Master Thesis

Fall 2010

Cobordism, Cohomology Theories and Formal Group Laws

Lev Kiwi

Supervised at
Northwestern University by:

Prof. Paul Goerss

Responsible Professor at
the Swiss Federal Institute
of Technology of Lausanne:

Prof. Kathryn Hess-Bellwald

NORTHWESTERN UNIVERSITY

ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE

Abstract

We first explore the notion of G-manifold, cobordism and then discuss the generalized Pontrjagin-Thom Theorem. We then compute the cobordism rings Ω_*^O and Ω_*^U using stable homotopy theory.
Secondly, we investigate the relations between cohomology theories and formal group laws. This is the very begininning of the so called *chromatic homotopy theory*. For any complex oriented cohomology theories, we associate a formal group law. We show that the formal group law associated to complex cobordism is universal.
Finally, for a suitable formal group law we give a way to realize it into a generalized homology theory.

Contents

Acknowledgments

I would like to thank Paul Goerss for his kindness and assistance during all my Master Thesis.
I am also grateful to Kathryn Hess for her help to realize this exchange semester.
I would also thank the United States of America for finally giving me the VISA.

1 Introduction

Let $G = (G_r)_r$ be a sequence of subgroups G_r of the real orthogonal group $O(r)$. By a G-manifold we mean a smooth manifold M with a G-structure on its stable normal bundle. Roughly speaking, this is a homotopy class of liftings $[\hat{\nu}]$ such that the following diagram commutes

$$\begin{array}{ccc} & & BG = \operatorname{colim} BG_r \\ & \overset{\hat{\nu}}{\nearrow} & \downarrow f \\ M & \xrightarrow[\nu]{} & BO = \operatorname{colim} BO(r). \end{array}$$

The map f is a fibration and ν is a map classifying the stable normal bundle of M.

In the framework of G-manifolds, there is a generalized Pontrjagin-Thom Theorem:

$$\Omega_*^G \cong \pi_*(\boldsymbol{MG}).$$

On the left hand side we have the cobordism ring of G-manifolds, while on the right hand side, $\boldsymbol{MG}$ denotes the Thom spectrum associated to G. Our purpose in Chapter 2 is to compute $\pi_*(\boldsymbol{MO})$ and $\pi_*(\boldsymbol{MU})$, and hence have a description of the rings Ω_*^O and Ω_*^U.

The first computation that we describe is due to Thom [Tho54]. In his paper, he show the following result.

Proposition 1.1. *The cohomology group $H^*(\boldsymbol{MO}; \mathbb{Z}/2)$ is a free module over the Steenrod algebra $\mathcal{A}_2$.*

Therefore, one deduce by the Whitehead Theorem that $\boldsymbol{MO}$ has the homotopy type of a product of Eilenberg-MacLane spectra $\prod \Sigma^h \boldsymbol{H}\mathbb{Z}/2$. Stable homotopy groups are then easy to compute. The full ring structure of the real cobordism ring is described in the following theorem.

Theorem 1.2. $\Omega_n^O \cong \mathbb{Z}/2[x_i : i \neq 2^j - 1]$

The computation of the complex cobordism ring Ω_*^U is a generalization of Thom's argument. This is due to Milnor and can be found in his celebrated paper [Mil60]. He show the following result.

Proposition 1.3. *The cohomology group $H^*(\boldsymbol{MU}; \mathbb{Z}/p)$ is a free module over $\mathcal{A}_p/(Q_0)$, where $\mathcal{A}_p$ denotes the Steenrod algebra and (Q_0) is the two-side ideal generated by the Bockstein homomorphism.*

Using the Adams spectral sequence, we can then compute the stable homotopy groups $\pi_*(\boldsymbol{MU})$. The ring structure of Ω_*^U is described in the following theorem.

Theorem 1.4. $\Omega_*^U \cong \mathbb{Z}[x_2, x_4, x_6, \dots]$

Then we consider complex oriented cohomology theories. This is a commutative multiplicative cohomology theory E^*, with a natural choice of Thom class $U_\xi \in \tilde{E}^{2n}(T\xi)$ for any n-dimensional complex vector bundle ξ. Main examples of such cohomology theories are:

- Singular cohomology with ring coefficients,
- K-Theory,
- Complex cobordism.

Consider the map $m : \mathbb{C}P^\infty \times \mathbb{C}P^\infty \to \mathbb{C}P^\infty$ classifying the tensor product of universal line bundles $\gamma^1 \otimes \gamma^1$. For a complex oriented cohomology theory E^*, the map m induces a ring homomorphism

$$E^*(pt)[\![x]\!] \cong E^*(\mathbb{C}P^\infty) \longrightarrow E^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty) \cong E^*(pt)[\![x, y]\!],$$

where $x \in E^*(pt)[\![x]\!]$ is the first Chern class of $\mathbb{C}P^\infty$. It turns out that the image of x is a formal group law $F(x, y) \in E^*(pt)[\![x, y]\!]$. This is a formal power series (in two variables) satisfying the following properties

- $F(F(x,y),z) = F(x,F(y,z))$,
- $F(x,y) = F(y,x)$,
- $F(x,0) = x = F(0,x)$.

Among formal group laws there exists a universal formal group law. This is a formal group law $F_u(x,y)$ over a ring L such that any formal group law $G(x,y)$ are obtained from $F_u(x,y)$ by a base change.

Theorem 1.5 (Lazard, Quillen). *The ring L is isomorphic to $\pi_*(\boldsymbol{MU}) \cong \mathbb{Z}[x_2, x_4, x_6, \dots]$ and the formal group law associated to complex cobordism is the universal formal group law.*

If we let MU^* denote the complex cobordism cohomology, i.e. the generalized cohomology theory associated to the Thom spectrum $\boldsymbol{MU}$. We can then say that it is in some sense the "most complicated" complex oriented cohomology theory. Using this, we can create new (co)homology theories by considering the tensor functor

$$X \mapsto MU_*(X) \otimes_{\pi_*(\boldsymbol{MU})} R.$$

If $F(x,y)$ is a formal group law over R, there exists a map $f : \pi_*(\boldsymbol{MU}) \to R$, which induces a structure of $\pi_*(\boldsymbol{MU})$-module over R. Considering a finite CW-complex X, we can see that $MU_*(X)$ is a finitely presented $\pi_*(\boldsymbol{MU})$-module. Moreover, $MU_*(X)$ has a $MU_*(MU)$-comodule structure. Hence, let $\mathcal{MU}$ denote the category of $MU_*(MU)$-comodules which are finitely presented $\pi_*(\boldsymbol{MU})$-modules.

In [Lan76], Landweber gives a sufficient and necessary condition on $F(x,y)$ such that the functor

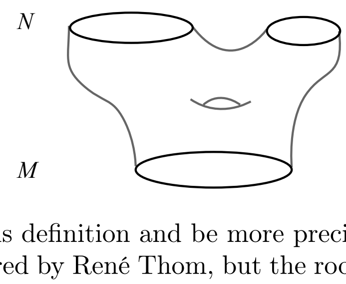
$$M \mapsto M \otimes_{\pi_*(\boldsymbol{MU})} R$$

is exact whenever M belongs to $\mathcal{MU}$.

At the end, we compute some stable homotopy groups of spheres using two different tools that we use throughout this paper. The first method uses the Serre exact sequence for a fibration. While the second method uses the Adams spectral sequence.

2 Cobordism

Roughly speaking, the idea of cobordism theory is to classify manifolds via surgery. To do this, we introduce the following equivalence relation: two manifolds M, N are *cobordant* if their disjoint union is the boundary of some manifold.



We will come back later on this definition and be more precise.

This idea was deeply explored by René Thom, but the roots of cobordism is due to Henry Poincaré in his *Analysis Situs* and in its complementary papers. His idea of homology is very close to the modern framework elaborated by Thom. The interested reader will find more informations about Poincaré's work in [Lef99].

In this section, we will give some great computations and beautiful results due mostly to Thom. The reader can evaluate how much they are important for stable algebraic topology in [May99b].

Throughout this paper all our manifolds are supposed smooth and compact.

2.1 (BG, f)-Manifolds

Let G_r be a subgroup of $O(r)$, the real orthogonal group. Then, one can form the classifying spaces BG_r and $BO(r)$. We have also a map $BG_r \to BO(r)$ induced by the inclusion that we can replace by a fibration $f_r : BG_r \to BO(r)$. Recall that $BO(r)$ is weakly equivalent to $G_r(\mathbb{R}^\infty)$, the Grassmann manifold of r-planes in $\mathbb{R}^\infty$. We have a natural r-plane bundle over $BO(r)$ consisting of pairs (X, v), where $X \in G_r(\mathbb{R}^\infty)$ is an r-plane and $v \in X$ is a vector. This bundle is called the *universal bundle* and is denoted by γ^r.

Let M be an n-dimensional manifold, and suppose we have an embedding $i : M \to \mathbb{R}^{n+r}$. By the Whitney embedding theorem, this map always exists for r big enough (depending only on n). Let $\nu(i)$ denote the normal bundle of M, this is the quotient of the pullback of the tangent bundle of $\mathbb{R}^{n+r}$ by the tangent bundle of M. The total space of the normal bundle can also be identified with the orthogonal complement of the tangent bundle. Moreover, if r is large enough the embedding is unique up to homotopy. Therefore, the normal bundle is unique up to isomorphism.

Definition. Let $\nu(i)$ be classified by the map $\nu : M \to BO(r)$. A (BG_r, f_r)-*structure* on $\nu(i)$ is a homotopy class of liftings $\hat{\nu}$ of ν to BG_r. This is an equivalence class of maps $\hat{\nu} : M \to BG_r$ such that the diagram

$$\begin{array}{ccc} & & BG_r \\ & \overset{\hat{\nu}}{\nearrow} & \downarrow{\scriptstyle f_r} \\ M & \xrightarrow{\nu} & BO(r) \end{array}$$

commutes. Two maps $\hat{\nu}$ and $\hat{\nu}'$ are equivalent, if there exists an homotopy H from $\hat{\nu}$ to $\hat{\nu}'$ and such that

$$\begin{array}{ccccc} & & & & BG_r \\ & & & \overset{H}{\nearrow} & \downarrow{\scriptstyle f_r} \\ M \times I & \xrightarrow{p_M} & M & \xrightarrow{\nu} & BO(r) \end{array}$$

commutes.

Suppose now given a sequence of groups $(G_r)_r$ with each G_r contained in $O(r)$, and let $f_r : BG_r \to BO(r)$ be a sequence of fibrations. Suppose also that we have maps $i_r : BG_r \to BG_{r+1}$ such that the diagram

$$\begin{array}{ccccccc} \cdots & \longrightarrow & BG_r & \xrightarrow{i_r} & BG_{r+1} & \xrightarrow{i_{r+1}} & \cdots \\ & & \downarrow{\scriptstyle f_r} & & \downarrow{\scriptstyle f_{r+1}} & & \\ \cdots & \longrightarrow & BO(r) & \xrightarrow{j_r} & BO(r+1) & \xrightarrow{j_{r+1}} & \cdots \end{array}$$

commutes, where $j_r : BO(r) \to BO(r+1)$ are the standard inclusions. Remark that a (BG_r, f_r)-structure on the normal bundle $\nu(i)$ defines a unique (BG_{r+1}, f_{r+1})-structure via the embedding $M \xrightarrow{i} \mathbb{R}^{n+r} \subset \mathbb{R}^{n+r+1}$.

Definition. A (BG, f)-*structure* on the stable normal bundle of M is a homotopy class of liftings $\hat{\nu}$ to $BG := \operatorname{colim}_r BG_r$. Hence, we have the following commutative diagram

$$\begin{array}{ccc} & & BG \\ & \overset{\hat{\nu}}{\nearrow} & \downarrow{\scriptstyle f} \\ M & \xrightarrow{\nu} & BO. \end{array}$$

In terms of BG_r's, this is can also be understood as an equivalence class of sequences of (BG_r, f_r)-structures $(\hat{\nu}_r)_r$ on the normal bundle of M. Two such sequences $(\hat{\nu}_r)_r$ and $(\hat{\nu}'_r)_r$ are equivalent if they agree for a sufficiently large r.

A (BG, f)-*manifold* is a pair consisting of a manifold M and a (BG, f)-structure on the stable normal bundle of M.

One may ask why this definition does not depend on the choice of the embedding $i : M \to \mathbb{R}^{n+r}$. In fact, if $j : M \to \mathbb{R}^{n+r}$ is another embedding, then there is a one-to-one correspondence between the (BG, f)-structures on the stable normal bundle $\nu(i)$ and those on the stable normal bundle $\nu(j)$ (see [Sto68, Chapter II, p.15]).

Example. Let us investigate first the most obvious example, i.e. $G_r = O(r)$. One can see easily that any manifold has a trivial (BO, id)-structure.

Example. Suppose now $G_r = SO(r)$, then $BSO(r)$ is weakly equivalent to the oriented Grassmann manifold, denoted $G_r^+(\mathbb{R}^\infty)$, consisting of oriented r-planes in $\mathbb{R}^\infty$. With this identification we have a natural map $f_r : BSO(r) \to BO(r)$ which is a 2-sheet covering of $BO(r)$. A lift $\hat{\nu}$ encodes the choice of an orientation of the normal bundle of our manifold. Therefore, (BSO, f)-manifolds are oriented manifolds.

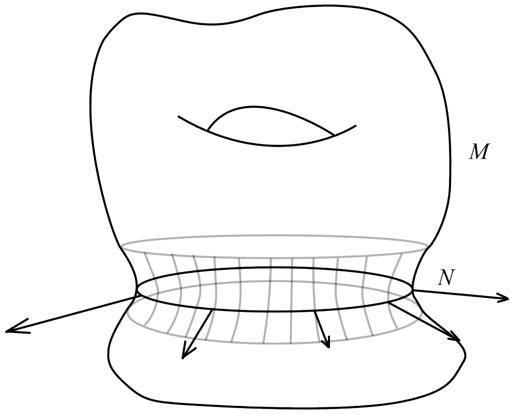
Example. Let now $G_{2r} = G_{2r+1} = U(r)$ the unitary group seen as a subgroup of $O(2r)$. The space $BU(r)$ can be identified with the complex Grassmann manifold, denoted $G_r(\mathbb{C}^\infty)$. In this case, we have also a natural fibration $f_r : BU(r) \to BO(2r)$. The reader can check that a (BU, f)-structure on the stable normal bundle is equivalent to giving it a complex structure. Hence (BU, f)-manifolds are stably almost-complex manifolds.

Example. For the last example, consider $G_r = 1$ be the trivial group. Then $BG_r \simeq EO(r)$ is weakly equivalent to the contractible space of r-frames in $\mathbb{R}^\infty$, denoted $V_r(\mathbb{R}^\infty)$. This is r-tuples of orthogonal vectors in $\mathbb{R}^\infty$. The canonical fibration $f_r : V_r(\mathbb{R}^\infty) \to G_r(\mathbb{R}^\infty)$ sends $(v_1, \ldots, v_r)$ to the r-plane generated by these vectors. One can now see that a $(B1, f)$-manifold is a manifold with a chosen trivialization of the stable normal bundle.

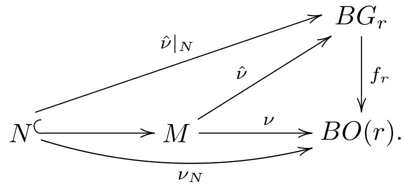
Notation. When the fibrations $f_r : BG_r \to BO(r)$ is implicitly understood, we will say G-*manifolds* (respectively G-*structure*) instead of (BG, f)-manifolds (resp. (BG, f)-structure).

2.2 The Cobordism Group Ω_*^G

Let M be a G-manifold, we want to specify how is defined the G-structure on the stable normal bundle of the boundary ∂M. We will make this explicit for any submanifold N of M with a trivialized normal bundle. Let n be the dimension of N and m the dimension of M. Suppose we have $i_N : N \to \mathbb{R}^{n+r}$ an embedding with r large enough. Using the trivializations, we can find an embedding of a neighborhood of N in M into $\mathbb{R}^{n+r} \times \mathbb{R}^{m-n}$ such that the neighborhood meets $\mathbb{R}^{n+r}$ orthogonally along N (e.g. by taking a tubular neighborhood in $\mathbb{R}^{n+r}$ of N and then take the intersection with M).



With this construction, one would have that the normal planes of N in $\mathbb{R}^{n+r}$ are the normal planes of M in $\mathbb{R}^{m+r}$ restricted to N. So if M has a G-structure on its stable normal bundle, then we have an induced G-structure on the stable normal bundle of N by restriction. Therefore, we have the following commutative diagram



In the specific case where $N = \partial M$, it can be shown by means of differential geometry that the boundary are *collared*. This is saying that there exists a tubular neighborhood of the boundary $(\partial M)_\varepsilon$ in M, which is diffeomorphic to $\partial M \times [0, \varepsilon)$. Therefore, we can suppose that ∂M has a trivialized normal bundle and the above discussion holds.

Definition. Two G-manifolds M, N are said *cobordant* if there exists G-manifolds U, V such that $M \sqcup \partial U \cong N \sqcup \partial V$. If M and N are cobordant, we say that M and N are in the same *cobordism class*.

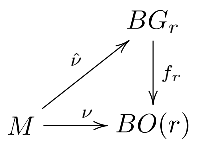
Definition. Let Ω_n^G denote the semigroup of cobordism classes of n-dimensional (compact) G-manifolds. The composition law is given by the disjoint union

$$[M] + [N] := [M \sqcup N].$$

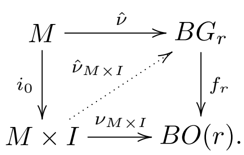
Proposition 2.1. *Ω_n^G is an abelian group.*

Proof. Let M be a G-manifold, one needs to find an inverse to M, i.e. N such that $[M \sqcup N] = [\emptyset]$. One can consider for this $M \times 1$ since $\partial(M \times I) = M \times 0 \sqcup M \times 1$. Note that $M \times 1$ might NOT be isomorphic to M as a G-manifold. For example, when $G = SO$, $M \times 1$ is the manifold M with the reversed orientation.

Define the G-structure on $M \times I$ by the following. Let $i : M \to \mathbb{R}^{n+r}$ be an embedding for some large r and let $\hat{\nu} : M \to BG_r$ be a lift such that the diagram



commutes. We now extend the embedding i for $M \times I$ into $\mathbb{R}^{n+r} \times \mathbb{R} = \mathbb{R}^{n+r+1}$ using the standard inclusion. With this embedding, the normal map of $M \times I$ is the projection on M followed by the normal map of M. We then have the commutative diagram



Such lift exists by the very definition of a fibration. This gives us a G-structure on $M \times I$, and by construction $M \times 0$ has the same G-structure as M. $\square$

2.3 Thom Space

Let ξ be an r-plane bundle with an Euclidian metric. The *Thom space* of ξ is the space obtained from the total space of ξ by collapsing all vectors of length at least one into a single basepoint, denoted ∞, i.e.

$$T\xi := E(\xi)/\{v \in E(\xi) : |v| \geq 1\}.$$

In the literature, the reader can find another description of the Thom space by the following construction. Starting with any fiber bundle ξ, consider the fiberwise one point compactification of ξ. Then identify all the new points ∞ to a single point. The resulting space is the Thom space. Remark that this previous construction might apply in a more general setting.

Notation. If ξ is a vector bundle over X, the Thom space $T\xi$ will sometimes be denoted TX (when there is no ambiguity).

Let $g : X \to Y$ be a continuous map and η be a bundle over Y. If $\xi = g^*\eta$, this is the pullback of η along g, which is a bundle over X, then one has an induced map

$$T\xi \xrightarrow{Tg} T\eta.$$

Let $j_r : BO(r) \to BO(r+1)$ be the canonical map induced by the inclusion. Considering the pullback of the universal bundle γ^{r+1} along j_r, one gets the following commutative diagram

$$\begin{array}{ccc} E(j_r^*\gamma^{r+1}) & \longrightarrow & E(\gamma^{r+1}) \\ \downarrow & & \downarrow \\ BO(r) & \xrightarrow{j_r} & BO(r+1). \end{array}$$

By a straightforward computation one can check that the bundle $j_r^*\gamma^{r+1}$ is isomorphic to the Whitney sum $\gamma^r \oplus \varepsilon^1$, where ε^1 denotes the trivial line bundle. Then the Thom space $T(\gamma^r \oplus \varepsilon^1)$ is isomorphic to the suspension $\Sigma T\gamma^r$. Therefore, the map $j_r : BO(r) \to BO(r+1)$ induces a map

$$\Sigma TBO(r) \xrightarrow{Tj_r} TBO(r+1).$$

Let now $(G_r)_r$ be a sequence of subgroups, with each G_r is contained in $BO(r)$. Suppose one has a commutative diagram

$$\begin{array}{ccccccc} \cdots & \longrightarrow & BG_r & \xrightarrow{i_r} & BG_{r+1} & \xrightarrow{i_{r+1}} & \cdots \\ & & \downarrow{\scriptstyle f_r} & & \downarrow{\scriptstyle f_{r+1}} & & \\ \cdots & \longrightarrow & BO(r) & \xrightarrow{j_r} & BO(r+1) & \xrightarrow{j_{r+1}} & \cdots \end{array}$$

We can then draw the commutative diagram

$$\begin{array}{ccccc} E((f_{r+1}i_r)^*\gamma^{r+1}) & \longrightarrow & E(f_{r+1}^*\gamma^{r+1}) & \longrightarrow & E(\gamma^{r+1}) \\ \downarrow & & \downarrow & & \downarrow \\ BG_r & \xrightarrow{i_r} & BG_{r+1} & \xrightarrow{f_{r+1}} & BO(r+1), \end{array}$$

and see that i_r induces a map

$$\Sigma TBG_r \xrightarrow{Ti_r} TBG_{r+1}.$$

Indeed,

$$\begin{aligned} T((f_{r+1}i_r)^*\gamma^{r+1}) =& T((j_r f_r)^*\gamma^{r+1}) = T(f_r^* \underbrace{j_r^*\gamma^{r+1}}_{\gamma^r \oplus \varepsilon^1}) \\ =& T((f_r^*\gamma^r) \oplus \varepsilon^1) \cong \Sigma T(f_r^*\gamma^r) = \Sigma TBG_r. \end{aligned}$$

At the end one gets the following commutative diagram

$$\begin{array}{ccc} \Sigma TBG_r & \xrightarrow{Ti_r} & TBG_{r+1} \\ \Big\downarrow{\scriptstyle \Sigma Tf_r} & & \Big\downarrow{\scriptstyle Tf_{r+1}} \\ \Sigma TBO(r) & \xrightarrow{Tj_r} & TBO(r+1). \end{array}$$

Definition. When $(G_r)_r$ is a sequence of subgroups as above, the Thom spaces TBG_r are denoted MG_r. The *Thom spectrum* of $G = (G_r)_r$, denoted by $\boldsymbol{MG}$, is the spectrum

$$(MG_0, MG_1, MG_2, \dots).$$

Examples. Here is a list of the associated Thom spectra to subgroups that we seen in 2.1.

$$\begin{aligned} \boldsymbol{MO} &= (*, MO(1), MO(2), MO(3), MO(4), \dots) \\ \boldsymbol{MSO} &= (*, MSO(1), MSO(2), MSO(3), MSO(4), \dots) \\ \boldsymbol{MU} &= (*, *, MU(1), \Sigma MU(1), MU(2), \Sigma MU(2), \dots) \\ \boldsymbol{M}1 &= (S^0, S^1, S^2, S^3, S^4, \dots) \end{aligned}$$

2.4 The Pontrjagin-Thom Theorem

Let $G = (G_r)_r$ be a sequence of groups, with each G_r contained in $O(r)$.

Theorem 2.2 (Pontrjagin-Thom)**.** *We have an isomorphism of groups*

$$\Omega_n^G \cong \pi_n(\boldsymbol{MG}).$$

Sketch of the proof. Let $[M] \in \Omega_n^G$, $i : M \to \mathbb{R}^{n+r}$ be a embedding with r large enough and $\nu_M : M \to BO(r)$ be the map classifying the normal bundle $\nu(i)$. Moreover, we denote $\hat{\nu}_M : M \to BG_r$ the lift defining the G-structure on M.

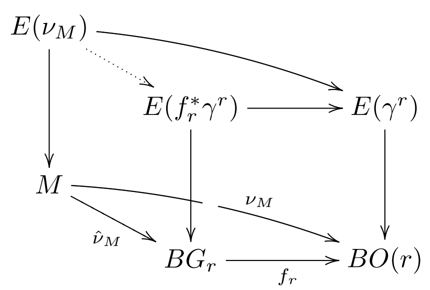
The total space of the normal bundle, denoted $E(\nu_M)$, can be seen as a subspace of the product $\mathbb{R}^{n+r} \times \mathbb{R}^{n+r}$. Let us consider the map

$$e : \mathbb{R}^{n+r} \times \mathbb{R}^{n+r} \to \mathbb{R}^{n+r} : (a, b) \mapsto a + b.$$

One can see that this map is smooth on $E(\nu_M)$. Furthermore, e restricted to $M = M \times 0 \subset E(\nu_M)$ is exactly our embedding $i : M \to \mathbb{R}^{n+r}$. By differential geometry, we can find an $\varepsilon > 0$ such that the map $e|_{E(\nu_M)} : E(\nu_M) \to \mathbb{R}^{n+r}$ gives us an embedding of the tubular neighborhood $E(\nu_M)_\varepsilon = \{n \in E(\nu_M) : |n| \leq \varepsilon\}$ in $\mathbb{R}^{n+r}$.

Considering S^{n+r} as the one point compactification $\mathbb{R}^{n+r} \cup \{\infty\}$, we define a map $S^{n+r} \xrightarrow{c} E(\nu_M)_\varepsilon / \partial E(\nu_M)_\varepsilon$ by collapsing every points outside or in the boundary of $E(\nu_M)_\varepsilon$ into ∞. Composing now by $E(\nu_M)_\varepsilon / \partial E(\nu_M)_\varepsilon \xrightarrow{\cdot 1/\varepsilon} T\nu_M$ the multiplication by $1/\varepsilon$, we get into the Thom space of the normal bundle of M, denoted $T\nu_M$.

The last step to define our map $\Omega_n^G \longrightarrow \pi_n(\boldsymbol{MG})$, is to find a map $T\nu_M \to MG_r$. By the universal property of the pullback



we have an induced map between $E(\nu_M)$ and $E(f_r\gamma^r)$. This map induces a map $T\nu_M \overset{T\hat{\nu}_M}{\longrightarrow} MG_r$.

Define the homomorphism $\theta : \Omega_n^G \to \pi_n(\boldsymbol{MG})$ is defined by sending $[M]$ to the homotopy class of the composition

$$\theta_M : S^{n+r} \overset{c}{\longrightarrow} E(\nu_M)_\varepsilon / \partial E(\nu_M)_\varepsilon \overset{\cdot 1/\varepsilon}{\longrightarrow} T\nu_M \overset{T\hat{\nu}_M}{\longrightarrow} MG_r.$$

At this stage, we can see that the homotopy type of this map does not depend on ε, or on the lift $\hat{\nu}_M$. Indeed, if we choose another ε', the map $1/\varepsilon' \circ c$ would be homotopic to $1/\varepsilon \circ c$. It is also true that if we choose another lift $\hat{\nu}'_M$ given the same G-structure on M, then one would have a homotopy between $T\hat{\nu}_M$ and $T\hat{\nu}'_M$. Nevertheless, we have still to show that $\theta[M]$ does not depend on the embedding $i : M \to \mathbb{R}^{n+r}$.

Claim 1. *The homotopy class of θ_M depends only on the cobordism class of M.*

Let W be a G-manifold. Consider the manifold $M \sqcup \partial W$ where the G-structure is the one induced by M and W. We want to show that

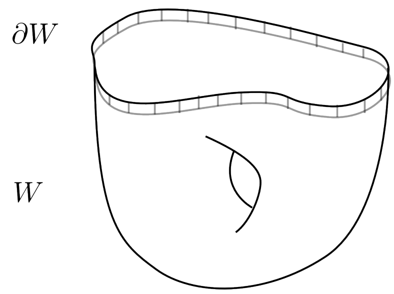
$$\theta_M \simeq \theta_{M \sqcup \partial W}.$$

Let $j : M \sqcup \partial W \to \mathbb{R}^{n+r}$ be any embedding for r large enough and let $\hat{\nu}_{M\sqcup\partial W} : M \sqcup \partial W \to BG_r$ give its G-structure. Moreover the restriction $\hat{\nu}_{M\sqcup\partial W}|_M$ gives the same G-structure as $\hat{\nu}_M$ (remark that $j|_M$ might not be equal to i) and the restriction $\hat{\nu}_{M\sqcup\partial W}|_{\partial W} = \hat{\nu}_W|_{\partial W} = \hat{\nu}_{\partial W}$.

Let $H : M \times I \to \mathbb{R}^{n+r}$ be a homotopy between $i : M \to \mathbb{R}^{n+r}$ and $j|_M : M \to \mathbb{R}^{n+r}$. Moreover, one can suppose by use of differential geometry that there exists $\delta_1, \delta_2 > 0$ such that

$$H(x,t) = i(x) \text{ for } 0 \le t < \delta_1, \text{ and } H(x,t) = j|_M(x) \text{ for } 1 - \delta_2 < t \le 1.$$

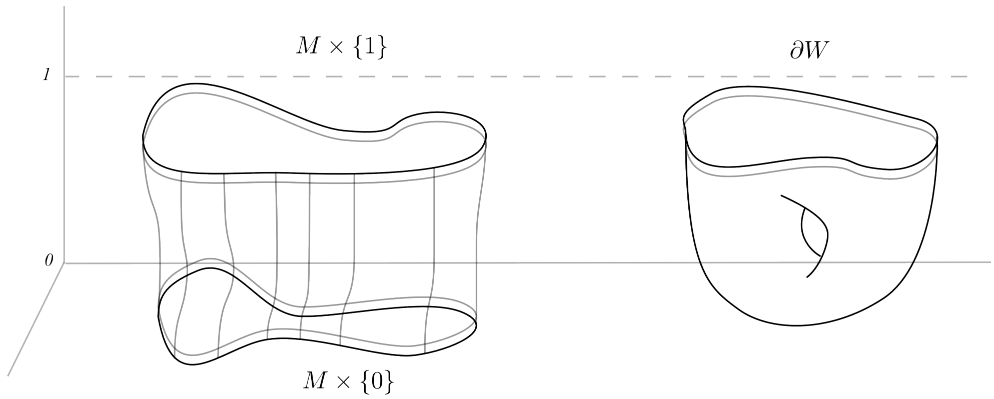
Let $k : W \to \mathbb{R}^{n+r} \times (0,1]$ be an embedding such that $k|_{\partial W} = j|_{\partial W} \times \{1\}$, and such that a tubular neighborhood of ∂W is embedded orthogonally to $j(\partial W) \times \{1\}$. The reader must have in mind the following cartoon.



If we look now at

$$\begin{aligned} F : M \times I \sqcup W &\longrightarrow \mathbb{R}^{n+r} \times I \\ (m,t) &\longmapsto (H(m,t), t) \\ w &\longmapsto k(w), \end{aligned}$$

one can see that this is an embedding on a closed neighborhood of the boundary. Without lost of generality, one can take it to be an embedding on the whole domain space. This can be represented in the following picture.



With our construction, we have the commutative solid diagram

$$\begin{array}{ccc} M \times \{0\} & \xrightarrow{\hat{\nu}_M} & BG_r \\ \big\downarrow & \nearrow & \big\downarrow f_r \\ M \times I & \xrightarrow[\nu(F|_{M\times I})]{} & BO(r). \end{array}$$

The lift is given by the very definition of a fibration. Moreover, we can modify the lift in order to have also

$$\begin{array}{ccc} M \times \{1\} & \xrightarrow{\hat{\nu}_{M \sqcup \partial W}|_M} & BG_r \\ \big\downarrow & \nearrow & \big\downarrow f_r \\ M \times I & \xrightarrow[\nu(F|_{M\times I})]{} & BO(r). \end{array}$$

If we now proceed to the construction of θ_M and $\theta_{M \sqcup \partial W}$ using embeddings i and j respectively, then we would get a homotopy between θ_M and $\theta_{M \sqcup \partial W}$ induced by F.

This homotopy is obtained by the following construction. At first, we have a collapse $S^{n+r} \times I \longrightarrow E(\nu_F)_\varepsilon / \partial E(\nu_F)_\varepsilon$ where $E(\nu_F)$ is the total space of the normal bundle ν_F of $M \times I \sqcup W$ induced by F. Secondly, the multiplication by $1/\varepsilon$ gives us a map $E(\nu_F)_\varepsilon / \partial E(\nu_F)_\varepsilon \to T\nu_F$. Finally, the G-structure defined above on $M \times I \sqcup W$ gives us an induced map $T\nu_F \overset{T\nu_F}{\longrightarrow} MG_r$. The reader can check that the composition of theses three maps is the desired homotopy.

We are now able to prove the claim. If we take W to be $\emptyset$, we solved the problem that $\theta[M]$ does not depend on the embedding of M. Moreover, if $[M] = [N]$, then there exists U, V such that $M \sqcup \partial U \cong N \sqcup \partial V$. Using the previous construction, we have

$$\theta_M \simeq \theta_{M \sqcup \partial U} \cong \theta_{N \sqcup \partial V} \simeq \theta_N.$$

Claim 2. *$\theta : \Omega_n^G \to \pi_n(\boldsymbol{M}(G))$ is a homomorphism of groups.*

If $[M], [N]$ are in Ω_n^G, we can choose embeddings $i : M \to \mathbb{R}^{n+r-1} \times \mathbb{R}_+$ and $j : N \to \mathbb{R}^{n+r-1} \times \mathbb{R}_-$ and such that the tubular neighborhood of each manifold lies in the same half space. Then $\theta_{[M]+[N]}$ is homotopic to

$$S^{n+r} \overset{m}{\longrightarrow} S^{n+r} \vee S^{n+r} \overset{\theta_M \vee \theta_N}{\longrightarrow} MG_r$$

where m is the pinch map.

Claim 3. *$\theta : \Omega_n^G \to \pi_n(\boldsymbol{M}G)$ is an isomorphism.*

The proof of this claim uses the great idea of transversality introduced by Thom. The reader shall find the proof in [Sto68, Chapter II, p.21]. ☐

2.5 Computation of Ω_*^O

In the last section we transformed a geometrical problem (understand the cobordism groups) into a homotopical problem. Generally, computing homotopy groups is often clueless and very hard. Nevertheless, it turns out that computing the homotopy groups of the Thom spectrum $\boldsymbol{MO}$ is possible. This is a very surprising fact.

The fundamental idea of this computation is to show that the mod 2 cohomology of the Thom spectrum is a free module over the Steenrod algebra. Using the Whitehead theorem, we will be able to identify the homotopy type of $\boldsymbol{MO}$ as some product of Eilenberg-MacLane spectra. The homotopy groups will then follow straightforward. This computation is due to Thom and can be founded in its celebrated paper [Tho54].

Let us first recall a few facts about the Grassmann manifold $G_n(\mathbb{R}^\infty) = BO(n)$.

Proposition 2.3. *The cohomology ring (with mod 2 coefficients) of the Grassmann manifold $G_n(\mathbb{R}^\infty)$ is the polynomial algebra over $\mathbb{Z}/2$ generated by the Stiefel-Whitney classes $w_1, \ldots, w_n$, i.e.*

$$H^*(G_n(\mathbb{R}^\infty); \mathbb{Z}/2) = \mathbb{Z}/2[w_1, \ldots, w_n].$$

Recall that each w_i is of degree i.

Proof. The reader can find the proof in [MS74, Chapter 7]. □

A useful way tot think of the Stiefel-Whitney classes w_k, is as the k-th symmetric polynomials in variables $t_1, \ldots, t_n$. This can be explained by looking at the subgroup of diagonal matrices in $O(n)$, denoted $T(n)$. One can then construct $BT(n)$, which is the n-fold product of real projective spaces $\mathbb{R}P^\infty \times \cdots \times \mathbb{R}P^\infty$. Its cohomology ring is a polynomial algebra over $\mathbb{Z}/2$ generated by $t_1, \ldots, t_n$ each of degree 1. The induced map $BT(n) \to BO(n)$ gives us in cohomology our desired identification. For more details about this, the reader can look in [BS53].

Thus, a $\mathbb{Z}/2$-basis of $H^k(G_n(\mathbb{R}^\infty); \mathbb{Z}/2)$ is given by all symmetric polynomials $\sum t_1^{a_1} \cdots t_r^{a_r}$, where $(a_1 \cdots a_r)$ is a partition of k (with r less than or equal to n). The sum is taken over all essential permutations, i.e. we mean that it is the "smallest" symmetric polynomial which contains $t_1^{a_1} \cdots t_r^{a_r}$.

Let us also recall how works the Thom isomorphism. Let ξ be an n-plane bundle over B with total space E, and let E_0 be non-zero vectors in E. We suppose also that our coefficient ring R is a field.

Proposition 2.4. *There exists a cohomology class U_ξ in $H^n(E, E_0; R)$ such that the map*

$$H^k(E; R) \xrightarrow{\cup U_\xi} H^{n+k}(E, E_0; R)$$

is an isomorphism. The class U_ξ is called the fundamental class *or* Thom class. *Moreover, the composition with the isomorphism π^*, i.e.*

$$H^k(B; R) \xrightarrow{\pi^*} H^k(E; R) \xrightarrow{\cup U_\xi} H^{n+k}(E, E_0; R)$$

is called the Thom isomorphism *and is generally denoted by ϕ.*

This allows us to compute the reduced cohomology of the Thom space $T\xi$, because $\tilde{H}^{n+k}(T\xi; R)$ can be identified to $H^{n+k}(E, E_0; R)$. Indeed, consider $D(E) := \{v \in E : |v| \leq 1\}$ the *disk bundle* of ξ and $S(E) := \{v \in E : |v| = 1\}$ the *sphere bundle* of ξ. Then the pair $(D(E), S(E))$ is a deformation retract of the pair (E, E_0), and since $(D(E), S(E))$ is a good pair, one has

$$\begin{aligned} H^{n+k}(E, E_0; R) &\cong H^{n+k}(D(E), S(E); R) \\ &\cong \tilde{H}^{n+k}(D(E)/S(E); R) = \tilde{H}^{n+k}(T\xi; R). \end{aligned}$$

Let us now use the Thom isomorphism to give a basis of $H^*(MO(n);\mathbb{Z}/2)$. In order to do this, consider the trivial cross-section $f : BO(n) \to E$, where $E = E(\gamma^n)$ is the total space of the universal bundle. Therefore, $f^* : H^*(E;\mathbb{Z}/2) \longrightarrow H^*(BO(n);\mathbb{Z}/2)$ is an inverse for π^*. Composing f^* by the natural restriction map $H^{n+k}(E, E_0 : \mathbb{Z}/2) \longrightarrow H^{n+k}(E;\mathbb{Z}/2)$, one can draw the commutative diagram

$$\begin{array}{ccc} & H^k(BO(n);\mathbb{Z}/2) & \\ {}^{\phi}_{\cong}\swarrow & & \searrow \\ \tilde{H}^{n+k}(MO(n);\mathbb{Z}/2) & \xrightarrow{\ f^*|\ } & H^{n+k}(BO(n);\mathbb{Z}/2). \end{array}$$

The arrow in the right hand side is defined as the composition. If we now chase in this diagram, we can identify the map on the right hand side to be $a \mapsto a \cdot w_n$ (where $\cdot$ denotes the cup product). But the map $a \mapsto a \cdot w_n$ is a monomorphism, thus $f^*|$ is also a monomorphism. Therefore, $\tilde{H}^*(MO(n);\mathbb{Z}/2)$ can be seen in $H^*(BO(n);\mathbb{Z}/2)$ as the ideal generated by w_n.

A $\mathbb{Z}/2$-basis of $H^{n+k}(MO(n);\mathbb{Z}/2)$ is given by all symmetric polynomials of the form

$$\sum t_1^{a_1+1} \cdots t_r^{a_r+1} \cdot t_{r+1} \cdots t_n$$

where $(a_1 \cdots a_r)$ runs over all partitions of k (with r less than or equal to n). The sum is taken over all essential permutations of $t_1^{a_1+1} \cdots t_r^{a_r+1} \cdot t_{r+1} \cdots t_n$. Remark that each permutation that is essential to $t_1^{a_1+1} \cdots t_r^{a_r+1} \cdot t_{r+1} \cdots t_n$, is also essential for $t_1^{a_1} \cdots t_r^{a_r}$.

Notation. If $\omega = (a_1 \cdots a_r)$ is a partition of i with length $r \leq n$, we write

$$s(\omega) = \sum t_1^{a_1} \cdots t_r^{a_r}.$$

Remark that $s(1 \cdots 1) = w_n$.

Definition. We say that a partition $\omega = (a_1, \dots, a_r)$ is *non-dyadic* if each $a_i \neq 2^m - 1$ and is *purely dyadic* if each a_i is of the form $2^m - 1$. Therefore, any partition ω can be uniquely factored as $\lambda\omega_R$, where λ is non-dyadic and $\omega_R = (r_1, r_2(2^2 - 1), \cdots, r_j(2^j - 1))$ is purely dyadic.

For a partition $\omega = (a_1, \dots, a_r)$, we define to quantities:

$$l(\omega) := r \quad \text{and} \quad w(\omega) := a_1 + \cdots + a_r.$$

For any two partitions $\omega = \lambda\omega_R$ and $\omega' = \lambda'\omega'_R$, we say that ω' is *less* than ω:

- if $l(\lambda') < l(\lambda)$, or
- if $l(\lambda') = l(\lambda)$, $w(\lambda) > w(\lambda')$.

Remark that we totally ignore the dyadic part of our partition ω.

We say that a monomial $t_1^{a_1} \cdots t_r^{a_r}$ is *non-dyadic* if each $a_i \neq 2^m$ and is *purely dyadic* if each a_i is of the form 2^m. We can also define an order on monomials in the same fashion as before by looking only the non-dyadic part.

Remark. It might be a little bit confusing, but one has to think that we arranged the definitions such that we can switch from $s(\omega) \cdot w_n$ and $\sum t_1^{a_1} \cdots t_r^{a_r}$. For instance, if λ is non-dyadic we have

$$s(\lambda) \cdot w_n = \sum t_1^{a_1+1} \cdots t_r^{a_r+1} \cdot t_{r+1} \cdots t_n,$$

then each monomial has exactly $l(\lambda) = r$ non-dyadic terms.

We will now define classes which will be used in the future to define a $\mathcal{A}_2$-basis for $H^*(\boldsymbol{MO};\mathbb{Z}/2)$.

For any $k \leq n$, we form the classes $\{Sq^{I_h}(s(\lambda_h) \cdot w_n)\}$ where h runs over all integers $0, \dots, k$, the λ_h runs over all non-dyadic partitions of h, and the Sq^{I_h} runs over all admissible sequences of total degree $(k - h)$. The definition of the Sq^I can be found in appendix A.

Lemma 2.5. *The classes $\{Sq^{I_h}(s(\lambda_h)\cdot w_n)\}$ are linearly independent.*

Proof. Let us first identify who are the "biggest" terms in each $Sq^{I_h}(s(\lambda_h)\cdot w_n)$ in the sense of the ordering described above.

If we look at $Sq^{I_h}(t_1^{a_1+1}\cdots t_r^{a_r+1}\cdot t_{r+1}\cdots t_n)$ then to maximize the number of non-dyadic terms we have to create the minimum of dyadic variables. Since the variables $t_{r+1}\cdots t_n$ are already dyadic, it is easy to see that applying Sq^{I_h} to them will never change their dyadic nature (they are already lost to us). This facts follows from the formula $Sq^m t^a = \binom{m}{a} t^{m+a}$. So in order to get the biggest terms, one need to consider the development of

$$t_1^{a_1+1}\cdots t_r^{a_r+1} Sq^{I_h}(t_{r+1}\cdots t_n). \tag{1}$$

Here the number of non-dyadic terms is equal to $l(\lambda_h)=r$. Another way to get r non-dyadic terms would be to take successive squares of the non-dyadic factors $t_1^{a_1+1}\cdots t_r^{a_r+1}$. But this maneuver would fail since the $w(\lambda)$ would be strictly bigger than in (1). Hence, we just proved the following.

Claim 1. *For each h and each λ_h, the sum of all the biggest terms in $Sq^{I_h}(s(\lambda_h)\cdot w_n)$ is exactly*

$$\sum t_1^{a_1+1}\cdots t_r^{a_r+1} Sq^{I_h}(t_{r+1}\cdots t_n)$$

where the sum is taken over the essential permutations of the polynomial

$$t_1^{a_1+1}\cdots t_r^{a_r+1} t_{r+1}\cdots t_n.$$

Remark now that a term cannot be expressed linearly by terms that are less than itself. Therefore, if one has a non-trivial linear combination, then it can be formed using only classes $Sq^{I_h}(s(\lambda_h)\cdot w_n)$ where the biggest terms has $l(\lambda)=r$ non-dyadic terms and of total non-dyadic degree $r+h$. That means that h cannot vary in our linear combination. One can also see that we cannot vary the partition λ_h, because our terms will be all different and making any non-trivial combination impossible.

Finally, we come to the fact that the only way to get a non-trivial linear combination is to make I vary with fixed h and λ_h. Let us write

$$0=\sum_\alpha c_\alpha t_1^{a_1+1}\cdots t_r^{a_r+1} Sq^{I_\alpha}(t_{r+1}\cdots t_n).$$

By extracting the terms $t_1^{a_1+1}\cdots t_r^{a_r+1}$ we get the relation:

$$0=t_1^{a_1+1}\cdots t_r^{a_r+1}\sum_\alpha c_\alpha Sq^{I_\alpha}(t_{r+1}\cdots t_n),$$

but this would imply $c_\alpha=0$. Indeed the Sq^I's (for admissible sequences) form a basis of the Steenrod algebra, thus $Sq^I(t_{r+1}\cdots t_n)$ are linearly independent whenever the total degree of Sq^{I_h}, i.e. $k-h$, is less than or equal to $n-r$ (cf. appendix A). This inequality is satisfied because $h\geq 2r$, and $k\leq n$. □

Corollary 2.6. *The classes $\{Sq^{I_h}(s(\lambda_h)\cdot w_n)\}$ form a basis of $H^{n+k}(MO(n);\mathbb{Z}/2)$ for $k\leq n$.*

Proof. We seen previously that the rank of $H^{n+k}(MO(n);\mathbb{Z}/2)$ is the number of partitions of k (where the number of terms is less or equal to n; this condition is always satisfied because we supposed $k\leq n$). Write $p(k)$ this number. We claim that $p(k)=\#\{Sq^{I_h}(s(\lambda_h)\cdot w_n)\}$.

Indeed, any partition can be factor into a non-dyadic partition and a purely dyadic. Thus

$$p(k)=\sum_{0\leq h\leq k}\#\{\text{purely dyadic partitions of } k-h\}\#\{\text{non-dyadic partitions of } h\}$$

and Serre showed in [Ser53a, p. 212] that $\#\{\text{purely dyadic partitions of } k-h\}$ is equal to the number of admissible sequences of total degree $k-h$. Thus our linearly independent system form a $\mathbb{Z}/2$-basis of $H^{n+k}(MO(n);\mathbb{Z}/2)$. □

Notation.

$$d(h) := \#\{\text{non-dyadic partitions of } h\}$$

Proposition 2.7. *The $\mathcal{A}_2$-module $H^*(\boldsymbol{MO};\mathbb{Z}/2)$ is free.*

Proof. Write $H^k(\boldsymbol{MO};\mathbb{Z}/2) = \lim_n H^{n+k}(MO(n);\mathbb{Z}/2)$. We previously showed that the $\{Sq^{I_h}(s(\lambda_h)\cdot w_n)\}$ is a $\mathbb{Z}/2$-basis of $H^{n+k}(MO(n);\mathbb{Z}/2)$ for n large enough ($n \geq k$). Therefore, if we let n go to infinity the classes "$s(\lambda_h)\cdot w_\infty$", seen in $H^*(\boldsymbol{MO};\mathbb{Z}/2)$, form a basis over the Steenrod algebra, $\mathcal{A}_2$. Moreover, the degree of $s(\lambda_h)\cdot\omega_\infty$ in $H^*(\boldsymbol{MO};\mathbb{Z}/2)$ is h. □

Proposition 2.8. *The spectrum $\boldsymbol{MO}$ has the same homotopy type as the product $\prod_{\lambda_h}\Sigma^h\boldsymbol{H}\mathbb{Z}/2$, where $\boldsymbol{H}\mathbb{Z}/2$ denotes the Eilenberg-MacLane spectrum, and λ_h runs over all non-dyadic partitions.*

Proof. In order to prove this proposition, we will use the Whitehead's theorem. This theorem states that if the cohomology of two spectra are isomorphic for any coefficient field $\mathbb{Z}/p$ and $\mathbb{Q}$, then they have the same homotopy type.

Claim 1.

$$H^*(\prod_{\lambda_h}\Sigma^h\boldsymbol{H}\mathbb{Z}/2;\mathbb{Z}/2) \cong \bigoplus_{\lambda_h}\Sigma^h\mathcal{A}_2.$$

By definition, $H^*(\prod_{\lambda_h}\Sigma^h\boldsymbol{H}\mathbb{Z}/2;\mathbb{Z}/2) = \lim_n H^{*+n}(\prod_{\lambda_h}K(\mathbb{Z}/2,h+n);\mathbb{Z}/2)$. Since these cohomology groups are finitely generated degreewise, we can apply the Künneth formula to get

$$H^{*+n}(\prod_{\lambda_h}K(\mathbb{Z}/2,h+n);\mathbb{Z}/2) = \bigoplus_{\lambda_h}H^{*+n}(K(\mathbb{Z}/2,h+n);\mathbb{Z}/2).$$

Indeed, if n is big enough the mixed terms will vanish. Let us observe this phenomena on $Y = K(\mathbb{Z}/2,h_1+n)\times K(\mathbb{Z}/2,h_2+n)$, the proof remains the same in our general case.

$$\begin{aligned} H^{*+n}(Y;\mathbb{Z}/2) &\cong \bigoplus_{p+q=*+n} H^p(K(\mathbb{Z}/2,h_1+n);\mathbb{Z}/2)\otimes H^q(K(\mathbb{Z}/2,h_2+n);\mathbb{Z}/2) \\ &= H^{*+n}(K(\mathbb{Z}/2,h_1+n);\mathbb{Z}/2)\oplus H^{*+n}(K(\mathbb{Z}/2,h_2+n);\mathbb{Z}/2) \\ &\bigoplus_{\substack{p+q=*+n\\ p,q>0}} H^p(K(\mathbb{Z}/2,h_1+n);\mathbb{Z}/2)\otimes H^q(K(\mathbb{Z}/2,h_2+n);\mathbb{Z}/2) \end{aligned}$$

In the sum with the tensor product, non-zero terms appears only when $*+n \geq h_1+h_2+2n$. So for n big enough, the sum gets trivial.

We can hence finish the computation that we start above,

$$H^*(\prod_{\lambda_h}\Sigma^h\boldsymbol{H}\mathbb{Z}/2;\mathbb{Z}/2) \cong \bigoplus_{\lambda_h}H^*(\Sigma^h\boldsymbol{H}\mathbb{Z}/2;\mathbb{Z}/2) \cong \bigoplus_{\lambda_h}\Sigma^h H^*(\boldsymbol{H}\mathbb{Z}/2;\mathbb{Z}/2) \cong \bigoplus_{\lambda_h}\Sigma^h\mathcal{A}_2.$$

By proposition 2.7, we get that the mod 2 cohomology of $\prod_{\lambda_h}\Sigma^h\boldsymbol{H}\mathbb{Z}/2$ and $\boldsymbol{MO}$ are equal.

Moreover, the mod $p>2$ cohomology of $\prod_{\lambda_h}\Sigma^h\boldsymbol{H}\mathbb{Z}/2$ is clearly trivial. To prove that $H^*(\boldsymbol{MO};\mathbb{Z}/p)$ is also trivial, we have to use the $\mathscr{C}$-theory of Serre (cf. appendix C).

Considering $\mathscr{C}_p$, the class of finite abelian groups such that each element has order prime to p, we deduce that $\pi_*(\boldsymbol{MO}) \in \mathscr{C}_p$ for any odd p because $\pi_*(\boldsymbol{MO})$ has no odd torsion. Indeed, every manifold M is cobordant to itself, $\partial(M\times I) = M \sqcup M$. By theorem C.3 we get that $H_*(\boldsymbol{MO};\mathbb{Z}) \in \mathscr{C}_p$. This implies that $H_*(\boldsymbol{MO};\mathbb{Z})$ has no odd torsion and hence $H_*(\boldsymbol{MO};\mathbb{Z}/p) = 0$ for any odd prime p.

Since both spectra has only 2-torsion, their cohomology with rational coefficients are trivial. So we are finally able to apply the Whitehead's theorem and get our desired result. □

Theorem 2.9.

$$\Omega_n^O \cong \underbrace{\mathbb{Z}/2 \times \cdots \times \mathbb{Z}/2}_{d(n)\ times}.$$

Proof. By the Pontrjagin-Thom theorem and proposition 2.8, we have

$$\Omega_n^O \cong \pi_n(\boldsymbol{MO}) \cong \pi_n(\prod_{\lambda_h} \Sigma^h \boldsymbol{H}\mathbb{Z}/2) \cong \prod_{\lambda_h} \pi_n(\Sigma^h \boldsymbol{H}\mathbb{Z}/2) = \underbrace{\mathbb{Z}/2 \times \cdots \times \mathbb{Z}/2}_{d(n) \text{ times}}. \qquad \square$$

If M and N are to G-manifolds, we can define a natural G-structure on the product $M \times N$. Since G is a group, we have a natural multiplication $BG \times BG \to BG$. Hence if $\hat{\nu}_M$ and $\hat{\nu}_N$ denotes liftings to BG of ν_M and ν_N, we have the following commutative diagram

$$\begin{array}{ccccc} & \overset{\hat{\nu}_M \times \hat{\nu}_N}{\nearrow} & BG \times BG & \longrightarrow & BG \\ & & \downarrow{\scriptstyle f \times f} & & \downarrow{\scriptstyle f} \\ M \times N & \xrightarrow[\nu_M \times \nu_N]{} & BO \times BO & \longrightarrow & BO. \end{array}$$

One can see easily that this induces a ring structure on Ω_*^G.

Theorem 2.10. *The graded ring Ω_*^O is isomorphic to the polynomial ring $\mathbb{Z}/2[x_2, x_4, x_5, \dots]$ on generators x_i for each $i \neq 2^j - 1$.*

We will not prove this theorem, however the reader can find the proof in [Swi02, Chapter 20].

2.6 Computation of Ω_*^U

One can see that in the previous computation, we were really lucky and it was somehow "too easy". More precisely, the fact that $H^*(\boldsymbol{MO}; \mathbb{Z}/2)$ turns out to be free over the Steenrod algebra is absolutely unhoped for any other Thom spectra. In our actual peculiar case, $H^*(\boldsymbol{MU}; \mathbb{Z}/p)$ is NOT free over the Steenrod algebra. However, $H^*(\boldsymbol{MU}; \mathbb{Z}/p)$ is free over the quotient $\mathcal{A}_p/(Q_0)$, where Q_0 denotes the Bockstein. This will allow us to compute an $\mathcal{A}_p$-free resolution of $H^*(\boldsymbol{MU}; \mathbb{Z}/p)$. Using then the Adams spectral sequence, we will be able to compute $\pi_*(\boldsymbol{MU})$. This performance is due to Milnor and can be found in its beautiful paper [Mil60].

Let us first give some useful facts about the Steenrod algebra $\mathcal{A}_p$ for any prime p. Before to go further, the reader should know some basic knowledge about it. A little summary take place in Appendix A.

Lemma 2.11. *The Steenrod algebra $\mathcal{A}_p$, for any prime number p, contains a subalgebra A_0 with the following properties:*

- *A_0 is a Grassmann algebra over $\mathbb{Z}/p$ with generators $Q_0, Q_1, \dots$ each of odd dimension, i.e. $A_0 = \mathbb{Z}/p[Q_0, Q_1, \dots]/I$, where I is the ideal generated by Q_iQ_i and $Q_iQ_j + Q_jQ_i$ for all i, j.*
- *$\mathcal{A}_p$ is free as a right A_0-module.*
- *Considering $\mathbb{Z}/p$ as a left A_0-module by $Q_i\mathbb{Z}/p = 0$ for all i, we have an isomorphism of left $\mathcal{A}_p$-modules $\mathcal{A}_p \otimes_{A_0} \mathbb{Z}/p \cong \mathcal{A}_p/(Q_0)$, where (Q_0) is the two side ideal generated by Q_0.*
- *An additive basis for the quotient $\mathcal{A}_p/(Q_0)$ is given by all elements $\mathcal{P}^R$.*

Since we cannot give a better proof as Milnor, we refer the reader to [Mil60] and [Mil58, Theorem 4a]. However, let us give some useful remarks to have a better understanding.

***p* odd:** In [Mil58], Milnor studies the dual of the Steenrod algebra $\mathcal{A}_p$ for an odd prime p. It might first be a weird idea, but it turns out that it is more easy to understand the dual rather than the Steenrod algebra itself. He obtains that the dual of the Steenrod algebra is the tensor product of a Grassmann algebra and a polynomial algebra. Therefore, we can get a basis of $\mathcal{A}_p$ by dualizing the obvious additive basis of the dual. Such a basis is given by elements

$$Q_0^{e_0} Q_1^{e_1} \dots \mathcal{P}^R$$

where $E = (e_0, e_1, \dots)$ is a sequence of 0 and 1, and $R = (r_1, r_2, \dots)$ is a sequence of non-negative integers such that, almost all terms in theses sequences are zero. Moreover, the elements Q_i satisfies our identities:

$$Q_i Q_i = 0 \text{ and } Q_i Q_j + Q_j Q_i = 0.$$

We have also the following facts:

- $\mathcal{P}^{(0,0,0,\dots)} = 1$, and $\mathcal{P}^{(r,0,0,\dots)} = P^r$ the Steenrod power.
- $\mathcal{P}^R Q_k - Q_k \mathcal{P}^R = Q_{k+1} \mathcal{P}^{R-(p^k,0,0,\dots)} + Q_{k+2} \mathcal{P}^{R-(0,p^k,0,\dots)} + \cdots$,
- Q_0 is the Bockstein, and $Q_{i+1} = P^{p^i} Q_i - Q_i P^{p^i}$.

By the difference $R - S$ we mean the sequence obtained by taking the difference term by term. If at least one of the term in the resulting sequence is less than zero, then $\mathcal{P}^{R-S}$ is understood to be zero. Thus, we can show by induction that the exact dimension of Q_i is $2p^i - 1$.

So far, we can understand who is A_0 and that $\mathcal{A}_p$ is free as a left (not right) A_0-module. Indeed, we can take for basis the elements $\mathcal{P}^R$.

Finally, we will try to understand the isomorphism $\mathcal{A}_p \otimes_{A_0} \mathbb{Z}/p \cong \mathcal{A}_p/(Q_0)$. Let us first consider $\mathcal{A}_p$ modulo the right ideal $Q_0 \mathcal{A}_p + Q_1 \mathcal{A}_p + \cdots$. It can be shown by use of the formula $\mathcal{P}^R Q_k = Q_k \mathcal{P}^R + Q_{k+1} \mathcal{P}^{R-(p^k,0,0,\dots)} + Q_{k+2} \mathcal{P}^{R-(0,p^k,0,\dots)} + \cdots$ that it is also a left ideal. Hence, $Q_0 \mathcal{A}_p + Q_1 \mathcal{A}_p + \cdots$ is a two side ideal containing (Q_0), but using the above formula with $k = 0$ and $R = (0, \dots, 0, 1, 0, \dots)$, shows us that each Q_i is contained in the ideal (Q_0). Hence the two ideals are isomorphic.

By the definition of the A_0 action on $\mathbb{Z}/p$, we can see that $\mathcal{A}_p \otimes_{A_0} \mathbb{Z}/p$ is the quotient of $\mathcal{A}_p$ by the left ideal $\mathcal{A}_p Q_0 + \mathcal{A}_p Q_1 + \cdots$. Hence, we shown that $\mathbb{Z}/p \otimes_{A_0} \mathcal{A}_p \cong \mathcal{A}_p/(Q_0)$ with the left and right interchanged. To obtain correctly the statement of 2.11 we need to use the anti-automorphism of $\mathcal{A}_p$.

One can easily see that the elements $\mathcal{P}^R$ forms a $\mathbb{Z}/p$-basis for $\mathcal{A}_p/(Q_0)$. Indeed, these elements form a basis of the left A_0-module $\mathcal{A}_p$. Therefore, it forms a $\mathbb{Z}/p$-basis for $\mathbb{Z}/p \otimes_{A_0} \mathcal{A}_p \cong \mathcal{A}_p/(Q_0)$.

$p = 2$: Milnor discuss this case in [Mil58] by studying the dual of the Steenrod algebra over $\mathbb{Z}/2$. By a similar procedure he finds out a basis of $\mathcal{A}_2$ given by elements Sq^R, where R runs over all sequences (where almost all terms are zero). We emphasis here that this basis has nothing to do with the Serre-Cartan basis. For example, the degree of Sq^R is $r_1 + 3r_2 + 7r_3 + \dots$ which is totally different from the the degree of Sq^I, where I runs over all admissible sequences.

Define now the $\mathcal{P}^R$ to be Sq^{2R} and the Q_{i-1} to be $Sq^{(0,\dots,0,1,0,\dots)}$, where the sequence $(0, \dots, 0, 1, 0, \dots)$ has a 1 in the i-th position. For convenience, let Δ_i denote such sequences. Moreover, we remark that $Q_0 = Sq^{\Delta_1} = Sq^1$, hence Q_0 is indeed the Bockstein.

In [Mil58], Milnor shows that if $E = (e_1, e_2, \dots)$ is a sequence of 0 and 1 and $R = (r_1, r_2, \dots)$ is a sequence of non-negative integers (such that almost all terms are equal to zero), then we have the following product formula $Sq^E Sq^R = (E, R) Sq^{E+R}$. The coefficient (E,R) denotes the product $\prod_i (e_i+r_i)!/e_i!r_i!$. Therefore, we can get $Q_iQ_j = Q_jQ_i$ (by setting $E=\Delta_{i+1}$ and $R=\Delta_{j+1}$) and $Q_iQ_i=0$.

By use of the above formula, we can prove by induction that $Q_0^{e_1}Q_1^{e_2}\cdots = Sq^E$. This leads us to the fact that since any sequence can be written uniquely in the form $E+2R$, the elements $Sq^ESq^{2R} = Sq^E\mathcal{P}^R$ form a $\mathbb{Z}/2$-basis for $\mathcal{A}_p$.

Moreover, we can prove the rest with the same argumentation as for the previous case, by use of the formula

$$\mathcal{P}^RQ_k - Q_k\mathcal{P}^R = Q_{k+1}\mathcal{P}^{R-(2^k,0,0,\dots)} + Q_{k+2}\mathcal{P}^{R-(0,2^k,0,\dots)} + \cdots$$

Let us now recall few facts about the complex Grassmann manifold $G_n(\mathbb{C}^\infty) = BU(n)$.

Proposition 2.12. *The cohomology ring of the complex Grassmann manifold $G_n(\mathbb{C}^\infty)$ is the free polynomial algebra over $\mathbb{Z}$ generated by the Chern classes $c_1,\dots,c_n$, i.e.*

$$H^*(G_n(\mathbb{C}^\infty);\mathbb{Z}) = \mathbb{Z}[c_1,\dots,c_n].$$

Recall that each c_i is of degree $2i$.

Proof. The reader can find the proof in [MS74, Chapter 14]. □

As for $\boldsymbol{MO}$, we can give an interpretation of Chern classes as symmetric polynomials by use of the subgroup of diagonal unitary matrices in $U(n)$. Using the Thom isomorphism (see proposition 2.4), one draw the commutative diagram

$$\begin{array}{ccc} & H^i(BU(n);\mathbb{Z}) & \\ {\scriptstyle\phi}\ \swarrow\ {\scriptstyle\cong} & & \searrow \\ \tilde{H}^{i+2n}(MU(n);\mathbb{Z}) & \xrightarrow{\ f^*|\ } & H^{i+2n}(BU(n);\mathbb{Z}). \end{array}$$

As in the section 2.5, we can chase on this diagram and find out that the map at the right hand side is $a\mapsto a\cdot c_n$. This implies that $f^*|$ is a monomorphism. Therefore, $\tilde{H}^{i+2n}(MU(n);\mathbb{Z})$ can be identified in $H^{i+2n}(BU(n);\mathbb{Z})$ as the ideal generated by c_n. Furthermore, we have that $H^*(BU(n);\mathbb{Z})$ is concentrated in even degrees, so is $\tilde{H}^*(MU(n);\mathbb{Z})$.

Therefore, an additive basis of $\tilde{H}^{i+2n}(MU(n);\mathbb{Z})$ when $i=2k$ is given by all symmetric polynomials

$$s(\omega)\cdot c_n = \sum t_1^{a_1+1}\cdots t_r^{a_r+1}t_{r+1}\cdots t_n,$$

where the sum is taken over all essential permutations of the monomial $t_1^{a_1+1}\cdots t_r^{a_r+1}\cdot t_{r+1}\cdots t_n$, and $\omega=(a_1,\dots,a_r)$ runs over all partitions of k with r less than or equal to n.

We can now discuss the maneuver of the computation. We will proceed in two steps which holds in the next two propositions.

Proposition 2.13. *The cohomology $H^*(\boldsymbol{MU};\mathbb{Z}/p)$ is a free module over $\mathcal{A}_p/(Q_0)$ with even dimensional generators.*

The proof of this proposition will be very similar to the proof given by Thom in [Tho54]. We will just have to extend his idea to the mod p Steenrod algebra and work with the $\mathcal{P}^R$'s instead of the Sq^I's.

Proposition 2.14. *If $H^*(\boldsymbol{Y};\mathbb{Z}/p)$ is a free $A/(Q_0)$-module with even dimensional generators, and if $C_*(\boldsymbol{Y};\mathbb{Z})$ is finitely generated and vanish for $*$ less than some constant, then the stable homotopy group $\{S^0,\boldsymbol{Y}\}_n$ contains no p-torsion.*

This proposition will be proven using the Adams spectral sequence. Since $H^*(\boldsymbol{Y};\mathbb{Z}/p)$ is "almost" free, we will be able to construct a not so complicated free resolution over $\mathcal{A}_p$.

Let us first investigate proposition 2.13. In order to argue in the same fashion as in the section 2.5, one needs to understand better how behaves the polynomial $\mathcal{P}^R$ on products and on 2-dimensional cohomology classes. Recall that Δ_i is the sequence $(0,\dots,0,1,0,\dots)$, where there is a 1 in the i-th position. We assume the convention, $\Delta_0 := (0,0,\dots)$.

Lemma 2.15. *If p is odd, or $p=2$ and $H^*(X;\mathbb{Z}/2)$ is annihilated by $Q_0 = Sq^1$, then the following holds:*

- *For $x, y \in H^*(X;\mathbb{Z}/p)$*

$$\mathcal{P}^R(xy) = \sum_{R_1+R_2=R} \mathcal{P}^{R_1}(x)\mathcal{P}^{R_2}(y).$$

- *For any class $t \in H^2(X;\mathbb{Z}/p)$*

$$\mathcal{P}^R t = \begin{cases} t^{p^i} & \text{if } R = \Delta_i \\ 0 & \text{if } R \neq \Delta_0, \Delta_1, \dots \end{cases}$$

Proof. The reader can find the proof in [Mil60]. □

If we assume this lemma, it will now be a piece of cake to generalize the Thom's argument. Recall that each partition $\omega = (a_1,\dots,a_r)$ can be written as $\lambda\omega_R$, where λ contains no terms of the form $p^j - 1$ and ω_R consists of r^j copies of $p^j - 1$ for each j, therefore write $R := (r_1, r_2, \dots)$. As in section 2.5 we define a partial ordering on partitions.

Definition. For any two partitions $\omega = \lambda\omega_R$ and $\omega' = \lambda'\omega'_R$, we say that ω' is *less* than ω:

- if $l(\lambda') < l(\lambda)$, or
- if $l(\lambda') = l(\lambda)$, $w(\lambda) > w(\lambda')$.

Recall that $l(\omega)$ is the number of terms and $w(\omega)$ is the total degree of a given partition ω.

Lemma 2.16. *Let $R = (r_1, r_2, \dots)$ be a sequence of non-negative integers almost all zero. Let λ be a partition with containing no terms of the form $p^j - 1$ and suppose n to be large (say $n \geq l(\lambda) + r_1 + r_2 + \cdots$). Then we have*

$$\mathcal{P}^R(s(\lambda)\cdot c_n) = s(\lambda\omega_R)\cdot c_n + \sum (cst) s(\lambda'\omega_{R'})$$

with $\lambda'\omega_{R'}$ less than $\lambda\omega_R$.

Proof. By definition,

$$s(\lambda)\cdot c_n = \sum t_1^{a_1+1}\cdots t_r^{a_r+1} t_{r+1}\cdots t_n$$

where the sum is taken over all essential permutations of $t_1^{a_1+1}\cdots t_r^{a_r+1}t_{r+1}\cdots t_n$. We want to understand what is the biggest terms when we apply $\mathcal{P}^R$ to this polynomial.

By the product formula given in lemma 2.15, we have that

$$\mathcal{P}^R(t_1^{a_1+1}\cdots t_r^{a_r+1}t_{r+1}\cdots t_n) = \sum_{R_1+\cdots+R_s=R} \mathcal{P}^{R_1}(t_1^{a_1+1})\cdots\mathcal{P}^{R_s}(t_n)$$

The bigger terms will have r terms which are not a power of p. Indeed, by setting R_i to be zero on the terms of the form $t_i^{a_i+1}$, we will not create new terms of the form t^{p^j}. We could also try to apply $\mathcal{P}^{R_i}$ to $t_i^{a_i+1}$ and get $(cst)t_i^{b_i+1}$ where b_i is not of the form p^j-1. Unfortunately, this would increase $w(\lambda)$ which in our partial order would say that we created a "smaller" term.

Therefore,

$$\mathcal{P}^R(s(\lambda)\cdot c_n) = \Big(\sum t_1^{a_1+1}\cdots t_r^{a_r+1}\mathcal{P}^R(t_{r+1}\cdots t_n)\Big) + \text{ (smaller symmetric polynomials)}$$

To finish the proof, one can easily check that $\sum t_1^{a_1+1}\cdots t_r^{a_r+1}\mathcal{P}^R(t_{r+1}\cdots t_n)$ is equal to $s(\lambda\omega_R)\cdot c_n$ using the product formula of lemma 2.15. □

Proof of Proposition 2.13. By the preceding lemma, we have a system of equations

$$\mathcal{P}^R(s(\lambda)\cdot c_n) = s(\lambda\omega_R)\cdot c_n + \sum (cst)s(\lambda'\omega_{R'})\cdot c_n,$$

with λ' less than λ. By induction, we can solve these equations to get the following system of equations:

$$s(\lambda\omega_R)\cdot c_n = \mathcal{P}^R(s(\lambda)\cdot c_n) + \sum (cst)\mathcal{P}^{R'}(s(\lambda')\cdot c_n).$$

One can see this as inverting some upper block triangular matrices (obtained by arranging the additive basis using the partial ordering).

Since the $s(\lambda\omega_R)\cdot c_n$ form a $\mathbb{Z}/p$-basis of $\tilde{H}^{*+2n}(MU(n);\mathbb{Z}/p)$ for n large enough, we get that the $\mathcal{P}^R(s(\lambda)\cdot c_n)$ form also a $\mathbb{Z}/p$-basis. If we let n go to infinity, we can see that the classes $s(\lambda)\cdot c_\infty$ form a basis of $H^*(\boldsymbol{MU};\mathbb{Z}/p)$ as an $\mathcal{A}_p/(Q_0)$-module. Moreover, the degree of $s(\lambda)\cdot c_\infty$ is $2\cdot w(\lambda)$ since each t_i is of degree 2. □

Scholium 2.17. The free $\mathcal{A}_p/(Q_0)$-module $H^*(\boldsymbol{MU};\mathbb{Z}/p)$ has generators $s(\lambda)\cdot c_\infty$, where $\lambda = (a_1, a_2, \dots)$ runs over all partitions such that $a_i \neq p^j - 1$. The degree of $s(\lambda)\cdot c_\infty$ is $2\cdot$(total degree of λ).

Let us now investigate the proof of proposition 2.14. To prove this, Milnor claims in [Mil60] that the Adams spectral sequence (see appendix B.2) remains true by replacing the finite CW-complex Y by a *stable object* $\boldsymbol{Y}$, i.e. a sequence of CW-complexes $\boldsymbol{Y} = \{Y_0, Y_1, Y_2, \dots\}$ such that the suspension ΣY_i is a subcomplex of Y_{i+1}. However, we need that our stable object $\boldsymbol{Y}$ satisfies a finiteness condition. This finiteness condition holds in the fact that $C_n(\boldsymbol{Y};\mathbb{Z})$ should be finitely generated and vanish for n less than some constant. These hypothesis are satisfied in our study case $\boldsymbol{MU}$.

Theorem 2.18 (Adams spectral sequence for a stable object)**.** *Let X be a based finite CW-complexe and let $\boldsymbol{Y}$ be a stable object as above. Then for any prime number p, we have a spectral sequence $\{E_r, d_r\}$ such that*

- *E_r is bigraded and the differental d_r is of bidegree $(r, r-1)$.*
- *$E_2^{s,t} = \mathrm{Ext}^s_{\mathcal{A}_p}(H^*(\boldsymbol{Y};\mathbb{Z}/p), \Sigma^t\tilde{H}^*(X;\mathbb{Z}/p))$, where $\mathcal{A}_p$ is the Steenrod algebra mod p.*
- *The spectral sequence converge to $[X,\boldsymbol{Y}]_n = \mathrm{colim}_i[\Sigma^i X, Y_i]_n$, i.e. there exists a sequence of subgroups*

  $$[X,\boldsymbol{Y}]_n = F^{0,n} \supset F^{1,n+1} \supset F^{2,n+2} \supset \cdots$$

  such that $E_\infty^{s,t} = F^{s,t}/F^{s+1,t+1}$. Moreover, the intersection $\bigcap_i F^{i,n+i}$ is equal to ${}_{(p)}[X,\boldsymbol{Y}]_n$.

We will now construct an $\mathcal{A}_p$-free resolution of $H^*(\boldsymbol{Y};\mathbb{Z}/p)$ providing that it is a free $\mathcal{A}_p/(Q_0)$-module and has even dimensional generators. This construction is purely homological algebra.

Lemma 2.19. *Let A_0 be a Grassmann algebra on generators $Q_0, Q_1, \dots,$ with $\dim(Q_i) = 2p^i - 1$, then there exists an A_0-free resolution of $\mathbb{Z}/p$ (seen as an A_0-module by $Q_i\mathbb{Z}/p = 0$)*

$$\cdots \longrightarrow F_2 \xrightarrow{d} F_1 \xrightarrow{d} F_0 \xrightarrow{\varepsilon} \mathbb{Z}/p \longrightarrow 0$$

where each F_s is a free A_0-module generated by symbols $b(r_0, r_1, \dots)$ with $r_0 + r_1 + \cdots = s$. The degree of each generator $b(r_0, r_1, \dots)$ is $\sum r_i(\dim Q_i + 1) = \sum 2r_ip^i$ and the degree of d is -1.

Proof. Let us first explore the case where A_0 is a Grassmann algebra on one generator Q_0, with $\dim(Q_0) = 1$. Consider the twisted polynomial algebra on one generator, namely $x^{(1)}$, this is the $\mathbb{Z}/p$-module generated by symbols $1, x^{(1)}, x^{(2)}, \dots$ with multiplication given by the following formula

$$x^{(k)}x^{(l)} = \frac{(k+l)!}{k!\,l!}x^{(k+l)}.$$

Moreover, we require that each $x^{(k)}$ has degree $2k$. We denote by P this twisted polynomial algebra.

If we now consider the tensor product $A_0 \otimes P$, we can define a differential by

$$d(Q_0) = 0, \quad d(x^{(k)}) = Q_0 d(x^{(k-1)}).$$

By a differential, we mean that $A_0 \otimes P$ is a DGA-Algebra in the sense of [Car54a], i.e. d is $\mathbb{Z}/p$-linear and has degree -1, $dd = 0$ and $d(xy) = d(x)y + (-1)^{|x|}xd(y)$. The reader can check that the (complicated) multiplication above, ensures that the differential is multiplicative (which would have been false with the standard polynomial algebra).

In [Car54b], Cartan claims that $A_0 \otimes P$ is acyclic, i.e. $\ker d = \operatorname{im} d$. However, this is easily verified by a straight forward computations. The reader must have in mind the following picture

$$\begin{array}{ccccccccc}
\cdots \stackrel{d}{\longmapsto} Q_0x^{(2)} & & & & & & & & \\
& x^{(2)} \stackrel{d}{\longmapsto} Q_0x^{(1)} & & & & & & & \\
& & x^{(1)} \stackrel{d}{\longmapsto} Q_0 & & & & & & \\
& & 1 \longmapsto 1 & & & & & & \\
\cdots \longrightarrow F_2 & \stackrel{d}{\longrightarrow} & F_1 & \stackrel{d}{\longrightarrow} & F_0 & \stackrel{\varepsilon}{\longrightarrow} & \mathbb{Z}/p & \longrightarrow & 0.
\end{array}$$

So considering these F_s as A_0-module over a generator $x^{(s)}$, this gives us our desired free resolution in the particular case where A_0 is a Grassmann algebra over one generator.

Let now investigate the case where A_0 is a Grassmann algebra on two generators, namely Q_0 of degree 1 and Q_1 of degree $2p-1$. As before, construct the twisted polynomial algebra P on two generators $x^{(1)}$ of degree 2 and $y^{(1)}$ of degree $2p$. The differential on the tensor product $A_0 \otimes P$ is defined on generators as before

$$d(Q_i) = 0, \quad d(x^{(k)}) = Q_0 d(x^{(k-1)}), \quad d(y^{(l)}) = Q_1 d(y^{(l-1)}).$$

In order to not be bothered by the signs we let the differential operate on the right, i.e. we force the relation

$$d(xy) = xd(y) + (-1)^{|y|}d(x)y.$$

Thus, we can deduce the general formula

$$d(a \otimes x^{(k)}y^{(l)}) = aQ_0 \otimes x^{(k-1)}y^{(l)} + aQ_1 \otimes x^{(k)}y^{(l-1)}$$

where $a \in A_0$. As before, we can show that $dd = 0$ and that $A_0 \otimes P$ is acyclic. Hence, one can construct an A_0-free resolution

$$\cdots \longrightarrow F_2 \stackrel{d}{\longrightarrow} F_1 \stackrel{d}{\longrightarrow} F_0 \stackrel{\varepsilon}{\longrightarrow} \mathbb{Z}/p \longrightarrow 0$$

where each F_s is a free A_0-module with basis $x^{(k)}y^{(s-k)}$.

In a similar way this construction can be generalized to our Grassmann algebra A_0 on generators $Q_0, Q_1, Q_2, \dots$. ☐

Lemma 2.20. *If $H^*(\boldsymbol{Y}; \mathbb{Z}/p)$ is a free $\mathcal{A}_p/(Q_0)$-module with generators $\{y_\alpha\}$, then there exists a free $\mathcal{A}_p$-resolution of $H^*(\boldsymbol{Y}; \mathbb{Z}/p)$*

$$\cdots \longrightarrow F'_2 \stackrel{d}{\longrightarrow} F'_1 \stackrel{d}{\longrightarrow} F'_0 \stackrel{\varepsilon}{\longrightarrow} H^*(\boldsymbol{Y}; \mathbb{Z}/p) \longrightarrow 0$$

where each F'_s is a free $\mathcal{A}_p$-module generated by symbols $b_\alpha(r_0, r_1, \dots)$ with $r_0 + r_1 + \cdots = s$. Moreover, the degree of each generator $b_\alpha(r_0, r_1, \dots)$ is $\dim y_\alpha + \sum 2r_i(p^i - 1) + s$.

Proof. It is enough to build a free $\mathcal{A}_p$-resolution of the free $\mathcal{A}_p/(Q_0)$-module generated by a single element y_α. The general resolution of $H^*(\boldsymbol{Y};\mathbb{Z}/p)$ will be given by taking direct sums of these resolutions.

Starting with the free A_0-resolution of the previous lemma

$$\cdots \longrightarrow F_2 \xrightarrow{d} F_1 \xrightarrow{d} F_0 \xrightarrow{\varepsilon} \mathbb{Z}/p \longrightarrow 0.$$

We apply to it the functor $\mathcal{A}_p \otimes_{A_0} (-)$ to get a free $\mathcal{A}_p$-resolution of $\mathcal{A}_p \otimes_{A_0} \mathbb{Z}/p$

$$\cdots \longrightarrow \mathcal{A}_p \otimes_{A_0} F_2 \xrightarrow{d} \mathcal{A}_p \otimes_{A_0} F_1 \xrightarrow{d} \mathcal{A}_p \otimes_{A_0} F_0 \xrightarrow{\varepsilon} \mathcal{A}_p \otimes_{A_0} \mathbb{Z}/p \longrightarrow 0.$$

Since $\mathcal{A}_p \otimes_{A_0} \mathbb{Z}/p \cong \mathcal{A}_p/(Q_0)$ as left $\mathcal{A}_p$-modules, this is a free resolution of $\mathcal{A}_p/(Q_0)$. By a shift of degrees, we get the desired resolution of the free $\mathcal{A}_p/(Q_0)$-module generated by y_α.

In order to ensure that the differential is of degree 0, we set that $b_\alpha(r_0, r_1, \dots)$ is of degree $\dim y_\alpha + \sum 2r_i(p^i - 1) + s$, where $s = r_1 + r_2 + \cdots$. Indeed, the explicit formula of the differential is given by

$$d(a \otimes b_\alpha(r_0, r_1, \dots)) = \sum_i aQ_i \otimes b_\alpha(r_0, \dots, r_i - 1, \dots).$$

The degree of the element on the left hand side is

$$|a| + |y_\alpha| + \sum 2r_j(p^j - 1) + s,$$

while each summand on the right hand side is of degree

$$|a| + (2p^i - 1) + |y_\alpha| + \sum 2r_j(p^j - 1) - 2(p^i - 1) + (s - 1). \qquad \square$$

We will use the above resolution to compute the spectral sequence with X; the space obtained by gluing a 2-cell on a circle with a map of degree p. X can also be seen as the pushout

$$\begin{array}{ccc} S^1 & \xrightarrow{p} & S^1 \\ \downarrow & & \downarrow \\ D^2 & \longrightarrow & X. \end{array}$$

This space gets particularly interesting by the following lemma.

Lemma 2.21. *There is an exact sequence of abelian groups*

$$0 \longrightarrow [S^1, \boldsymbol{Y}]_n \otimes \mathbb{Z}/p \longrightarrow [X, \boldsymbol{Y}]_n \longrightarrow \operatorname{Tor}([S^1, \boldsymbol{Y}]_{n-1}, \mathbb{Z}/p) \longrightarrow 0.$$

Moreover, if $[S^0, \boldsymbol{Y}]_n$ contains p-torsion, then $[X, \boldsymbol{Y}]_m$ has to be non-trivial for two consecutive values of m.

Proof. The exact sequence follows by factorizing the following long exact sequence

$$\begin{array}{ccccccccc} [\Sigma^{n+1}S^1, Y_i] & \xrightarrow{\cdot p} & [\Sigma^{n+1}S^1, Y_i] & \longrightarrow & [\Sigma^n X, Y_i] & \longrightarrow & [\Sigma^n S^1, Y_i] & \xrightarrow{\cdot p} & [\Sigma^n S^1, Y_i]. \\ & & \downarrow & \nearrow & & \searrow & \uparrow & & \\ & & [\Sigma^{n+1}S^1, Y_i] \otimes \mathbb{Z}/p & & & & \operatorname{Tor}([\Sigma^n S^1, Y_i], \mathbb{Z}/p) & & \end{array}$$

Take the colimit to obtain the desired exact sequence.

The rest of the lemma follows easily. Suppose that $[S^0, \boldsymbol{Y}]_n$ contains p-torsion for a certain n, then $[S^1, \boldsymbol{Y}]_{n-1} = [S^0, \boldsymbol{Y}]_n$ also contains p-torsion. Hence the groups $[S^1, \boldsymbol{Y}]_{n-1} \otimes \mathbb{Z}/p$ and $\operatorname{Tor}([S^1, \boldsymbol{Y}]_{n-1}, \mathbb{Z}/p)$ are non-trival, thus $[X, \boldsymbol{Y}]_m$ is non-trivial for $m = n - 1, n$. $\square$

Proof of Proposition 2.14. We will now compute the spectral sequence and show that $[X, \boldsymbol{Y}]_m$ cannot be trivial for two consecutive values of m. Hence, by the previous lemma one would have that $[S^0, \boldsymbol{Y}]_n$ contains no p-torsion.

First, let us compute $\tilde{H}^*(X;\mathbb{Z}/p)$. The cellular chain complex of X is

$$\cdots \longrightarrow 0 \longrightarrow \mathbb{Z} \xrightarrow{p} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \longrightarrow 0.$$

Hence its mod p reduced cohomology groups have generators

$$x \in \tilde{H}^1(X;\mathbb{Z}/p), \quad Q_0 x \in \tilde{H}^2(X;\mathbb{Z}/p).$$

Let us now consider the $\mathcal{A}_p$-free resolution given in lemma 2.20, and apply to it the functor $\mathrm{Hom}_{\mathcal{A}_p}(-, \Sigma^t \tilde{H}^*(X;\mathbb{Z}/p))$. This construction gives us a cochain complex

$$0 \longrightarrow \mathrm{Hom}_{\mathcal{A}_p}(F_0', \Sigma^t \tilde{H}^*(X;\mathbb{Z}/p)) \xrightarrow{d^*} \mathrm{Hom}_{\mathcal{A}_p}(F_1', \Sigma^t \tilde{H}^*(X;\mathbb{Z}/p)) \xrightarrow{d^*} \cdots$$

By linear algebra, we are able to describe a basis of $\mathrm{Hom}_{\mathcal{A}_p}(F_s', \Sigma^t \tilde{H}^*(X;\mathbb{Z}/p))$. This basis consists in the following homomorphisms:

- for each $b_\alpha(r_0, r_1, \dots)$ of degree $t+1$, we have the map $h_\alpha(r_0, r_1, \dots)$ defined on the basis by sending $b_\alpha(r_0, r_1, \dots)$ to x and all other elements of the basis to 0,
- for each $b_\alpha(r_0, r_1, \dots)$ of degree $t+2$, we have the map $h'_\alpha(r_0, r_1, \dots)$ defined on the basis by sending $b_\alpha(r_0, r_1, \dots)$ to $Q_0 x$ and all other elements of the basis to 0.

By the very definition of the differential, the reader can check that

$$d^*(h_\alpha(r_0, r_1, \dots)) = h'_\alpha(r_0 + 1, r_1, \dots)$$

and that

$$d^*(h'_\alpha(r_0, r_1, \dots)) = 0.$$

Therefore, the $E_2^{s,t}$ term of the spectral sequence has a basis composed by all elements $h'_\alpha(0, r_1, r_2, \dots)$ of degree $t + 2 = \dim y_\alpha + \sum_i 2r_i(p^i - 1) + s$. If we rewrite this equality, we have that

$$t - s = \dim y_\alpha + \sum_i 2r_i(p^i - 1) - 2.$$

Therefore, $E_2^{s,t} = 0$ if $t-s$ is odd. Thus $[X, \boldsymbol{Y}]_n$ is concentrated in even degrees and is trivial whenever n is odd. We conclude by the previous lemma that $[S^0, \boldsymbol{Y}]_n$ has no p-torsion □

We will now use Serre $\mathscr{C}$-theory and the Pontrjagin-Thom theorem to get the description of Ω^U_*.

Theorem 2.22.

$$\Omega^U_n \cong \begin{cases} \underbrace{\mathbb{Z} \times \cdots \times \mathbb{Z}}_{p(k)\ times} & \text{if } n = 2k \\ 0 & \text{if } n \neq 2k \end{cases}$$

Proof. By Serre $\mathscr{C}$-theory, we have that the stable Hurewicz homomorphism

$$[S^0, \boldsymbol{MU}]_n \longrightarrow H_n(\boldsymbol{MU};\mathbb{Z})$$

is a $\mathscr{C}_F$-isomorphism, where $\mathscr{C}_F$ to be the class of finite abelian groups. Using that $[S^0, \boldsymbol{MU}]_n$ has no torsion, we deduce by the Thom isomorphism (in homology) that $[S^0, \boldsymbol{MU}]_n$ is isomorphic to a $p(k)$-fold product $\mathbb{Z} \times \cdots \times \mathbb{Z}$ for $n = 2k$. Here $p(k)$ denotes the number of partitions of k. □

Remark. The Thom isomorphism in homology is given by the composition

$$\tilde{H}_{n+i}(T\xi;\mathbb{Z}) \xrightarrow{U_\xi\cap} H_i(E;\mathbb{Z}) \xrightarrow{\pi_*} H_i(B;\mathbb{Z})$$

where ξ is an oriented n-plane bundle over B with total space E. In our study case, $BU(r)$ is covered by a complex r-plane bundle which gives it a natural orientation as a real plane bundle. The Thom isomorphism induces by taking colimits the following isomorphism

$$H_n(\boldsymbol{MU};\mathbb{Z}) \xrightarrow{\cong} H_n(BU;\mathbb{Z}).$$

Moreover, the cell structure of BU is well known, so is $H_*(BU;\mathbb{Z})$. The number of n-cells for $n = 2k$ is exactly $p(k)$, and there is no odd dimensional cells. This can be found in [MS74].

We will now investigate the ring structure of Ω_*^U. We shall prove the following theorem.

Theorem 2.23. *The graded ring Ω_*^U is isomorphic to the polynomial ring $\mathbb{Z}[x_2, x_4, x_6, \dots]$ on generators x_n for each $n = 2i$.*

We recall that the definition of the ring structure on Ω_*^U was given in the end of last section. Moreover, the Pontrjagin-Thom isomorphism becomes an isomorphism of graded rings (the ring structure of $\pi_*(\boldsymbol{MU})$ is described in section 3.3). To prove this theorem, we will compute the (multiplicative) Adams spectral sequence for $\boldsymbol{Y} = \boldsymbol{MU}$ and $X = S^0$, this is the spectral sequence where the E_2-term is

$$E_2^{s,t} = \operatorname{Ext}^s_{\mathcal{A}_p}(H^*(\boldsymbol{MU};\mathbb{Z}/p), \Sigma^t\mathbb{Z}/p)$$

and the spectral sequence converge to $\pi_{t-s}(\boldsymbol{MU})$.

Proposition 2.24. *The multiplicative $E_2^{*,*}$-term of this spectral sequence is isomorphic to a polynomial ring $\mathbb{Z}/p[h_0, x_2, x_4, \dots]$. Furthermore, the bidegree of h_0 is $(1,1)$ and the bidegree of x_{2i} is $(0, 2i)$ if $i \neq p^j - 1$ and is $(1, 2i+1)$ if $i = p^j - 1$.*

We break the computation of $\operatorname{Ext}^s_{\mathcal{A}_p}(H^*(\boldsymbol{MU};\mathbb{Z}/p), \Sigma^t\mathbb{Z}/p)$ into two claims.

Claim 1. *The cohomology ring $H^*(\boldsymbol{MU};\mathbb{Z}/p)$ is isomorphic to the tensor product $\mathcal{A}_p/(Q_0)\otimes P^*$, such that the dual of P^* is isomorphic to the polynomial ring $P_* = \mathbb{Z}/p[x_2, x_4, \dots]$ where the x_{2i} runs over all $i \neq p^j - 1$ and are of degree $2i$.*

Claim 2. *We have an isomorphism of rings $\operatorname{Ext}^s_{A_0}(\mathbb{Z}/p, \Sigma^t\mathbb{Z}/p) \cong \mathbb{Z}/p[h_0, h_1, h_2, \dots]$ where the bidegree of h_j is $(1, 2p^i - 1)$.*

Proof of Claim 1. We already shown that we have an isomorphism of $\mathbb{Z}/p$-modules

$$H^*(\boldsymbol{MU};\mathbb{Z}/p) \cong \mathcal{A}_p/(Q_0) \otimes \underbrace{\operatorname{span}\{s(\lambda)\cdot c_\infty : \lambda = (a_1, a_2, \dots), a_i \neq p^j - 1\}}_{=:P^*}.$$

Moreover, we have a coalgebra structure on $H^*(\boldsymbol{MU})$ induced by the Thom isomorphism, i.e.

$$\begin{array}{ccc} H^*(BU;\mathbb{Z}/p) & \xrightarrow{c_k\mapsto \sum_{i+j=k} c_i\otimes c_j} & H^*(BU;\mathbb{Z}/p)\otimes H^*(BU;\mathbb{Z}/p) \\ \downarrow{\scriptstyle\phi} & & \downarrow{\scriptstyle\phi\otimes\phi} \\ H^*(\boldsymbol{MU}) & \longrightarrow & H^*(\boldsymbol{MU};\mathbb{Z}/p)\otimes H^*(\boldsymbol{MU};\mathbb{Z}/p), \end{array}$$

where $H^*(BU;\mathbb{Z}/p)$ is identified with the ring $\mathbb{Z}/p[c_1, c_2, \dots]$. In our terms, ϕ is described by

$$c_k \mapsto s(\underbrace{1, 1, \dots, 1}_{k\text{ times}})\cdot c_\infty.$$

Therefore, the coproduct on $H^*(\boldsymbol{MU};\mathbb{Z}/p)$ is given by

$$s(\lambda)\cdot c_\infty \longmapsto \sum_{\xi\zeta=\lambda} s(\xi)\cdot c_\infty \otimes s(\zeta)\cdot c_\infty,$$

where $\xi\zeta$ denotes the juxtaposition of the two partitions.

Considering now the dual of $H^*(\boldsymbol{MU};\mathbb{Z}/p)$, this is the algebra $\mathbb{Z}/p[b_1, b_2, \dots]$ where the b_i's are of degree $2i$. By [Mil58], we know that the dual of $\mathcal{A}_p/(Q_0)$ is a polynomial algebra $\mathbb{Z}/p[\xi_1, \xi_2, \dots]$ where the ξ_j have degree $2(p^j-1)$. And so the dual of P^* is our polynomial ring $P_* = \mathbb{Z}/p[x_{2i} : i \neq p^j - 1]$. □

Proof of Claim 2. Recall that we constructed an A_0-free resolution of $\mathbb{Z}/p$ of the form

$$\cdots \longrightarrow F_2 \xrightarrow{d} F_1 \xrightarrow{d} F_0 \xrightarrow{\varepsilon} \mathbb{Z}/p \longrightarrow 0$$

where each F_s is a free A_0-module generated by symbols $b(r_0, r_1, \dots)$ with $r_0 + r_1 + \cdots = s$. The degree of each generator $b(r_0, r_1, \dots)$ is $\sum 2r_j(p^j-1)+s$. Moreover, the graded $\mathbb{Z}/p$-module $\oplus F_s$ has a multiplicative structure given on generators by

$$b(r_0, r_1, \dots)\cdot b(s_0, s_1, \dots) = b(r_0+s_0, r_1+s_1, \dots).$$

Since $\mathbb{Z}/p$ has an A_0-module structure given by $Q_i\mathbb{Z}/p = 0$, all our differential are trivial and $h(0,0,\dots)$ is sent on 1 via ε. If we now apply the functor $\mathrm{Hom}_{A_0}(-, \Sigma^t\mathbb{Z}/p)$ to our resolution, one can see that $\mathrm{Hom}_{A_0}(F_s, \Sigma^t\mathbb{Z}/p)$ is a $\mathbb{Z}/p$ -module generated by elements

$$\begin{aligned} h(r_0, r_1, \dots) : F_s &\longrightarrow \Sigma^t\mathbb{Z}/p \\ b(s_0, s_1, \dots) &\longmapsto \begin{cases} 1, & \text{if } (s_0, s_1, \dots) = (r_0, r_1, \dots) \\ 0, & \text{otherwise} \end{cases} \end{aligned}$$

and $|b(r_0, r_1, \dots)| = \sum 2r_j(p^j-1)+s = t$. Thus, the induced multiplicative structure is given by

$$h(r_0, r_1, \dots)\cdot h(s_0, s_1, \dots) = h(r_0+s_0, r_1+s_1, \dots),$$

and since our differentials are all trivial on the F_s, so they are on the $\mathrm{Hom}_{A_0}(F_s, \Sigma^t\mathbb{Z}/p)$.

Finally, we define the ring isomorphism by the following

$$\begin{aligned} \mathbb{Z}/p[h_0, h_1, h_2, \dots] &\longrightarrow \mathrm{Ext}^*_{A_0}(\mathbb{Z}/p, \Sigma^*\mathbb{Z}/p) \\ h_j &\longmapsto h\underbrace{(0, \dots, 0, 1, 0, \dots)}_{j\text{-th position}}. \end{aligned}$$

□

Proof of Proposition 2.24. With these two claims, we have the followings isomorphisms

$$\begin{aligned} E_2^{*,*} &= \mathrm{Ext}^*_{\mathcal{A}_p}(H^*(\boldsymbol{MU};\mathbb{Z}/p), \Sigma^*\mathbb{Z}/p) \\ &\cong \mathrm{Ext}^*_{\mathcal{A}_p}(\mathcal{A}_p/(Q_0)\otimes P^*, \Sigma^*\mathbb{Z}/p) \\ &\cong \mathrm{Ext}^*_{\mathcal{A}_p}(\mathcal{A}_p\otimes_{A_0}\mathbb{Z}/p\otimes P^*, \Sigma^*\mathbb{Z}/p) \\ &\cong \mathrm{Ext}^*_{A_0}(\mathbb{Z}/p\otimes P^*, \Sigma^*\mathbb{Z}/p) \\ &\cong P_*\otimes \mathrm{Ext}^*_{A_0}(\mathbb{Z}/p, \Sigma^*\mathbb{Z}/p) \\ &\cong \mathbb{Z}/p[x_{2i} : i\neq p^j-1]\otimes \mathbb{Z}/p[h_0, h_1, h_2, \dots] \\ &\cong \mathbb{Z}/p[h_0, x_2, x_4, x_6, \dots] \end{aligned}$$

where the $x_{2i} := h_j$, whenever $i = p^j - 1 > 0$. This implies that the spectral sequence looks like (where $p = 3$ for instance)

s								
$\vdots$	$\vdots$		$\vdots$		$\vdots$	$\vdots$		
4	h_0^4		$h_0^4x_2$		$h_0^4x_2^2$	$h_0^3x_4$		
3	h_0^3		$h_0^3x_2$		$h_0^3x_2^2$	$h_0^2x_4$		
2	h_0^2		$h_0^2x_2$		$h_0^2x_2^2$	h_0x_4		
1	h_0		h_0x_2		$h_0x_2^2$	x_4		
0	1		x_2		x_2^2			
	0	1	2	3	4		5	$\cdots$ $t-s$

This implies that the spectral sequence is trivial, since the differentials d_r of the spectral sequence are of bidegree $(r, r-1)$. $\square$

Proof of Theorem 2.23. Let $Q(\pi_*(\boldsymbol{MU}))$ denote the module of indecomposable. One can show that

$$Q(\pi_*(\boldsymbol{MU}))_m \otimes \mathbb{Z}/p \cong \begin{cases} \mathbb{Z}/p & \text{if } m = 2i, i > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Indeed, this follows by our previous computation of the E_2-term and a technical lemma concerning the convergence of the Adams spectral sequence (cf. [Ada74, Chapter 2, Lemma 8.7]). If we let x^I denote any monomial in the x_{2i}, then each tower x^I, h_0x^I, $h_0^2x^I$, ... will give us a summand in $\pi_*(\boldsymbol{MU})$. This is because all multiples h_0^s are represented in $\pi_*(\boldsymbol{MU})$ by multiples of p^s.

Since the above result is true for all prime number p, we have that

$$Q(\pi_*(\boldsymbol{MU}))_m \cong \begin{cases} \mathbb{Z} & \text{if } m = 2i, i > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Thus, $\pi_*(\boldsymbol{MU}) \cong \mathbb{Z}[x_2, x_4, x_6, \ldots]$. $\square$

3 Cohomology Theories and FGL

In this chapter we explain the very beginning of the so called "chromatic" approach to stable homotopy theory. Recall that the latter is the study of spectra, or equivalently generalized (co)homology theories.

At first, we will study formal group laws in its generality. We shall talk about the *Lazard* ring L and its *universal formal group law.* This formal group law is particularly interesting because any formal group law can be obtain by base change from it.

Secondly, we will study complex oriented cohomology theories. We shall see how to create formal group laws from them. It turns out that the formal group law associated to complex cobordism is isomorphic to the universal formal group law. Since the latter was the "most complicated" formal group law, we can deduce that complex cobordism is the "most complicated" cohomology theory among complex oriented cohomology theories. Moreover, we will see how from a formal group law, one can *realize* a cohomology theory (whenever it is possible).

A more modern approach would use the language of stacks and a lot of algebraic geometry. Since we give here only an introduction, we preferred to stick to the classical formulations of this theory.

Let $\mathscr{C}$ be the category of pairs of topological spaces (X, A), i.e. $A \subset X$.

Definition. Let $E^* : \mathscr{C} \to \mathcal{A}b^{\mathbb{Z}}$ be a contravariant functor, with a *coboundary operator*, i.e. for any $(X, A) \in \mathscr{C}$ we have a homomorphism

$$\delta^* : E^*(A) \to E^{*+1}(X, A)$$

natural in (X, A). We say that E^* is a *generalized cohomology theory* if the following axioms are satisfied:

Homotopy axiom If $f, g : (X, A) \to (Y, B)$ are homotopic, then $f^* = g^*$.

Exactness axiom For any pair (X, A), we have a long exact sequence

$$\cdots \to E^{*-1}(A) \xrightarrow{\delta} E^*(X, A) \to E^*(X) \to E^*(A) \to \cdots$$

Excision axiom If U is an open subset of X such that $\overline{U} \subset intA$, then we have a natural isomorphism

$$E^*(X, A) \xrightarrow{\cong} E^*(X - U, A - U).$$

Additivity axiom If $(X, A) = \coprod (X_i, A_i)$, then we have a natural isomorphism

$$E^*(X, A) \xrightarrow{\cong} \prod E^*(X_i, A_i).$$

In a similar fashion, one can define the notion of a *generalized homology theory.* This can be specified using the axioms above in their dual statement. For a precise definition, we refer the reader to [ES52].

Sometimes, we require our cohomology theory to satisfy the **dimension axiom**; this is $E^*(pt) = 0$ for all $* \neq 0$. However, we will have any use of it because the cohomology theories that we will consider does not satisfy this axiom. For more details, we invite the reader to look at [ES52]. This is the original text where these generalized (co)homology theories appeared.

From this definition one can talk about *reduced* (co)homology theory, the long exact sequence of a triple or even the Mayer-Vietoris exact sequence. Moreover, all results with singular (co)homology that can be proven without using chain complexes are good candidates to be interpreted in this general framework. For instance, we will see later how we can talk about the Thom isomorphism with some extra conditions on the generalized cohomology theory.

The major result involving generalized cohomology theories is the Brown representability theorem. In order to give a precise statement, let us define some objects.

Definition. An Ω-*spectrum* is a sequence of CW-complexes $\boldsymbol{E}_*$ with basepoints and homotopy equivalences $h_q : \Omega E_q \to E_{q+1}$.

Let (X, A) be a pair in $\mathscr{C}$, we define the based space $Z(X, A)$ to be

$$X \cup A \times I \cup \{p\}/\sim$$

where we identify $a \sim (a, 1)$ and $(a, 0) \sim p$ for any $a \in A$. The basepoint of $Z(X, A)$ is p.

Proposition 3.1. *Let $\boldsymbol{E}_*$ be an Ω-spectrum, then the functor $[Z(-,-), \boldsymbol{E}_*]$ is a generalized cohomology theory.*

Theorem 3.2 (E. H. Brown)**.** *Let E^* be a generalized cohomology theory, then there exists an Ω-spectrum $\boldsymbol{E}_*$ unique up to homotopy such that $[Z(-,-), \boldsymbol{E}_*]$ and E^* are naturally equivalent on the category of based path-connected CW-pairs.*

The proof of this theorem can be found in [Bro62]. However, the reader will prefer the more modern (and revisited) one from [Swi02].

Proposition 3.3. *The singular cohomology $H^*(-; A)$ with coefficient in an abelian group A is represented by the Eilenberg-MacLane spectrum $\boldsymbol{H}A := \{K(A, n)\}_n$.*

3.1 Complex Oriented Cohomology Theories

Definition. Let E^* be a generalized cohomology theory. We say that E^* is *multiplicative* if there is a natural homomorphisms of graded abelian groups

$$E^*(X,A) \otimes E^*(X,A) \xrightarrow{\cup} E^*(X,A), \qquad \mathbb{Z} \xrightarrow{\eta} E^*(pt),$$

such that the following diagrams commutes

$$\begin{array}{ccc} E^*(X,A)\otimes E^*(X,A)\otimes E^*(X,A) & \xrightarrow{1\otimes\cup} & E^*(X,A)\otimes E^*(X,A) \\ \downarrow{\scriptstyle \cup\otimes 1} & & \downarrow{\scriptstyle \cup} \\ E^*(X,A)\otimes E^*(X,A) & \xrightarrow{\cup} & E^*(X,A) \end{array}$$

$$\begin{array}{ccccc} \mathbb{Z}\otimes E^*(pt) & \xrightarrow{\eta\otimes 1} & E^*(pt)\otimes E^*(pt) & \xleftarrow{1\otimes\eta} & E^*(pt)\otimes\mathbb{Z} \\ & \searrow{\scriptstyle \cong} & \downarrow{\scriptstyle \cup} & \swarrow{\scriptstyle \cong} & \\ & & E^*(pt). & & \end{array}$$

The homomorphisms $\cup$ and η are called respectively the *cup product* and the *unit*.

Moreover, a multiplicative cohomology theory E^* is said *commutative* if we have in addition that the diagram

$$\begin{array}{ccc} E^*(X,A)\otimes E^*(X,A) & & \\ \downarrow{\scriptstyle T} & \searrow{\scriptstyle \cup} & \\ & & E^*(X,A) \\ & \nearrow{\scriptstyle \cup} & \\ E^*(X,A)\otimes E^*(X,A) & & \end{array}$$

commutes. Here T denotes the twist.

From the cup product, we can deduce the existence of a *cross product*, i.e. a natural homomorphism of graded abelian groups

$$E^*(X,A) \otimes E^*(Y,B) \xrightarrow{\times} E^*((X,A)\times(Y,B)).$$

Recall that the product of pairs $(X,A)\times(Y,B)$ is the pair $(X\times Y, X\times B\cup A\times Y)$. This cross product induces a product in reduced cohomology, called the *Pontrjagin product*

$$\tilde{E}^*(X)\otimes\tilde{E}^*(Y) \xrightarrow{\wedge} \tilde{E}^*(X\wedge Y).$$

Remark. It is a matter of fact that the existence of any of these products implies the existence of the others. Also, if the reader has in mind that cohomology theories comes from spectra, one may ask what kind of additional structure we have (on the associated spectrum) if the cohomology theory is multiplicative. It turns out that these are *ring spectra*. We will not go into details, but the reader can find some answers in [Swi02, Chapter 13].

Definition. Let E^* be a generalized cohomology theory. We say that E^* is a *complex oriented* cohomology theory if it is commutative multiplicative and for any finite dimensional complex vector bundle ξ, we have a natural choice of a Thom class. More precisely, for any complex vector bundle ξ over X with projection map $\pi : V \to X$ and dimension n, there is a natural class $U_\xi \in E^{2n}(V, V_0)$ such that the following properties holds:

- For any $x \in X$, the image of U_ξ under the composition

$$E^{2n}(V, V_0) \xrightarrow{\bar{i}^*} E^{2n}(F, F_0) \cong \tilde{E}^{2n}(S^{2n}) \xrightarrow{\Sigma^{2n}} \tilde{E}^0(S^0) \cong E^0(pt)$$

  is a unit. Here, i denote the inclusion of $\{x\}$ into X.

- The classes U_ξ are natural under pullbacks, i.e. $U_{f^*\xi} = f^*(U_\xi)$.
- The classes U_ξ are multiplicative, i.e. $U_{\xi \times \zeta} = U_\xi \times U_\zeta$.

The reader should have in mind the case where E^* is singular cohomology. Hence, these Thom classes already appeared in Proposition 2.4. However, we made the assumption that cohomology had its coefficients in a field. Whenever our (real) vector bundle is *oriented*, the proposition remains true for coefficients in any ring. Since complex vector bundles have a natural orientation for their underlying real vector bundles structures, Proposition 2.4 holds for any ring coefficients considering complex vector bundles.

Examples. We will see three main examples of complex oriented cohomology theories, these are:

1. Singular cohomology with ring coefficients,
2. K-Theory,
3. Complex cobordism.

We just discussed the first example. The two other deserve their own section (cf. 3.2 and 3.3).

We will now see that complex oriented cohomology theories allows us to talk about the Thom isomorphism, Euler classes and Chern classes. We will give here some general results and computation with such cohomology theories, such that the reader can have some intuition of what one can do in this framework.

Proposition 3.4. *Let E^* be a complex oriented cohomology theory and let ξ be a n-dimensional complex vector bundle, then there is an isomorphism*

$$E^*(V) \xrightarrow{\cup U_\xi} E^{*+2n}(V, V_0),$$

that gives rise to a Thom isomorphism given by the composition

$$E^*(X) \xrightarrow{\pi^*} E^*(V) \xrightarrow{\cup U_\xi} E^{*+2n}(V, V_0).$$

This isomorphism is denoted by ϕ.

Proof. Let us break the proof into several cases.

Case (ξ is a trivial bundle): Write $V = X \times \mathbb{C}^n$, hence we have the following commutative diagram

$$\begin{array}{ccccccc}
E^{2n}(X \times \mathbb{C}^n, (X \times \mathbb{C}^n)_0) & \xrightarrow{\cong} & \tilde{E}^{2n}(\Sigma^{2n} X_+) & \xrightarrow{\Sigma^{2n}} & \tilde{E}^0(X_+) & \longrightarrow & E^0(X) \\
\downarrow{\scriptstyle \bar{i}^*} & & \downarrow{\scriptstyle \Sigma^{2n} i_+^*} & & \downarrow{\scriptstyle (i_+)^*} & & \downarrow{\scriptstyle i^*} \\
E^{2n}(\mathbb{C}^n, \mathbb{C}^n_0) & \xrightarrow{\cong} & \tilde{E}^{2n}(S^{2n}) & \xrightarrow{\Sigma^{2n}} & \tilde{E}^0(S^0) & \xrightarrow{\cong} & E^0(pt).
\end{array}$$

Using the fact that the Thom class U_ξ is sent on a unit, we get that the top suspension isomorphism sends U_ξ unto a unit. Hence, one has that

$$E^*(X) \xrightarrow{\cong} \tilde{E}^{*+2n}(\Sigma^{2n} X_+) \cong E^{*+2n}(X \times \mathbb{C}^n, (X \times \mathbb{C}^n)_0)$$

is given by $z \mapsto z \times U_\xi$.

Case (ξ is a vector bundle over a compact space): The proof follows by induction on n, where $X = X^1 \cup \cdots \cup X^n$, and $\xi|_{X^i}$ are trivial. Let us do the proof for $n = 2$, the general case is proved similarly.

Write $V^1 = X^1 \times \mathbb{C}^n$, $V^2 = X^2 \times \mathbb{C}^n$ and $V^\cap = X^\cap \times \mathbb{C}^n$, where $X^\cap$ denotes the intersection of X^1 and X^2. Now draw the Mayer-Vietoris exact sequences of (V^1, V^2) and $((V^1, V^1_0), (V^2, V^2_0))$. By naturality of the Thom class, this gives us the following commutative diagram

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & E^*(V) & \longrightarrow & E^*(V^1) \oplus E^*(V^2) & \longrightarrow & E^*(V^\cap) & \longrightarrow & \cdots \\ & & \downarrow{\scriptstyle \times U_\xi} & & \downarrow{\scriptstyle (-)\times U_{\xi|} \oplus (-)\times U_{\xi|}} & & \downarrow{\scriptstyle \times U_{\xi|}} & & \\ \cdots & \longrightarrow & E^{*+2n}(V, V_0) & \longrightarrow & E^{*+2n}(V^1, V^1_0) \oplus E^{*+2n}(V^2, V^2_0) & \longrightarrow & E^{*+2n}(V^\cap, V^\cap_0) & \longrightarrow & \cdots \end{array}$$

By the 5-lemma, we deduce our desired result.

Case (ξ is any complex vector bundle): The result follows by the $\lim^1$ exact sequence where the limit is taken over all compact subspaces of X. □

Definition. If E^* is a complex oriented cohomology theory, we define the *Euler class*, denoted $e(\xi) \in E^{2n}(X)$, to be the image of U_ξ via

$$E^{2n}(V, V_0) \longrightarrow E^{2n}(V) \xrightarrow[\cong]{(\pi^*)^{-1}} E^{2n}(X) .$$

If ξ is a complex line bundle, then the Euler class is called the *first Chern class* and is denoted $c_1(\xi)$.

Remark. We will not develop the whole theory of Chern classes, although it is possible to do it in this general framework. One idea would be to consider the projective bundle $\mathbb{P}(\xi)$ of ξ, this is the fiber bundle over X whose fiber over any point x is the projective space of $\pi^{-1}(x)$. With this construction there is also a tautological line bundle over $\mathbb{P}(\xi)$, denoted $\mathbb{L}_\xi$. Using the Atiyah-Hirzebruch spectral sequence, one can show that $E^*(\mathbb{P}(\xi))$ is free over $E^*(X)$ with basis $1, t, \ldots, t^{n-1}$, where t denotes the first Chern class $c_1(\mathbb{L}_\xi) \in \tilde{E}^2(\mathbb{P}(\xi))$. Therefore, there exists unique elements $c_i(\xi) \in E^*(X)$ such that

$$t^n = c_1(\xi)t^{n-1} - c_2(\xi)t^{n-2} + \cdots + (-1)^{n-1}c_n(\xi).$$

Proposition 3.5. *Let E^* be a complex oriented cohomology theory. There exists a class x in $\tilde{E}^2(\mathbb{C}P^\infty)$ such that its image under the composition*

$$\tilde{E}^2(\mathbb{C}P^\infty) \longrightarrow \tilde{E}^2(\mathbb{C}P^1) \cong \tilde{E}^2(S^2) \cong E^0(pt)$$

is a unit.

Proof. Let γ^1 be the universal (complex) line bundle over $\mathbb{C}P^\infty$. We can hence consider its first Chern class $c_1(\gamma^1) \in E^2(\mathbb{C}P^\infty)$. Remark that this class is send on 0 in the following exact sequence

$$0 \longrightarrow \tilde{E}^2(\mathbb{C}P^\infty) \longrightarrow E^2(\mathbb{C}P^\infty) \longrightarrow E^2(pt) \longrightarrow 0.$$

Indeed, by naturality of the Chern classes we have that $c_1(\gamma^1)$ is sent on $c_1(i^*\gamma^1) = c_1(\varepsilon^1)$; which is trivial. Hence $c_1(\gamma^1)$ comes from a reduced class $x \in \tilde{E}^2(\mathbb{C}P^\infty)$. The reader can easily check that its image under the composition

$$\tilde{E}^2(\mathbb{C}P^\infty) \longrightarrow \tilde{E}^2(\mathbb{C}P^1) \cong \tilde{E}^2(S^2) \cong E^0(pt)$$

is a unit. □

Remark. In the literature, some authors assume the above proposition as the definition of a complex oriented cohomology theory (see for example [Ada74, Chapter 2]). In fact, this proposition is equivalent to our definition of orientability in a generalized multiplicative cohomology theory.

Once we have the first Chern class x (in the sense of the above proposition), we can recover all the Chern classes of $BU(n)$ (see the previous remark). We can then extend by naturality theses classes to any complex vector bundle. So a class $x \in \tilde{E}^2(\mathbb{C}P^\infty)$ with the above property will induce a unique orientation on our generalized multiplicative cohomology theory.

Let $x \in \tilde{E}^2(\mathbb{C}P^\infty)$ be a class given in the above proposition. We can hence define a map (for each n)

$$E^*(pt)[x] \longrightarrow E^*(\mathbb{C}P^n).$$

One can see that x^{n+1} is sent on zero. Indeed, let us cover $\mathbb{C}P^n$ by the canonical $n+1$ contractible open sets U_i. Since x is a reduced class, its restrict to zero on each U_i. Therefore, $x \in E^*(\mathbb{C}P^n, U_i)$ for any i, and by the very definition of the cross product $x^{n+1} \in E^*(\mathbb{C}P^n, U_1 \cup \cdots \cup U_{n+1}) = E^*(\mathbb{C}P^n, \mathbb{C}P^n) = 0$.

Proposition 3.6. *The induced map*

$$E^*(pt)[x]/(x^{n+1}) \longrightarrow E^*(\mathbb{C}P^n)$$

is an isomorphism.

Proof. Consider the Atiyah-Hirzebruch spectral sequence. Recall that its $E_2^{p,q}$ terms are equal to $H^p(\mathbb{C}P^n; E^q(pt))$ and the spectral sequence converge to $E^{p+q}(\mathbb{C}P^n)$. Since, this spectral sequence is natural and multiplicative, by the universal coefficient theorem we can deduce that the E_2 term is isomorphic to $E^*(pt)[x]/(x^{n+1})$. Moreover, x and all elements in $E^*(pt)$ are permanent cycles, i.e. has trivial image under the differentials d_r, for all $r \geq 2$. Therefore, the whole spectral sequence is trivial, and this gives us our desired result. □

Corollary 3.7. *Let E^* be a complex oriented cohomology theory, then $E^*(\mathbb{C}P^\infty)$ is isomorphic to the ring of power series $E^*(pt)[\![x]\!]$.*

In order to prove this, let us recall some results about $\lim^1$. Let

$$\cdots \xrightarrow{p} A_3 \xrightarrow{p} A_2 \xrightarrow{p} A_1$$

be a sequence of abelian groups. We define a homomorphism of abelian groups

$$d : \prod A_i \to \prod A_i : (a_i)_i \mapsto (a_i - pa_{i+1})_i.$$

Then one can see that its kernel is exactly $\lim_i A_i$. We define $\lim_i^1 A_i$ as the cokernel of d.

In [Mil62], Milnor proved that if we have a sequence of CW-complexes $K_1 \subset K_2 \subset \cdots$ with union K, then the following sequence of abelian groups

$$0 \longrightarrow \lim_i{}^1 E^{*-1}(K_i) \longrightarrow E^*(K) \longrightarrow \lim_i E^*(K_i) \longrightarrow 0$$

is exact.

Proof of corollary 3.7. The infinite complex projective space $\mathbb{C}P^\infty$ has a natural filtration $\mathbb{C}P^1 \subset \mathbb{C}P^2 \subset \cdots$. Thus one has a short exact sequence

$$0 \longrightarrow \lim_n{}^1 E^{*-1}(\mathbb{C}P^n) \longrightarrow E^*(\mathbb{C}P^\infty) \longrightarrow \lim_n E^*(\mathbb{C}P^n) \longrightarrow 0$$

and since each map $E^*(\mathbb{C}P^n) = E^*(pt)[x]/(x^{n+1}) \to E^*(pt)[x]/(x^n) = E^*(\mathbb{C}P^{n-1})$ is surjective, we have that $\lim_n^1 E^{*-1}(\mathbb{C}P^n) = 0$. So

$$E^*(\mathbb{C}P^\infty) \cong \lim_n E^*(pt)[x]/(x^{n+1}) \cong E^*(pt)[\![x]\!].$$ □

Similarly, one can prove the following proposition.

Proposition 3.8. *Let E^* be a complex oriented cohomology theory, then $E^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty)$ is isomorphic to the ring of power series $E^*(pt)[\![x_1, x_2]\!]$, where $x_1 = x \otimes 1$ and $x_2 = 1 \otimes x$.*

3.2 Example: K-Theory

We will see in this section a very nice example of a complex oriented cohomology theory; which is K-theory. We will first recall some general facts about it and then discuss its orientability. A very similar discussion can hold for KO-theory (for real vector bundle) and KSp-theory (for quaternionic vector bundle). But since these two last examples are NOT complex oriented cohomology theory, we will vanish them from our discussion. However, the interested reader can always look in [Swi02, Chapter 11] for a very precise discussion about all of this.

Let us start by recall the Grothendieck construction. For any abelian semigroup S, there exists an abelian group $K(S)$ and a homomorphism of semigroups $\phi : S \to K(S)$ satisfying the following universal property: for any abelian group A and homomorphism of semigroups $\psi : S \to A$, there exists a unique homomorphism of groups $\bar{\psi} : K(S) \to A$ such that the following diagram commutes

$$\begin{array}{ccc} S & \xrightarrow{\psi} & A \\ {\scriptstyle\phi}\downarrow & \exists ! \nearrow {\scriptstyle\bar{\psi}} & \\ K(S). & & \end{array}$$

The group $K(S)$ is obtained by adding formal inverse to S.

Let $V_{\mathbb{C}}(X)$ denote the set of isomorphism classes of complex vector bundles over X, where X is a compact space. This set is an abelian semigroup for the Whitney sum. The *K-theory* of X is defined as the abelian group $K(X) := K(V_{\mathbb{C}}(X))$ obtained via the Grothendieck construction. Moreover, elements of $K(X)$ can be written as $\{\xi\} - \{\eta\}$, where ξ and η are complex vector bundles and $\{\xi\}$, $\{\eta\}$ denotes their isomorphism classes.

Considering the homomorphism of semigroups $V_{\mathbb{C}}(X) \to \mathbb{Z} : \{\xi\} \mapsto \dim \xi$, we have an induced group homomorphism $K(X) \to \mathbb{Z}$. When X is a point, this map becomes an isomorphism, hence it can be identified with the induced map

$$K(X) \to K(*).$$

We define the *reduced K-theory* of a (compact) based space X to be the kernel of the above map, where $*$ denote the based point. This abelian group is denoted $\tilde{K}(X)$. Moreover, we have the following isomorphism $K(X) \cong \tilde{K}(X) \times \mathbb{Z}$.

Another useful way to think about $\tilde{K}$ is to talk about stable equivalence classes.

Definition. Let ξ and η be two complex vector bundles over a compact base space X. They are said *stably equivalent* if there exists trivial bundles ε and ε' over B such that $\xi \oplus \varepsilon \cong \eta \oplus \varepsilon'$.

Proposition 3.9. *If ξ is a complex vector bundle over a compact space X, then there exists a complex vector bundle ζ such that $\xi \oplus \zeta$ is equivalent to the trivial bundle ε^q for some q.*

Proof. The reader shall find the proof in [May99a, Chapter 24]. □

This proposition implies that the set of stable equivalence classes of complex vector bundles over X has an abelian group structure. Moreover, we have the following isomorphism

$$\{\text{Stable equivalence classes of complex vector bundles over } X\} \xrightarrow{\cong} \tilde{K}(X)$$

given by $\{\xi\}_s \longmapsto \{\xi\} - \{\varepsilon^d\}$, where $d = \dim \xi$ and $\{\xi\}_s$ denotes the stable equivalence class of ξ.

Notation. Let BU denote $\bigcup_k BU(k)$.

Theorem 3.10. *For a based path-connected compact space X, we have a natural isomorphism*

$$[X, BU] \xrightarrow{\sim} \tilde{K}(X).$$

Proof. The reader can find the proof in [Swi02, Theorem 11.56]. □

Corollary 3.11. *For a path-connected compact space X, we have a natural isomorphism*

$$[X_+, \mathbb{Z} \times BU] \xrightarrow{\sim} K(X),$$

where $\mathbb{Z}$ is given the discrete topology.

One can now start to think of K as a cohomology theory. The following theorem will make explicit which Ω-spectrum we must considered to have the right definition of K^*.

Theorem 3.12 (Bott periodicity theorem)**.** *There is a homotopy equivalence*

$$\mathbb{Z} \times BU \simeq \Omega^2 BU.$$

This theorem allows us to construct a 2-periodic Ω-spectrum $\boldsymbol{K}$ defined as

$$K_n := \begin{cases} \mathbb{Z} \times BU & \text{if } n \text{ is even} \\ \Omega BU & \text{if } n \text{ is odd.} \end{cases}$$

This spectrum induces a cohomology theory K^* called (complex) *K-theory.*

Let us compute the coefficient groups $\tilde{K}(S^n)$ for $n \geq 0$. For $n = 0$ one can easily see that $\tilde{K}(S^0) \cong \mathbb{Z}$. For $n > 0$, $\tilde{K}(S^n) \cong [S^n, \mathbb{Z} \times BU] = \pi_n(BU)$ and by the Bott periodicity theorem we only need to compute $\pi_1(BU)$, since

$$\pi_{q+2}(BU) \cong \pi_q(\Omega^2 BU) \cong \pi_q(\mathbb{Z} \times BU) \cong \begin{cases} \mathbb{Z} & \text{if } q = 0 \\ \pi_q(BU) & \text{if } q > 0. \end{cases}$$

Moreover, $\pi_1(BU) \cong \pi_0(U) \cong \pi_0(U(1)) = \pi_0(S^1) = 0$. Hence

$$\tilde{K}(S^n) \cong \begin{cases} \mathbb{Z} & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$

If X is a compact space, one can define a bilinear map

$$V_{\mathbb{C}}(X) \times V_{\mathbb{C}}(X) \to V_{\mathbb{C}}(X) : (\{\xi\}, \{\zeta\}) \mapsto \{\xi \otimes \zeta\}$$

where $\xi \otimes \zeta$ denotes the fiber bundle whose fibers are the tensor product (over $\mathbb{C}$) of the fibers of ξ and ζ. By the universal property of the Grothendieck construction, this induces a unique homomorphism of abelian groups

$$K(X) \otimes K(X) \to K(X).$$

which is associative, commutative and natural in X (cf. [Swi02, Proposition 13.91]).

It is now reasonable to expect that one can give a multiplicative structure to the cohomology theory K^*. Indeed, we can define suitable maps $\mu_{nm} : K_n \wedge K_m \to K_{n+m}$ and $\eta_n : S^n \to K_n$ which will induce a ring structure on the spectrum $\boldsymbol{K}$ (cf. [Swi02, p. 300]). Moreover, when we restrict our attention to path connected compact spaces, the resulting product on the cohomology theory K^* will correspond to the product structure defined above.

Let us now show that K^* is complex oriented. It is enough to provide a class $x \in \tilde{K}^2(\mathbb{C}P^\infty)$ which satisfies the property of proposition 3.5. Define x to be $\{\gamma^1\} - 1 \in \tilde{K}(\mathbb{C}P^\infty) \cong \tilde{K}^0(\mathbb{C}P^\infty) = \tilde{K}^2(\mathbb{C}P^\infty)$. Such x is send on $\{\xi\} - 1 \in \tilde{K}(\mathbb{C}P^1)$ where ξ is the Hopf line bundle over $\mathbb{C}P^1 \simeq S^2$ and hence satisfies the property of proposition 3.5. In fact, the first Chern class is $c_1(L) = L - 1$ for any complex line bundle.

The interested reader can find in [May99a, Chapter24, Section 3] the whole theory of Chern classes and the Thom isomorphism in the particular case of K-theory.

3.3 Example: Complex cobordism

We define *complex cobordism* MU^* as the cohomology theory associated to the spectrum $\boldsymbol{MU}$. For pointed space X, the (reduced) cohomology group are given by

$$\widetilde{MU}^k(X) := \operatorname{colim}_n[\Sigma^n X, \boldsymbol{MU}_{n+k}].$$

By [Whi62] we know that this is a well defined cohomology theory. Remark that in chapter 2, we computed $MU^*(pt) = \pi_*(\boldsymbol{MU})$. This suggest that this cohomology theory is multiplicative since $\pi_*(\boldsymbol{MU})$ turns out to be a graded ring.

To define a multiplicative structure on MU^*, we will define a ring structure on $\boldsymbol{MU}$, i.e. we will give maps of spectra

$$\mu : \boldsymbol{MU} \wedge \boldsymbol{MU} \to \boldsymbol{MU}, \qquad \eta : S^0 \longrightarrow \boldsymbol{MU}$$

such that μ is associative up to homotopy and η is a unit for μ up to homotopy.

It is enough to define continuous maps $\mu_{nm} : MU(n) \wedge MU(m) \to MU(n+m)$ and $\eta_{2n} : S^{2n} \to MU(n)$ satisfying some coherence relations (cf. [Swi02, Proposition 13.80]). Define the map $BU(n) \times BU(m) \to BU(n+m)$ to be the map classifying the Whitney sum $\gamma^n \oplus \gamma^m$, where γ^k denotes the universal fiber bundle over $BU(k)$. Looking at the Thom spaces, we have an induced map

$$M(\gamma^n) \wedge M(\gamma^m) \cong M(\gamma^n \oplus \gamma^m) \longrightarrow M(\gamma^{n+m}).$$

This is our map $\mu_{nm} : MU(n) \wedge MU(m) \to MU(n+m)$. Moreover, we have the inclusion $* \to BU(n)$. Taking the Thom spaces, we have an induced map $S^{2n} = M(\varepsilon^{2n}) \to MU(n)$.

We show now that MU^* is complex oriented. Let ξ be a complex vector bundle over X of dimension n. Then there exists a map $X \to BU(n)$ classifying ξ, which induces a map $T\xi \to MU(n)$. By definition, this map gives us a class $U_\xi \in \widetilde{MU}^{2n}(T\xi)$. We show that U_ξ is our desired Thom class. Indeed, let $i : \{x\} \to X$ be the inclusion for any $x \in X$, then one has to check that the composition

$$\widetilde{MU}^{2n}(T\xi) \xrightarrow{i^*} \widetilde{MU}^{2n}(Ti^*\xi) \cong \widetilde{MU}^{2n}(S^{2n}) \xrightarrow{\Sigma^{2n}} \widetilde{MU}^{0}(S^0)$$

sends U_ξ to our preferred generator of $\widetilde{MU}^0(pt)$. First, remark that U_ξ is sent on $U_{i^*\xi}$ by the first map. This class is represented by a map

$$S^{2n*}\xi \longrightarrow T\xi \longrightarrow MU(n).$$

Since $BU(n)$ is path connected, the diagram

$$\begin{array}{ccc} x & \longrightarrow & * \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle j} \\ X & \longrightarrow & BU(n) \end{array}$$

commutes up to homotopy for any $x \in X$. Therefore, the previous map factors up to homotopy through $Tj^*\gamma^n$

$$\begin{array}{ccc} S^{2n} = Ti^*\xi & \xrightarrow{\cong} & Tj^*\gamma^n \cong \Sigma^{2n}S^0 \\ \downarrow & & \downarrow \\ T\xi & \longrightarrow & MU(n), \end{array}$$

and $\Sigma^{2n}S^0 \to MU(n)$ represent the class of our preferred generator in $\widetilde{MU}^0(S^0) = \pi_0(\boldsymbol{MU})$.

3.4 Formal group laws

We already seen that for a complex oriented cohomology theory E^*, power series arises naturally by looking at its value on $\mathbb{C}P^\infty$. If we now consider the map $t : \mathbb{C}P^\infty \times \mathbb{C}P^\infty \to \mathbb{C}P^\infty$ classifying the complex fiber bundle $\gamma^1 \otimes \gamma^1$ over $\mathbb{C}P^\infty \times \mathbb{C}P^\infty$, then we have an induced map in cohomology

$$E^*(pt)[\![x]\!] = E^*(\mathbb{C}P^\infty) \longrightarrow E^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty) = E^*(pt)[\![x_1, x_2]\!].$$

Since $m : \mathbb{C}P^\infty \times \mathbb{C}P^\infty \to \mathbb{C}P^\infty$ defines an H-space structure on $\mathbb{C}P^\infty$, we have that the power series $F(x, y)$ defined by $m^*(x) = F(x_1, x_2) \in E^*(pt)[\![x_1, x_2]\!]$ is a formal group law in the following sense:

Definition. Let R be a ring. A *formal group law* over R is a power series $F(x, y) \in R[\![x, y]\!]$ such that:

- $F(x, y) = F(y, x)$,
- $F(x, 0) = x = F(0, x)$,
- $F(F(x, y), z) = F(x, F(y, z))$.

We will come back on the topological aspects later. Now we will discuss formal group laws in a purely algebraic way and talk about the so called *universal* formal group law.

Let us write $F(x, y) = \sum_{i,j \geq 0} a_{ij} x^i y^j$, then the conditions above can be interpreted in terms of relations among the a_{ij}. For example, the first condition implies that $a_{ij} = a_{ji}$. The second say that $a_{10} = 1$ and $a_{i0} = 0$ whenever $i \neq 1$. The associativity can also be interpreted by some relation, but is very complicated to write down.

Definition. Let F and G be formal group laws over R. By a *map of formal group laws* $f : F \to G$, we understand a formal power series $f(x) \in R[\![x]\!]$ such that:

- $f(0) = 0$, and
- $f(F(x, y)) = G(f(x), f(y))$

Lemma 3.13. *Let $f : F \to G$ be a map of formal group laws over R. f is an isomorphism if and only if $f'(0)$ is a unit in R.*

If $f'(0) = 1$ we say that f is a *strict isomorphism.*

Examples. Define $\mathbb{G}_a(x, y) := x + y$ and $\mathbb{G}_m(x, y) := (1 + x)(1 + y) - 1 = x + y + xy$. One can see that these are obviously formal group laws. $\mathbb{G}_a$ is called the *additive formal group law* and $\mathbb{G}_m$ is called the *multiplicative formal group law.*

One can see that these two formal group laws are isomorphic, if our ring is a $\mathbb{Q}$-algebra. Indeed, the isomorphism $f : \mathbb{G}_a \to \mathbb{G}_m$ is given by the formal power series of the logarithm

$$f(x) = \log(1 + x) = \sum_{n>0} (-1)^n \frac{x^n}{n}.$$

Another interesting example is given by starting with an invertible power series $g(x)$. Form the formal group law

$$G(x, y) := g^{-1}(g(x) + g(y)).$$

Moreover, we have an isomorphism $G \to \mathbb{G}_a$ given by the power series $g(x)$.

Remark. Let $f : R \to S$ be a ring homomorphism and let $F(x, y) = \sum a_{ij} x^i y^j$ be a given formal group law over R. Then one can define the formal group law over S, denoted f^*F by

$$f^*F(x, y) := \sum_{i,j \geq 0} f(a_{ij}) x^i y^j.$$

This construction is called a *base change.*

We are now in our way to talk about the Lazard theorem.

Proposition 3.14. *There exists a universal formal group law in the following sense: there is a ring R and a group law F_u over R such that the map*

$$\begin{aligned} Ring(R, S) &\longrightarrow FGL/S \\ f &\longmapsto f^* F_u \end{aligned}$$

is an isomorphism.

Proof. Let us first consider the free polynomial rings $\mathbb{Z}[a_{ij}]$ among symbols a_{ij}. We define the ring R to be the quotient $\mathbb{Z}[a_{ij}]/\sim$ where the relations are given by the formal group laws conditions. Hence, we define $F_u := \sum a_{ij}x^i y^j$. By definition of R, it is obviously a formal group law and the reader can check straight forward that the above map is an isomorphism. □

Let us say that the a_{ij}'s have degree $2(i+j-1)$, then one can see that $R = \oplus R_{2n}$ becomes a graded ring (where the homogeneous part of degree $2n$ is denoted R_{2n}). Moreover, the following properties holds:

- $R_0 = \mathbb{Z}$ and $R_{2n} = 0$ for any $n < 0$ (to be fancy, one can say that R is *connected*).
- R_{2n} is finitely generated as an abelian group.

Notation.

$$C_n(x,y) := \frac{1}{d_n}((x+y)^n - x^n - y^n)$$

where d_n is the greatest common devisor of the binomial coefficient $\binom{n}{k}$ for $1 \leq k \leq n-1$, i.e. $d_n = p$ if $n = p^j$ and $d_n = 1$ otherwise.

Theorem 3.15 (Lazard)**.** *Let L be the graded ring $\mathbb{Z}[x_1, x_2, \dots]$ where each x_i is of degree $2i$. Then there is a formal group law F over L, such that*

$$F(x,y) \equiv \sum x_n C_{n+1}(x,y) \mod I^2 \tag{2}$$

where I is the ideal $(x_1, x_2, \dots)$, and the map $R \to L$ classifying F is an isomorphism of graded rings.

In order to prove this theorem, we need a technical lemma called the *symmetric 2-cocycle lemma.*

Let A be an abelian group. We consider the graded ring $\mathbb{Z} \oplus A$ where $ab = 0$ for any $a, b \in A$ and elements of A are all in degree $2n$. For convenience, we will write it $\mathbb{Z} \oplus A_{2n}$. A formal group law over this rings must look like $x+y+f(x,y)$, where $f(x,y)$ is a homogeneous polynomial in $A[x,y]$ of degree $n-1$. Moreover, $f(x,y)$ satisfy the following properties:

- $f(x,y) = f(y,x)$,
- $f(x,0) = 0$,
- $f(x,y) + f(x, y+z) = f(y,z) + f(x+y, z)$ (2-cocycle condition)

Such homogeneous polynomial $f(x,y)$ is called a *symmetric* 2*-cocycle* with values in A.

Lemma 3.16 (symmetric 2-cocycle)**.** *For any symmetric* 2*-cocycle $f(x,y)$ with values in A there exists an $a \in A$ such that $f(x,y) = aC_n(x,y)$.*

Proof. The reader can find the proof in [Rav86, Appendix 2, lemma A.2.1.29]. □

Remark. There exists a much nicer and conceptual proof of this lemma involving homological algebra. The idea is to consider the cochain complex

$$A \xrightarrow{d^0} A[x] \xrightarrow{d^1} A[x,y] \xrightarrow{d^2} A[x,y,z].$$

where $d^0(a) = a$, $d^1(f(x)) = f(x+y) - f(x) - f(y)$, $d^2(f(x,y)) = f(x,y) - f(x+y,z) + f(x,y+z) - f(y,z)$. Therefore, a polynomial satisfying the 2-cocycle condition is an element in $\ker d^2$, i.e. is a 2-dimensional cocycle.

Moreover, remark that I is the augmentation ideal of L, hence the equality (2) is a result involving the module of indecomposable $QL = I/I^2$. To be more specific, for any connected graded ring S, we define the *augmentation ideal* to be $I(S) := \{s \in S : \deg(s) > 0\}$. The module of *indecomposables* is defined as $QS := I(S)/I(S)^2$.

Proof of the Lazard theorem. Let us consider the formal group law over $\mathbb{Z} \oplus \mathbb{Z}_{2n}$ defined by $x + y + C_n(x,y)$. By universality, there is a ring homomorphism $R \to \mathbb{Z} \oplus \mathbb{Z}_{2n}$ classifying this formal group law.

Claim 1. *There is a canonical isomorphism $(QR)_{2n} \to \mathbb{Z}$ induced by the map above.*

Indeed, one can see that $Ring(R, \mathbb{Z} \oplus \mathbb{Z}_{2n}) \cong Ab((QR)_{2n}, \mathbb{Z})$. But we know by the symmetric 2-cocycle lemma that formal group laws over $\mathbb{Z} \oplus \mathbb{Z}_{2n}$ are in bijection with $\mathbb{Z} \cong Ab(\mathbb{Z}, \mathbb{Z})$. By Yoneda lemma, we have that $(QR)_{2n}$ and $\mathbb{Z}$ are canonically isomorphic.

If we know consider the Lazard ring $L = \mathbb{Z}[x_1, x_2, \dots]$, we can construct a map $L \to R$ using the previous isomorphism. More precisely, let $r_n \in R$ be an element whose image in QR_{2n} maps to 1 under the isomorphism in claim 1. The map $L \to R$ is defined on generators by sending x_n to r_n. Remark that with these choices, the map $L \to R$ is not canonical.

Claim 2. *The map $L \to R$ is surjective.*

This follows from the fact that a map $S \to T$ of connected, graded rings is surjective if and only if its induced maps on the module of indecomposable $QS \to QT$ is surjective. This result can be found in [MM65]. Although, the proof follows easily by induction on the degree of the map.

Claim 3. *The map $L \to R$ is injective.*

Let us consider the graded ring $U = \mathbb{Z}[m_1, m_2, \dots]$ where each m_n has degree $2n$, and let $g(x)$ be the power series $x + m_1x^2 + m_2x^3 + \cdots$. Thus, define a formal group law G over U, by

$$G(x,y) := g^{-1}(g(x) + g(y)).$$

This formal group law induces a map $R \to U$.

In order to prove that $L \to R$ is injective, we will prove that the composition $L \to R \to U$ is injective. Since L and U are polynomial rings, it is enough to show injectivity on the indecomposables, i.e. we will prove that the composition

$$(QL)_{2n} \xrightarrow{\cong} (QR)_{2n} \longrightarrow (QU)_{2n}$$

is injective. Recall that this map is induced by a base change, i.e. x_n is sent on the coefficient of the term $C_{n+1}(x,y)$ of the formal group law G, which is seen in the quotient $\mathbb{Z} \oplus (QU)_{2n}$ of U.

We compute this coefficient by looking first at $g(x)$ and $g^{-1}(x)$ in $\mathbb{Z} \oplus QU_{2n}$. The reader can easily see that $g(x) \equiv x + m_nx^{n+1}$ and $g^{-1}(x) \equiv x - m_nx^{n+1}$. Therefore, on this quotient G looks like

$$\begin{aligned} G(x,y) &= g^{-1}(g(x) + g(y)) \\ &\equiv g^{-1}(x + y + m_n(x^{n+1} + y^{n+1})) \\ &\equiv x + y + m_n(x^{n+1} + y^{n+1}) - m_n(x + y + \cdots)^{n+1} \\ &\equiv x + y - d_{n+1}m_nC_{n+1}(x,y). \end{aligned}$$

This implies that the map $(QL)_{2n} \cong (QR)_{2n} \to (QU)_{2n}$ sends x_n on $-d_{n+1}m_n$ which is obviously injective. $\square$

3.5 Quillen's theorem

Let us now study the formal group laws associated to our complex oriented cohomology theories.

The first easy example to understand is the formal group law associated to singular cohomology. The map $m : \mathbb{C}P^\infty \times \mathbb{C}P^\infty \to \mathbb{C}P^\infty$ induces in singular cohomology a map $m^* : H^*(\mathbb{C}P^\infty) \to H^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty)$. Identifying $H^*(\mathbb{C}P^\infty) \cong \mathbb{Z}[x]$ and $H^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty) \cong \mathbb{Z}[x_1, x_2]$, we have that $m^*(x) = x_1 + x_2$. Therefore, the formal group law associated to singular cohomology is the additive formal group law.

Let us now investigate the formal group law associated to K-theory. Recall that $m : \mathbb{C}P^\infty \times \mathbb{C}P^\infty \to \mathbb{C}P^\infty$ is the map classifying the tensor product of universal line bundles, i.e. $m^*\gamma^1 = \gamma^1_1 \otimes \gamma^1_2$. Therefore, passing to K-theory we have that $m^*(1+x) = (1+x_1)(1+x_2)$, thus $m^*(x) = x_1 + x_2 + x_1x_2$

The most interesting case is the formal group law associated to MU^*.

Theorem 3.17 (Quillen)**.** *The ring homomorphism*

$$\theta_{MU} : L \longrightarrow MU^*(pt) = \pi_*(\boldsymbol{MU})$$

classifying the formal group law associated to MU^* is an isomorphism, and the formal group law associated to MU^* is the universal formal group law.

Let $U = \mathbb{Z}[m_1, m_2, \dots]$ be the ring considered in the proof of the Lazard theorem.

Lemma 3.18. *We have a natural bijection of sets*

$$Ring(U, S) \xrightarrow{\sim} \{(\phi, F) |\, \phi : \mathbb{G}_a \to F \text{ is a strict isomorphism}, F \text{ is a FGL over } S\}.$$

This follows easily by the fact that a map of rings $f : U \to S$ is completely determined by its value on the m_i's. Thus $\phi(x) := \sum f(m_i)x^{i+1}$. Defining $F := \phi^{-1}(\phi(x) + \phi(y))$, we obviously have our bijection.

Let E^* and M^* be two complex oriented cohomology theories. Let x_E and x_M be our preferred classes in $\tilde{E}^2(\mathbb{C}P^\infty)$ and $\tilde{M}^2(\mathbb{C}P^\infty)$. Then the generalized cohomology theory $(E \wedge M)^*$ has two different orientations induced from x_E and x_M. Let us write the corresponding preferred classes $\hat{x}_E$ and $\hat{x}_M$, and their associated formal group laws $\hat{F}$ and $\hat{G}$ respectively.

Moreover, we know that $(E \wedge M)^*(\mathbb{C}P^\infty) \cong (E \wedge M)^*(pt)[\![\hat{x}_E]\!]$. In particular,

$$\hat{x}_M = \sum t_i \hat{x}_E^{i+1}$$

where the t_i's are in $(E \wedge M)^*(pt)$. Since $(E \wedge M)^*$ is complex oriented, t_0 has to be invertible. Let $g(x)$ denotes the above power series. Thus

$$\begin{aligned} m^* : (E \wedge M)^*(\mathbb{C}P^\infty) &\longrightarrow (E \wedge M)^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty) \\ \hat{x}_E &\longmapsto \hat{F}(\hat{x}_E, \hat{y}_E) \\ g(\hat{x}_E) &\longmapsto g(\hat{F}(\hat{x}_E, \hat{y}_E)) \\ \hat{x}_M &\longmapsto \hat{G}(\hat{x}_M, \hat{y}_M) \end{aligned}$$

which implies that $g(\hat{F}(x_E, y_E)) = \hat{G}(g(\hat{x}_E), g(\hat{y}_E))$.

Restricting our attention to $E^* = H^*$ and $M^* = MU^*$, we have that $(H \wedge MU)^*(pt) \cong H_*(\boldsymbol{MU}) \cong \mathbb{Z}[b_1, b_2, \dots]$, and the above relation becomes

$$\hat{x}_{MU} = \sum b_i \hat{x}_H^{i+1}. \tag{3}$$

The reader can find the proof in [Rav86, Chapter 4, Lemma 4.1.8].

Since the formal group law associated to H^* is $\mathbb{G}_a$, we deduce that

$$g^{-1}(g(x) + g(y)) = \hat{F}_{MU}(x, y) = h^* F_{MU}(x, y) \tag{4}$$

where $h : \pi_*(\boldsymbol{MU}) \to H_*(\boldsymbol{MU})$ is the Hurewicz homomorphism. Therefore, we have a strict isomorphism of formal group laws $\mathbb{G}_a \to h^* F_{MU}$ over $H_*(\boldsymbol{MU})$ given by the power series $g(x) = \sum b_i x^{i+1}$. Lemma 3.18 gives us a ring homomorphism $\Phi : U \to H_*(\boldsymbol{MU})$.

Proposition 3.19. *Let $\theta : L \to U$ be the map considered in the proof of the Lazard theorem. Then the following diagram*

$$\begin{array}{ccc} L & \xrightarrow{\theta} & U \\ {\scriptstyle \theta_{MU}}\downarrow & & \downarrow{\scriptstyle \Phi} \\ \pi_*(\boldsymbol{MU}) & \xrightarrow{h} & H_*(\boldsymbol{MU}) \end{array}$$

commutes.

Proof. We will show that

$$\begin{array}{ccc} Ring(L, S) & \xleftarrow{\theta^*} & Ring(U, S) \\ {\scriptstyle \theta^*_{MU}}\uparrow & & \uparrow{\scriptstyle \Phi^*} \\ Ring(\pi_*(\boldsymbol{MU}), S) & \xleftarrow{h^*} & Ring(H_*(\boldsymbol{MU}), S) \end{array}$$

commutes for all ring S. By the Yoneda lemma, the result will follow.

Claim 1. *We have a natural bijection of sets*

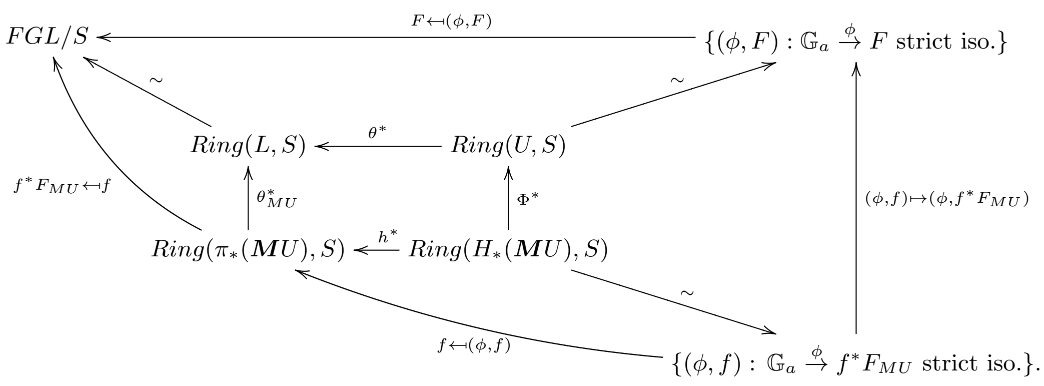
$$Ring(H_*(\boldsymbol{MU}), S) \xrightarrow{\sim} \left\{ (\phi, f) : \begin{array}{l} \phi : \mathbb{G}_a \to f^* F_{MU} \textit{ is a strict isomorphism,} \\ f : \pi_*(\boldsymbol{MU}) \to S \textit{ is a ring homomorphism} \end{array} \right\}.$$

Let $\lambda : H_*(\boldsymbol{MU}) \to S$ be a ring homomorphism, then define $f := \lambda \circ h$. By (4) we have a strict isomorphism given by $g(x)$ between $\mathbb{G}_a$ and $h^* F_{MU}$ over $H_*(\boldsymbol{MU})$. Hence, by a base change we have a strict isomorphism

$$\phi_\lambda : \mathbb{G}_a \longrightarrow (\lambda \circ h)^* F_{MU} = f^* F_{MU}.$$

Hence, we defined a map $\lambda \mapsto (\phi_\lambda, \lambda \circ h)$. Conversely, a map $\lambda : H_*(\boldsymbol{MU}) = \mathbb{Z}[b_1, b_2, \dots] \to S$ is completely determined by a power series $\phi(x) = \sum \lambda(b_i) x^{i+1}$. Thus the result follows by (3).

Since we now know which functors L, U and $H_*(\boldsymbol{MU})$ corepresents, we can consider the following diagram



The outer diagram is commutative which implies our result. ☐

Proof of Theorem 3.17. Let us consider again the diagram of proposition 3.19.

$$\begin{array}{ccc} L & \xrightarrow{\theta} & U \\ {\scriptstyle \theta_{MU}}\downarrow & & \downarrow{\scriptstyle \Phi} \\ \pi_*(\boldsymbol{MU}) & \xrightarrow{h} & H_*(\boldsymbol{MU}). \end{array}$$

Recall that in the proof of Lazard theorem, we showed that the map θ is injective. In addition, we showed in the first chapter that the Hurewicz homomorphism is also injective. Since Φ is an isomorphism, we can conclude that θ_{MU} is injective.

We now need to show that θ_{MU} is surjective. Since we are working with polynomial rings, it is enough to show this on indecomposable. We already know that the map θ on indecomposables sends x_n to $-d_{n+1}m_n$. Using the Adams spectral sequence, one can show that the Hurewicz homomorphism behaves the same way on indecomposables, i.e.

$$h : Q(\pi_*(\boldsymbol{MU}))_{2n} \longrightarrow Q(H_*(\boldsymbol{MU}))_{2n}$$
$$x_n \longmapsto -d_{n+1}b_n.$$

Thus θ_{MU} is surjective on indecomposables. □

Remark. The computation of the Hurewicz homomorphism is fully done in [Rav86, Theorem 3.1.5]. However, it is not such hard result since we already know that $h : \pi_*(\boldsymbol{MU}) \to H_*(\boldsymbol{MU})$ is injective. Therefore, on indecomposables h looks like $x_n \mapsto (cst)\, b_n$, so one only needs to show that the coefficient $(cst) = -d_{n+1}$. This can be done easily since the Hurewicz homomorphism is the edge homomorphism of the Adams spectral sequence of Proposition 2.24.

The multiplication by p is a consequence of the fact that the generator x_{2n} in the $E_2^{*,*}$-term $\mathrm{Ext}^*_{\mathcal{A}_p}(H^*(\boldsymbol{MU};\mathbb{Z}/p), \Sigma^*\mathbb{Z}/p)$ is of bidegree $(1, 2n+1)$ whenever $n = p^j - 1$, while the other generators x_{2n} are of bidegree $(0, 2n)$ for $n \neq p^j - 1$.

3.6 $MU_*(\boldsymbol{MU})$

In this section, we will see that $MU_*(\boldsymbol{MU})$ has a very rich structure. In fact, the reader has to think about it as an analogue to $H^*(\boldsymbol{H}\mathbb{Z}/p;\mathbb{Z}/p)$, where $\boldsymbol{H}\mathbb{Z}/p$ denotes the Eilenberg-MacLane spectrum associated to $\mathbb{Z}/p$.

Recall that the cohomology group $H^*(\boldsymbol{H}\mathbb{Z}/p;\mathbb{Z}/p)$ is am algebra over $\mathbb{Z}/p$ isomorphic to the mod p Steenrod algebra, hence it has a Hopf algebra structure (see for this the Milnor's celebrated paper [Mil58]). Moreover, $H^*(X;\mathbb{Z}/p)$ has a module structure over this algebra.

Thus, this gives us the motivation to study the homology group $MU_*(\boldsymbol{MU})$ as a $\pi_*(\boldsymbol{MU})$-module. Our purpose is first to understand the structure of $MU_*(\boldsymbol{MU})$ and then show that $MU_*(X)$ has a comodule structure over this coalgebra.

First recall that for any spectrum $\boldsymbol{E}$, we can associate a generalized homology theory, denoted E_*. The (reduced) homology groups for pointed CW-complexes are given by

$$\tilde{E}_k(X) := \pi_k(\boldsymbol{E} \wedge X).$$

The reader can find in [Whi62], that such functor is a well defined generalized homology theory.

Let us now consider the generalized reduced homology theory MU_* (for convenience we will write MU_* instead of $\widetilde{MU}_*$ and all our spaces are going to be pointed). Recall that the spectrum $\boldsymbol{MU}$ is a commutative ring spectrum, i.e there are maps of spectra $\mu : \boldsymbol{MU} \wedge \boldsymbol{MU}$ and $\eta : S^0 \to \boldsymbol{MU}$ such that μ is associative and commutative up to homotopy and η is a homotopy unit for μ.

Lemma 3.20. *$MU_*(X)$ is a left $\pi_*(\boldsymbol{MU})$-module for any pointed space X.*

Proof. The map $\mu \wedge 1 : \boldsymbol{MU} \wedge \boldsymbol{MU} \wedge X \to \boldsymbol{MU} \wedge X$ induces a natural group homomorphism $\pi_*(\boldsymbol{MU}) \otimes MU_*(X) \longrightarrow MU_*(X)$. This is our $\pi_*(\boldsymbol{MU})$-action on $MU_*(X)$. □

Remark. The reader might notice that $MU_*(\boldsymbol{MU})$ has also a right $\pi_*(\boldsymbol{MU})$-module structure induced by

$$\boldsymbol{MU} \wedge \boldsymbol{MU} \wedge \boldsymbol{MU} \stackrel{1\wedge\mu}{\longrightarrow} \boldsymbol{MU} \wedge \boldsymbol{MU}.$$

We will now discuss the full structure of $MU_*(\boldsymbol{MU}) = \pi_*(\boldsymbol{MU} \wedge \boldsymbol{MU})$. We refer the reader to [Ada69, Lecture 3] for the proofs of the following statements. $MU_*(\boldsymbol{MU})$ has the following structure maps:

1. A *product*
$$\varphi : MU_*(\boldsymbol{MU}) \otimes MU_*(\boldsymbol{MU}) \longrightarrow MU_*(\boldsymbol{MU})$$
induced by
$$\boldsymbol{MU} \wedge \boldsymbol{MU} \wedge \boldsymbol{MU} \wedge \boldsymbol{MU} \stackrel{1\wedge T\wedge 1}{\longrightarrow} \boldsymbol{MU} \wedge \boldsymbol{MU} \wedge \boldsymbol{MU} \wedge \boldsymbol{MU} \stackrel{\mu\wedge\mu}{\longrightarrow} \boldsymbol{MU} \wedge \boldsymbol{MU}$$
where T is the twist. The product map φ is an associative homomorphism of $\pi_*(\boldsymbol{MU})$-bimodules.

2. A *left unit* and *right unit*
$$\eta_L : \pi_*(\boldsymbol{MU}) \longrightarrow MU_*(\boldsymbol{MU}), \qquad \eta_R : \pi_*(\boldsymbol{MU}) \longrightarrow MU_*(\boldsymbol{MU})$$
induced respectively by $\eta \wedge 1 : S^0 \wedge \boldsymbol{MU} \to \boldsymbol{MU} \wedge \boldsymbol{MU}$ and $1 \wedge \eta : \boldsymbol{MU} \wedge S^0 \to \boldsymbol{MU} \wedge \boldsymbol{MU}$. The maps η_L, η_R are homomorphisms of $\pi_*(\boldsymbol{MU})$-bimodules, and η_L (resp. η_R) is a left (resp. right) unit with respect to the product φ.

3. A *counit*
$$\varepsilon : MU_*(\boldsymbol{MU}) \longrightarrow \pi_*(\boldsymbol{MU})$$
induced by $\mu : \boldsymbol{MU} \wedge \boldsymbol{MU} \to \boldsymbol{MU}$. The counit ε is a homomorphism of $\pi_*(\boldsymbol{MU})$-bimodules. Moreover, $\varepsilon\eta_L = 1$, $\varepsilon\eta_R = 1$, and ε is a homomorphism of graded algebra.

4. A *conjugation*
$$c : MU_*(\boldsymbol{MU}) \longrightarrow MU_*(\boldsymbol{MU})$$
induced by the twist $T : \boldsymbol{MU} \wedge \boldsymbol{MU} \longrightarrow \boldsymbol{MU} \wedge \boldsymbol{MU}$. The conjugation c is a homomorphism of $\pi_*(\boldsymbol{MU})$-bimodules and satisfies: $c\eta_L = \eta_R$, $c\eta_R = \eta_L$, $\varepsilon c = \varepsilon$ and $c^2 = 1$. Moreover, c is a homomorphism of graded algebra.

5. A *coproduct*
$$\psi : MU_*(\boldsymbol{MU}) \longrightarrow MU_*(\boldsymbol{MU}) \otimes_{\pi_*(\boldsymbol{MU})} MU_*(\boldsymbol{MU})$$
We will come back later on the definition of this map. The coproduct ψ is a coassociative homomorphism of $\pi_*(\boldsymbol{MU})$-bimodules, and ε is counital for ψ. Furthermore, ψ is a homomorphism of graded algebra. We also have $\psi\eta_L(\lambda) = \eta_L(\lambda) \otimes 1$, and $\psi\eta_R(\lambda) = 1 \otimes \eta_R(\lambda)$ for any $\lambda \in \pi_*(\boldsymbol{MU})$.

Remark. This kind of structure is called a *Hopf algebroid* (see [Rav86, Appendix 1]). Remark that in such structure if $\eta_L = \eta_R$, then it turns out to be a Hopf algebra.

We will now define a natural coproduct map

$$\psi_X : MU_*(X) \longrightarrow MU_*(\boldsymbol{MU}) \otimes_{\pi_*(\boldsymbol{MU})} MU_*(X).$$

The coproduct map ψ will therefore be defined as the special case where $X = \boldsymbol{MU}$

Lemma 3.21. *There is a natural isomorphism of left $\pi_*(\boldsymbol{MU})$-modules*

$$m : MU_*(\boldsymbol{MU}) \otimes_{\pi_*(\boldsymbol{MU})} MU_*(X) \stackrel{\cong}{\longrightarrow} MU_*(\boldsymbol{MU} \wedge X).$$

The reader can find the proof in [Ada69, Lecture 3].

Let $h : MU_*(X) \to MU_*(\boldsymbol{MU} \wedge X)$ be the natural map induced by

$$1 \wedge \eta \wedge 1 : \boldsymbol{MU} \wedge S^0 \wedge X \longrightarrow \boldsymbol{MU} \wedge \boldsymbol{MU} \wedge X.$$

Therefore, we define ψ_X to be the composition

$$\psi_X : MU_*(X) \xrightarrow{h} MU_*(\boldsymbol{MU} \wedge X) \xrightarrow{m^{-1}} MU_*(\boldsymbol{MU}) \otimes_{\pi_*(\boldsymbol{MU})} MU_*(X).$$

Remark that ψ_X is a map of left $\pi_*(\boldsymbol{MU})$-modules.

Lemma 3.22. *The map ψ_X defines on $MU_*(X)$ a left $MU_*(\boldsymbol{MU})$-comodule structure in the following sense:*

- *The map ψ_X is coassociative with respect to ψ, i.e. we have $(\psi \otimes 1)\psi_X = (1 \otimes \psi_X)\psi_X$.*
- *The map ε is counital with respect to ψ_X, i.e. we have $(\varepsilon \otimes 1)\psi_X = \mathrm{id}$.*

The reader can find the proof in [Ada69, Lecture 3].

Lemma 3.23. *If X is a finite pointed CW-complex, then $MU_*(X)$ is a finitely presented module over $\pi_*(\boldsymbol{MU})$.*

The reader can proved this easily by induction on the number of cells.

3.7 Landweber Exact Functor Theorem

We now know how to a complex oriented cohomology theory E^* associate a formal group law over $E^*(pt)$. The construction that we are about to explain in this section is to create generalized homology theories from formal group laws.

Lemma 3.24. *Let $\phi : F \to G$ be a homomorphism of formal group laws over R. If p is a prime number such that $p = 0$ in R and $\phi'(0) = 0$, then there exists a power series $\psi(x) \in R[\![x]\!]$ such that $\phi(x) = \psi(x^p)$.*

Proof. Since ϕ is a homomorphism of formal group laws, we have the following equality

$$\phi(F(x, y)) = G(\phi(x), \phi(y)).$$

Applying the partial differential $\frac{\partial}{\partial x}$, we get

$$\phi'(F(x, y))\frac{\partial F}{\partial x}(x, y) = \frac{\partial G}{\partial x}(\phi(x), \phi(y))\phi'(x).$$

Now set $x = 0$,

$$\phi'(y)\frac{\partial F}{\partial x}(0, y) = \phi'(\underbrace{F(0, y)}_{=y})\frac{\partial F}{\partial x}(0, y) = \frac{\partial G}{\partial x}(\phi(0), \phi(y))\underbrace{\phi'(0)}_{=0} = 0.$$

Moreover, remark that the power series $\frac{\partial F}{\partial x}(0, y) = 1 + a_{11}y + \cdots$ is invertible, because the leading term is a unit. Therefore, $\phi'(y) = 0$. □

Notation. If F is a formal group law, we shall write $x +_F y := F(x, y)$.

Definition. Let F be a formal group law over R. For $n \in \mathbb{N}$, the *n-series* of F is the power series

$$[n]_F(x) := \underbrace{x +_F \cdots +_F x}_{n \text{ times}}.$$

Moreover, it is easy to see that the n-series defines a homomorphism of formal group laws $[n]_F : F \to F$.

Let p be a prime number and F be a formal group law over R. Then one can see that

$$[p]_F(x) = px + \cdots \in R[\![x]\!].$$

Therefore, over $R/(p)$ we have $[p]'_F(0) = 0$. By the lemma, we have that

$$[p]_F(x) = \psi_1(x^p) = v_1 x^p + \cdots \in R/(p)[\![x]\!]$$

For a power series $\psi(x) \in R/(p)[\![x]\!]$. Remark that such $\psi(x)$ is also a homomorphism of formal group laws. Indeed, one needs to check that $\psi_1(x +_F y) = \psi_1(x) +_F \psi_1(y)$. Recall that in characteristic p, one has the following property $(x + y)^p = x^p + y^p$. This formula holds as well for a formal group law, i.e. we have $(x +_F y)^p = x^p +_F y^p$. Thus,

$$\psi_1(x^p +_F y^p) = \psi_1((x +_F y)^p) = [p]_F(x +_F y) = [p]_F(x) +_F [p]_F(y) = \psi_1(x^p) +_F \psi_1(y^p),$$

and this implies that $\psi_1(x +_F y) = \psi_1(x) +_F \psi_1(y)$ over $R/(p)$.

Now let us look at $\psi_1'(0) = v_1 + \cdots$. Therefore, $\psi_1'(0) = 0$ over $R/(p, v_1)$. Applying the lemma, we have that $\psi_1(x) = \psi_2(x^p) = v_2 x^p + \cdots$ for a certain power series $\psi_2(x) \in R/(p, v_1)[\![x]\!]$.

By induction one can repeat the process and obtain for any prime p and integer n, ideals

$$I(p, n) := (p, v_1, v_2, \ldots, v_{n-1}).$$

Let $\mathcal{MU}$ be the category of $MU_*(\boldsymbol{MU})$-comodules (in the sense of lemma 3.22) which are finitely presented as $\pi_*(\boldsymbol{MU})$-modules.

Theorem 3.25 (Landweber)**.** *Let F be a formal group law over R, then the functor*

$$M \longmapsto M \otimes_{\pi_*(\boldsymbol{MU})} R$$

on $\mathcal{MU}$ is exact if and only if the multiplication by p on R and by v_n on $R/I(p, n)$ is monic for each prime number p and $n > 0$.

We refer the reader to [Lan76] for a detailed proof of this statement.

Corollary 3.26. *In the above situation, the functor*

$$X \longmapsto MU_*(X) \otimes_{\pi_*(\boldsymbol{MU})} R$$

is a generalized homology theory on the category of finite CW-complexes.

Example. Let $R = \mathbb{Q}$, and consider the additive formal group law

$$\mathbb{G}_a(x, y) := x + y$$

over R. One can see straightforward that the condition of the Landweber exact functor theorem are satisfied since the p-series are $[p]_{\mathbb{G}_a}(x) = px$. Thus the functor

$$X \mapsto MU_*(X) \otimes_{\pi_*(\boldsymbol{MU})} \mathbb{Q}$$

is a well defined homology theory. In fact, this is singular homology with rational coefficients.

Example. Let $R = \mathbb{Z}[t^{\pm 1}]$ where t is of degree -2, and consider the multiplicative formal group law

$$\mathbb{G}_m(x, y) := x + y + txy = t^{-1}((tx + 1)(ty + 1) - 1)$$

over R. We want to show that the condition of the Landweber exact functor theorem are satisfied. We claim that the n-series of $\mathbb{G}_m$ are

$$[n]_{\mathbb{G}_m}(x) = t^{-1}\big((tx + 1)^n - 1\big).$$

Indeed, for $n = 2$ the statement holds, and by induction one can see that

$$\begin{aligned}[n]_{\mathbb{G}_m}(x) +_{\mathbb{G}_m} x &= t^{-1}\left(\left[t\Big(t^{-1}\big((tx+1)^n - 1\big)\Big) + 1\right](tx+1) - 1\right)\\ &= t^{-1}\big((tx+1)^n(tx+1) - 1\big)\\ &= t^{-1}\big((tx+1)^{n+1} - 1\big) = [n+1]_{\mathbb{G}_m}(x).\end{aligned}$$

So for any prime number p, the p-series have the following form

$$[p]_{\mathbb{G}_m}(x) = t^{-1}\big((tx+1)^p - 1\big) = t^{-1}\left(\sum_{k=1}^{p}\binom{p}{k}t^k x^k\right).$$

Thus over $R/(p)$ this power series has only one term, namely $t^{p-1}x^p$. Hence $v_1 = t^{p-1}$, and $v_n = 0$ for $n > 1$.

Since multiplication by p on $\mathbb{Z}[t^{\pm}]$ and multiplication by t^{p-1} on $\mathbb{Z}/p[t^{\pm}]$ are monic, we conclude that

$$X \mapsto MU_*(X) \otimes_{\pi_*(\boldsymbol{MU})} \mathbb{Z}[t^{\pm}]$$

is a homology theory. In fact, this homology theory is isomorphic to K-theory.

Example. Elliptic curves give us a great source of formal group laws. Using the Landweber exact functor theorem, we can create a big family of cohomology theories called *elliptic cohomology theories*. The reader can find in [Sil09, Chapter 4] the description of the maneuver to create formal group laws from elliptic curves.

4 Some Homotopy Groups of Spheres

So far, we solved two nice homotopical problems: computing the homotopy groups of $\boldsymbol{MO}$ and $\boldsymbol{MU}$. In this section, we will give some clues to the greatest homotopical problem. This is the computation of stable homotopy groups of spheres.

Our first computation of $\pi_{n+1}(S^n)$ and $\pi_{n+2}(S^n)$ will use the Serre exact sequence (cf Appendix B.1). For more details and further computations of $\pi_{n+k}(S^n)$ using this method, we refer the reader to [MT68, Chapter 12]. The second computation is based on the Adams spectral sequence.

4.1 First computation: using Serre's exact sequence

The main idea in this section is to approximate the spheres using the Eilenberg-MacLane spaces and the path fibrations. Let us also assume n to be sufficiently large (say $n = 593$). We start our first computation by setting $X_0 := K(\mathbb{Z}, n)$. Recall that its mod 2 cohomology groups were computed by Serre in [Ser53a] using its spectral sequence. Some ideas of this computations can be founded in appendix B.1.

Recall that by the Brown representability theorem there is a map $Sq^2 : X_0 \to K(\mathbb{Z}/2, n+2)$ corresponding to the class $Sq^2\iota_n \in H^{n+2}(X_0; \mathbb{Z}/2)$. Consider the path fibration ev_1 over $K(\mathbb{Z}/2, n+2)$ and take its pullback along Sq^2.

$$\begin{array}{ccccc} F_1 & \longrightarrow & X_1 & & \mathcal{P}K(\mathbb{Z}/2, n+2) \\ & & \downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle \mathrm{ev}_1} \\ & & X_0 & \xrightarrow{Sq^2} & K(\mathbb{Z}/2, n+2) \end{array}$$

This gives us a fibration p_1 over X_0 and its fiber F_1 is homeomorphic to the fiber of ev_1, i.e. $F_1 = \Omega K(\mathbb{Z}/2, n+2) = K(\mathbb{Z}/2, n+1)$. We will now compute the mod 2 cohomology of X_1

using the Serre exact sequence associated to the fibration X_1. This one looks like

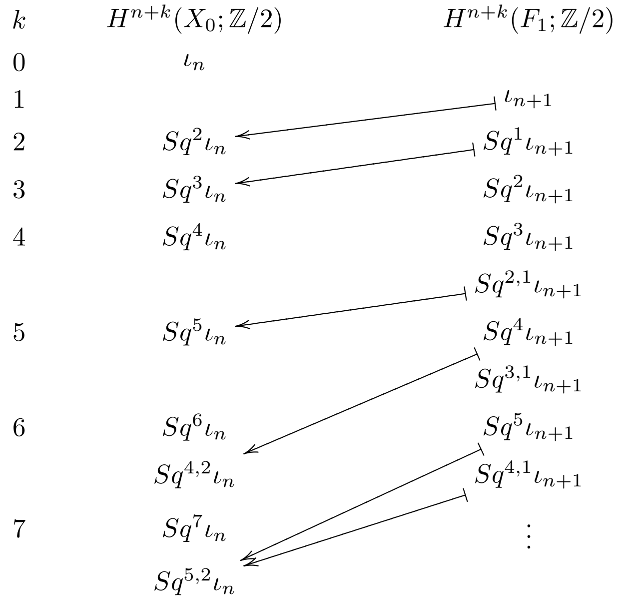
$$\cdots \xrightarrow{\tau} H^n(X_0;\mathbb{Z}/2) \xrightarrow{p_1^*} H^n(X_1;\mathbb{Z}/2) \xrightarrow{i^*} H^n(F_1;\mathbb{Z}/2) \xrightarrow{\tau}$$

$$H^{n+1}(X_0;\mathbb{Z}/2) \xrightarrow{p_1^*} H^{n+1}(X_1;\mathbb{Z}/2) \xrightarrow{i^*} H^{n+1}(F_1;\mathbb{Z}/2) \xrightarrow{\tau}$$

$$H^{n+2}(X_0;\mathbb{Z}/2) \xrightarrow{p_1^*} H^{n+2}(X_1;\mathbb{Z}/2) \xrightarrow{i^*} H^{n+2}(F_1;\mathbb{Z}/2) \xrightarrow{\tau}$$

$$H^{n+3}(X_0;\mathbb{Z}/2) \xrightarrow{p_1^*} \cdots$$

So as far as we can compute the transgression τ we will be able to know the cohomology groups of X_1. Here is a sample of these computations:



The computation of this transgression holds on the two following facts: $\tau(\iota_{n+1}) = Sq^2\iota_n$ and the transgression commutes with the Steenrod squares. Indeed, the equality $\tau(\iota_{n+1}) = Sq^2\iota_n$ follows by construction, and the other fact is explained in appendix B.1. Besides, this is a good exercise to practice the Adem relations.

Therefore, we can compute the mod 2 cohomology groups of X_1 using the exact sequence above.

k	0	1	2	3	4	5	6
$H^{n+k}(X_1;\mathbb{Z}/2)$	ι_n			α	$Sq^4\iota_n, \beta$	γ	$Sq^6\iota_n, \delta$

These classes are described by: $i_n := p_1^*(i_n)$, $i^*(\alpha) = Sq^2(\iota_{n+1})$, $Sq^4(\iota_n) := p_1^*(Sq^4\iota_n)$, $i^*(\beta) = Sq^3\iota_{n+1}$, ...

Now let $f_0 : S^n \to X_0$ be the map representing the generator of $\pi_n(X_0) = \pi_n(K(\mathbb{Z}, n))$. Composing with Sq^2, we get a null-homotopic map. Hence there exists a homotopy $H : S^n \times I \to K(\mathbb{Z}/2, n+2)$ such that $H(-, 0) = *$ and $H(-, 1) = Sq^2 \circ f_0$. This gives us a map $S^n \to \mathcal{P}K(\mathbb{Z}/2, n+2) : x \mapsto H(x, -)$. By the universal property of the pullback, we have a lift $f_1 : S^n \to X_1$ such that the following diagram commutes

$$\begin{array}{ccccc} & & X_1 & \longrightarrow & \mathcal{P}K(\mathbb{Z}/2, n+2) \\ & {\scriptstyle f_1} \nearrow & \downarrow {\scriptstyle p_1} & & \downarrow \\ S^n & \xrightarrow{f_0} & X_0 & \xrightarrow{Sq^2} & K(\mathbb{Z}/2, n+2). \end{array}$$

So far, we have constructed a space X_1 and a map $f_1 : S^n \to X_1$ such that the induced maps in mod 2 cohomology $f_1^* : H^i(X_1; \mathbb{Z}/2) \to H^i(S^n; \mathbb{Z}/2)$ are isomorphisms for $i < n+2$ and is a monomorphism for $i = n+2$. By the $\mathscr{C}_p$ approximation theorem (cf. theorem C.5) we have that the 2-components of $\pi_i(X_1)$ and $\pi_i(S^n)$ are equal for $i < n+2$. By the long exact sequence of a fibrations, the reader can easily deduce that $\pi_n(X_1) = \mathbb{Z}$, $\pi_{n+1}(X_1) = \mathbb{Z}/2$ and the other homotopy groups are trivial. Therefore we have that the 2-components of $\pi_{n+1}(S^n)$ are $\mathbb{Z}/2$.

Let us now go further and give a better approximation by "killing" the homology class $\alpha \in H^{n+3}(X_1; \mathbb{Z}/2)$. By Brown representability theorem, we have a map $\alpha : X_1 \to K(\mathbb{Z}/2, n+3)$. As above, consider the path fibration ev_1 over $K(\mathbb{Z}/2, n+3)$. If we now take the pullback along α, we get a fibration $p_2 : X_2 \to X_1$ with fiber $F_2 = K(\mathbb{Z}/2, n+2)$.

$$\begin{array}{ccccc} F_2 & \longrightarrow & X_2 & & \\ & & \downarrow {\scriptstyle p_2} & & \\ & & X_1 & \xrightarrow{\alpha} & K(\mathbb{Z}/2, n+3) \end{array}$$

We need now to compute the homology groups of X_2. These can be obtained by using the Serre exact sequence associated to p_2. So as far as we can compute the transgression, as far we will know $H^{n+k}(X_2; \mathbb{Z}/2)$. However, we will see that this is not that easy anymore. Here is the beginning of the computation of the transgression.

k	$H^{n+k}(X_1; \mathbb{Z}/2)$	$H^{n+k}(F_2; \mathbb{Z}/2)$
0	ι_n	
1		
2		ι_{n+2}
3	α	$Sq^1 \iota_{n+2}$
4	$Sq^4 \iota_n$	$Sq^2 \iota_{n+2}$
	β	
5	γ	$Sq^3 \iota_{n+2}$
		$Sq^2 Sq^1 \iota_{n+2}$
6	$Sq^6 \iota_n$	$\cdots$
	δ	

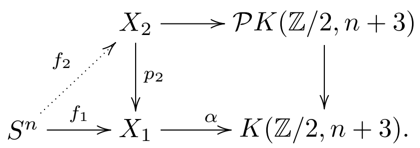
Let us give some explanations for these computations. By construction, we have that $\tau(\iota_{n+2}) = \alpha$. Moreover, by naturality of the Steenrod squares we have

$$i^*(Sq^1 \alpha) = Sq^1 i^*(\alpha) = Sq^1 Sq^2(\iota_{n+1}) = Sq^3(\iota_{n+1}).$$

But we have also by definition that $i^*(\beta) = Sq^3(\iota_{n+1})$. Since $i^*(Sq^4(\iota_{n+1})) = 0$ we have that $\beta = Sq^1(\alpha) + (cst)Sq^4(\iota_n)$. One can see that when we defined β, we had a choice to make. One could claim that $\beta = Sq^1(\alpha)$ and so that is how we justify the above relation.

For $\tau(Sq^2(\iota_{n+2}))$ and $\tau(Sq^2 Sq^1 \iota_{n+2})$ things become more complicated, and we refer the reader to [MT68, Chapter 12]. However, even without these results we shown that X_2 is a better approximation to S^n that X_1. This will allow us to compute the 2-components of $\pi_{n+2}(S^n)$.

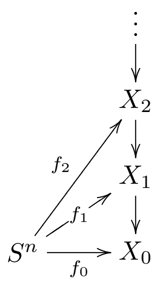
Let $f_1 : S^n \to X_1$ be the map defined above. Composing with α we get a null-homotopic map, hence there is a lift $f_2 : S^n \to X_2$ such that the following diagram commutes



One can see that f_2 induces a map $f^* : H^i(X_2; \mathbb{Z}/2) \to H^i(S^n; \mathbb{Z}/2)$ which is an isomorphism for $i < n+3$ and a monomorphism for $i = n+3$. Therefore, by the $\mathscr{C}_p$ approximation theorem (cf. C.5) we have that the 2-components of $\pi_i(X_2)$ and $\pi_i(S^n)$ are equal for $i < n+3$. This shows us that the 2-components of $\pi_{n+2}(S^n)$ is equal to $\mathbb{Z}/2$.

For $\pi_{n+3}(S^n)$ things will not work that easily, because we cannot kill $Sq^4(\iota_n)$ using an Eilenberg-MacLane space $K(\mathbb{Z}/2, n+4)$. The obtained space X_3 will not have trivial $(n+4)$-th cohomology group.

Remark. Let us summarize what we have done above. To compute the homotopy groups $\pi_{n+k}(S^n)$, we used successive approximation of S^n. This was building a *tower* of space



such that $f_\# : \pi_{n+i}(X_k) \xrightarrow{\cong} \pi_{n+i}(S^n)$ mod $\mathscr{C}_p$, for $i \leq k$. This idea has a general framework where such towers are called *Postnikov systems*. For more details, we refer the reader to [MT68, Chapter 12].

4.2 Second computation: using Adams spectral sequence

Let us now computes the stable homotopy groups $\pi_{n+k}(S^n)$ (for small k) using the Adams spectral sequence. Recall that the E_2-terms are $\mathrm{Ext}^s_{\mathcal{A}_p}(\tilde{H}^*(Y;\mathbb{Z}/p), \Sigma^t \tilde{H}^*(X;\mathbb{Z}/p))$. Moreover the spectral sequence converge to ${}_{(p)}\{X, Y\}_{t-s}$ (cf. Appendix B.2). If we let $Y = X = S^0$, it is now a matter to understand $\mathrm{Ext}^s_{\mathcal{A}_p}(\mathbb{Z}/p, \Sigma^t \mathbb{Z}/p)$. In fact, the Adams spectral sequence reduce the computation of the homotopy groups of sphere into three steps:

1. Compute the E_2-terms.

2. Compute the differentials.

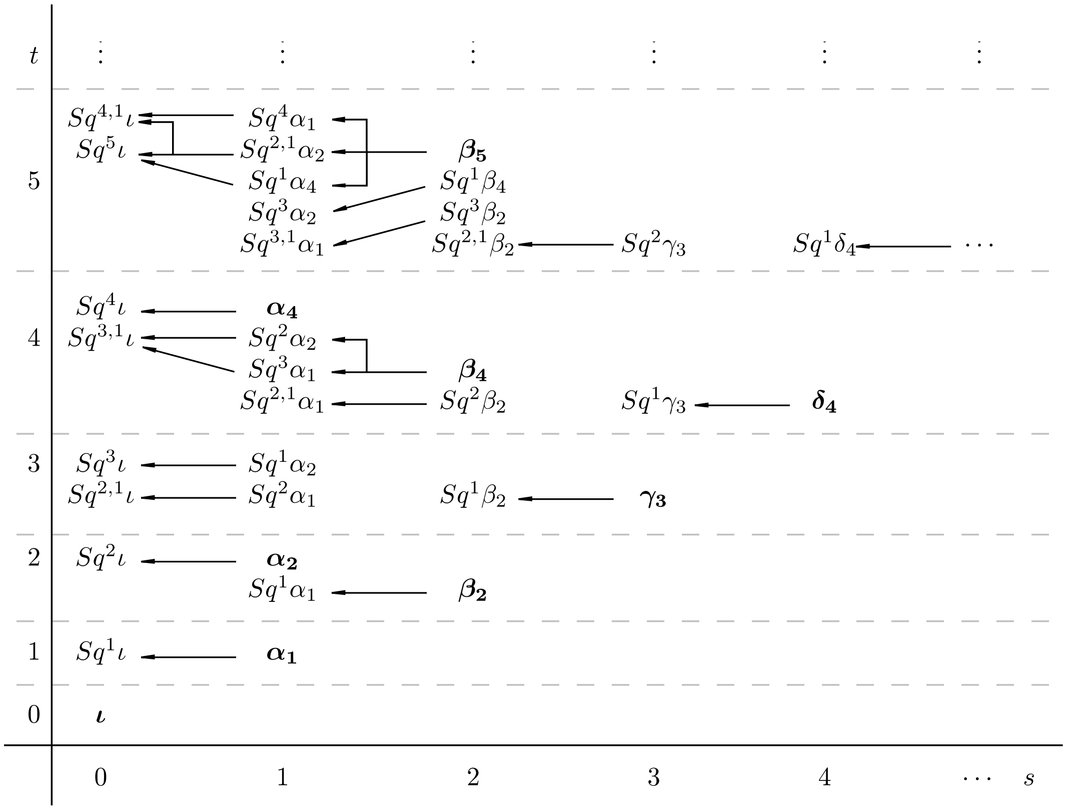
3. Solve the group extension problem.

As in the previous section, we will compute the 2-components of $\pi_{n+k}(S^n)$ (for n sufficiently large), i.e. we are interested in $\mathrm{Ext}^s_{\mathcal{A}_2}(\mathbb{Z}/2, \Sigma^t \mathbb{Z}/2)$. Therefore, we have to compute an $\mathcal{A}_2$-free resolution of $\mathbb{Z}/2$. This is an exact sequence

$$0 \longleftarrow \mathbb{Z}/2 \longleftarrow F_0 \xleftarrow{d_1} F_1 \xleftarrow{d_2} F_2 \xleftarrow{d_3} F_3 \longleftarrow \cdots$$

where each F_s are free $\mathcal{A}_2$-modules.

The modules F_0 and F_1 can be easily described. $F_0 \cong \mathcal{A}_2$ is the Steenrod algebra on one generator ι, while F_1 is the free $\mathcal{A}_2$-module generated by $\{\alpha_{2^i}\}_{i\geq 0}$, where α_{2^i} is of degree 2^i. The differential d_1 sends α_{2^i} on $Sq^{2^i}\iota$.



The reader can find more of this computation in [Hat04, Chapter 2].

Remark that in our resolution, all differentials satisfies $\ker(d_s) \subset I(\mathcal{A}_2)F_{s-1}$, where $I(\mathcal{A}_2)$ is the augmentation ideal. Such resolution are said *minimal.*

Lemma 4.1. *If the $\mathcal{A}_2$-free resolution*

$$0 \longleftarrow \mathbb{Z}/2 \longleftarrow F_0 \xleftarrow{d_1} F_1 \xleftarrow{d_2} F_2 \xleftarrow{d_3} F_3 \longleftarrow \cdots$$

is minimal, then the differentials

$$d_s^* : \mathrm{Hom}_{\mathcal{A}_2}(F_{s-1}, \Sigma^t\mathbb{Z}/2) \longrightarrow \mathrm{Hom}_{\mathcal{A}_2}(F_s, \Sigma^t\mathbb{Z}/2)$$

are all zero. Thus

$$\mathrm{Ext}^s_{\mathcal{A}_2}(\mathbb{Z}/2, \Sigma^t\mathbb{Z}/2) = \mathrm{Hom}_{\mathcal{A}_2}(F_s, \Sigma^t\mathbb{Z}/2).$$

The reader can find the proof of this statement in [MT68, Chapter 18, Proposition 7].

Therefore, with the above resolution we described $E_2^{s,t} = \mathrm{Ext}^s_{\mathcal{A}_2}(\mathbb{Z}/2, \Sigma^t\mathbb{Z}/2)$ for $t, s \leq 5$. Moreover, this spectral sequence has a multiplicative structure for which the differential is multiplicative.

Theorem 4.2. **$s = 1$:** *There is a $\mathbb{Z}/2$-basis of elements $h_i \in E_2^{1,2^i}$, for $i \geq 0$.*

$s = 2$: *$E_2^{2,t}$ is generated by the products h_ih_j modulo the relation $h_ih_{i+1} = 0$, for $i \geq 0$.*

$s = 3$: *In $E_2^{3,t}$, the products $h_ih_jh_k$ are subject to the relations $h_ih_{i+2}^2 = 0$, $h_i^3 = h_{i-2}^2h_{i+1}$ and the relations induced by $h_ih_{i+1} = 0$.*

The elements $h_0, h_0^2, h_0^3, \ldots$ are dual of the elements $\alpha_1, \beta_2, \gamma_3, \ldots$ While the elements $h_1, h_2, h_3, \ldots$ are dual of $\alpha_2, \alpha_4, \alpha_8, \ldots$

Remark. We emphasis that $E_2^{3,t}$ is not generated by products $h_i h_j h_k$ only. There is also other generators (the first appears for $t = 11$ and is commonly called c_0). The reader can find the statement of this theorem in [MT68, Chapter 18].

Using the above theorem, let us draw the multiplicative E_2-terms.

s	$\vdots$						
4	h_0^4						
3	h_0^3			$h_1^3 = h_0^2 h_2$			
2	h_0^2		h_1^2	$h_0 h_2$			
1	h_0	h_1		h_2			
0	1						
	0	1	2	3	4	5	$\cdots$ $t-s$

In [MT68, Chapter 18], the reader can find a complete set of generators of this spectral sequence for $t - s \leq 17$.

For dimensional reasons, the only differential that can be non trivial is $d_r(h_1)$. Since $h_0 h_1 = 0$ we can deduce that $d_r(h_1) = 0$. Therefore, the above E_2-terms are permanent cycles. Moreover, we can identify h_0 via the map $2\iota : S^n \to S^n$. This shows that the 2-components of $\pi_{n+k}(S^n)$ are the following:

k	0	1	2	3	4	5
$\pi_{n+k}(S^n)$	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/8$	0	0

A Steenrod Algebra

A.1 Steenrod Squares and Powers

We give here a compilation of the fundamental properties of the squaring operations.

Theorem A.1. *For any $i \geq 0$, there exists natural homomorphisms*

$$Sq^i : H^*(X, A; \mathbb{Z}/2) \longrightarrow H^{*+i}(X, A; \mathbb{Z}/2)$$

with the following properties:

- *Whenever $i > *$, $Sq^i = 0$.*
- *$Sq^i(x) = x^2$ for all $x \in H^i(X, A; \mathbb{Z}/2)$.*
- *$Sq^0 = \mathrm{id}$, and Sq^1 is the Bockstein homomorphism.*
- *$\delta Sq^i = Sq^i \delta$, where δ is the connecting homomorphism in the long exact sequence of a pair.*
- *$\Sigma Sq^i = Sq^i \Sigma$, where Σ is the suspension isomorphism, given by the reduced cross product $x \mapsto x \times \iota$, with ι a generator of $\tilde{H}^1(S^1; \mathbb{Z}/2)$.*
- *$Sq^i(xy) = \sum_j Sq^j(x) Sq^{i-j}(y)$, this formula is called the* Cartan formula.
- *for $a < 2b$, $Sq^a Sq^b = \sum_{c=0}^{[a/2]} \binom{b-c-1}{a-2c} Sq^{a+b-c} Sq^c$, this formula is called the* Adem relation.

The reader can find the proof of the existence of the Steenrod squares in [MT68].

Let us now give some useful results for the squares in $\mathbb{R}P^\infty \times \cdots \times \mathbb{R}P^\infty$. Recall that the cohomology ring $H^*(\mathbb{R}P^\infty;\mathbb{Z}/2)$ can be identified as the polynomial ring $\mathbb{Z}/2[t]$. By the Künneth formula, the n-fold product $\mathbb{R}P^\infty \times \cdots \times \mathbb{R}P^\infty$ has for cohomology with mod 2 coefficient, the polynomial ring $\mathbb{Z}/2[t_1,\ldots,t_n]$, where each t_i is of degree 1. This ring contains a subring $\mathbb{Z}/2[s_1,\ldots,s_n]$, where s_k is the k-th symmetric polynomial in variables $t_1,\ldots,t_n$.

Lemma A.2. *For any $x \in H^1(X,A;\mathbb{Z}/2)$, we have $Sq^i(x^j) = \binom{j}{i}x^{i+j}$.*

Proof. Consider the *total squares* $Sq = \sum_i Sq^i$. For x we have $Sq(x) = x + x^2$, and by the Cartan formula, one can check that Sq is a ring homomorphism. Therefore

$$Sq(x^j) = (x+x^2)^j = x^j(1+x)^j = x^j\sum_k \binom{j}{k}x^k = \sum_k \binom{j}{k}x^{j+k}. \qquad \square$$

Until now, we worked with mod 2 coefficients. We give now a compilation of the fundamental properties of the Steenrod powers, which is an extension of the squaring operations for any prime coefficients $p > 2$.

Theorem A.3. *For any $i \geq 0$, there exists natural homomorphisms*

$$P^i : H^*(X,A;\mathbb{Z}/p) \longrightarrow H^{*+2i(p-1)}(X,A;\mathbb{Z}/p)$$

with the following properties:

- *Whenever $2i > *$, $P^i = 0$.*
- *$P^i(x) = x^p$ for all $x \in H^{2i}(X,A;\mathbb{Z}/2)$.*
- *$P^0 = \mathrm{id}$.*
- *$\delta P^i = P^i\delta$, where δ is the connecting homomorphism in the long exact sequence of a pair.*
- *$\Sigma P^i = P^i\Sigma$, where Σ is the suspension isomorphism, given by the reduced cross product $x \mapsto x \times \iota$, with ι a generator of $\tilde{H}^1(S^1;\mathbb{Z}/p)$.*
- *$P^i(xy) = \sum_j P^j(x)P^{i-j}(y)$, this formula is called the* Cartan formula.

As for the Steenrod squares, we have Adem relations for powers.

Theorem A.4 (Adem relations)**.** *Let $\beta : H^*(X,A;\mathbb{Z}/p) \to H^{*+1}(X,A;\mathbb{Z}/p)$ be the Bockstein homomorphism (mod p). If $a < pb$, then the following equality holds:*

$$P^aP^b = \sum_j (-1)^{a+j}\binom{(p-1)(b-j)-1}{a-pj}P^{a+b-j}P^j,$$

and if $a \leq pb$, then the following equality holds:

$$P^a\beta P^b = \sum_j (-1)^{a+j}\binom{(p-1)(b-j)}{a-pj}\beta P^{a+b-j}P^j - \sum_j (-1)^{a+j}\binom{(p-1)(b-j)-1}{a-pj-1}P^{a+b-j}\beta P^j.$$

A.2 Steenrod Algebra

We will give here a short summary about the Steenrod algebra and some of its properties.

Definition. The *Steenrod Algebra* in mod 2 coefficients, denoted $\mathcal{A}_2$, is defined as the $\mathbb{Z}/2$-graded module generated by the symbols Sq^i, for each $i \geq 0$, modulo the Adem relations and the relation that $1 \sim Sq^0$. Naturally, it has a canonical multiplication which allows us to think about it as an algebra.

When we have iterated squares $Sq^{i_1}Sq^{i_2}\cdots Sq^{i_r}$, it is convenient to write it simply as Sq^I, where $I = (i_1 \cdots i_r)$. We say that I is an *admissible sequence* if $i_1 \geq 2i_2 \geq \cdots \geq 2i_r$, where each i_k is a non-negative integer.

Theorem A.5. *A basis of $\mathcal{A}_2$ as a graded $\mathbb{Z}/2$-module is given by the $\{Sq^I\}$, where I runs over all admissible sequences. This basis is called the* Serre-Cartan basis.

As for the Steenrod Algebra $\mathcal{A}_2$, we define $\mathcal{A}_p$.

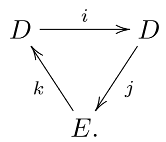
Definition. The *Steenrod Algebra* in mod p coefficients, denoted $\mathcal{A}_p$, is defined as the $\mathbb{Z}/p$-graded module generated by the symbols β and P^i, for each $i \geq 0$, modulo the Adem relations and the relation that $1 \sim P^0$. Naturally, it has a canonical multiplication which allows us to think about it as an algebra.

There would be much more to say about the Steenrod Algebra. For this, we refer the reader to the beautiful paper of Milnor [Mil58].

B Spectral Sequences

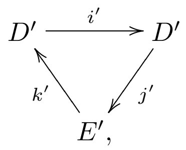
Let us introduce first the general situation of how arise a spectral sequence. We will then see two particular examples which are the Serre spectral sequence and the Adams spectral sequence.

Definition. An *exact couple* is a pair of modules D, E (which will be generally bigraded), together with homomorphisms i, j, k such that the following diagram is exact



Notice that $d := jk : E \to E$ is a differential on E, i.e. $dd = 0$.

Beginning with an exact couple, we can define its *derived couple*



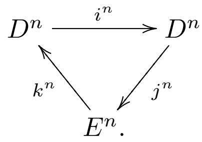
by setting $D' = i(D)$, $E' = \ker d/\operatorname{im} d = H(E, d)$, $i' = i|_{D'}$, $j' = ji^{-1}$ and $k'[y] := k(y)$ for any $y \in E$.

Lemma B.1. *A derived couple is a well defined exact couple.*

The proof of this lemma follows from an easy diagram chase.

Definition. A *spectral sequence* associated to an exact couple is the sequence of its iterated derived couples.

The n-th derived couple will be denoted



We refer the reader to [MT68, Chapter 7] for more details about general theory of spectral sequencess. Since the most interesting part of the sequence is the E part, we will abbreviate this sequence by $\{E^r, d^r\}$. We will see later that the E^2-term plays generally an important role. Moreover, we will see also that the spectral sequences we will consider are going to "converge" in a suitable way.

B.1 Serre Spectral Sequence

The spectral sequence we are now going to talk about was a real breakthrough in algebraic topology. We will talk about its consequences after we stated the theorem. All of this can be found in Serre's original paper [Ser51].

Theorem B.2 (Serre Spectral Sequence in Homology)**.** *Let $p : E \to B$ be a Serre fibration with fiber F. Suppose B and F are path connected and B simply connected. Then the following holds:*

- *there is a spectral sequence $\{E^r, d^r\}$, where the E^r are bigraded and d^r is of bidegree $(-r, r-1)$.*
- $E^2_{p,q} = H_p(B; H_q(F))$.
- *The spectral sequence converges to $H_*(E)$, i.e. for each $n \geq 0$ there exists a sequence of modules such that*

  $$H_n(E) = F_{n,0} \supset F_{n-1,1} \supset \cdots \supset F_{0,n}$$

  with $E^\infty_{p,q} := F_{p,q}/F_{p-1,q+1}$. By saying that the spectral sequence converge, *we means that the $E^r_{p,q}$ gets stabilized for a certain (large) r, and in the stable range $E^r_{p,q} \cong E^\infty_{p,q}$.*
- *$i_* : H_n(F) \to H_n(E)$ corresponds to the map*

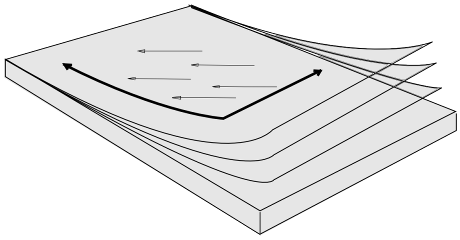
  $$H_n(F) = E^2_{0,n} \twoheadrightarrow E^r_{0,n} \cong F_{0,n} \subset H_n(E),$$

  with r sufficiently large.
- *$p_* : H_n(E) \to H_n(B)$ corresponds to the map*

  $$H_n(E) = F_{n,0} \twoheadrightarrow E^\infty_{n,0} \cong E^r_{n,0} \subset E^2_{n,0} = H_n(B),$$

  with r sufficiently large.

This 3-graded object has to be thought as sheets of a bloc note with a p-axis and a q-axis. Each sheet representing $E^r_{*,*}$.



Example. This spectral sequence, is a very efficient way to compute the homology of the loop space of a space B by using the path fibration

$$\begin{array}{ccc} \Omega B & \longrightarrow & \mathcal{P}B \simeq * \\ \downarrow & & \downarrow{\scriptstyle \mathrm{ev}_1} \\ * & \longrightarrow & B. \end{array}$$

One can show easily for example that $H^*(\Omega S^n) = \mathbb{Z}$ when $* = k(n-1)$ and 0 otherwise, see [MT68, Chapter 8].

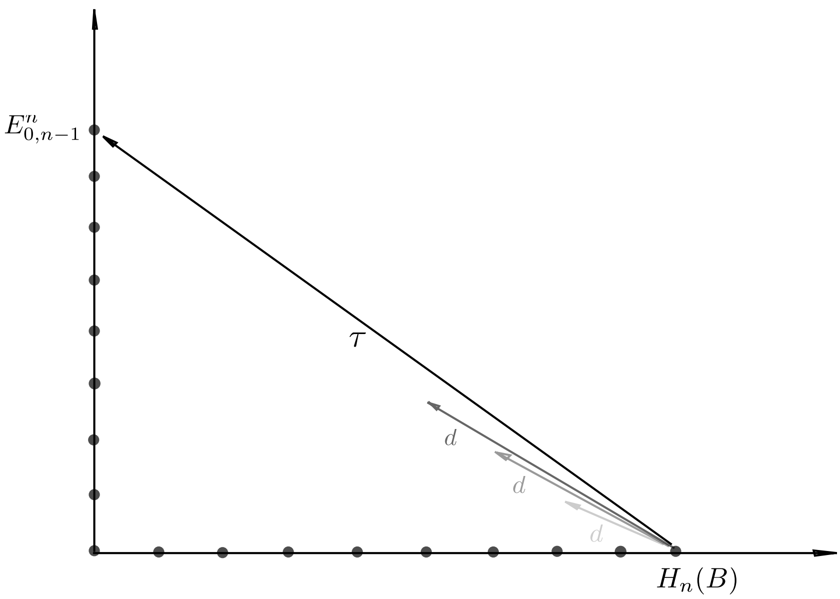
The Serre spectral sequence is very powerful and allows us to prove great theorems like the Hurewicz theorem and the Whitehead's theorem, see [MT68, Chapter 8]. Another consequence of this is the exact sequence in homology for a fibration.

Theorem B.3. *Let $p : E \to B$ be a Serre fibration with fiber F, with B simply connected. Suppose $H_i(B) = 0$ for $0 < i < p$, and $H_j(F) = 0$ for $0 < j < q$. Then we have a finite exact sequence*

$$H_{p+q-1}(F) \xrightarrow{i_*} H_{p+q-1}(E) \xrightarrow{p_*} H_{p+q-1}(B) \xrightarrow{\tau}$$
$$\to H_{p+q-2}(F) \xrightarrow{i_*} H_{p+q-2}(E) \xrightarrow{p_*} H_{p+q-2}(B) \xrightarrow{\tau}$$
$$\to \cdots\cdots \xrightarrow{i_*} H_1(E) \xrightarrow{p_*} 0.$$

The map $\tau : H_n(B) \to H_{n-1}(F)$ is called the transgression *and is the map corresponding to $d^n_{n,0} : H_n(B) = E^2_{n,0} \subset E^n_{n,0} \to E^n_{0,n-1} = E^2_{0,n-1} = H_{n-1}(F)$. This transgression is well defined when $n < p+q$ because all the previous differential starting from $H_n(B)$ are all zero. Remark also that normally we don't have $E^n_{0,n-1} = E^2_{0,n-1}$, but this follows also by the fact that all arriving differentials (until d^n) in $E^2_{0,n-1}$ are zero.*

In a more general way, we say that $x \in H_n(B)$ is *transgressive* if $\tau(x)$ is well defined. τ can be seen as the maximum differential sending $x \in H_n(B)$ to $\tau(x) \in E^n_{0,n-1}$. This makes sense only if the previous $d(x)$ are all zero. If we draw the E^* terms (all superposed on the E^2 term), then the transgression can be seen somehow in the following cartoon



We will see that transgressive elements will play a central role in our favorites computations. Let us now introduce the Serre spectral sequence in cohomology. This one is even better than its homological analogue, since it respects the multiplicative structure.

Theorem B.4 (Serre Spectral Sequence in Cohomology)**.** *Let $p : E \to B$ be a Serre fibration with fiber F. Suppose B and F are path connected and B simply connected, and let R be a ring. Then the following holds:*

- *There is a spectral sequence $\{E_r, d_r\}$, where the E_r are bigraded ring and d_r is an anti-derivation of bidegree $(r, 1-r)$, i.e. we have*

$$d_r(ab) = d_r(a) \cdot b + (-1)^{|a|} a \cdot d_r(b).$$

- *The multiplicative structure of E_{r+1} is induced by the multiplicative structure of E_r.*
- $E_2^{p,q} = H^p(B; H^q(F; R))$.
- *The spectral sequence converges to $H^*(E; R)$, i.e. for each $n \geq 0$ there exists a sequence of modules such that*

$$H^n(E; R) = F^{0,n} \supset F^{1,n-1} \supset \cdots \supset F^{n,0}$$

  with $E_\infty^{p,q} := F^{p,q}/F^{p+1,q-1}$.
- *The multiplicative structure of E_∞ is induced by the cup product in $H^*(E; R)$.*
- *$i^* : H^n(E; R) \to H^n(F; R)$ corresponds to the map*

$$H^n(E; R) = F^{0,n} \twoheadrightarrow E_r^{0,n} \subset E_2^{0,n} = H_n(F; R),$$

  with r sufficiently large.
- *$p^* : H^n(B; R) \to H^n(E; R)$ corresponds to the map*

$$H^n(B; R) = E_2^{n,0} \twoheadrightarrow E_r^{n,0} \cong E_\infty^{n,0} = F^{n,0} \subset H^n(E; R),$$

  with r sufficiently large.

As for homology, we have a finite exact sequence in cohomology associated to a fibration. More precisely, if $p : E \to B$ be a Serre fibration with fiber F, with B simply connected such that $H^i(B; R) = 0$ for $0 < i < p$, and $H^j(F; R) = 0$ for $0 < j < q$, then the following sequence is exact:

$$\cdots \xrightarrow{\tau} H^{p+q-2}(B; R) \xrightarrow{p^*} H^{p+q-2}(E; R) \xrightarrow{i^*} H^{p+q-2}(F; R) \xrightarrow{\tau} H^{p+q-1}(B; R) \xrightarrow{p^*} H^{p+q-1}(E; R) \xrightarrow{i^*} H^{p+q-1}(F; R).$$

Here also, τ is the transgression, i.e. the maximal differential $d_n^{0,n-1}$ going from $H^{n-1}(F; R)$ to $H^n(B; R)$. More generally, we say that $x \in H^{n-1}(F; R)$ is *transgressive* if $\tau(x)$ is defined, i.e. $d_r(x) = 0$ for all $r < n$.

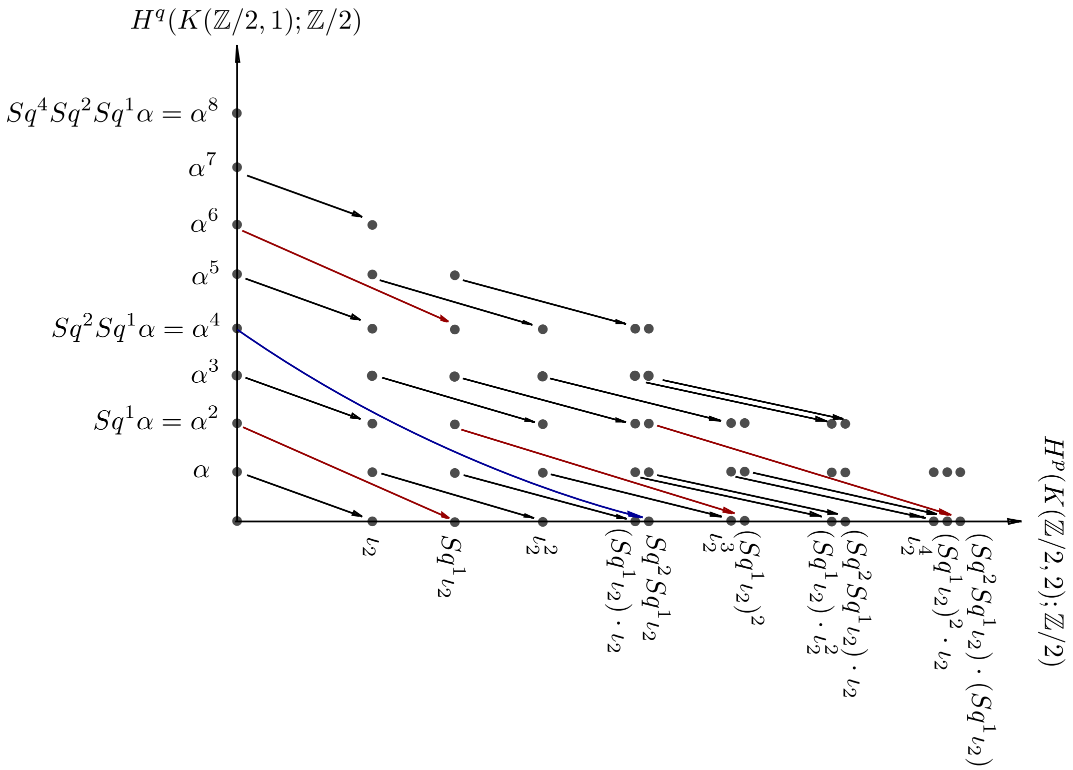
Proposition B.5. *Let $R = \mathbb{Z}/2$. If $x \in H^{n-1}(F; \mathbb{Z}/2)$ is transgressive, then $Sq^i(x) \in H^{n+i-1}(F; \mathbb{Z}/2)$ is also transgressive. Moreover, if $y \in H^n(B; \mathbb{Z}/2)$ such that $y \in \tau(x)$, then $Sq^i(y) \in \tau(Sq^i(x))$.*

In the special case where $E^2_{n,0} = E^n_{n,0}$, we have $\tau Sq^i(x) = Sq^i(\tau x)$.

Let us compute the first terms of $H^*(K(\mathbb{Z}/2,2);\mathbb{Z}/2)$ using the Serre spectral sequence associated to the fibre map

$$\begin{array}{ccc} K(\mathbb{Z}/2,2) \simeq \Omega K(\mathbb{Z}/2,1) & \longrightarrow & \mathcal{P}K(\mathbb{Z}/2,1) \\ \downarrow & & \downarrow {\scriptstyle \mathrm{ev}_1} \\ * & \longrightarrow & K(\mathbb{Z}/2,1). \end{array}$$

First, the cohomology ring (with mod 2 coefficients) of $K(\mathbb{Z}/2,1)$ is well known as the polynomial ring over $\mathbb{Z}/2$ on one generator that we will denote α. Since $\mathcal{P}K(\mathbb{Z}/2,1) \simeq *$, the $E_\infty^{p,q} = 0$ for all p,q except the case $p = q = 0$. Hence $E_\infty^{0,0} \cong \mathbb{Z}$. The following figure represent the $E_*^{p,q}$ terms.



Let us try to understand the art of chasing in spectral sequence of this kind. First, by the Hurewicz theorem, we have an element $\iota_2 \in H^2(K(\mathbb{Z}/2,2);\mathbb{Z}/2)$. Since the total space of the fiber is contractible, α must hit ι_2 via d_2. By the Künneth formula, we know that the $E_2^{2,q}$ terms are $\iota_2 \otimes \alpha^q$, and by the product formula for the differential, we know that $d_2(\alpha^2) = 2\iota_2 \otimes \alpha = 0$. More generally, we deduce that $d_2(\alpha^{2k}) = 0$ and $d_2(\alpha^{2k+1}) = \iota_2 \otimes \alpha^{2k}$.

Now, consider $\alpha^2 = Sq^1(\alpha)$. By the proposition above, $d_3(Sq^1(\alpha)) = Sq^1(d_2(\alpha)) = Sq^1\iota_2$. So we have an element $Sq^1\iota_2$ in $H^3(K(\mathbb{Z}/2,2);\mathbb{Z}/2)$. From these elements, we can deduce $\iota_2^2 = d_2(\iota_2 \otimes \alpha)$ and $(Sq^1\iota_2)\cdot\iota_2 = d_2(Sq^1\iota_2 \otimes \alpha)$.

A new element will appear in $H^6(K(\mathbb{Z}/2,2);\mathbb{Z}/2)$, this one is given by the transgression $d_5(Sq^2Sq^1\alpha) = Sq^2Sq^1d_2(\alpha) = Sq^2Sq^1\iota_2$. Indeed, $d_3(\alpha^4) = 2Sq^1\iota_2 \otimes \alpha^2 = 0$, and $d_4(\alpha^4) = 0$ because there is nothing in $E_4^{4,1}$. The reader can continue this computation few steps further, but to get the following theorem one needs to arrange information in some way. The proof can be found in [MT68].

Theorem B.6. *The cohomology ring $H^*(K(\mathbb{Z}/2,q);\mathbb{Z}/2)$ is a polynomial ring over $\mathbb{Z}/2$ generated by symbols $\{Sq^I\iota_q\}$ where I runs over all admissible sequences of excess less than q.*

Remark. The *excess* of $I = (i_1, i_2, \cdots, i_r)$ is the difference $i_1 - i_2 - \cdots - i_r$.

Theorem B.7. *The cohomology ring $H^*(K(\mathbb{Z}, q); \mathbb{Z}/2)$ (for $q > 1$) is a polynomial ring over $\mathbb{Z}/2$ generated by symbols $\{Sq^I \iota_q\}$ where I runs over all admissible sequences of excess less than q and which the last non-zero term of I is not 1.*

These results were originally proved by Serre and they can be founded in its original paper [Ser53a].

B.2 Adams Spectral Sequence

One of the most efficient tool to compute stable homotopy groups is the Adams spectral sequence. This spectral sequence was introduced in the late fifty's by Adams to solve the Hopf invariant one problem. Today, it has also a lot of different applications as for example the computation of the homotopy groups of spheres.

Let X, Y be based finite CW-complexes. Define the *stable track group* $\{X, Y\}_n$ as

$$\operatorname{colim}_i [\Sigma^{i+n} X, \Sigma^i Y].$$

For any prime number p, let ${}_{(p)}\{X, Y\}_n$ denote the subgroup of $\{X, Y\}_n$ consisting of elements whose order are finite and prime to p.

Theorem B.8 (Adams spectral sequence)**.** *For based finite CW-complexes X, Y and any prime number p, we have a spectral sequence $\{E_r, d_r\}$ such that*

- *E_r is bigraded and the differential d_r is of bidegree $(r, r-1)$.*
- *$E_2^{s,t} = \operatorname{Ext}^s_{\mathcal{A}_p}(\tilde{H}^*(Y; \mathbb{Z}/p), \Sigma^t \tilde{H}^*(X; \mathbb{Z}/p))$, where $\mathcal{A}_p$ is the Steenrod algebra mod p.*
- *The spectral sequence converge to $\{X, Y\}_n$, i.e. there exists a sequence of subgroups*

$$\{X, Y\}_n = F^{0,n} \supset F^{1,n+1} \supset F^{2,n+2} \supset \cdots$$

such that $E_\infty^{s,t} = F^{s,t}/F^{s+1,t+1}$. Moreover, the intersection $\bigcap_i F^{i,n+i}$ is equal to ${}_{(p)}\{X, Y\}_n$.

We recall briefly the definition of Ext_R. If M and N are graded R-modules. Choose a projective resolution of M, i.e. an exact sequence of projective R-modules

$$\cdots \xrightarrow{d} P_2 \xrightarrow{d} P_1 \xrightarrow{d} P_0 \xrightarrow{\varepsilon} M \longrightarrow 0,$$

then apply the functor $\operatorname{Hom}_R(-, N)$ (degree preserving R-linear maps) to get a cochain complex

$$0 \longrightarrow \operatorname{Hom}_R(P_0, N) \xrightarrow{d^*} \operatorname{Hom}_R(P_1, N) \xrightarrow{d^*} \operatorname{Hom}_R(P_2, N) \longrightarrow \cdots$$

The s-th homology group of this complex is $\operatorname{Ext}^s_R(M, N)$. Moreover the *suspension* of a graded module is defined by $(\Sigma N)_t := N_{t-1}$.

C Serre's $\mathscr{C}$-theory

In [Ser53b], Serre developed a very nice way to generalize the classical theorems of algebraic topology. The idea is to look maps mod $\mathscr{C}$, where $\mathscr{C}$ is a suitable class of abelian groups. For the following, we refer the reader to Serre's original paper or [MT68, Chapter 10].

Definition. A *class of abelian groups* is a collection $\mathscr{C}$ of abelian groups which satisfying the following axiom:

Axiom 1 For any short exact sequence $0 \to A' \to A \to A'' \to 0$ of abelian groups, we have that $A \in \mathscr{C}$ if and only if $A', A'' \in \mathscr{C}$.

Let us give us some additional axioms that a class of abelian groups may satisfy.

Axiom 2 If $A, B \in \mathscr{C}$, then $A \otimes B \in \mathscr{C}$ and $\mathrm{Tor}(A, B) \in \mathscr{C}$.

Axiom 2 (bis) If $A \in \mathscr{C}$, then $A \otimes B \in \mathscr{C}$ for any abelian group B.

Axiom 3 If $A \in \mathscr{C}$, then $H_n(K(A, 1); \mathbb{Z}) \in \mathscr{C}$ for all $n > 0$.

Examples. Obviously, the class containing only the trivial group satisfies these axioms. We give here a short list some non-trivial classes that are interesting and that we might use in this paper. Other classes might be found in the original Serre's paper.

$\mathscr{C}_F$: The class of finite abelian groups satisfies axioms 1, 2 and 3.

$\mathscr{C}_{FG}$: The class of finitely generated abelian groups satisfies also axioms 1,2 and 3.

$\mathscr{C}_p$: The class of abelian torsion groups such that every element has exponent prime to p, this class satisfies axiom 1, 2 (bis) and 3.

Definition. Let $f : A \to B$ be a homomorphism of abelian grous. We say that f is a *$\mathscr{C}$-monomorphism* if $\ker f \in \mathscr{C}$, and *$\mathscr{C}$-epimorphism* if $\operatorname{coker} f \in \mathscr{C}$. A *$\mathscr{C}$-isomorphism* is both a $\mathscr{C}$-monomorphism and a $\mathscr{C}$-epimorphism.

Theorem C.1 (Hurewicz theorem mod $\mathscr{C}$)**.** *Let X be a 1-connected space. If $\mathscr{C}$ is a class of abelian groups satisfying axioms 1,2 and 3, and if $\pi_i(X) \in \mathscr{C}$ for all $i < n$, then $H_i(X) \in \mathscr{C}$ for all $i < n$ and the Hurewicz homomorphism $\pi_n(X) \to H_n(X)$ is a $\mathscr{C}$-isomorphism.*

Theorem C.2 (Relative Hurewicz theorem mod $\mathscr{C}$)**.** *Let (X, A) be a pair of 1-connected topological space such that $i_\# : \pi_2(A) \to \pi_2(X)$ is an epimorphism. If $\mathscr{C}$ is a class of abelian groups satisfying axioms 1, 2 (bis) and 3 and if $\pi_i(X, A) \in \mathscr{C}$ for all $i < n$, then $H_i(X, A) \in \mathscr{C}$ for all $i < n$ and the relative Hurewicz homomotphism $\pi_n(X, A) \to H_n(X, A)$ is a $\mathscr{C}$-isomorphism.*

Theorem C.3 (Hurewicz theorem mod $\mathscr{C}$ for a spectrum)**.** *Let $\boldsymbol{E}$ be a spectrum. if $\mathscr{C}$ is a class of abelian groups satisfying axioms 1,2 and 3, and if $\pi_i(\boldsymbol{E}) \in \mathscr{C}$ for all $i < n$, then $H_i(\boldsymbol{E}) \in \mathscr{C}$ for $i < n$ and the stable Hurewicz homomorphism $\pi_n(\boldsymbol{E}) \to H_n(\boldsymbol{E})$ is a $\mathscr{C}$-isomorphism.*

In particular, if $\pi_i(\boldsymbol{E}) \in \mathscr{C}$ for all i, then $H_i(\boldsymbol{E}) \in \mathscr{C}$ for all i.

Theorem C.4 (Whitehead's theorem mod $\mathscr{C}$)**.** *Let $f : A \to B$ be a map between 1-connected spaces and suppose $f_\# : \pi_2(A) \to \pi_2(X)$ is an isomorphism. Suppose also that $\mathscr{C}$ is a class of abelian groups satisfying axioms 1, 2 (bis) and 3. Then $f_\# : \pi_i(A) \to \pi_i(X)$ is a $\mathscr{C}$-isomorphism for all $i < n$ and is a $\mathscr{C}$-epimorphism for $i = n$, if and only if $f_* : H_n(A) \to H_n(X)$ is a $\mathscr{C}$-isomorphism for all $i < n$ and a $\mathscr{C}$-epimorphism for $i = n$.*

Theorem C.5 ($\mathscr{C}_p$ Approximation theorem)**.** *Let (X, A) be a pair 1-connected spaces such that $H_i(A)$ and $H_i(X)$ are finitely generated for all i. Moreover, let $f : A \to X$ denote the inclusion map and suppose that $f_\# : \pi_2(A) \to \pi_2(X)$ is an epimorphism. Then the following conditions are equivalent:*

1. *$f^* : H^i(X; \mathbb{Z}/p) \to H^i(A; \mathbb{Z}/p)$ is an isomorphism for $i < n$ and a monomorphism for $i = n$.*
2. *$f_* : H_i(A; \mathbb{Z}/p) \to H_i(X; \mathbb{Z}/p)$ is an isomorphism for $i < n$ and an epimorphism for $i = n$.*
3. *$H_i(X, A; \mathbb{Z}/p) = 0$ for $i \leq n$.*
4. *$H_i(X, A; \mathbb{Z}) \in \mathscr{C}_p$ for $i \leq n$.*
5. *$\pi_i(X, A) \in \mathscr{C}_p$ for $i \leq n$.*
6. *$f_\# : \pi_i(A) \to \pi_i(X)$ is a $\mathscr{C}_p$-isomorphism for $i < n$ and a $\mathscr{C}_p$-epimorphism for $i = n$*

If these conditions holds, then $\pi_i(A)$ and $\pi_i(X)$ have isomorphic p-components for all $i < n$.

The proof of Theorems C.1, C.2, C.4 and C.5 can be found in Serre's original paper [Ser53b] and in [MT68]. The proof of Theorem C.3 can be found in [Rud98, Theorem 4.24].

References

[Ada69] J. F. Adams. Lectures on generalised cohomology. In *Category Theory, Homology Theory and their Applications, III (Battelle Institute Conference, Seattle, Wash., 1968, Vol. Three)*, pages 1–138. Springer, Berlin, 1969.

[Ada74] J. F. Adams. *Stable homotopy and generalised homology.* University of Chicago Press, Chicago, Ill., 1974. Chicago Lectures in Mathematics.

[Bro62] Edgar H. Brown, Jr. Cohomology theories. *Ann. of Math. (2)*, 75:467–484, 1962.

[BS53] A. Borel and J.-P. Serre. Groupes de Lie et puissances réduites de Steenrod. *Amer. J. Math.*, 75:409–448, 1953.

[Car54a] Henri Cartan. Sur les groupes d'Eilenberg-Mac Lane $H(\Pi, n)$. I. Méthode des constructions. *Proc. Nat. Acad. Sci. U. S. A.*, 40:467–471, 1954.

[Car54b] Henri Cartan. Sur les groupes d'Eilenberg-Mac Lane. II. *Proc. Nat. Acad. Sci. U. S. A.*, 40:704–707, 1954.

[ES52] Samuel Eilenberg and Norman Steenrod. *Foundations of algebraic topology.* Princeton University Press, Princeton, New Jersey, 1952.

[Hat04] Allen Hatcher. *Spectral Sequence in Algebraic Topology.* `http://www.math.cornell.edu/~hatcher/SSAT/SSATpage.html`, 2004.

[Lan76] Peter S. Landweber. Homological properties of comodules over $M\mathrm{U}_*(M\mathrm{U})$ and $\mathrm{BP}_*(\mathrm{BP})$. *Amer. J. Math.*, 98(3):591–610, 1976.

[Lef99] Solomon Lefschetz. The early development of algebraic topology. In *History of topology*, pages 531–559. North-Holland, Amsterdam, 1999.

[May99a] J. P. May. *A concise course in algebraic topology.* Chicago Lectures in Mathematics. University of Chicago Press, Chicago, IL, 1999.

[May99b] J. P. May. Stable algebraic topology, 1945–1966. In *History of topology*, pages 665–723. North-Holland, Amsterdam, 1999.

[Mil58] John Milnor. The Steenrod algebra and its dual. *Ann. of Math. (2)*, 67:150–171, 1958.

[Mil60] J. Milnor. On the cobordism ring Ω^* and a complex analogue. I. *Amer. J. Math.*, 82:505–521, 1960.

[Mil62] J. Milnor. On axiomatic homology theory. *Pacific J. Math.*, 12:337–341, 1962.

[MM65] John W. Milnor and John C. Moore. On the structure of Hopf algebras. *Ann. of Math. (2)*, 81:211–264, 1965.

[MS74] John W. Milnor and James D. Stasheff. *Characteristic classes.* Princeton University Press, Princeton, N. J., 1974. Annals of Mathematics Studies, No. 76.

[MT68] Robert E. Mosher and Martin C. Tangora. *Cohomology operations and applications in homotopy theory.* Harper & Row Publishers, New York, 1968.

[Rav86] Douglas C. Ravenel. *Complex cobordism and stable homotopy groups of spheres*, volume 121 of *Pure and Applied Mathematics.* Academic Press Inc., Orlando, FL, 1986.

[Rud98] Yuli B. Rudyak. *On Thom spectra, orientability, and cobordism.* Springer Monographs in Mathematics. Springer-Verlag, Berlin, 1998. With a foreword by Haynes Miller.

[Ser51] Jean-Pierre Serre. Homologie singulière des espaces fibrés. Applications. *Ann. of Math. (2)*, 54:425–505, 1951.

[Ser53a] Jean-Pierre Serre. Cohomologie modulo 2 des complexes d'Eilenberg-MacLane. *Comment. Math. Helv.*, 27:198–232, 1953.

[Ser53b] Jean-Pierre Serre. Groupes d'homotopie et classes de groupes abéliens. *Ann. of Math. (2)*, 58:258–294, 1953.

[Sil09] Joseph H. Silverman. *The arithmetic of elliptic curves*, volume 106 of *Graduate Texts in Mathematics*. Springer, Dordrecht, second edition, 2009.

[Sto68] Robert E. Stong. *Notes on cobordism theory.* Mathematical notes. Princeton University Press, Princeton, N.J., 1968.

[Swi02] Robert M. Switzer. *Algebraic topology—homotopy and homology.* Classics in Mathematics. Springer-Verlag, Berlin, 2002. Reprint of the 1975 original [Springer, New York; MR0385836 (52 #6695)].

[Tho54] René Thom. Quelques propriétés globales des variétés différentiables. *Comment. Math. Helv.*, 28:17–86, 1954.

[Whi62] George W. Whitehead. Generalized homology theories. *Trans. Amer. Math. Soc.*, 102:227–283, 1962.