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A global branch of solutions to a semilinear equation on an unbounded interval

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Synopsis

For a semilinear second order differential equation on $(0, \infty)$, conditions are given for the bifurcation and asymptotic bifurcation in L^p of solutions to the Neumann problem. Bifurcation occurs at the lowest point of the spectrum of the linearised problem. Under stronger hypotheses, there is a global branch of solutions. These results imply similar conclusions for the same equation on R with appropriate symmetry.

1. Introduction

We consider the following Neumann problem:

$$\left.\begin{aligned} &u''(x)+\lambda u(x)+q(x)f(u(x), u'(x))=0 \quad \text{for} \quad x>0,\\ &u'(0)=\lim_{x\to\infty} u(x)=0, \end{aligned}\right\} \tag{N}$$

where the functions q and f satisfy:

(H1) $q \in C(R_+, R)$ and $\lim_{x\to\infty} q(x)=L$ with $L>0$;

(H2) $f\in C^1(R^2, R)$ with $f(0, 0)=0$ and $\operatorname{grad} f(0, 0)=(0, 0)$;

(H3) there exist positive constants a and A such that

$$k^{-2}D_1f(k^a s, k^{a+1}t)\to A\,|s|^{2/a},$$
$$k^{-1}D_2f(k^a s, k^{a+1}t)\to 0 \quad \text{as} \quad k\to 0^+,$$

uniformly for (s, t) in bounded subsets of R^2.

A classical solution to (N) is a pair (λ, u) where $\lambda\in R$, $u\in C^2(R_+, R)$ and (N) is satisfied. It is convenient to reformulate this problem using Sobolev spaces [**1**, Chap. VIII]. Let

$$X=\{u\in W^{2,1}((0,\infty)) : u'(0)=0\}$$

with the norm

$$\|u\|_X=|u|_1+|u'|_1+|u''|_1, \quad \text{for} \quad u\in X.$$

where $|u|_p$ denotes the usual norm in $L^p((0,\infty))=L^p$ for $1\leqq p\leqq\infty$. Then $X\subset C^1(R_+, R)\cap L^p$ for $1\leqq p\leqq\infty$ and $\lim_{x\to\infty} u(x)=\lim_{x\to\infty} u'(x)=0$ for all $u\in X$.

Hence we see that (λ, u) is a classical solution to (N) provided that $(\lambda, u)\in R\times X$ and (N) is satisfied almost everywhere on R_+.

This paper is concerned with the bifurcation of solutions to (N) from the point $(0, 0)$ in $R\times L^p$. The question has already been studied in several contexts when f is independent of u'. Variational methods are used in [**6**, **7**], the topological degree is used in [**9**, **10**] and in [**12**] the case where $q(x)=1$ for all $x\geqq 0$ is solved by quadrature. From these contributions we know that when

(i) q is non-increasing and satisfies (H1),
(ii) $f(s, t)=|s|^{2/a}s$ for all $(s, t)\in R^2$,

there is bifurcation in L^p if and only if $ap>1$.

We show how a suitable scaling of the variables can be used to reduce (N) to a situation where the implicit function theorem establishes the bifurcation of a continuous branch of solutions to (N) in $R\times L^p$ provided that $ap>1$. Scaling has been used in a similar way to deal with bifurcation for certain integral equations involving convolutions [**3**, **4**]. Recently scaling has also been used to establish bifurcation at an eigenvalue where the linearisation is not a Fredholm operator [**5**]. We recall, however, that the linearisation of (N) has no eigenvalues in L^p for $p\geqq 1$.

Under the more restrictive assumptions (L1) to (L3), we are able to prove that this branch of solutions can be extended to a curve parametrised by λ for all $\lambda\in(-\infty, 0)$. Furthermore, the functions u corresponding to solutions on this curve are positive.

To state our results for (N), we first introduce an "asymptotic limit for (N)":

$$\left.\begin{aligned} &v''(x)-v(x)+B\,|v(x)|^{2/a}\,v(x)=0 \quad \text{for} \quad x>0,\\ &v'(0)=0 \quad \text{and} \quad \lim_{x\to\infty} v(x)=0, \end{aligned}\right\} \qquad (\mathrm{N})_\infty$$

where $B=aAL/(2+a)$ and L, a and A are the constants appearing in (H1) and (H3). An elementary phase-plane analysis shows that $(\mathrm{N})_\infty$ has a unique solution, denoted by v_0, and that v_0 is positive and decreasing.

THEOREM 1. (Bifurcation). *Let the conditions* (H1), (H2) *and* (H3) *be satisfied. Then there exist* $\eta>0$ *and* $v\in C([0, \eta), X)$ *such that* $v(0)=v_0$ *and, for* $0<k<\eta$, $(-k^2, u_k)$ *is a non-trivial classical solution to* (N) *where* $u_k(x)=k^a v(k)(kx)$ *for* $x\geqq 0$.

Remark. For $1\leqq p\leqq\infty$, $|u_k|_p=k^{a-(1/p)}\,|v(k)|_p$ and $|v(k)|_p\to|v_0|_p$ as $k\to 0^+$. Furthermore, $k\mapsto(-k^2, u_k)$ is a continuous curve in $R\times L^p$ for $1\leqq p\leqq\infty$ and $(-k^2, u_k)\to(0, 0)$ as $k\to 0^+$ provided that $ap>1$.

Under stronger assumptions, we can improve this local result and show that the above curve extends globally.

(L1) $q\in C^1(R_+, R)$ with $q'(x)\leqq 0$ for all $x\geqq 0$ and $\lim_{x\to\infty} q(x)=L$ where $L>0$.

(L2) $h\in C^1(R_+, R)$ with $h(0)=h'(0)=0$ and $s^2h'(s)>sh(s)>rH(s)>0$ for all $s>0$ where $r>2$ and $H(s)=\int_0^s h(t)\,dt$.

(L3) There exist positive constants a and A such that $k^{-2}h'(k^a s)\to As^{2/a}$ as $k\to 0^+$.

Apart from permitting a global analysis, these hypotheses also ensure that the solutions on the branch have the same qualitative behaviour as v_0. Let

$$K=\{u\in X\colon u(x)>0 \text{ and } u'(x)<0 \text{ for all } x>0\}.$$

THEOREM 2. (Global continuation). *Let the conditions* (L1), (L2) *and* (L3) *be satisfied and set* $f(s,t)=h(|s|)$ *for all* $(s,t)\in R^2$. *Then there exists* $u\in C^1((-\infty,0),X)$ *such that for all* $\lambda<0$, $(\lambda,u(\lambda))$ *is a (non-trivial) solution to* (N) *and* $u(\lambda)\in K$. *Furthermore, for* $0<\sqrt{-\lambda}<\eta$, $u(\lambda)=u_{\sqrt{-\lambda}}$ *where* u_k *is the solution given in Theorem* 1.

The local result is proved in Section 2 and the global continuation is established in Section 3.

Remarks. 1. Solutions to the Neumann problem (N) can be used to construct solutions to the following related problem:

$$\left.\begin{aligned}&u''(x)+\lambda u(x)+q(x)f(u(x),u'(x))=0 \quad \text{for} \quad x\in R,\\ &\lim_{x\to-\infty}u(x)=\lim_{x\to\infty}u(x)=0,\end{aligned}\right\}\tag{D}$$

provided that

(a) $q\in C(R,R)$ is even and satisfies (H1),
(b) $f\in C^1(R^2,R)$ satisfies (H2) and (H3) with $f(s,t)=f(s,-t)$ for all $(s,t)\in R^2$.

In fact, under these conditions a solution (λ,u) to (N) is made into a solution to (D) by simply extending u to be an even function on R. Thus results similar to Theorems 1 and 2 hold for the problem (D) under the assumptions (a) and (b). When q is not even, the problem (D) cannot be reduced to (N) and the situation is much more complicated [**8**].

2. The method of scaling can also be applied to the N-dimensional generalisation of (D):

$$\left.\begin{aligned}&\Delta u(x)+\lambda u(x)+q(x)f(u(x))=0 \quad \text{for} \quad x\in R^N,\\ &\lim_{|x|\to\infty}u(x)=0,\end{aligned}\right\}$$

provided that q is radially symmetric. This remark will be amplified elsewhere.

2. Bifurcation by scaling

In this section we prove Theorem 1 by reducing the problem (N) to a situation in which the implicit function can be applied.

LEMMA 2.1. *Let the function f satisfy the conditions* (H2) *and* (H3). *Given* $\varepsilon>0$ *and a bounded subset D of* R^2, *there exists* $\delta>0$ *such that*

$$\left|k^{-(2+a)}f(k^a s,k^{a+1}t)-\frac{aA\,|s|^{2/a}s}{2+a}\right|<\varepsilon\{|s|+|t|\}$$

for $0<k<\delta$ *and* $(s,t)\in D$.

Proof. For $k>0$ and $(s,t)\in R^2$,

$$\left|k^{-(2+a)}f(k^a s, k^{a+1}t)-\frac{aA|s|^{2/a}s}{2+a}\right|$$
$$=\left|\int_0^1\left\{k^{-(2+a)}\frac{d}{dr}f(rk^a s, rk^{a+1}t)-A|rs|^{2/a}s\right\}dr\right|$$
$$\leqq\int_0^1|k^{-2}D_1f(rk^a s, rk^{a+1}t)-A|rs|^{2/a}|\,dr\,|s|$$
$$+\int_0^1|k^{-1}D_2f(rk^a s, rk^{a+1}t)|\,dr\,|t|.$$

The result now follows from (H3).

For $(k,u)\in R\times X$, we define a function F as follows:

$$F(k,u)(x)=\begin{cases}q(x/|k|)\,|k|^{-(2+a)}f(|k|^a u(x),|k|^{a+1}u'(x)) & \text{if } k\neq 0,\\ B\,|u(x)|^{2/a}u(x) & \text{if } k=0,\end{cases}$$

where $B=aAL/(2+a)$.

LEMMA 2.2. *Let the conditions* (H1), (H2) *and* (H3) *be satisfied.*

(a) *F maps $R\times X$ continuously into L^1.*

(b) *For each $k\in R$, $F(k,\ ):X\to L^1$ is Fréchet differentiable and for $u,v\in X$,*

$$D_uF(k,u)v(x)=\begin{cases}q(x/|k|)\{|k|^{-2}D_1f(|k|^a u(x),|k|^{a+1}u'(x))v(x)\\ +|k|^{-1}D_2f(|k|^a u(x),|k|^{a+1}u'(x))v'(x)\} & \text{if } k\neq 0,\\ AL\,|u(x)|^{2/a}v(x) & \text{if } k=0.\end{cases}$$

(c) *D_uF maps $R\times X$ continuously into the Banach space of all bounded linear operators from X into L^1.*

Proof. We recall that X is continuously embedded in $W^{1,\infty}(0,\infty)$. The lemma is then established in a standard manner using (H3) and Lemma 2.1.

LEMMA 2.3. *Let the conditions* (H1), (H2) *and* (H3) *be satisfied. Let $u(x)=k^a v(kx)$ for $x\geqq 0$ and $k>0$. The following statements are equivalent.*

(i) *$(-k^2,u)\in R\times X$ is a solution to* (N).

(ii) *$v\in X$ and $v''(x)-v(x)+F(k,v)(x)=0$ for $x>0$.*

Proof. Trivial.

Proof of Theorem 1. Let $Tu=u''-u$ and let $G(k,u)=Tu+F(k,u)$. Then $T:X\to L^1$ is an isomorphism and $G:R\times X\to L^1$ is continuous. Furthermore, for $u,v\in X$ and $k\in R$, $D_uG(k,u)v=Tv+D_uF(k,u)v$. Thus $G(0,v_0)=0$ where v_0 is the unique solution to $(N)_\infty$ in X.

Hence, by Lemma 2.2 and the implicit function theorem (see for example [**2**, p. 222]), it is sufficient to show that $D_uG(0,v_0):X\to L^1$ is an isomorphism.

Let $Cv(x)=Al\,|\,v_0(x)|^{2/a}v(x)$ for $v\in X$. Then $C:X\to L^1$ is a compact linear operator and $D_uG(0,v_0)=T+C$. Since $T:X\to L^1$ is an isomorphism, we need only show that $T+C$ is injective. For this, we suppose that $v\in X$ and $(T+C)v=0$.

Then

$$v''(x)-v(x)+AL\,|v_0(x)|^{2/a}\,v(x)=0 \quad \text{for} \quad x>0. \tag{2.1}$$

From (2.1) and $(N)_\infty$, we have that $\int_0^\infty |v_0(x)|^{2/a}\,v_0(x)v(x)\,dx=0$ and so v has at least one zero, denoted by z, in $(0,\infty)$. On setting $w(x)=v_0'(x)$, we have that

$$w''(x)-w(x)+AL\,|v_0(x)|^{2/a}\,w(x)=0 \quad \text{for} \quad x>0 \tag{2.2}$$

since v_0 satisfies $(N)_\infty$. From (2.1), we obtain

$$-v'(z)w(z)+\int_z^\infty \{-v'(x)w'(x)-v(x)w(x)+AL\,|v_0(x)|^{2/a}\,v(x)w(x)\}\,dx=0$$

and from (2.2),

$$\int_z^\infty \{-v'(x)w'(x)-v(x)w(x)+AL\,|v_0|^{2/a}\,v(x)w(x)\}\,dx=0.$$

Hence, $v'(z)=0$ and so, by (2.1), we must have $v(x)=0$ for all $x\geqq 0$.

This completes the proof of the theorem.

3. Global continuation

Throughout this section we suppose that $f: R^2\to R$ is defined by

$$f(s,t)=\begin{cases} h(|s|)\,|s|/s & \text{for} \quad (s,t)\in R^2 \quad \text{with} \quad s\neq 0,\\ 0 & \text{for} \quad s=0 \quad \text{and} \quad t\in R,\end{cases}$$

where the function h satisfies the conditions (L2) and (L3). It is easily seen that f satisfies the hypotheses (H2) and (H3) and the problem (N) can be written as

$$\left.\begin{aligned} &u''(x)+\lambda u(x)+q(x)g(|u(x)|)u(x)=0 \quad \text{for} \quad x>0,\\ &u'(0)=0 \quad \text{and} \quad \lim_{x\to\infty} u(x)=0,\end{aligned}\right\} \tag{N}$$

where $g(s)=s^{-1}h(s)$ for $s>0$. For the proofs which follow, we note some simple consequences of the assumptions (L2) and (L3).

(i) The function $s^{-r}H(s)$ is increasing for s in R_+.

(ii) There is an increasing function $g\in C(R_+,R)$ such that $sg(s)=h(s)$ for all

$$s\geqq 0, \quad g(0)=0 \quad \text{and} \quad \lim_{s\to\infty} g(s)=\infty.$$

(iii) On setting $\theta=r/(r-2)$, we have that

$$0<g(s)<\theta\{g(s)-2s^{-2}H(s)\} \quad \text{for all} \quad s>0.$$

(iv) Setting $j(s)=s^{-2}H(s)$ for $s>0$, we have that j is increasing on $(0,\infty)$ with $\lim_{s\to 0} j(s)=0$ and $\lim_{s\to\infty} j(s)=\infty$.

(v) $f(s,t)=g(|s|)s$ for all $(s,t)\in R^2$.

We begin by showing that the branch of solutions of (N) given by Theorem 1 lies in the set K. Then we show that a branch of solutions cannot leave K. Finally, by *a priori* estimates and the implicit function theorem, we prove that the branch can be extended globally to cover $(-\infty, 0)$.

LEMMA 3.1. *Let the conditions* (L1), (L2) *and* (L3) *be satisfied and let* $v \in C([0,\eta), X)$ *be the function given by Theorem* 1. *There exists* $k_0 > 0$ *such that* $v(k) \in K$ *for* $0 < k < k_0$.

Proof. From the phase-plane for $(N)_\infty$, we see that $v(0) = v_0 \in K$. For $0 < k < \eta$, $v(k) \in X$ and satisfies

$$v''(x) + \{-1 + q(x/k)k^{-2}g(k^a\,|v(x)|)\}v(x) = 0 \quad \text{for} \quad x > 0. \tag{3.1}$$

By adapting Lemma 2.1 to the stronger hypotheses we obtain the following. Given $\varepsilon > 0$ and a bounded subset D of R, there exists $\delta > 0$ such that

$$\left|k^{-2}g(k^a\,|s|) - \frac{aA}{2+a}\,|s|^{2/a}\right| < \varepsilon$$

for $0 < k < \delta$ and $s \in D$.

Thus, since $\lim_{x\to\infty} v_0(x) = 0$, there exists $z > 0$ such that

$$\left\{-1 + \frac{q(0)aA\,|v_0(x)|^{2/a}}{2+a}\right\} < -\tfrac{1}{2} \quad \text{for all} \quad x \geqq z.$$

Hence there is an open neighbourhood U of $(0, v_0)$ in $R \times X$ such that

$$\{-1 + q(x/k)k^{-2}g(k^a\,|v(x)|)\} < -\tfrac{1}{4} \quad \text{for all} \quad x \geqq z.$$

provided that $(k, v) \in U$.

If $(k, v) \in U$ and (3.1) is satisfied, it follows that

$$v'(x)v(x) + \int_x^\infty v'(y)^2\,dy = \int_x^\infty \{-1 + q(y/k)k^{-2}g(k^a\,|v(y)|)\}v(y)^2\,dy$$

$$< -\tfrac{1}{4}\int_x^\infty v(y)^2\,dy \quad \text{for all} \quad x \geqq z.$$

This proves that $v'(x)v(x) < 0$ for all $x \geqq z$, and since $v_0 \in K$, we can conclude that $v(x) > 0$ and $v'(x) < 0$ for all $x \geqq z$, provided that $(k, v) \in U$ and satisfies (3.1). On choosing a sufficiently small neighbourhood U of $(0, v_0)$, the result now follows from the continuous embedding of X in $C^1(R_+, R)$.

LEMMA 3.2. *Let the conditions* (L1), (L2) *and* (L3) *be satisfied.*

(a) *The solutions to* (N) *form a closed subset of* $R \times X$.
(b) *If* (λ, u) *is a solution to* (N) *with* $\lambda < 0$ *and* $u \in \bar{K}$ (*the closure of* K *in* X), *then* $u \in K \cup \{0\}$.
(c) *If* (λ, u) *is a solution to* (N) *with* $\lambda < 0$ *and* $u \in K$, *there is an open neighbourhood* U *of* (λ, u) *in* $R \times X$ *such that* $v \in K$ *whenever* $(\mu, v) \in U$ *and satisfies* (N).

Proof. (a) Trivial.

(b) Suppose that $u \neq 0$. Since u satisfies (N) it can only have simple zeros. This implies that $u(x)>0$ for all $x \geqq 0$. On setting $w(x)=u'(x)/u(x)$, we find that

$$w'(x)=\frac{u''(x)}{u(x)}-\left(\frac{u'(x)}{u(x)}\right)^2$$

and

$$u'(x)^2+\lambda u(x)^2+2q(x)H(u(x))=-\int_x^\infty 2q'(y)H(u(y))\,dy \quad \text{for} \quad x\geqq 0.$$

Thus,

$$\begin{aligned} w'(x)&=q(x)\left\{\frac{2H(u(x))}{u(x)^2}-g(u(x))\right\}+\frac{1}{u(x)^2}\int_x^\infty 2q'(y)H(u(y))\,dy \\ &<-\frac{1}{\theta}q(x)g(u(x)) \text{ (by (iii))} \\ &<0. \end{aligned}$$

Since $w(0)=0$, it follows that $u'(x)<0$ for all $x>0$ and so $u \in K$.

(c) For $u \in K$, we have $u(x)>0$ for $x \geqq 0$ and since u satisfies (N) we also have that $u''(x)+\{\lambda+q(x)g(u(x))\}u(x)=0$ for $x>0$. Furthermore, $\lim_{x\to\infty} u(x)=0$ and so there exists $z>0$ such that $\lambda+q(x)g(u(x)) \leqq \frac{1}{2}\lambda<0$ for all $x \geqq z$. The result is now established in the same way as Lemma 3.1.

LEMMA 3.3. *Let the conditions* (L1), (L2) *and* (L3) *be satisfied. Let* (λ, u) *be a solution to* (N) *with* $\lambda<0$ *and* $u \in K$. *Set* $k=\sqrt{-\lambda}$.

(a) $0<2LH(u(x)) \leqq k^2u(x)^2-u'(x)^2 \leqq 2q(x)H(u(x))$ *for* $x \geqq 0$.
(b) $\lim_{x\to\infty} u'(x)/u(x)=-k$ *and, for all* $\varepsilon>0$, $\lim_{x\to\infty} e^{(k-\varepsilon)x}u(x)=0$.

Proof. (a) By (N), $u'(x)^2-k^2u(x)^2=\int_x^\infty 2q(y)H(u(y))'\,dy$ for $x \geqq 0$. But $q'(y) \leqq 0$ and $H(u(y))' \leqq 0$ by (L1), (L2) and the assumption that $u \in K$. Hence we obtain, $-2q(x)H(u(x)) \leqq \int_x^\infty 2q(y)H(u(y))'\,dy \leqq -2LH(u(x))$. This proves (a).

(b) By (L1) and (L2), $\lim_{x\to\infty} q(x)=L$ and $\lim_{s\to 0} j(s)=0$. From (a) it now follows that $\lim_{x\to\infty} u'(x)/u(x)=-k$ and, given $\varepsilon>0$, there exists $z \geqq 0$ such that $u'(x) \leqq (-k+\varepsilon)u(x)$ for all $x \geqq z$. This implies that $e^{(k-\varepsilon)x}u(x)$ is a decreasing function of x on $[z,\infty)$. The proof is complete.

LEMMA 3.4. *Let the conditions* (L1), (L2) *and* (L3) *be satisfied. There exist increasing functions* A *and* $B \in C((-\infty, 0), R)$ *such that*

$$A(\lambda) \leqq B(\lambda) \quad \textit{for all} \quad \lambda<0,$$
$$0<A(\lambda) \leqq |u|_\infty = u(0) \leqq B(\lambda)$$

and

$$A(\lambda) \leqq \|u\|_X \leqq B(\lambda)\left\{1+\frac{\theta}{\sqrt{-\lambda}}+\theta 2\sqrt{-\lambda}\right\}$$

whenever (λ, u) *is a solution to* (N) *with* $\lambda<0$ *and* $u\in K$. *Furthermore,* $\lim_{\lambda\to-\infty} A(\lambda)=+\infty$ *and* $\lim_{\lambda\to 0} B(\lambda)=0$.

Proof. Let $A(\lambda)=j^{-1}(-\lambda/2q(0))$ and $B(\lambda)=j^{-1}(-\lambda/2L)$. From Lemma 3.3(a), we see that $2Lj(u(0))\leqq-\lambda\leqq 2q(0)j(u(0))$ and hence $A(\lambda)\leqq u(0)\leqq B(\lambda)$ whenever (λ, u) is a solution to (N) with $\lambda<0$ and $u\in K$.

Now setting $w(x)=u'(x)/u(x)$ as in Lemma 3.2(b), we obtain $w'(x)<-(1/\theta)q(x)g(u(x))$ for all $x>0$, with $w(0)=0$ and $\lim_{x\to\infty} w(x)=-\sqrt{-\lambda}$, by Lemma 3.3(b). Hence, $\int_0^\infty q(x)g(u(x))\,dx\leqq\theta\sqrt{-\lambda}$ and, by (N),

$$-\lambda\int_0^\infty u(x)\,dx=\int_0^\infty q(x)g(u(x))u(x)\,dx\leqq u(0)\theta\sqrt{-\lambda}\leqq B(\lambda)\theta\sqrt{-\lambda}.$$

Thus $|u|_1\leqq B(\lambda)\theta/\sqrt{-\lambda}$ and

$$|u''|_1\leqq-\lambda\,|u|_1+\int_0^\infty q(x)g(u(x))u(x)\,dx\leqq-2\lambda\,|u|_1\leqq 2B(\lambda)\theta\sqrt{-\lambda}.$$

On the other hand,

$$A(\lambda)\leqq u(0)=-\int_0^\infty u'(x)\,dx=|u'|_1\leqq B(\lambda).$$

Thus, we have that

$$\|u\|_X=|u|_1+|u'|_1+|u''|_1\geqq|u'|_1\geqq A(\lambda)$$

and

$$\|u\|_X\leqq\frac{B(\lambda)\theta}{\sqrt{-\lambda}}+B(\lambda)+2B(\lambda)\theta\sqrt{-\lambda}$$
$$=B(\lambda)\theta\left\{\frac{1}{\sqrt{-\lambda}}+\frac{1}{\theta}+2\sqrt{-\lambda}\right\}.$$

Proof of Theorem 2. For $\lambda<0$ and $u\in X$, let $N(\lambda, u)(x)=u''(x)+\lambda u(x)+q(x)g(|u(x)|)u(x)$. Then $N\in C^1((-\infty,0)\times X, L^1)$ and, for $u, v\in X$, $D_uN(\lambda, u)v=Sv+P(u)v$ where $Sv=v''+\lambda v$ and $P(u)v(x)=q(x)h'(|u(x)|)v(x)$. Since $\lambda<0$, the mapping $S: X\to L^1$ is an isomorphism. Furthermore, for $u\in X$, we have that $\lim_{x\to\infty} u(x)=0$ and hence $P(u): X\to L^1$ is a compact linear operator. It follows that $D_uN(\lambda, u)=S+P(u): X\to L^1$ is an isomorphism if and only if it is injective.

To use the implicit function theorem to prove Theorem 2, we must show that $D_uN(\lambda, u): X\to L^1$ is injective whenever (λ, u) is a solution to (N) with $\lambda<0$ and $u\in K$.

If (λ, u) satisfies (N) and $u\in K$, we have

$$u''(x)+\lambda u(x)+q(x)h(u(x))=0 \quad\text{for all}\quad x>0 \tag{3.2}$$

and if $v\in X\backslash\{0\}$ is such that $D_uN(\lambda, u)v=0$, we have

$$v''(x)+\lambda v(x)+q(x)h'(u(x))v(x)=0 \quad\text{for all}\quad x>0. \tag{3.3}$$

Hence,

$$\int_0^\infty q(x)\{h(u(x))v(x)-h'(u(x))v(x)u(x)\}\,dx=0.$$

Since $sh'(s)>h(s)$ for $s>0$ and $u(x)>0$ for $x>0$, it follows that there exists $z>0$ such that $v(z)=0$. Furthermore, as in the proof of Lemma 3.1, there exists $z_1>0$ such that $v'(x)v(x)<0$ for all $x>z_1$. Thus, replacing v by $-v$ if necessary, we can suppose that $v(z)=0$, $v'(z)>0$ and $v(x)>0$ for all $x>z$. On setting $w(x)=u'(x)$, we have that $w(x)<0$ for all $x>0$ and

$$w''(x)+\lambda w(x)+q(x)h'(u(x))w(x)+q'(x)h(u(x))=0, \tag{3.4}$$

From (3.3) and (3.4), it follows that

$$-v'(z)w(z)+\int_z^\infty[-v'(x)w'(x)+\lambda v(x)w(x)+q(x)h'(u(x))v(x)w(x)]\,dx=0$$

and

$$-w'(z)v(z)+\int_z^\infty[-v'(x)w'(x)+\lambda v(x)w(x)+q(x)h'(u(x))v(x)w(x)]\,dx$$
$$=-\int_z^\infty q'(x)h(u(x))v(x)\,dx.$$

Thus, $v'(z)w(z)=-\int_z^\infty q'(x)h(u(x))v(x)\,dx\geqq 0$ and so $v'(z)\leqq 0$. This contradicts the fact that $v'(z)>0$ and we conclude that $D_uN(\lambda,u)\colon X\to L^1$ must be injective.

In view of Lemmas 3.1 to 3.4, the proof of Theorem 2 is completed by establishing the following fact. A subsequence converging in $R\times X$ can be extracted from any sequence $\{(\lambda_n, u_n)\}$ of solutions to (N) such that

$$\lambda_n\to\lambda \quad\text{with}\quad \lambda<0,$$
$$u_n\in K,$$
$$|u_n|_\infty\leqq\|u_n\|_X\leqq C \quad\text{for all}\quad n.$$

To prove this fact, we note first that since $u_n\in K$,

$$xu_n(x)\leqq\int_0^x u_n(y)\,dy\leqq\|u_n\|_X\leqq C \quad\text{for all}\quad x>0. \tag{3.5}$$

By Lemma 3.3(a) and the property (i) of the function H,

$$-\lambda_n u_n(x)^2-u_n'(x)^2\leqq 2q(x)H(u_n(x))$$
$$\leqq 2q(0)u_n(x)^rC^{-r}H(C) \quad\text{where}\quad r>2.$$

Hence, using (3.5), we obtain

$$-\lambda_n-u_n'(x)^2/u_n(x)^2\leqq 2q(0)x^{-r+2}C^{-2}H(C).$$

It follows that there exist m and z such that

$$u_n'(x)/u_n(x)\leqq-\tfrac{1}{2}\sqrt{-\lambda} \quad\text{for all}\quad x\geqq z \quad\text{and}\quad n\geqq m.$$

and consequently,

$$0 < u_n(x) \leqq C \exp\{-\tfrac{1}{2}\sqrt{-\lambda}(x-z)\} \quad \text{for all} \quad x \geqq z \quad \text{and} \quad n \geqq m.$$

By using this estimate and the equation (N), the existence of a subsequence of $\{u_n\}$ converging in X is easily established.

This completes the proof of Theorem 2.

Remark. From the results stated in [**11**], it follows that all positive solutions to (N) belong to the branch given by Theorem 2.

References

1 H. Brézis. *Analyse Functionnelle, Théorie et Applications* (Paris: Masson, 1983).
2 L. Kantorovitch and G. Akilov. *Analyse Fonctionnelle* Vol. II (Moscow: Mir, 1981).
3 Y. Demay, Bifurcation d'un soliton pour une équation de la physique des plasmas. *C.R. Acad. Sci. Paris* **285** (1977), 769–772.
4 M. Robert and C. A. Stuart. Intrinsic structure of the critical liquid-gas interface. *Phys. Rev. Letters* **49** (1982), 1434–1437.
5 R. J. Magnus. The transformation of vector-functions scaling and bifurcation. *Trans. Amer. Math. Soc.* **286** (1984), 689–714.
6 C. A. Stuart. Bifurcation pour des problèmes de Dirichlet et de Neumann sans valeurs propres. *C.R. Acad. Sci. Paris* **288** (1979), 761–764.
7 C. A. Stuart. Bifurcation for Neumann problems without eigenvalues. *J. Differential Equations* **36** (1980), 391–407.
8 C. A. Stuart. Bifurcation in $L^p(R)$ for a semilinear equation, to appear.
9 J. F. Toland, Global bifurcation for Neumann problems without eigenvalues. *J. Differential Equations* **44** (1982), 82–101.
10 J. F. Toland. Singular elliptic eigenvalue problems for equations and systems. In *Systems of Nonlinear Partial Differential Equations* (ed. J. M. Ball) (New York: Reidel, 1983).
11 J. F. Toland. Uniqueness of positive solutions of some Sturm–Liouville problems on the half-line. *Proc. Roy. Soc. Edinburgh Sect. A* **97** (1984), 259–263.
12 T. Küpper and D. Reimer. Necessary and sufficient conditions for bifurcation from the essential spectrum. *Nonlinear Anal. T.M.A.* **3** (1979), 555–561.

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