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Hausdorff dimension of some groups acting on the binary tree

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Abstract. Based on the work of Abercrombie [1], Barnea and Shalev [4] gave an explicit formula for the Hausdorff dimension of a group acting on a rooted tree. We focus here on the binary tree $\mathscr{T}$. Abért and Virág [2] showed that there exist finitely generated (but not necessarily level-transitive) subgroups of $\operatorname{Aut}\mathscr{T}$ of arbitrary dimension in $[0, 1]$.

In this article we explicitly compute the Hausdorff dimension of the level-transitive spinal groups. We then give examples of 3-generated spinal groups which have transcendental Hausdorff dimension, and construct 2-generated groups whose Hausdorff dimension is 1.

1 Introduction

Although it is known [2] that finitely generated subgroups of $\operatorname{Aut}\mathscr{T}$ may have arbitrary Hausdorff dimension, there are only very few explicit computations in the literature. Further, all known examples have rational dimension, starting with the first Grigorchuk group, whose dimension is $5/8$, as we will see below. In this article we give two explicit constructions of finitely generated groups. We obtain groups of dimension 1 on the one hand, and groups whose dimension is transcendental on the other hand. This is achieved by computing the dimension of the so-called spinal groups acting on the binary tree, which are generalizations of the Grigorchuk groups.

We begin by recalling the definition of Hausdorff dimension in the case of groups acting on the binary tree. In Section 3, we define the spinal groups we are interested in. Section 4 is devoted to the statement of Theorem 4.4, which gives a formula for computing the Hausdorff dimension of any spinal group acting level-transitively on the binary tree; the proof is deferred to Section 7. Finally, the construction of 3-generated spinal groups of irrational Hausdorff dimension is given in Section 5, and groups with Hausdorff dimension 1 are exhibited in Section 6.

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2 Hausdorff dimension

Let $\mathscr{T}$ be the infinite binary rooted tree, and let $\operatorname{Aut}\mathscr{T}$ denote its automorphism group. It is known [4] that the Hausdorff dimension of a closed subgroup G of $\operatorname{Aut}\mathscr{T}$ is

$$\dim_H G = \liminf_{m\to\infty} \frac{\log|G \bmod m|}{\log|\operatorname{Aut}\mathscr{T} \bmod m|},$$

where the '$\bmod m$' notation stands for the action on the first m levels of the tree (i.e. $(G \bmod m) = G/\operatorname{Stab}_G(m)$, where $\operatorname{Stab}_G(m)$ is the pointwise stabilizer of the mth level of the tree). Moreover, one easily computes that $|\operatorname{Aut}\mathscr{T} \bmod m| = 2^{2^m-1}$. This yields the more explicit formula

$$\dim_H G = \liminf_{m\to\infty} \frac{\log_2|G \bmod m|}{2^m}.$$

Below we will identify the vertices of $\mathscr{T}$ with the set of finite words over the alphabet $X = \{0, 1\}$. We recall that there is a canonical decomposition of the element $g \in \operatorname{Aut}\mathscr{T}$ as

$$g = \langle\!\langle g@0, g@1 \rangle\!\rangle \sigma,$$

with $g@x \in \operatorname{Aut}\mathscr{T}$ and $\sigma \in \operatorname{Sym}(X)$. We will often identify $\operatorname{Sym}(X)$ with C_2, the cyclic group of order 2, or with the additive group of $\mathbb{Z}/2\mathbb{Z}$, the finite field with 2 elements.

3 Spinal groups

We only deal here with a specific case of the more general definition of spinal groups which can be found in [5].

Before defining the spinal group G_ω, we need a *root group* A which we will always take to be $A = \langle a \rangle = C_2$, and a *level group* B, which will be the n-fold direct power of C_2. We think of B as an n-dimensional vector space over $\mathbb{Z}/2\mathbb{Z}$. Let $\omega = \omega_1\omega_2\ldots$ be a fixed infinite sequence of non-trivial elements from B^*, the dual space of B, and let Ω denote the set of such sequences. We let the non-trivial element a of A act on $\mathscr{T}$ by exchanging the two maximal subtrees. Next, we let each element $b \in B$ act via the recursive formulæ $b = \langle\!\langle \omega_1(b), b@1 \rangle\!\rangle$ and $b@1^k = \langle\!\langle \omega_{k+1}(b), b@1^{k+1} \rangle\!\rangle$. The *spinal group* G_ω is the group generated by $A \cup B$. Note that we require each ω_i to be non-trivial, and this implies that G_ω is level-transitive (i.e. G_ω acts transitively on X^n for all $n \in \mathbb{N}$).

The *syllable form* of ω is $\omega_1^{a_1}\omega_2^{a_2}\ldots$ where $\omega_i \neq \omega_{i+1}$ and the exponents a_i denote multiplicities. In contrast to this we say that $\omega = \omega_1\omega_2\ldots$ is in *developed form*. We designate by s_k the sum of the k first terms of the sequence (a_i), i.e. the length of the k-syllable prefix of ω.

4 Main theorem

Before stating the main theorem, we need a few more definitions.

Definition 4.1. The *shift* $\sigma : \Omega \to \Omega$ is defined on a sequence $\omega = \omega_1\omega_2\dots$ by $\sigma\omega = \omega_2\omega_3\dots$.

Definition 4.2. For a positive integer m and a sequence $\omega = \omega_1^{a_1}\omega_2^{a_2}\dots$ in syllable form, we define $s^{-1}(m)$ as the unique integer that satisfies $s_{s^{-1}(m)-1} + 1 < m \leqslant s_{s^{-1}(m)} + 1$.

In other words, $s^{-1}(m)$ is the number of syllables of the prefix of ω of length $m-1$. If we write $\omega = \omega_1^{a_1}\omega_2^{a_2}\dots$ in syllable form, then $\{\omega_1, \dots, \omega_{s^{-1}(m)}\}$ is the set of elements of B^* which appear in $(G \bmod m)$.

Definition 4.3. Let $\omega = \omega_1^{a_1}\omega_2^{a_2}\dots$ be in syllable form. Given $m \in \mathbb{N}$, we define $\dim(\omega \bmod m)$ as the dimension of the vector space spanned by $\omega_1, \dots, \omega_{s^{-1}(m)}$. We also define $\underline{\dim}(\omega)$ by

$$\underline{\dim}(\omega) = \liminf_{n\to\infty}\left\{\lim_{m\to\infty}\dim(\sigma^n\omega \bmod m)\right\}.$$

The next theorem expresses the Hausdorff dimension of any level-transitive spinal group acting on the binary tree. Its proof is given in Section 7.

Theorem 4.4. *Consider $\omega = \omega_1^{a_1}\omega_2^{a_2}\dots$ in syllable form with $\underline{\dim}(\omega) = n \geqslant 2$. The Hausdorff dimension of G_ω is given by*

$$\dim_H G_\omega = \frac{1}{2}\liminf_{k\to\infty}\left(\frac{\Sigma_k}{2^{s_k}} + \frac{1}{2^{s_k}}\sum_{i=2}^{n-1}2^{s_{\lambda_i}}\left(2 - \frac{1}{2^{a_{\lambda_i}}}\right) + \frac{1}{2^{a_k}}\left(1 - \frac{1}{2^{a_{k-1}}}\right)\right),$$

where $\Sigma_k = \sum_{j=1}^{k} 2^{s_{j-1}}a_j$, and for each $i \in \{2, \dots, n-1\}$ we let $\lambda_i(k)$ be the smallest integer such that

$$\dim(\sigma^{s_{\lambda_i(k)}}\omega \bmod (s_{k+1} - s_{\lambda_i(k)})) = i.$$

Remark 4.5. In the case when the sequence ω is eventually constant, i.e. when $\underline{\dim}(\omega) = 1$, then Proposition 7.2 can be used to show that $\log_2|G_\omega \bmod m|$ grows linearly with m, whence $\dim_H G_\omega = 0$.

In the special case where $B = C_2 \times C_2$ and ω is not eventually constant, one can use the following corollary to compute $\dim_H G_\omega$:

Corollary 4.6. *If $\underline{\dim}(\omega) = 2$, then*

$$\dim_H G_\omega = \frac{1}{2}\liminf_{k\to\infty}\left(\frac{\Sigma_k}{2^{s_k}} + \frac{1}{2^{a_k}}\left(1 - \frac{1}{2^{a_{k-1}}}\right)\right).$$

Example 4.7. Consider $B = C_2 \times C_2$. Let ω_1, ω_2, ω_3 be the three non-trivial elements of B^*, and consider $\omega = \omega_1\omega_2\omega_3\omega_1\omega_2\omega_3\ldots$. The group G_ω is the 'first Grigorchuk group', first introduced in [6] (see also [3]). Since the corresponding integer sequence is given by $a_k = 1$ for all k, then $s_k = k$ and $\Sigma_k = 2^k - 1$. The last corollary yields

$$\dim_H G_\omega = \tfrac{5}{8}.$$

Example 4.8. The Hausdorff dimension of some spinal groups is computed in [7]. Consider $B = C_2^n$, and fix a functional $\phi \in B^*$ and an automorphism ρ (i.e. an invertible linear transformation) of B. We consider the sequence $\omega = \omega_1\omega_2\ldots$ defined by $\omega_1 = \phi$ and $\omega_n = \rho^*(\omega_{n-1})$ for $n > 1$, where ρ^* denotes the adjoint automorphism of ρ. We restrict attention to the case where ρ and ϕ are such that $\underline{\dim}(\omega) = n$ (this is a rephrasing of the condition in [7, Proposition 2]). This implies that every sequence of n consecutive terms of ω generates B^*. Each syllable of ω has length 1, and so we can apply Theorem 4.4 with $a_i = 1$ and $s_i = i$ for all i, and $\lambda_i(k) = k - i$ for $i \in \{2, \ldots, n-1\}$ and $k \geqslant n$. This yields

$$\dim_H G_\omega = 1 - \frac{3}{2^{n+1}},$$

as was found in [7].

5 Finitely generated groups of irrational Hausdorff dimension

Throughout this section we restrict to the case $\underline{\dim}(\omega) = 2$. Let $D \subset [0,1]$ be the set of possible Hausdorff dimensions for groups G_ω. More precisely,

$$D = \{\lambda \in \mathbb{R} \mid \exists \omega \in \Omega \text{ with } \underline{\dim}(\omega) = 2 \text{ and } \dim_H G_\omega = \lambda\}.$$

Although it is not easy to determine which numbers lie in D, we are able to show the following. Let C denote the Cantor set constructed by removing the second quarter of the unit interval, and iterating this process on the obtained intervals. Thus C is compact and totally disconnected, and contains transcendental elements.

Theorem 5.1. *The set D contains several copies of C, each being the image of C under an affine map with rational coefficients.*

Corollary 5.2. *The set D contains transcendental elements.*

Proof of Theorem 5.1. We first define the functions

$$f_{a,n} : \mathbb{R} \to \mathbb{R}, \quad x \mapsto \frac{x}{2^a} + \frac{a+n}{2^a}.$$

We will only consider the case where a is a strictly positive integer and $n \in \mathbb{Z}$, and we will simply write f_a instead of $f_{a,0}$.

Consider $\omega = \omega_1^{a_1}\omega_2^{a_2}\dots$. Starting from Corollary 4.6, we can see that

$$\begin{aligned}\dim_H G_\omega &= \frac{1}{2}\liminf_{k\to\infty}\left(\frac{\Sigma_k}{2^{s_k}} + \frac{1}{2^{a_k}}\left(1 - \frac{1}{2^{a_{k-1}}}\right)\right)\\ &= \frac{1}{2}\liminf_{k\to\infty}(f_{a_k,1}\circ f_{a_{k-1},-1}\circ f_{a_{k-2}}\circ\cdots\circ f_{a_1})(0). \qquad (1)\end{aligned}$$

Observe that the functions f_1 and f_2 define an iterated function system whose invariant set C' is the image of C under the map $x\mapsto \frac{1}{3}(x+2)$. Indeed $x_1 = 1$ and $x_2 = \frac{2}{3}$ are the fixed points of f_1 and f_2 respectively. Write $\Delta = x_1 - x_2$. Then $f_1([x_2, x_2+\Delta]) = \left[x_2 + \frac{1}{2}\Delta, x_2+\Delta\right]$ and $f_2([x_2, x_2+\Delta]) = \left[x_2, x_2+\frac{1}{4}\Delta\right]$.

Now fix a point $\hat{x}\in C'$. There exists a sequence $(b_i)\in\{1,2\}^{\mathbb{N}}$ such that

$$\hat{x} = \lim_{k\to\infty}(f_{b_0}\circ\cdots\circ f_{b_k})(y) \qquad (2)$$

for any point $y\in\mathbb{R}$. We call the sequence $(b_0, b_1,\dots)$ the *code* of $\hat{x}$. Notice that the main differences between (1) and (2) are the ordering of the factors, and the limit which is a lim inf in (1).

Fix $s\in\mathbb{N}$ with $s>2$. We define the sequence $(a_i)\in\{1,2,s\}^{\mathbb{N}}$ by

$$(a_1, a_2,\dots) = (b_0, s, b_1, b_0, s, b_2, b_1, b_0, s,\dots).$$

The sequence (a_i) thus consists of prefixes of the sequence (b_i) of increasing length, written backwards, and separated by s. We set $\omega = \omega_1^{a_1}\omega_2^{a_2}\omega_1^{a_3}\omega_2^{a_4}\dots$. We will show that $\dim_H G_\omega$ is the image of $\hat{x}$ under an affine map with rational coefficients. Recall that $\dim_H G_\omega$ is the lim inf of

$$\tfrac{1}{2}(f_{a_k,1}\circ f_{a_{k-1},-1}\circ f_{a_{k-2}}\circ\cdots\circ f_{a_1})(0). \qquad (3)$$

Notice that the maps $f_{a,n}$ are order-preserving, and observe that $(f_{\alpha,1}\circ f_{\beta,-1})(0)\geqslant\frac{3}{4}$ if $\alpha\in\{1,2\}$ and $b\geqslant 1$, while $(f_{\alpha,1}\circ f_{\beta,-1})(1) < \frac{3}{4}$ if $\alpha = s$ and $\beta\geqslant 1$. Therefore the lowest values in (3) are attained when $a_k = s$. We conclude that

$$\dim_H G_\omega = \frac{1}{2}\lim_{k\to\infty}(f_{s,1}\circ f_{b_0,-1}\circ f_{b_1}\circ\cdots\circ f_{b_k})(0) = \frac{1}{2}(f_{s,1}\circ f_{b_0,-1}\circ f_{b_0}^{-1})(\hat{x}).$$

It should be noted that the maps $f_{s,1}\circ f_{b_0,-1}\circ f_{b_0}^{-1}$ do not have disjoint images. Nevertheless, it can be checked that

$$f_{1,-1}\circ f_1 = f_{2,-1}.$$

This implies that a point $x\in C'$ whose code is $2b_1b_2\dots$ is mapped to the same point as the point whose code is $11b_1b_2\dots$. On the other hand, the set of points whose code starts with a 1 is just $f_1(C')$. Under the maps $f_{s,1}\circ f_{b_0,-1}\circ f_{b_0}^{-1}$, the set $f_1(C')$ is sent to $(f_{s,1} \circ f_{1,-1})(C')$, and the maps $f_{s,1} \circ f_{1,-1}$ are affine with rational coefficients and have disjoint images for all $s > 2$. Thus D contains a countable infinity of disjoint copies of C'. □

6 Construction of full-dimensional finitely generated groups

We begin with a few easy statements which will be useful.

Proposition 6.1. *Let $H \leqslant G$ be subgroups of* $\operatorname{Aut} X^*$. *Then* $\dim_H H \leqslant \dim_H G$. *Moreover,* $\dim_H H = \dim_H G$ *if the index of H in G is finite.*

Proof. $H \leqslant G$ implies $|H \bmod n| \leqslant |G \bmod n|$ for all $n \geqslant 0$. Thus $\dim_H H \leqslant \dim_H G$. Moreover, if $k = |G : H|$ is finite then $k|H \bmod n| \geqslant |G \bmod n|$ for all $n \geqslant 0$. This yields the second claim. □

Definition 6.2. Let H and G be subgroups of $\operatorname{Aut} X^*$ and n be a positive integer. We write $H^{2^n} \preceq G$ if G contains 2^n copies of H acting on the 2^n subtrees of level n.

Proposition 6.3. *Let H and G be subgroups of* $\operatorname{Aut} X^*$, *such that $H^{2^n} \preceq G$. Then* $\dim_H H \leqslant \dim_H G$.

Proof. It is straightforward to check that the hypothesis $H^{2^n} \preceq G$ implies that $|H \bmod (m-n)|^{2^n} \leqslant |G \bmod m|$ for $m \geqslant n$. The conclusion follows. □

We now turn to the construction of a full-dimensional group. Let $a_1 = \sigma \in \operatorname{Aut} X^*$ be the permutation exchanging the two maximal subtrees and let a_n be defined recursively as

$$a_n = \langle\!\langle 1, a_{n-1} \rangle\!\rangle \sigma.$$

It is easy to see that a_n is of order 2^n. It can be viewed as a finite-depth version of the familiar adding machine $t = \langle\!\langle 1, t \rangle\!\rangle \sigma$. The important thing is that a_n acts as a full cycle on the nth level vertices of the tree, but has no activity below the nth level.

Next, for any element $g \in \operatorname{Aut} X^*$ and any word $w \in X^*$, we define $w * g \in \operatorname{Aut} X^*$ as the element which acts as g on the subtree wX^*, and trivially everywhere else. The following identity can be checked directly:

$$(w * g)^h = w^h * g^{(h@w)}.$$

Let $g_1, \ldots, g_n$ be any elements in $\operatorname{Aut} X^*$. We define the element

$$\delta(g_1, \ldots, g_n) = \prod_{i=0}^{n-1} (1^i 0^{n-i}) * g_{i+1}.$$

Notice that $1^i 0^{n-i} = (1^n)^{(a_n)^{2^i}}$, and that the product above can be taken in any order as the elements all commute (since they act non-trivially on different subtrees).

Lemma 6.4. *Consider* $G = \langle g_1, \dots, g_n \rangle$ *and* $H = \langle a_n, \delta(g_1, \dots, g_n) \rangle$. *Then* $(G')^{2^n} \preceq H'$.

Proof. The following equalities are immediate consequences of the definitions:

$$\delta(g_1, \dots, g_n)^{(a_n)^k} = \prod_{i=0}^{n-1} (1^n)^{(a_n)^{2^i+k}} * g_{i+1};$$

$$[\delta(g_1, \dots, g_n), a_n^{2^i - 2^j}][\delta(g_1, \dots, g_n)^{-1}, a_n^{2^i - 2^j}] = (1^n)^{(a_n)^{2^i}} * [g_{j+1}, g_{i+1}]^{g_{j+1}^{-1}}.$$

These relations hold for all $i, j \in \{0, \dots, n-1\}$ and all positive integer k. These two equalities imply that

$$w * [g_i, g_j] \in H'$$

for all $w \in X^n$ and $i, j \in \{1, \dots, n\}$. This in turn implies that $(G')^{2^n} \preceq H'$. $\square$

Let $B_n = C_2^n$ be the direct product of n copies of C_2. Let $\omega_1, \dots, \omega_n$ be a basis of B_n^* and consider the spinal group G_n defined through the sequence $\omega_1 \dots \omega_n \omega_1 \dots \omega_n \dots$. In other words, $G_n = \langle a, b_{(1,n)}, \dots, b_{(n,n)} \rangle$, with

$$\begin{aligned} a &= a_1 = \sigma, \\ b_{(i,n)} &= \langle\!\langle 1, b_{(i+1,n)} \rangle\!\rangle \quad \text{for } i = 1, \dots, n-1, \\ b_{(n,n)} &= \langle\!\langle a, b_{(1,n)} \rangle\!\rangle. \end{aligned}$$

It follows from Corollary 4.6 that $\dim_H G_n = 1 - 3/2^{n+1}$. Define the elements

$$\tilde{b}_n = \delta(a, b_{(1,n-3)}, \dots, b_{(n-3,n-3)}, a_{n+1}, \tilde{b}_{n+1})$$

for $n \geqslant 3$, and write $H_n = \langle a_n, \tilde{b}_n \rangle$.

Theorem 6.5. H_n *has Hausdorff dimension equal to* 1, *for all* $n \geqslant 3$.

Proof. Lemma 6.4 yields that

$$(H'_{m+1})^{2^m} \preceq H'_m, \quad (G'_{m-3})^{2^m} \preceq H'_m,$$

for all $m \geqslant 3$. Thus $\dim_H H'_m \geqslant \dim_H G'_{m-3}$ by Proposition 6.3, and

$$\dim_H H'_n \geqslant \dim_H H'_m \quad \text{for } 3 \leqslant n \leqslant m.$$

Proposition 6.1 allows us to state that

$$\dim_H H_n \geqslant \dim_H H'_n \geqslant \dim_H G'_m = \dim_H G_m$$

for $3 \leqslant n \leqslant m+3$. The last equality holds because G_m is generated by m elements of order 2, so $|G_m : G'_m|$ is finite. This yields that $\dim_H H_n = 1$ for all $n \geqslant 3$, since $\dim_H G_m = 1 - 3/2^{m+1}$. □

Remark 6.6. We could easily extend this construction by taking any sequence of finitely generated groups G_n such that $\limsup(\dim_H G'_n) = 1$.

7 Proof of the main theorem

The remainder of the article is devoted to the proof of Theorem 4.4. Our first goal is to find a recursive formula for $|G_\omega \bmod m|$. We begin with a few simple but very useful lemmata.

Let $\pi : \operatorname{Aut}\mathcal{T} \to (\operatorname{Aut}\mathcal{T} \bmod 1) \cong C_2$ be the natural epimorphism. The following lemma is folklore.

Lemma 7.1. *The map* $\phi_n : \operatorname{Aut}\mathcal{T} \to C_2$ *given by* $g \mapsto \prod_{w\in X^n} \pi(g@w)$ *is an epimorphism for all* $n \in \mathbb{N}$.

For a group $G \leqslant \operatorname{Aut}\mathcal{T}$, we let $\operatorname{Stab}_G(n)$ be the subgroup of G consisting of the elements that fix the first n levels of the tree. When $G = \operatorname{Aut}\mathcal{T}$ we simply write $\operatorname{Stab}(n)$. For $v \in X^m$, we define $v\phi_n : \operatorname{Stab}(m) \to C_2$ by $v\phi_n(g) = \prod_{w\in X^n} \pi(g@(vw))$.

Corollary 7.2. *The map* $v\phi_n : \operatorname{Stab}(m) \to C_2$ *is an epimorphism for all* $v \in X^m$ *and* $m, n \in \mathbb{N}$.

Proof. This is straightforward because $\operatorname{Stab}(m) \cong (\operatorname{Aut}\mathcal{T})^{X^m}$. □

In the following we let $\langle \omega \bmod m\rangle$ (resp. $\langle\omega\rangle$) designate the vector space spanned by $\omega_1, \dots, \omega_{s^{-1}(m)}$ (resp. $\omega_1, \omega_2, \dots$).

Consider $\psi \in \langle\omega\rangle$ and $x \in X$. We define homomorphisms $\bar{\psi}^x : \operatorname{Stab}_{G_\omega}(1) \to C_2$ as follows. Write $\psi = \omega_{i_1} + \dots + \omega_{i_k}$ where $\omega_{i_1}, \dots, \omega_{i_k}$ are pairwise distinct and each of them appears at least once in ω. Let $n_j + 1$ be the position of the first occurrence of ω_{i_j} in ω. If $n_j > 0$ for all j then we define $\bar{\psi}^x = \sum_{j=1}^k x\phi_{n_j}$ (we implicitly identify C_2 with $\mathbb{Z}/2\mathbb{Z}$). Otherwise if $n_1 = 0$ we set $\bar{\psi}^x = \bar{x}\phi_0 + \sum_{j=2}^k x\phi_{n_j}$, with $\bar{x} = 1 - x$.

By construction, $\bar{\psi}^x$ is a homomorphism. The following lemma is less obvious. (Note that $\operatorname{Stab}_{G_\omega}(1)$ is generated by $\{b, b^a : b \in B\}$.)

Lemma 7.3. *Suppose that* $\psi \in \langle\omega\rangle$. *Then the map* $\bar{\psi}^0 : \operatorname{Stab}_{G_\omega}(1) \to C_2$ *(resp.* $\bar{\psi}^1$*) is the homomorphism induced by* $b \mapsto \psi(b)$ *and* $b^a \mapsto 1$ *(resp.* $b \mapsto 1$ *and* $b^a \mapsto \psi(b)$*) for* $b \in B$. *In particular,* $\bar{\psi}^x$ *is surjective.*

Proof. This is easy to check in the case when ψ appears in the sequence ω. The general case is just a linear combination of the terms of ω. □

Remark 7.4. One can define $\bar{\psi}^x : (\mathrm{Stab}_{G_\omega}(1) \bmod m) \to C_2$ for any $\psi \in \langle \omega \bmod m \rangle$ in the same way and the lemma still holds.

To proceed further we need to define some specific subgroups of G_ω. Let ψ be an element of B^*. We define the subgroup $T_\omega(\psi) = \langle \ker \psi \rangle^{G_\omega}$, where the superscript designates normal closure in G_ω. It should be noted that $T_\omega(\psi) \leqslant \mathrm{Stab}_{G_\omega}(1)$ for every $\psi \in B^*$. We now state and prove two technical lemmata, which lead to Proposition 7.7.

Lemma 7.5. *Let $\omega = \omega_1^{a_1}\omega_2^{a_2}\ldots$ be in syllable form, let $m \geqslant 1$ be an integer and let ψ_0 be a non-trivial element in B^* such that $\psi_0 \neq \omega_1$. Then*

$$\log_2|(G_\omega \bmod m)/(T_\omega(\psi_0) \bmod m)| = \begin{cases} 1 & \text{if } m \leqslant s_k + 1, \\ 3 & \text{if } m > s_k + 1, \end{cases}$$

where k is the greatest integer such that ψ_0 is linearly independent from $\omega_1, \ldots, \omega_k$.

Proof. First suppose that $m \leqslant s_k + 1$. Let $\{\psi_1, \ldots, \psi_\lambda\}$ be a basis of $\langle \omega \bmod m \rangle$. Since $\psi_0 \notin \langle \omega \bmod m \rangle$, the set $\{\psi_0, \ldots, \psi_\lambda\}$ is a basis of some subspace of B^*. Let $\{b_0, \ldots, b_\lambda\} \subset B$ be a dual basis, i.e. $b_0, \ldots, b_\lambda$ satisfy $\psi_i(b_j) = \delta_{ij}$ for all $i, j \in \{0, \ldots, \lambda\}$.

We have $b_1, \ldots, b_\lambda \in \ker \psi_0$, and so $b_1, \ldots, b_\lambda \in T_\omega(\psi_0)$. Since $(\mathrm{Stab}_{G_\omega}(1) \bmod m)$ is generated as a normal subgroup by the images of $b_1, \ldots, b_\lambda$, we have

$$(T_\omega(\psi_0) \bmod m) = (\mathrm{Stab}_{G_\omega}(1) \bmod m).$$

Therefore $(G_\omega \bmod m)/(T_\omega(\psi_0) \bmod m) = C_2$.

Now suppose that $m > s_k + 1$. Let $\{\psi_0, \ldots, \psi_\lambda\}$ be a basis of $\langle \omega \rangle$ and let $\{b_0, \ldots, b_\lambda\}$ be a dual basis. Write $H = \mathrm{Stab}_{G_\omega}(1) = \langle b_0, \ldots, b_\lambda \rangle^{G_\omega}$. Then obviously $(G_\omega \bmod m)/(H \bmod m) = C_2$.

We now prove that $(H \bmod m)/(T_\omega(\psi_0) \bmod m) = C_2 \times C_2$. This group is generated by the images of b_0 and b_0^a. A straightforward computation shows that $b_0^2 = [b_0, b_0^a] = 1$. Therefore $(H \bmod m)/(T_\omega(\psi_0) \bmod m)$ is a quotient of $C_2 \times C_2$. Consider the map $\Psi : g \mapsto (\overline{\psi_0}^0(g), \overline{\psi_0}^1(g))$. It is a surjective group homomorphism $(H \bmod m) \to C_2 \times C_2$, but $\Psi(T_\omega(\psi_0))$ is trivial. Therefore $\Psi(T_\omega(\psi_0) \bmod m)$ has index 4 in $\Psi(H \bmod m)$. This finishes the proof. □

Lemma 7.6. *Let $\omega = \omega_1^{a_1}\omega_2^{a_2}\ldots$ be in syllable form and set $m \geqslant 1$. Then*

$$\log_2|(G_\omega \bmod m)/(T_\omega(\omega_1) \bmod m)| = \begin{cases} 1 & \text{if } m = 1, \\ m + 1 & \text{if } 1 < m \leqslant a_1 + 1, \\ a_1 + 2 & \text{if } a_1 + 1 < m \leqslant s_k + 1, \\ a_1 + 3 & \text{if } s_k + 1 < m, \end{cases}$$

where k is the greatest integer such that ω_1 is linearly independent from $\omega_2, \ldots, \omega_k$.

Proof. The case $m = 1$ is very simple because $(T_\omega(\omega_1) \bmod 1)$ is the trivial group and $(G_\omega \bmod 1) = C_2$.

Define $H = \mathrm{Stab}_{G_\omega}(a_1 + 1)$. If $1 < m \leqslant a_1 + 1$, since $T_\omega(\omega_1) \leqslant H$, we know that $(T_\omega(\omega_1) \bmod m)$ is trivial. It is clear that $(G_\omega \bmod m)$ is isomorphic to a dihedral group of order 2^{m+1} because $(G_\omega \bmod m)$ is generated by two involutions a and b_1, and ab_1 has order 2^m in $(G_\omega \bmod m)$.

Suppose now that $a_1 + 1 < m \leqslant s_k + 1$. Let $\{\omega_{i_1}, \dots, \omega_{i_\lambda}\}$ be a basis of $\langle \omega \rangle$, with $\omega_{i_1} = \omega_1$. Let $\{b_1, \dots, b_\lambda\}$ be a dual basis. It is readily checked that the group $\langle a, b_1 \rangle$ is dihedral of order 2^{a_1+3}, and a straightforward computation shows that $a^2 = b_1^2 = 1$ and $(ab_1)^{2^{a_1+1}} \in \mathrm{Stab}_{G_\omega}(s_k + 1)$. Since $(G_\omega \bmod m)/(T_\omega(\omega_1) \bmod m)$ is generated by the images of a and b_1, we conclude that this group is dihedral of order 2^{a_1+2}.

Finally, if $s_k + 1 < m$, it is sufficient to prove that

$$(H \bmod m)/(T_\omega(\omega_1) \bmod m) \cong C_2.$$

By the above we know that this group is generated by the image of $(ab_1)^{2^{a_1+1}}$, and that this element is of order 2. Hence $(H \bmod m)/(T_\omega(\omega_1) \bmod m)$ is a quotient of C_2. Express ω_1 as a linear combination of $\omega_2, \dots, \omega_{k+1}$: say $\omega_1 = \omega_{i_1} + \cdots + \omega_{i_\lambda}$. Let n_j be the position of the last occurence of ω_{i_j} in $(\omega \bmod m)$. Fix $v = 0^{a_1+1}$ and define $\overline{\omega_1} : (H \bmod m) \to C_2$ by $\overline{\omega_1} = \sum_{j=1}^{\lambda} v\phi_{n_j}$. Then $\overline{\omega_1}$ is surjective but $\overline{\omega_1}(T_\omega(\omega_1) \bmod m)$ is trivial. This completes the proof. □

Proposition 7.7. *Let $\omega = \omega_1^{a_1}\omega_2^{a_2}\dots$ be in syllable form and set $m > a_1$. Then*

$$\log_2|G_\omega \bmod m| = 2 + a_1 + \delta(m) + 2^{a_1}(\log_2|G_{\sigma^{a_1}\omega} \bmod (m - a_1)| - 2\delta(m) - 1),$$

with

$$\delta(m) = \begin{cases} 0 & \text{if } \omega_1 \text{ is linearly independent from } \omega_2, \dots, \omega_{s^{-1}(m)}, \\ 1 & \text{otherwise.} \end{cases}$$

Proof. For $m > a_1$, Lemma 7.6 gives

$$\log_2|(G_\omega \bmod m)/(T_\omega(\omega_1) \bmod m)| = \begin{cases} a_1 + 2 & \text{if } m \leqslant s_k + 1, \\ a_1 + 3 & \text{if } m > s_k + 1, \end{cases}$$

where k is the greatest integer such that ω_1 is linearly independent from $\omega_2, \dots, \omega_k$. We can rewrite this equation as

$$\log_2|G_\omega \bmod m| = a_1 + 2 + \delta(m) + \log_2|T_\omega(\omega_1) \bmod m|. \tag{4}$$

Next, iteration of the relation $T_\omega(\omega_1) = T_{\sigma\omega}(\omega_1) \times T_{\sigma\omega}(\omega_1)$ gives

$$T_\omega(\omega_1) = \underbrace{T_{\sigma^{a_1}\omega}(\omega_1) \times \cdots \times T_{\sigma^{a_1}\omega}(\omega_1)}_{2^{a_1}}.$$

Therefore

$$\log_2|T_\omega(\omega_1) \bmod m| = 2^{a_1}\log_2|T_{\sigma^{a_1}\omega}(\omega_1) \bmod (m-a_1)|. \tag{5}$$

But Lemma 7.5 yields

$$\log_2|(G_{\sigma^{a_1}\omega} \bmod (m-a_1))/(T_{\sigma^{a_1}\omega}(\omega_1) \bmod (m-a_1))| = \begin{cases} 1 & \text{if } m \leqslant s_{k'}+1, \\ 3 & \text{if } m > s_{k'}+1, \end{cases}$$

where k' is the greatest integer such that ω_1 is linearly independent from $\omega_2, \ldots, \omega_{k'}$, i.e. $k = k'$. Therefore we can rewrite the preceding equation as

$$\log_2|G_{\sigma^{a_1}\omega} \bmod (m-a_1)| = 1 + 2\delta(m) + \log_2|T_{\sigma^{a_1}\omega}(\omega_1) \bmod (m-a_1)|. \tag{6}$$

Equations (4), (5) and (6) give the result. □

Now the technical part is over, and the following statements and their proof, including the proof of Theorem 4.4, are easy consequences of what has been shown above.

Proposition 7.8. *Let $\omega = \omega_1^{a_1}\omega_2^{a_2}\ldots$ be in syllable form and consider $\lambda \in \mathbb{N}$ and $m > s_\lambda$. Then*

$$\begin{aligned}\log_2|G_\omega \bmod m| = 3 &+ 2^{s_0}(1 + a_1 + \delta_1 - 2\delta_0) + \cdots + 2^{s_{\lambda-1}}(1 + a_\lambda + \delta_\lambda - 2\delta_{\lambda-1}) \\ &+ 2^{s_\lambda}(\log_2|G_{\sigma^{s_\lambda}\omega} \bmod (m-s_\lambda)| - 2\delta_\lambda - 1),\end{aligned}$$

with

$$\delta_j = \delta_j(m) = \begin{cases} 0 & \textit{if } \omega_j \textit{ is linearly independent from } \omega_{j+1}, \ldots, \omega_{s^{-1}(m)}, \\ 1 & \textit{otherwise}. \end{cases}$$

(*We set $\delta_0 = 1$ and $s_0 = 0$.*)

Proof. This follows directly from λ applications of Proposition 7.7. □

Corollary 7.9. *Let λ be such that the space spanned by $\omega_{\lambda+1}, \ldots, \omega_{s^{-1}(m)}$ contains all elements ω_j with $1 \leqslant j \leqslant s^{-1}(m)$. Then*

$$\log_2|G_\omega \bmod m| = 3 + \Sigma_\lambda + 2^{s_\lambda}(\log_2|G_{\sigma^{s_\lambda}\omega} \bmod (m-s_\lambda)| - 3),$$

with

$$\Sigma_\lambda = 2^{s_0}a_1 + \cdots + 2^{s_{\lambda-1}}a_\lambda.$$

Lemma 7.10. *Let* $\omega = \omega_1^{a_1}\omega_2^{a_2}\dots$ *be in syllable form. Given* $m > a_1 + 1$, *if* λ *is the smallest integer such that* ω_λ *is linearly independent from* $\omega_{\lambda+1}, \dots, \omega_{s^{-1}(m)}$, *then*

$$\log_2|G_\omega \bmod m| = 3 + \Sigma_\lambda + 2^{s_\lambda+1} - 2^{s_{\lambda-1}} + 2^{s_\lambda}(\log_2|G_{\sigma^{s_\lambda}\omega} \bmod (m - s_\lambda)| - 3).$$

Proof. This is just a consequence of Proposition 7.8 and Corollary 7.9. □

Remark 7.11. *If* $1 < m \leqslant a_1 + 1$ *then* $(G_\omega \bmod m)$ *is just a dihedral group of order* $m + 1$, *whence*

$$\log_2|G_\omega \bmod m| = m + 1.$$

We are naturally led to the following proposition.

Proposition 7.12. *Consider* $\omega = \omega_1^{a_1}\omega_2^{a_2}\dots$ *in syllable form with* $\dim(\omega \bmod m) = n$. *Then*

$$\log_2|G_\omega \bmod m| = 3 + \Sigma_{s^{-1}(m)-1} + \sum_{i=2}^{n-1}(2^{s_{\lambda_i}+1} - 2^{s_{\lambda_i-1}}) - 2^{s_{\lambda_1-1}} + 2^{s_{\lambda_1}}(m - s_{\lambda_1}), \quad (7)$$

where λ_j *is the smallest integer such that* $\dim(\sigma^{s_{\lambda_j}}\omega \bmod (m - s_{\lambda_j})) = j$, *for each* $j \in \{1, \dots, n-1\}$.

Proof. We simply apply the previous lemma $n - 1$ times to obtain

$$\log_2|G_\omega \bmod m| = 3 + \Sigma_{\lambda_1} + \sum_{i=1}^{n-1}(2^{s_{\lambda_i}+1} - 2^{s_{\lambda_i-1}})$$
$$+ 2^{s_{\lambda_1}}(\log_2|G_{\sigma^{s_{\lambda_1}}\omega} \bmod (m - s_{\lambda_1})| - 3).$$

Next, we have $\dim(\sigma^{s_{\lambda_1}}\omega \bmod (m - s_{\lambda_1})) = 1$ and $\lambda_1 = s^{-1}(m) - 1$. Remark 7.11 yields

$$\log_2|G_\omega \bmod m| = 3 + \Sigma_{s^{-1}(m)-1} + \sum_{i=1}^{n-1}(2^{s_{\lambda_i}+1} - 2^{s_{\lambda_i-1}}) + 2^{s_{\lambda_1}}(m - s_{\lambda_1} - 2).$$

The result is obtained by extracting the first term of the sum. □

We are now ready to prove Theorem 4.4, which we restate here.

Theorem 7.13. *Consider* $\omega = \omega_1^{a_1}\omega_2^{a_2}\dots$ *in syllable form with* $\underline{\dim}(\omega) = n \geqslant 2$. *The Hausdorff dimension of* G_ω *is equal to*

$$\dim_H G_\omega = \frac{1}{2}\liminf_{k\to\infty}\left(\frac{\Sigma_k}{2^{s_k}} + \frac{1}{2^{s_k}}\sum_{i=2}^{n-1}2^{s_{\lambda_i}}\left(2 - \frac{1}{2^{a_{\lambda_i}}}\right) + \frac{1}{2^{a_k}}\left(1 - \frac{1}{2^{a_{k-1}}}\right)\right),$$

where for each $i \in \{2, \ldots, n-1\}$ we let $\lambda_i(k)$ be the smallest integer such that

$$\dim(\sigma^{s_{\lambda_i(k)}}\omega \bmod (s_{k+1} - s_{\lambda_i(k)})) = i.$$

Proof. Starting with equation (7), we write $k = s^{-1}(m)$. Recalling that $\lambda_1 = k - 1$, we compute that

$$\frac{\log_2 |G_\omega \bmod m|}{2^m} = \frac{1}{2^{m-s_k}}\left(\frac{3}{2^{s_k}} + \frac{\Sigma_{k-1}}{2^{s_k}} + \frac{1}{2^{s_k}}\sum_{i=2}^{n-1}(2^{s_{\lambda_i}+1} - 2^{s_{\lambda_i - 1}}) - \frac{1}{2^{s_k - s_{k-2}}} + \frac{1}{2^{a_k}}(m - s_{k-1})\right).$$

If we fix k and consider m such that $s_{k-1} + 1 < m \leqslant s_k + 1$, the numbers λ_i do not depend on m. We easily check that the expression is minimal when $m = s_k + 1$. Therefore

$$\dim_H G_\omega = \frac{1}{2}\liminf_{k\to\infty}\left(\frac{\Sigma_k}{2^{s_k}} + \frac{1}{2^{s_k}}\sum_{i=2}^{n-1}(2^{s_{\lambda_i}+1} - 2^{s_{\lambda_i - 1}}) - \frac{1}{2^{a_k + a_{k-1}}} + \frac{1}{2^{a_k}}\right).$$

The result follows. □

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