) \ ,$$

where $\sim$ denotes Witt equivalence. Since $\mathbf{q}$ is hyperbolic over X_f, condition (b) implies that $\mathrm{Tr}^{\mathcal{E}(1)}(\mathbf{q}|_{\pi^{-1}(1)}) \sim 0$ and from condition (c) we get that

$$\mathrm{Tr}^{\mathcal{E}(0)}(\mathbf{q}|_{\{x\}\amalg D}) \sim \mathrm{Tr}^{\mathcal{E}(x)}(\mathbf{q}|_{\{x\}}) \perp \mathrm{Tr}^{\mathcal{E}(D)}(\mathbf{q}|_D) \sim \mathrm{Tr}^{\mathcal{E}(x)}(\mathbf{q}|_{\{x\}}) \ ,$$

which shows that $\mathrm{Tr}^{\mathcal{E}(x)}(\mathbf{q}|_{\{x\}})$ is hyperbolic. On the other hand, since x is a rational point, by (5) of §12 $\mathrm{Tr}^{\mathcal{E}(x)}(\mathbf{q}|_{\{x\}})$ is proportional to $\mathbf{q}|_{\{x\}}$ and thus $\mathbf{q}|_{\{x\}}$ is hyperbolic as well.

3. The specialization lemma for relative curves

Let R be a local ring of a smooth variety over a field k of characteristic different from 2. Let $U = \mathrm{Spec}(R)$ and let u be the closed point of U. Let $p : \mathcal{X} \to U$ be an affine U-scheme, smooth over k. Let f be a regular element of $k[\mathcal{X}]$ such that $k[\mathcal{X}]/(f)$ is finite over R. We denote by $\mathcal{X}_f$ the principal open set defined by $f \neq 0$.

Lemma 3.1. *Let* $\mathbf{q}$ *be a quadratic space over* $\mathcal{X}$ *which is hyperbolic over* $\mathcal{X}_f$. *Assume that the canonical bundle* $\omega_{\mathcal{X}/k}$ *is trivial, that there exists a finite surjective morphism* $\mathcal{X} \to U \times \mathbb{A}^1_k$ *of* U*-schemes and that there exists a section* $\Delta : U \to \mathcal{X}$ *of* p *such that* p *is smooth along* $\Delta(U)$. *Then the restriction* $\Delta^*\mathbf{q}$ *of* $\mathbf{q}$ *to* $\Delta(U)$ *is hyperbolic.*

Proof. Note that the existence of a finite surjective U-morphism $\mathcal{X} \to U \times \mathbb{A}^1_k$ implies that $\mathcal{X}$ is affine, flat over $U \times \mathbb{A}^1_k$ ([6], Corollary 18.17) and that every component of $p^{-1}(u)$ is one-dimensional.

Using the finite map $\mathcal{X} \to U \times \mathbb{A}^1_k$, by Lemma 10.1 we can construct a finite surjective morphism

$$\pi : \mathcal{X} \to U \times \mathbb{A}^1_k$$

of U-schemes with the following properties:

(a) π is étale along $\Delta(U)$.
(b) $\pi^{-1}(U \times \{1\})$ is in X_f.
(c) $\pi^{-1}(U \times \{0\}) = \Delta(U) \amalg \mathcal{D}$, where $\mathcal{D} \subset X_f$.

This morphism π induces a finite homomorphism of R-algebras

$$\pi^* : R[t] \to R[\mathcal{X}]$$

which, by [6], Corollary 18.17, is flat. The canonical bundle of $\mathcal{X}$ is trivial by assumption and the canonical bundle of $U \times \mathbb{A}^1_k$ is trivial becouse U is local. Hence Corollary 11.2 yields an Euler trace

$$\mathcal{E} : R[\mathcal{X}] \to R[t]$$

for which the associated map $\lambda : R[\mathcal{X}] \to \mathrm{Hom}_{R[t]}(R[\mathcal{X}], R[t])$ is an isomorphism. By the results of §12 this Euler trace induces a transformation $\mathrm{Tr}^{\mathcal{E}}$ of quadratic spaces over $\mathcal{X}$ into quadratic spaces over $U \times \mathbb{A}^1_k$. By Karoubi's theorem ([9] or [10], VII, §4) the canonical map $\mathrm{W}(R) \to \mathrm{W}(R[t])$ is an isomorphism. Hence, as in the proof of Lemma 2.1 (but omitting the obvious superscripts)

$$\mathrm{Tr}(\mathbf{q}|_{\Delta(U)}) \sim \mathrm{Tr}(\mathbf{q}|_{\Delta(U)\amalg\mathcal{D}}) = \mathrm{Tr}(\mathbf{q})|_{U\times\{0\}} \sim \mathrm{Tr}(\mathbf{q})|_{U\times\{1\}} \sim \mathrm{Tr}(\mathbf{q}|_{\pi^{-1}(U\times 1)}) \sim 0\ .$$

This shows that $\mathrm{Tr}(\mathbf{q}|_{\Delta(U)})$ is stably hyperbolic. Since U is local, $\mathrm{Tr}(\mathbf{q}|_{\Delta(U)})$ is hyperbolic by Witt's cancellation theorem ([10], Corollary 5.7.5). On the other hand (see §12, (5)) $\mathbf{q}|_{\Delta(U)}$ is a multiple of $\mathrm{Tr}(\mathbf{q})|_{\Delta(U)}$ and thus it is hyperbolic too.

4. The specialization lemma for hermitian spaces over constant algebras

By an *Azumaya algebra with involution* over a commutative ring S we mean a pair (A, σ) consisting of an S-algebra A and an S-linear involution σ on A such that:

(1) A is an Azumaya algebra over its center Z.
(2) Z is either S or an étale quadratic extension of S.
(3) $Z^\sigma = S$.

This definition extends in an obvious way to that of an algebra over a scheme.

Let R, U, $p : \mathcal{X} \to U$ and $f \in k[\mathcal{X}]$ be as in the previous section.

Lemma 4.1. *Let (A, σ) be an Azumaya algebra with involution on U and $\mathbf{h}$ an ϵ-hermitian space over $p^*(A, \sigma)$ which is hyperbolic over $\mathcal{X}_f$. Assume that the canonical bundle $\omega_{\mathcal{X}/k}$ is trivial, that there exists a finite surjective morphism $\mathcal{X} \to U \times \mathbb{A}^1_k$ of U-schemes and that there exists a section $\Delta : U \to \mathcal{X}$ of p such that p is smooth along $\Delta(U)$. Then $\Delta^*\mathbf{h}$ is hyperbolic.*

Proof. Exactly the same as that of Lemma 3.1, provided Witt's cancellation theorem holds in this more general situation. Since U is local, $A/\mathrm{rad}(A)$ is either simple or hyperbolic. In both cases we can apply Theorem 5.7.2 of [10].

5. Rationally trivial quadratic spaces are locally trivial

The next result is known to be true (see [11], [14] and [4]) for any base field of characteristic $\neq 2$. We now reprove it by a different method, which we will then extend to the case of hermitian spaces over non-constant Azumaya algebras.

Theorem 5.1. *Let R be a local ring of a smooth variety over a field k of characteristic different from 2 and K the field of fractions of R. Let $\mathbf{q}$ be a quadratic space over R. If $\mathbf{q}_K$ is hyperbolic, then $\mathbf{q}$ is hyperbolic.*

Proof. By assumption there exist a smooth d-dimensional k-algebra A and a prime ideal $\mathfrak{p}$ of A such that $R = A_\mathfrak{p}$. We first reduce the proof to the case in which $\mathfrak{p}$ is maximal. To do this, choose a maximal ideal $\mathfrak{m}$ containing $\mathfrak{p}$. Since k is infinite, by a standard general position argument we can find d algebraically independent elements $X_1, \dots, X_d$ such that A is finite over $k[X_1, \dots, X_d]$ and étale at $\mathfrak{m}$. After a linear change of coordinates we may assume that $A/\mathfrak{p}$ is finite over $B = k[X_1, \dots, X_m]$, where m is the dimension of $A/\mathfrak{p}$. Clearly A is smooth over B at $\mathfrak{m}$ and thus, for some $h \in A \setminus \mathfrak{m}$, the localization A_h is smooth over B. Let S be the set of nonzero elements of B, $k' = S^{-1}B$ the field of fractions of B and $A' = S^{-1}A_h$. The prime ideal $\mathfrak{p}' = S^{-1}\mathfrak{p}_h$ is maximal in A', the k'-algebra A' is smooth and $R = A'_{\mathfrak{p}'}$.

From now on we assume that $R = \mathcal{O}_{X,x}$ is the local ring of a closed point x of a smooth d-dimensional affine variety X over k.

Replacing X by a sufficiently small affine neighbourhood of x we may assume that $\mathbf{q}$ is a quadratic space over X and that $\omega_{X/k}$ is trivial. We can choose an $f \in k[X]$ such that $\mathbf{q}|_{X_f}$ is hyperbolic. Let Z be the closed subscheme of X defined by $f = 0$. By Quillen's trick (see [18], Lemma 5.12) we can find a morphism $q : X \to \mathbb{A}^{d-1}_k$ with the following properties:

(1) q is smooth at x.
(2) $q|_Z : Z \to \mathbb{A}^{d-1}_k$ is finite.
(3) q factors as

$$\begin{tikzcd} X \arrow[rr, "q_1"] \arrow[dr, "q"'] & & \mathbb{A}^d_k \arrow[dl, "pr"] \\ & \mathbb{A}^{d-1}_k & \end{tikzcd}$$

with q_1 finite and surjective.

Consider the cartesian square

$$\begin{tikzcd} \mathcal{X} \arrow[r, "p_X"] \arrow[d, "p"'] & X \arrow[d, "q"] \\ U \arrow[r, "r"'] & \mathbb{A}_k^{d-1} \end{tikzcd}$$

where $U = \mathrm{Spec}(\mathcal{O}_{X,x})$, $r = q|_U$, $\mathcal{X} = U \times_{\mathbb{A}_k^{d-1}} X$ and p is the first projection. Let $\Delta : U \to \mathcal{X}$ be the diagonal. Denote by $\mathfrak{q}$ the quadratic space $p_X^* \mathbf{q}$ and by $\mathfrak{f}$ the composition of f with p_X. We first check that $\mathfrak{q}$, p, Δ and $\mathfrak{f}$ satisfy the hypotheses of Lemma 3.1.

Since r is smooth, p_X is also smooth and since X is smooth over k, so is $\mathcal{X}$. By base change, condition (3) implies that $\mathcal{X}$ is an affine relative curve over U. Since U is local and q is smooth at x, p is smooth along $\Delta(U)$. From (3), by base change via $r : U \to \mathbb{A}_k^{d-1}$, we get a commutative triangle

$$\begin{tikzcd} \mathcal{X} \arrow[rr, "p_1"] \arrow[dr, "p"'] & & U \times \mathbb{A}_k^1 \arrow[dl] \\ & U & \end{tikzcd}$$

with p_1 finite. Again by the same base change we see that $R[\mathcal{X}]/(\mathfrak{f})$ is finite over R and that $\mathfrak{q}|_{\mathcal{X}_\mathfrak{f}}$ is hyperbolic. To see that $\omega_{\mathcal{X}/k}$ is trivial, observe that

$$\omega_{\mathcal{X}/k} \simeq p_X^*(\omega_{X/k}) \otimes_{\mathcal{O}_X} \omega_{\mathcal{X}/X}$$

and that $\omega_{\mathcal{X}/X} \simeq p^* \omega_{U/\mathbb{A}_k^{d-1}}$. Since U is essentially smooth over $\mathbb{A}_k^{d-1}$, $\omega_{U/\mathbb{A}_k^{d-1}}$ is locally free of rank one, hence trivial because U is local. Thus $p^* \omega_{U/\mathbb{A}_k^{d-1}}$ is trivial and, since $\omega_{X/k}$ is trivial by assumption, we conclude that $\omega_{\mathcal{X}/k}$ is trivial.

We can now apply Lemma 3.1, which says that $\Delta^* \mathfrak{q}$ is hyperbolic. Since $\mathfrak{q} = p_X^* \mathbf{q}$ and $\Delta^* p_X^* = (p_X \Delta)^* = \mathrm{id}$, the space $\Delta^* \mathfrak{q}$ coincides with $\mathbf{q}$ and the theorem is proved.

6. The specialization lemma for hermitian spaces over absolute curves

Let X be a smooth affine curve over a field k of characteristic $\neq 2$ and let $(\mathfrak{A}, \mathfrak{s})$ be an Azumaya algebra with involution over X. Let $\mathfrak{h}$ be an ϵ-hermitian space over $(\mathfrak{A}, \mathfrak{s})$.

Lemma 6.1. *Let $x \in X(k)$ be a rational point of X and put $(A, \sigma) = (\mathfrak{A}, \mathfrak{s}) \otimes_{k[X]} k(x)$. If $\mathfrak{h} \otimes_{k[X]} k(X)$ is hyperbolic over $(\mathfrak{A}, \mathfrak{s}) \otimes_{k[X]} k(X)$, then its fibre $\mathfrak{h} \otimes_{k[X]} k(x)$ at x is hyperbolic over (A, σ).*

For the proof we need the following result, which will be generalized in the next section.

Lemma 6.2. *Let $(\mathfrak{A}, \mathfrak{s})$ and $(\mathfrak{B}, \mathfrak{t})$ be Azumaya algebras with involution over X and suppose that there exists an isomorphism $\theta : (A, \sigma) \to (B, \tau)$ between their fibres at the rational point x. Then there exist a commutative triangle*

$$\begin{tikzcd} \widetilde{X} \arrow[rr, "\pi"] \arrow[dr, "\widetilde{p}"'] & & X \arrow[dl, "p"] \\ & \operatorname{Spec}(k) & \end{tikzcd} ,$$

a rational point $\widetilde{x} : \operatorname{Spec}(k) \to \widetilde{X}$ and an isomorphism $\Theta : (\widetilde{\mathfrak{A}}, \widetilde{\mathfrak{s}}) \to (\widetilde{\mathfrak{B}}, \widetilde{\mathfrak{t}})$ between the inverse images of $(\mathfrak{A}, \mathfrak{s})$ and $(\mathfrak{B}, \mathfrak{t})$ over $\widetilde{X}$, such that:

(a) *π is étale (but not necessarily finite).*
(b) *$\pi \circ \widetilde{x} = x$.*
(c) *$\widetilde{x}^*(\Theta) = \theta$, i.e. Θ induces θ on the fibers.*

Proof of Lemma 6.2. Let $R = \mathcal{O}^h_{X,x}$ be the henselization of the local ring of X at x. Lifting θ to an isomorphism $A_R \to B_R$ of Azumaya algebras we are reduced to the case $A_R = B_R$, with two possibly different involutions σ and τ. The automorphism $\sigma\tau$ is the conjugation by an invertible element $u \in A$ and the condition for (A, σ) to be isomorphic to (B, τ) is the existence of a $v \in A$ such that $\tau(v)v = u$. For any given u this condition defines a smooth R-subscheme of the group scheme A^* of invertible elements of A. Indeed, over the strict henselization of R, A becomes a matrix algebra and the condition $\tau(v)v = u$ defines a scheme isomorphic to the orthogonal, symplectic or unitary group. Since $\tau(v)v = u$ has a solution $\overline{v}$ over the residue field of R, it has a solution v over R which lifts $\overline{v}$. Since R is an inductive limit of quasi-finite étale extensions of $\mathcal{O}_{X,x}$ with residue field k, there exists an étale extension $\widetilde{\mathcal{O}}$ of $\mathcal{O}_{X,x}$ such that $(A, \sigma)_{\widetilde{\mathcal{O}}} \simeq (B, \tau)_{\widetilde{\mathcal{O}}}$. We can thus take $\widetilde{X} = \operatorname{Spec}(\widetilde{\mathcal{O}})$.

Proof of Lemma 6.1. Let $(\mathfrak{B}, \mathfrak{t}) = p^*(A, \sigma)$ be the constant extension of (A, σ) to X and let $\pi : \widetilde{X} \to X$ be as in Lemma 6.2. The hermitian space $\pi^*\mathfrak{h}$ is hyperbolic over $k(\widetilde{X})$, hence, replacing X by $\widetilde{X}$, we may assume that $(\mathfrak{A}, \mathfrak{s})$ is the constant algebra $p^*(A, \sigma)$. In this case we can repeat the proof given for Lemma 2.1.

7. How to make Azumaya algebras isomorphic

In this section k is a field of arbitrary characteristic.

Proposition 7.1. *Let $S = \operatorname{Spec}(R)$ be a regular semilocal scheme and T a closed subscheme of S. Let (A, σ) and (B, τ) be two Azumaya algebras with involution over S, of the same rank. Assume that there exists an isomorphism $\varphi : (A, \sigma)|_T \to (B, \tau)|_T$. Then there exists a finite étale covering $\pi : \widetilde{S} \to S$, a section $\delta : T \to \widetilde{S}$ of π over T and an isomorphism $\Phi : \pi^*(A, \sigma) \to \pi^*(B, \tau)$ such that $\delta^*(\Phi) = \varphi$.*

To prove this proposition we need a variant of Bertini's theorem (see also [8]).

Lemma 7.2. *Let $S = \operatorname{Spec}(R)$ be a regular semilocal scheme and T a closed subscheme of S. Let $\overline{X}$ be a closed subscheme of $\mathbb{P}^d_S = \operatorname{Proj}(S[X_0, \ldots, X_d])$ and $X = \overline{X} \cap \mathbb{A}^d_S$, where $\mathbb{A}^d_S$ is the affine space defined by $X_0 \neq 0$. Let $X_\infty = \overline{X} \setminus X$ be the intersection of $\overline{X}$ with the hyperplane at infinity $X_0 = 0$. Assume that over T there exists a section $\delta : T \to X$ of the canonical projection $X \to S$. Assume further that*

(1) *X is smooth and equidimensional over S, of relative dimension r.*
(2) *For every closed point $s \in S$ the closed fibres of X_∞ and X satisfy*

$$\dim(X_\infty(s)) < \dim(X(s)) = r\ .$$

Then there exists a closed subscheme $\widetilde{S}$ of X which is finite étale over S and contains $\delta(T)$.

Proof. Since S is semilocal, after a linear change of coordinates we may assume that δ maps T into the closed subscheme of $\mathbb{P}^d_T$ defined by $X_1 = \cdots = X_d = 0$. For each closed fibre $\mathbb{P}^d_s$ of $\mathbb{P}^d_S$ we can choose a family of quadratic polynomials $H_1(s), \ldots, H_r(s)$ such that the subscheme $Y(s)$ of $\mathbb{P}^d_S(s)$ defined by the equations

$$H_1(s) = 0\,,\ \ldots\ ,\ H_r(s) = 0$$

intersects $X(s)$ transversally, contains the point $(1 : 0 : \cdots : 0)$ and avoids $X_\infty(s)$ (see [3], XI, Théorème 2.1). By the chinese remainders' theorem there exists a common lift $H_i \in R[X_0, \ldots, X_d]$ of all polynomials $H_i(s)$, $s \in \mathrm{Max}(R)$. We may choose this common lift H_i such that $H_i(1, 0, \ldots, 0) = 0$. Let Y be the closed subscheme of $\mathbb{P}^d_S$ defined by

$$H_1 = 0\,, \ldots, H_r = 0\ .$$

We claim that the subscheme $\widetilde{S} = Y \cap X$ has the requiered properties. Note first that $X \cap Y$ is finite over S. In fact, $X \cap Y = \overline{X} \cap Y$, which is projective over S and such that every closed fibre (hence every fibre) is finite. Since the closed fibres of $X \cap Y$ are finite étale over the closed points of S, to show that $X \cap Y$ is finite étale over S it only remains to show that it is flat over S. Noting that $X \cap Y$ is defined in every closed fibre by a regular sequence of equations and localizing at each closed point of S, we see that flatness follows from the next, purely algebraic, lemma.

Lemma 7.3. *Let R be a regular local ring with maximal ideal m. Let A be a commutative flat R-algebra and let "bar" denote reduction modulo m. Let $a_1, \ldots, a_n \in A$ be such that $\overline{a}_1, \ldots, \overline{a}_n$ form a regular sequence. If the R-module $A/(a_1, \ldots, a_n)$ is of finite type over R, then it is flat.*

Proof.. The quotient $A/(a_1, \ldots, a_n)$ is semilocal, hence, replacing A by a suitable semilocal algebra we may assume that $a_1, \ldots, a_n$ and m are contained in the radical of A. Since the R-module $A/(a_1, \ldots, a_n)$ is of finite type, it suffices to check that $\mathrm{Tor}^1_R(A/(a_1, \ldots, a_n), \overline{R})$ vanishes. We show by induction on i that $\mathrm{Tor}^1_R(A/(a_1, \ldots, a_i), \overline{R}) = 0$ for $0 \le i \le n$. If $i = 0$, the quotient $A/(a_1, \ldots, a_n)$ is just A, which is flat by assumption.

Sublemma 7.4. *For $0 \le i < n$ the sequence*

$$0 \to A/(a_1, \ldots, a_i) \xrightarrow{a_{i+1}} A/(a_1, \ldots, a_i) \to A/(a_1, \ldots, a_{i+1}) \to 0$$

is exact.

Granting the sublemma, consider the long exact sequence associated to the short sequence above:

$$\begin{aligned}\mathrm{Tor}^1_R(\overline{A}/(\overline{a}_1,\dots,\overline{a}_i),\overline{R}) \to \mathrm{Tor}^1_R(\overline{A}/(\overline{a}_1,\dots,\overline{a}_i),\overline{R}) \to \\ \to \mathrm{Tor}^1_R(\overline{A}/(\overline{a}_1,\dots,\overline{a}_{i+1}),\overline{R}) \xrightarrow{\partial} \overline{A}/(\overline{a}_1,\dots,\overline{a}_i) \xrightarrow{\overline{a}_{i+1}} \\ \xrightarrow{\overline{a}_{i+1}} A/(\overline{a}_1,\dots,\overline{a}_i) \to A/(\overline{a}_1,\dots,\overline{a}_{i+1}) \to 0\ .\end{aligned}$$

By induction hypothesis $\mathrm{Tor}^1_R(\overline{A}/(\overline{a}_1,\dots,\overline{a}_i),\overline{R}) = 0$ and, by assumption, multiplication by $\overline{a}_{i+1}$ is injective, hence $\mathrm{Tor}^1_R(\overline{A}/(\overline{a}_1,\dots,\overline{a}_{i+1}),\overline{R}) = 0$

To prove the sublemma, let $(r_1,\dots,r_m)$ be a regular system of parameters of R. By assumption $(r_1,\dots,r_m,a_1,\dots,a_n)$ is a regular sequence in $\mathrm{rad}(A)$, hence the sequence $(a_1,\dots,a_n)$ is also regular (see for instance [21], IV,§4). This proves the sublemma and thus we are finished with the proof of Lemma 7.2.

Proof of Proposition 7.1. We identify the affine space associated to the free R-module $\mathrm{Hom}_S(A,B)$ with $\mathbb{A}^d_S$. Let X be the subscheme of $\mathrm{Hom}_S(A,B)$ consisting of those F that satisfy the system of equations

$$(\star)\qquad \begin{cases} F(aa') = F(a)F(a') & \forall a,a' \in A \\ F(\sigma(a)) = \tau(F(a)) & \forall a \in A \\ F(1) = 1\ . \end{cases}$$

Let $p : X \to S$ be the canonical projection. On X there exists a tautological homomorphism $F : p^*(A,\sigma) \to p^*(B,\tau)$. Since A and B have the same rank, F is in fact an isomorphism of algebras with involutions. For each S-scheme $g : Y \to S$ there exists a funtorial bijection

$$\mathrm{Mor}_S(Y,X) \to \mathrm{Isom}_Y(g^*(A,\sigma), g^*(B,\tau))\ .$$

We denote by F_j the image of an S-morphism $j : Y \to X$ under this bijection. Note that the tautological isomorphism corresponds indeed to the identity of X. In particular, the given $\varphi : (A,\sigma)|_T \to (B,\tau)|_T$ corresponds to a section $\delta : T \to X$ as in Lemma 7.2, which we now want to apply for showing the existence of $\widetilde{S}$. The scheme X defined by $(\star)$ is smooth over S because it is a principal homogeneous space under the smooth S-group $\mathrm{Aut}_S(B,\tau)$. Let $\overline{X}$ be the closed subscheme of $\mathbb{P}^d_S$ defined by the homogenization of the system $(\star)$. To check that X and $\overline{X}$ satisfy the second assumption of Lemma 7.2 we can replace S by it strict henselization S^{sh}. Over S^{sh} the algebras A and B are isomorphic to a matrix algebra $\mathrm{M}_n(S^{sh})$, with the same involution $a \mapsto ua^tu^{-1}$, where u is either the identity matrix or the standard skew-symmetric matrix. Thus the system $(\star)$ is equivalent to a system of equations with coefficients in $\mathbb{Z}$ and (2) is obviously satisfied. By Lemma 7.2, X contains a closed subscheme $\widetilde{S}$ which is finite étale over S and contains $\delta(T)$. The inclusion $j : \widetilde{S} \to X$ defines, as we mentioned above, an isomorphism

$$F_j : (A,\sigma)|_{\widetilde{S}} \to (B,\tau)|_{\widetilde{S}}$$

which is the Φ we were looking for.

8. The specialization lemma for hermitian spaces over relative curves

We keep the notations and the assumptions made at the beginning of §3. We suppose, further, that $(\mathfrak{A},\mathfrak{s})$ is an Azumaya algebra with involution over $\mathcal{X}$.

Lemma 8.1. *Let $\mathfrak{h}$ be an ϵ-hermitian space over $(\mathfrak{A}, \mathfrak{s})$ which is hyperbolic over $(\mathfrak{A}, \mathfrak{s})_f$. Assume that the canonical bundle $\omega_{\mathcal{X}/k}$ is trivial, that there exists a finite surjective morphism $\mathcal{X} \to U \times \mathbb{A}^1_k$ of U-schemes and that there exists a section $\Delta : U \to \mathcal{X}$ of p such that p is smooth along $\Delta(U)$. Then the restriction $\Delta^*\mathfrak{h}$ of $\mathfrak{h}$ to $\Delta(U)$ is hyperbolic over $\Delta^*(\mathfrak{A}, \mathfrak{s})$.*

We need a few auxiliary results.

Lemma 8.2. *Let $q : \mathcal{X} \to U \times \mathbb{A}^1_k$ be a finite surjective morphism. Let $\mathcal{Y}$ be a closed nonempty subscheme of $\mathcal{X}$, finite over U. Let $\mathcal{V}$ be an open subset of $\mathcal{X}$ containing $q^{-1}(q(\mathcal{Y}))$. There exists an open set $\mathcal{W} \subseteq \mathcal{V}$ still containing $q^{-1}(q(\mathcal{Y}))$ and endowed with a finite surjective morphism (in general $\neq q$) $\mathcal{W} \to U \times \mathbb{A}^1_k$.*

Proof. Let $\mathcal{Z} = \mathcal{X} \setminus \mathcal{V}$. The sets $Z = q(\mathcal{Z})$ and $Y = q(\mathcal{Y})$ are closed because q is finite. If $I(Y)$ and $I(Z)$ are the ideals defining Y and Z, we have

$$I(Y) + I(Z) = R[t]$$

because Y and Z are obvioulsly disjoint. Let $\varphi \in I(Z)$ and $\rho \in I(Y)$ be such that $\varphi + \rho = 1$ Let $W = \mathrm{Spec}(R[t, 1/\varphi])$ be the principal open set defined by $\varphi \neq 0$ and put $\mathcal{W} = q^{-1}(W)$. It is clear that $q^{-1}(q(\mathcal{Y})) \subset \mathcal{W}$ and that $\mathcal{W} \subseteq \mathcal{V}$. Note that the reduction of φ modulo the maximal ideal of R is not zero because $\varphi|_Y \equiv 1$ and Y is finite over U. Since $q : \mathcal{W} \to W$ is finite, it suffices to construct a finite surjective morphism $W \to U \times \mathbb{A}^1_k$. This amounts to finding an $s \in R[t, 1/\varphi]$ such that $R[t, 1/\varphi]$ is finite over $R[s]$. To do this it suffices to take $s = \psi/\varphi$ where $\psi \in R[t]$ is a monic polynomial of degree larger than that of φ and comaximal with φ. In fact, in this case, on the one hand t satisfies the relation of integral dependence over $R[s]$

$$\psi(t) - \varphi(t)s = 0$$

and on the other hand, since $a\varphi + b\psi = 1$ for some polynomials a, b, its inverse $1/\varphi = a + bs$ is also integral over $R[s]$.

To show the existence of such a ψ we use the following result, in which "bar" denotes the reduction modulo the maximal ideal of R.

Lemma 8.3. *Let $\varphi, \psi \in R[t]$ and suppose that ψ is monic of positive degree and that $\overline{\varphi} \neq 0$. If $\overline{\varphi}$ and $\overline{\psi}$ are comaximal, so are φ and ψ.*

Proof. Choose a positive integer N such that $\varphi_1 = \varphi + \psi^N$ is monic and consider the natural homomorphism of finite R-modules

$$\alpha : R[t]/(\varphi_1, \psi) \to R[t]/(\varphi_1) \times R[t]/(\psi) .$$

Since $\overline{\varphi}_1$ and $\overline{\psi}$ are comaximal, $\overline{\alpha}$ is an isomorphism and hence, by Nakayama's lemma, α is an isomorphism too and the assertion follows immediately.

To finish the construction of s observe that since $\overline{\varphi} \neq 0$ we can choose a monic polynomial ψ' over $\overline{R}$, coprime with $\overline{\varphi}$ and of degree larger than that of φ. By the lemma above any monic lift $\psi \in R[t]$ of ψ' yields a suitable $s = \psi/\varphi$.

Proof of Lemma 8.1. Let $q : \mathcal{X} \to U \times \mathbb{A}^1_k$ be a finite surjective U-morphism. The following diagram summarizes the situation:

$$\begin{tikzcd} & \mathcal{Z} \arrow[d] & \\ \mathcal{X}_f \arrow[r, hook] & \mathcal{X} \arrow[r, "q"] \arrow[d, shift left, "p"] & U \times \mathbb{A}^1_k \\ & U \arrow[u, shift left, "\Delta"] & \end{tikzcd}$$

Here $\mathcal{Z}$ is the closed subscheme defined by the equation $f = 0$. By assumption, $\mathcal{Z}$ is finite over U. Let $\mathcal{Y} = q^{-1}(q(\mathcal{Z} \cup \Delta(U)))$. Since $\mathcal{Z}$ and $\Delta(U)$ are finite over U and since q is a finite morphism of U-schemes, $\mathcal{Y}$ is finite over U. Denote by $y_1, \dots, y_n$ its closed points and let $S = \mathrm{Spec}(\mathcal{O}_{\mathcal{X}, y_1, \dots, y_n})$ and $T = \Delta(U) \subseteq S$. Let $(A, \sigma) = \Delta^*(\mathfrak{A}, \mathfrak{s})$ and let $(\mathfrak{B}, \mathfrak{t})$ be the inverse image of (A, σ) on S. We denote by $\varphi : (\mathfrak{A}, \mathfrak{s})|_T \to (\mathfrak{B}, \mathfrak{t})|_T$ the canonical isomorphism. Recall that $\mathcal{X}$ is k-smooth by assumption and thus S is regular. By Proposition 7.1 there exists a finite étale covering $\pi_0 : \widetilde{S} \to S$, a section $\delta : T \to \widetilde{S}$ of π_0 over T and an isomorphism

$$\Phi_0 : \pi_0^*(\mathfrak{A}, \mathfrak{s}) \to \pi_0^*(\mathfrak{B}, \mathfrak{t})$$

such that $\delta^* \Phi_0 = \varphi$. We can extend all these data to a neighbourhood $\mathcal{V}$ of $\{y_1, \dots, y_n\}$ and get a diagram

$$\begin{tikzcd} & \widetilde{S} \arrow[r, hook] \arrow[d, "\pi_0"] & \widetilde{\mathcal{V}} \arrow[d, "\pi"] & \\ T \arrow[ur, "\delta"] \arrow[r, hook] & S \arrow[r, hook] & \mathcal{V} \arrow[r, hook] & \mathcal{X} \end{tikzcd}$$

with $\pi : \widetilde{\mathcal{V}} \to \mathcal{V}$ finite étale, and an isomorphism $\Phi : \pi^*(\mathfrak{A}, \mathfrak{s}) \to \pi^*(\mathfrak{B}, \mathfrak{t})$. Since T projects isomorphically onto U, T is still a closed subscheme of $\mathcal{V}$. Note that $\mathcal{V}$ contains $q^{-1}(q(\mathcal{Y})) = \mathcal{Y}$ because $\mathcal{Y}$ is semilocal and $\mathcal{V}$ contains all of its closed points. By Lemma 8.2 there exists an open subset $\mathcal{W} \subseteq \mathcal{V}$ containing $\mathcal{Y}$ and endowed with a finite surjective U-morphism $r : \mathcal{W} \to U \times \mathbb{A}^1_k$. Let $\widetilde{\mathcal{X}} = \pi^{-1}(\mathcal{W})$, $\widetilde{p} : \widetilde{\mathcal{X}} \to U$ the structural morphism and $\widetilde{\Delta} : U \to \widetilde{\mathcal{X}}$ the section of $\widetilde{p}$ obtained by composing δ with Δ. We still denote by Φ the restriction of Φ to $\widetilde{\mathcal{X}}$. Since $\pi^*(\mathfrak{B}, \mathfrak{t})|_{\widetilde{\mathcal{X}}} = \widetilde{p}^*(A, \sigma)$, the space $\widetilde{\mathfrak{h}} = \pi^* \mathfrak{h}$ can be considered — through Φ — as a space over the constant algebra $\widetilde{p}^*(A, \sigma)$. Since $\Delta = \pi \circ \widetilde{\Delta}$ and $\widetilde{\Delta}^*(\Phi) : (A, \sigma) \to (A, \sigma)$ is the identity map, $\widetilde{\Delta}^* \widetilde{\mathfrak{h}} = \Delta^* \mathfrak{h}$ as spaces over (A, σ). Thus, to show that $\Delta^* \mathfrak{h}$ is hyperbolic it suffices to show that $\widetilde{\Delta}^* \widetilde{\mathfrak{h}}$ is hyperbolic. Denoting by $\widetilde{f} \in k[\widetilde{\mathcal{X}}]$ the composition of f with π and by $\widetilde{\mathcal{Z}}$ the vanishing locus of $\widetilde{f}$, we only have to check that $\widetilde{\mathcal{X}}$, $\widetilde{f}$, $\widetilde{\mathcal{Z}}$, $\widetilde{\mathfrak{h}}$ and (A, σ) satisfy the hypotheses of Lemma 4.1.

The morphism $\pi : \widetilde{\mathcal{X}} \to \mathcal{W}$ is finite and surjective and we have constructed a finite surjective morphism $r : \mathcal{W} \to U \times \mathbb{A}^1_k$, hence $r \circ \pi : \widetilde{\mathcal{X}} \to U \times \mathbb{A}^1_k$ is finite and surjective and therefore $\widetilde{\mathcal{X}}$ is affine over U. The scheme $\widetilde{\mathcal{X}}$ is smooth over k because $\widetilde{\mathcal{X}} \to \mathcal{X}$ is étale and $\mathcal{X}$ is smooth over k. Since $\widetilde{\mathcal{X}} \to \mathcal{X}$ is étale, it is flat and therefore, f being regular in $k[\mathcal{X}]$, $\widetilde{f}$ is regular in $k[\widetilde{\mathcal{X}}]$. Since $\mathcal{Z} \subset \mathcal{W}$ and $\pi : \widetilde{\mathcal{X}} \to \mathcal{W}$ is finite, $\widetilde{\mathcal{Z}}$ is finite over $\mathcal{Z}$ and hence also over U. By assumption $\omega_{\mathcal{X}/k}$ is trivial and hence, since $\widetilde{\mathcal{X}}$ is étale over $\mathcal{X}$, $\omega_{\widetilde{\mathcal{X}}/k}$ is trivial. We already mentioned the existence of a finite surjective morphism of U-schemes $\widetilde{\mathcal{X}} \to U \times \mathbb{A}^1_k$. The space $\widetilde{\mathfrak{h}}_{\widetilde{f}}$ is clearly hyperbolic. It remains to check that $\widetilde{p}$ is smooth along $\widetilde{\Delta}(U)$. This is clear because $\widetilde{p}$ is smooth along $\pi^{-1}(\Delta(U)) \supseteq \widetilde{\Delta}(U)$.

This shows that all the hypotheses of Lemma 4.1 are satisfied. We conclude that $\widetilde{\Delta}^*\widetilde{\mathfrak{h}}$ is hyperbolic and this finishes the proof of Lemma 8.1.

9. Rationally trivial hermitian spaces are locally trivial

Theorem 9.1. *Let R be a local ring of a smooth variety over a field k of characteristic different from 2 and K the field of fractions of R. Let (A, σ) be an Azumaya algebra with involution over R and $\mathfrak{h}$ an ϵ-hermitian space over (A, σ). If $\mathfrak{h}_K$ is hyperbolic, then $\mathfrak{h}$ is hyperbolic.*

Proof. It is the same as the proof of Theorem 5.1, using Lemma 8.1 instead of Lemma 3.1.

Theorem 9.2. *Let R be a regular local ring containing a field k of characteristic different from 2 and let K be the field of fractions of R. Let (A, σ) be an Azumaya algebra with involution over R and $\mathfrak{h}$ an ϵ-hermitian space over (A, σ). If $\mathfrak{h}_K$ is hyperbolic, then $\mathfrak{h}$ is hyperbolic.*

Proof. By Popescu's theorem (see [15], [16] and [17] or [2] or, for a self-contained proof, [23]) R is a limit of essentially smooth local algebras over the prime field of k. Using this result and Theorem 9.1 the proof follows from the formal arguments given in [12], §8.

10. The geometric presentation lemma

Lemma 10.1. *Let R be a local essentially smooth algebra over an infinite field k, m its maximal ideal and A an essentially smooth k-algebra, which is finite over the polynomial algebra $R[t]$. Suppose that $\epsilon : A \to R$ is an R-augmentation and let $I = \ker \epsilon$. Assume that A is smooth over R at every prime containing I. Given $f \in A$ such that A/Af is finite over R we can find an $s \in A$ such that*

1. *A is finite over $R[s]$.*
2. *$A/As = A/I \times A/J$ for some ideal J of A.*
3. *$J + Af = A$.*
4. *$A(s-1) + Af = A$.*

Proof. Replacing t by $t - \epsilon$ we may assume that $t \in I$. We denote by "bar" the reduction modulo m. By the assumptions made on A the quotient $\overline{A}$ is smooth over $\overline{R}$ at its maximal ideal $\overline{I}$. Choose an $\alpha \in A$ such that $\overline{\alpha}$ is a local parameter of the localization $\overline{A}_{\overline{I}}$ of $\overline{A}$ at $\overline{I}$. By the chinese remainders' theorem we may assume that $\overline{\alpha}$ does not vanish at the zeros of $\overline{f}$ different from $\overline{I}$. Without changing $\overline{\alpha}$ we may replace α by $\alpha - \epsilon(\alpha)$ and assume that $\alpha \in I$. Since A is integral over $R[t]$ there exists a relation of integral dependence

$$\alpha^n + p_1(t)\alpha^{n-1} + \cdots + p_n(t) = 0 .$$

For any $r \in k^*$ and any N larger than the degree of each $p_i(t)$, putting $s = \alpha - rt^N$ we see that from the equation above that t is integral over $R[s]$. Hence A, which is integral over $R[t]$, is integral over $R[s]$. Clearly $s \in I$. To insure that $\overline{s}$ is also a local parameter of $\overline{A}_{\overline{I}}$ it suffices to take $N \geq 2$. By assumption A and $R[s]$ are both regular and since A is finite over $R[s]$, by Corollary 18.17 of [6], A is locally free over $R[s]$. Since $\overline{s}$ is a local parameter of $\overline{A}_{\overline{I}}$, $\overline{A}/\overline{s}\overline{A}$ is étale over $\overline{R}$ at the augmentation ideal $\overline{I}$ and so we can find a $g \notin I + mA$ such that $(A/As)_g$ is étale over R. By the next sublemma A/As splits as in (2).

Sublemma 10.2. *Let B be a commutative ring, $\gamma : B \to C$ a finite commutative B-algebra and $\lambda : C \to B$ an augmentation with augmentation ideal I. Let $h \in C$ be such that*

(a) *C_h is étale over B.*
(b) *$\lambda(h)$ is invertible in B.*

Then C splits as $C/I \times C/J$ for some ideal J of C.

Proof. Since $B \to C_h$ is étale and the composite map $B \xrightarrow{\gamma} C_h \xrightarrow{\lambda} B$ is the identity of B, by Prop. 4.7 of [1] $C_h \to B$ is étale. But $C \to C_h$ is étale, hence $\lambda : C \to B$ is étale and in particular it induces an open morphism $\lambda^* : \mathrm{Spec}(B) \to \mathrm{Spec}(C)$. Its image $\lambda^*(\mathrm{Spec}(B)) = \mathrm{Spec}(C/I)$ is therefore open and since it is also closed, C splits as claimed.

To finish the proof of Lemma 10.1 we still have to choose $r \in k^*$ so that conditions (3) and (4) are satisfied. Since A/Af is semilocal, there are only finitely many maximal ideals of A containing f. We denote by $m_1, \dots, m_p$ those which, in case $f \in I + mA$, are different from $I + m$. Recalling that α was chosen outside $m_1 \cup \cdots \cup m_p$, we have $s \notin m_1 \cup \cdots \cup m_p$ for almost any choice of $r \in k^*$. To see that condition (3) is satisfied it suffices to show that $J \not\subseteq m_i$ for $1 \leq i \leq p$ and that $J \not\subseteq mA + I$. The first assertion is clear because $s \in J \setminus m_i$ for $1 \leq i \leq p$. For the second one note that, since $A/As = A/I \times A/J$, we have $I + J = A$ and therefore $J \not\subseteq mA + I$. It remains to satisfy (4). Since A/Af is semilocal there exists a $\lambda \in k$ such that $s - \lambda$ is invertible in A/Af. Without perturbing conditions (1), (2) and (3) we may replace s by $\frac{1}{\lambda}s$ and thus satisfy (4) as well.

11. The Euler trace

Let k be any field and $A \hookrightarrow B$ a finite extension of smooth k-algebras of dimension d. Let Ω_A and Ω_B be the modules of Kähler differentials of A and B over k and let $\Omega_{B/A}$ be the module of relative differentials of B over A. Let $\omega_A = \bigwedge^d \Omega_A$, $\omega_B = \bigwedge^d \Omega_B$.

Proposition 11.1. *There exists an isomorphism of B-modules*

$$\omega_B \to \mathrm{Hom}_A(B, \omega_A) .$$

Proof. Let R be the polynomial algebra $A[X_1, \dots, X_n]$ and $\rho : R \to B$ a surjective homomorphism of A-algebras. Let $I = \ker(\rho)$. Since B is a local complete intersection over A, by Lemma 4.4 of [20] there exists an isomorphism of B-modules

$$(*) \qquad \mathrm{Hom}_A(B, A) \simeq \bigwedge^n \left(\mathrm{Hom}_B(I/I^2, B)\right) .$$

On the other hand, from the canonical exact sequence of projective B-modules (see [1], VII, Theorem 5.8)

$$0 \to I/I^2 \to B \otimes_R \Omega_R \to \Omega_B \to 0 \ ,$$

we deduce, taking maximal exterior powers, that

$$\omega_B \otimes_B \bigwedge^n \left(I/I^2\right) \simeq B \otimes_A \omega_A \ . \tag{$\dagger$}$$

From (†) we get, using the fact that I/I^2 is a finitely generated projective B-module,

$$\omega_B \simeq (B \otimes_A \omega_A) \otimes_B \mathrm{Hom}_B \left(\bigwedge^n (I/I^2), B \right) \simeq (B \otimes_A \omega_A) \otimes_B \bigwedge^n \left(\mathrm{Hom}_B(I/I^2, B)\right)$$

and then, from (∗),

$$(B \otimes_A \omega_A) \otimes_B \bigwedge^n \left(\mathrm{Hom}_B(I/I^2, B)\right) \simeq \omega_A \otimes_A \mathrm{Hom}_A(B, A) \simeq \mathrm{Hom}_A(B, \omega_A) \ .$$

Corollary 11.2. *If ω_A and ω_B are trivial, then there exists an isomorphism of B-modules*

$$\lambda : B \simeq \mathrm{Hom}(B, A) \ .$$

The isomorphism λ induces an A-linear map

$$\mathcal{E} : B \to A$$

defined by $\mathcal{E}(x) = \lambda(1)(x)$. Conversely, from $\mathcal{E}$ we get back λ by $\lambda(x)(y) = \mathcal{E}(xy)$.

We call $\mathcal{E}$ *the Euler trace*, because Euler used a special case of it (see [7] and also [22], Chap. III). If $B = A[t]/(f)$, f a monic polynomial, $\mathcal{E}$ may be defined by $\mathcal{E}(b) = \mathrm{tr}(b/f'(\tau))$, where τ is the class of t in B and tr is the usual trace for the A-algebra B.

Proposition 11.3. *Let B be a finite locally free A-algebra and $\mathcal{E} : B \to A$ an A-linear map such that the bilinear map*

$$\lambda : B \to \mathrm{Hom}_A(B, A) \quad \textit{given by} \quad \lambda(x)(y) = \mathcal{E}(xy)$$

is an isomorphism. Then, for every $A \to A'$, we have an A'-linear map

$$\mathcal{E}' = \mathcal{E} \otimes_A A' : B' = B \otimes_A A' \to A'$$

such that the associated $\lambda' : B' \to \mathrm{Hom}_{A'}(B', A')$ is an isomorphism of B'-modules. If $B = B_1 \times B_2$, λ decomposes as $\lambda_1 \times \lambda_2$, where $\lambda_i : B_i \to \mathrm{Hom}_A(B_i, A)$ is the map associated to $\mathcal{E}|_{B_i}$. In particular, if $B = B_1 \times A$, then the map $\lambda_2 : A \to A$ is the multiplication by a unit of A.

12. Traces and hermitian spaces

We recall a few notions and constructions concerning hermitian spaces. We refer to [10] for unexplained terminology and omitted proofs. By projective module we always mean a projective module of finite type.

Let $\epsilon = \pm 1$ and let (A, σ) be an Azumaya algebra with involution (see §4). An ϵ-sesquilinear form over (A, σ) is a pair $\mathbf{h} = (V, h)$ consisting of a projective right A-module V and a biadditive map $h : V \times V \to A$ satisfying

$$h(va, v'a') = \sigma(a)h(v, v')a' \quad \text{and} \quad h(v, v') = \epsilon\sigma(h(v', v)) .$$

for all $a, a' \in A$ and $v, v' \in V$. The pair $\mathbf{h}$ is an ϵ-hermitian space if, further, the map $\lambda : V \to \mathrm{Hom}_A(V, A)$ given by $\lambda(v)(v') = h(v, v')$ is an isomorphism.

An isometry of hermitian spaces $(V, h) \to (V', h')$ is an A-linear isomorphism $\varphi : V \to V'$ such that $h(v, w) = h'(\varphi(v), \varphi(w))$ for all $v, w \in V$.

Let W be a projective right A-module. The dual $W^* = \mathrm{Hom}_A(W, A)$ has a natural structure of right A-module given by $(fa)(x) = \sigma(a)f(x)$. The hyperbolic space associated to W is

$$H^\epsilon(W) = (W \oplus W^*, h) \quad \text{where} \quad h((w, f), (w', f')) = f(w') + \epsilon\sigma(f'(w)) .$$

An ϵ-hermitian space is said to be hyperbolic if it is isometric to a space of the form $H^\epsilon(W)$.

Let now $A \hookrightarrow B$ be a finite flat extension of commutative rings and let $(\mathfrak{A}, \mathfrak{s})$ be an Azumaya algebra with involution over A. Let $\mathcal{E} : B \to A$ be an A-linear map such that the associated $\lambda : B \to \mathrm{Hom}_A(B, A)$ is an isomorphism. To every ϵ-hermitian space $\mathbf{h} = (V, h)$ over $(\mathfrak{A}, \mathfrak{s}) \otimes_A B$ we associate the ϵ-sesquilinear form $\mathrm{Tr}^{\mathcal{E}}(\mathbf{h}) = (V_A, \mathcal{E} \circ h)$, where V_A denotes V considered as an A-module. This sesquilinear form is in fact an ϵ-hermitian space and it is easy to check (see [10], I, §7) that Tr has the following properties:

(1) $\mathrm{Tr}^{\mathcal{E}}(\mathbf{h} \perp \mathbf{h}') = \mathrm{Tr}^{\mathcal{E}}(\mathbf{h}) \perp \mathrm{Tr}^{\mathcal{E}}(\mathbf{h}')$.

(2) If $\mathbf{h}$ is hyperbolic, $\mathrm{Tr}^{\mathcal{E}}(\mathbf{h})$ is hyperbolic.

(3) For any homomorphism of commutative rings $A \to A'$ we have

$$\mathrm{Tr}^{\mathcal{E}}(\mathbf{h} \otimes_A A') = \mathrm{Tr}^{\mathcal{E}}(\mathbf{h}) \otimes_A A' ,$$

where $\mathcal{E}' = \mathcal{E} \otimes_A A'$.

(4) If, as at the end of §11, $B = B_1 \times B_2$ and $\mathcal{E}_i = \mathcal{E}|_{B_i}$,

$$\mathrm{Tr}^{\mathcal{E}}(\mathbf{h}) = \mathrm{Tr}^{\mathcal{E}_1}(\mathbf{h}_1) \perp \mathrm{Tr}^{\mathcal{E}_2}(\mathbf{h}_2) ,$$

where $\mathbf{h}_i = \mathbf{h} \otimes_B B_i$.

(5) If, as in (4), $B = B_1 \times B_2$ but $B_2 = A$, there exists a unit $u \in A^*$ such that, for any $\mathbf{h}$,

$$\mathrm{Tr}^{\mathcal{E}_2}(\mathbf{h}_2) = u \cdot \mathbf{h}_2 .$$

(6) The linear map $\mathcal{E} : B \to A$ induces a homomorphism of Witt groups

$$\mathrm{Tr}^{\mathcal{E}} : \mathrm{W}^\epsilon((\mathfrak{A}, \mathfrak{s}) \otimes_A B) \to \mathrm{W}^\epsilon((\mathfrak{A}, \mathfrak{s})) .$$

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