mathematics

MDPI

Article

Survey on the Figà–Talamanca Herz algebra †

Antoine Derighetti

EPFL SB-DO, MAA1, Station 8, CH-1015 Lausanne, Switzerland; antoine.derighetti@epfl.ch

† This is an amplified version of a doctoral course (three hours) given at the international meeting "2017 Banach algebras and Applications, Oulu, Finland, 3–11 July 2017".

Received: 16 May 2019; Accepted: 2 July 2019; Published: 24 July 2019

check for updates

Abstract: This paper presents a self contained approach to the theory of convolution operators on locally compact groups (both commutative and non commutative) based on the use of the Figà –Talamanca Herz algebras. The case of finite groups is also considered.

Keywords: convolution operators; Figà–Talamanca Herz algebras; Fourier multipliers; spectrum; support of convolution operators

---

1. Introduction

A p-convolution operator T on a locally compact group G is a continuous linear operator of $L^p(G)$ such that T commutes with left translations. The set of all p-convolution operators is denoted $CV_p(G)$. If G is abelian, $CV_2(G)$ is isomorphic to $L^\infty(\widehat{G})$; moreover, $CV_p(G) \subset CV_2(G)$. We show, using the Figà–Talamanca Herz algebra $A_p(G)$, that the relation $CV_p(G) \subset CV_2(G)$ also holds for amenable G (Theorem 17). A more general statement is obtained for non-amenable groups (Theorem 18).

Every bounded function on a LCA can be recovered from his spectrum (Corollary 4). To every $T \in CV_p(G)$, we associate a closed subset of G called the support of T. Then, using once again the algebra $A_p(G)$, we show that T can be recovered from the support of T if G is amenable (Theorem 13). We obtain therefore a kind of harmonic analysis of p-convolutions operators strongly similar to the harmonic analysis of bounded functions on LCA.

2. Convolution Operators. The Fourier Transform of a Convolution Operator

In the following, G is a locally compact group. For $\varphi : G \to \mathbb{C}$, we put

$$\check{\varphi}(x) = \varphi(x^{-1}),$$

for $a, b \in G$ we also put

$${}_a\varphi(x) = \varphi(ax) \ \text{ and } \ \varphi_b(x) = \varphi(xb).$$

We choose a left invariant Haar measure m_G on G. We recall that m_G is a positive Radon measure, with $m_G \neq 0$,

$$m_G({}_a\varphi) = m_G(\varphi) = m_G(\varphi_a)\Delta_G(a) = m_G(\check{\varphi}\check{\Delta}_G)$$

for every $\varphi \in C_{00}(G)$ and for every $a \in G$. We set

$$m_G(\varphi) = \int_G \varphi(x)dx\,.$$

Definition 1. *Let G be a locally compact group and $1 < p < \infty$. A bounded linear operator T of $L^p(G)$ ($T \in \mathcal{L}(L^p(G))$ such that $T({}_a\varphi) = {}_aT(\varphi)$ for every $\varphi \in C_{00}(G)$ and for every $a \in G$, is said to be a "p-convolution operator of the group G". The set of all p-convolution operators of G is denoted $CV_p(G)$.*

Let μ be a bounded Radon measure on G ($\mu \in M^1(G)$). We put

$$\lambda_G^p(\mu)\varphi = \varphi * \left(\Delta_G^{1/p'}\check{\mu}\right)$$

for every $\varphi \in C_{00}(G)$, more explicitly

$$\left(\lambda_G^p(\mu)\varphi\right)(x) = \int_G \varphi(xy)\Delta_G(y)^{1/p}d\mu(y)$$

for every $x \in G$. Recall that $1/p + 1/p' = 1$ and that

$$\check{\mu}(\varphi) = \mu(\check{\varphi})$$

for every $\varphi \in C_{00}(G)$. We have $\lambda_G^p(\mu) \in CV_p(G)$ and

$$\left\|\lambda_G^p(\mu)\right\|_p \le \|\mu\|,$$

where $\|T\|_p$ denotes the norm of every $T \in \mathcal{L}(L^p(G))$. Observe that

$$\lambda_G^p(\delta_a)\varphi = \varphi_a\Delta_G(a)^{1/p}.$$

We also have

$$\lambda_G^p(\delta_a * \delta_b) = \lambda_G^p(\delta_{ab}),$$

the map $a \mapsto \lambda_G^p(\delta_a)$ is a faithful representation of G into $L^p(G)$. This representation is called the right-regular representation of G, and is denoted λ_G^p.

We have

$$\lambda_{\mathbb{Z}}^p(M^1(\mathbb{Z})) \neq CV_p(\mathbb{Z}).$$

Consider indeed $f : \mathbb{Z} \to \mathbb{R}$ defined by $f(n) = 1/n$ for $n \in \mathbb{Z} \setminus \{0\}$ and with $f(0) = 0$. Then, the map $\varphi \mapsto f * \varphi$ belongs to $CV_p(\mathbb{Z})$ (see Titchmarsh [1] Theorem A page 321) but $f \notin M^1(\mathbb{Z})$.

If a locally compact group G contains an infinite abelian subgroup, then

$$\lambda_G^p(M^1(G)) \neq CV_p(G).$$

Let $\widehat{G}$ be the Pontrjagin dual of a locally compact abelian group G. For $\mu \in M^1(G)$, we put

$$\widehat{\mu}(\chi) = \int_G \overline{\chi(x)}d\mu(x).$$

For $f \in \mathcal{L}^1(G)$, we set $\widehat{f} = \widehat{fm_G}$; more explicitly, for every $\chi \in \widehat{G}$, we have

$$\widehat{f}(\chi) = \int_G \overline{\chi(x)}f(x)dx.$$

There is a unique Haar measure on $\widehat{G}$, denoted $m_{\widehat{G}}$, such that, for every $f \in \mathcal{L}^1(G) \cap C(G)$ with $\widehat{f} \in C_{00}(\widehat{G})$, we have

$$f(x) = \int_{\widehat{G}} \widehat{f}(\chi)\chi(x)dm_{\widehat{G}}(\chi)$$

for every $x \in G$.

The map $f \mapsto [\widehat{f}]$ is a linear isometry of $L^1(G) \cap L^2(G)$ into $L^2(\widehat{G})$ $(= L^2(\widehat{G}, m_{\widehat{G}}))$. We denote by $\mathcal{F}$ his continuous extension to $L^2(G)$. We recall (Plancherel theorem) that $\mathcal{F}$ is a Banach isometry of

$L^2(G)$ onto $L^2(\widehat{G})$. Precisely, for $f: G \to \mathbb{C}$, $[f]$ denotes the set of all $g: G \to \mathbb{C}$ with $g(x) = f(x)$ almost everywhere.

Observe that for G, an arbitrary locally compact group, $CV_p(G)$ with the composition of operators, as a product, is a unital Banach algebra:

$$\|AB\|_p \leq \|A\|_p \, \|B\|_p \,,$$

the identity operator I is the unit. Moreover, $T \mapsto T^*$ is an involution of $CV_2(G)$, we have $\|T^*\|_2 = \|T\|_2$.

Theorem 1. *Let G be a locally compact abelian group. We put for* $\varphi \in L^\infty(\widehat{G})$

$$\Lambda_{\widehat{G}}(\varphi)(f) = \mathcal{F}^{-1}\Big(\varphi \mathcal{F}(f)\Big)$$

for every $f \in L^2(G)$*. Then,* $\Lambda_{\widehat{G}}$ *is an involutive Banach algebra isometry of* $L^\infty(\widehat{G})$ *onto* $CV_2(G)$*.*

The main difficulty is to verify that the above map is surjective. This is a consequence of the following statement (spectral theorem): let Ω be a bounded linear map of $L^2(G)$ such that $\Omega(rf) = r\Omega(f)$ for every $r \in C_{00}(G)$ and for every $f \in L^2(G)$, then there is $\varphi \in L^\infty(G)$ such that $\Omega(f) = \varphi f$ for every $f \in L^2(G)$. For details, see [2] page 13.

Corollary 1. *Let G be a locally compact abelian group. Then,* $CV_2(G)$ *is a commutative Banach algebra.*

The following result is straightforward.

Let G be a general locally compact group (not necessarily commutative), $1 < p < \infty$ and $T \in \mathcal{L}(L^p(G))$. Then, $T \in CV_p(G)$ if and only if

$$T(f * \varphi) = f * T\varphi$$

for every $f \in L^1(G)$ and for every $\varphi \in L^p(G)$.

We are going to show that. for G, an abelian locally compact group, we have

$$CV_p(G) \subset CV_2(G)$$

(for every $1 < p < \infty$) and the corresponding inequality for the norms

$$\|T\|_2 \leq \|T\|_p \ .$$

Theorem 2. *Let G be a locally compact abelian group,* $1 < p < \infty$ *and* $T \in CV_p(G)$*. For every* $\varphi \in L^p(G) \cap L^2(G)$*, we have* $T\varphi \in L^2(G)$ *and*

$$||T\varphi||_2 \leq \|T\|_p \, ||\varphi||_2 \,.$$

Proof. (1) For every $\varphi \in L^1(G) \cap L^p(G) \cap L^{p'}(G)$, we have $T\varphi \in L^{p'}(G)$ and

$$||T\varphi||_{p'} \leq \|T\|_p \, ||\varphi||_{p'} \,.$$

Letting $s \in T\varphi$, we verify that

$$\left| \int_G s(x)\psi(x)dx \right| \leq ||\varphi||_{p'} \, \|T\|_p \, N_p(\psi)$$

for every $\psi \in C_{00}(G)$.

We have
$$\int_G s(x)\psi(x)dx = s * \check{\psi}(e) = \Big(T\varphi * [\psi]\check{\ }\Big)(e).$$
However, using strongly the commutativity of G, we get
$$T\varphi * [\psi]\check{\ } = [\psi]\check{\ } * T\varphi = T\Big([\psi]\check{\ } * \varphi\Big) = T\Big(\varphi * [\psi]\check{\ }\Big) = \varphi * T([\psi]\check{\ }).$$
Therefore,
$$\int_G s(x)\psi(x)dx = \Big(\varphi * T([\psi]\check{\ })\Big)(e).$$
We choose now $r \in \varphi$ and $t \in T([\psi]\check{\ })$ and obtain
$$\Big(\varphi * T([\psi]\check{\ })\Big)(e) = \int_G r(x)t(x^{-1})dx = \int_G r(x^{-1})t(x)dx,$$
which implies
$$\left|\int_G s(x)\psi(x)dx\right| \leq N_{p'}(\check{r})N_p(t) = \|\varphi\|_{p'}\|T[\psi]\check{\ }\|_p \leq \|\varphi\|_{p'} \,|\!|\!|T|\!|\!|_p\, N_p(\psi).$$

(2) It suffices to show that, for $\varphi \in \Big\{[r]\Big| r \in \mathcal{L}^1(G), r \text{ step function}\Big\}$, we have $T\varphi \in L^2(G)$ and
$$\|T\varphi\|_2 \leq |\!|\!|T|\!|\!|_p \|\varphi\|_2\,.$$
Suppose $p < 2$. Put
$$t = \frac{1/2 - 1/p'}{1/p - 1/p'}.$$
We have
$$\frac{1}{2} = (1-t)\frac{1}{p'} + t\frac{1}{p}.$$
According to (1)
$$\|T\varphi\|_{p'} \leq |\!|\!|T|\!|\!|_p \|\varphi\|_{p'}.$$
By Riesz–Thorin, we get $T\varphi \in L^2(G)$ and
$$\|T\varphi\|_2 \leq |\!|\!|T_p|\!|\!|^{1-t+t} \|\varphi\|_2\,.$$
For $p > 2$, we proceed similarly. □

Remark 1. *Even though we use in a very strong way the commutativity of G, this theorem extends to* **arbitrary locally compact amenable groups.** *This is shown in Section 5. We recall that a locally compact group G is said to be amenable if there is a linear functional $\mathcal{M}$ on the vector space $C^b(G)$ of all continuous bounded complex valued functions on G such that $\mathcal{M}(\varphi) \geq 0$ if $\varphi \geq 0$, $\mathcal{M}(1_G) = 1$ and $\mathcal{M}({}_a\varphi) = \mathcal{M}(\varphi)$ for every $a \in G$ (see [3] Chapter 8 § 5).*

Definition 2. *The unique bounded operator S of $L^2(G)$ with $S\varphi = T\varphi$ for every $\varphi \in L^2(G) \cap L^p(G)$ is denoted $\alpha_p(T)$.*

Theorem 3. *Let G be a locally compact abelian group and $1 < p < \infty$. Then, α_p is a contractive monomorphism of the Banach algebra $CV_p(G)$ into the Banach algebra $CV_2(G)$. For every $\mu \in M(G)$, we have $\alpha_p(\mu) = \mu$.*

Definition 3. *For every $T \in CV_p(G)$, we put $\widehat{T} = \Lambda_{\widehat{G}}^{-1}(\alpha_p(T))$, $\widehat{T}$ is called the* **"Fourier Transform of** *T***"**.

The Banach algebra $CV_p(\mathbb{R}^n)$ is directly related to the L^p-Theory of Fourier series of n variables. For $m = (m_1, ..., m_n) \in \mathbb{Z}^n$, we put

$$\chi_m(e^{i\theta_1}, ..., e^{i\theta_n}) = e^{im_1\theta_1 + ... + m_n\theta_n}.$$

Let K be a compact neighborhood of 0 in $\mathbb{R}^n$. For $f \in L^1(\mathbb{T}^n)$ and $\lambda > 0$, we put

$$s_\lambda^{(K)} f = \sum_{m \in \mathbb{Z}^n \cap \lambda K} \widehat{f}(m)\chi_m .$$

For $f \in L^1(\mathbb{T})$ $s_N^{([-1,1])} f$ is the Fourier sum of order N.

Theorem 4. *Let K be a compact convex neighborhood of* 0 *in* $\mathbb{R}^n$ *and* $1 < p < \infty$. *The following statements are equivalent.*

(1) $\lim_{\lambda\to\infty} \|f - s_\lambda^{(K)} f\|_p = 0$ *for every* $f \in L^p(\mathbb{T}^n)$.
(2) *There is a unique* $T \in CV_p(\mathbb{R}^n)$ *with* $\widehat{T} = [1_K]$.

According to Marcel Riesz [4] $\lim_{\lambda\to\infty} \|f - s_\lambda^{([-1,1])} f\|_p = 0$ for every $f \in L^p(\mathbb{T})$ and every $1 < p < \infty$. Consequently, for every interval I of $\mathbb{R}$, there is $T \in CV_p(T)$ with $\widehat{T} = 1_I$. It is not difficult to deduce from this that, for $n > 1$, for every convex polyhedral set C of $\mathbb{R}^n$ and for every $1 < p < \infty$, there is $T \in CV_p(\mathbb{R}^n)$ with $\widehat{T} = [1_C]$.

Let D be the unit ball of $\mathbb{R}^n$ for $n > 1$. According to Laurent Schwartz [5] and Charles Fefferman [6] for every $1 < p < \infty$, with $p \neq 2$, **there is no** $T \in CV_p(\mathbb{R}^n)$ with $\widehat{T} = [1_D]$. In [5] Schwartz proved that for

$$1 < p < \frac{2n}{n+1} \text{ and for } \frac{2n}{n-1} < p < \infty$$

there no $T \in CV_p(\mathbb{R}^n)$ with $\widehat{T} = [1_D]$.

For a detailed exposition of Fefferman's result, see [7] Chap. 10, Section 10.1, pages 734–744.

3. The Figà–Talamanca Herz algebra $A_p(G)$, the dual of $A_p(G)$

Definition 4. *Let G be a locally compact group and* $1 < p < \infty$. *We denote by* $\mathcal{A}_p(G)$ *the set of all pairs* $((k_n), (l_n))$ *where* (k_n) *is a sequence of* $\mathcal{L}^p(G)$ *and* (l_n) *is a sequence of* $\mathcal{L}^{p'}(G)$ *with*

$$\sum_{n=1}^{\infty} N_p(k_n) N_{p'}(l_n) < \infty$$

Definition 5. *We denote by* $A_p(G)$ *the set*

$$\Big\{ u : G \to \mathbb{C} \,\Big|\, \text{there is } ((k_n), (l_n)) \in \mathcal{A}_p(G) \text{ with}$$

$$u(x) = \sum_{n=1}^{\infty} \Big(\overline{k_n} * \check{l}_n\Big)(x) \text{ for every } x \in G \Big\}.$$

For $u \in A_p(G)$, *we put*

$$\|u\|_{A_p(G)} = inf \Big\{ \sum_{n=1}^{\infty} N_p(k_n) N_{p'}(l_n) \,\Big|\, ((k_n), (l_n)) \in \mathcal{A}_p(G)$$

$$\text{such that } u = \sum_{n=1}^{\infty} \overline{k_n} * \check{l}_n \Big\}.$$

Theorem 5. *Let G be a locally compact group and $1 < p < \infty$. Then:*

(1) $A_p(G)$ *is a linear subspace of* $C_0(G)$,

(2) $\| \|_{A_p(G)}$ *is a norm on* $A_p(G)$, *with respect to this norm* $A_p(G)$ *is a Banach space. For every* $u \in A_p(G)$, *we have*

$$\|u\|_\infty \leq \|u\|_{A_p(G)}.$$

(3) $A_p(G) \cap C_{00}(G)$ *is dense in* $A_p(G)$.

Definition 6. *Let G be a abelian locally compact group. For every* $f \in L^1(\widehat{G})$, *we put*

$$\Phi_{\widehat{G}}(f)(x) = \int_{\widehat{G}} f(\chi)\chi(x)d\chi$$

for every $x \in G$.

Theorem 6. *Let G be an abelian locally compact group. Then,* $A_2(G)$ *is a involutive Banach algebra for the complex conjugation and the pointwise product. The map* $\Phi_{\widehat{G}}$ *is an involutive isometric isomorphism of the Banach algebra* $L^1(\widehat{G})$ *onto* $A_2(G)$. *For every* $u \in A_2(G)$, *there is* $k, l \in \mathcal{L}^2(G)$ *with* $u = \overline{k} * \check{l}$ *and*

$$\|u\|_{A_2(G)} = N_2(k)N_2(l).$$

Remark 2. *The functions k, l are not unique. They are obtained in a canonical way, using the map* $\mathcal{F}$ *of page 2 (see [2] page 40).*

We present now a generalization of the first part of Theorem 6, to every $1 < p < \infty$ and to every locally compact group.

Theorem 7. *Let G be a locally compact group and* $1 < p < \infty$. *For the pointwise product and the complex conjugation,* $A_p(G)$ *is an involutive Banach algebra.*

Proof. Consider $k, l, r, s \in C_{00}(G)$. For every $h \in G$ put $f(h) = kr_h$, $g(h) = ls_h$ and $j(h) = \overline{f(h)} * g(h)\check{}$. We have $j \in C_{00}(G; A_p(G))$,

$$\int_G j(h)dh \in A_p(G)$$

and

$$\left\| \int_G j(h)dh \right\|_{A_p(G)} \leq \int_G \|j(h)\|_{A_p(G)}dh.$$

But

$$\int_G \|j(h)\|_{A_p(G)}dh \leq \int_G N_p(kr_h)N_{p'}(ls_h)dh$$

$$\leq \left(\int_G N_p(kr_h)^p dh \right)^{1/p} \left(\int_G N_{p'}(ls_h)^{p'} dh \right)^{1/p'}.$$

We have moreover

$$\left(\int_G N_p(kr_h)^p dh \right)^{1/p} = N_p(k)N_p(r)$$

and

$$\left(\int_G N_{p'}(ls_h)^{p'} dh \right)^{1/p} = N_{p'}(l) N_{p'}(s)$$

and consequently

$$\left\| \int_G j(h) dh \right\|_{A_p(G)} \leq N_p(k) N_p(r) N_{p'}(l) N_{p'}(s).$$

Let $x \in G$. We know that $\delta_x \in A_p(G)'$, thus

$$\delta_x \left(\int_G j(h) dh \right) = \int_G j(h)(x) dh.$$

But for every $h \in G$ we have

$$j(h) = \int_G \overline{k(xt) r(xth)} l(t) s(th) dt$$

and therefore

$$\int_G j(h)(x) dh = (\bar{k} * \check{l})(x)(\bar{r} * \check{s})(x).$$

□

Remark 3. *This result is due to Carl Herz. For details, generalizations and references, see ([2] pages 41–44). This proof is based on the use of Bochner integral.*

Definition 7. *Let G be a locally compact group and $1 < p < \infty$. The involutive Banach algebra $A_p(G)$ (see Definitions 4 and 5; see also Theorems 5 and 7) is called the Figà–Talamanca Herz algebra of G.*

We show that the dual of $A_p(G)$ is $CV_p(G)$ for a large class of locally compact groups G including the amenable groups.

Definition 8. *Let G be a locally compact group and $1 < p < \infty$. The topology on $\mathcal{L}(L^p(G))$ associated to the family of seminorms*

$$T \mapsto \left| \sum_{n=1}^{\infty} \left\langle T[k_n], [l_n] \right\rangle \right|$$

with $((k_n), (l_n)) \in \mathcal{A}_p(G)$ is called the "ultraweak topology".

Remark 4. *This topology is locally convex and Hausdorff.*

Definition 9. *Let G be a locally compact group and $1 < p < \infty$. The closure of $\lambda_G^p(M^1(G))$ in $\mathcal{L}(L^p(G))$, with respect to the ultraweak topology, is denoted $PM_p(G)$. Every element of $PM_p(G)$ is called a "p-pseudomeasure".*

Remark 5. *1. $PM_p(G) \subset CV_p(G)$ for every locally compact group G and for every $1 < p < \infty$.*

2. $PM_2(G) = CV_2(G)$ for every locally compact group G. This is a consequence of a deep result of Jacques Dixmier (1952) [8]. We present a self-contained proof of this result for unimodular groups in [2] (Section 2.3 Theorem 5 page 32).

3. $PM_p(G) = CV_p(G)$ for every amenable locally compact group G and for every $1 < p < \infty$.

4. $PM_p(G) = CV_p(G)$ for every $1 < p < \infty$, for a large class of nonamenable locally compact groups including $SO(1,n)$, $SU(1,n)$ and $S_p(1,n)$ (Michael Cowling, (1998)).

5. It is unknown whether $PM_p(G) = CV_p(G)$ for every $1 < p < \infty$ and for every locally compact group G.

Lemma 1. *Let G be a locally compact group, $1 < p < \infty$, $T \in PM_p(G)$, $((k_n),(l_n)), ((k'_n),(l'_n)) \in \mathcal{A}_p(G)$ with*

$$\sum_{n=1}^{\infty} \overline{k_n} * \check{l}_n = \sum_{n=1}^{\infty} \overline{k'_n} * \check{l'_n}.$$

Then,

$$\sum_{n=1}^{\infty} \left\langle T[\tau_p k_n], [\tau_{p'} l_n] \right\rangle_{L^p(G),L^{p'}(G)} = \sum_{n=1}^{\infty} \left\langle T[\tau_p k'_n], [\tau_{p'} l'_n] \right\rangle_{L^p(G),L^{p'}(G)},$$

where $\tau_p \varphi(x) = \varphi(x^{-1})\Delta_G(x^{-1})^{1/p}$.

This Lemma permits writing the following definition.

Definition 10. *Let G be a locally compact group and $1 < p < \infty$. For every $T \in PM_p(G)$, we put*

$$\Psi_G^p(T)(u) = \sum_{n=1}^{\infty} \overline{\left\langle T[\tau_p k_n], [\tau_{p'} l_n] \right\rangle},$$

for every $u \in A_p(G)$ and for every $((k_n),(l_n)) \in \mathcal{A}_p(G)$ with

$$u = \sum_{n=1}^{\infty} \overline{k_n} * \check{l}_n .$$

We are now able to give a description of the dual of $A_p(G)$.

Theorem 8. *(Eymard, Figà–Talamanca, Herz) Let G be a locally compact group and $1 < p < \infty$. Then:*

1. *Ψ_G^p is a conjugate linear isometry of $PM_p(G)$ onto $A_p(G)'$.*
2. *$\Psi_G^p(\lambda_G^p(\tilde{\mu})) = \mu$ for every $\mu \in M^1(G)$, where $\tilde{\varphi}(x) = \overline{\varphi(x^{-1})}$ and $\tilde{\mu}(\varphi) = \overline{\mu(\tilde{\varphi})}$.*
3. *Ψ_G^p is an homeomorphism of $PM_p(G)$, with the ultraweak topology, onto $A_p(G)'$, with the weak topology $\sigma(A_p(G)', A_p(G))$.*

See [2] Section 4.1 Theorem 6 pages 49–51.
We now define a pairing between $A_p(G)$ and $PM_p(G)$.

Definition 11. *Let G be a locally compact group and $1 < p < \infty$. For every $T \in PM_p(G)$ and and for every $u \in A_p(G)$, we put*

$$\left\langle u, T \right\rangle_{A_p(G),PM_p(G)} = \sum_{n=1}^{\infty} \overline{\left\langle T\left[\tau_p k_n\right], \left[\tau_{p'} l_n\right] \right\rangle}$$

for every $((k_n),(l_n)) \in \mathcal{A}_p(G)$ such that

$$u = \sum_{n=1}^{\infty} \overline{k_n} * \check{l}_n .$$

Remark 6. *The map $(u, T) \mapsto \left\langle u, T \right\rangle_{A_p(G),PM_p(G)}$ is a sesquilinear form on $A_p(G) \times PM_p(G)$.*

Corollary 2. *Let G be a locally compact* **abelian** *group. Then:*

1. *$CV_2(G) = PM_2(G)$ see [2] Section 4.2 Theorem 1 page 52. This is obtained without using [8].*
2. *For $p = 2$, the pairing $\left\langle u, T \right\rangle_{A_2(G),PM_2(G)}$ is the concrete pairing $\left\langle f, \phi \right\rangle_{L^1(\widehat{G}),L^\infty(\widehat{G})}$. The precise relation between these pairings is*

$$\left\langle u, T \right\rangle_{A_2(G),PM_2(G)} = \left\langle \Phi_{\widehat{G}}^{-1}(u), \widehat{T} \right\rangle_{L^1(\widehat{G}),L^\infty(\widehat{G})}$$

for every $u \in A_2(G)$ and for every $T \in CV_2(G)$.

3. *We also get the following important complement to Theorem 1: the map $\Lambda_{\widehat{G}}$ is an homeomorphism of $L^\infty(\widehat{G})$, with the topology $\sigma(L^\infty, L^1)$, onto $CV_2(G)$ with the ultraweak topology.*
4. *For every $u \in \mathcal{L}^\infty(\widehat{G})$ there is a net (u_α) of trigonometric polynomials such that $\lim_\alpha u_\alpha = u$ with respect to the $\sigma(L^\infty(\widehat{G}), L^1(\widehat{G}))$ topology and such that $\|u_\alpha\|_\infty \le \|u\|_\infty$ for every α.*
5. *For every $u \in \mathcal{L}^\infty(\widehat{G})$, there is a net (f_α) of integrable functions on G such that $\lim_\alpha \widehat{f_\alpha} = u$ with respect to the $\sigma(L^\infty(\widehat{G}), L^1(\widehat{G}))$ topology and such that $\|\widehat{f_\alpha}\|_\infty \le \|u\|_\infty$ for every α.*
6. *We finally obtain a complement to Theorem 3. For every $1 < p < \infty$, we have $A_2(G) \subset A_p(G)$ and $\|u\|_{A_p} \le \|u\|_{A_2}$ for every $u \in A_2(G)$. Moreover, for every $T \in CV_p(G)$, we also have*

$$\Big\langle u, T \Big\rangle_{A_p, PM_p} = \Big\langle u, \alpha_p(T) \Big\rangle_{A_2, PM_2}.$$

We give some hints for the proof of 6. (see [2] Section 4.2 Theorem 7 page 55).

Proof. Let u be an element of $A_2(G)$. According to Theorem 6, there is $k, l \in \mathcal{L}^2(G)$ with $\|u\|_{A_2(G)} = N_2(k)N_2(l)$. For every $F \in A_p(G)'$, we put

$$\omega(F) = \overline{\Big\langle \alpha_p((\Psi_G^p)^{-1}(F))\big[\check{k}\big], \big[\check{l}\big] \Big\rangle}_{L^2(G), L^2(G)}.$$

Using the bounded weak topology (N. Dunford and J. T. Schwartz, Linear Operators. Part I, page 428), we obtain the existence of $v \in A_p(G)$ with $\omega(F) = F(v)$ for every $F \in A_p(G)'$. This implies $v = u$ and therefore $u \in A_p(G)$ with $\|u\|_{A_p} \le \|u\|_{A_2}$. □

Proposition 1. *Let G be a locally compact group, $1 < p < \infty$, $u \in A_p(G)$, $a \in G$, U a neighborhood of a, $\alpha > 1$ and $\varepsilon > 0$. Then, there is $v \in A_p(G) \cap C_{00}(G)$ and V neighborhood of a such that:*

1. *$0 \le v(x) \le 1$ for every $x \in G$.*
2. *$v(x) = 1$ on V.*
3. *$\|v\|_{A_p} < \alpha$.*
4. *$supp\, v \subset U$.*
5. *$\|uv - u(a)v\|_{A_p} < \varepsilon$.*

Remark 7. *We refer to [2] Section 4.3 Proposition 3 page 58. For G abelian and $p = 2$ the classical proof uses in a very strong way the amenability of G ([3] Chapter 5, Section 2. 3. (ii) page 114). Consequently, from point of view of the Figà–Talamanca Herz algebra $A_p(G)$, every locally compact group is amenable! We are more explicit below.*

A consequence of this is the fact that holomorphic functions operate on $A_p(G)$.

Theorem 9. *Let G be a locally compact group, $1 < p < \infty$, $u \in A_p(G)$, K a compact subset of G, and $F : U \to \mathbb{C}$, an holomorphic function on an open neighborhood U of $u(K)$ in $\mathbb{C}$. Then, there is $v \in A_p(G)$ with $F(u(x)) = v(x)$ for every $x \in K$.*

We need the following corollary.

Corollary 3. *Let G be a locally compact group, $1 < p < \infty$, $u \in A_p(G)$ and K a compact subset of G. Suppose that $u(x) \neq 0$ for every $x \in K$. Then, there is $v \in A_p(G)$ such that $v(x) = \dfrac{1}{u(x)}$ for every $x \in K$.*

4. The Spectrum of a Bounded Function, the Support of a Convolution Operator

Definition 12. *Let G be a locally compact abelian group. We denote by ε_G the canonical map of G into $\widehat{\widehat{G}}$.*

Definition 13. *Let G be a locally compact abelian group and $u \in L^\infty(\widehat{G})$. We call "spectrum of u"the set of all $x \in G$ such that $\varepsilon_G(x)$ belongs to the closure of*

$$\left\{ \sum_{j=1}^{n} c_j \ {\chi_j}u \ \middle| \ n \in \mathbb{N}, \ c_1, ..., c_n \in \mathbb{C}, \ \chi_1, ..., \chi_n \in \widehat{G} \right\}$$

in $L^\infty(\widehat{G})$ with respect to the topology $\sigma(L^\infty, L^1)$. This set is denoted $sp\,u$.

Using the bipolar theorem, it is not difficult to obtain the following two characterizations of the spectrum of u.

Proposition 2. *Let G be a locally compact abelian group, $u \in L^\infty(\widehat{G})$ and $x \in G$. Then, the following properties are equivalent:*

1. *$x \in sp\,u$.*
2. *For every $f \in L^1(\widehat{G})$ with $f * u = 0$ we have $\widehat{f}(\varepsilon_G(x)) = 0$.*
3. *For every open neighborhood W of x there is $h \in L^1(\widehat{G})$ with $supp\left(\widehat{h} \circ \varepsilon_G\right) \subset W$ and $\left\langle h, u \right\rangle_{L^1, L^\infty} \neq 0$.*

We intend to recover ("spectral synthesis problem") every $u \in L^\infty(\widehat{G})$ from $\mathrm{sp}\,u$. The subject of harmonic analysis of bounded functions is not very old, see A. Zygmund [9] vol. II page 335, Notes of Chap. XVI. For $\mathbb{R}$, the above definition (property 2. of the Proposition 2) of the spectrum is **from 1953** [10] .

In fact, we want to do much more.

Definition 14. *Let G be an arbitrary locally compact group, $1 < p < \infty$ and $T \in CV_p(G)$. We call "support of the convolution operator T", the set of all $x \in G$ such that, for every neighborhood U of e and for every neighborhood V of x, there is $\varphi, \psi \in C_{00}(G)$ with $supp\,\varphi \subset U$ $supp\,\psi \subset V$ and $\left\langle T[\varphi], [\psi] \right\rangle \neq 0$. The support of T is denoted $supp\,T$.*

We try to recover every $T \in CV_p(G)$ from $\mathrm{supp}\,T$.

This set is closed. For $\mu \in M^1(G)$, we easily have $\mathrm{supp}\,\lambda_G^p(\mu) = (\mathrm{supp}\,\mu)^{-1}$. Using Condition 3 above, it is not difficult to verify that for G abelian $(\mathrm{supp}\,T)^{-1} = \mathrm{sp}\,\widehat{T}$ for every $T \in CV_p(G)$. Using the duality of $A_p(G)$ with $PM_p(G)$ (Section 3, Theorem 8), we at first obtain a generalization of Property 3.

Theorem 10. *Let G be an arbitrary locally compact group, $1 < p < \infty$, $T \in PM_p(G)$ and $x \in G$. Then, $x \in supp\,T$ if and only if for every open neighborhood V of x there is $u \in A_p(G)$ with $supp\,u \subset V$ and $< u, T >_{A_p, PM_p} \neq 0$.*

Proof. Suppose that for every open neighborhood Z of x there is $u \in A_p(G)$ with $\mathrm{supp}\,u \subset Z$ and $< u, T >_{A_p, PM_p} \neq 0$. We prove that $x \in \mathrm{supp}\,T$. Let U, V open subsets of G with $e \in U$ and $x \in V$. Let W an open neighborhood of x relatively compact with $\overline{W} \subset V$. Choose $u \in A_p(G)$ with $\mathrm{supp}\,u \subset W$, $< u, T > \neq 0$ and $((k_n), (l_n)) \in \mathcal{A}_p(G)$ with $u = \sum_{n=1}^{\infty} \overline{k_n} * \check{l}_n$. Put $\varepsilon = \dfrac{| < u, T > |}{2 \| T \|_p}$. There is $N \in \mathbb{N}$ such that $\sum_{n=N+1}^{\infty} N_p(k_n) N_{p'}(l_n) < \dfrac{\varepsilon}{4}$. There is also $\varphi \in C_{00}(G)$ with $\varphi \geq 0$, $\int_G \varphi(x)dx = 1$, $\mathrm{supp}\,\varphi \subset U^{-1}$ and

$$N_p(\varphi * k_n - k_n) < \frac{\varepsilon}{2 \cdot 2^n (1 + N_{p'}(l_n))}$$

for every $1 \leq n \leq N$. Then, $((\varphi * k_n), (l_n)) \in \mathcal{A}_p(G)$ and

$$\|u - \sum_{n=1}^{\infty} \overline{(\varphi * k_n)} * \check{l}_n\|_{A_p} < \varepsilon.$$

From $\varphi * u = \sum_{n=1}^{\infty} \overline{(\varphi * k_n)} * \check{l}_n$, we get $< \varphi * u, T > \neq 0$; consequently, $\left\langle T[\tau_p \varphi], [\tau_{p'} \check{u}] \right\rangle \neq 0$ with supp $\tau_p \varphi \subset U$ and supp $\tau_{p'} \check{u} \subset V$. □

It is also possible to generalize Property 2. To achieve this goal, we have to introduce on $CV_p(G)$ a structure of left normed $A_p(G)$-module: $(u, T) \mapsto uT$. We give some hints on the definition of uT.

For $k \in \mathcal{L}^p(G)$, $l \in \mathcal{L}^{p'}(G)$ and $T \in CV_p(G)$ there is a unique bounded operator of $L^p(G)$, denoted $(\bar{k} * \check{l})T$, such that

$$< (\bar{k} * \check{l})[\varphi], [\psi] >= \int_G \left\langle T[{}_{t^{-1}}(\check{k})\varphi], [{}_{t^{-1}}(\check{l})\psi] \right\rangle dt$$

for every $\varphi, \psi \in C_{00}(G)$. We have $(\bar{k} * \check{l})T \in CV_p(G)$. Then, for $u \in A_p(G)$, we put $uT = \sum_{n=1}^{\infty} (\overline{k_n} * \check{l}_n)T$ for every $((k_n), (l_n)) \in \mathcal{A}_p(G)$ such that $u = \sum_{n=1}^{\infty} (\overline{k_n} * \check{l}_n)$. See [2] Chapter 5 for a detailed exposition.

If G is abelian for $u \in A_2(G)$ and $T \in CV_p(G)$, we have

$$\widehat{uT} = \Phi_{\widehat{G}}^{-1}(\overline{u}) * \widehat{T}.$$

Consequently, $\widehat{uT}$ is a smoothing of $\widehat{T}$. We can also consider uT as a non commutative smoothing of T: for $T \in CV_p(G)$ and $u \in A_p(G)$ we always have $uT \in PM_p(G)$ ([2] Section 5.2 Theorem 7 page 73). Moreover, for every $u, v \in A_p(G)$ and $T \in PM_p(G)$, we have $< u, vT >=< uv, T >$.

Using the Corollary 3 of Section 3 and this structure of left normed $A_p(G)$-module on $CV_p(G)$, we get the following statement ([2] Section 6.2 Theorem 2 page 89).

Theorem 11. *Let G be an arbitrary locally compact group, $1 < p < \infty$, $T \in CV_p(G)$ and $u \in A_p(G)$. Then, we have $\{x \in G | x \in supp T, u(x) \neq 0\} \subset supp\, uT \subset supp\, T \cap supp\, u$.*

Then, the following extension of Property 2. to all locally compact groups is verified.

Theorem 12. *Let G be an arbitrary locally compact group, $1 < p < \infty$, $T \in CV_p(G)$ and $x \in G$. Then, $x \in supp\, T$ if and only if for every $u \in A_p(G)$ with $uT = 0$, we have $u(x) = 0$.*

We can now make Remark 7 of Section 3 more precise: there is always $m \in PM_p(G)'$ such that $m(uT) = u(e)T$ for every $u \in A_p(G)$ and every $T \in PM_p(G)$. For $L^\infty(G)$ of a locally compact amenable group G, see [3] Ch. 8 § 5.4. page 179.

Suppose G amenable. Then, every $T \in CV_p(T)$ is the limit of a concrete net of compact p-convolution operators with respect to the ultraweak topology. For G abelian every $\varphi \in L^\infty(\widehat{G})$ is the weak * limit of an explicit net of bounded functions having compact spectrum.

Theorem 13. *Let G be a locally compact amenable group, $1 < p < \infty$, $T \in CV_p(G)$ and U a neighborhood of $supp\, T$. Then, there is a net (f_α) of $C_{00}(G)$ such that*

1. *$\lim \lambda_G^p(f_\alpha) = T$ ultraweakly.*
2. *$\left\| \lambda_G^p(f_\alpha) \right\|_p \leq \|T\|_p$ for every α.*
3. *$supp\, f_\alpha \subset U$ for every α.*

The proof of this result ([2] Section 6.5 Corollary 4 page 98) shows that the **construction of the approximating net is explicit**.

We can now **improve** Assertion 5 of Corollary 2.

Corollary 4. *Let G be a locally compact abelian group, $\varphi \in L^\infty(\widehat{G})$ and U a neighborhood of $sp\,\varphi$. Then, there is a net (f_α) of $C_{00}(G)$ such that*

1. $\lim \widehat{f_\alpha} = \varphi$ *with respect to the topology* $\sigma(L^\infty, L^1)$.
2. $\|\widehat{f_\alpha}\|_\infty \leq \|\varphi\|_\infty$ *for every* α.
3. *supp* $f_\alpha \subset U$ *for every* α.

For many locally compact groups including $\mathbb{R}^n, T^n$, the group $O(2)$ of symmetries of the plane and the Heisenberg group of real 3×3 matrices, we can improve the Theorem 13 (and consequently Corollary 4).

Theorem 14. *Let G one the groups $\mathbb{R}^n$ or $\mathbb{T}^n$, $1 < p < \infty$, $T \in CV_p(G)$ and U a neighborhood of supp T. Then, there is a net (μ_α) of finitely supported measures on G such that*

1. $\lim \lambda_G^p(\mu_\alpha) = T$ *ultraweakly.*
2. $\left|\!\left|\!\left| \lambda_G^p(\mu_\alpha) \right|\!\right|\!\right|_p \leq |\!|\!| T |\!|\!|_p$ *for every* α.
3. *supp* $\mu_\alpha \subset U$ *for every* α.

For more on Theorem 14, see [11].

Remark 8. *It is* **not** *possible to replace U by supp T in Theorem 13 and U by $sp\,\varphi$ in Corollary 4. A similar improvement is also excluded for Theorem 14.*

5. $CV_p(G)$ as a Subspace of $CV_2(G)$

We extend to amenable groups the relations $CV_p(G) \subset CV_2(G)$ (see Theorem 3) and $A_2(G) \subset A_p(G)$ (see point 6. of Corollary 2).

Let μ be a positive Radon measure on a locally compact Hausdorff space X, $\mathcal{H}$ a complex Hilbert space and $1 \leq p < \infty$. To every $T \in \mathcal{L}(L^p(X,\mu))$ we associate the unique bounded operator, denoted $T_\mathcal{H}$, of $L^p_\mathcal{H}(X,\mu)$ such that $T_\mathcal{H}(fv) = T(f)v$ for every $f \in L^p(X,\mu)$ and every $v \in \mathcal{H}$.

Theorem 15. *Let μ be a positive Radon measure on a locally compact Hausdorff space X, $\mathcal{H}$ a nonzero complex Hilbert space and $1 \leq p < \infty$. Then, the map $T \mapsto T_\mathcal{H}$ is a linear isometry of $\mathcal{L}(L^p(X,\mu))$ into $\mathcal{L}(L^p_\mathcal{H}(X,\mu))$.*

For the proof and classical motivations, see [2] Chap. 8, Section 8.1, pages 145–150. Using Theorem 15, we obtain a generalization of Theorem 3 to **all locally compact groups.**

Theorem 16. *Let G be a locally compact group, $1 < p < \infty$, $u \in A_p(G)$, $T \in CV_p(G)$ and $\varphi \in L^2(G) \cap L^p(G)$. Then:*

1. $\tau_p\Big(uT(\tau_p\varphi)\Big) \in L^2(G)$.
2. $\|\tau_p\Big(uT(\tau_p\varphi)\Big)\|_2 \leq |\!|\!| T |\!|\!|_p \, \|u\|_{A_p} \|\varphi\|_2$.

Proof. We need some notations.

For $\varphi : G \mapsto \mathbb{C}$, we set $\varphi^*(x) = \overline{\varphi(x^{-1})}\Delta_G(x^{-1})$ for every $x \in G$.

For $F : G \times G \mapsto \mathbb{C}$, we put $\tau(F)(x,y) = F(x, x^{-1}y)$for every $x, y \in G$.

For $F : G \times G \mapsto \mathbb{C}$ and $x \in G$, we denote by $\omega(F)(x)$ the map of G into $\mathbb{C}$ defined by $y \mapsto F(x,y)$.

We denote by $\mathcal{N}_{L^2(G)}$ the subspace of all maps f of G into $L^2(G)$ for which $\{x \in G | f(x) \neq 0\}$ is negligible.

For $F \in C_{00}(G \times G)$, we finally denote by $\varpi(F)$ the element of $L^p(G; L^2(G))$ $f + \mathcal{N}_{L^2(G)}$ where $f(x) = [\omega(F)(x)]$ for every $x \in G$.

Consider k, l, φ, ψ and $\alpha \in C_{00}(G)$. We have

$$\left\langle (\bar{k} * \check{l})\left(T\lambda_G^p(\overline{\alpha^*})\right)[\tau_p\varphi], [\tau_{p'}\psi]\right\rangle = \left\langle \left(T\lambda_G^p(\overline{\alpha^*})\right)_{L^2(G)} \varpi(\tau(F)), \varpi(\tau(F'))\right\rangle$$

where $F = (\tau_p k) \otimes \varphi$ and $F' = (\tau_{p'} l) \otimes \psi$. We refer to [2] Section 8.2 Theorem 4 page 155 for the proof of this relation. Theorem 15 implies that

$$\left|\left\langle \left(T\lambda_G^p(\overline{\alpha^*})\right)_{L^2(G)} \varpi(\tau(F)), \varpi(\tau(F'))\right\rangle\right|$$

$$\leq \left|\!\left| T\lambda_G^p(\overline{\alpha^*})\right|\!\right|_p N_p(k) N_{p'}(l) N_2(\varphi) N_2(\psi)$$

and, consequently,

$$\left|\left\langle (\bar{k} * \check{l})\left(T\lambda_G^p(\overline{\alpha^*})\right)[\tau_p\varphi], [\tau_{p'}\psi]\right\rangle\right| \leq \left|\!\left| T\lambda_G^p(\overline{\alpha^*})\right|\!\right|_p N_p(k) N_{p'}(l) N_2(\varphi) N_2(\psi).$$

Let $\varepsilon > 0$. There is $\alpha \in C_{00}(G)$ with $\alpha \geq 0$, $\int_G \alpha(y)dy = 1$ and

$$\left|\left\langle (\bar{k} * \check{l})\left(T\lambda_G^p(\overline{\alpha^*})\right)[\tau_p\varphi], [\tau_{p'}\psi]\right\rangle - \left\langle (\bar{k} * \check{l})T[\tau_p\varphi], [\tau_{p'}\psi]\right\rangle\right| < \varepsilon$$

and, therefore,

$$\left|\left\langle (\bar{k} * \check{l})T[\tau_p\varphi], [\tau_{p'}\psi]\right\rangle\right| < \varepsilon + |\!|\!|T|\!|\!|_p N_p(k) N_{p'}(l) N_2(\varphi) N_2(\psi).$$

To finish the proof, it suffices to verify that, for $T \in CV_p(G)$, $u \in A_p(G)$ and $\varphi, \psi \in C_{00}(G)$, we have $\left|\left\langle (uT)[\tau_p\varphi], [\tau_{p'}\psi]\right\rangle\right| \leq |\!|\!|T|\!|\!|_p \|u\|_{A_p} N_2(\varphi) N_2(\psi)$.

Let $\varepsilon > 0$. There exist sequences (k_n) and (l_n) of $C_{00}(G)$ with $u = \sum_{n=1}^{\infty} \overline{k_n} * \check{l}_n$ and

$$\sum_{n=1}^{\infty} N_p(k_n) N_{p'}(l_n) < \|u\|_{A_p} + \frac{\varepsilon}{1 + |\!|\!|T|\!|\!|_p N_2(\varphi) N_2(\psi)}.$$

We finally obtain

$$\left|\left\langle (uT)[\tau_p\varphi], [\tau_{p'}\psi]\right\rangle\right| \leq \sum_{n=1}^{\infty} N_p(k_n) N_{p'}(l_n) \, |\!|\!|T|\!|\!|_p N_2(\varphi) N_2(\psi)$$

$$< \varepsilon + N_2(\varphi) N_2(\psi) \|u\|_{A_p}.$$

□

As a corollary, we get for amenable groups the inclusion of $CV_p(G)$ in $CV_2(G)$. This result is due to Herz and Rivière [12].

Corollary 5. *Let G be an amenable locally compact group, $1 < p < \infty$, $T \in CV_p(G)$ and $\varphi \in L^2(G) \cap L^p(G)$. Then:*

1. $\tau_p T \tau_p \varphi \in L^2(G)$,
2. $\|\tau_p T \tau_p \varphi\|_2 \leq |\!|\!|T|\!|\!|_p \|\varphi\|_2$.

Proof. Let $\psi \in C_{00}(G)$ with $N_2(\psi) \leq 1$ and $\varepsilon > 0$. Using the amenability of G, we can find (see [2] Section 5.4 Lemma 1 page 80) $k, l \in C_{00}(G)$ with $N_p(k) = N_{p'}(l) = 1$ and

$$| < (\overline{k} * \check{l})T[\tau_p\varphi], [\tau_{p'}\psi] > - < T[\tau_p\varphi].[\tau_{p'}\psi] > | < \varepsilon.$$

Using Theorem 16, we obtain

$$| < \tau_p T \tau_p[\varphi], [\psi] > | < \varepsilon + \|\tau_p\Big((\overline{k} * \check{l})T\Big)\tau_p[\varphi]\|_2$$

$$\leq \varepsilon + \|\overline{k} * \check{l}\|_{A_p} \,|\!|\!| T |\!|\!|_p\, N_2(\varphi) \leq \varepsilon + |\!|\!| T |\!|\!|_p\, N_2(\varphi).$$

□

Definition 15. *Let G be a locally compact group and $1 < p < \infty$. We denote by E_p the set of all $T \in CV_p(G)$ such that:*

1. *$\tau_p T \tau_p \varphi \in L^2(G)$ for every $\varphi \in L^p(G) \cap L^2(G)$.*
2. *There is $C > 0$ with $\|\tau_p T \tau_p \varphi\|_2 \leq C\|\varphi\|_2$ for every $\varphi \in L^p(G) \cap L^2(G)$.*

Lemma 2. *Let G be a locally compact group and $1 < p < \infty$. Then:*

1. *E_p is a sub algebra of $CV_p(G)$.*
2. *$\lambda_G^p(M^1(G)) \subset E_p$.*
3. *$A_p(G)CV_p(G) \subset E_p$.*

Lemma 3. *Let G be a locally compact group and $1 < p < \infty$. For every $T \in E_p$, there is a unique $S \in \mathcal{L}(L^2(G))$ such that $S\varphi = \tau_p T \tau_p \varphi$ for every $\varphi \in L^p(G) \cap L^2(G)$. We have $\tau_2 S \tau_2 \in CV_2(G)$.*

Definition 16. *Let G be a locally compact group, $1 < p < \infty$, $T \in E_p$ and S as in Lemma 3. We put $\alpha_p(T) = \tau_2 S \tau_2$.*

Lemma 4. *Let G be a locally compact group and $1 < p < \infty$. Then:*

1. *α_p is an injective homomorphism of the algebra E_p into $CV_2(G)$.*
2. *$\alpha_p(\lambda_G^p(\mu)) = \lambda_G^2(\mu)$ for every $\mu \in M^1(G)$.*

We now generalize Theorem 3 to the class of amenable groups.

Theorem 17. *Let G be an amenable locally compact group and $1 < p < \infty$. Then:*

1. *α_p is contractive Banach algebra monomorphism of $CV_p(G)$ into $CV_2(G)$.*
2. *$\Big|\!\Big|\!\Big| \lambda_G^2(\mu) \Big|\!\Big|\!\Big|_2 \leq \Big|\!\Big|\!\Big| \lambda_G^p(\mu) \Big|\!\Big|\!\Big|_p$ for every $\mu \in M^1(G)$.*

Remark 9. *Even for* **a finite group**, *Statement 2 is not trivial !*

Theorem 18. *Let G be a locally compact group, $1 < p < \infty$, $u \in A_2(G)$ and $v \in A_p(G)$. Then, $uv \in A_p(G)$ and $\|uv\|_{A_p} \leq \|u\|_{A_2}\|v\|_{A_p}$.*

Proof. Let $((k_n), (l_n)) \in \mathcal{A}_2(G)$ with $\displaystyle\sum_{n=1}^{\infty} \overline{k_n} * \check{l}_n = u$. For every $F \in A_p(G)'$, we put

$$\omega(F) = \sum_{n=1}^{\infty} \overline{\Big\langle \alpha_p(v(\Psi_G^p)^{-1}(F))[\tau_2 k_n], [\tau_2 l_n] \Big\rangle}$$

for Ψ_G^p (see Definition 10). As in the proof of Point 6 of Corollary 2, we get the existence of $w \in A_p(G)$ such that $\omega(F) = F(w)$ for every $F \in A_p(G)'$. This implies $w = uv$ and consequently $uv \in A_p(G)$. □

We finally generalize Point 6 of Corollary 2 to the class of amenable groups.

Theorem 19. *Let G be an amenable locally compact group and* $1 < p < \infty$. *Then:*

1. $A_2(G) \subset A_p(G)$.
2. *For every* $u \in A_2(G)$ *one has* $\|u\|_{A_p} \leq \|u\|_{A_2}$.

Proof. Let u be an element of $A_2(G)$. Choose $((k_n),(l_n)) \in \mathcal{A}_p(G)$ such that $\sum_{n=1}^{\infty} \overline{k_n} * \check{l}_n = u$. For every $F \in A_p(G)'$, we put

$$\omega(F) = \sum_{n=1}^{\infty} \overline{\left\langle \alpha_p((\Psi_G^p)^{-1}(F))[\tau_2 k_n], [\tau_2 l_n] \right\rangle}.$$

There is $v \in A_p(G)$ such that $\omega(F) = F(v)$ for every $F \in A_p(G)'$. This implies $v = u$ and consequently $u \in A_p(G)$ with $\|u\|_{A_p} \leq \|u\|_{A_2}$.
□

The inclusion of $CV_p(G)$ into $CV_2(G)$, for G abelian is a consequence of the relation

$$\left\| \lambda_G^p(\mu) \right\|_p = \left\| \lambda_G^{p'}(\mu) \right\|_{p'} \ (^*)$$

for every $\mu \in M_{\mathbb{C}}^1(G)$ and the Riesz–Thorin interpolation theorem. For the dihedral group D_4, there is $f : D_4 \to \mathbb{C}$ [13] such that

$$\left\| \lambda_{D_4}^4(f) \right\|_4 \neq \left\| \lambda_{D_4}^{4/3}(f) \right\|_{4/3}.$$

Herz [14] obtained the same statement for every finite nonabelian group. We could prove this for all locally compact groups containing such a subgroup. For specific non-amenable groups, more precise results have been obtained. In [15], Lohoué constructed, for any $1 < p_\circ < \infty$, a positive measure on $SL_2(\mathbb{R})$, which convolves $L^{p_\circ}$ but does not convolve any other L^p. Consequently, for every $1 < p < \infty$ with $p \neq 2$, there is a positive measure which convolves L^p but not $L^{p'}$.

To obtain the inclusion $A_2(G) \subset A_p(G)$ for amenable nonabelian groups, we need substitutes to the relation $(^*)$. Two results are necessary. The first is Theorem 15. This result seems to be an innocuous Banach space property! This theorem requires the use of random variables. It is another formulation of a result due to Marcinkiewics and Zygmund [16]. This permitted them to solve a famous problem raised by Paley. Observe that, in the formulation of Theorem 15, random variables do not appear. Nevertheless, the proof requires them!

The second tool, needed to prove the inclusion of $A_2(G)$ into $A_p(G)$ for amenable groups, is Theorem 16. In fact, it permits to obtain much more. We put, for an arbitrary locally compact group G, the following definition

$$MA_p(G) = \{u : G \to \mathbb{C} \,|\, uv \in A_p(G) \text{ for every } v \in A_p(G)\}$$

and we also set for every $u \in MA_p(G)$

$$\|u\|_{MA_p(G)} = \sup\{\|uv\|_{A_p(G)} \,|\, v \in A_p(G), \|v\|_p \leq 1\}.$$

Then, Theorem 18 implies that $A_2(G) \subset MA_p(G)$ and that

$$\|u\|_{MA_p(G)} \leq \|u\|_{A_2(G)},$$

for every $u \in A_2(G)$. Hence, the coefficients of the regular representation in $L^2(G)$ and of the regular representation in $L^{p'}(G)$ belong to $MA_p(G)$. It is a **general fact**: the coefficients of any bounded representation of G in a Banach space $L^{p'}$ belong to $MA_p(G)$.

We finally improve Theorem 19, obtaining relations between $A_p(G)$ and $A_q(G)$ for arbitrary $p, q > 1$.

Theorem 20. *Let G be a locally compact unimodular amenable group and $p, q \in \mathbb{R}$. Suppose $1 < p < q < 2$ or $2 < q < p$. Then, $CV_p(G) \subset CV_q(G)$ and $A_q(G) \subset A_p(G)$. Moreover, we have the following two inequalities:*

(1) $\|T\|_q \leq \|T\|_p$ *for every $T \in CV_p(G)$.*
(2) $\|u\|_{A_p(G)} \leq \|u\|_{A_q(G)}$ *for every $u \in A_q(G)$.*

Proof. (1) We show that $\left\|\lambda_G^q(\mu)\right\|_q \leq \left\|\lambda_G^p(\mu)\right\|_p$ for every $\mu \in M_{\mathbb{C}}^1(G)$.

Let $\mathcal{S}$ be the set $\left\{[r] \,\middle|\, r \text{ is a complex integrable step function}\right\}$ and $\mathcal{M}$ the set $\left\{[f] \,\middle|\, f : G \to \mathbb{C}\ m_G\text{-measurable}\right\}$. For every $\mu \in M_{\mathbb{C}}^1(G)$ and every $[\varphi] \in \mathcal{S}$, we have $\lambda_G^q(\mu)([\varphi]) = \lambda_G^p(\mu)([\varphi]) = \lambda_G^2(\mu)([\varphi]) = [\varphi * \check{\mu}]$. For every $[\varphi] \in \mathcal{S}$, we put $T([\varphi]) = [\varphi * \check{\mu}]$.

Suppose that $1 < p < q < 2$.

Choosing $t = \dfrac{2(q-p)}{q(2-p)}$, we get
$$\frac{1}{q} = \frac{(1-t)}{p} + \frac{t}{2}.$$

Let $\varphi \in \mathcal{S}$, we have $T\varphi \in L^p(G) \cap L^2(G)$ with $\|T\varphi\|_p \leq \left\|\lambda_G^p(\mu)\right\|_p \|\varphi\|_p$ and $\|T\varphi\|_2 \leq \left\|\lambda_G^2(\mu)\right\|_2 \|\varphi\|_2$. Riesz–Thorin's theorem implies $T\varphi \in L^q(G)$ and
$$\|T\varphi\|_q \leq \left\|\lambda_G^p(\mu)\right\|_p^{1-t} \left\|\lambda_G^2(\mu)\right\|_2^t \|\varphi\|_q.$$

Then, Theorem 17 implies
$$\|T\varphi\|_q \leq \left\|\lambda_G^p(\mu)\right\|_p^{1-t} \left\|\lambda_G^p(\mu)\right\|_p^t \|\varphi\|_q$$

and, finally, $\left\|\lambda_G^q(\mu)\right\|_q \leq \left\|\lambda_G^p(\mu)\right\|_p$. If $2 < q < p$, we proceed similarly.

(2) We have $A_q(G) \subset A_p(G)$ and $\|u\|_{A_p(G)} \leq \|u\|_{A_q(G)}$ for every $u \in A_q(G)$.

Suppose at first $u \in A_2(G)$. We have $u \in A_p(G)$ and
$$\|u\|_{A_p(G)} = \sup\left\{|m(u)| \,\middle|\, \left\|\lambda_G^p(\tilde{\mu})\right\|_p \leq 1, \mu \in M_{\mathbb{C}}^1(G)\right\}$$

and, consequently, $\|u\|_{A_p(G)} \leq \|u\|_{A_q(G)}$.

Suppose now that u is an arbitrary element of $A_q(G)$.

There is a sequence (u_n) of $A_2(G)$ such that $\|u - u_n\|_{A_q(G)} \to 0$. The sequence (u_n) is Cauchy in $A_p(G)$, the space $A_p(G)$ being complete, the function u belongs to $A_p(G)$ and $\|u\|_{A_p(G)} \leq \|u\|_{A_q(G)}$.

(3) We have $CV_p(G) \subset CV_q(G)$ and $\|T\|_q \leq \|T\|_p$ for every $T \in CV_p(G)$.

Let T be a convolution operator of $CV_p(G)$. For every $u \in A_q(G)$, we put $L(u) = \left\langle u, T\right\rangle_{A_p(G), PM_p(G)}$, and we have $|L(u)| \leq \|u\|_{A_p(G)} \|T\|_p \leq \|u\|_{A_q(G)} \|T\|_p$. This implies $T \in CV_q(G)$ with $\|T\|_q \leq \|T\|_p$. □

Funding: This research received no external funding.

Conflicts of Interest: The author declares no conflict of interest.

References

1. Titchmarsh, E.C. Reciprocal formulae involving series and integrals. *Math. Zeit.* **1926**, *25*, 321–347. [CrossRef]
2. Derighetti, A. *Convolution Operators on Groups, Lecture Notes of the Unione Matematica Italiana 11*; Springer: Berlin, Germany, 2011.
3. Reiter, H. *Classical Harmonic Analysis and Locally Compact Groups, Oxford Mathematical Monographs*; Oxford at the Clarendon Press: Wotton-under-Edge, UK, 1968.
4. Riesz, M. Sur les fonctions conjuguées. *Math. Zeit.* **1927**, *27*, 218–244. [CrossRef]
5. Schwartz, L. Sur les multiplicateurs de FL^p. *Proc. R. Physiog. Soc. Lund.* **1953**, *22*, 124–128. See also L. Schwartz, Oeuvres Scientifiques (I), Documents Mathématiques 9, pp. 347–351. Société Mathématique de France (2011).
6. Fefferman, C. The multiplier problem for the ball. *Ann. Math.* **1971**, *94*, 330–336. [CrossRef]
7. Grafakos, L. *Classical and Modern Fourier Operators*; Pearson Prentice Hall: Upper Saddle River, NJ, USA, 2004.
8. Dixmier, J. Algèbres quasi-unitaires. *Comment. Math. Helv.* **1952**, *26*, 275–322. [CrossRef]
9. Zygmund, A. *Trigonometric Series. Volume II*; Cambridge University Press: Cambridge, MA, USA, 1977.
10. Pollard, H. The harmonic analysis of bounded functions. *Duke Math. J.* **1953**, *20*, 499–512. [CrossRef]
11. Delmonico, C. Atomization Process for convolution operators on locally compact groups. *Proc. Am. Math. Soc.* **2006**, *134*, 3231–3241. [CrossRef]
12. Herz, C.; Rivière, N. Estimates for translation-invariant operators on spaces with mixed norms. *Stud. Math. XLIV* **1972**, 44, 511–515. [CrossRef]
13. Oberlin, D.M. $M_p(G) \neq M_q(G)(p^{-1} + q^{-1} = 1)$. *Israel J. Math.* **1975**, *22*, 175–179. [CrossRef]
14. Herz, C. On the Asymmetry of Norms of Convolution Operators, I. *J. Funct. Anal.* **1976**, *23*, 11–22. [CrossRef]
15. Lohoué, N. Estimations L^p des coefficients de représentations et opérateurs de convolution. *Adv. Math.* **1980**, *38*, 178–221.
16. Marcinkiewics, J.; et Zygmund, A. Quelques inégalités pour les opérations linéaires. *Fund. Math.* **1939**, *32*, 115–221. [CrossRef]