Efficient Computation of Sensitivity Coefficients of Node Voltages and Line Currents in Unbalanced Electrical Networks

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***Abstract*—The problem of optimal control of power distribution systems is becoming increasingly compelling due to the progressive penetration of distributed energy resources in this specific layer of the electrical infrastructure. Distribution systems are, indeed, experiencing significant changes in terms of operation philosophies that are often based on optimal control strategies relying on the computation of linearized dependencies between controlled (e.g. voltages) and control variables (e.g. power injections). As the implementation of these strategies in real-time controllers imposes stringent time constraints, the derivation of analytical dependency between controlled and control variables becomes a non-trivial task to be solved. With reference to optimal voltage and power flow controls, this paper aims at providing an analytical derivation of node voltage and line current flows as a function of the nodal power injections. Compared to other approaches presented in the literature, the one proposed here is based on the use of $[\mathbf{Y}]$ compound matrix of a generic multi-phase unbalanced network. In order to estimate the computational benefits of the proposed approach, the relevant improvements are also quantified versus traditional methods. The validation of the proposed method is carried out by using both IEEE 13 and 34 node test feeders. The paper finally shows the use of the proposed method for the problem of optimal voltage control applied to the IEEE 34 node test feeder.**

***Index Terms*—Voltage/current sensitivity coefficients, unbalanced electrical networks, power systems optimal operation, smart grids.**

I. Introduction

Optimal controls of power systems are often based on the solution of linear problems that link control variables to controlled quantities by means of sensitivity coefficients. Typical optimization problems refer to scheduling of generators, voltage control, losses reduction etc. So far, these categories of problems have been commonly investigated in the domain of high voltage transmission networks. However, during the past years, the increased penetration of distributed energy resources (DERs) in power distribution systems has raised the importance of developing optimal control strategies specifically applied to the operation of these networks (e.g. [1], [2], [3], [4], [5], [6]). Within this context, it is worth noting that the solution of optimal problems becomes of interest only if it meets the stringent time constraints required by real-time controls and imposed by the higher dynamics of these networks compared to the transmission ones.

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Typical examples of optimal controls that are not yet deployed in active distribution networks (ADNs) are voltage and power flow controls. Usually this category of problems has been addressed in the literature by means of linear-approaches applied to the dependency between voltages and power flows as a function of the power injections (e.g. [4], [5], [7], [8]).

The typical approach for the solution of this class of control problems is the use of the sensitivity coefficients through an updated Jacobian matrix derived from the load flow problem [9], [10], [11], [12], [13]. However, from the computational point of view, the main disadvantage of such a method is that, for every change in the operation conditions of the network, an updated Jacobian matrix needs to be built on the basis of the network state and needs, then, to be inverted. This procedure involves non-trivial computation constraints for the implementation in real-time centralized or decentralized controllers.

For this reason, the authors of [14] have proposed the direct computation of voltages and network losses sensitivity coefficients, based on the Gauss-Seidel formulation of the load flow problem, by making use of the $[\mathbf{Z}]$ matrix of a balanced network. Also, in [7] it has been proposed the use of the $[\mathbf{Z}]$ matrix along with the constant-current model for loads and generators. In [8] the sensitivity coefficients are proposed to be calculated starting from the network branch currents.

In order to increase the computational efficiency of this category of approaches, and to extend it to the inherent multi-phase unbalanced configuration of distribution networks, this paper aims at providing a straightforward derivation of node voltages and line currents sensitivities as a function of the power injections. To this end, we propose to use the network $[\mathbf{Y}]$ compound matrix, which has the advantage of being sparse. The main contribution of this paper is the analytical derivation of the sensitivity coefficients. A further contribution of the paper is the estimation of the computational benefits of the proposed method quantified versus the traditional approach. Finally the paper shows an application of the proposed method to optimal voltage control applied to the IEEE test feeders unbalanced distribution networks [15].

The structure of the paper is the following: Section II focuses on the problem formulation by describing, in detail, the analytical procedure at the base of the proposed method. It also includes a uniqueness proof of the solution of the linear system used to calculate the sensitivity coefficients. The same section also provides a computational cost analysis of the proposed method versus traditional approaches. Section III validates the proposed method using the IEEE 13 and 34 node test feeders. Section IV shows a typical application example of sensitivity coefficients related to the optimal voltage control in unbalanced distribution systems. Section V provides the final remarks and future applications of the proposed method.

II. Problem Formulation

A. Classical Computation of Sensitivity Coefficients in Power Networks

In this paragraph we make reference to a balanced network composed by K busses.

Traditionally, there are three proposed ways to calculate the sensitivity coefficients of our interest. The first method consists of estimating them by a series of load flow calculations each performed for a small variation of a single control variable (i.e. nodal power injections, P_l, Q_l) [4]:

$$\frac{\partial|\bar{E}_i|}{\partial P_l} = \left.\frac{\Delta|\bar{E}_i|}{\Delta P_l}\right|_{\substack{\Delta P_{i,i\neq l}=0\\ \Delta Q_{i,i\neq l}=0}} \qquad \frac{\partial|\bar{I}_{ij}|}{\partial P_l} = \left.\frac{\Delta|\bar{I}_{ij}|}{\Delta P_l}\right|_{\substack{\Delta P_{i,i\neq l}=0\\ \Delta Q_{i,i\neq l}=0}}$$
$$\frac{\partial|\bar{E}_i|}{\partial Q_l} = \left.\frac{\Delta|\bar{E}_i|}{\Delta Q_l}\right|_{\substack{\Delta P_{i,i\neq l}=0\\ \Delta Q_{i,i\neq l}=0}} \qquad \frac{\partial|\bar{I}_{ij}|}{\partial Q_l} = \left.\frac{\Delta|\bar{I}_{ij}|}{\Delta Q_l}\right|_{\substack{\Delta P_{i,i\neq l}=0\\ \Delta Q_{i,i\neq l}=0}} \tag{1}$$

where $\bar{E}_i$ is the direct sequence phase-to-ground voltage of node i and $\bar{I}_{ij}$ is the direct sequence current flow between nodes i and j $(i, j \in \{1 \cdots K\})$.

The second method uses the Newton Raphson formulation of the load flow calculation to directly infer the voltage sensitivity coefficients as submatrices of the inverted Jacobian matrix (e.g.[9], [10], [11], [12], [13]):

$$J = \begin{bmatrix} \dfrac{\partial \mathbf{P}}{\partial|\bar{\mathbf{E}}|} & \dfrac{\partial \mathbf{P}}{\partial \boldsymbol{\theta}} \\ \\ \dfrac{\partial \mathbf{Q}}{\partial|\bar{\mathbf{E}}|} & \dfrac{\partial \mathbf{Q}}{\partial \boldsymbol{\theta}} \end{bmatrix}. \tag{2}$$

As known, the submatrix $\dfrac{\partial \mathbf{Q}}{\partial|\bar{\mathbf{E}}|}$ is usually adopted to express voltage variations as a function of reactive power injections when the ratio of longitudinal line resistance versus reactance is negligible. It is worth noting that such an assumption is no longer applicable to distribution systems that require in addition to take into account active power injections.

A third approach typically derived from circuit theory is based on the use of the so-called adjoint network (e.g.[16], [17], [18], [19], [20]).

B. Analytical Derivation of Voltage and Current Sensitivity Coefficients

This subsection contains the main analytical development of this paper related to the derivation of the voltage sensitivity coefficients [1].

[1]as shown in subsection II-B2 the current sensitivities can be straightforwardly derived from the voltage ones

1) Voltage Sensitivity Coefficients: the analysis starts with the voltage sensitivity coefficients. To this end, we derive mathematical expressions that link bus voltages to bus active and reactive power injections. For this purpose, a K-bus 3-phase generic electrical network is considered. The following analysis treats each phase of the network separately and, thus, it can be applied to unbalanced networks.

As known, the equations that link the voltage of each phase of the busses to the corresponding injected current are in total $M = 3K$ and they are given by:

$$[\bar{\mathbf{I}}_{\mathbf{abc}}] = [\bar{\mathbf{Y}}_{\mathbf{abc}}] \cdot [\bar{\mathbf{E}}_{\mathbf{abc}}] \tag{3}$$

where $[\bar{\mathbf{I}}_{\mathbf{abc}}] = [\bar{I}_a^1, \bar{I}_b^1, \bar{I}_c^1 ..., \bar{I}_a^K, \bar{I}_b^K, \bar{I}_c^K]^T$, $[\bar{\mathbf{E}}_{\mathbf{abc}}] = [\bar{E}_a^1, \bar{E}_b^1, \bar{E}_c^1 ..., \bar{E}_a^K, \bar{E}_b^K, \bar{E}_c^K]^T$. We denoted by a, b, c the three network phases. The $[\bar{\mathbf{Y}}_{\mathbf{abc}}]$ matrix is formed by using the so-called compound admittance matrix (e.g. [21]) as follows:

$$\left[\bar{\mathbf{Y}}_{\mathbf{abc}}\right] = \begin{bmatrix} \bar{Y}_{aa}^{11} & \bar{Y}_{ab}^{11} & \bar{Y}_{ac}^{11} & \cdots & \bar{Y}_{aa}^{1K} & \bar{Y}_{ab}^{1K} & \bar{Y}_{ac}^{1K} \\ \bar{Y}_{ba}^{11} & \bar{Y}_{bb}^{11} & \bar{Y}_{bc}^{11} & \cdots & \bar{Y}_{ba}^{1K} & \bar{Y}_{bb}^{1K} & \bar{Y}_{bc}^{1K} \\ \bar{Y}_{ca}^{11} & \bar{Y}_{cb}^{11} & \bar{Y}_{cc}^{11} & \cdots & \bar{Y}_{ca}^{1K} & \bar{Y}_{cb}^{1K} & \bar{Y}_{cc}^{1K} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \bar{Y}_{aa}^{K1} & \bar{Y}_{ab}^{K1} & \bar{Y}_{ac}^{K1} & \cdots & \bar{Y}_{aa}^{KK} & \bar{Y}_{ab}^{KK} & \bar{Y}_{ac}^{KK} \\ \bar{Y}_{ba}^{K1} & \bar{Y}_{bb}^{K1} & \bar{Y}_{bc}^{K1} & \cdots & \bar{Y}_{ba}^{KK} & \bar{Y}_{bb}^{KK} & \bar{Y}_{bc}^{KK} \\ \bar{Y}_{ca}^{K1} & \bar{Y}_{cb}^{K1} & \bar{Y}_{cc}^{K1} & \cdots & \bar{Y}_{ca}^{KK} & \bar{Y}_{cb}^{KK} & \bar{Y}_{cc}^{KK} \end{bmatrix}.$$

In order to simplify the notation, in what follows we will assume the following correspondences: $[\bar{\mathbf{I}}_{\mathbf{abc}}] = [\bar{I}_1, ..., \bar{I}_M]^T$, $[\bar{\mathbf{E}}_{\mathbf{abc}}] = [\bar{E}_1, ..., \bar{E}_M]^T$ and

$$\left[\bar{\mathbf{Y}}_{\mathbf{abc}}\right] = \begin{bmatrix} \bar{Y}_{11} & \cdots & \bar{Y}_{1M} \\ \vdots & \cdots & \vdots \\ \bar{Y}_{1M} & \cdots & \bar{Y}_{MM} \end{bmatrix}.$$

For the rest of the analysis we will consider the network as composed by S slack busses and N busses with PQ injections, (i.e.$\{1, 2, \cdots M\} = \mathcal{S} \cup \mathcal{N}$ with $\mathcal{S} \cap \mathcal{N} = \emptyset$).

It is well known that the i-th element of the above system of equations (3) can be expressed as:

$$\bar{I}_i = \sum_{j \in \mathcal{S} \cup \mathcal{N}} \bar{Y}_{ij} \bar{E}_j \quad , i \in \mathcal{N} \tag{4}$$

and, the link between power injections and bus voltages reads:

$$\underline{S}_i = \underline{E}_i \sum_{j \in \mathcal{S} \cup \mathcal{N}} \bar{Y}_{ij} \bar{E}_j \quad , i \in \mathcal{N}. \tag{5}$$

The derived system of equations (5) holds for all the phases of each bus of the network. Since the objective is to calculate the partial derivatives of the voltage magnitude over the active and reactive power injected in the other busses, we have to consider separately the slack bus of the system. As known, the assumptions for the slack bus equations are to keep its voltage constant and equal to the network rated value, by also fixing its phase equal to zero. Hence, for the three phases of the slack bus, it holds that:

$$\frac{\partial \bar{E}_i}{\partial P_l} = 0 \quad , \forall i \in \mathcal{S}. \tag{6}$$

At this point, by using equation (5) as a starting point one can derive closed-form mathematical expressions to define and quantify voltage sensitivity coefficients with respect to active and reactive power variations in correspondence of the N busses of the network. To derive voltage sensitivity coefficients, the partial derivatives of the voltages with respect to the active and reactive power P_l and Q_l of a bus $l \in \mathcal{N}$ have to be computed. The partial derivatives with respect to active power satisfy the following system of equations:

$$\mathbf{1}_{\{i=l\}} = \frac{\partial \underline{E}_i}{\partial P_l} \sum_{j \in \mathcal{S} \cup \mathcal{N}} \bar{Y}_{ij} \bar{E}_j + \underline{E}_i \sum_{j \in \mathcal{N}} \bar{Y}_{ij} \frac{\partial \bar{E}_j}{\partial P_l} \tag{7}$$

where it has been taken into account that:

$$\frac{\partial \underline{S}_i}{\partial P_l} = \frac{\partial \{P_i - jQ_i\}}{\partial P_l} = \mathbf{1}_{\{i=l\}}. \tag{8}$$

The system of equations (7) is not linear over complex numbers, but it is linear with respect to $\frac{\partial \bar{E}_i}{\partial P_l}, \frac{\partial \underline{E}_i}{\partial P_l}$, therefore it is linear over real numbers with respect to rectangular coordinates. As we show next, it has a unique solution and can therefore be used to compute the partial derivatives in rectangular coordinates to reduce the computational effort.

A similar system of equations holds for the sensitivity coefficients with respect to the injected reactive power Q_l. With the same reasoning, by taking into account that:

$$\frac{\partial \underline{S}_i}{\partial Q_l} = \frac{\partial \{P_i - jQ_i\}}{\partial Q_l} = -j\mathbf{1}_{\{i=l\}} \tag{9}$$

we obtain that:

$$-j\mathbf{1}_{\{i=l\}} = \frac{\partial \underline{E}_i}{\partial Q_l} \sum_{j \in \mathcal{S} \cup \mathcal{N}} \bar{Y}_{ij} \bar{E}_j + \underline{E}_i \sum_{j \in \mathcal{N}} \bar{Y}_{ij} \frac{\partial \bar{E}_j}{\partial Q_l}. \tag{10}$$

By observing the above linear systems of equations (7) and (10), we can see that the matrix that needs to be inverted in order to solve the system is fixed independently of the power of the l-th bus with respect to which we want to compute the partial derivatives. The only element that changes is the left hand side of the equations.

Once $\frac{\partial \bar{E}_i}{\partial P_l}, \frac{\partial \underline{E}_i}{\partial P_l}$ are obtained, the partial derivatives of the voltage magnitude can be expressed as:

$$\frac{\partial |E_i|}{\partial P_l} = \frac{1}{|E_i|} Re(\underline{E}_i \frac{\partial \bar{E}_i}{\partial P_l}) \tag{11}$$

and similar equations hold for derivatives with respect to reactive power injections.

Theorem 1: The system of equations (7), which is linear with respect to rectangular coordinates, has a unique solution for every electrical network. The same holds for the system of equations (10).

Proof: Since the linear system of equations has as many unknowns as equations, the theorem is equivalent to showing that the corresponding homogeneous system of equations has only the trivial solution. The homogeneous system can be written as:

$$0 = \bar{\Delta}_i \sum_{j \in \mathcal{S} \cup \mathcal{N}} \bar{Y}_{ij} \bar{E}_j + \underline{E}_i \sum_{j \in \mathcal{N}} \bar{Y}_{ij} \underline{\Delta}_j \tag{12}$$

where $\bar{\Delta}_i$ are the unknown complex numbers, defined for $i \in \mathcal{N}$. We want to show that $\bar{\Delta}_i = 0$ for all $i \in \mathcal{N}$. Let us consider two electrical networks with the same topology, i.e. same $[\bar{\mathbf{Y}}_{\mathbf{abc}}]$ matrix; in the first network, the voltage at the ith non slack bus is $\bar{E}_i - \bar{\Delta}_i$; in the second network, it is $\bar{E}_i + \bar{\Delta}_i$. Voltage profiles at slack busses are identical. Let $\underline{S}'_i$ be the absorbed power at the ith bus in the first network, and $\underline{S}''_i$ in the second. Apply equation (5) to each network, subtract and apply equation (12). It follows that $\underline{S}'_i = \underline{S}''_i$ for all $i \in \mathcal{S} \cup \mathcal{N}$. Thus the two networks have the same active and reactive powers at all busses. It follows that the voltage profile of these networks must be exactly the same, i.e. $\bar{E}_i - \bar{\Delta}_i = \bar{E}_i + \bar{\Delta}_i$ and thus $\bar{\Delta}_i = 0$ for all $i \in \mathcal{N}$. ■

2) Current Sensitivity Coefficients: From the previous analysis, the sensitivity coefficients linking the power injections to the voltage variations are known. Thus, it is straightforward to express the branch current sensitivities with respect to the same power injections. Assuming to represent the lines that compose the network by means of π models, the current flow $\bar{I}_{ij}$ between nodes i and j can be expressed as a function of the phase-to-ground voltages of the relevant i, j nodes as follows:

$$\bar{I}_{ij} = \bar{Y}_{ij}(\bar{E}_i - \bar{E}_j) \tag{13}$$

where $\bar{Y}_{ij}$ is the generic element of $[\bar{\mathbf{Y}}_{\mathbf{abc}}]$ matrix between node i and node j.

Since the voltages can be expressed as a function of the power injections into the network busses, the partial derivatives of the current with respect to the active and reactive power injections in the network can be expressed as:

$$\begin{aligned} \frac{\partial \bar{I}_{ij}}{\partial P_l} &= \bar{Y}_{ij}(\frac{\partial \bar{E}_i}{\partial P_l} - \frac{\partial \bar{E}_j}{\partial P_l}) \\ \frac{\partial \bar{I}_{ij}}{\partial Q_l} &= \bar{Y}_{ij}(\frac{\partial \bar{E}_i}{\partial Q_l} - \frac{\partial \bar{E}_j}{\partial Q_l}) \end{aligned}. \tag{14}$$

Applying the same reasoning as earlier, the branch current sensitivity coefficients with respect to an active power P_l can be computed using the following expressions:

$$\frac{\partial |I_{ij}|}{\partial P_l} = \frac{1}{|I_{ij}|} Re(\underline{I}_{ij} \frac{\partial \bar{I}_{ij}}{\partial P_l}). \tag{15}$$

Similar expressions can be derived for the current coefficients with respect to the reactive power in the busses as:

$$\frac{\partial |I_{ij}|}{\partial Q_l} = \frac{1}{|I_{ij}|} Re(\underline{I}_{ij} \frac{\partial \bar{I}_{ij}}{\partial Q_l}). \tag{16}$$

C. Computational Cost Analysis

The aim of this subsection is to show the computational advantage of the proposed method compared to the classical approach with respect to the network size. Furthermore, the two methods are applied to the IEEE 13 and 34 node test feeders and compared in terms of CPU time necessary to calculate the voltage sensitivity coefficients.

We are assuming that:

1) there are loads/injections in all three phases of the system
2) the phasors of phase-to-ground voltages in all the network are known (e.g. coming from a state estimation process)

In the following table, Algorithm 1 shows the steps required to calculate the voltage sensitivity coefficients using the traditional method and Algorithm 2 shows the corresponding steps using the analytical method proposed here.

For the traditional method an updated Jacobian needs to be built, and its inverse will provide the desired voltage sensitivities. For the analytical method the corresponding steps refer to invert a square matrix of size $2N$ and multiply the inverse matrix with one column vector for each PQ bus in the network.

In Table I the mean value of the CPU time necessary to calculate the voltage sensitivity coefficients is presented for the IEEE 13 and 34 node test feeders respectively, when 1000 iterations of the method are executed. It can be observed that the analytical approach is more than 2.5 times faster than the traditional method. In Fig. 1 the mean CPU time necessary to calculate the voltage sensitivity coefficients is depicted for the two feeders. One can observe the advantage of the proposed analytical method as the number of busses in the network increases.

Algorithm 1 Computation of voltage sensitivity coefficients using the Jacobian method

1: build Jacobian matrix associated to the Newton Raphson method
2: invert matrix J of size $2N \times 2N$
3: extract the sub-matrices corresponding to the desired sensitivity coefficients

Algorithm 2 Computation of voltage sensitivity coefficients using the analytical method

1: build the matrix of the linear system of equations
2: invert matrix of size $2N \times 2N$
3: do N multiplications of the inverse matrix with vectors of size $2N \times 1$

Table I
CPU TIME NECESSARY FOR CALCULATING VOLTAGE SENSITIVITY COEFFICIENTS IN THE IEEE 13 AND THE 34 NODE TEST FEEDERS WHEN ALL PHASES OF ALL BUSSES HAVE LOADS

	Jacobian	**Analytical**
13 node test feeder	29.6msec	12.1msec
34 node test feeder	207.9msec	78.1msec

III. NUMERICAL VALIDATION

The numerical validation of the proposed method for the computation of voltage/current sensitivities is performed with two different approaches. In particular, as the inverse of the load flow Jacobian matrix provides the voltage sensitivities, the comparison reported below makes reference to such a method for the voltage sensitivity only.

On the contrary, as the inverse of the load flow Jacobian matrix does not provide current sensitivity coefficients, their accuracy is evaluated by using a numerical approach where

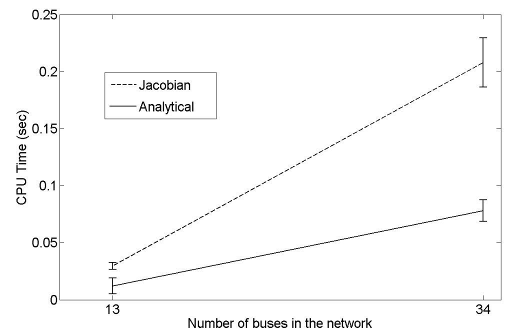


Figure 1. Mean CPU time necessary to calculate the voltage sensitivities as a function of the network size, the error bars show the relevant 95% confidence intervals.

the load flow problem is solved by applying small injection perturbations into a given network (see Section II-A).

Fig.2 shows the IEEE 13 nodes test feeder implemented in the EMTP-RV simulation environment [22], [23], [24] adopted to perform the multiphase load flow.

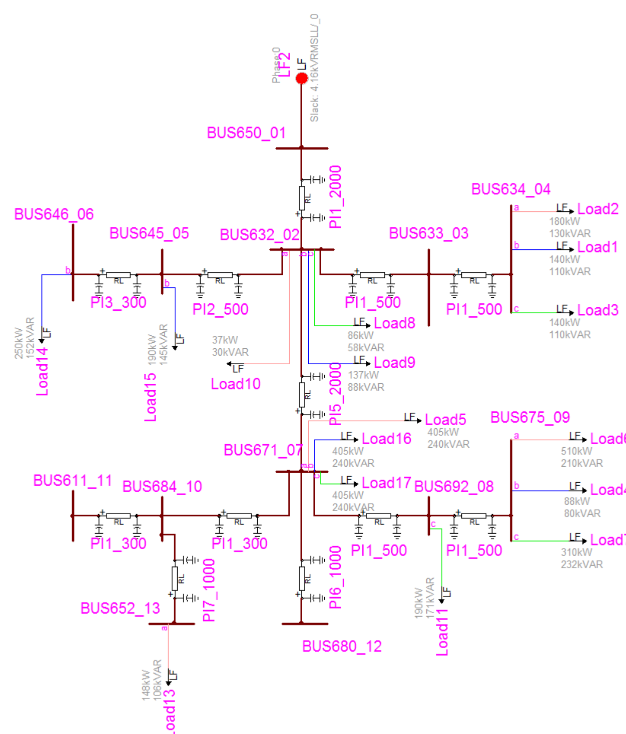
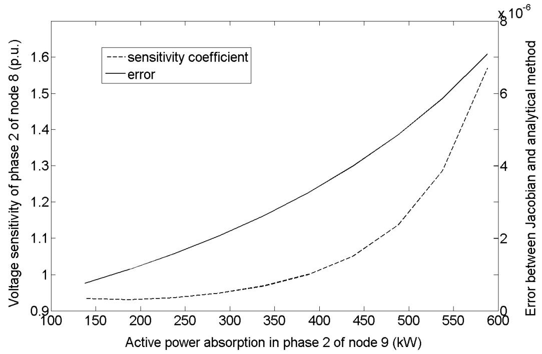


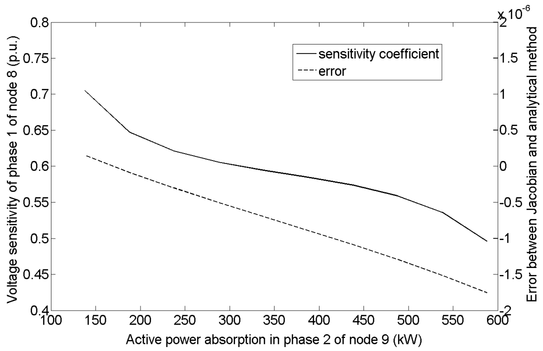
Figure 2. IEEE 13 node test feeder represented in the EMTP-RV simulation environment.

For the sake of brevity we limit the validation of the proposed method to a reduced number of busses exhibiting the largest voltage sensitivity against PQ load/injections. In particular, we refer to the variation of voltages at bus 8 with respect to load/injection in bus 9, i.e.

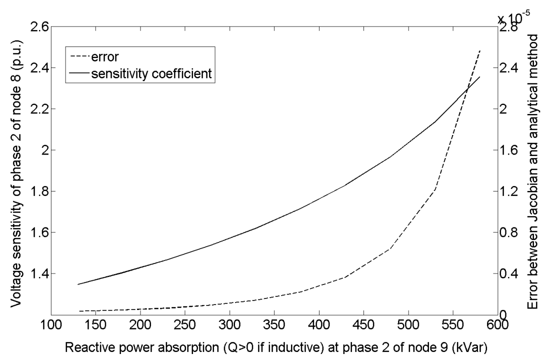
$$\frac{\partial |E_8^a|}{\partial P_9^b}, \frac{\partial |E_8^b|}{\partial P_9^b}, \frac{\partial |E_8^a|}{\partial Q_9^b}, \frac{\partial |E_8^b|}{\partial Q_9^b}$$



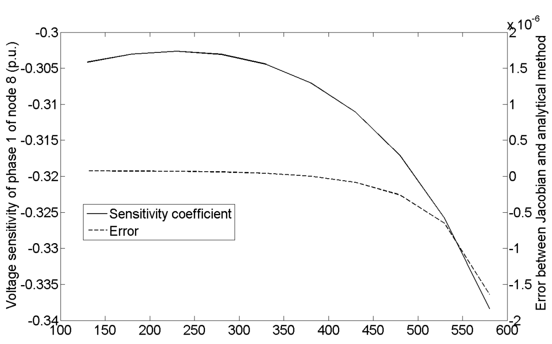
(a) Voltage sensitivity coefficient of phase a of bus 8 with respect to active power absorption at phase a of bus 9



(b) Voltage sensitivity coefficient of phase a of bus 8 with respect to active power absorption at phase b of bus 9

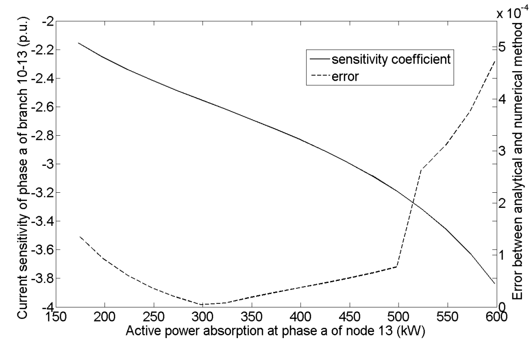


(c) Voltage sensitivity coefficient of phase a of bus 8 with respect to reactive power absorption at phase a of bus 9

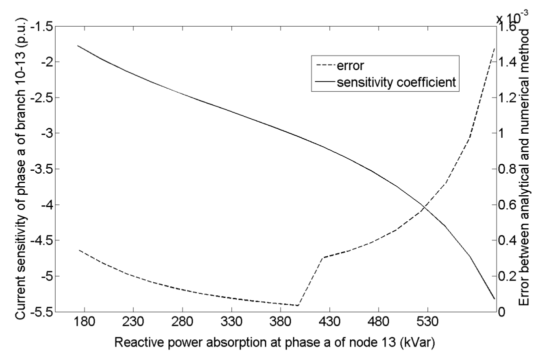


(d) Voltage sensitivity coefficient of phase a of bus 8 with respect to reactive power absorption at phase b of bus 9

Figure 3. Voltage sensitivity coefficient of phase a of bus 8 with respect to active and reactive power absorption at phase a and b of bus 9



(a) Current sensitivity coefficient of phase a of branch 10-13 with respect to active power absorption at phase a of bus 13



(b) Current sensitivity coefficient of phase a of branch 10-13 with respect to reactive power absorption at phase a of bus 13

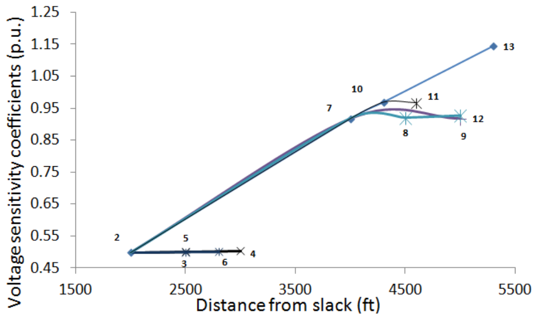
Figure 4. Current sensitivity coefficients with respect to power absorption at phase a of node 13

In Fig.3(a) the voltage sensitivity of phase a bus 8 is shown with respect to active power absorption at phase a of bus 9. Fig.3(b) shows for the same busses as Fig.3(a), the same sensitivity but referring to voltage and power belonging to different phases. Additionally, Fig. 3(c) and 3(d) show the voltage sensitivity of bus 8 with respect to reactive power absorption at bus 9. In all these four figures the dashed line represents the relative error between the traditional approach (i.e. based on the inverse of the Jacobian matrix) and the analytical method proposed here. As it can be observed, it is in the order of 10^{-6}.

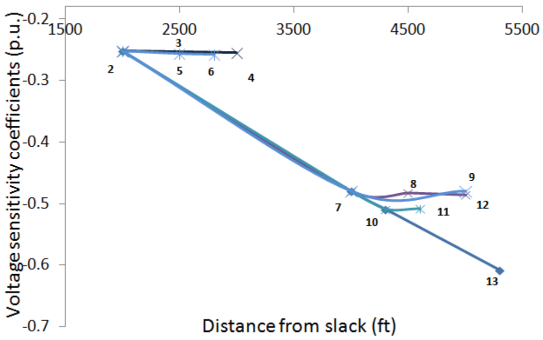
In Fig.4(a) and Fig.4(b) the current sensitivity coefficient of phase a of branch $10-13$ is presented with respect to active and reactive power absorption in phase a of bus 13. In the same figures, the dashed lines represent the relative error between the analytical values and the numerical ones. Even for these coefficients extremely low errors are obtained.

Finally, Fig.5 depicts the variation of voltage sensitivity coefficients in all the network with respect to active and reactive power absorption at phase a of node 13 as a function of the distance from the slack bus in feet.

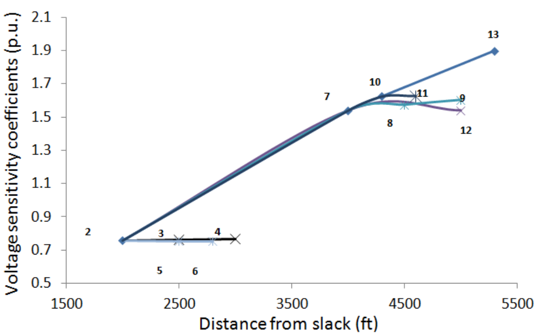
This type of representation allows to observe the overall network behavior against specific PQ nodes absorptions/injections. In particular, we can see that larger sensitivities are observed when the distance between the considered voltage and the slack bus increases. Furthermore, a lower, but quantified dependency between coefficients related to different



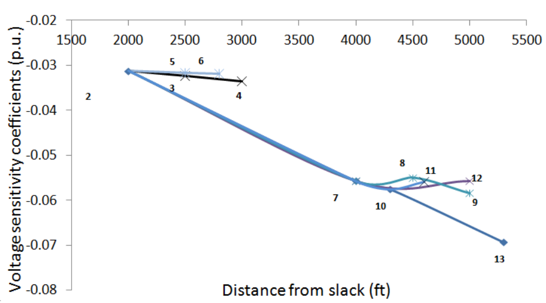
(a) Voltage sensitivity coefficients $\frac{\partial|\mathrm{E}_i^a|}{\partial P_{13}^a}$ with respect to active power absorption at phase a of node 13 as a function of the distance from the slack bus



(b) Voltage sensitivity coefficients $\frac{\partial|\mathrm{E}_i^b|}{\partial P_{13}^a}$ with respect to active power absorption at phase a of node 13 as a function of the distance from the slack bus



(c) Voltage sensitivity coefficients $\frac{\partial|\mathrm{E}_i^a|}{\partial Q_{13}^a}$ with respect to reactive power absorption at phase a of node 13 as a function of the distance from the slack bus



(d) Voltage sensitivity coefficients $\frac{\partial|\mathrm{E}_i^b|}{\partial Q_{13}^a}$ with respect to reactive power absorption at phase a of node 13 as a function of the distance from the slack bus

Figure 5. Voltage sensitivity coefficients with respect to power absorption at phase a of node 13 as a function of the distance from the slack bus

phases, can be observed. Also, as expected, reactive power has a larger influence on voltage variations although the active power exhibits a non negligible influence. From the operational point of view it is worth observing that, figures as Fig.5, provide to network operators an immediate view of the response of the electrical network against specific loads/injections that could also be used for closed loop control or contingency analysis.

IV. Application of the Proposed Procedure to the Problem of Optimal Voltage Control

For the application part, the IEEE 34 test node feeder is considered as depicted in Fig.6. In busses 18, 23, 24 and 33 we assume to have distributed energy resources that the Distribution Network Operator (DNO) can control in terms of active and reactive power. Their initial operating values, as well as their rated power outputs, are shown in Table II.

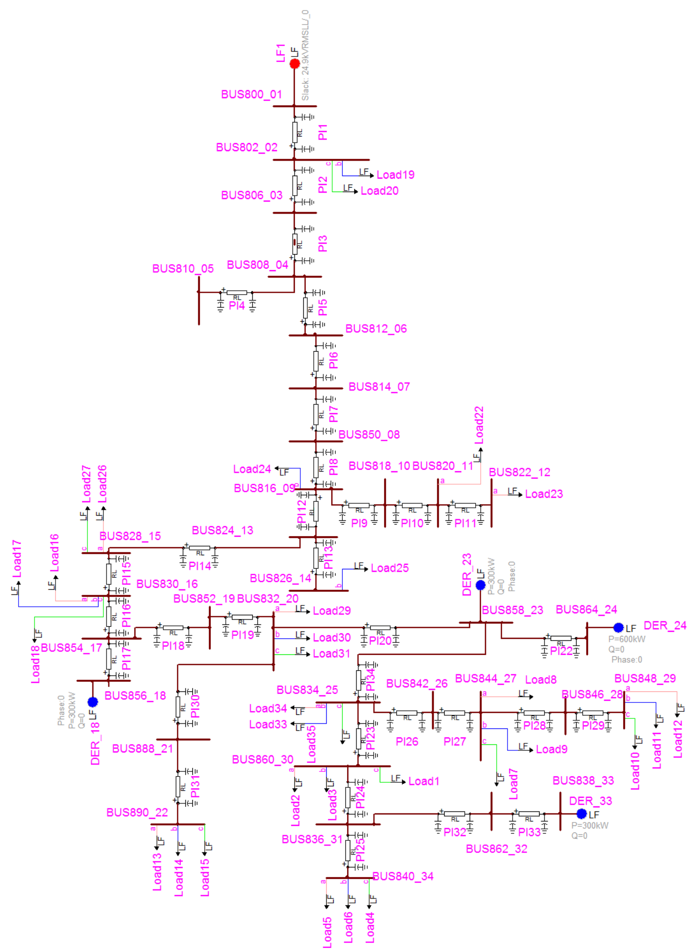


Figure 6. IEEE 34 node test feeder represented in the EMTP-RV simulation environment

Table II
Initial and maximum operational set points of the DERs in the 34 test node feeder

	Pinitial(kW)	Pmax(kW)
$\mathbf{DER_{18}}$	210	300
$\mathbf{DER_{23}}$	100	600
$\mathbf{DER_{24}}$	250	600
$\mathbf{DER_{33}}$	150	300

The optimal control problem is formulated as a linear one taking advantage of the voltage sensitivity coefficients. The controlled variables are the bus node voltages and the control variables are the active and reactive power injections of the DER under the control of the DNO, $\mathbf{\Delta x} = [\mathbf{\Delta P}_{DER}, \mathbf{\Delta Q}_{DER}]$. The objective of the linear optimization problem relevant to the problem is:

$$\min_{\mathbf{\Delta x}} \parallel \overline{E}_i - \overline{E} \parallel \tag{17}$$

The linearized relationship that links bus voltages with control variables is expressed in the following way (e.g. [4]):

$$\Delta|\overline{E}_i| = \mathbf{K_P}_i \mathbf{\Delta P}_i + \mathbf{K_Q}_i \mathbf{\Delta Q}_i \tag{18}$$

where $\mathbf{K_P}_i$ is the vector of sensitivity coefficients with respect to the active powers of the DERs and $\mathbf{K_Q}_i$ is the vector of sensitivity coefficients with respect to the reactive powers of the DERs. The imposed constraints on the operational points of the DERs are the following:

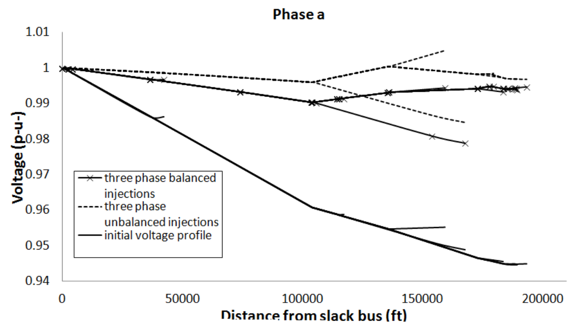
$$\begin{aligned} 0 \leq P_{DER_i} &\leq P_{DER_{i_{max}}} \\ Q_{DER_{i_{min}}} \leq Q_{DER_i} &\leq Q_{DER_{i_{max}}} \end{aligned} \tag{19}$$

In order to simplify the analysis, we have assumed that the DER capability curves are rectangular ones in the PQ plane.

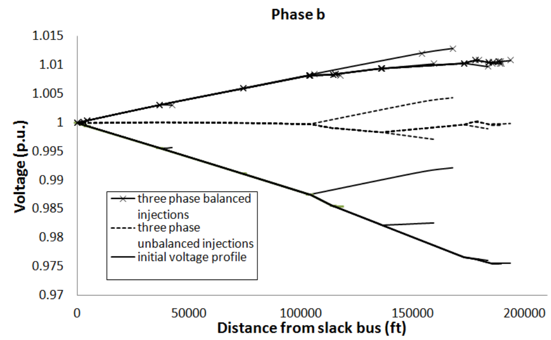
The formulated linearized problem is solved by using the linear least squares method. The method used to calculate analytically the sensitivity coefficient allows us to consider two different optimization scenarios. In the first (opt_1), the operator of the system is assumed to control the set points of the DERs considering that they are injecting equal powers into the three phases, whereas in the second case (opt_2) it is assumed to have a more sophisticated control on each of the phases independently. It is worth noting that this second option, although far from a realistic implementation, allows us to show the capability of the proposed method to deal with the inherent unbalanced nature of distribution networks. Table III and Table IV show the optimal operational set points corresponding to these cases. Additionally, in Fig.7 the voltage profile of the busses of the system is presented in the initial and the optimal cases. The solid line in the figures shows the initial voltage profile, the solid line with the markers the first case optimal scenario (opt_1) and the dashed line represents the second case where the DNO has full control in each of the phases of the DERS (opt_2). What can be observed is that, when the operator has control on each of the three phases of the DERs output, the optimal voltage profile is better than the one corresponding to control of the 3-phase output of the set points of the DERs.

Table III
OPTIMAL OPERATIONAL SET POINTS OF THE DERS IN THE 34 TEST NODE FEEDER WHEN THE SYSTEM OPERATOR HAS CONTROL ON THEIR 3-PHASE OUTPUT

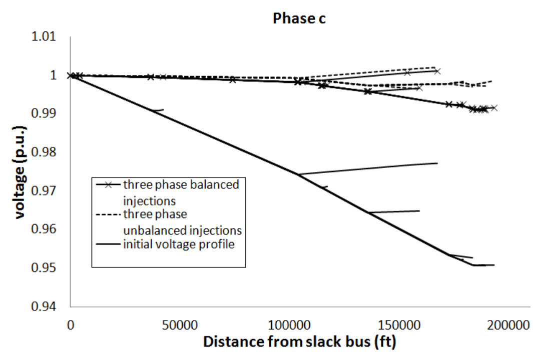
	$\mathbf{Popt_1(kW)}$	$\mathbf{Qopt_1(kVar)}$
$\mathbf{DER_{18}}$	300	120.66
$\mathbf{DER_{23}}$	600	600
$\mathbf{DER_{24}}$	600	19.5
$\mathbf{DER_{33}}$	300	228.48



(a) IEEE 34 node test feeder - Voltage profile of phase a of the busses



(b) IEEE 34 node test feeder - Voltage profile of phase b of the busses



(c) IEEE 34 node test feeder - Voltage profile of phase c of the busses

Figure 7. Initial and optimized voltage profile of the IEEE 34 node test feeder

Table IV
OPTIMAL OPERATIONAL SET POINTS OF THE DERS IN THE 34 TEST NODE FEEDER WHEN THE SYSTEM OPERATOR HAS CONTROL ON EACH OF THE THREE PHASES INDEPENDENTLY

	$\mathbf{Popt_2(kW)}$	$\mathbf{Qopt_2(kVar)}$
$\mathbf{DER^a_{18}}$	99.77	99.32
$\mathbf{DER^b_{18}}$	96.4	-83.43
$\mathbf{DER^c_{18}}$	0.01	77.2
$\mathbf{DER^a_{23}}$	196	137.77
$\mathbf{DER^b_{23}}$	162.88	199.97
$\mathbf{DER^c_{23}}$	146.93	153.94
$\mathbf{DER^a_{24}}$	178.33	130.28
$\mathbf{DER^b_{24}}$	111.66	200
$\mathbf{DER^c_{24}}$	114.23	150.10
$\mathbf{DER^a_{33}}$	92.17	-38.56
$\mathbf{DER^b_{33}}$	75.13	96.51
$\mathbf{DER^c_{33}}$	99.77	77.97

V. Conclusion

In this paper we have proposed a new method for the analytical computation of voltages and currents sensitivity coefficients as a function of the nodal power injections. The innovative aspects of the proposed method are the following: (i) it is based on the use of the $[\mathbf{Y}]$ compound matrix and (ii) it supports the computation of these sensitivities for a generic unbalanced electrical network and is thus suitable for distribution systems.

Compared to the traditional use of the jacobian load-flow matrix, it allows us to reduce the computation time by almost a factor of three, thus enabling in principle its implementation in real-time optimal controllers.

The paper has also validated the proposed method by making reference to typical IEEE 13 and 34 nodes distribution test feeders. The former has been used to numerically validate the computation of the coefficients whilst the latter has been used to show an application example related to a possible integration of the proposed method for the problem of optimal voltage control in unbalanced distribution systems.

It is worth observing that the analytical computation of voltages and currents sensitivities will reduce the computational time of several traditional power systems problems involving non-negligible computational efforts, such as: real-time centralized controls, contingency analysis or optimal planning.

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