QUADRATIC FORMS OVER COMPLETE LOCAL RINGS

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1. Introduction

Let A be a complete excellent local domain of Krull dimension 2 and K its field of fractions. We further assume that 2 is invertible in A and that the residue field of A is algebraically closed. We first show that the unramified Brauer group of K (with respect to all discrete valuations of K) vanishes. Using this result we prove that every rank 4 quadratic form which is isotropic in all completions of K with respect to discrete valuations, is isotropic. For $K = \mathbb{C}((x, y))$ this was announced by P. Jaworski in 1999, at a conference on quadratic forms in Oberwolfach. The results presented here, obtained immediately after the conference, are a consequence of our efforts to give a short proof of Jaworski's announcement. They subsequently led to various generalizations, notably in the case when the residue field of A is real closed (see [1] and, for further developments, [2] and [3]). Jaworski's proof has now appeared as well [6].

2. The unramified Brauer group

Let A be as above and m its maximal ideal. Let $\pi : X \to \mathrm{Spec}(A)$ be a desingularization [8] of $\mathrm{Spec}(A)$. Since X is obtained from $\mathrm{Spec}(A)$ by a sequence of blowing ups and normalizations, the map π is proper. We denote by X_n the fibre of $\mathrm{Spec}(A/m^{n+1})$.
We prove a result (Lemma 2.2) in the spirit of Lemma 3.3 of [5].

Lemma 2.1. *The natural maps*

$$Pic(X_n) \to Pic(X_{n+1})$$

are surjective.

Proof. This follows from the exact sequence of sheaves

$$0 \longrightarrow \frac{m^n\mathcal{O}_X}{m^{n+1}\mathcal{O}_X} \longrightarrow \left(\frac{\mathcal{O}_X}{m^{n+1}\mathcal{O}_X}\right)^* \longrightarrow \left(\frac{\mathcal{O}_X}{m^{n}\mathcal{O}_X}\right)^* \longrightarrow 1 ,$$

noting that

$$H^2\big(X, \frac{m^n\mathcal{O}_X}{m^{n+1}\mathcal{O}_X}\big) = H^2\big(X_0, \frac{m^n\mathcal{O}_X}{m^{n+1}\mathcal{O}_X}\big) = 0$$

because X_0 is of dimension 1.

Lemma 2.2. *The canonical homomorphism*

$$Br(X) \to \lim_{\leftarrow} Br(X_n)$$

is injective.

Proof. Let $\mathcal{A}$ be an Azuamya algebra over X. Denote by $\mathcal{A}_n$ the algebra obtained from $\mathcal{A}$ under base change from X to X_n and suppose that it is trivial for each n. Let

$$u_n : \mathcal{A}_n \xrightarrow{\sim} \mathcal{E}nd(V_n)$$

be an isomorphism, where V_n is a locally free sheaf on X_n. The sheaf V_n is determined by $\mathcal{A}_n$ up to a line bundle. By Lemma 2.1 we can successively modify each V_n in such a way that V_n is isomorphic to $V_{n+1} \otimes_{\mathcal{O}_{X_{n+1}}} \mathcal{O}_{X_n}$. In this case the isomorphisms u_n also form a projective system. By [5], 5.1.4, the projective system $(V_n, n \in \mathbb{N})$ gives a locally free $\mathcal{O}_X$-module V and an isomorphism

$$u : \mathcal{A} \xrightarrow{\sim} \mathcal{E}nd(V) .$$

Corollary 2.3. *The Brauer group of X is trivial. In particular, the unramified Brauer group of K is trivial.*

Proof. In fact, since X_n is a curve over $\mathrm{Spec}(A/m^{n+1})$ and A/m is algebraically closed, $Br(X_n) = 0$ (See [5], page 101). By a well-known purity theorem ([4], Proposition 2.3) an unramified element of $Br(K)$ is in the image of $Br(X) \to Br(K)$ and hence is zero.

3. Quadratic forms

Theorem. *Let A be a complete excellent local domain of Krull dimension 2 and K its field of fractions. Assume that 2 is invertible in A and that the residue field of A is algebraically closed. Every rank 4 quadratic form q over K which is isotropic over every completion of K at a discrete valuation, is isotropic.*

Proof. After scaling we may assume that $q =< 1, a, b, abd >$ with $a, b, d \in K^*$. If d is a square, then q is the norm form of the quaternion algebra $\mathcal{A} = \left(\frac{a,b}{K}\right)$. The condition that q is isotropic at all completions implies that $\mathcal{A}$ is split at all completions of K. In particular $\mathcal{A}$ is unramified in $Br(K)$ and hence, by 2.3, is zero. In particular, q is hyperbolic.
Suppose now that d is not a square. Let $L = K(\sqrt{d})$. The field L satisfies the same assumptions as K. The form q_L over L has trivial discriminant and is isotropic at all completions of L at discrete valuations. By the previous case, q_L is hyperbolic. The form q therefore contains a multiple of $< 1, -d >$ ([7], Ch. 7, Lemma 3.1) and, being of discriminant d, also contains a subform of discriminant 1. Hence q is isotropic.

References

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