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Bootstrapping superconformal field theories in four dimensions

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Abstract

The conformal bootstrap is a non-perturbative technique designed to study conformal field theories using only first principles, such as unitarity, crossing symmetry and the existence of an Operator Product Expansion. In this thesis we discuss an application of the bootstrap method in four dimensional conformal field theories. We also consider in detail the special case where the theory is supersymmetric. In particular we focus on the case study of four abelian currents. The non-supersymmetric setup applies to all conformal field theories with a global abelian symmetry group. When we include the assumption of supersymmetry, the current is taken to be the generator of the R-symmetry, which is tied to the stress tensor due to the superconformal algebra. The supersymmetric setup therefore applies to all local superconformal field theories. We start by introducing all the necessary ingredients. In particular, we discuss the formalism of the embedding space and of the conformal frame to study conformal kinematics. We also give a supersymmetric generalization of the conformal frame formula to count three-point tensor structures. Then we address the important problem of expanding superspace correlators in their components. To this aim we introduce a set of differential operators that act in superspace. Using this formalism we are able to compute the linear relations among the operators in the same superconformal multiplet. This is a necessary step in the computation of superconformal blocks, but it will also be useful for other purposes that we discuss before passing to the bootstrap analysis. First we use it to impose the averaged null energy condition on arbitrary superconformal field theories. This will lead to interesting consequences on their local operator spectrum. Next we focus on the case of local superconformal field theories with eight supercharges and we prove that a certain class of operators termed "exotic primaries" cannot exist. Finally, after a pedagogical introduction to the notion of the conformal bootstrap, we carry out a detailed study of the correlator of four conserved currents. In particular, we compute the conformal and superconformal blocks and the crossing equations. We conclude by proposing several numerical studies and strategies and by showing some preliminary results for the non-supersymmetric case.

Keywords: Conformal field theory, superconformal field theory, superspace, averaged null energy condition, conformal bootstrap, conserved current.

Riassunto

Il bootstrap conforme è una tecnica non-perturbativa ideata per studiare le teorie di campo conformi usando solamente principi primi, come l'unitarietà, la simmetria di crossing e l'esistenza di una Operator Product Expansion. In questa tesi discutiamo un'applicazione del metodo del bootstrap per teorie di campo conformi in quattro dimensioni. Consideriamo in dettaglio anche il caso speciale in cui la teoria è supersimmetrica. In particolare, ci focalizziamo sull'esempio di quattro correnti abeliane. La formulazione non supersimmetrica si applica a tutte le teorie di campo conformi con un gruppo di simmetria globale abeliano. Quando includiamo l'assunzione di supersimmetria, prendiamo come corrente il generatore dell'R-simmetria, la quale è legata al tensore energia-impulso per via dell'algebra superconforme. Di conseguenza la formulazione supersimmetrica si applica a tutte le teorie di campo superconformi locali. Iniziamo introducendo tutti gli ingredienti necessari. In particolare, discutiamo il formalismo dello spazio di embedding e del frame conforme al fine di studiare la cinematica conforme. Inoltre forniamo una generalizzazione supersimmetrica della formula del frame conforme per contare le strutture tensoriali a tre punti. Successivamente consideriamo l'importante problema di espandere i correlatori in superspazio nelle loro componenti. A tal fine introduciamo un insieme di operatori differenziali che agiscono nel superspazio. Usando questo formalismo siamo in grado di calcolare le relazioni lineari tra gli operatori nello stesso multipletto superconforme. Questo è un passaggio necessario nel calcolo dei blocchi superconformi, ma sarà anche utile per altri scopi che discutiamo prima di passare all'analisi del bootstrap. Per prima cosa lo usiamo per imporre la condizione di energia nulla integrata su teorie di campo superconformi arbitrarie. Successivamente ci concentriamo su teorie di campo superconformi con otto supercariche e dimostriamo che una certa classe di operatori chiamati "primari esotici" non può esistere. Finalmente, dopo un'introduzione pedagogica alla nozione del bootstrap conforme, effettuiamo uno studio dettagliato del correlatore di quattro correnti conservate. In particolare, calcoliamo i blocchi conformi e superconformi e le equazioni di crossing. Concludiamo proponendo diversi studi numerici e strategie e mostrando alcuni risultati preliminari per il caso non supersimmetrico.

Parole chiave: Teorie di campo conformi, teorie di campo superconformi, superspazio, condizione di energia nulla integrata, bootstrap conforme, corrente conservata.

Foreword

This thesis is based on three published papers

Paper I A. Manenti, A. Stergiou and A. Vichi, *R-current three-point functions in 4d $\mathcal{N} = 1$ superconformal theories*, *JHEP* **12** (2018) 108 [1804.09717].

Paper II A. Manenti, A. Stergiou and A. Vichi, *Implications of ANEC for SCFTs in four dimensions*, *JHEP* **01** (2020) 093 [1905.09293].

Paper III A. Manenti, *Differential operators for superconformal correlation functions*, *JHEP* **04** (2020) 145 [1910.12869].

And a fourth paper to appear

Paper IV D. Karateev, P. Kravchuk. A. Manenti, A. Stergiou and A. Vichi, *Conformal Bootstrap of Abelian Currents in 4d*, to appear.

Chapter 1 is mainly introductory, but has a small overlap with **Paper II**. Chapter 2 is partly introductory and partly based on **Paper III**; it also has an overlap with **Paper I**. Chapter 3 is entirely based on **Paper III**. Chapter 4 is based on **Paper II**. Chapter 5 is based on a part of **Paper III**. Chapter 6 is again introductory. Finally Chapter 7 and 8 are based on **Paper IV**, except for Section 7.4 which is based on **Paper I**.

The content of Section 2.3 is an adaptation of a part of [1] to the case of three-point functions. It is therefore partly original.

Contents

Introduction

Quantum field theory is a remarkably powerful framework, capable of describing a vast class of phenomena in fundamental physics. We can find theories that admit a formulation in terms of quantum fields across a wide range of energy scales. Notable examples are the theories that describe the critical behavior of quantum and statistical systems and the theories that study the properties of the fundamental interactions for energies below the Planck scale. There are also numerous idealized models that do not describe nature, but are useful for better understanding the underlying mathematical structures of quantum field theory. Superconformal theories in various dimensions are an example of this. Not only they constitute a theoretical laboratory for quantum field theorists, but they also provide a dual description of String Theory in a negatively curved background. Such a duality is often referred to as AdS/CFT correspondence, and it is our most powerful tool aimed at understanding the quantum aspects of gravity.

Despite the importance of quantum field theory throughout nearly all branches of theoretical physics, our understanding of it in a non-perturbative sense is still unsatisfactory. Naturally, most of the interesting phenomena take place in a strongly coupled regime, where our intuition and our computational tools fall short. This means that we need a robust and general framework in which we can study theories that exhibit a strongly coupled behavior. In this thesis we will argue that a good candidate of such a formalism is the so-called conformal bootstrap and we will showcase an application of it in four dimensions. However, before introducing it, let us discuss some general aspects of quantum field theory in order to understand better the context and the motivations.

Renormalization group flow

An important aspect of quantum field theory (QFT) is the dependence on the energy scale. At large distances we cannot resolve the microscopic details of the system under consideration. Therefore the high energy excitations become less and less important as we "zoom out" and can be consistently neglected. In doing so, the Hamiltonian restricted to only the low lying degrees of freedom is modified in order to account for the excitations that we removed. This induces a transformation flow in the space of parameters of the theory which takes the name of Wilsonian renormalization group (RG)

flow.

QFTs at different energy scales may look very different. A notorious example of this is quantum chromodynamics: the theory of strong interactions. At high energy it behaves like a weakly interacting theory of partons and at low energies it is a strongly coupled theory of hadrons, exhibiting confinement and chiral symmetry breaking. When the RG flow makes the couplings grow, we quickly exit the perturbative regime and have fewer tools at our disposal to follow the evolution of a theory. We can try to focus on special points with enhanced symmetry, namely the fixed points: those where the flow is stationary. The theories at the fixed points are by definition scale invariant. Furthermore, in nearly all cases of interest scale invariance is enhanced to a larger group that includes all transformations that preserve angles.[1] This is called the conformal group. QFTs that enjoy the conformal group as a symmetry are called conformal field theories (CFT). They will be the subject of the present thesis.

The modern perspective sees a QFT as the theory that lives along an RG flow between an ultraviolet and an infrared CFT. This interpretation does not encompass all cases as the UV theory need not be a CFT. However we will adopt this viewpoint here.

From an experimentalist's perspective CFTs are quite boring: the physics looks exactly the same at all energy scales and all particles are massless.[2] On the other hand theorists find them interesting for essentially two reasons. First, according to the viewpoint given above, they are in a sense the fundamental objects that give rise to QFTs. Secondly, CFTs are a good approximations of a system that exhibits a separation of scales from the UV to the IR, in an energy range equidistant from the two scales. This feature is present by assumptions in all effective field theories and, in particular, in the Standard Model as well.

Symmetries

Symmetries are our most powerful tool to study QFTs. We already encountered the conformal symmetry, whose consequences will be described at length in this work. Symmetries are of central importance in the context of critical phenomena. They are part of what characterizes a universality class, namely the set of microscopic descriptions of a system which flow to the same theory in the infrared. For example, the Ising universality class is characterized by having two relevant operators (temperature and magnetization) and a $\mathbb{Z}_2$ global symmetry group. Furthermore we can use symmetries to constrain the

[1] To be more precise, scale symmetry is always enhanced to conformal symmetry in two dimensional unitary CFTs [2]. In four dimensions instead it is still an open problem [3–5].

[2] This statement depends on our common way of thinking about high energy physics in terms of particles. However there has been a proposal of a type of matter that is not comprised of particles. This type of excitations go under the name of "unparticles" [6,7]. With this scenario in mind, a conformal sector would actually be very interesting experimentally.

local observables by means of selection rules and Ward identities.[3] Both are very robust results of representation theory and therefore are independent on the coupling.

A widely studied type of symmetry involves transformations that relate objects with different spins and statistics to each other. It goes under the name of supersymmetry. It is not associated to a Lie group in the usual sense because the generators satisfy graded commutation relations. Nevertheless it shares many properties with ordinary Lie groups.

Supersymmetry was first introduced for phenomenological reasons. A possible scenario that explains the Higgs mass naturalness is offered by a supersymmetric extension of the Standard Model. The contributions to the Higgs mass given by loops of bosons and fermions cancel if the theory is supersymmetric. This lead to the conjecture that there should be new physics above ~ 1 TeV which consists in superpartners to the particles of the Standard Model. This also requires that supersymmetry must be broken as we do not observe it at lower energies. However, the modern developments in supersymmetry are less oriented to phenomenology and more oriented to understanding pure QFT or quantum gravity.

The main reason why supersymmetry is still a very active area of research is that String Theory requires it in order to be free of quantum instabilities. As a consequence, when we compactify strings on a manifold that preserves a certain amount of supercharges, we obtain a supersymmetic QFT at low energies (upon taking the appropriate decoupling limit).

Furthermore the rare occasions where we can compute quantities exactly in an interacting QFT typically require some amount of supersymmetry as, in that case, we have additional tools at our disposal, like supersymmetric localization [8–10]. This is essentially a consequence of the existence of protected observables. In an ordinary QFT protected operators can only be conserved currents, otherwise the theory would be free. Whereas in supersymmetric QFTs we have a much richer variety of protected operators, giving rise to many beautiful mathematical structures, which we will not review in this thesis.

When we impose supersymmetry on a CFT we obtain a superconformal field theory (SCFT). The combination of conformal symmetry and supersymmetry implies very strong constraints on the resulting theory. This is because the representation theory of superconformal algebras is very stringent. The algebras have been classified by Nahm [11] and their unitary representations have been studied in great detail leading to a classification for all dimensions and all amounts of supercharges [12–15].

[3] By selection rules we mean statements like $\langle \mathcal{O}_1 \dots \mathcal{O}_n \rangle = 0$ if no singlet appears in the tensor product of representations $\bigotimes_{i=1}^{n} \rho(\mathcal{O}_i)$. By Ward identities we mean the class of identities that involve the integral of a correlator containing a conserved current. Namely $\int \mathrm{d}\Omega \, \langle JX \rangle \sim \langle Q[X] \rangle$, Q being the charge operator.

Conformal field theories in four dimensions

Physics in four dimensions is undoubtedly the most interesting one to study since we live in a universe that is four dimensional (at least at the energy scales probed so far). It is disappointing that not much is known about interacting non-supersymmetric $4d$ CFTs. The only class of candidates available to us so far are the so-called Caswell-Banks-Zaks fixed points [16, 17]. They are obtained as the low energy limit of an asymptotically free gauge theory coupled to matter, such as quantum chromodynamics. In the case of an $\mathrm{SU}(N_c)$ gauge theory with fundamental fermions transforming under the flavor group $\mathrm{SU}(N_f)$, the fixed point exists only when N_f and N_c lie in a specific range termed conformal window, whose precise lower limit is still under debate.[4] At $N_c = 3$ different estimates of the lower bound on N_f range from 10 to 12. See [18] for a recent study, [19] for a review of the lattice results and [20–24] for computations via other methods.

Naturally it is difficult to get quantitative results as the IR limit of these theories is notoriously strongly interacting. This has to be contrasted with the situation in three dimensions where interacting fixed points are more abundant. That is because the Lagrangians are easier to construct since they typically consist in theories of scalars and at most fermions. The prime example is given by the $\mathrm{O}(N)$ vector models. Due to the simplicity of these theories, the perturbative ϵ-expansion followed by resummation is a viable approach and it gives correct results. Moreover, at large N one can also study these theories in a $1/N$ perturbative expansion. Later we will say more about three dimensions in the context of the conformal bootstrap.

If we allow for supersymmetry, the situation in four dimensions is not so dire anymore. The amount of supersymmetry[5] $\mathcal{N}$ can take values from 1 to 4 in four dimensions. If $\mathcal{N} = 1$ there is recent evidence of a so-called minimal interacting SCFT, whose Lagrangian description is still unknown [25, 26]. Another example is analogous to the Caswell-Banks-Zaks fixed point: it is a supersymmetric extension of QCD, which is believed to have a conformal window for $\frac{3}{2}N_c < N_f < 3N_c$ [27].

The story for $\mathcal{N} = 2$ supersymmetry is much richer. There are several geometric constructions inspired by the String Theory duals. Some examples are the so-called class $\mathcal{S}$ SCFTs [28, 29], a generalization of the Argyres-Douglas theories [30, 31]. In addition, there exists a classification of $\mathcal{N} = 2$ SCFTs with a two-dimensional Coulomb branch (also called rank 1 SCFTs) [32, 33]. All $\mathcal{N} = 2$ SCFTs are associated to a two-dimensional vertex operator algebra through a cohomological construction [34].[6] This may give insights towards a possible more general classification in the future.

On the contrary, the class of $\mathcal{N} = 4$ SCFTs is believed to contain only one theory:[7] $\mathcal{N} = 4$

[4] The upper limit of $N_f \leqslant \frac{11}{2}N_c$ is more robust because it is accessible in perturbation theory.

[5] In four dimensions the number of supercharges N_Q is $N_Q = 4\mathcal{N}$.

[6] A very similar construction also works for six dimensional SCFTs with maximal supersymmetry [35].

[7] By one theory we mean one conformal manifold of theories. That is, one family of theories obtained by

super Yang-Mills (SYM).[8] It is dual to String Theory on $AdS_5 \times S^5$ [36] and has since been one of the most studied examples of this duality. The only case left to discuss is $\mathcal{N} = 3$. Surprisingly, examples of such SCFTs have been found only recently [37–39] and a bootstrap study was carried out in [40]. The reason is that any Lagrangian construction will automatically fall back to $\mathcal{N} = 4$ supersymetry due to CPT invariance. In this thesis we will not be interested in $\mathcal{N} = 4$ or 3 supersymmetry.

It seems that the situation for $\mathcal{N} = 1$ supersymmetry is closer to the one of non-supersymmetric CFTs. Namely our knowledge of the landscape of consistent theories is still rather limited. This is due to the absence of all the sophisticated mathematical structures that make an appearance after $\mathcal{N} \geqslant 2$ — like vertex operator algebras, string constructions etc. It is worth then to adopt a more explorative approach. One possibility is the numerical bootstrap and we will explain it shortly in the context of general (not necessarily supersymmetric) CFTs.

1 The conformal bootstrap

Axiomatic approach to conformal field theories

In the axiomatic approach to CFTs we are interested in studying the properties of correlation functions of local operators. Conformal symmetry completely fixes the functional form of each three-point function up to a finite number of real numbers called operator product expansion (OPE) coefficients. The collection of OPE coefficients, conformal dimension and spin of all the operators in the theory is referred to as CFT data. Higher-point correlation functions may be obtained in terms of the lower ones by means of the so-called operator product expansion, which we will define shortly. Therefore, if we want to fully specify all observables of local operators, it is sufficient to simply provide the set of CFT data. Since in this context we do not care about nonlocal operators — such as Wilson lines or defects — this is enough to fully characterize the theory.

A distinctive feature of CFTs is that the operators live in an Hilbert space because they can be put in a one-to-one correspondence with the states in radial quantization. As a consequence we can always expand the product of two local operators at points x and $x + \varepsilon$ as an infinite sum over local operators at x. This is the so-called operator product expansion. It is possible to show that the OPE always converges in a CFT [41–43].

By taking the OPE of different pairs of operators inside a correlation function we can eventually reduce it to sums of two-point functions. However, for $n \geqslant 4$-point functions

varying the marginal coupling τ.

[8] It should be emphasized that there is no rigorous evidence supporting this lore yet, except for the fact that it holds true when restricted to Lagrangian theories. The uniqueness of super Yang Mills as an $\mathcal{N} = 4$ SCFT is still a very important matter to be settled.

there is more than one way to pair the operators and they all have to agree. This puts strong, non-perturbative, constraints on the CFT under study. The axiomatic approach consists in imposing this constraint on the correlators and "bootstrapping" the theory from it. In this context bootstrapping refers to obtaining something out of nothing, that is, obtaining a fully fledged quantum theory just from its consistency conditions. Clearly crossing is not the only constraint that can be imposed. One can also require unitarity — which will turn out to be crucial for the numerical methods — and any global symmetry that is shared by the theories under investigation.

This axiomatic approach was initiated in the seventies with the seminal works of Polyakov, Mack, Ferrara and several other authors [44–48].[9] Unfortunately, the approach in its original formulation was unable to produce concrete results for many years. The reason is that the crossing equations are too complicated to be studied analytically in the case of $d > 2$ dimensions. The case of two dimensions is special because the conformal symmetry is enhanced to the Virasoro algebra. Indeed there are some instances where crossing has been solved explicitly in closed analytic form. An example is the solution of Liouville theory with the DOZZ formula [49, 50]. In higher dimensions this quickly becomes unfeasible. However, in 2008 there was a rebirth of this program when it was realized that the crossing equations could be studied by means of a numerical method [51].

A numerical method

The modern revival of the conformal bootstrap was motivated by a question of naturalness in the context of the theory of conformal technicolor. The details of this theory are outside the scope of this thesis. However the concrete question that needed to be addressed boils down to: "how high can we make the conformal dimension of ϕ^2 if we fix the dimension of ϕ to be Δ_ϕ?"

This calls for an axiomatic approach: we want to be completely agnostic about the theory and consider the correlator $\langle\phi\phi\phi\phi\rangle$. Since ϕ^2 is part of the OPE $\phi \times \phi$, we will be able to constrain its dimension by imposing the crossing equations. Unitarity implies that the OPE coefficients squared are positive numbers. This turns the crossing equation in a particular type of convex optimization problems, for which there exist efficient numerical algorithms in the mathematical literature.

One of the first results of the numerical bootstrap program was the determination of the critical exponents of the $3d$ Ising model [52, 53]. The Ising model was recognized in a feature or "kink" present in the exclusion plots.[10] By kink we mean a noticeable

[9] We will not attempt to give a detailed historical account of the early developments of the bootstrap in this introduction.

[10] It is often correct to interpret a kink as a physical theory. In this case it definitely is. An exception is when the kink heavily depends on some additional, ad hoc, assumptions or it is numerically unstable.

change in the slope of the exclusion bounds. Further studies we able to improve on this result and to isolate the theory in a closed region in parameter space that is referred to as "island." In the latter, a crucial assumption was that there was only one $\mathbb{Z}_2$ even relevant scalar (in the RG sense).

Other noteworthy results in $3d$ are the $\mathrm{O}(N)$ models [54–56]. In this case the authors obtained a family of islands, one for each value of N. More recent studies have been focused in particular to the $\mathrm{O}(2)$ case because of its phenomenological relevance [57, 58].

There were also several attempts to bootstrap gauge theories. See e.g. [59] where the authors bootstrapped correlators of monopoles in QED_3. In this axiomatic formulation it is impossible to identify with certainty a gauge theory, because its physical correlators will be made of gauge invariant operators. Nevertheless one can gain sufficient evidence by comparing with available perturbative or large N computations.

Another interesting application of the bootstrap in $3d$ is the study of M-theory on a stack of N coincident M2-branes. This can lead to a maximally supersymmetric SCFT in the decoupling limit [60–62] (see also [63]). Different bootstrap studies were made in [64–66]. In particular, the results of [66] show a large variety of high precision islands.

All attempts in four dimensions have so far been unsuccessful in finding islands. The main targets of the numerical studies are either gauge theories — inspired by the hope of finding a Caswell-Banks-Zaks-like fixed point — or supersymmetric theories. The seminal paper and its follow-ups were set in four dimensions and aimed at finding general bounds [51, 67–69]. Later, a more detailed study followed [25]. Further investigations assumed the presence of global symmetries [70] such as $\mathrm{SO}(N)$ [71, 72], $\mathrm{SO}(N) \times \mathrm{SO}(M)$ [73], $\mathrm{SU}(N)$ [73,74] and $\mathrm{SU}(N) \times \mathrm{SU}(N)$ [72,75]. Other studies instead assumed $\mathcal{N} = 1$ supersymmetry [25, 26, 76].

Minimal supersymmetry and extended supersymmetry are fairly different in terms of their bootstrap setups. The former is similar to the non-supersymmetric setup. The only difference is that the contributions of operators in the same multiplets need to be grouped together in what are known as the superconformal blocks. On the other hand the setup for extended supersymmetry also needs to take into account the existence of protected operators. The contributions of the protected operators can be computed in a model independent way and can thus be input in the crossing equations. Furthermore, the superconformal blocks are never computed explicitly as in the $\mathcal{N} = 1$ case, rather they are obtained by alternative methods which work only for protected external operators (which are typically the cases of interest). Examples of such bootstrap studies for $\mathcal{N} = 2, 4$ supersymmetry are [77–81].

Boostrap of spinning operators

The works we cited so far consisted in applying crossing symmetry on a correlator of four scalar operators. The bootstrap method however does not have such a limitation and one is free to consider operators with arbitrary spin as well. The only example considered so far in four dimensions is the bootstrap of Weyl fermions transforming under a $\mathrm{U}(1)$ global symmetry [82]. Unfortunately, the bounds look rather smooth and there is no conclusive evidence of new theories. This might be due to the fact that the interesting theories lie well inside the allowed region and so have little effect on the shape of the bound. In three dimensions on the other hand there have been already several works that considered a spinning setup [58, 83, 84].

The computations quickly become very challenging and intensive as the spin is increased. But of course there are theoretical motivations for considering other spinning operators as well. There are two operators which are special: the conserved current (vector of dimension $d-1$) and the stress-energy tensor (rank-two symmetric traceless tensor of dimension d). The former appears in any theory with a global symmetry, due to the Noether theorem, and the latter appears in any local theory. Their correlators are tightly constrained by the Ward identities and the conformal dimensions of the external operators are fixed. This leaves us with fewer free parameters and, as a consequence, we can obtain rigorous, theory independent, bounds on a large class of theories. For example, an upper bound $\Delta^\star$ on the dimension of the lightest scalar in the OPE of two currents implies that *any* theory with a global symmetry must have a scalar operator of dimension less than $\Delta^\star$. An example of this can be found in three dimensions [84].

Another motivation behind considering spinning operators is that in dimensions bigger than three there are some CFT data which are inaccessible by bootstrapping only scalars. Those are the data associated to operators with non-vanishing transverse spin.[11] In the OPE of spinning operators instead we can find exchanged operators of nonzero transverse spin. Imposing assumptions on them gives us more constraining power and, in turn, may lead to stronger bounds.

Currents are particularly interesting in the case of supersymmetric theories. Every SCFT comes with a global symmetry:[12] the R-symmetry. Representation theory dictates that the current of the R-symmetry must be part of the same multiplet containing the stress tensor. In supersymmetric theories with at least 8 supercharges we are in luck because the lowest component of this multiplet is a scalar, so we do not have to resort to an expensive spinning setup. Meanwhile for lower amounts of supersymmery the bottom component has nonzero spin. In particular, for exactly 4 supercharges the bottom component is precisely the R-current. The present thesis will be focused on this case.

[11] In four dimensions the transverse spin of $(j, \bar{j})$ is $|j - \bar{j}|$. In general, it is given by the number of boxes in the Young tableaux that belong to the rows after the first one.

[12] Except $3d$ $\mathcal{N} = 1$.

It should be noted, however, that one has the possibility of regarding the current as part of a flavor multiplet as well. In that case the analysis would be dependent on the assumption that the theories under investigation have a global symmetry other than the R-symmetry.

As we remarked in the previous paragraphs, considering a bootstrap setup with spinning operators comes with many challenges. The additional indices that appear in the correlator give rise to different tensor structures, which will most likely mix under crossing. Furthermore, the OPE becomes considerably more involved. Computations of this sort require the use of a powerful and general formalism that allows for automation on a computer.

2 Studying conformal kinematics

The many formalisms in conformal field theory

In a CFT the functional form of two and three-point functions is fixed. Imposing conformal invariance on a correlator however is not straightforward, especially if the operators have nonzero spin. The formalism for symmetric traceless tensors for general dimension d was developed in the nineties by Osborn and Petkou [85]. Soon after the 2008 revival it became necessary to have a lighter and more efficient formalism, in order to open up the possibility of bootstrapping spinning operators.

If the d dimensional space is embedded in a $d+2$ dimensional space with signature $(2,d)$ the conformal group acts linearly. Dirac was the first one to use this fact to study wave equations [86]. This idea goes under the name of embedding formalism. In its more recent developments it is combined with the so-called index-free formalism, where one introduces polarizations to contract all open indices. The formulation for general d initially dealt only with symmetric traceless representations — since they are not d dependent [87,88]. In four dimensions or higher there exist also non symmetric traceless representations. If one is interested in those representations, it is probably best to make a separate treatment for each number of dimensions. The only case studied in depth so far is that of $d=4$ [89–91]. With more work, however, it is possible to develop a theory for general d as well [92].

Another formalism that is used goes under the name of conformal frame [93]. Unlike the embedding formalism, here covariance is lost. However the frame is chosen such that as many coordinates as possible are fixed to some convenient value, so that the tensor structures become very simple objects. It is also useful for counting the number of independent tensor structures.

The embedding formalism and the conformal frame are both indispensable for setting

up the bootstrap of a correlator of spinning operators. There are several steps that need to be performed: the three-point function analysis, the computation of the conformal blocks and the crossing equations. The first two are best done with the embedding formalism and the last one is best done in conformal frame.

The most important elements required for performing a bootstrap study are the conformal blocks. They represent a contribution of a single primary operator to a four-point function, analogously to the partial waves for a scattering amplitude. The simplest conformal blocks are those of four external scalars and they were computed in the pioneering works of Dolan and Osborn [94–96]. They found explicit formulas for even spacetime dimension. Later, with the aid of the so-called recursion relations, the conformal blocks were computed for arbitrary complex values of d [97,98]. They are not given in a closed for expression, but it is possible to present them in an expansion that converges exponentially fast [99]. These recursion relations have recently been generalized for the general spinning case in $d = 3$ [100], see also [101]. A general strategy for computing spinning conformal blocks — which we will also adopt here — is to first compute the conformal blocks of a minimal four-point function,[13] which are referred to as seeds. From the seeds one can then obtain the desired blocks by the action of some conformally covariant differential operators [88,90,102,103].

There are several other approaches available in the literature. One example is the shadow formalism [104,105]. There exists also an alternative formulation of the embedding space which works in full generality for all representations of the spin group [106–111]. There the conformal blocks are constructed by applying a specific set of substitution rules on a compact expression given in terms of Gegenbauer polynomials. The elementary building blocks of these substitution rules are a generalization of Exton G-functions. The theory is very rich but, unfortunately, it is outside the scope of this thesis. Yet another interesting approach consists in translating the problem of computing conformal blocks in that of finding a solution to a quantum mechanical model called Calogero-Sutherland model [112,113].

Superspace

The supersymmetry algebra mixes with the conformal algebra, it is therefore necessary to introduce a new formalism when dealing with SCFTs. Since supersymmetry entails the presence of spinors, all formalisms for superconformal symmetry tend to be dimension specific, most commonly for $d = 4$.

Superconformal multiplets can be conveniently grouped in a single field that lives in superspace. Superspace is an extension of ordinary Minkowsky spacetime by the addition of Grassmannian coordinates θ_i. The action of the supersymmetry algebra can be seen

[13] Minimal means the simplest possible four-point function that exchanges the desired representation.

as a transformation acting on these Grassmann variables. Since each coordinate squares to zero, the fields can be written as a terminating Taylor expansion. The coefficients of this polynomial in the θ_i are the various operators in the multiplet.

The correlation function of n superfields $\Phi_1, \ldots, \Phi_n$ automatically contains, in its Taylor expansion in θ, all correlators of all operators in the multiplets associated to the Φ_i's. In particular, this can be used to relate among each other the OPE coefficients of the operators in the same multiplets. These relations, while being purely kinematic, cannot be obtained by simple group theoretic arguments. The only technical obstacle is to actually perform the expansion in θ because the expressions of the superspace correlators are rather involved. Part of the thesis is devoted to address this problem in great detail as it is a vital step in the computation of superconformal blocks. The superconformal blocks are sums of conformal blocks with appropriate coefficients dictated by the supersymmetry algebra, and they represent the contribution of an entire superconformal multiplet to a four-point function.

The first appearance of a complete formalism for superconformal symmetry focused on $4d$ $\mathcal{N} = 1$ [114, 115]. It was later generalized to extended supersymmetry [116], in particular $\mathcal{N} = 2$ [117], and to six dimensions [118]. The case of extended supersymmetry can also be specialized to protected operators, where the Ward identities are very strong and greatly constrain the functional form of the four point function. As a consequence, the blocks can be computed almost directly [119–121]. There also exists a supersymmetric version of the embedding formalism [122, 123]. The only disadvantage of this method is that it has not been found yet a way to fully characterize the redundancies among the various structures. It has however been successfully used for the computation of superconformal blocks for general scalars in $\mathcal{N} = 1$ [124] and the stress tensor multiplet in $\mathcal{N} = 2$ [125].

Most of the results involving supersymmetric conformal blocks focus on scalar external operators [120, 124–130]. This does not go hand in hand with the huge progress made for spinning operators in non-supersymmetric CFTs, as we reviewed previously. Recently a general theory has been developed [1]. In the formalism of [1] the superconformal blocks are computed by solving a perturbation of the Calogero-Sutherland model [112] which becomes exact at a finite order. Another result involving general superconformal blocks is the study of their pole structure given in [131].

In this thesis we will address the problem of computing spinning superconformal blocks. Specifically, we will study the blocks of four currents in $4d$ $\mathcal{N} = 1$ SCFTs. We will however utilize a different formalism than the one mentioned above. It consists in the introduction of differential operators which act in superspace and can be used to simplify the task of expanding the superspace correlators in the Grassmann variables.

3 Locality

Locality implies that the energy-momentum measurements follow Gauss' law. Namely, the flux of energy-momentum on a codimension-one surface Σ does not depend on continuous deformations of Σ. This requires the existence of a spin two operator which is conserved and has dimension d: the stress-energy tensor $T_{\mu\nu}$.

The OPE coefficients of $T_{\mu\nu}$ are tightly constrained by the conservation equations and the Ward identities. In setting up a bootstrap problem, it is important to carefully analyze all such constraints in order to prevent unphysical solutions being part of the search space. As we will argue later, sometimes Ward identities alone, without any dynamical input, may have interesting consequences on their own.

Supersymmetric Ward identities

A possible interesting problem where the Ward identities may have some impact regards the existence of a class of superconformal multiplets called exotic primaries [132]. This is precisely an instance where Ward identities alone are strong enough to completely fix the three-point functions of a stress tensor multiplet and two exotic primaries. Remarkably, however, the solution of the Ward identity turns out to be inconsistent with supersymmetry. As a result the exotic primaries cannot appear in any local theory.

Energy conditions

Ward identities are not the only constraints that a correlator of the stress tensor has to satisfy. Three-point function, in particular, are subject to a class of inequalities that go under the name of averaged null energy condition (ANEC).

The averaged null energy is an observable that has a long history in jet physics — see for example [133–135] — but it was first examined in the CFT context in the seminal work [136]. There, it was shown that an energy-positivity condition implies constraints on the coefficients in the three-point function of the stress-energy tensor. More precisely, the expectation value of the stress tensor in a state $|\psi\rangle$ integrated along a null geodesic must be a non-negative quantity. In [136] this was viewed as a positivity requirement for the energy measured by a hypothetical "calorimeter" placed at a large distance from the region where $|\psi\rangle$ is localized. It was later proved with several different approaches [137–139]. In holographic CFTs this inequality has a simple interpretation. It is a direct consequence of the causality of signals that dip into the bulk.

One might naively think that the only consequence of the ANEC is a system of inequalities on the OPE coefficients. Indeed the first applications of it did have results of this sort [136, 140]. However, it was later noted that a more careful study of the

ANEC constraints leads to bounds on the conformal dimensions as well [141]. In this thesis we study an analogous set of bounds that follow from further assuming $\mathcal{N} = 1$ supersymmetry, thus extending the results of [141].

4 Outline

This thesis will be aimed at setting up the bootstrap problem for abelian currents in four dimensional CFTs. All the steps leading up to the final setup will be explained in detail and the pertinent literature will be reviewed along the way. For completeness, when possible, we will explain the main concepts with more generality than what is needed for the final result.

We subdivide the material in three parts, Part I studies the consequences of conformal and superconformal symmetry on the local observables of the theory. Part II is an interlude: it concerns the implications of locality — i.e. the existence of a stress tensor — on any SCFT. Finally Part III contains the study of the conformal blocks of four currents.

Outline of Part I

In more detail, Chapter 1 introduces CFTs and the correlation functions of local operators. The formalism of embedding space and conformal frame are reviewed. The material is somewhat standard and can be skipped by an expert audience, except for Subsection 1.3.2 which is needed for Section 2.3.

Chapter 2 instead is about SCFTs. The concept of superspace is introduced, mainly focusing on four dimensions with $\mathcal{N} = 1$ and 2. Section 2.2 is important in order to familiarize with the notation. Section 2.3 instead contains some partially novel results about counting of superconformal tensor structures.

Chapter 3 introduces a set of differential operators in SCFTs for expanding superconformal correlators in the Grassmann variables. This is needed for the computation of superconformal blocks and, in general, to relate the OPE coefficients of the operators in the same multiplet. The analysis has been done for general $4d$ $\mathcal{N} = 1, 2$ SCFTs.

Outline of Part II

Chapter 4 discusses the constraints of the ANEC on general four dimensional $\mathcal{N} = 1$ SCFTs. The analysis consists in expanding a superspace correlator in the Grassmann variables, applying all Ward identities and imposing the ANEC. The solution of the inequalities is obtained by means of numerical techniques.

Chapter 5 discusses the proof of the absence of the exotic primaries from any local

$\mathcal{N} = 2$ SCFT.

Outline of Part III

Chapter 6 is an introduction to the concept of the conformal bootstrap. In particular, Section 6.2 discusses the method of semidefinite programming and Section 6.4 reviews all the tools available in four dimensions for computing conformal blocks of spinning operators. These concepts will be all used in the subsequent chapter. This chapter can be safely skipped by readers who are already familiar with the literature.

Finally Chapter 7 contains the computation of the conformal blocks. This is original, but still unpublished, work. Section 7.1 contains the analysis of non-supersymmetric three-point functions, Section 6.3 instead contains the analysis of the four-point function $\langle JJJJ \rangle$. Section 7.3 shows the computation of the conformal blocks for all exchanged operators. Section 7.4 and 7.5 address the necessary modifications to the setup to include the assumption of $\mathcal{N} = 1$ supersymmetry. Section 8.1 shows the final result: the set of crossing equations that need to be analyzed by the numerical bootstrap machinery. Finally Section 8.2 discusses some concrete proposals for the future numerical investigations.

Outline of the appendices

The appendices are numerous and some are rather technical. Appendix A contains all notations and conventions used throughout the manuscript. Appendix B has more details regarding the superspace formalism and has some identities useful for Chapter 3. In particular, Appendix B.1 will turn out to be useful for the computation of superconformal blocks. Appendix C has all the intermediate results needed for the ANEC analysis. Appendix D contains all the results relative to the bootstrap analysis. In particular Appendix D.1 contains necessary data for the conformal blocks, Appendix D.2 has the definition of our choice of four-point structures and finally Appendix D.3 contains all the necessary information to compute superconformal blocks starting from the non-supersymmetric ones.

Superconformal symmetry Part I

1 Conformal symmetry

1.1 The conformal group

The aim of this chapter is to introduce conformal symmetry in quantum field theories. It is far from being a self-consistent and complete exposition. A classic reference is the textbook [142], which mainly focuses on $d = 2$ but introduces the conformal group for general d. Some of the original works on the foundations of conformal field theory and its representation theory are [44, 45, 47, 143]. We also refer the reader to the more modern introductions [144–146], which are thought for $d > 2$ and focus on the conformal bootstrap. A review on the subject of the bootstrap can be found in [147]. We will address this topic later, in Chapter 6.

The conformal group in d dimensional flat space is the set of all transformations of $\mathbb{R}^d$ that preserve angles. In Lorentzian signature it is isomorphic to $\mathrm{SO}(d, 2)$. It extends the Poincaré group $\mathrm{SO}(d-1, 1) \ltimes \mathbb{R}^d$ by $d+1$ additional generators, namely the dilatation D and the special conformal transformations K_μ. The former generates scale transformations

$$x^\mu \mapsto \lambda\, x^\mu\,, \qquad \lambda > 0\,, \tag{1.1}$$

while the latter generate a nonlinear transformation that takes the form

$$x^\mu \mapsto \frac{x^\mu - b^\mu x^2}{1 - 2(b\cdot x) + b^2 x^2}\,, \qquad b \in \mathbb{R}^d\,. \tag{1.2}$$

The remaining generator are, of course, translations P_μ and rotations/boosts $M_{\mu\nu}$. It is obvious that the generators P_μ, $M_{\mu\nu}$ and D do preserve angles. The result for K_μ on the other hand is slightly more involved, but after a small computation one can show that indeed the transformation (1.2) induced on the tangent vectors[1] is proportional to an

[1] Given a transformation $x^\mu \to f^\mu(x)$, one can define the induced transformation on the tangent vectors $v^\mu_x \to \frac{\partial f^\mu(x)}{\partial x^\nu} v^\nu_x$. It is sometimes referred to as pushforward.

orthogonal matrix

$$f^\mu(x) := \frac{x^\mu - b^\mu x^2}{1 - 2(b\cdot x) + b^2 x^2} \quad \Rightarrow \quad \frac{\partial f^\mu}{\partial x^\rho} g_{\mu\nu} \frac{\partial f^\nu}{\partial x^\lambda} = \frac{g_{\rho\lambda}}{(1 - 2(b\cdot x) + b^2 x^2)^2}\,. \tag{1.3}$$

The algebra of the generators reads:

$$\begin{aligned}
[D, P_\mu] &= iP_\mu\,, & [D, K_\mu] &= -iK_\mu\,,\\
[P_\mu, K_\nu] &= 2i(g_{\mu\nu} D - M_{\mu\nu})\,,\\
[M_{\mu\nu}, P_\rho] &= i(g_{\mu\rho} P_\nu - g_{\nu\rho} P_\mu)\,, & [M_{\mu\nu}, K_\rho] &= i(g_{\mu\rho} K_\nu - g_{\nu\rho} K_\mu)\,,\\
[M_{\mu\nu}, M_{\rho\lambda}] &= i(g_{\mu\rho} M_{\nu\lambda} + g_{\nu\lambda} M_{\mu\rho} - g_{\nu\rho} M_{\mu\lambda} - g_{\mu\lambda} M_{\nu\rho})\,,
\end{aligned} \tag{1.4}$$

all the other commutators being zero. As one can see, the generator D defines a grading, making P of weight 1 and K of weight -1. We can thus regard them as "ladder" operators and construct representations by diagonalizing D. More precisely, we use a strategy similar to the method of induced representation normally used for the Poincaré group. However, instead of choosing a frame in momentum space, we will stay in position space, picking as a reference point the origin $x = 0$. The group that leaves the origin invariant — i.e. the little group — is generated by K_μ, $M_{\mu\nu}$ and D. Local operators at the origin are required to be finite dimensional irreducible representations of this group. We further assume that D can be diagonalized. Since K_μ lowers the D weight, at some point its action must be trivial. We thus always have a lowest weight state which is annihilated by K_μ. We call such a state a *conformal primary* $\mathcal{O}_{\Delta,\ell}(0)$

$$K_\mu \mathcal{O}_{\Delta,\ell}(0) = 0\,, \quad M_{\mu\nu}\mathcal{O}_{\Delta,\ell}(0) = \mathcal{S}_{\mu\nu}(\ell)\mathcal{O}_{\Delta,\ell}(0)\,, \quad D\mathcal{O}_{\Delta,\ell}(0) = -i\Delta\mathcal{O}_{\Delta,\ell}(0)\,, \tag{1.5}$$

where Δ is the conformal dimension, ℓ denotes the spin Dynkin labels and $\mathcal{S}_{\mu\nu}$ is a spin ℓ matrix representation of $\mathrm{SO}(d-1,1)$. By acting on $\mathcal{O}_{\Delta,\ell}(0)$ with P_μ we can generate all the other states of the representation, which are called *descendants*. This allows us to move away from the origin:

$$\mathcal{O}_{\Delta,\ell}(x) = e^{-i\,x^\mu P_\mu}\,\mathcal{O}_{\Delta,\ell}(0)\,e^{i\,x^\mu P_\mu}\,. \tag{1.6}$$

The whole representation is then infinite dimensional as it is spanned by arbitrary products of P_μ acting on the primary. We will call such a representation a *conformal multiplet*. In unitary theories, the quantum numbers Δ and ℓ must satisfy some inequalities which are called unitarity bounds. In particular, in four dimensions ℓ is a pair of integers $(j, \bar{j})$

and the bounds are[2]

$$\begin{aligned} &\Delta \geqslant 1 && (j=\bar{j}=0)\,, \\ &\Delta \geqslant \tfrac{1}{2}j+1 && (j>0,\,\bar{j}=0)\,, \\ &\Delta \geqslant \tfrac{1}{2}\bar{j}+1 && (j=0,\,\bar{j}>0)\,, \\ &\Delta \geqslant \tfrac{1}{2}(j+\bar{j})+2 && (j,\bar{j}>0)\,. \end{aligned} \tag{1.7}$$

For symmetric traceless representations in any d the bound instead is

$$\begin{aligned} &\Delta \geqslant \tfrac{d-2}{2} && (\ell=0)\,, \\ &\Delta \geqslant \ell+d-2 && (\ell>0)\,. \end{aligned} \tag{1.8}$$

When a unitarity bound is saturated, one or more states in the representation become orthogonal to the whole multiplet (and thus also null). We can therefore consistently mod these states out. The result is still an infinite dimensional representation but with fewer states, we will thus refer to is as a *short multiplet*. The simplest example of a short multiplet is a free scalar ϕ. It saturates the bound of $(d-2)/2$ and indeed it has a null state

$$[P^2,\phi]=\Box\phi=0\,, \tag{1.9}$$

as dictated by its equation of motion. Another example are the conserved currents of any spin ℓ. They saturate the bound $\ell+d-2$ and, by definition, satisfy the equation

$$[P^\mu, J_{(\mu\mu_2\cdots\mu_\ell)}]=0\,. \tag{1.10}$$

The vanishing of a descendant holds as an operator equation, in the sense that it still remains true when inserted in any correlation function, up to contact terms.

$$\langle(\partial\cdot J)(x)\mathcal{O}_1(x_1)\cdots\mathcal{O}_n(x_n)\rangle=\sum_{i=1}^{n}c_i(x_1,\ldots,x_n)\,\delta(x-x_i)\,. \tag{1.11}$$

The functions c_i are not arbitrary but may be fixed in terms of the correlation function $\langle\mathcal{O}_1\cdots\mathcal{O}_n\rangle$ and the transformation properties of the $\mathcal{O}_i$'s under the symmetry generated by J. The relation (1.11) goes under the name of Ward identity. In the next section we will discuss them more quantitatively. The two most common examples are $\ell=1$ where the current is associated to an ordinary global symmetry or $\ell=2$ where the current is the stress tensor and it is obviously associated to the conformal symmetry.

[2]It is also possible to have $\Delta=j=\bar{j}=0$, which would correspond to the identity operator.

1.2 Conformal correlators

1.2.1 General notions

The main focus of this thesis are conformal correlation functions of primary local operators. The correlation function is defined as an expectation value in a conformally invariant vacuum $|\Omega\rangle$, which we will keep implicit. Conformal symmetry is strong enough to completely fix two-point functions and fix three-point functions up to a finite number of coefficients, referred to as *OPE coefficients*, where OPE stands for operator product expansion. If we consider only scalar operators ϕ_i of conformal dimensions Δ_i then one simply has

$$\langle \phi_1(x_1)\phi_2(x_2)\rangle = \begin{cases} (x_{12}^2)^{-\Delta} & \text{if } \Delta_1 = \Delta_2 \equiv \Delta\,, \\ 0 & \text{otherwise}\,. \end{cases} \tag{1.12}$$

where $x_{ij} \coloneqq x_i - x_j$. For generic operators the two-point function is also uniquely fixed. However, for spin representation which are not real $\mathcal{O}$ has to be paired with $\bar{\mathcal{O}}$, which is the operator whose conformal dimension is the same as $\mathcal{O}$ and whose spin representation is the complex conjugate.[3]

The three-point function of three scalars is given by

$$\langle \phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\rangle = \frac{\lambda_{\phi_1\phi_2\phi_3}}{|x_{12}|^{\Delta_{123}}|x_{13}|^{\Delta_{132}}|x_{23}|^{\Delta_{231}}}\,, \tag{1.13}$$

where $\Delta_{ijk} \coloneqq \Delta_i + \Delta_j - \Delta_k$. The real constant $\lambda_{\phi_1\phi_2\phi_3}$ is the OPE coefficient, which in the case of scalar operators is unique. For spinning operators there will be in general more than one tensor structure, each of them multiplied by an independent λ coefficient. Note that the value of $\lambda_{\phi_1\phi_2\phi_3}$ is physical since we normalized the operators ϕ_i in such a way that their two-point function is (1.12). A general treatment of two and three-point functions for arbitrary spinning operators in general spacetime dimensions was initiated in [85]. We will however use a more modern method that goes under the name of *embedding formalism*, which will be introduced in the next subsection. A simple case, however, that does not require any heavy formalism it that of two scalars and a symmetric traceless tensor of spin ℓ. Their correlator reads

$$\langle \phi(x_1)\phi(x_2)\mathcal{O}(x_3)\rangle = \frac{\lambda_{\phi\phi\mathcal{O}}}{|x_{12}|^{\Delta_{123}+\ell}|x_{13}|^{\Delta_{132}-\ell}|x_{23}|^{\Delta_{231}-\ell}}\left(\frac{x_{13}^\mu}{x_{13}^2} - \frac{x_{23}^\mu}{x_{23}^2}\right)^\ell\,, \tag{1.14}$$

where, with a slight abuse of notation, we defined $(Z^\mu)^\ell$ to mean $Z^{\mu_1}\cdots Z^{\mu_\ell}$ − traces. Also in this case only one tensor structure appears. We should comment on the name

[3] In four dimensions $(j,\bar{j})^* = (\bar{j},j)$. If there are other quantum numbers such as a global charge, then they should also be conjugated.

"OPE coefficient." Its origin stems from the fact that we can always take the operator product expansion (or OPE) between two operators in a CFT and the contribution of $\mathcal{O}_3$ in the OPE $\mathcal{O}_1 \times \mathcal{O}_2$ is entirely fixed by they three-point correlator, by simply taking the expectation value of both sides multiplied by $\overline{\mathcal{O}}_3$. Concretely

$$\phi(x) \times \phi(0) = \sum_{\Delta,\ell} \frac{\lambda_{\phi\phi\mathcal{O}_{\Delta,\ell}}}{|x|^{2\Delta_\phi - \Delta}} C(\partial_\mu, x)_{\mu_1 \cdots \mu_\ell} \mathcal{O}^{\mu_1 \cdots \mu_\ell}_{\Delta,\ell}(0)\,, \tag{1.15}$$

with the functions C fixed by (1.14) (we will not show it explicitly). In the above formula the sum over operators, as well as the coefficients, are theory dependent and thus not known a priori. The OPE can be precisely seen as an algebra on the space of local operators, the $\lambda_{\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3}$ being the structure constants. One of the axioms of conformal field theory requires this algebra to be associative. This yields very strong nontrivial constraints which will be explained in Chapter 6.

In general, a three-point function of operators $\mathcal{O}_i$ which may have nonzero spin can be written as a linear combination of tensor structures $\mathrm{t}^a_{\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3}$. Each structure is multiplied by its own OPE coefficient

$$\langle \mathcal{O}_1(\mathbf{x}_1)\mathcal{O}_2(\mathbf{x}_2)\mathcal{O}_3(\mathbf{x}_3)\rangle = \sum_{a=1}^{n_{123}} \lambda^{(a)}_{\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3} \mathrm{t}^a_{\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)\,, \tag{1.16}$$

where $\mathbf{x}_i$ is a shorthand to denote x_i together with the spin polarizations that are carried by $\mathcal{O}_i$. We will address the problem of enumerating these structures in the subsequent sections.

The last correlator that we will need is the four-point function. For simplicity we will only illustrate the case of four not necessarily identical scalars in this subsection. Even in this simple case the kinematics starts being nontrivial, in the sense that we cannot fix the correlator anymore but we need to allow for an arbitrary function $f(u,v)$ of the so-called *cross ratios*

$$u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}\,, \qquad v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}\,. \tag{1.17}$$

In terms of a theory-dependent f, one has [95]

$$\Big\langle \prod_{i=1}^4 \phi_i(x_i) \Big\rangle = \frac{1}{|x_{12}|^{\Delta_1+\Delta_2}|x_{34}|^{\Delta_3+\Delta_4}} \left(\frac{x_{24}^2}{x_{14}^2}\right)^{\frac{1}{2}\Delta_{12}} \left(\frac{x_{14}^2}{x_{13}^2}\right)^{\frac{1}{2}\Delta_{34}} f(u,v)\,, \tag{1.18}$$

with $\Delta_{ij} \coloneqq \Delta_i - \Delta_j$. By using the OPE in (1.15) inside the four-point correlator, together with the fact that two-point functions are diagonal, one can express f as a sum over the operators in the spectrum of $\phi_1 \times \phi_2 \cap \phi_3 \times \phi_4$. Other permutations are equivalently valid and their equality stems from the OPE associativity which we mentioned before

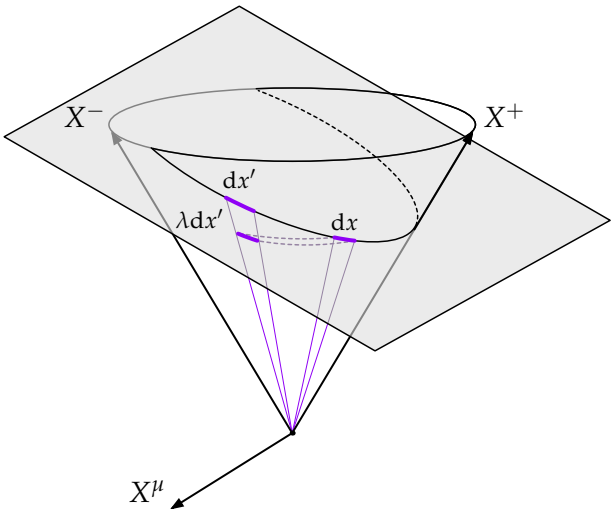


Figure 1.1. The null cone and the Poincaré section. An infinitesimal interval dx is mapped to its conformal transformation dx' by a $\mathbb{R}^{d,2}$ isometry and a rescaling.

and will be addressed later on in this thesis. More precisely one can write

$$f(u,v) = \sum_{\mathcal{O}_{\Delta,\ell} \in \phi_1 \times \phi_2} \lambda_{\phi_1\phi_2\mathcal{O}_{\Delta,\ell}} \lambda_{\overline{\mathcal{O}}_{\Delta,\ell}\phi_3\phi_4} \, G^{(\Delta_{12},\Delta_{34})}_{\Delta,\ell}(u,v)\,,$$

where the function g is called *conformal block*. We put a bar on $\mathcal{O}$ in the second OPE coefficient to agree with the general case. When all external operators are scalars, the only operators that are exchanged are symmetric traceless tensors. Thus in this case $\mathcal{O} = \overline{\mathcal{O}}$ and the bar would not be necessary. Even for external scalars the conformal blocks are somewhat complicated functions. For even dimensions they can be taken as combinations of ${}_2F_1$ hypergeometric functions [95, 96]. Whereas in any other dimension they can be obtained with a variety of methods, which were briefly reviewed in the Introduction.

If, on the other hand, the external operators have nonzero spin, equation (1.18) will be replaced by a sum over the possible tensor structures, as in (6.4). The conformal blocks for spinning operators will be discussed in more detail in Section 6.4.

1.2.2 Embedding formalism

In the previous subsection we called this formalism modern. Although, in fact, it dates back to Dirac [86]. We actually referred to its more recent formulation as appeared in [87, 88, 92] and, more specifically, to its four dimensional specialization [89, 91, 148]. There exists an even more general theory which is able to deal in an uniform way with all representations of the spin group in arbitrary dimensions [106, 107, 110, 111]. For the present thesis, which is focused on $d = 4$, we will adopt the formalism of [89, 91].

The idea behind the embedding formalism is to extend the d dimensional space to a $d+2$ dimensional space where the conformal group $\mathrm{SO}(d,2)$ acts linearly. It is then much easier to write down conformally covariant tensor structures. The wanted result is obtained by projecting down to d dimensions. This projection is made in two steps. First we restrict ourselves on the null cone

$$X^2 = X^+X^- + X^\mu X_\mu = 0\,, \qquad X^\pm := X^d \pm X^{d+1}\,. \tag{1.19}$$

and then we identify points that differ by a scale factor

$$X^M \sim \lambda\, X^M\,, \qquad \lambda \in \mathbb{R}^+\,. \tag{1.20}$$

This last condition may be enforced by choosing a section $X^+ = f(X^\mu)$. A standard choice is the Poincaré section which yields a flat metric in d dimensions

$$(X^+, X^-, X^\mu)\big|_{\text{Poincaré}} \equiv (1, x^2, x^\mu)\,. \tag{1.21}$$

It is then easy to check that linear isometries in $\mathbb{R}^{d,2}$ yield conformal transformations in $\mathbb{R}^{d-1,1}$. This is because, after a generic transformation, the null condition $X^2 = 0$ is preserved but we may have to rescale the coordinates in order to fall back into the section

$$X^M \mapsto \Lambda^M{}_N X^N \sim \lambda^{-1}\Lambda^M{}_N X^N\,, \qquad \lambda := \Lambda^+{}_N X^N(x)\,. \tag{1.22}$$

Due to the null cone constraint, this results into an x dependent factor that multiplies the d dimensional metric $\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu$

$$\mathrm{d}s^2 = \mathrm{d}X^M \mathrm{d}X_M\big|_{\text{Poin.}} \;\mapsto\; \lambda^2 \mathrm{d}X^M\mathrm{d}X_M + 2\mathrm{d}\lambda\, X^M\mathrm{d}X_M + (\mathrm{d}\lambda)^2 X^2\big|_{\text{Poin.}}\,, \tag{1.23}$$

the last two term vanish due to $X^2 = 0 = \mathrm{d}(X^2)$, so we are left only with the first one which reads $\lambda^2 g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu$.

After this general introduction we will specialize to $d = 4$. The projective null cone is therefore embedded in six dimensions. We now need to have a convenient formalism to deal with tensors. In four dimensions the indices are contracted with spinor polarizations $\eta, \bar\eta$ (see Appendix A.1). In the same fashion here we introduce twistor polarizations $S, \bar S$. The conventions follow the literature but for the reader's convenience they are summarized in Appendix A.2. An operator in six dimensions is taken to be a homogeneous function satisfying

$$O(\lambda X, \mu S, \bar\mu \bar S) = \lambda^{-\Delta-\frac{1}{2}(j+\bar\jmath)}\,\mu^j\,\bar\mu^{\bar\jmath}\,O(X,S,\bar S)\,, \tag{1.24}$$

where Δ is the conformal dimension and $(j,\bar\jmath)$ the spin of O's four dimensional counterpart $\mathcal{O}(x,\eta,\bar\eta) \equiv O(X,S,\bar S)|_{\text{Poincaré}}$. The projection down to the Poincaré patch is

defined as (1.21) for X^M, while for S and $\bar{S}$ it is defined as follows

$$S_b\big|_{\text{Poincaré}} = \eta^{\alpha}\mathbf{X}_{\alpha b}\big|_{\text{Poincaré}}\,, \qquad \bar{S}^b\big|_{\text{Poincaré}} = \bar{\eta}_{\dot{\alpha}}\bar{\mathbf{X}}^{\dot{\alpha}b}\big|_{\text{Poincaré}}\,. \tag{1.25}$$

These projections, together with $X^2 = \bar{\mathbf{X}}\mathbf{X} = \mathbf{X}\bar{\mathbf{X}} = 0$, induce an equivalence relation on the operators in six dimensions. Namely two operators O and O' are considered equivalent if they differ by terms proportional to $\bar{\mathbf{X}}S$, $\bar{S}\mathbf{X}$, $\bar{S}S$ or X^2. We can think of this as a "gauge freedom" which we can use to reduce the number of independent tensor structures.

In this thesis the embedding formalism will be applied to three and four-point functions. For this reason we will only list the building blocks that arise in the construction of correlators with up to four points. After imposing the gauge conditions there can be only eight different classes of building blocks. The first arises for $n \geqslant 2$ points, the next four arise for $n \geqslant 3$ points and the last three only for $n \geqslant 4$ points[4]

$$\begin{aligned} X_{ij} &:= -2(X_i \cdot X_j)\,, \\ I^{ij} &:= \bar{S}_i S_j\,, & K_k^{ij} &:= \sqrt{\frac{X_{ij}}{X_{ik}X_{kj}}}\, S_i \bar{\mathbf{X}}_k S_j\,, \\ J^i_{jk} &:= \frac{1}{X_{jk}} \bar{S}_i \mathbf{X}_j \bar{\mathbf{X}}_k S_i\,, & \bar{K}_k^{ij} &:= \sqrt{\frac{X_{ij}}{X_{ik}X_{kj}}}\, \bar{S}_i \mathbf{X}_k \bar{S}_j\,, \end{aligned} \tag{1.26a}$$

$$\begin{aligned} I^{ij}_{kl} &:= \frac{1}{X_{kl}} \bar{S}_i \mathbf{X}_k \bar{\mathbf{X}}_l S_j\,, & L^i_{jkl} &:= \frac{S_i \bar{\mathbf{X}}_j \mathbf{X}_k \bar{\mathbf{X}}_l S_i}{\sqrt{X_{jk}X_{kl}X_{lj}}}\,, \\ \bar{L}^i_{jkl} &:= \frac{\bar{S}_i \mathbf{X}_j \bar{\mathbf{X}}_k \mathbf{X}_l \bar{S}_i}{\sqrt{X_{jk}X_{kl}X_{lj}}}\,. \end{aligned} \tag{1.26b}$$

The three-point tensor structures $\mathrm{t}^a_{\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3}$ of a given correlator (see (1.16)) are built as products of the terms in (1.26a) such that they satisfy the correct scaling as in (1.24). Furthermore, not all products are independent as they are subject to various nonlinear identities. It is possible to obtain a basis of independent structures by requiring that

i) There are either only K or only $\bar{K}$ terms (or none of either)

ii) Only at most two distinct J terms out of three can appear

iii) K_i^{kl} and J^i_{mn} cannot appear together. $\bar{K}_i^{kl}$ and J^i_{mn} cannot appear together.

With these simple rules writing down tensor structures becomes just an enumeration problem. We first deal with the scaling property (1.24) for $X \to \lambda X$ by defining a

[4]The labels i, j, k are all distinct and there is no sum over repeated labels.

prefactor

$$\mathcal{K}_3 = \prod_{\substack{i<j \\ k\neq i,j}}^{3} X_{ij}^{\frac{1}{2}(\kappa_k-\kappa_i-\kappa_j)}\,, \qquad \kappa_i = \Delta_i + \tfrac{1}{2}(j_i+\bar{\jmath}_i)\,. \tag{1.27}$$

Then the most general tensor structure can be written as

$$\mathrm{t}^a_{\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3} = \mathcal{K}_3 \prod_{i\neq j}^{3} (I^{ij})^{m_{ij}} (K_1^{23})^{k_1} (K_2^{13})^{k_2} (K_3^{12})^{k_3} (\bar{K}_1^{23})^{\bar{k}_1} (\bar{K}_2^{13})^{\bar{k}_2} (\bar{K}_3^{12})^{\bar{k}_3} (J_{23}^1)^{p_1} (J_{13}^2)^{p_2} (J_{12}^3)^{p_3}\,, \tag{1.28}$$

for some numbers $m_{ij}, k_i, \bar{k}_i, p_i$ that must satisfy the constraints following from points *i*), *ii*) and *iii*) listed above and the scaling property (1.24) for $S, \bar{S} \to \mu S, \bar{\mu}\bar{S}$. For example *i*) implies that for all i, $k_i = 0$ or $\bar{k}_i = 0$. Whereas *ii*) implies that for at least one i, $p_i = 0$. The constraints from scaling of S and $\bar{S}$ can be written as the following linear system

$$\begin{cases} j_1 = m_{21} + m_{31} + k_2 + k_3 + p_1\,, \\ \bar{\jmath}_1 = m_{12} + m_{13} + \bar{k}_2 + \bar{k}_3 + p_1\,, \\ \quad \text{cyclic permutations of } 1,2,3. \end{cases} \tag{1.29}$$

From (1.28) we can obtain the tensor structures in four dimension via the projection on the Poincaré patch. See (A.22) for the projections of the building blocks defined in this section $\mathbb{I}^{ij}$, $\mathbb{J}^i_{jk}$, $\mathbb{K}^{ij}_k$ and $\bar{\mathbb{K}}^{ij}_k$.

Four-point tensor structures instead are built out of the terms in (1.26a) and (1.26b). It is much harder to find a minimal set of independent structures in this case because there are additional identities that arise at higher order and it is not known how to classify them or how to find some general rules like *i*), *ii*), *iii*) listed before. We will defer the study of four-point function to Sections 6.3 and 6.4. The projection to four dimensions $\mathbb{I}^{ij}_{kl}$, $\mathbb{L}^i_{jkl}$ and $\bar{\mathbb{L}}^i_{jkl}$ can be found in (A.23).

1.3 Counting tensor structures

The tool we are going to describe in this section goes under the name of conformal frame. It can be used to count the number of allowed tensor structures in a given correlator without having to actually construct them. By tensor structures we mean the number of independent OPE coefficient in three-point correlators and the number of independent functions of the cross ratios for $n \geqslant 4$-point correlators. A possible application is to check that the structures obtained by other methods (such as the embedding formalism) are indeed linearly independent and complete. It can also be used to construct the tensor structures, but we will discuss this aspect in Section 6.3, in the context of four-point functions. In Section 2.3 we will also describe a supersymmetric version of it. This discussion is based on [93, 113] (see also [149]). Since the signature of spacetime does

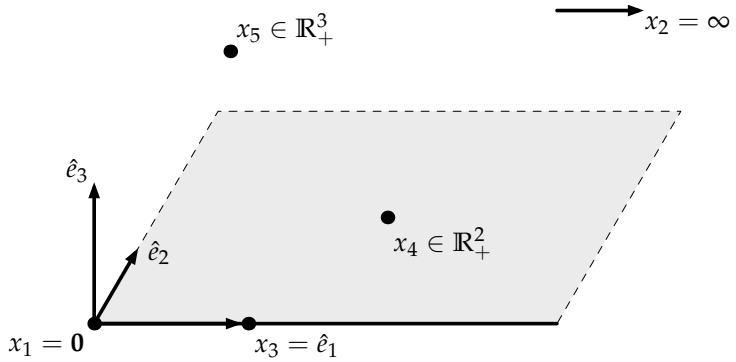


Figure 1.2. Example of the conformal frame configuration for $d \geqslant 3$ and $n = 5$. In that case the tensor structures multiply arbitrary functions of five "cross ratios," similar to u and v in the four point case. $\mathbb{R}^k_+$ denotes the upper half hyperplane.

not play any role, we will switch to Euclidean for the moment.

1.3.1 Conformal frame

The advantage of this method is that it is purely group theoretical, so it is possible to make statements about complicated representations in any number of dimensions without any additional difficulty. Given an n-point correlator

$$\langle \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) \rangle \,, \tag{1.30}$$

one can use the conformal symmetry to put the points x_i in some preferred positions. First we can use the d translations to set $x_1 = 0$, and then the d special conformal transformations (1.2) to set $x_2 = \infty$.[5] If $n > 2$ we use a dilatation to set $|x_3| = 1$ and $d-1$ rotations to set $x_3 = \hat{e}_1$ (or any other preferred direction). Then, continuing with the same reasoning for k steps, if $n > k$ we can use $d - k + 1$ rotations to put x_{k+1} in the hyperplane spanned by $\hat{e}_1, \ldots, \hat{e}_{k-1}$. This process clearly stops as soon as $k = n$ or $k = d + 2$ because we either run out of points or of available hyperplanes. Therefore the final configuration is

$$\begin{aligned} &x_1 = 0,\ x_2 = \infty,\ x_3 = \hat{e}_1,\ x_{k\geqslant 4} \in \mathrm{Span}\{\hat{e}_1, \ldots, \hat{e}_{k-2}\} && \text{if } n \leqslant d+2\,, \\ &\left.\begin{aligned} &x_1 = 0,\ x_2 = \infty,\ x_3 = \hat{e}_1,\ x_{4\leqslant k\leqslant d+2} \in \mathrm{Span}\{\hat{e}_1, \ldots, \hat{e}_{k-2}\}, \\ &x_{k\geqslant d+3} \text{ unchanged} \end{aligned}\right\} && \text{if } n > d+2\,. \end{aligned} \tag{1.31}$$

Define H to be the stability group of such a configuration. If we are in the second case ($n > d + 2$) that means that we used up all symmetries so $H = \{1\}$. The same is true for the boundary case $n = d + 2$. If on the other hand we are in the first case with $n < d + 2$,

[5] We imagine to work in the conformal compactification of $\mathbb{R}^d = S^d$, where the point at infinity is the north pole.

then we have $\frac{1}{2}(d-n+1)(d-n+2)$ rotations left and indeed they make up the group $\mathrm{SO}(d-n+2)$. So the final result for $n>2$ may be expressed as[6]

$$H=\mathrm{SO}(d-m+2)\,,\qquad m\coloneqq\min(n,d+2)\,. \tag{1.32}$$

The case $n=2$ is different because we never use dilatations. Therefore H becomes $\mathrm{SO}(d)\times\mathrm{SO}(1,1)$. In what follows, we will be mostly interested in three-point functions. In that case the configuration reads $(0,\infty,\hat{e}_1)$ and the stability group is $H=\mathrm{SO}(d-1)$.

The reason why we emphasized so much the stability group H is because the number of tensor structures can be seen as the number of H-singlets in the tensor product of the representations of the operators $\mathcal{O}_i$. Define ρ_i as the $\mathrm{SO}(d)$ representation of $\mathcal{O}_i$, Res^G_H as the decomposition of a G representation into H irreducible representations and ρ^H as the space of H-singlets in ρ. Calling $n(\dots)$ the number of tensor structures in the correlator $\langle\dots\rangle$ one can write the following general formula

$$n(\mathcal{O}_1\cdots\mathcal{O}_n)=\dim\left(\mathrm{Res}^{\mathrm{SO}(d)}_{\mathrm{SO}(d-m+2)}\bigotimes_{k=1}^{n}\rho_k\right)^{\mathrm{SO}(d-m+2)}\,, \tag{1.33}$$

where, recall, $m=\min(n,d+2)$. For $n=2$ the formula is a bit different because H has the extra factor due to dilatations. Letting (ρ,Δ) be a representation of $\mathrm{SO}(d)\times\mathrm{SO}(1,1)$, we have

$$n(\mathcal{O}_1\mathcal{O}_2)=\dim\big((\rho_1,\Delta_1)\otimes(\rho_2^*,-\Delta_2)\big)^{\mathrm{SO}(d)\times\mathrm{SO}(1,1)}\,, \tag{1.34}$$

where ρ^* is the conjugate representation of ρ.[7] One can clearly see that the result is 1 if $\Delta_1=\Delta_2$ and $\rho_1\cong\rho_2^*$ and zero otherwise. The fact that we need to take this conjugation might seem confusing at first. Why is this special only for the two-point function? The reason is that we actually need to take this conjugate for higher points too, but in that case it does not make any difference as far as the number of singlets is concerned. To see why we need it, let us derive this formula using the approach of [113]. This will require to define conformal correlators as functions on the conformal group. The discussion of [113] is focused on four-point functions. We will adapt it to three-point functions as it will be useful later.

1.3.2 Group theory of tensor structures

Conformal correlators can be thought of as functions on the conformal group. Let $\mathcal{O}_i$ transform in the representation $V_i\coloneqq(\rho_i,\Delta_i)$ of $\mathrm{SO}(d)\times\mathrm{SO}(1,1)$. Then it can be

[6] The stability group for $n\geqslant 3$ is actually $\mathrm{O}(d+2-m)$. However, since we will not be interested in the properties of the correlator under parity, we will ignore this detail and only work with the connected component $\mathrm{SO}(d+2-m)$.

[7] Recall, in four dimensions $(j,\bar{\jmath})^*=(\bar{\jmath},j)$.

associated to the space of functions

$$\Gamma^{V_i} := \left\{ \varphi \in C^\infty(G, V_i) \;\middle|\; \begin{array}{ll} \varphi(e^{\lambda D} g) = e^{\lambda \Delta_i} \varphi(g) & \forall\, e^{\lambda D} \in \mathrm{SO}(1,1) \\ \varphi(rg) = \rho_i(r)\varphi(g) & \forall\, r \in \mathrm{SO}(d) \\ \varphi(kg) = \varphi(g) & \forall\, k = e^{ib^\mu K_\mu} \end{array} \right\}, \tag{1.35}$$

where we denoted the conformal group as $G = \mathrm{SO}(d+1,1)$ (recall that for the moment we are in Euclidean signature). For convenience let us also define

$$G_0 = \mathrm{SO}(1,1) \times \mathrm{SO}(d)\,, \qquad G_P = \mathrm{SO}(d-1,1)_P\,, \qquad G_K = \mathrm{SO}(d-1,1)_K\,. \tag{1.36}$$

The subscripts P and K denote the two different Poincaré subgroups of the conformal group: the first being the standard one and the second being the one where translations are replaced by special conformal transformations. The definition (1.35) is very intuitive: we want functions that transform as primaries under the left action of G_K. The product of two such spaces can be characterized as follows

$$\Gamma^{V_1} \otimes \Gamma^{V_2} = \Gamma^{V_1, V_2} := \\ \left\{ \varphi \in C^\infty(G, V_1 \otimes V_2^*) \;\middle|\; \begin{array}{ll} \varphi(e^{\lambda D} g) = e^{\lambda(\Delta_1 - \Delta_2)} \varphi(g) & \forall\, e^{\lambda D} \in \mathrm{SO}(1,1) \\ \varphi(rg) = \rho_1(r) \otimes \rho_2^*(r)\varphi(g) & \forall\, r \in \mathrm{SO}(d) \end{array} \right\}, \tag{1.37}$$

where we denoted $V^* = (\rho^*, -\Delta)$. In words, the resulting space is a space of functions from the group to the tensor product $V_1 \otimes V_2^*$ which satisfy some covariance properties. Namely the subgroup G_0 needs to act covariantly on the left. This is the step where the conjugate that we encountered in (1.34) comes about. The result (1.37) is proved in Theorem 9.2 of [150]. Here we just sketch an intuition. The map $*$ is a Weyl reflection of the conformal group $w : G \to G$. It flips the sign of D, therefore, according to the grading discussed below (1.4), it exchanges the roles of P_μ and K_μ. Since w is also an inner automorphism one has

$$\Gamma^V \cong \bar{\Gamma}^{V^*}\,.$$

where $\bar{\Gamma}$ is defined in the same way as Γ, except that we require the action of P_μ to be trivial instead of that of K_μ. Thus, in order to prove (1.37), we need to exhibit an isomorphism

$$\Gamma^{V_1, V_2} \longrightarrow \Gamma^{V_1} \otimes \bar{\Gamma}^{V_2^*}$$
$$\psi(g) \longrightarrow \varphi_1(g) \otimes \varphi_2(g)$$

If $g \in G_0$ we know how to recover both φ_1 and φ_2 as they are fixed by the covariance properties. If instead g is a translation, $\varphi_2(g) = \varphi_2(1)$, thus we can recover φ_1 by solving $\psi(g) = \varphi_1(g) \otimes \varphi_2(1)$. Similarly, if g is a special conformal transformation, $\varphi_1(g) = \varphi_1(1)$ and we can recover φ_2 by $\psi(g) = \varphi_1(1) \otimes \varphi_2(g)$. Essentially the role

played by the Weyl inversion is to make sure that the subgroups that act trivially on φ_1 and on φ_2 do not overlap. The proof then consists in showing that this map is invertible, but we will refer the reader to [150] for a complete derivation.

The space of three-point conformal correlators can be realized as the space of G-invariants in the tensor product of the three representations

$$\left(\Gamma^{V_1} \otimes \Gamma^{V_2} \otimes \Gamma^{V_3}\right)^G = \left(\Gamma^{V_1,V_2} \otimes \Gamma^{V_3}\right)^G . \tag{1.38}$$

The last product can be written as the space of functions $\varphi : G \times G \to V_1 \otimes V_2^* \otimes V_3$ which satisfy the covariance properties of (1.35) in the first G factor and of (1.37) in the second one. We will not write this out explicitly hoping that it is clear enough. In other words, the space of functions mentioned above is fully specified by the group elements on the coset $G_0\backslash G \times G/G_K$.[8] Passing to the G-invariants yields a double coset $G_0\backslash G/G_K$ (we will not show this). Let us call $(\Gamma^{V_1,V_2,V_3})^G$ the space obtained, namely

$$\begin{aligned}&\left(\Gamma^{V_1} \otimes \Gamma^{V_2} \otimes \Gamma^{V_3}\right)^G = \left(\Gamma^{V_1,V_2,V_3}\right)^G := \\ &\left\{\varphi \in C^\infty(G, V_1 \otimes V_2^* \otimes V_3) \;\middle|\; \begin{array}{ll} \varphi(ag) = \pi_1(a) \otimes \pi_2(a)\ \varphi(g) & \forall\ a \in G_0 \\ \varphi(gb) = \pi_3(b)\ \varphi(g) & \forall\ b \in G_K \end{array}\right\} ,\end{aligned} \tag{1.39}$$

where π_i are the V_i representations of the group to which their argument belongs. The space of conformal correlators is precisely the codomain of these functions, i.e. the set of values that φ can assume. However, claiming that the result is $V_1 \otimes V_2^* \otimes V_3$ would be too fast: the decomposition of an element of G into an element of its double coset is not unique, therefore there are some compatibility conditions of the covariance properties that need to be satisfied. The ambiguity of the coset decomposition is characterized precisely by the stability group H. So the final result, as we expected, is

$$n(\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3) = \dim\left(\mathrm{Res}_H^{G_0}\ V_1 \otimes V_2^* \otimes V_3\right)^H , \tag{1.40}$$

as in (1.33) for $n = 3$, except for the $*$ in V_2, which, as we argued earlier, does not change the final answer if $n > 2$.[9] All we need to do now is to motivate that indeed H is the stability group of the coset decomposition $G_0\backslash G/G_K$. We will again present just a sketch of the argument. It is easier to think in terms of the conformal algebra (this may miss some discrete identifications which we do not care about). The generators of $\mathrm{SO}(d+1,1)$ are L_{MN}, where $D = L_{01}$ and $M_{\mu\nu} = L_{\mu\nu}$. The algebra $\mathfrak{g}_0$ is spanned by L_{01} and $L_{\mu\nu}$ for $\mu, \nu = 2, \ldots, d+1$. While $\mathfrak{g}_K$ is spanned by the same generators and in addition by $L_{1\mu}$.

[8] To see this, we can make the third factor covariant with respect to the *right* action, instead of the left as in (1.35). The space obtained is clearly isomorphic to Γ^{V_3}.

[9] In (1.33) we have representations of $\mathrm{SO}(d)$ whereas here they are representations of G_0. This makes no difference because we can drop the $\mathrm{SO}(1,1)$ factor since it does not appear in H. Therefore the V_i's simply become ρ_i's.

We then single out an element which is not in any of these algebras, let us take $a = L_{02}$. Any element $g \in G$ may be written as

$$g = g_0\, e^{\tau\, a} g_K\,, \qquad g_0 \in G_0\,,\ g_K \in G_K\,. \tag{1.41}$$

as a consequence of the Cartan decomposition. This decomposition is ambiguous: all rotations M_{ij}, with $i, j = 3, \dots d+1$ commute with a and can thus be transferred from the g_0 factor to the g_K factor. Therefore, at last, the stability group of the decomposition (1.41) is $H = \mathrm{SO}(d-1)$ as claimed.

1.4 Ward identities

In this brief subsection we will write down the Ward identities for the stress tensor and a general abelian current and present a simple counting argument that appeared in [82] to compute how many equations stem from the Ward identities of a given correlator. Part of this subsection overlaps with the content of **Paper II**.

Let us denote three-point functions as

$$\mathsf{t}_{\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3}(x_i;\eta_i,\bar\eta_i) \coloneqq \langle \mathcal{O}_1(\mathbf{x}_1)\mathcal{O}_2(\mathbf{x}_2)\mathcal{O}_3(\mathbf{x}_3)\rangle\,, \tag{1.42}$$

and two-point functions as

$$\mathsf{n}_{\bar{\mathcal{O}}\mathcal{O}}(x_{12},\eta_{12},\bar\eta_{12}) \coloneqq \langle \bar{\mathcal{O}}(\mathbf{x}_1)\mathcal{O}(\mathbf{x}_2)\rangle = i^{j+\bar j} c_{\mathcal{O}}\frac{(\eta_2 \mathsf{x}_{12}\bar\eta_1)^j(\eta_1\mathsf{x}_{12}\bar\eta_2)^{\bar j}}{{x_{12}}^{2\Delta+j+\bar j}}\,. \tag{1.43}$$

We start from the abelian current case. J is a Noether current associated to a certain $U(1)$ global symmetry. We take $\mathcal{O}$ to have charge r under this group and $\bar{\mathcal{O}}$ to be its conjugate. Define then Σ as a codimension-one surface enclosing x_2 and x_3 but not x_1. The Ward identity states

$$\frac{i}{2}\int_\Sigma \mathrm{d}\Omega(x_{23})\, x_{23}^2\, \partial_{\eta_2}\mathsf{x}_{23}\partial_{\bar\eta_2}\,\mathsf{t}_{\bar{\mathcal{O}}J\mathcal{O}}(x_i;\eta_i,\bar\eta_i) = 2\,r\,\mathsf{n}_{\bar{\mathcal{O}}\mathcal{O}}(x_{13},\eta_{1,3},\bar\eta_{1,3})\,. \tag{1.44}$$

The factor $i/2$ on the left hand side comes from the $-1/2$ obtained from $x^\mu J_\mu = -\frac{1}{2}\bar{\mathsf{x}}^{\dot\alpha\alpha}J_{\alpha\dot\alpha}$ and a $-i$ from the Wick rotation. Indeed the integral in the above equation is in Euclidean signature and the right prescription for the Wick rotation is the one that keeps the operators radially ordered as indicated, namely if $x_i^0 = -i\tau_i$, then $\tau_1 > \tau_2 > \tau_3$. The factor of 2 on the right hand side is a normalization for J. Since this integral depends only topologically on the points we can evaluate it in the simplified limit $x_1 \to \infty$, $x_{23} \to 0$.

We proceed by considering the Ward identities for the conformal group. To each conformal Killing vector ε^a_μ is associated a possibly independent identity. In the cases we will consider in this thesis it is sufficient to impose only $\varepsilon_\mu = x_\mu$ (dilatations) and $\varepsilon^\nu_\mu = \delta^\nu_\mu$

(translations). Dilatations and translations imply respectively the identities

$$\begin{aligned} -\frac{i}{8}\int_\Sigma \mathrm{d}\Omega(x_{23})\, x_{23}^2\, \partial_{\eta_2}\mathrm{x}_{23}\partial_{\bar\eta_2}\, \partial_{\eta_2}\mathrm{x}_2\partial_{\bar\eta_2}\, \mathrm{t}_{\bar{\mathcal{O}}T\mathcal{O}}(x_i,\eta_i,\bar\eta_i) \\ = -2i\,(\Delta + x_3\cdot\partial_3)\;\mathrm{n}_{\bar{\mathcal{O}}\mathcal{O}}(x_{13},\eta_{1,3},\bar\eta_{1,3})\,, \end{aligned} \tag{1.45a}$$

$$\begin{aligned} -\frac{i}{8}\int_\Sigma \mathrm{d}\Omega(x_{23})\, x_{23}^2\, \partial_{\eta_2}\mathrm{x}_{23}\partial_{\bar\eta_2}\, \partial_{\eta_2}\mathrm{y}\partial_{\bar\eta_2}\, \mathrm{t}_{\bar{\mathcal{O}}T\mathcal{O}}(x_i,\eta_i,\bar\eta_i) \\ = -2i\, y\cdot\partial_3\, \mathrm{n}_{\bar{\mathcal{O}}\mathcal{O}}(x_{13},\eta_{1,3},\bar\eta_{1,3})\,, \end{aligned} \tag{1.45b}$$

where y^μ is an arbitrary vector used to contract the free index of the translation Killing vector.

The integrals appearing in (1.44) and (1.45) may be computed by taking a convenient limit, namely $x_{23} \ll x_{13}$. In this limit we must only keep the terms that scale as $O(x_{23}^0)$. It is possible that one also obtains terms of order $O(x_{23}^{-1})$, but those are necessarily zero by parity as they must have an odd number of x_{23}^μ's. Taking $x_{23} \ll x_{13}$ does not affect generality because we can always make a conformal transformation to achieve any desired configuration. Then we specialize Σ to be a three-sphere, so that we can use rotational symmetry to simplify the integrand as follows

$$\begin{aligned} \int \mathrm{d}\Omega(x)\,\frac{x^\mu x^\nu}{x^2} &= \frac{\delta^{\mu\nu}}{4}2\pi^2\,, \\ \int \mathrm{d}\Omega(x)\,\frac{x^\mu x^\nu x^\rho x^\lambda}{x^4} &= \frac{\delta^{\mu\nu}\delta^{\rho\lambda}+\delta^{\mu\rho}\delta^{\lambda\nu}+\delta^{\mu\lambda}\delta^{\rho\nu}}{4\cdot 6}2\pi^2\,, \\ \int \mathrm{d}\Omega(x)\,\frac{x^{\mu_1}\cdots x^{\mu_{2n}}}{x^{2n}} &= \frac{\delta^{\mu_1\mu_2}\cdots\delta^{\mu_{2n-1}\mu_{2n}}+\text{permutations}}{2^n(2)_n}2\pi^2\,, \end{aligned} \tag{1.46}$$

where $(2)_n$ is the Pochhammer symbol and "permutations" stands for all inequivalent permutations of the μ_i indices.

An approach similar to the conformal frame defined in the previous subsection allows us to compute the expected number of independent linear equations that stem from a Ward identity. The topological operator that enters the stress tensor Ward identity is given by the integral of T contracted with a conformal Killing vector ε^μ_{MN}

$$L_{MN} = \int_\Sigma \mathrm{d}\Omega\, x^2\, \partial_\eta \mathrm{x}\partial_{\bar\eta_2}\partial_\eta(\sigma\cdot\varepsilon_{MN})\partial_{\bar\eta}\, T(\mathrm{x})\,. \tag{1.47}$$

It is possible to contract this again with the Killing vector, thus obtaining an object that transforms like a primary of dimension -1

$$\mathcal{Q}^\mu(x) := \varepsilon^{\mu MN}(x) L_{MN}\,. \tag{1.48}$$

$\mathcal{Q}$ depends on the point x but it is not a local operator. It is instead a finite dimensional representation of the conformal group: the adjoint [103]. Under $G_0 = \mathrm{SO}(d)\times\mathrm{SO}(1,1)$

it decomposes as

$$\begin{array}{rcccccccc} \mathrm{Res}^{G}_{G_0}\,\mathbf{adj}^{G} & = & (\mathbf{fund},-1) & \oplus & (\mathbf{adj},0) & \oplus & (\bullet,0) & \oplus & (\mathbf{fund},1) \\ & = & K_\mu & \oplus & M_{\mu\nu} & \oplus & D & \oplus & P_\mu\,, \end{array} \tag{1.49}$$

where the $\bullet$ represents a singlet and **fund** and **adj** in the right hand side refer to the fundamental and the adjoint of $\mathrm{SO}(d)$, respectively. We can imagine a Ward identity as a three-point function involving such an operator. Therefore we can count the allowed tensor structures by the arguments presented in the previous subsection. The stability group H is the group of two points, namely G_0. Calling $n_W(\mathcal{O})$ the number of Ward identities in $\langle\overline{\mathcal{O}}T\mathcal{O}\rangle$ one has

$$n_W(\mathcal{O}) = \left(\mathrm{Res}^{G}_{G_0} V_{\mathcal{O}}\otimes V^{*}_{\mathcal{O}}\otimes \mathbf{adj}^{G}\right)^{G_0} = \left(\rho_{\mathcal{O}}\otimes\rho^{*}_{\mathcal{O}}\otimes(\bullet\oplus\mathbf{adj})\right)^{\mathrm{SO}(d)}\,,$$

where $V_{\mathcal{O}} = (\rho_{\mathcal{O}},\Delta)$ is the G_0 representation of $\mathcal{O}$. Since we are in four dimensions

$$\mathbf{adj} = (2,0)\oplus(0,2)\,. \tag{1.50}$$

We will use this formula in Chapter 4 to compute the number of Ward identities for $\rho_{\mathcal{O}} = (j,0)$ and $(j,1)$.

2 Supersymmetry

2.1 The superconformal group

2.1.1 Representations of the superconformal algebra

The conformal group can be extended by the addition of some fermionic generators which satisfy a $\mathbb{Z}_2$ graded Lie algebra, in the same spirit as Poincaré supersymmetry. The grading is referred to as fermion number and the supercharges have fermion number 1. In the non conformal case one introduces some supercharges Q that square to a translation, namely

$$\{Q_a, Q_b\} \sim \gamma^\mu_{ab} P_\mu \,, \tag{2.1}$$

where γ^μ is a Dirac matrix $\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}$. Moreover the Q's transform under a spinorial representation of $\mathrm{SO}(d)$. Since in the conformal group we have two Poincaré subgroups, the other being G_K (see (1.36)), one naturally has to introduce additional supercharges that square to a special conformal transformation

$$\{S_a, S_b\} \sim \gamma^\mu_{ab} K_\mu \,. \tag{2.2}$$

In order to respect also the grading realized by the dilatation generator, Q must have dimension $1/2$ and S dimension $-1/2$. The algebra is thus characterized by the number of spacetime dimensions d and the number of supercharges N_Q. Typically one quotes the number $\mathcal{N}$ instead, which is N_Q divided by the dimension of a minimal spinor in d dimensions. In addition to Q and S one often has to introduce additional bosonic generators R that commute with the conformal group and transform the supercharges among themselves. They constitute the so-called R-symmetry algebra.

The superconformal algebras have been classified long ago [11] and their unitary representations have been studied extensively, leading to a classification in four dimensions [12, 13] and in any dimension [14, 15]. Surprisingly, only a finite number of values for $\mathcal{N}$ and d are allowed: a mathematical obstruction prevents superconformal algebras

to exist in dimensions greater than six; moreover, locality of the resulting superconformal field theory (SCFT) puts an upper bound on the amount of supercharges $\mathcal{N}$.[1] The complete list of all superconformal algebras is given in Table A.1 in Appendix A.3.1. This thesis is focused on the case $4d$, $\mathcal{N}=1$ and $4d$, $\mathcal{N}=2$ which have respectively R-symmetry $\mathfrak{u}(1)$ and $\mathfrak{su}(2)\oplus\mathfrak{u}(1)$. In Appendix A.3.1 the reader can also find the superconformal algebra and the conventions used.

The classification of unitary superconformal multiplets shares some similarities with the non-supersymmetric case. Since S behaves like a "square root" of K one can impose a condition stronger than primality. Namely we define superconformal primary (or superprimary) an operator $\mathcal{O}$ that satisfies

$$S_I^\alpha \mathcal{O}(0) = \bar{S}^{I\dot{\alpha}}\mathcal{O}(0) = 0\,. \tag{2.3}$$

Clearly a superprimary is also a primary, but the converse is not necessarily true. By applying products of the supercharges Q on $\mathcal{O}$ we can build the superdescendants. Schematically

$$(Q^\ell \bar{Q}^{\bar{\ell}}\mathcal{O})(0) \sim \prod_{i=1}^{\ell} Q_{\alpha_i}^{I_i} \prod_{j=1}^{\bar{\ell}} \bar{Q}_{I_j\dot{\alpha}_j}\mathcal{O}(0)\,. \tag{2.4}$$

These operators will not be superconformal primaries, but appropriate linear combinations of them and the descendants $P_{\mu_1}\cdots P_{\mu_n}\mathcal{O}(0)$ will be conformal primaries. The Q's and the $\bar{Q}$'s are nilpotent and they can be put into a canonical order by using the anticommutation relation $\{Q,\bar{Q}\}\sim P$. This means that the superdescendants will only contain a finite number of primaries among them. Each of them can then form a full conformal multiplet upon acting with arbitrary products of P_μ. The conclusion is that a superconformal multiplet is a finite sum of conformal multiplets with various spins and dimensions obtained by acting with Q and $\bar{Q}$ on a superconformal primary operator.

Just like the non-supersymmetric case, also here unitarity places constraints on the allowed quantum numbers of a superprimary operator. Together with the dimension Δ and the spin $(j,\bar{\jmath})$ we also have the $\mathfrak{u}(1)$ R-charge r and (only for $\mathcal{N}=2$) the $\mathfrak{su}(2)$ Dynkin label R.[2] Let us first define the $q,\bar{q}$ charges as :

$$\begin{aligned} &\Delta = q+\bar{q}\,, && r=\tfrac{2}{3}(q-\bar{q})\,, && \text{for } \mathcal{N}=1\,,\\ &\Delta = q+\bar{q}\,, && r=2(q-\bar{q})\,, && \text{for } \mathcal{N}=2\,. \end{aligned} \tag{2.5}$$

The unitarity bounds for long multiplets in these variables read

$$\begin{aligned} &2q \geqslant j+2\,, && 2\bar{q}\geqslant \bar{\jmath}+2\,, && \text{for } \mathcal{N}=1\,,\\ &2q \geqslant j+2+R\,, && 2\bar{q}\geqslant \bar{\jmath}+2+R\,, && \text{for } \mathcal{N}=2\,. \end{aligned} \tag{2.6}$$

[1] In three dimensions there actually exist theories for any $\mathcal{N}$, but those for $\mathcal{N}>8$ are necessarily free.
[2] This means that the Cartan R_3 ranges in $-R/2,\ldots,R/2$.

However, unlike the non-supersymmetric case, here there are additional nontrivial allowed values of q and $\bar{q}$ which are disconnected from the above bounds.[3] Namely one can have

$$\begin{aligned} &q = j = 0 && \text{and/or} && \bar{q} = \bar{\jmath} = 0\,, && \text{for } \mathcal{N} = 1\,, \\ &q = \tfrac{R}{2}\,,\ j = 0 && \text{and/or} && \bar{q} = \tfrac{R}{2}\,,\ \bar{\jmath} = 0\,, && \text{for } \mathcal{N} = 2\,. \end{aligned} \tag{2.7}$$

Also in this case, when one or more of these bounds are saturated, some states become null and we obtain a short multiplet. However now there are two kinds of short multiplets: those which are connected to the long multiplet unitarity bound (type A shortening) and those that satisfy one of the conditions in (2.7) (type B shortening). The story is analogous also in other superconformal algebras: there are one or more unitary values that are disconnected from the spectrum of long multiplets. These give rise to shortening conditions of type A, B, etc. in some cases up to D.

Let us be more precise for the case at hand, we will follow the notation of [14]. Since we have two sets of supercharges Q and $\overline{Q}$, the shortening condition is specified by a pair of letters $\mathcal{X}_L\overline{\mathcal{X}}_R$, where $\mathcal{X}_{L,R} \in \{L, A_1, A_2, B_1\}$. The subscript indicates the level at which one can find the null state. Furthermore we denote as $[j;\bar{\jmath}]^{(R;r)}$ an $\mathcal{N} = 2$ multiplet and as $[j;\bar{\jmath}]^{(r)}$ an $\mathcal{N} = 1$ multiplet. The shortening $\mathcal{X}_L$ can be

L: Unconstrained action of Q on $\mathcal{O}$. Bound $2q \geqslant j + 2 + R$.

A_1: Null state $[j-1;\bar{\jmath}]^{(R+1;r-1)}_{\Delta+\frac{1}{2}} = Q^{\alpha(I}\mathcal{O}^{I_1\cdots I_R)}_{\alpha\alpha_2\cdots\alpha_j}$. Saturates $2q = j + 2 + R$.

A_2: Null state $[0;\bar{\jmath}]^{(R+2;r-2)}_{\Delta+1} = \epsilon^{\alpha\beta}Q^{(I}_{\alpha}Q^{J}_{\beta}\mathcal{O}^{I_1\cdots I_R)}$. Saturates $2q = 2 + R$, has $j = 0$.

B_1: Null state $[1;\bar{\jmath}]^{(R+1;r-1)}_{\Delta+\frac{1}{2}} = Q^{(I}_{\alpha}\mathcal{O}^{I_1\cdots I_R)}$. Disconnected from L: $q = 0$, $j = 0$.

An analogous table can be done for the $\overline{Q}$ charges, upon changing $q, j \to \bar{q}, \bar{\jmath}$. The $\mathcal{N} = 1$ case can be simply obtained by setting R to zero and dropping the I, J indices.

2.1.2 Embedding of $\mathcal{N} = 1$ into $\mathcal{N} = 2$

Clearly the $\mathcal{N} = 2$ superconformal algebra contains an $\mathcal{N} = 1$ subalgebra. The embedding is not unique, but any choice is equivalent. We will pick the subalgebra generated by Q^1_α and $\overline{Q}_{\underline{1}\dot{\alpha}}$. This leads to

$$q_{\mathcal{N}=1} = q_{\mathcal{N}=2} - R_3\,, \qquad \bar{q}_{\mathcal{N}=1} = \bar{q}_{\mathcal{N}=2} + R_3\,, \tag{2.8}$$

where R_3 is the Cartan of the $\mathfrak{su}(2)$ R-symmetry.

[3] In a non-supersymmetric CFT the only operator disconnected from the unitarity bound is the identity ($\Delta = 0$).

For $\mathcal{N}=1$ we denote as $\mathcal{O}$ the superconformal multiplet, as O the superprimary and as $(Q^\ell \bar{Q}^{\bar{\ell}} O)$, with $\ell, \bar{\ell} = 0, 1, 2$, its superdescendants, which are taken to be conformal primaries by definition. This is a slight abuse of notation because, as we mentioned earlier, the operator obtained by acting with Q and $\bar{Q}$ on O is not, in general, a primary. We need to take appropriate linear combinations with the conformal descendants. This will be discussed later. When there are multiple choices for the spin, those operators will be distinguished by a superscript. For example $(Q\bar{Q}O)^{+-}$ has spin $(j+1, \bar{\jmath}-1)$.

For $\mathcal{N}=2$ instead we define $(Q^{\underline{2}\,\ell} \bar{Q}_{\underline{2}}{}^{\bar{\ell}} \mathcal{O})$ to be a family of $\mathcal{N}=1$ superconformal primaries, with the same abuse of notation we did before. In fact, due to the anticommutation relations[4]

$$\{Q^I_\alpha, S^\beta_J\} = -4\delta^\beta_\alpha R^I{}_J\,, \qquad \{\bar{S}^{I\dot{\alpha}}, \bar{Q}_{J\dot{\beta}}\} = -4\delta^{\dot{\beta}}_{\dot{\alpha}} R^I{}_J\,, \qquad I \neq J\,, \tag{2.9}$$

in general such operators will not be obtained by simply acting with $Q^{\underline{2}}$ and $\bar{Q}_{\underline{2}}$ on $\mathcal{O}$ and thus some subtractions might be needed. Each of these families will decompose in $R+1$ multiplets with the following charges

$$(q+m,\, \bar{q}-m)\,, \qquad m = -\tfrac{1}{2}R,\, -\tfrac{1}{2}R+1, \ldots, \tfrac{1}{2}R\,, \tag{2.10}$$

where we denoted the $\mathcal{N}=2$ charges with $q, \bar{q}$ and the $\mathfrak{su}(2)$ representation of $\mathcal{O}$ with R. The set of all conformal multiplets inside a long $\mathcal{N}=1$ supermultiplet is illustrated in Figure 2.1. The $\mathcal{N}=1$ decomposition of a long $\mathcal{N}=2$ according to the embedding defined above can instead be found in Figure 2.2.

2.1.3 Introducing superspace

One multiplet of great interest is the one that contains the stress tensor. Any local theory has one, by definition. Since T itself is a short conformal multiplet, its superconformal version must be short too. For $\mathcal{N}=1$ it goes under the name of the Ferrara-Zumino multiplet [151] and its superprimary is a conserved vector $J_{\alpha\dot{\alpha}}$, which is the R-symmetry Noether current. For $\mathcal{N}=2$ instead the superprimary is a scalar $\mathcal{J}$ of dimension 2 [152]. The shortening conditions are of A type on either side. So, to summarize,

$$\begin{aligned} J_{\alpha\dot{\alpha}} &= A_1 A_1[1;1]^{(0)}_3\,, \\ \mathcal{J} &= A_2 A_2[0;0]^{(0;0)}_2 = \widehat{\mathcal{C}}_{0(0,0)}\,, \end{aligned}$$

where in the second line we also included the notation of [13]. At order $(Q^{\underline{2}}\bar{Q}_{\underline{2}}\mathcal{J})^{++}$ we can identify precisely the Ferrara-Zumino multiplet J. Inside J, other than the R-current $J^{(R)}$, we have the stress tensor at order $(Q\bar{Q}J)^{++}$ and the supersymmetry currents[5] $S_{\mu\alpha}$

[4] The generators $R^I{}_J$ for $I \neq J$ are the $\mathfrak{su}(2)_R$ ladder operators $R_\pm$.

[5] Not to be confused with the conformal supercharges S^α and $\bar{S}^{\dot{\alpha}}$.

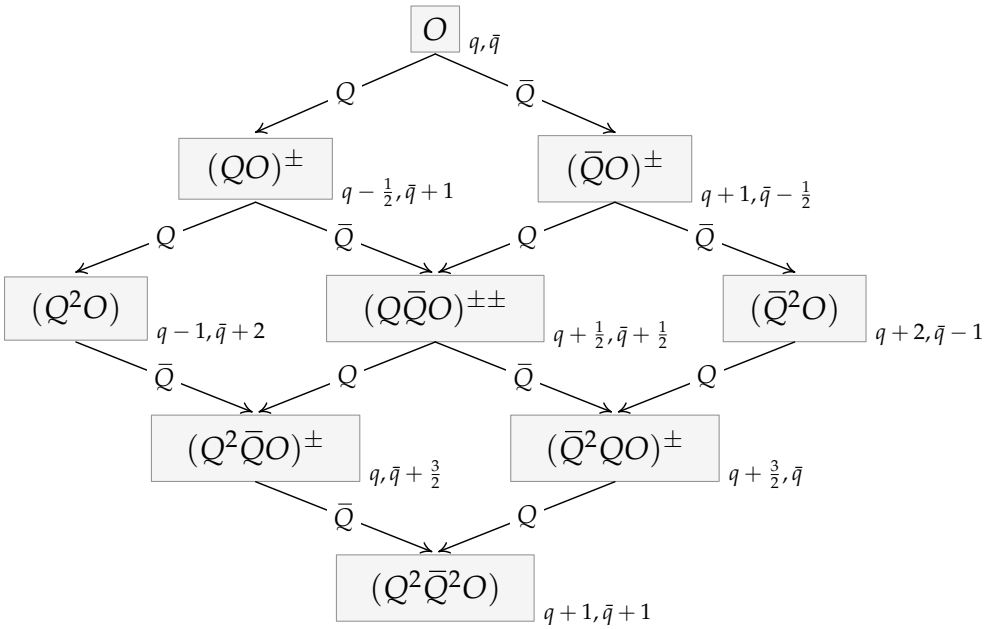


Figure 2.1. Diagram of all operators in an $\mathcal{N}=1$ long multiplet. Superscripts of $\pm$ indicate the choice of spin $j\pm1$ or $\bar{\jmath}\pm1$. Subscripts indicate the q, $\bar{q}$ charges. The R-charge grows from left to right by 1 and the conformal dimension grows from top to bottom by $1/2$. A box represents the full tower of descendants $O, \partial_\mu O, \partial^2 O, \ldots$

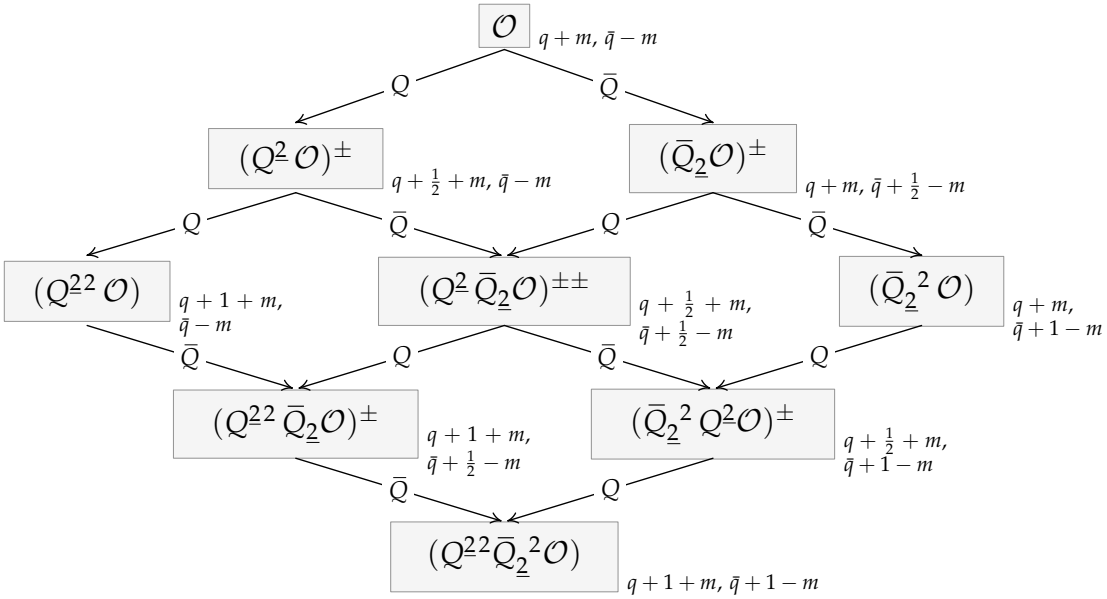


Figure 2.2. Diagram of all $\mathcal{N}=1$ multiplets in an $\mathcal{N}=2$ long multiplet. Superscripts of $\pm$ indicate the choice of spin $j\pm1$ or $\bar{\jmath}\pm1$. Subscripts indicate the $\mathcal{N}=1$ q, $\bar{q}$ charges. Here $m=R_3$ takes integer spaced values between $-R/2$ and $R/2$. The $\mathfrak{u}(1)_{\mathcal{N}=2}$ R-charge grows from left to right by 1 and the conformal dimension grows from top to bottom by $1/2$. A box represents a family of long $\mathcal{N}=1$ multiplets. Underlined indices are $\mathfrak{su}(2)_R$ indices.

and $\bar{S}_{\mu\dot{\alpha}}$ at order $(QJ)^+$ and $(\bar{Q}J)^+$, respectively.

Another interesting multiplet is the chiral scalar Φ. In $\mathcal{N}=1$ Lagrangian theories it is a multiplet that represents the matter content: it contains a complex scalar, a Weyl fermion and a scalar auxiliary field. In $\mathcal{N}=2$ instead the chiral scalar is an $\mathfrak{su}(2)_R$ singlet and it is also called vector multiplet as it contains the gauge sector of the theory. The expectation values of the scalars in the multiplet parametrize the so-called Coulomb branch. The shortening condition of both the $\mathcal{N}=1$ and the $\mathcal{N}=2$ case is of $L\bar{B}_1$ type (or $B_1\bar{L}$) and therefore they have $\bar{q}=0$ (or, respectively, $q=0$). That means

$$\begin{aligned}\Phi_{\mathcal{N}=1} &= L\bar{B}_1[0;0]^{(r)}_{3r/2}\,,\\ \Phi_{\mathcal{N}=2} &= L\bar{B}_1[0;0]^{(0;r)}_{r/2} = \bar{\mathcal{E}}_{r/2(0,0)}\,,\end{aligned} \tag{2.11}$$

and similarly for their complex conjugate partners. In the second line we wrote the multiplet in the notation of [13] as well.

A useful tool for combining together all operators inside a multiplet is superspace. All definitions can be found in Appendix A.3.2. An operator in superspace depends on the point x^μ and also on Grassmann variables $\theta^\alpha_I, \bar{\theta}^I_{\dot{\alpha}}$, where I is only present in the $\mathcal{N}=2$ case. We use a shorthand z to denote the tuple $x, \theta_I, \bar{\theta}^I$, $\mathbf{x}$ to denote $x, \eta, \bar{\eta}$ and finally $\mathbf{z}$ to denote $z, \eta, \bar{\eta}$.

$$\mathcal{O}(\mathbf{z}) \coloneqq \mathcal{O}(x, \theta_I, \bar{\theta}^I, \eta, \bar{\eta}) = e^{i\theta_I Q^I + i\bar{\theta}^I \bar{Q}_I}\, O(\mathbf{x})\,. \tag{2.12}$$

The shortening conditions can be realized in superspace as some differential equations. In particular, the A_1 type shortening signifies a D_α, with α contracted and a A_2 type a $D^\alpha D_\alpha$. So for the stress tensor multiplet we have

$$D_\alpha J^{\alpha\dot{\alpha}} = 0\,, \qquad \bar{D}_{\dot{\alpha}} J^{\alpha\dot{\alpha}} = 0\,, \tag{2.13a}$$

$$\epsilon^{\alpha\beta} D^I_\alpha D^J_\beta \mathcal{J} = 0\,, \qquad \epsilon^{\dot{\alpha}\dot{\beta}} \bar{D}_{I\dot{\alpha}} \bar{D}_{J\dot{\beta}} \mathcal{J} = 0\,. \tag{2.13b}$$

The B_1 type, on the other hand, consists in imposing $D_\alpha \mathcal{O}_{\dot{\alpha}_1\cdots\dot{\alpha}_{\bar{j}}} = 0$, without contracting any Lorentz indices. Therefore the chiral scalar needs to satisfy

$$\bar{D}^I_{\dot{\alpha}} \Phi = 0\,. \tag{2.14}$$

The expansion of $J_{\alpha\dot{\alpha}}$ in components reads (see, e.g. [153])

$$\begin{aligned}-\tfrac{1}{2}\bar{\sigma}^{\dot{\alpha}\alpha}_\mu J_{\alpha\dot{\alpha}}(z) = J^{(R)}_\mu(x) &+ \tfrac{i}{2}\theta^\alpha S_{\mu\alpha}(x) - \tfrac{i}{2}\bar{S}_{\mu\dot{\alpha}}(x)\bar{\theta}^{\dot{\alpha}}\\ &+ \theta^\alpha \sigma^\nu_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}} \left(T_{\mu\nu}(x) - \tfrac{1}{2}\epsilon_{\mu\nu\rho\lambda}\partial^\rho J^{(R)\lambda}(x)\right)\\ &- \tfrac{1}{8}\theta^2\, \partial_\nu S_\mu(x)\sigma^\nu\bar{\theta} - \tfrac{1}{8}\bar{\theta}^2\, \theta\sigma^\nu \partial_\nu \bar{S}_\mu(x) - \tfrac{1}{4}\theta^2\bar{\theta}^2 \partial^2 J^{(R)}_\mu(x)\,.\end{aligned} \tag{2.15}$$

The shortening condition (2.13a) implies the following conservation and irreducibility

conditions:

$$\partial_\mu J^\mu = \partial_\mu T^{\mu\nu} = T^\mu{}_\mu = \partial_\mu S^\mu_\alpha = T_{[\mu\nu]} = \partial_\mu \bar{S}^\mu_{\dot\alpha} = \bar\sigma^{\dot\alpha\alpha}_\mu S^\mu_\alpha = \bar{S}^\mu_{\dot\alpha}\,\bar\sigma^{\dot\alpha\alpha}_\mu = 0\,. \tag{2.16}$$

The first four equations impose the conservation of the currents that generate the superconformal transformations, while the remaining ones simply follow from the fact that the operators involved are irreducible representation of the Lorentz group. In order to keep this discussion compact, we will not show explicitly the decomposition of $\mathcal{J}$ into $\mathcal{N}=1$ multiplets.

In the chiral scalar case the shortening condition does not imply any conservation equation, but rather it removes some operators from the multiplet. Indeed, by defining $y^\mu = x^\mu + i\theta\sigma^\mu\bar\theta$, it is easy to see that $\bar{D}_{\dot\alpha} y^\mu = 0$ and thus, in $\mathcal{N}=1$ superspace,

$$\Phi_{\mathcal{N}=1}(z) = \phi(y) + \theta^\alpha \psi_\alpha(y) + \theta^2\, F(y)\,. \tag{2.17}$$

This means that there are only half the terms in the expansion. The $\mathcal{N}=2$ chiral multiplet can easily be expanded in $\mathcal{N}=1$ following the results of Subsection 2.1.2. Defining now $z_0 = z|_{\theta_{\underline{2}}=\bar\theta^{\underline{2}}=0}$ one has

$$\begin{aligned}
\Phi_{\mathcal{N}=2}(z) &= \Phi(z_0) + \theta^\alpha_{\underline{2}}\Psi_\alpha(z_0) + \theta^2_{\underline{2}}\mathcal{F}(z_0)\,,\\
\Phi &= [0;0]^{(r/3)}_{r/2}\,,\\
\Psi &= [1;0]^{((r+1)/3)}_{(r+1)/2}\,,\\
\mathcal{F} &= [0;0]^{((r+2)/2)}_{(r+2)/2}\,,
\end{aligned} \tag{2.18}$$

where Φ is just the chiral multiplet of (2.17), Ψ is an $\mathcal{N}=1$ vector multiplet and finally $\mathcal{F}$ is another chiral multiplet. When r hits the A_2 unitarity bound, $r=2$, there is an additional conservation condition. As a consequence Ψ has at level one a self-dual free field of spin $(2,0)$, namely a field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, hence the name vector multiplet. By extension, the $\mathcal{N}=2$ chiral multiplet $\Phi_{\mathcal{N}=2}$ is often referred to as vector multiplet as well.

Lastly we would like to mention another important multiplet in $\mathcal{N}=2$ theories: the hypermultiplet. Using the same notations as before the hypermultiplet Q can be written as

$$H = B_1\bar{B}_1\,[0;0]^{R;0}_R = \widehat{\mathcal{B}}_R\,. \tag{2.19}$$

When $R=1$ (doublet representation) the multiplet becomes a free field. In $\mathcal{N}=1$ language it can be seen as a chiral and an anti-chiral multiplet. This is particularly easy to see from Figure 2.2 because the shortening conditions remove all the boxes except the one at the root. So we get two $\mathcal{N}=1$ multiplets: one with $q,\bar{q} = 1,0$ and one with

$q, \bar{q} = 0, 1$

$$H^{I=\underline{1}} = \Phi \ \in \ [0;0]_1^{(2/3)}\,, \qquad H^{I=\underline{2}} = \overline{\Phi} \ \in \ [0;0]_1^{(-2/3)}\,.$$

The above equality is only schematic. For example, we purposely left unclear the dependence on the Grassmann variables $\theta_{\underline{2}}$, $\bar{\theta}_{\underline{2}}$ as it is fixed by the shortening condition.

Both the hypermultiplet $\widehat{\mathcal{B}}_R$ and the chiral multiplet $\bar{\mathcal{E}}_{r/2(0,0)}$ play a key role in all $\mathcal{N} = 2$ quantum field theories. Their expectation values parametrize two types of moduli spaces of vacua which are called respectively Higgs branch and Coulomb branch. These moduli spaces of vacua are present in most Lagrangian $\mathcal{N} = 2$ QFTs and their names come from the degrees of freedom that can be found in the low energy effective field theory. In the Higgs branch the hypermultiplet gets a vacuum expectation value thus breaking the gauge symmetry and, as a consequence, the scalars in the multiplet are eaten by the gauge vector which becomes massive. In the Coulomb branch, on the other hand, the chiral multiplet is the one that gets a vacuum expectation value. Since it transforms in the adjoint of the gauge group, this leaves a number of massless photons in the IR theory, hence the name Coulomb. An introduction to $\mathcal{N} = 2$ supersymmetry can be found in [154].

This concludes our introduction to superspace. The interested reader can find the $\mathcal{N} = 1$ superspace expansion of a generic long multiplet in Appendix B.2.

2.2 Superconformal correlators

We will follow the conventions of [155] for four dimensional spinors and utilize the formalism of [114, 115] for $\mathcal{N} = 1$ superspace and its generalization [116, 117] for $\mathcal{N} = 2$ superspace. See Appendix A.3.2 for more details.

2.2.1 $\mathcal{N} = 1$ superspace

Given three superconformal primaries $\mathcal{O}_1, \mathcal{O}_2$ and $\mathcal{O}_3$ whose sum of R-charges is $0, 1$ or 2 in absolute value, one can define a three-point function as

$$\langle \mathcal{O}_1(\mathbf{z}_1)\, \mathcal{O}_2(\mathbf{z}_2)\, \mathcal{O}_3(\mathbf{z}_3) \rangle = \mathcal{K}_{\mathcal{O}_1\mathcal{O}_2}(\mathbf{z}_{1,2}, z_3; \partial_{\chi_{1,2}}, \partial_{\bar{\chi}_{1,2}})\, t_{\mathcal{O}_3}^{\mathcal{O}_1\mathcal{O}_2}(Z_3; \chi_{1,2}, \bar{\chi}_{1,2}; \eta_3, \bar{\eta}_3)\,. \quad (2.20)$$

The $\mathcal{K}_{\mathcal{O}_1\mathcal{O}_2}$ is an universal prefactor and the $t_{\mathcal{O}_3}^{\mathcal{O}_1\mathcal{O}_2}$ encodes all the information of the three-point function and can be expressed as a linear combination of tensor structures. The commuting spinors $\chi_i, \bar{\chi}_i$ are auxiliary polarizations that are removed by the derivatives in the prefactor. The variable Z_3 collectively denotes the superconformally covariant variables X_3, Θ_3 and $\overline{\Theta}_3$ whose definition can be found in (A.30). Clearly, since here the operator $\mathcal{O}_3$ is treated differently, there are two other equivalent representation related by cyclic permutations. For more details see Appendix B.1.

The general form of the prefactor $\mathcal{K}_{\mathcal{O}_1\mathcal{O}_2}$ is the following

$$\mathcal{K}_{\mathcal{O}_1\mathcal{O}_2} = \frac{1}{j_1!\,\bar{j}_1!\,j_2!\,\bar{j}_2!}\frac{\prod_{i=1}^{2}(\eta_i \mathrm{x}_{i\bar{3}}\partial_{\bar{\chi}_i})^{j_i}\,(\partial_{\chi_i}\mathrm{x}_{3\bar{i}}\bar{\eta}_i)^{\bar{j}_i}}{\prod_{i=1}^{2}\mathrm{x}_{\bar{3}i}{}^{2q_i+j_i}\,\mathrm{x}_{\bar{i}3}{}^{2\bar{q}_i+\bar{j}_i}}\,. \tag{2.21}$$

For the definition of $x_{i\bar{j}}$ see (A.26). Note that, as the name suggests, the prefactor only depends on the quantum numbers of the first two operators. The $t^{\mathcal{O}_1\mathcal{O}_2}_{\mathcal{O}_3}$ can contain all Lorentz invariant combinations of its arguments as tensor structures. They need to be homogeneous functions of the auxiliary spinors — with the degree dictated by the spins j_i and $\bar{j}_i$ — and also satisfy a scaling property illustrated in (B.3). Due to the Schouten identities[6] these tensor structures can be hard to enumerate. The expected number can be computed by a group theoretic formula which will be introduced in Section 2.3. Moreover, since the problem is essentially analogous to listing tensor structures in embedding space, one can easily obtain them by using the results of [89,91] (reviewed in Section 1.2.2). The idea is to first define the mapping

$$\begin{aligned}
\hat{\mathbb{K}}^{ij}_{k} &\longrightarrow \eta_i\eta_j\,, & \hat{\bar{\mathbb{K}}}^{ij}_{k} &\longrightarrow \bar{\eta}_i\bar{\eta}_j\,,\\
\hat{\mathbb{J}}^{i}_{jk} &\longrightarrow U_3^{-1}\,\eta_i \mathrm{U}_3\,\bar{\eta}_i & \hat{\mathbb{I}}^{ij} &\longrightarrow U_3^{-1}\,\eta_i \mathrm{U}_3\,\bar{\eta}_j\,,
\end{aligned} \tag{2.22}$$

where we have renamed χ_i and $\bar{\chi}_i$ to η_i and $\bar{\eta}_i$ respectively for simplicity of notation and used U_3 defined in (A.31). Then we proceed order by order in $\Theta_3, \overline{\Theta}_3$. The order zero is trivial as it suffices to apply (2.22) to the three-point function. Now say we want to compute the order $\Theta_3\overline{\Theta}_3$. We simply consider the three-point functions with all four combinations of spin[7]

$$(j_1,\bar{j}_1),\,(j_2,\bar{j}_2),\,(j_3\pm 1,\bar{j}_3\pm 1)\,, \tag{2.23}$$

and $\Delta_3 \to \Delta_3 + 1$. Then, after applying the mapping (2.22), we remove the extra $\eta_3, \bar{\eta}_3$ spinors with $\Theta_3\partial_{\eta_3}$, $\overline{\Theta}_3\partial_{\bar{\eta}_3}$ and attach missing $\eta_3, \bar{\eta}_3$ spinors with $\Theta_3\eta_3$, $\overline{\Theta}_3\bar{\eta}_3$. For quadratic orders it suffices to attach an overall Θ_3^2 or $\overline{\Theta}_3^2$ to the three-point function with no shifts in the spins. The same logic applies to the other orders. The constraints of multiplet shortening and conservation can be applied directly on the t by using the fact that the shortening differential operators always annihilate the prefactor with the appropriate quantum numbers. This is a consequence of

$$\begin{aligned}
&A_1 \text{ shortening:} && \frac{\partial}{\partial\eta_{1\alpha}}D_{1\alpha}\frac{(\eta_1\mathrm{x}_{1\bar{3}}\bar{\eta}_1)^j}{\mathrm{x}_{\bar{3}1}{}^{2j+2}}\,f(x_{\bar{1}3}) = 0\,,\\
&A_2 \text{ shortening:} && D_1{}^2\frac{1}{\mathrm{x}_{\bar{1}3}{}^2}\,f(x_{\bar{1}3}) = 0\,,\\
&B_1 \text{ shortening:} && D_{1\alpha}\,f(x_{\bar{1}3}) = 0\,,
\end{aligned} \tag{2.24}$$

[6] The Schouten identities are explained in Appendix A.1.

[7] If j_3 or $\bar{j}_3$ is zero the corresponding negative shift is omitted. The choice of shifting the spin labels of the third operator is unimportant and equivalent to any other choice, even when $j_3\bar{j}_3 = 0$. In that case, correlators of operators with negative spin are defined to be identically zero.

for $x_{13} \neq 0$. Similar identities hold for $\bar{D}$. If $x_1 = x_3$ we have the usual contact term singularity. Once the differential operator is past the prefactor we can use (A.34) to act on $t_{\mathcal{O}_3}^{\mathcal{O}_1\mathcal{O}_2}$.

2.2.2 $\mathcal{N} = 2$ superspace

With a formalism similar to the previous case one can construct superconformal three-point functions of $\mathcal{N} = 2$ primaries. First we need to introduce the following unitary matrix

$$u_I{}^J(z_{ij}) = \delta_I{}^J - 4i\,\frac{\theta_{ij\,I}\,\mathrm{x}_{\bar{\imath}j}\,\bar{\theta}_{ij}^J}{x_{\bar{\imath}j}{}^2}\,. \tag{2.25}$$

By rescaling $u(z_{ij})$ appropriately we obtain a unimodular matrix

$$\hat{u}_I{}^J(z_{ij}) = \left(\frac{x_{\bar{\jmath}i}{}^2}{x_{\bar{\imath}j}{}^2}\right)^{\frac{1}{2}} u_I{}^J(z_{ij})\,, \qquad \hat{u}(z_{ij}) \in \mathrm{SU}(2)\,. \tag{2.26}$$

Let us consider three superconformal primaries $\mathcal{O}_1^{\mathcal{I}_1}$, $\mathcal{O}_2^{\mathcal{I}_2}$ and $\mathcal{O}_3^{\mathcal{I}_3}$. Here $\mathcal{I}_i$ is an $\mathfrak{su}(2)$ index transforming under the representation R_i. Let us denote as

$$\mathcal{T}^R{}_{\mathcal{I}}{}^{\mathcal{J}}(u)\,, \qquad u \in \mathrm{SU}(2)\,, \tag{2.27}$$

the representation with Dynkin label R of $\mathrm{SU}(2)$. The simplest cases are[8]

$$\mathcal{T}^{\mathbf{1}}{}_I{}^J(u) = u_I^J\,, \qquad \mathcal{T}^{\mathbf{2}}{}_A{}^B(u) = \frac{1}{4}(\sigma^A\epsilon)^{J_1I_1}(\epsilon\sigma^B)_{J_2I_2}\big(u_{I_1}^{\ I_2}\,u_{J_1}^{\ J_2} + u_{I_1}^{\ J_2}\,u_{J_1}^{\ I_2}\big)\,. \tag{2.28}$$

The most general three-point function then has the following form

$$\begin{aligned}\langle \mathcal{O}_1^{\mathcal{I}_1}(\mathbf{z}_1)\,\mathcal{O}_2^{\mathcal{I}_2}(\mathbf{z}_2)\,\mathcal{O}_3^{\mathcal{I}_3}(\mathbf{z}_3)\rangle = {}& \mathcal{K}_{\mathcal{O}_1\mathcal{O}_2}(\mathbf{z}_{1,2}, z_3; \partial_{\chi_{1,2}}, \partial_{\bar{\chi}_{1,2}})\times \\ & \mathcal{T}^R{}_{\mathcal{I}_1}{}^{\mathcal{J}_1}(\hat{u}(z_{13}))\,\mathcal{T}^R{}_{\mathcal{I}_2}{}^{\mathcal{J}_2}(\hat{u}(z_{23}))\times \\ & t_{\mathcal{O}_3}^{\mathcal{O}_1\mathcal{O}_2|\,\mathcal{J}_1\mathcal{J}_2\mathcal{I}_3}(Z_3; \chi_{1,2}, \bar{\chi}_{1,2}; \eta_3, \bar{\eta}_3)\,,\end{aligned} \tag{2.29}$$

with $\mathcal{K}$ defined in (2.21) and Z_3 denoting $X_3, \Theta_3^I, \overline{\Theta}_{3\,I}$ (see Appendix A.3.2). The t has the same scaling properties as the $\mathcal{N} = 1$ case (see (B.3)). In addition it has to transform as a tensor with the indices in the appropriate representations. The dependence on the $\mathfrak{su}(2)$ indices may come from Θ^I, $\overline{\Theta}_I$ or explicit ϵ_{IJ} and δ_J^I tensors.

[8] $(\sigma^A)^J{}_I$ are the usual three-dimensional Pauli matrices and $\epsilon_{\underline{12}} = \epsilon^{\underline{21}} = 1$ is the Levi-Civita tensor.

2.3 Counting superconformal correlators

In this section we will derive a formula analogous to (1.40) to count the number of superconformally invariant tensor structures. The argument is identical to the one in [1] but adapted to the case of three-point functions. The result is general and applies to all superconformal algebras. Ultimately we will be interested in $4d$, $\mathcal{N} = 1$, for which a formula has already been derived in **Paper I**. At the end of this chapter we will obtain it as a particular case.

2.3.1 Group theory of superconformal tensor structures

One immediate obstacle to the construction of Section 1.3 is that it is quite hard to construct functions on a superconformal group. The algebra is an easier object to deal with. The first thing we are going to do then is to pass at the level of the algebra so that we will not have to worry about the meaning of a supergroup.

The starting point was the space of functions from the supergroup $\mathcal{G}$ to a certain representation space V

$$\Gamma^V := C^\infty(\mathcal{G}, V)\,. \tag{2.30}$$

V is taken to be a representation of $G_0 = \mathrm{SO}(d) \times \mathrm{SO}(1,1) \times R$, where R is the R-symmetry group. We call $\mathfrak{g}$ the Lie superalgebra of $\mathcal{G}$. Namely

$$\begin{aligned} \mathfrak{g} &= \mathfrak{su}(2,2|1)\,, \qquad \text{for } 4d,\ \mathcal{N} = 1\,, \\ \mathfrak{g} &= \mathfrak{su}(2,2|2)\,, \qquad \text{for } 4d,\ \mathcal{N} = 2\,. \end{aligned} \tag{2.31}$$

Define $U(\mathfrak{g})$ to be the universal enveloping algebra of $\mathfrak{g}$, namely the "freest" algebra of all generators in $\mathfrak{g}$ subject to the commutation relations only. For any element $A \in U(\mathfrak{g})$ we can write down a differential operator $\mathcal{L}_A$ that acts on the functions $\varphi \in \Gamma^V$. By evaluating this derivative at, say, the origin, we obtain something that could be called a Taylor coefficient:

$$\varphi_A := \mathcal{L}_A \varphi(e) \in V\,. \tag{2.32}$$

By varying $A \in U(\mathfrak{g})$ we can trade the knowledge of φ for the knowledge of its Taylor coefficients. φ_A is effectively a map from $U(\mathfrak{g})$ to V. So we will get rid of Γ^V and focus on Θ^V composed by all linear maps from $U(\mathfrak{g})$ to V

$$\Theta^V := \mathrm{Hom}\,(U(\mathfrak{g}), V)\,. \tag{2.33}$$

We now write a decomposition of the algebra in the grading dictated by the dilatation operator. Namely $\mathfrak{g}_n$ is the eigenspace of D with eigenvalue n.

$$\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_{-\frac{1}{2}} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{\frac{1}{2}} \oplus \mathfrak{g}_1\,. \tag{2.34}$$

The space $\mathfrak{g}_{-1}$ contains the special conformal transformations, $\mathfrak{g}_{-\frac{1}{2}}$ the supercharges S, $\mathfrak{g}_0$ contains dilatations, rotations and the R-symmetry algebra $\mathfrak{r}$, $\mathfrak{g}_{\frac{1}{2}}$ has the supercharges Q and finally $\mathfrak{g}_1$ has the translations. With a notation similar to (1.36) we define $\mathfrak{g}_P$ as the algebra with non-negative grading and $\mathfrak{g}_K$ as the algebra with non-positive grading. Namely

$$\mathfrak{g}_P = \mathfrak{g}_0 \oplus \mathfrak{g}_{\frac{1}{2}} \oplus \mathfrak{g}_1\,, \qquad \mathfrak{g}_K = \mathfrak{g}_{-1} \oplus \mathfrak{g}_{-\frac{1}{2}} \oplus \mathfrak{g}_0\,. \tag{2.35}$$

As before, we want to refine the definition of Θ so that it represents a primary operator. It suffices to put covariance properties under the left action of $\mathfrak{g}_K$. Therefore we redefine Θ as

$$\Theta^V := \left\{ \varphi \colon U(\mathfrak{g}) \to V \;\middle|\; \varphi(xA) = (-1)^{|x||\varphi|}\pi(x)\varphi(A) \quad \forall\, x \in \mathfrak{g}_K,\, A \in U(\mathfrak{g}) \right\}\,, \tag{2.36}$$

where $\pi : \mathfrak{g}_0 \to V$ is the representation of V that is extended to act trivially on $\mathfrak{g}_K \ominus \mathfrak{g}_0$ and $|\cdot|$ is the fermion number grading. Our goal now is to compute the tensor product of two such vector spaces. The proof of the result that we are about to give follows the same lines used to show the form of Γ^{V_1,V_2} explained in Section 1.3. Furthermore, a complete proof is given in [1]. For these reasons we will not repeat it here.

The tensor products of two Θs reads

$$\begin{aligned}&\Theta^{V_1} \otimes \Theta^{V_2} = \Theta^{V_1,V_2} := \\ &\quad \left\{ \varphi \colon U(\mathfrak{g}) \to V_1 \otimes V_2^* \;\middle|\; \varphi(xA) = (-1)^{|x||\varphi|}\pi_1(x) \otimes \pi_2^*(x)\varphi(A) \quad \forall\, x \in \mathfrak{g}_0,\, A \in U(\mathfrak{g}) \right\}.\end{aligned} \tag{2.37}$$

Notice the difference: now we ask for covariance only under $\mathfrak{g}_0$. The representation V_2^* has the same meaning as before: it is the representation obtained by conjugating π_2 with the Weyl reflection w. It simply conjugates the spin, changes sign to $-\Delta$ and leaves the R-symmetry untouched. Note that the Weyl reflection w maps $\mathfrak{g}_K \leftrightarrow \mathfrak{g}_P$ because $w(\mathfrak{g}_n) = \mathfrak{g}_{-n}$.

Let us continue with the third operator. We need to multiply by Θ^{V_3} and then take the $\mathfrak{g}$-invariants of the result. This will yield the space of superconformal correlators. The product is given by functions $\varphi : U(\mathfrak{g}) \otimes U(\mathfrak{g}) \to V_1 \otimes V_2^* \otimes V_3$ which satisfy suitable covariance properties. These functions are determined by the values that they assume in the coset $U(\mathfrak{g}_0)\backslash U(\mathfrak{g}) \otimes U(\mathfrak{g})/U(\mathfrak{g}_K)$. Like before, taking the $\mathfrak{g}$ invariants turns this into $U(\mathfrak{g}_0)\backslash U(\mathfrak{g})/U(\mathfrak{g}_K)$. Therefore we obtain

$$\begin{aligned}&\left(\Theta^{V_1} \otimes \Theta^{V_2} \otimes \Theta^{V_3}\right)^G = \left(\Theta^{V_1,V_2,V_3}\right)^G := \\ &\left\{ \varphi \colon U(\mathfrak{g}) \to V_1 \otimes V_2^* \otimes V_3 \;\middle|\; \begin{array}{ll} \varphi(aA) = (-1)^{|a||\varphi|}\pi_1(a) \otimes \pi_2^*(a)\ \varphi(A) & \forall\, a \in \mathfrak{g}_0,\, A \in U(\mathfrak{g}) \\ \varphi(Ab) = (-1)^{|b||\varphi|}\pi_3(b)\ \varphi(A) & \forall\, b \in \mathfrak{g}_K,\, A \in U(\mathfrak{g}) \end{array} \right\}.\end{aligned} \tag{2.38}$$

Later we will denote this space again as $\mathrm{Hom}(U(\mathfrak{g}), V_1 \otimes V_2^* \otimes V_3)$, leaving the covariance properties implicit.

Now the choice of going to the Taylor coefficient space pays off. The universal enveloping algebra of any superalgebra can be decomposed as follows

$$U(\mathfrak{g}) = U(\mathfrak{g}_{\text{even}}) \otimes \Lambda\mathfrak{g}_{\text{odd}}\,, \tag{2.39}$$

where $\mathfrak{g}_{\text{even}}$ is the direct sum of $\mathfrak{g}_n$ for n even. Similarly for $\mathfrak{g}_{\text{odd}}$. Equivalently $\mathfrak{g}_{\text{even}}$ is the set of bosonic generators and $\mathfrak{g}_{\text{odd}}$ the set of fermionic ones. The space $\Lambda\mathfrak{g}_{\text{odd}}$ denotes the alternating algebra of $\mathfrak{g}_{\text{odd}}$, namely the set of all antisymmetrized products of elements in $\mathfrak{g}_{\text{odd}}$. We can use this to pass from functions on $U(\mathfrak{g})$ to functions on $U(\mathfrak{g}_{\text{even}})$

$$\text{Hom}(U(\mathfrak{g}_{\text{even}}) \otimes \Lambda\mathfrak{g}_{\text{odd}}, V) \cong \text{Hom}(U(\mathfrak{g}_{\text{even}}), \Lambda\mathfrak{g}^*_{\text{odd}} \otimes V)\,. \tag{2.40}$$

The dual space on the left hand side $\Lambda\mathfrak{g}^*_{\text{odd}}$ is actually isomorphic to $\Lambda\mathfrak{g}_{\text{odd}}$. This is nice because functions from $U(\mathfrak{g}_{\text{even}})$ to any complicated vector space can be interpreted as functions on the bosonic part of the group $G \times R$, where R represents the R-symmetry group and G the conformal group. Indeed we can do the argument that led us from Γ^V to Θ^V in the opposite direction. Therefore

$$\text{Hom}(U(\mathfrak{g}_{\text{even}}), \Lambda\mathfrak{g}_{\text{odd}} \otimes V) \cong C^\infty(G \times R, \Lambda\mathfrak{g}_{\text{odd}} \otimes V)\,. \tag{2.41}$$

We are now almost done. The space of tensor structures is given by the set of values that the functions φ can assume. To claim that it is $\Lambda\mathfrak{g}_{\text{odd}} \otimes V_1 \otimes V_2^* \otimes V_3$ would be wrong because the covariance properties on the left and on the right do not act freely. There is again a stability group $\mathcal{H}$. The bosonic part of this subgroup is $H = \text{SO}(d-1) \times R$. The reason for the first factor was explained in the non-supersymmetric case. The second factor follows immediately since R commutes with the whole conformal group, so in the decomposition $U(\mathfrak{g}_0)\backslash U(\mathfrak{g})/U(\mathfrak{g}_K)$ it can be taken from one side to the other. There is however an extra set of generators that we have to mod out, namely the conformal supercharges in $\mathfrak{g}_K$. This reduces the alternating algebra $\Lambda\mathfrak{g}_{\text{odd}}$ to the alternating algebra of the Q's only, namely $\Lambda\mathfrak{g}_{\frac{1}{2}}$.

The final result is that the space of superconformally invariant tensor structures in the correlator $\langle \mathcal{O}_1\mathcal{O}_2\mathcal{O}_3 \rangle$, which we call $\mathfrak{n}(\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3)$, is given by

$$\mathfrak{n}(\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3) = \dim\left(\text{Res}_H^{G_0}\, \Lambda\mathfrak{g}_{\frac{1}{2}} \otimes V_1 \otimes V_2^* \otimes V_3\right)^H\,, \tag{2.42}$$

where $H = \text{SO}(d-1) \times R$ and $G_0 = \text{SO}(d) \times \text{SO}(1,1) \times R$. Since the $\text{SO}(1,1)$ factor does not appear in H, the conformal dimension does not play any role and we get

$$\mathfrak{n}(\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3) = \dim\left(\text{Res}_H^{\text{SO}(d)\times R}\, \Lambda\mathfrak{g}_{\frac{1}{2}} \otimes (\rho_1, r_1) \otimes (\rho_2^*, r_2) \otimes (\rho_3, r_3)\right)^H\,, \tag{2.43}$$

where we use r to denote a representation of R.

2.3.2 Rederiving the formula for the case $4d$, $\mathcal{N} = 1$

We want to now specialize (2.43) for the case of four dimensional $\mathcal{N} = 1$ supersymmetry as it will be the main focus of this thesis. This particular case of the formula first appeared in **Paper I**. Here we will rederive it using the general formalism explained before.

By looking at the function t in (2.20) we can get a better intuition of the formula. Indeed t contains all terms that can be built out of contractions of $\eta_i, \bar{\eta}_i, X_3$ and $\Theta_3, \overline{\Theta}_3$. Now we can separate the terms according to the various orders in Θ_3 and $\overline{\Theta}_3$. At order zero we have the usual formula in conformal frame for the non-supersymmetric case. At order, e.g., Θ_3 instead we can replace all occurrences of the Grassmann variable with an extra polarization η_4. This can be interpreted as having an operator, say $\mathcal{O}_1$, transforming in the *reducible* representation $\rho_1 \otimes (1,0)$. That would mean that the first j α indices are contracted with η_1 and the fictitious $(j+1)$th is contracted by η_4. The structures in this case are counted by

$$n_\Theta(\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3) = \dim\left(\mathrm{Res}^{\mathrm{SO}(4)}_{\mathrm{SO}(3)}\,(1,0)\otimes\rho_1\otimes\rho_2^*\otimes\rho_3\right)^{\mathrm{SO}(3)}\,. \tag{2.44}$$

Similarly, for all other possible orders in $\Theta_3, \overline{\Theta}_3$ we can think of adding extra indices to one of the operators and writing down formulae such as the one above. Now the comparison to (2.43) is clear: the possible orders of Θ_3 and $\overline{\Theta}_3$ are precisely the elements of $\Lambda\mathfrak{g}_{\frac{1}{2}}$. The Grassmann variables anticommute and transform in the same representations as the Q's. So they make up the same alternating algebra.

Let us be more explicit. Consider $\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3$ to transform in the representations $V_i = (\rho_i, r_i)$, where ρ_i are representations of $\mathrm{SO}(4)$ and r_i are $\mathfrak{u}(1)$ charges. Define now the sum of the R-charges δ

$$\delta = r_1 + r_2 + r_3\,. \tag{2.45}$$

As mentioned in Subsection 2.2.1, δ can be $0, \pm 1$ or ± 2. The function $t^{\mathcal{O}_1\mathcal{O}_2}_{\mathcal{O}_3}$ contains a subset of the following monomials (we will drop the subscript 3 in Θ_3 for brevity)

$$\Theta^0\overline{\Theta}^0\,,\quad \Theta^\alpha\,,\quad \overline{\Theta}^2\Theta^\alpha\,,\quad \overline{\Theta}^{\dot\alpha}\,,\quad \Theta^\alpha\overline{\Theta}^{\dot\alpha}\,,\quad \Theta^2\overline{\Theta}^{\dot\alpha}\,,\quad \Theta^2\,,\quad \overline{\Theta}^2\,,\quad \Theta^2\overline{\Theta}^2\,. \tag{2.46}$$

Which ones are present depends on the R-charges of the operators. Let us denote as $n_{\mathcal{X}}(\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3)$ the number of structures of a given order $\mathcal{X}$ in $\Theta, \overline{\Theta}$, where $\mathcal{X}$ is any monomial in (2.46). Following the discussion above we have

$$\begin{aligned} n_1(\mathcal{O}_i) = n_{\Theta^2\overline{\Theta}^2}(\mathcal{O}_i) = n_{\Theta^2}(\mathcal{O}_i) = n_{\overline{\Theta}^2}(\mathcal{O}_i) &= \mathrm{Res}\,\rho_1\otimes\rho_2\otimes\rho_3\,,\\ n_\Theta(\mathcal{O}_i) = n_{\Theta\overline{\Theta}^2}(\mathcal{O}_i) &= \mathrm{Res}\,\rho_1\otimes\rho_2\otimes\rho_3\otimes(1,0)\,,\\ n_{\overline{\Theta}}(\mathcal{O}_i) = n_{\Theta^2\overline{\Theta}}(\mathcal{O}_i) &= \mathrm{Res}\,\rho_1\otimes\rho_2\otimes\rho_3\otimes(0,1)\,,\\ n_{\Theta\overline{\Theta}}(\mathcal{O}_i) &= \mathrm{Res}\,\rho_1\otimes\rho_2\otimes\rho_3\otimes(1,1)\,. \end{aligned} \tag{2.47}$$

where $\text{Res} \equiv \text{Res}^{\text{SO}(4)}_{\text{SO}(3)}$ and a superscript SO(3) in all terms is understood. Then the general formula for the number $\mathfrak{n}(\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3)$ of tensor structures in the three-point function $\langle\mathcal{O}_1\mathcal{O}_2\mathcal{O}_2\rangle$ may be written as

$$\mathfrak{n}(\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3) = \begin{cases} 2n_1(\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3) + n_{\Theta\bar{\Theta}}(\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3) & \delta = 0\,, \\ n_1(\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3) & \delta = \pm 2\,, \\ n_{\Theta}(\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3) + n_{\bar{\Theta}}(\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3) & \delta = \pm 1\,. \end{cases} \tag{2.48}$$

Now we show how this is an immediate consequence of (2.43). It suffices to notice that, due to the factor R in H, we need to take singlets in the $\mathfrak{u}(1)$ R-group as well. This means that only the terms with R-charge $-\delta$ in $\Lambda\mathfrak{g}_{\frac{1}{2}}$ will survive. For $4d$, $\mathcal{N} = 1$ the alternating algebra is given explicitly as

$$\begin{aligned} \Lambda\mathfrak{g}_{\frac{1}{2}} = {} & [0;0]^{(0)}_0 \oplus [0;1]^{(1)}_{1/2} \oplus [1;0]^{(-1)}_{1/2} \oplus [0;0]^{(-2)}_1 \oplus [0;0]^{(2)}_1 \\ & \oplus [1;1]^{(0)}_1 \oplus [0;1]^{(-1)}_{3/2} \oplus [1;0]^{(1)}_{3/2} \oplus [0;0]^{(0)}_2\,. \end{aligned} \tag{2.49}$$

where the notation $[j;\bar{\jmath}]^{(r)}_{\Delta}$ represents $\rho = (j,\bar{\jmath})$ and R-charge r. The conformal dimension Δ was included for completeness but it does not play any role here. The reader may check that the representations with R-charge equal to $-\delta$ match precisely those included in (2.48) for all cases $\delta = 0, \pm 1, \pm 2$. For example: the case $\delta = 0$ selects $[0;0]^{(0)}_1 \oplus [1;1]^{(0)}_1 \oplus [0;0]^{(0)}_2$. According to the definitions in (2.47) this selects two n_1's and one $n_{\Theta\bar{\Theta}}$. Similarly $\delta = 1$ selects $[1;0]^{(-1)}_{1/2} \oplus [0;1]^{(-1)}_{3/2}$, which is again in agreement with the formula as it yields $n_{\Theta} + n_{\bar{\Theta}}$.

It is also possible to use this formalism to account for kinematic constraints such as permutation of the operators and conservation. It is however quite difficult to develop a general formula that encompasses all possible cases. We will therefore postpone this discussion to the case of two Ferrara-Zumino multiplets and a general long multiplet $\mathcal{O}$, which will be the main focus of Chapter 7.

3 Differential operators

This chapter is based on **Paper III**.

3.1 Introduction

This chapter's goal is to introduce a convenient formalism to expand a superspace correlation function in its components. The final output of the formalism shown here will be a set of linear relations among the OPE coefficients and norms of the operators in the same multiplet[1]

$$\lambda^{(a)}_{(Q^\ell \bar{Q}^{\bar{\ell}} \mathcal{O}_1)\mathcal{O}_2\mathcal{O}_3} = M^a{}_b \, \lambda^{(b)}_{\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3}\,, \qquad \mathcal{C}_{(Q^\ell \bar{Q}^{\bar{\ell}} \mathcal{O}_1)} = N \, \mathcal{C}_{\mathcal{O}_1}\,,$$

for some, in general rectangular, matrix M and some complex number N. Once the conformal blocks of all operators in a multiplet are known, it suffices to take the appropriate linear combination following from the knowledge of M and N.

The main motivation behind seeking these kinds of relations is to be able to compute superconformal blocks. Their knowledge is indispensable if one wants to carry a conformal bootstrap program for studying SCFTs. A detailed account of the current state of the art in terms of superconformal blocks is given in the Introduction and thus will not be repeated here. The explanation of how to compute them given the results of this section is in Section 7.5. On the other hand, there are also other interesting applications of the formalism shown here which are explored in Chapter 4 and Chapter 5. Namely, we can impose on the whole multiplet the constraints stemming from locality.[2] This often has implications which are stronger than the ones obtained by studying the component fields separately.

[1] We call λ the coefficient appearing in the three-point function in some standard basis and $\mathcal{C}$ the normalization of the two-point function relative to some standard convention. See Section 3.4.1 for more details.

[2] A theory is local if it admits a conserved stress tensor.

The approach adopted here consists in defining a set of superconformally covariant differential operators that can be applied to any correlator in superspace. In $\mathcal{N}=1$, by setting the Grassmann variables to zero, one obtains a superconformal primary. Whereas in $\mathcal{N}=2$, by setting to zero only the Grassmann variables $\theta_{\underline{2}}, \bar{\theta}^{\underline{2}}$, one obtains an $\mathcal{N}=1$ superconformal primary. The advantage of these operators is that they have nice covariant properties when acting on the prefactor of three-point functions, thus allowing us to define their action directly on $t_{\mathcal{O}_3}^{\mathcal{O}_1\mathcal{O}_2}$ in (2.20).

In order to define such differential operators we need to carefully subtract the conformal descendants — or the superdescendants of the other supercharges in the $\mathcal{N}=2$ case — that are generated when acting with Q, $\bar{Q}$ on $\mathcal{O}$. This is done in full generality to all orders in $\mathcal{N}=1$. Whereas only some cases have been considered in $\mathcal{N}=2$. Specifically we considered all operators with vanishing $\mathfrak{su}(2)$ R-charge up to quadratic order in the $Q^{\underline{2}}$, $\bar{Q}_{\underline{2}}$ supercharges and all operators with $\mathfrak{su}(2)$ R-charge 1 and 1/2 up to linear order. By expanding the differentiated three-point function in a standard basis one can read out the linear relations among the OPE coefficients of $\mathcal{O}$ and $Q^{\ell}\bar{Q}^{\bar{\ell}}\mathcal{O}$. Similarly, by acting on two-point functions, one can obtain the relative norms, even though they are already known in general for $\mathcal{N}=1$ [156]. We also introduce a Mathematica package to work on four dimensional superspace. It will be briefly explained at the end of this chapter.

The differential operators that we will define in what follows will be denoted with the letter "D" in several different fonts. In order not to generate confusion we summarize here their meaning. See Subsection 2.2.1 for the definition of the Z variables.

Symbol	Supercharges	Acts on
$D_{Q^{\ell}\bar{Q}^{\ell}}$	$\mathcal{N}=1$	z_1 / z_2
$\mathcal{D}_{Q^{\ell}\bar{Q}^{\ell}}$ / $\mathcal{Q}_{Q^{\ell}\bar{Q}^{\ell}}$	$\mathcal{N}=1$	Z_3
$\mathbb{D}_{Q^{\ell}\bar{Q}^{\ell}}$	$\mathcal{N}=2$	z_1 / z_2
$\mathfrak{D}_{Q^{\ell}\bar{Q}^{\ell}}$ / $\mathfrak{Q}_{Q^{\ell}\bar{Q}^{\ell}}$	$\mathcal{N}=2$	Z_3

Table 3.1. Reminder for the notation of differential operators. The alternatives in the first column represent the differential operator obtained by acting on the first / second operator, respectively.

3.2 Constructing the differential operators

3.2.1 $\mathcal{N}=1$ case

Strategy

In this section we derive a set of superconformally covariant differential operators D that extract a given order in $\theta, \bar{\theta}$ from an $\mathcal{N}=1$ superconformal multiplet $\mathcal{O}(\mathbf{z})$. That is,

we want them to satisfy the following property

$$D_{Q^\ell \bar{Q}^{\bar{\ell}}} \mathcal{O}(\mathbf{z})\big|_0 = (Q^\ell \bar{Q}^{\bar{\ell}} O)(\mathbf{x})\,. \tag{3.1}$$

where $|_0$ means evaluating at $\theta = \bar{\theta} = 0$ after taking the derivative. The set of all operators in a long multiplet is illustrated in Figure 2.1. Clearly, by the definition of $\mathcal{O}(\mathbf{z})$ (2.12), the first orders will be simply a derivative with respect to θ or $\bar{\theta}$. However, the situation becomes more complicated when both ℓ and $\bar{\ell}$ are nonzero. In this case the term multiplying $\theta^\ell \bar{\theta}^{\bar{\ell}}$ is not a conformal primary, but a linear combination of $(Q^\ell \bar{Q}^{\bar{\ell}} O)$ and the descendants of the previous orders. Thus we need to be able to disentangle these contributions. Furthermore we will not be content with any form of the differential operator. We will need to express it as a combination of chiral and antichiral derivatives D_α, $\bar{D}_{\dot{\alpha}}$ (A.29). The reason will be evident in the next section: these derivative have nice covariant properties that allow us to pass them through the prefactor $\mathcal{K}_{\mathcal{O}_1\mathcal{O}_2}$ of a three-point function and then their action on $t_{\mathcal{O}_3}^{\mathcal{O}_1\mathcal{O}_2}$ can be fully recast as a derivative with respect to $X_3, \Theta_3, \overline{\Theta}_3$.

Firstly we need to compute the exact linear combinations of descendants that appear in the $\ell\bar{\ell} \neq 0$ terms. This has been done already in [156] by analyzing superconformal two-point functions. We summarize their results in Appendix B.2. Then we need an ansatz for the differential operator that we wish to compute. Schematically we have

$$D_Q \sim D_\alpha\,, \qquad D_{\bar{Q}} \sim \bar{D}_{\dot{\alpha}}\,. \tag{3.2}$$

Therefore an ansatz for $D_{Q^\ell \bar{Q}^{\bar{\ell}}}$ will be something of the form

$$D_{Q^\ell \bar{Q}^{\bar{\ell}}} \sim a_1\, (D_Q)^\ell (D_{\bar{Q}})^{\bar{\ell}} + \text{ permutations}\,, \tag{3.3}$$

and the coefficients $a_1, \ldots$ need to be fixed in terms of the c_i's in (B.13–B.15). This matching could be done by simply working out the algebra of the chiral derivatives, namely $\{D_\alpha, \bar{D}_{\dot{\alpha}}\} = 2\sigma^\mu_{\alpha\dot{\alpha}}\, \partial_\mu$. However, we opted for a more convenient method. The strategy is to define a functional that acts on the non-supersymmetric operators $O(\mathbf{x})$ and turns them into an explicit function $\varphi[O](\mathbf{x})$. It is then possible to implement the rules for derivatives and index contractions in a computer algebra system and impose the following equality.

$$\varphi\big[D_{Q^\ell \bar{Q}^{\bar{\ell}}} \mathcal{O}(\mathbf{z})\big|_0\big](\mathbf{x}) = \varphi\big[(Q^\ell \bar{Q}^{\bar{\ell}} O)\big](\mathbf{x})\,. \tag{3.4}$$

The functional φ can be chosen arbitrarily as long as it is generic enough to make (3.4) imply (3.1).[3] There are a few advantages to this method. First, it can be easily

[3] Another way to say this is the following: we want to prove some identities between differential operators. The identities should hold for any choice of functions to which the operators are applied. Therefore we need to find a set of functions which are generic enough to completely fix the ansatz, but also as easy as possible to manipulate.

implemented using the package introduced in Section 3.4. Second, it is possible to choose among many functionals thus obtaining an overconstrained system of equations like (3.4). The existence of a solution serves as a check for our results.

A possible choice is $\varphi[O](\mathbf{x}) = \langle XO(\mathbf{x})\rangle$. However if X is a local operator then φ becomes a quadratic functional (because necessarily $X = \bar{O}$) and it is hard to solve the constraints. If X is the product of two local operators we have a linear functional but there is no choice that gives a nonzero three-point function with all possible Os. Naturally, there is no reason why φ needs to be a physical correlator. This is then our choice:

$$\varphi[O](\mathbf{x}) = \frac{(\chi \mathrm{x} \bar{\eta})^l (\eta \mathrm{x} \bar{\chi})^k}{x^{\Delta_O + l + k}} (\eta\chi)^{j-k} (\bar{\eta}\bar{\chi})^{\bar{\jmath}-l} . \tag{3.5}$$

The parameters k, l can be varied between 0 and, respectively, j and $\bar{\jmath}$ to obtain a family of functionals. The only identity that needs to be considered when comparing derivatives of the above expression is the following

$$\chi \mathrm{x} \bar{\chi}\ \eta \mathrm{x} \bar{\eta} = \chi \mathrm{x} \bar{\eta}\ \eta \mathrm{x} \bar{\chi} + x^2\, \eta\chi\ \bar{\eta}\bar{\chi} . \tag{3.6}$$

This is particularly convenient because the main obstacle in solving (3.4) is finding all linearly independent tensors. In this case, thanks to (3.6), a basis of independent tensors can be simply taken to be

$$(\chi \mathrm{x} \bar{\chi})^n (\eta \mathrm{x} \bar{\eta})^m (\chi \mathrm{x} \bar{\eta})^a (\eta \mathrm{x} \bar{\chi})^b (\eta\chi)^c (\bar{\eta}\bar{\chi})^d , \qquad \text{with } mn = 0 . \tag{3.7}$$

Now the task of fixing the ansatz for the differential operators $D_{Q^\ell \bar{Q}^{\bar{\ell}}}$ is tedious but entirely straightforward.

First order

At first order in θ and $\bar{\theta}$ no descendants need to be subtracted. The differential operators are simply ∂_θ or $\partial_{\bar{\theta}}$, which can be then completed to chiral derivatives (A.29)

$$\begin{aligned} D^+_Q &= -\frac{i}{j+1}\, \eta^\alpha D_\alpha , & D^-_Q &= -\frac{i}{j}\, \frac{\partial}{\partial \eta_\alpha} D_\alpha , \\ D^+_{\bar{Q}} &= -\frac{i}{\bar{\jmath}+1}\, \bar{\eta}^{\dot{\alpha}} \bar{D}_{\dot{\alpha}} , & D^-_{\bar{Q}} &= -\frac{i}{\bar{\jmath}}\, \frac{\partial}{\partial \bar{\eta}_{\dot{\alpha}}} \bar{D}_{\dot{\alpha}} . \end{aligned} \tag{3.8}$$

As needed, these operators satisfy

$$D^\pm_Q\, \mathcal{O}(\mathbf{z})\big|_0 = (QO)^\pm(\mathbf{x}) , \qquad D^\pm_{\bar{Q}}\, \mathcal{O}(\mathbf{z})\big|_0 = (\bar{Q}O)^\pm(\mathbf{x}) . \tag{3.9}$$

They will be used as building blocks for the subsequent differential operators.

Orders Q^2 and $\bar{Q}^2$

For the quadratic order we may use the operators $D^\alpha D_\alpha$ and $\bar{D}_{\dot\alpha}\bar{D}^{\dot\alpha}$. However in the next section we want to prove that all $D_{Q^\ell\bar{Q}^{\bar\ell}}$ commute with the prefactor of the three-point function. This can be done easily only if all operators are expressed in terms of $D^\pm_Q$ and $D^\pm_{\bar{Q}}$. Since we act on homogeneous functions of η and $\bar\eta$ this amounts to only an overall factor.[4] The result is

$$D_{Q^2} = \frac{2j(j+1)}{j+2}\, D^-_Q\, D^+_Q\,, \qquad D_{\bar{Q}^2} = -\frac{2\bar{\jmath}(\bar{\jmath}+1)}{\bar{\jmath}+2}\, D^-_{\bar{Q}}\, D^+_{\bar{Q}}\,. \tag{3.10}$$

The factors $j(j+1)$ in the numerator simply cancel the denominator of (3.8). For scalar operators they can be omitted and D^-_Q is used without the j at the denominator.

Order $Q\bar{Q}$

This is the first order where we need to subtract the descendants. Schematically we have

$$\mathcal{O}\big|_{\theta\bar\theta} = \theta\bar\theta\left((Q\bar{Q}O) - ic\,\partial_\mu O\right)\,, \tag{3.11}$$

see (B.13) for the full expression. The needed ansatz is simple. Letting s and r represent either a plus or a minus sign we have

$$D^{sr}_{Q\bar{Q}} = a^{sr}\, D^s_Q D^r_{\bar{Q}} + b^{sr}\, D^r_{\bar{Q}} D^s_Q\,. \tag{3.12}$$

The coefficients a^{sr} and b^{sr} are a function of c_1 if $sr = ++$, c_2 if $sr = -+$, c_3 if $sr = +-$ and c_4 if $sr = --$. We will directly give the final expression by replacing the values of c_i computed in [156].

$$\begin{aligned}
a^{++} &= \frac{2q+j}{2(q+\bar{q})+j+\bar{\jmath}}\,, & b^{++} &= -\frac{2\bar{q}+\bar{\jmath}}{2(q+\bar{q})+j+\bar{\jmath}}\,,\\
a^{-+} &= \frac{2q-j-2}{2(q+\bar{q}-1)-j+\bar{\jmath}}\,, & b^{-+} &= -\frac{2\bar{q}+\bar{\jmath}}{2(q+\bar{q}-1)-j+\bar{\jmath}}\,,\\
a^{+-} &= \frac{2q+j}{2(q+\bar{q}-1)+j-\bar{\jmath}}\,, & b^{+-} &= -\frac{2\bar{q}-\bar{\jmath}-2}{2(q+\bar{q}-1)+j-\bar{\jmath}}\,,\\
a^{--} &= \frac{2q-j-2}{2(q+\bar{q}-2)-j-\bar{\jmath}}\,, & b^{--} &= -\frac{2\bar{q}-\bar{\jmath}-2}{2(q+\bar{q}-2)-j-\bar{\jmath}}\,.
\end{aligned} \tag{3.13}$$

[4] Indeed for an homogeneous function $f_\ell = \eta_{\alpha_1}\cdots\eta_{\alpha_\ell} f^{\alpha_1\ldots\alpha_\ell}$ one has

$$\begin{aligned}
\partial_\eta D\,\eta D\, f_\ell &= \tfrac{1}{2}\, D^2 \partial_{\eta_\beta}\,\eta_\beta\eta_{\alpha_1}\cdots\eta_{\alpha_\ell} f^{\alpha_1\ldots\alpha_\ell}\\
&= \tfrac{1}{2}(\ell+2)\, D^2 f_\ell\,.
\end{aligned}$$

We should remark that these expressions are valid for a generic long multiplet. When the multiplet is short (e.g. $2q = j + 2$) the differential operators associated to null superdescendants should be discarded.

As a quick example we can take the $Q\bar{Q}$ descendant of the Ferrara-Zumino multiplet $J(\mathbf{z})$, namely the energy-momentum tensor. The result is simply one half the commutator

$$T(\mathbf{x}) = \frac{1}{2}\left(D^+_Q D^+_{\bar{Q}} - D^+_{\bar{Q}} D^+_Q\right) J(\mathbf{z})\big|_0\,, \tag{3.14}$$

as can be easily seen by letting $j = \bar{\jmath} = 1$ and $q = \bar{q} = 3/2$.

Orders $Q^2\bar{Q}$ and $\bar{Q}^2 Q$

The contribution at this order is shown in equation (B.14). Letting $s = \pm$, the ansatz is

$$\begin{aligned} D^s_{\bar{Q}^2 Q} &= c^s\, D^s_Q\, D^-_{\bar{Q}}\, D^+_{\bar{Q}} + d^s\, D^-_{\bar{Q}}\, D^s_Q\, D^+_{\bar{Q}} + e^s\; D^-_{\bar{Q}}\, D^+_{\bar{Q}}\, D^s_Q\,, \\ D^s_{Q^2\bar{Q}} &= \bar{c}^s\, D^s_{\bar{Q}}\, D^-_Q\, D^+_Q + \bar{d}^s\, D^-_Q\, D^s_{\bar{Q}}\, D^+_Q + \bar{e}^s\; D^-_Q\, D^+_Q\, D^s_{\bar{Q}}\,. \end{aligned} \tag{3.15}$$

The various coefficients are a function of $c_{5,6,7,8}$ and $\bar{c}_{5,6,7,8}$. We have the following simple relation

$$c^s,\, d^s,\, e^s = -\bar{c}^s,\, -\bar{d}^s,\, -\bar{e}^s\,\big|_{j\leftrightarrow\bar{\jmath},\, q\leftrightarrow\bar{q}}\,, \tag{3.16}$$

It will then suffice to quote the result for the coefficients of $D^\pm_{\bar{Q}^2 Q}$ only

$$\begin{aligned} c^+ &= \frac{2\bar{\jmath}(\bar{\jmath}+1)(2q+j)(j+\bar{\jmath}+2q+2\bar{q}+2)}{(\bar{\jmath}+2)(\bar{\jmath}-j-2q-2\bar{q}+2)(j+\bar{\jmath}+2q+2\bar{q})}\,, \\ c^- &= \frac{2\bar{\jmath}(\bar{\jmath}+1)(2q-j-2)(\bar{\jmath}-j+2q+2\bar{q})}{(\bar{\jmath}+2)(j+\bar{\jmath}-2q-2\bar{q}+4)(\bar{\jmath}-j+2q+2\bar{q}-2)}\,, \\ d^+ &= \frac{4\bar{\jmath}(\bar{\jmath}+1)(2q+j)}{(\bar{\jmath}-j-2q-2\bar{q}+2)(j+\bar{\jmath}+2q+2\bar{q})}\,, \\ d^- &= \frac{4\bar{\jmath}(\bar{\jmath}+1)(2q-j-2)}{(j+\bar{\jmath}-2q-2\bar{q}+4)(\bar{\jmath}-j+2q+2\bar{q}-2)} \\ e^+ &= -\frac{2\bar{\jmath}(\bar{\jmath}+1)(2\bar{q}+\bar{\jmath})}{(\bar{\jmath}+2)(j+\bar{\jmath}+2q+2\bar{q})}\,, \\ e^- &= -\frac{2\bar{\jmath}(\bar{\jmath}+1)(2\bar{q}+\bar{\jmath})}{(\bar{\jmath}+2)(\bar{\jmath}-j+2q+2\bar{q}-2)}\,. \end{aligned} \tag{3.17}$$

Order $Q^2\bar{Q}^2$

At last we have the highest order in θ and $\bar{\theta}$. The subtractions needed are six: c_9 through c_{14}. This means that our ansatz will need seven terms obtained by permuting

$D^+_Q, D^-_Q, D^+_{\bar{Q}}$ and $D^-_{\bar{Q}}$. In total there are fourteen permutations after taking into account $\{D, D\} = \{\bar{D}, \bar{D}\} = 0$. Not all of these are independent and the choice of seven out of these is not unique. We made this ansatz

$$\begin{aligned} D_{Q^2\bar{Q}^2} = \quad & f_1\, D^-_{\bar{Q}}\, D^+_{\bar{Q}}\, D^-_Q\, D^+_Q + f_2\, D^-_{\bar{Q}}\, D^-_Q\, D^+_{\bar{Q}}\, D^+_Q + f_3\, D^-_{\bar{Q}}\, D^-_Q\, D^+_Q\, D^+_{\bar{Q}} \\ & + f_4\, D^-_Q\, D^-_{\bar{Q}}\, D^+_{\bar{Q}}\, D^+_Q + f_5\, D^-_Q\, D^-_{\bar{Q}}\, D^+_Q\, D^+_{\bar{Q}} \\ & + f_6\, D^-_Q\, D^+_Q\, D^-_{\bar{Q}}\, D^+_{\bar{Q}} + f_7\, D^+_Q\, D^-_{\bar{Q}}\, D^+_{\bar{Q}}\, D^-_Q\,. \end{aligned} \tag{3.18}$$

These are the values of the coefficients f_i

$$\begin{aligned} f_1 &= \frac{4j(j+1)\bar{\jmath}(\bar{\jmath}+1)(2\bar{q}-\bar{\jmath}-2)(\bar{\jmath}+2\bar{q})(j+\bar{\jmath}+2q+2\bar{q}+2)}{(j+2)(\bar{\jmath}+2)(j-\bar{\jmath}-2q-2\bar{q}+2)(j-\bar{\jmath}+2q+2\bar{q}-2)(j+\bar{\jmath}+2q+2\bar{q})}\,, \\ f_2 &= \frac{16j(j+1)\bar{\jmath}(\bar{\jmath}+1)(2\bar{q}-\bar{\jmath}-2)(2\bar{q}+\bar{\jmath})}{(j-\bar{\jmath}-2q-2\bar{q}+2)(j+\bar{\jmath}-2q-2\bar{q}+4)(j-\bar{\jmath}+2q+2\bar{q}-2)(j+\bar{\jmath}+2q+2\bar{q})}\,, \\ f_3 &= -\frac{8j(j+1)\bar{\jmath}(\bar{\jmath}+1)(2q+j)}{(j+2)} \times \\ &\quad \times \frac{\left(j^2+4j\bar{q}+2j+\bar{\jmath}^2+2\bar{\jmath}-4q^2-8q\bar{q}+4q-4\bar{q}^2+12\bar{q}\right)}{(j-\bar{\jmath}-2q-2\bar{q}+2)(j+\bar{\jmath}-2q-2\bar{q}+4)(j-\bar{\jmath}+2q+2\bar{q}-2)(j+\bar{\jmath}+2q+2\bar{q})}\,, \\ f_4 &= \frac{4j\bar{\jmath}(\bar{\jmath}+1)(2q-j-2)(2\bar{q}+\bar{\jmath})(j-\bar{\jmath}-2q-2\bar{q})}{(\bar{\jmath}+2)(j-\bar{\jmath}-2q-2\bar{q}+2)(j+\bar{\jmath}-2q-2\bar{q}+4)(j+\bar{\jmath}+2q+2\bar{q})}\,, \\ f_5 &= \frac{16j(j+1)\bar{\jmath}(\bar{\jmath}+1)(2q-j-2)(2q+j)}{(j-\bar{\jmath}-2q-2\bar{q}+2)(j+\bar{\jmath}-2q-2\bar{q}+4)(j-\bar{\jmath}+2q+2\bar{q}-2)(j+\bar{\jmath}+2q+2\bar{q})}\,, \\ f_6 &= \frac{4j(j+1)\bar{\jmath}(\bar{\jmath}+1)(2q-j-2)(2q+j)(j+\bar{\jmath}+2q+2\bar{q}+2)}{(j+2)(\bar{\jmath}+2)(j-\bar{\jmath}-2q-2\bar{q}+2)(j-\bar{\jmath}+2q+2\bar{q}-2)(j+\bar{\jmath}+2q+2\bar{q})}\,, \\ f_7 &= \frac{4j\bar{\jmath}(\bar{\jmath}+1)(2q+j)(2\bar{q}+\bar{\jmath})(j+\bar{\jmath}+2q+2\bar{q}+2)}{(\bar{\jmath}+2)(\bar{\jmath}-j+2q+2\bar{q}-2)(j-\bar{\jmath}+2q+2\bar{q}-2)(j+\bar{\jmath}+2q+2\bar{q})}\,. \end{aligned} \tag{3.19}$$

All these expressions are available within the Mathematica package that we introduce in Section 3.4.

3.2.2 $\mathcal{N} = 2$ case

General remarks

Now we want to define differential operators that extract full $\mathcal{N} = 1$ superconformal multiplets inside an $\mathcal{N} = 2$ multiplet. The embedding of the $\mathcal{N} = 1$ subalgebra was explained in Section 2.1.2. There we also commented on the necessity of taking suitable linear combinations of superdescendants in order to define the superconformal primary $(Q^{2\underline{\ell}}\bar{Q}_{\underline{2}}{}^{\bar{\ell}}\mathcal{O})$. We only considered operators with $R = \{0, 1, 2\}$ and give results up to the first nontrivial order in the Grassmann variables with $I = 2$, leaving the general analysis

for future work.

Zero R-charge

If the superconformal primary is an $\mathfrak{su}(2)$ singlet, no computation is needed at the linear order. Indeed from (2.9) and from the fact that $R_{\pm}$ on a singlet yields zero, one can see that these operators are automatically $\mathcal{N}=1$ superconformal primaries

$$ {}_{\underline{2}}D^{\pm}_{\underline{Q}}\,\mathcal{O}(\mathbf{z})\big|_0 = (Q^{\underline{2}}O)^{\pm}(\mathbf{z})\,, \qquad {}_{\underline{2}}D^{\pm}_{\underline{\bar{Q}}}\,\mathcal{O}(\mathbf{z})\big|_0 = (\bar{Q}_{\underline{2}}O)^{\pm}(\mathbf{z})\,, \tag{3.20} $$

where now we defined $|_0$ as $|_{\theta_{\underline{2}}=\theta^{\underline{2}}=0}$ and the prefix on the differential operator as an $\mathfrak{su}(2)$ index

$$ {}_{I}D_{Q^\ell\bar{Q}^{\bar{\ell}}} \coloneqq D_{Q^\ell\bar{Q}^{\bar{\ell}}}\big|_{D\to D^I,\,\bar{D}\to\bar{D}_I}\,. \tag{3.21} $$

Similarly, the order, Q^2 and $\bar{Q}^2$ require no subtractions as well and can be defined by attaching an index $I=2$ to (3.10).

More interesting is the order $Q\bar{Q}$. We expect a single superconformal primary at this level. The correct differential operator is a linear combination of the operators in (3.12) for $I=1$ and 2. We will not prove this result here but postpone the discussion to Section 3.3.2. Let us denote with a boldface $\mathbb{D}$ the $\mathcal{N}=2$ differential operators. Letting s and t represent a sign $\pm$, the result is

$$ \mathbb{D}^{st}_{Q\bar{Q}} = {}_{\underline{2}}D^{st}_{Q\bar{Q}} + A^{st}\,{}_{\underline{1}}D^{st}_{Q\bar{Q}}\,, \tag{3.22} $$

with

$$ \begin{aligned} A^{++} &= -\frac{2}{2(q+\bar{q}+1)+j+\bar{\jmath}}\,, & A^{-+} &= -\frac{2}{2(q+\bar{q})-j+\bar{\jmath}}\,,\\ A^{--} &= -\frac{2}{2(q+\bar{q}-1)-j-\bar{\jmath}}\,, & A^{+-} &= -\frac{2}{2(q+\bar{q})+j-\bar{\jmath}}\,. \end{aligned} \tag{3.23} $$

As a quick example we can reproduce the result of [117] for the stress tensor multiplet. Let us denote with $\mathcal{J}(z)$ the $\mathcal{N}=2$ superconformal primary and with $J(\mathbf{z})$ the Ferrara-Zumino multiplet. Recalling that for $\mathcal{J}$ we have $q=\bar{q}=1$ and $j=\bar{\jmath}=0$, the result is

$$ J(\mathbf{z}) = \mathbb{D}^{++}_{Q\bar{Q}}\,\mathcal{J}\big|_0 = \left({}_{\underline{2}}D^{++}_{Q\bar{Q}}\,\mathcal{J} - \frac{1}{3}\,{}_{\underline{1}}D^{++}_{Q\bar{Q}}\,\mathcal{J}\right)\big|_0\,, \tag{3.24} $$

with, from (3.12) and (3.8),

$$ {}_{I}D^{++}_{Q\bar{Q}}\,\mathcal{J} = -\frac{1}{2}\,[D^I_\alpha,\,\bar{D}_{\dot{\alpha}I}]\,\mathcal{J}\,. \tag{3.25} $$

Apart from an overall minus sign, which simply reflects in a different normalization, we obtain the same linear combination.

R-charge 1/2

When the $\mathfrak{su}(2)$ R-charge is non-zero, the differential operators are nontrivial starting from the first order. The simplest case is that of the doublet, for which we expect two $\mathcal{N}=1$ superconformal primaries. The operator $\mathcal{O}$ on which they act will have an I index. Letting $s=\pm$, we can write general ansatze as

$$ {}_J\mathbb{D}^s_Q\,\mathcal{O}_J = {}_ID^s_Q\,\mathcal{M}^{IJ}_s\,\mathcal{O}_J\,, \qquad {}_J\mathbb{D}^s_{\bar{Q}}\,\mathcal{O}_J = {}_ID^s_{\bar{Q}}\,\overline{\mathcal{M}}^{IJ}_s\,\mathcal{O}_J\,. \tag{3.26} $$

We denoted with $\mathbb{D}$ the $\mathcal{N}=2$ differential operator. In order to have a superconformally covariant operator we need to contract the index of $\mathcal{O}$ with an appropriate matrix. Since we expect two multiplets to arise at each order, the solutions for $\mathcal{M}_s$ and $\overline{\mathcal{M}}_s$ must have two degrees of freedom each. We denote the two classes of solutions as $\mathcal{A}$ and $\mathcal{B}$. The matrix $\mathcal{M}$ may be an arbitrary linear combination of those solutions, but the basis that we chose is the one that projects into $\mathcal{N}=1$ multiplets of definite R-charge. In order to avoid confusion we indicate here what term is represented by each entry

$$ \mathcal{M}_s = \begin{pmatrix} {}_{\underline{1}}D^s_Q\,\mathcal{O}_{\underline{1}} & {}_{\underline{1}}D^s_Q\,\mathcal{O}_{\underline{2}} \\ {}_{\underline{2}}D^s_Q\,\mathcal{O}_{\underline{1}} & {}_{\underline{2}}D^s_Q\,\mathcal{O}_{\underline{2}} \end{pmatrix}, \qquad \text{similarly for } \overline{\mathcal{M}}_s\,. \tag{3.27} $$

The $\mathcal{A}$ solution for $\mathcal{M}$ will be $\mathcal{M}_s=\mathcal{A}_s$, $\overline{\mathcal{M}}_s=\bar{\mathcal{A}}_s$, with

$$ \mathcal{A}_s = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad \bar{\mathcal{A}}_s = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \tag{3.28} $$

The solutions does not depend on s. Whereas the $\mathcal{B}$ solution for $\mathcal{M}$ will be $\mathcal{M}_s=\mathcal{B}_s$, $\overline{\mathcal{M}}_s=\bar{\mathcal{B}}_s$, with

$$ \begin{aligned} \mathcal{B}_+ &= \begin{pmatrix} -\frac{2}{2q+j+1} & 0 \\ 0 & 1 \end{pmatrix}, & \mathcal{B}_- &= \begin{pmatrix} -\frac{2}{2q-j-1} & 0 \\ 0 & 1 \end{pmatrix}, \\ \bar{\mathcal{B}}_+ &= \begin{pmatrix} 0 & \frac{2}{2\bar{q}+\bar{\jmath}+1} \\ 1 & 0 \end{pmatrix}, & \bar{\mathcal{B}}_- &= \begin{pmatrix} 0 & \frac{2}{2\bar{q}-\bar{\jmath}-1} \\ 1 & 0 \end{pmatrix}. \end{aligned} \tag{3.29} $$

In this case a nontrivial linear combination is needed to obtain an $\mathcal{N}=1$ superconformal primary and the solution does depend on the sign s. The resulting operators will have charges and spin dictated by Figure 2.2. As before, we defer the proof of these results to Section 3.3.2.

R-charge 1

The case of $\mathfrak{su}(2)$ R-charge 1 is not conceptually different from the last section. Now the operator will have an adjoint index A and the matrices $\mathcal{M}$ in (3.26) will be rectangular.

$$ {}_J\mathbb{D}^s_Q\,\mathcal{O}_J = {}_I D^s_Q\,\mathcal{M}^{IA}_s\,\mathcal{O}_A\,, \qquad {}_J\mathbb{D}^s_{\bar{Q}}\,\mathcal{O}_J = {}_I D^s_{\bar{Q}}\,\overline{\mathcal{M}}^{IA}_s\,\mathcal{O}_A\,. \tag{3.30} $$

We expect three degrees of freedom for the choice of $\mathcal{M}_s$ and $\overline{\mathcal{M}}_s$. Thus we can span the basis by three classes of solutions $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$. For the reader's convenience we will show all of them at once by taking an arbitrary linear combination of them[5]

$$ \begin{aligned} a\,\mathcal{A}_+ + b\,\mathcal{B}_+ + c\,\mathcal{C}_+ &= \begin{pmatrix} \frac{2c}{2q+j+2} & \frac{2ic}{2q+j+2} & -\frac{2ib}{2q+j} \\ \frac{1}{2}(a+ib) & \frac{i}{2}(a-ib) & c \end{pmatrix}, \\ a\,\mathcal{A}_- + b\,\mathcal{B}_- + c\,\mathcal{C}_- &= \begin{pmatrix} \frac{2c}{2q-j} & \frac{2ic}{2q-j} & -\frac{2ib}{2q-j-2} \\ \frac{1}{2}(a+ib) & \frac{i}{2}(a-ib) & c \end{pmatrix}, \\ a\,\overline{\mathcal{A}}_+ + b\,\bar{\mathcal{B}}_+ + c\,\bar{\mathcal{C}}_+ &= \begin{pmatrix} \frac{2c}{2\bar{q}+\bar{j}+2} & -\frac{2ic}{2\bar{q}+\bar{j}+2} & \frac{2ib}{2\bar{q}+\bar{j}} \\ \frac{1}{2}(a-ib) & -\frac{i}{2}(a+ib) & c \end{pmatrix}, \\ a\,\overline{\mathcal{A}}_- + b\,\bar{\mathcal{B}}_- + c\,\bar{\mathcal{C}}_- &= \begin{pmatrix} \frac{2c}{2\bar{q}-\bar{j}} & -\frac{2ic}{2\bar{q}-\bar{j}} & \frac{2b}{2\bar{q}-\bar{j}-2} \\ \frac{1}{2}(a-ib) & -\frac{i}{2}(a+ib) & c \end{pmatrix}. \end{aligned} \tag{3.31} $$

As before, the various $\mathcal{N}=1$ superconformal primaries are obtained by (3.30) by replacing $\mathcal{M}$ with either $\mathcal{A}$, $\mathcal{B}$ or $\mathcal{C}$. The proof these results is postponed to Section 3.3.2.

3.3 Acting on three-point functions

3.3.1 $\mathcal{N}=1$ case

Idea

The main goal is to fix a basis of non-supersymmetric three-point functions for a given triplet of representations $\mathrm{t}^a_{O_1O_2O_3}$, where $a=1,\dots n_{123}$, and to expand the three-point function of a superdescendant in that basis. Namely we want to find the coefficients $\lambda^{(a)}$ such that

$$ D_{Q^\ell\bar{Q}^{\bar{\ell}}}\,\langle\textstyle\prod_{i=1}^3\mathcal{O}_i(\mathbf{z}_i)\rangle\big|_0 = \sum_{a=1}^{n_{(1^{\ell\bar{\ell}})23}}\lambda^{(a)}\,\mathrm{t}^a_{(Q^\ell\bar{Q}^{\bar{\ell}}O_1)O_2O_3}(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3)\,, \tag{3.32} $$

[5] As before, this choice of basis is not arbitrary but it is the one that projects on $\mathcal{N}=1$ multiplets with definite R-charges.

provided that the full superconformal three-point function is known.[6] A three-point function can be decomposed as (2.20). When we act on it with, say, D^+_Q we get two terms. But since $\mathcal{K}_{\mathcal{O}_1\mathcal{O}_2}$ is bosonic, when the Grassmann variables are set to zero only one survives

$$D^+_Q\langle\prod_{i=1}^3 \mathcal{O}_i(\mathbf{z}_i)\rangle\big|_0 = \mathcal{K}_{\mathcal{O}_1\mathcal{O}_2}\, D^+_Q\, t^{\mathcal{O}_1\mathcal{O}_2}_{\mathcal{O}_3}(Z_3)\big|_0\,. \tag{3.33}$$

This is certainly convenient as we do not have to worry about the prefactor when taking derivatives, but it is not yet what we need. If we want to compare the above expression with a chosen basis of three-point functions we still need to expand the definitions of Z_3 and to act with the spinor derivatives inside $\mathcal{K}_{\mathcal{O}_1\mathcal{O}_2}$. It would be much better if we could express (3.33) as

$$D^+_Q\langle\prod_{i=1}^3 \mathcal{O}_i(\mathbf{z}_i)\rangle\big|_0 = \mathcal{K}_{(Q\mathcal{O}_1)^+\mathcal{O}_2}\, \mathcal{D}^+_{\bar{Q}}\, t^{\mathcal{O}_1\mathcal{O}_2}_{\mathcal{O}_3}(Z_3)\big|_0\,, \tag{3.34}$$

following [117]. Here $\mathcal{K}_{(Q\mathcal{O}_1)^+\mathcal{O}_2}$ is the prefactor of an hypothetical three-point function of $(Q\mathcal{O}_1)^+$, $\mathcal{O}_2$ and $\mathcal{O}_3$, if $(Q\mathcal{O}_1)^+$ were a superconformal primary. It is simply a $\mathcal{K}_{\mathcal{O}_1\mathcal{O}_2}$ with shifted arguments

$$\mathcal{K}_{(Q\mathcal{O}_1)^\pm\mathcal{O}_2} = \mathcal{K}_{\mathcal{O}_1\mathcal{O}_2}\Big|^{q_1\to q_1-1/2\,,}_{\bar{q}_1\to\bar{q}_1+1}_{j_1\to j_1\pm1} \qquad \mathcal{K}_{(\bar{Q}\mathcal{O}_1)^\pm\mathcal{O}_2} = \mathcal{K}_{\mathcal{O}_1\mathcal{O}_2}\Big|^{\bar{q}_1\to\bar{q}_1-1/2\,.}_{q_1\to q_1\pm1}_{\bar{j}_1\to\bar{j}_1+1} \tag{3.35}$$

And $\mathcal{D}^+_{\bar{Q}}$ will be defined later together with all the details, but the important point is that it is a differential operator with respect to the variables X_3, Θ_3 and $\overline{\Theta}_3$. Now the problem is drastically simplified. We can choose a basis of non-nilpotent tensor structures in $t^{\mathcal{O}_1\mathcal{O}_2}_{\mathcal{O}_3}$,

$$t^{\mathcal{O}_1\mathcal{O}_2}_{\mathcal{O}_3;\,a}(X_3)\,, \qquad a = 1,\dots,n_{123}\,, \tag{3.36}$$

which in turn will induce a basis of non-supersymmetric three-point functions. Then the comparison can be done at the level of the t. Assuming for now that this reasoning works for all superdescendants one has[7]

$$\mathcal{D}_{Q^{\bar{\ell}}\bar{Q}^{\ell}}\, t^{\mathcal{O}_1\mathcal{O}_2}_{\mathcal{O}_3}(Z_3)\big|_0 = \sum_{a=1}^{n_{(1^{\ell\bar{\ell}})23}} \lambda^{(a)}\, t^{(Q^{\ell}\bar{Q}^{\bar{\ell}}\mathcal{O}_1)\mathcal{O}_2}_{\mathcal{O}_3;\,a}(X_3)\,, \tag{3.37}$$

in place of (3.32). It is evident that (3.37) is easier to solve for $\lambda^{(a)}$. But we went too fast in all the steps involved. First we need to show that $D_{Q^{\ell}\bar{Q}^{\bar{\ell}}}$ actually commutes with the prefactor for any $\ell, \bar{\ell}$. Then we also need to prove that (3.34) is always possible and define the $\mathcal{D}_{Q^{\bar{\ell}}\bar{Q}^{\ell}}$ operators that arise from it.

The proof of (3.34) is not hard. We need to make use of the formulas (A.34), which are valid for any function of $Z_3 = X_3, \Theta_3, \overline{\Theta}_3$. The extra factors of $\mathrm{x}_{i\bar{j}}$ that appear can be

[6] We implicitly defined $n_{(1^{\ell\bar{\ell}})23}$ as the number of tensor structures in $\langle(Q^{\ell}\bar{Q}^{\bar{\ell}}O_1)O_2O_3\rangle$.

[7] Notice the swap $\ell \leftrightarrow \bar{\ell}$.

reabsorbed in the prefactor and they give automatically the right shifts in the quantum numbers. Then one can define the derivatives

$$\begin{aligned} \mathcal{D}^+_Q &= \frac{1}{\bar{\jmath}+1}\,\chi^\alpha \mathcal{D}_\alpha\,, \qquad & \mathcal{D}^-_Q &= -\frac{1}{\bar{\jmath}}\,\frac{\partial}{\partial\chi_\alpha}\mathcal{D}_\alpha\,, \\ \mathcal{D}^+_{\bar{Q}} &= \frac{1}{j+1}\,\bar{\chi}^{\dot\alpha} \bar{\mathcal{D}}_{\dot\alpha}\,, \qquad & \mathcal{D}^-_{\bar{Q}} &= -\frac{1}{j}\,\frac{\partial}{\partial\bar{\chi}_{\dot\alpha}}\bar{\mathcal{D}}_{\dot\alpha}\,, \end{aligned} \tag{3.38}$$

in complete analogy with (3.8).[8] The detailed expressions are given in Appendix B.3. Clearly one can also define, by repeated application, the following operators

$$\mathcal{D}_{Q^{\bar\ell}\bar{Q}^\ell} = D_{Q^\ell\bar{Q}^{\bar\ell}}\big|_{D\to\bar{\mathcal{D}},\,\bar{D}\to\mathcal{D}}\,. \tag{3.39}$$

Checking for commutativity with $\mathcal{K}_{\mathcal{O}_1\mathcal{O}_2}$ requires us to show

$$\left(D_{Q^\ell\bar{Q}^{\bar\ell}}\,\mathcal{K}_{\mathcal{O}_1\mathcal{O}_2}\,t^{\mathcal{O}_1\mathcal{O}_2}_{\mathcal{O}_3} - \mathcal{K}_{(Q^\ell\bar{Q}^{\bar\ell})\mathcal{O}_1\mathcal{O}_2}\,\mathcal{D}_{Q^{\bar\ell}\bar{Q}^\ell}\,t^{\mathcal{O}_1\mathcal{O}_2}_{\mathcal{O}_3}\right)\big|_0 = 0\,. \tag{3.40}$$

This is trivially true if $\ell\bar\ell$ vanishes: If ℓ or $\bar\ell$ is 1 then we can do the same argument as the example of before: the derivative acting on $\mathcal{K}_{\mathcal{O}_1\mathcal{O}_2}$ is necessarily fermionic and thus vanishing if the Grassman variables are set to zero. When, on the other hand, ℓ or $\bar\ell$ is 2, there are two pieces. One is fermionic an thus vanishing and the other must be proportional to θ^2 or $\bar\theta^2$ due to its R-charge scaling. If $\ell\bar\ell \neq 0$ the result is non-trivial and will be proven explicitly in the next paragraphs. We can however argue that the commutativity property must hold without any computation. Indeed it is easy to convince oneself that the terms in (3.40) that survive after setting the θ's to zero cannot recombine to form an expression with the right prefactor and a function of X_3. Therefore, if they did not vanish, the result of $D_{Q^\ell\bar{Q}^{\bar\ell}}$ applied on a correlator would not be a three-point function of conformal primaries but of a combination of primaries and descendants. This is a contradiction by construction of the operator $D_{Q^\ell\bar{Q}^{\bar\ell}}$. We will nevertheless carry an explicit computation in order to have a non-trivial check of our results. The rest of this section will be devoted to show that (3.40) holds and thus complete the proof of (3.37).

Order $Q\bar{Q}$

We want to show (3.40) for $\ell = \bar\ell = 1$. There are two kinds of terms: those where one derivative acts on $\mathcal{K}$ and one on t and those where both act on $\mathcal{K}$. After setting the Grassmann variables to zero only the latter may survive, so we need to focus on them. Concretely we need to show

$$(a^{sr}\,D^s_Q D^r_{\bar{Q}} + b^{sr}\,D^r_{\bar{Q}} D^s_Q)\;\mathcal{K}_{\mathcal{O}_1\mathcal{O}_2} = 0\,. \tag{3.41}$$

[8] Notice the swap $j \leftrightarrow \bar\jmath$.

We now use (B.19) for the first derivative and (B.20) for the second one. The expressions obtained for different values of $s, r = \pm$ will be proportional to the following factors:

$$\begin{aligned} s = r = 1 \ : \quad & a^{++}\,(2\bar{q} + \bar{\jmath}) + b^{++}\,(2q + j)\,, \\ -s = r = 1 \ : \quad & a^{-+}\,(2\bar{q} + \bar{\jmath}) + b^{-+}\,(2q - j - 2)\,, \\ s = -r = 1 \ : \quad & a^{+-}\,(2\bar{q} - \bar{\jmath} - 2) + b^{+-}\,(2q + j)\,, \\ s = r = -1 \ : \quad & a^{--}\,(2\bar{q} - \bar{\jmath} - 2) + b^{--}\,(2q - j - 2)\,. \end{aligned} \tag{3.42}$$

From (3.13) it is easy to verify that all these quantities are zero and thus the derivative commutes with the prefactor as needed. As we commented earlier, this depends crucially on the fact that the differential operators do not yield conformal descendants.

Orders $Q^2\bar{Q}$ and $\bar{Q}^2 Q$

For this order, only the terms with one derivative on t and two derivatives on $\mathcal{K}$ can survive. Furthermore, the derivatives on $\mathcal{K}$ must be with respect to Q and $\bar{Q}$. Since also applying derivatives on t shifts the quantum numbers (see (B.21)), one needs to be careful with the ordering. For the derivatives that act on $\mathcal{K}$ we use (B.19) and (B.20) as before. There are in total eight different cases (see (3.15)): in $D^s_{Q^2\bar{Q}}$ either the D^+_Q or the D^-_Q may act on the t and s may be $\pm$. Similarly in $D^s_{\bar{Q}^2 Q}$, $D^+_{\bar{Q}}$ or $D^-_{\bar{Q}}$ may act on the t and s may be $\pm$. For brevity we only illustrate two cases. If $D^+_{\bar{Q}}$ acts on the t and $s = +$, the result is proportional to

$$(2q + j)\,e^+ - (2q + j + 2)\,d^+ - (2\bar{q} - \bar{\jmath} - 4)\,c^+\,. \tag{3.43}$$

If, on the other hand, $D^-_{\bar{Q}}$ acts on the t and $c = -$, the result is proportional to

$$(2\bar{q} - \bar{\jmath} - 2)\,\bar{e}^- - (2\bar{q} - \bar{\jmath})\,\bar{d}^- - (2q - j - 4)\,\bar{c}^-\,. \tag{3.44}$$

In all cases it can be checked that the resulting expressions vanish when one replaces the coefficients with (3.17).

Order $Q^2\bar{Q}^2$

This is the last and most challenging order. The terms that can survive are of two kinds: those where two derivatives (D_Q and $D_{\bar{Q}}$) act on the t and the other two on the $\mathcal{K}$ and those where all the derivatives act on $\mathcal{K}$. Working out these cases in the same way as we did before requires deriving formulas for repeated applications of the differential operators, similar to those appearing in Appendix B.3. We preferred resorting to a "brute force" approach instead. We used the explicit definition of $\mathcal{K}$ and applied the derivatives on it using the Mathematica package introduced in Section 3.4. The case

where all derivatives act on $\mathcal{K}$ is straightforward and can be done with the functions defined in the package. The case where only two derivatives act on $\mathcal{K}$ requires a small explanation first. Since t is a generic function, we cannot take explicit derivatives of it. But we can always modify (A.34) as follows

$$\begin{aligned} D_{1\alpha}\, t(Z_3) &= -i\frac{(\mathrm{x}_{1\bar{3}})_{\alpha\dot{\alpha}}}{{x_{\bar{1}3}}^2}\,\bar{\xi}^{\dot{\alpha}}\,\overleftarrow{\partial}_{\bar{\xi}}\partial_{\bar{\xi}'}\,\bar{\xi}'\overline{\mathcal{D}}\,t(Z_3)\,, \\ \overline{D}_{1\dot{\alpha}}\, t(Z_3) &= -i\,\xi^{\alpha}\frac{(\mathrm{x}_{3\bar{1}})_{\alpha\dot{\alpha}}}{{x_{\bar{3}1}}^2}\,\partial_{\xi'}\overleftarrow{\partial}_{\xi}\,\xi'\mathcal{D}\,t(Z_3)\,, \end{aligned} \tag{3.45}$$

where $\xi, \xi', \bar{\xi}, \bar{\xi}'$ are other auxiliary polarization. In this way we can factorize either a $\xi'\mathcal{D}\,\bar{\xi}'\overline{\mathcal{D}}t$ or a $\bar{\xi}'\overline{\mathcal{D}}\,\xi'\mathcal{D}t$ and focus on the rest. Now the problem becomes explicit and one can check whether the resulting expressions vanish.

We performed this computation and observed that, with the values of f_i given in (3.19), all expressions identically vanish. This completes the proof of (3.40).

3.3.2 $\mathcal{N} = 2$ case

Lowest order

The lowest order $\theta_{\underline{2}} = \bar{\theta}^{\underline{2}} = 0$ is almost entirely trivial. The matrices ${\hat{u}_I}^J$ appearing in the prefactor of (2.29) reduce to

$${\hat{u}_1}^1(z_{13})\big|_{\theta_{\underline{2}}=\bar{\theta}^{\underline{2}}=0} = \left(\frac{{x_{\bar{1}3}}^2}{{x_{\bar{3}1}}^2}\right)^{\frac{1}{2}}, \qquad {\hat{u}_2}^2(z_{13})\big|_{\theta_{\underline{2}}=\bar{\theta}^{\underline{2}}=0} = \left(\frac{{x_{\bar{3}1}}^2}{{x_{\bar{1}3}}^2}\right)^{\frac{1}{2}}, \tag{3.46}$$

the off-diagonal components being zero. It is also obvious from (A.26) that all $\mathcal{N} = 2$ quantities that depend on $x_{\bar{\imath}j}$ simply reduce to the same quantity but with the $\mathcal{N} = 1$ definition of $x_{\bar{\imath}j}$. By looking at the prefactor (2.21) it is easy to see that the factors of $\hat{u}$ can be absorbed by shifting the $q, \bar{q}$ labels as follows

$${\hat{u}_1}^1(z_{13})\,\mathcal{K}_{\mathcal{O}_1\mathcal{O}_2} = \mathcal{K}_{\mathcal{O}_1\mathcal{O}_2}\Big|_{\substack{q_1\to q_1+1/2\\ \bar{q}_1\to\bar{q}_1-1/2}}, \qquad {\hat{u}_2}^2(z_{13})\,\mathcal{K}_{\mathcal{O}_1\mathcal{O}_2} = \mathcal{K}_{\mathcal{O}_1\mathcal{O}_2}\Big|_{\substack{q_1\to q_1-1/2\\ \bar{q}_1\to\bar{q}_1+1/2}}. \tag{3.47}$$

The component of $\mathcal{O}$ with $\mathfrak{su}(2)$ R-charge $R_3 = m$ will have a prefactor containing $({\hat{u}_1}^1)^{\frac{1}{2}R-m}\,({\hat{u}_2}^2)^{\frac{1}{2}R+m}$. This contributes to an $\mathcal{N} = 1$ superconformal primary with $q, \bar{q}$ charges equal to $(q-m, \bar{q}+m)$, consistently with (2.10).

Zero R-charge

If the superconformal primary is an $\mathfrak{su}(2)$ singlet, the commutativity with the prefactor follows immediately from the $\mathcal{N} = 1$ case. Indeed the structure of $\mathcal{K}_{\mathcal{O}_1\mathcal{O}_2}$ is identical

except for the fact that $x_{i\bar{j}}$ has more Grassmann variables. The crucial observation is that only $x_{3\bar{\imath}}$ or $x_{i\bar{3}}$ appear. We can thus write

$$\begin{aligned} x_{i\bar{3}} = \big(x_{i3} - 2i\theta_{\underline{1}i}\,\bar{\theta}^{\underline{1}}_{\bar{\imath}} - 2i\theta_{\underline{1}3}\,\bar{\theta}^{\underline{1}}_{3} + 4i\theta_{\underline{1}i}\,\bar{\theta}^{\underline{1}}_{3}\big) \\ - 2i\theta_{\underline{2}i}\,\bar{\theta}^{\underline{2}}_{\bar{\imath}} - 2i\theta_{\underline{2}3}\,\bar{\theta}^{\underline{2}}_{3} + 4i\theta_{\underline{2}i}\,\bar{\theta}^{\underline{2}}_{3}\,, \end{aligned} \tag{3.48}$$

Since there is no term mixing $\theta_{\underline{1}}$ and $\theta_{\underline{2}}$ we can simply rename the quantity inside the parentheses as x'_{i3} and carry the same exact computation as the $\mathcal{N}=1$ case. The same argument applies to $x_{3\bar{\imath}}$. The necessary shifts that need to be applied to the prefactor differ sightly from (3.35). They follow directly from (A.36)

$$\begin{aligned} \mathcal{K}_{(Q^I\mathcal{O}_1)^{\pm}\mathcal{O}_2} &= \hat{u}_J{}^I(z_{31})\,\mathcal{K}_{\mathcal{O}_1\mathcal{O}_2}\Big|_{\substack{\bar{q}_1\to\bar{q}_1+1/2\\ j_1\to j_1\pm 1/2}}\,, \\ \mathcal{K}_{(\bar{Q}_I\mathcal{O}_1)^{\pm}\mathcal{O}_2} &= \hat{u}_I{}^J(z_{13})\,\mathcal{K}_{\mathcal{O}_1\mathcal{O}_2}\Big|_{\substack{q_1\to q_1+1/2\\ \bar{\jmath}_1\to\bar{\jmath}_1\pm 1/2}}\,. \end{aligned} \tag{3.49}$$

Then, using (A.36) followed by setting the $\theta_{\underline{2}}, \bar{\theta}^{\underline{2}}$ Grassmann variables to zero results in an $\mathcal{N}=1$ superconformal correlator. The identities presented in (3.47), (3.49) imply that the resulting superconformal primary has the desired $q, \bar{q}$ charges: $(q+1/2,\,\bar{q})$ for the $Q^{\underline{2}}$ descendant and $(q,\,\bar{q}+1/2)$ for the $\bar{Q}_{\underline{2}}$ descendant.[9] The result is consistent with Figure 2.2.

At order $Q^{\underline{2}}\bar{Q}_{\underline{2}}$ instead we need to use the operator $\mathbb{D}_{Q\bar{Q}}$ defined in (3.22). Here we will adopt a different strategy from the $\mathcal{N}=1$ case. We will actually use the commutativity with the prefactor to derive the form of the differential operator. The reason why this is a valid proof is that, thanks to the formulas (A.36), we can show that the action of such an operator on a three-point function yields a correlator of a superconformal primary. We could have followed the same approach for the $\mathcal{N}=1$ case, of course. However in the way we did it the prefactor commutativity served as an important nontrivial check of our results. The computation is a bit more involved than that of Section 3.3.1 because we are not setting all θ's to zero but only $\theta_{\underline{2}}$ and $\bar{\theta}^{\underline{2}}$. In particular, there are non-vanishing contributions also from terms where only one derivative acts on $\mathcal{K}$. We follow a "brute force" approach similar to that of Section 3.3.1: we act either with both differential operators D and $\bar{D}$ on the $\mathcal{K}$ or we act with one of them on the t and we factorize it away using (3.45). Letting $|_0 \equiv |_{\theta_{\underline{2}}=\bar{\theta}^{\underline{2}}=0}$, the following equation has a unique solution for A^{st}, given by (3.23):

$$\begin{aligned} &\big({}_{\underline{2}}D^{st}_{Q\bar{Q}} + A^{st}\,{}_{\underline{1}}D^{st}_{Q\bar{Q}}\big)\,\mathcal{K}_{\mathcal{O}_1\mathcal{O}_2}\,t^{\mathcal{O}_1\mathcal{O}_2}_{\mathcal{O}_3}\,\big|_0 = \\ &\qquad = \mathcal{K}_{(Q^{\underline{2}}\bar{Q}_{\underline{2}}\mathcal{O}_1)\mathcal{O}_2}\,\hat{u}_{\underline{2}}{}^{\underline{2}}(z_{13})\,\hat{u}_{\underline{1}}{}^{\underline{1}}(z_{13})\,\big({}_{\underline{2}}\mathcal{D}^{st}_{Q\bar{Q}} + A^{st}\,{}_{\underline{1}}\mathcal{D}^{st}_{Q\bar{Q}}\big)t^{\mathcal{O}_1\mathcal{O}_2}_{\mathcal{O}_3}\,\big|_0\,, \end{aligned} \tag{3.50}$$

[9] To see this, one needs to use the property $\hat{u}_{\underline{2}}{}^{\underline{2}}(z_{31}) = \hat{u}_{\underline{1}}{}^{\underline{1}}(z_{13})$.

where we used the property in footnote 9. In analogy with (3.21) and (3.39) we defined

$$ {}_I\mathcal{D}_{Q^\ell\bar{Q}^\ell} \equiv \mathcal{D}_{Q^\ell\bar{Q}^\ell}\big|_{\mathcal{D}\to\mathcal{D}_I,\,\bar{\mathcal{D}}\to\bar{\mathcal{D}}^I}\,. \tag{3.51} $$

According to the shifts defined in (3.49) and the definitions in (3.46) the result of the action on the t is a superconformal primary with charges $(q+1/2,\bar{q}+1/2)$ as expected from Figure 2.2. We can then define the analog of the $\mathbb{D}_{Q\bar{Q}}$ when acting on the t as

$$ \mathfrak{D}^{st}_{Q\bar{Q}} = {}_{\underline{2}}\mathcal{D}^{st}_{Q\bar{Q}} + A^{st}\,{}_{\underline{1}}\mathcal{D}^{st}_{Q\bar{Q}}\,. \tag{3.52} $$

R-charge 1/2

Following an approach similar to the last subsection here we will claim that (3.26) annihilates the prefactor. The difference now is that the prefactor has indices: $\mathcal{K}_{\mathcal{O}_1\mathcal{O}_2}\hat{u}_I{}^J$ one of which is contracted with the matrix $\mathcal{M}$. By explicitly computing the action of the differential operators on the prefactor we can impose that it vanishes and use this to fix the matrix $\mathcal{M}$. We discover that there are two possible solutions for each case: $\mathcal{A}$ defined in (3.28) and $\mathcal{B}$ defined in (3.29). These two choices will give rise to two independent $\mathcal{N}=1$ multiplets when acting on the t. They will have charges $(q+1/2\pm1/2,\bar{q}\mp1/2)$ for the Q descendant and $(q\pm1/2,\bar{q}+1/2\mp1/2)$ for the $\bar{Q}$ descendant, as described by Figure 2.2. If we choose $\mathcal{M}=\mathcal{A}$ we have (a $|_0$ is implicit in all the following formulas)

$$ \begin{aligned} {}_ID^s_Q\,\mathcal{A}^{IJ}_s\,\mathcal{K}_{\mathcal{O}_1\mathcal{O}_2}\hat{u}_J{}^K(z_{13})\,t^{\mathcal{O}_1\mathcal{O}_2|K}_{\mathcal{O}_3} &= \mathcal{K}_{(Q^{\underline{2}}\mathcal{O}_1)\mathcal{O}_2}\,\hat{u}_1{}^1(z_{13})\,\hat{u}_2{}^2(z_{31})\,{}_{\underline{2}}\mathcal{D}^s_{\bar{Q}}\,t^{\mathcal{O}_1\mathcal{O}_2|1}_{\mathcal{O}_3}\,,\\ {}_ID^s_{\bar{Q}}\,\bar{\mathcal{A}}^{IJ}_s\,\mathcal{K}_{\mathcal{O}_1\mathcal{O}_2}\hat{u}_J{}^K(z_{13})\,t^{\mathcal{O}_1\mathcal{O}_2|K}_{\mathcal{O}_3} &= \mathcal{K}_{(\bar{Q}_{\underline{2}}\mathcal{O}_1)\mathcal{O}_2}\,\hat{u}_2{}^2(z_{13})\,\hat{u}_2{}^2(z_{13})\,{}_{\underline{2}}\mathcal{D}^s_{Q}\,t^{\mathcal{O}_1\mathcal{O}_2|2}_{\mathcal{O}_3}\,. \end{aligned} \tag{3.53} $$

The first line corresponds to the $Q^{\underline{2}}$ descendant with charges $(q+1,\bar{q}-1/2)$, while the second line corresponds to the $\bar{Q}_{\underline{2}}$ descendant with charges $(q-1/2,\bar{q}+1)$, as can be seen from (3.47), (3.49) and the property in footnote 9. Similarly, if we choose $\mathcal{M}=\mathcal{B}$ we have

$$ \begin{aligned} {}_ID^s_Q\,\mathcal{B}^{IJ}_s\,\mathcal{K}_{\mathcal{O}_1\mathcal{O}_2}\hat{u}_J{}^K(z_{13})\,t^{\mathcal{O}_1\mathcal{O}_2|K}_{\mathcal{O}_3} &= \mathcal{K}_{(Q^{\underline{2}}\mathcal{O}_1)\mathcal{O}_2}\,\hat{u}_1{}^1(z_{13})\,\hat{u}_2{}^2(z_{13})\,\times\\ &\qquad\times\left({}_{\underline{2}}\mathcal{D}^s_{\bar{Q}}\,t^{\mathcal{O}_1\mathcal{O}_2|2}_{\mathcal{O}_3}+\mathcal{B}^{11}_s\,{}_{\underline{1}}\mathcal{D}^s_{\bar{Q}}\,t^{\mathcal{O}_1\mathcal{O}_2|1}_{\mathcal{O}_3}\right),\\ {}_ID^s_{\bar{Q}}\,\bar{\mathcal{B}}^{IJ}_s\,\mathcal{K}_{\mathcal{O}_1\mathcal{O}_2}\hat{u}_J{}^K(z_{13})\,t^{\mathcal{O}_1\mathcal{O}_2|K}_{\mathcal{O}_3} &= \mathcal{K}_{(\bar{Q}_{\underline{2}}\mathcal{O}_1)\mathcal{O}_2}\,\hat{u}_1{}^1(z_{13})\,\hat{u}_2{}^2(z_{13})\,\times\\ &\qquad\times\left({}_{\underline{2}}\mathcal{D}^s_{Q}\,t^{\mathcal{O}_1\mathcal{O}_2|1}_{\mathcal{O}_3}+\bar{\mathcal{B}}^{12}_s\,{}_{\underline{1}}\mathcal{D}^s_{Q}\,t^{\mathcal{O}_1\mathcal{O}_2|2}_{\mathcal{O}_3}\right), \end{aligned} \tag{3.54} $$

where again we have used the property of footnote 9. Now the first line corresponds to the $Q^{\underline{2}}$ descendant with charges $(q,\bar{q}+1/2)$ and the second line corresponds to the $\bar{Q}_{\underline{2}}$ descendant with charges $(q+1/2,\bar{q})$.

R-charge 1

There are no qualitative differences between the cases with $R = 1$ and $R = 2$. As before we act on the prefactor, impose that it vanishes, and solve for the matrix $\mathcal{M}$. This will give rise to three different choices, $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$ defined in (3.31). We will now list all possible ways of acting on the t and show the charges of the $\mathcal{N} = 1$ superconformal primaries that are produced. In order to streamline the notation we will denote with $\mathcal{T} \equiv \mathcal{T}^{\mathbf{2}}$ the R-charge 1 ($R = 2$) representation that appears in (2.28). We will also define

$$t_{\mathcal{O}_3}^{\mathcal{O}_1\mathcal{O}_2|\pm} = \frac{1}{2}\left(t_{\mathcal{O}_3}^{\mathcal{O}_1\mathcal{O}_2|1} \pm i t_{\mathcal{O}_3}^{\mathcal{O}_1\mathcal{O}_2|2}\right) . \tag{3.55}$$

Let us start from $\mathcal{A}$

$$\begin{aligned} {}_I D_{\mathcal{Q}}^s \, \mathcal{A}_s^{IA} \, \mathcal{K}_{\mathcal{O}_1\mathcal{O}_2} \mathcal{T}(\hat{u}(z_{13}))_A{}^B \, t_{\mathcal{O}_3}^{\mathcal{O}_1\mathcal{O}_2|B} &= \mathcal{K}_{(\mathcal{Q}\underline{^2}\mathcal{O}_1)\mathcal{O}_2} \, (\hat{u}_1{}^1(z_{13}))^3 \, \underline{_2}\mathcal{D}_{\bar{\mathcal{Q}}}^s \, t_{\mathcal{O}_3}^{\mathcal{O}_1\mathcal{O}_2|+} , \\ {}_I D_{\bar{\mathcal{Q}}}^s \, \bar{\mathcal{A}}_s^{IA} \, \mathcal{K}_{\mathcal{O}_1\mathcal{O}_2} \mathcal{T}(\hat{u}(z_{13}))_A{}^B \, t_{\mathcal{O}_3}^{\mathcal{O}_1\mathcal{O}_2|B} &= \mathcal{K}_{(\bar{\mathcal{Q}}\underline{_2}\mathcal{O}_1)\mathcal{O}_2} \, (\hat{u}_2{}^2(z_{13}))^3 \, \underline{_2}\mathcal{D}_{\mathcal{Q}}^s \, t_{\mathcal{O}_3}^{\mathcal{O}_1\mathcal{O}_2|-} . \end{aligned} \tag{3.56}$$

These represent $\mathcal{N} = 1$ superconformal primaries with charges, respectively, $(q + 3/2, \bar{q} - 1)$ and $(q - 1, \bar{q} + 3/2)$. Then we continue with $\mathcal{B}$

$$\begin{aligned} {}_I D_{\mathcal{Q}}^s \, \mathcal{B}_s^{IA} \, \mathcal{K}_{\mathcal{O}_1\mathcal{O}_2} \mathcal{T}(\hat{u}(z_{13}))_A{}^B \, t_{\mathcal{O}_3}^{\mathcal{O}_1\mathcal{O}_2|B} &= \mathcal{K}_{(\mathcal{Q}\underline{^2}\mathcal{O}_1)\mathcal{O}_2} \, \hat{u}_2{}^2(z_{13}) \times \\ &\quad \times i\left(\underline{_2}\mathcal{D}_{\bar{\mathcal{Q}}}^s \, t_{\mathcal{O}_3}^{\mathcal{O}_1\mathcal{O}_2|-} + \mathcal{B}_s^{13} \, \underline{_1}\mathcal{D}_{\bar{\mathcal{Q}}}^s \, t_{\mathcal{O}_3}^{\mathcal{O}_1\mathcal{O}_2|3}\right) , \\ {}_I D_{\bar{\mathcal{Q}}}^s \, \bar{\mathcal{B}}_s^{IA} \, \mathcal{K}_{\mathcal{O}_1\mathcal{O}_2} \mathcal{T}(\hat{u}(z_{13}))_A{}^B \, t_{\mathcal{O}_3}^{\mathcal{O}_1\mathcal{O}_2|B} &= \mathcal{K}_{(\bar{\mathcal{Q}}\underline{_2}\mathcal{O}_1)\mathcal{O}_2} \, \hat{u}_1{}^1(z_{13}) \times \\ &\quad \times (-i)\left(\underline{_2}\mathcal{D}_{\mathcal{Q}}^s \, t_{\mathcal{O}_3}^{\mathcal{O}_1\mathcal{O}_2|+} + \bar{\mathcal{B}}_s^{13} \, \underline{_1}\mathcal{D}_{\mathcal{Q}}^s \, t_{\mathcal{O}_3}^{\mathcal{O}_1\mathcal{O}_2|3}\right) . \end{aligned} \tag{3.57}$$

These represent $\mathcal{N} = 1$ superconformal primaries with charges, respectively, $(q - 1/2, \bar{q} + 1)$ and $(q + 1, \bar{q} - 1/2)$. Finally we have $\mathcal{C}$

$$\begin{aligned} {}_I D_{\mathcal{Q}}^s \, \mathcal{C}_s^{IA} \, \mathcal{K}_{\mathcal{O}_1\mathcal{O}_2} \mathcal{T}(\hat{u}(z_{13}))_A{}^B \, t_{\mathcal{O}_3}^{\mathcal{O}_1\mathcal{O}_2|B} &= \mathcal{K}_{(\mathcal{Q}\underline{^2}\mathcal{O}_1)\mathcal{O}_2} \, \hat{u}_1{}^1(z_{13}) \times \\ &\quad \times \left(\underline{_2}\mathcal{D}_{\bar{\mathcal{Q}}}^s \, t_{\mathcal{O}_3}^{\mathcal{O}_1\mathcal{O}_2|3} + 2\mathcal{C}_s^{11} \, \underline{_1}\mathcal{D}_{\bar{\mathcal{Q}}}^s \, t_{\mathcal{O}_3}^{\mathcal{O}_1\mathcal{O}_2|+}\right) , \\ {}_I D_{\bar{\mathcal{Q}}}^s \, \bar{\mathcal{C}}_s^{IA} \, \mathcal{K}_{\mathcal{O}_1\mathcal{O}_2} \mathcal{T}(\hat{u}(z_{13}))_A{}^B \, t_{\mathcal{O}_3}^{\mathcal{O}_1\mathcal{O}_2|B} &= \mathcal{K}_{(\bar{\mathcal{Q}}\underline{_2}\mathcal{O}_1)\mathcal{O}_2} \, \hat{u}_2{}^2(z_{13}) \times \\ &\quad \times \left(\underline{_2}\mathcal{D}_{\mathcal{Q}}^s \, t_{\mathcal{O}_3}^{\mathcal{O}_1\mathcal{O}_2|3} + 2\bar{\mathcal{C}}_s^{11} \, \underline{_1}\mathcal{D}_{\mathcal{Q}}^s \, t_{\mathcal{O}_3}^{\mathcal{O}_1\mathcal{O}_2|-}\right) . \end{aligned} \tag{3.58}$$

Here we also used $\mathcal{C}^{11} = -i\mathcal{C}^{12}$ and $\bar{\mathcal{C}}^{11} = i\bar{\mathcal{C}}^{12}$. These last operators represent $\mathcal{N} = 1$ superconformal primaries with charges, respectively, $(q + 1/2, \bar{q})$ and $(q, \bar{q} + 1/2)$.

3.4 A *Mathematica* package

Computations in superspace, in particular those necessary to solve (3.37), might be hard to do by hand. We introduce a Mathematica package as a convenient tool to perform such tasks. It can be found in the repository `gitlab.com/maneandrea/spinoralgebra`. There is also a version of this package that only deals with commuting variables, which can be used for any tensor computation in four dimensions. A complete documentation is made available in the form of a notebook.

The package works with the index-free formalism following the same conventions as this thesis. There are different input and output notations available: one for improved readability and one to write code more easily. It is possible to reduce, Taylor expand and compare expressions. Furthermore, many differential operators have been defined, including the chiral derivatives (A.29, A.32, A.33) and all the operators appearing in (B.10 – B.15). It contains a precomputed two-point function for general values of $q, \bar{q}, j, \bar{j}$. By including the package `CFTs4D` [91] it is also possible to use the functionalities for $\mathcal{N} = 1$ superspace three-point functions. For any three given operators, the package gives a basis of $t^{\mathcal{O}_1\mathcal{O}_2}_{\mathcal{O}_3}$ tensor structures.

3.4.1 Note on the conventions

For two-point function we follow the conventions of [156]. Namely if a superprimary $\mathcal{O}$ has a two-point function given by

$$\langle \bar{\mathcal{O}}(\mathbf{z}_1)\mathcal{O}(\mathbf{z}_2)\rangle = i^{j+\bar{j}}\, n_{\mathcal{O}} \frac{(\eta_1 \mathrm{x}_{1\bar{2}}\bar{\eta}_2)^j (\eta_2 \mathrm{x}_{\bar{1}2}\bar{\eta}_1)^{\bar{j}}}{{x_{1\bar{2}}}^{2q+j} {x_{\bar{1}2}}^{2\bar{q}+\bar{j}}}\,, \qquad n_{\mathcal{O}} > 0\,, \tag{3.59}$$

then any of its descendants will have a non-supersymmetric two-point function given by

$$\langle \bar{O}'(\mathbf{x}_1)\, O'(\mathbf{x}_2)\rangle = i^{j'+\bar{j}'}\, n_{(Q^\ell \bar{Q}^{\bar{\ell}} O)} \frac{(\eta_1 \mathrm{x}_{12}\bar{\eta}_2)^{j'} (\eta_2 \mathrm{x}_{12}\bar{\eta}_1)^{\bar{j}'}}{{x_{12}}^{2(q'+\bar{q}')+j'+\bar{j}'}}\,, \qquad O'(\mathbf{x}) \equiv (Q^\ell \bar{Q}^{\bar{\ell}} O)(\mathbf{x})\,, \tag{3.60}$$

with the ratio of the respective normalizations fixed. From the package, it can be obtained as follows

$$\frac{i^{j'+\bar{j}'}\, n_{(Q^\ell \bar{Q}^{\bar{\ell}} O)}}{i^{j+\bar{j}}\, n_{\mathcal{O}}} = \texttt{operatorNorm["}Q^\ell \bar{Q}^{\bar{\ell}} O\texttt{", \{q,qb\}, \{j,jb\}]}\,. \tag{3.61}$$

Explicitly, the values of $n_{(Q^\ell \bar{Q}^{\bar{\ell}} O)}$ for $\ell = \bar{\ell} \equiv k$ and $q = \bar{q} = \Delta/2$ are given in Table 3.2.[10] Supersymmetric three-point functions instead can be computed by the function `SUSY3pf`. The expressions are given directly in the space where $t^{\mathcal{O}_1\mathcal{O}_2}_{\mathcal{O}_3}(X, \Theta, \bar{\Theta})$ lives, which we

[10] We show explicitly only the values for $\ell = \bar{\ell}$ and $q = \bar{q}$ because they will be useful later on.

$(Q^k\bar{Q}^k\mathcal{O})$	$n_{(Q^k\bar{Q}^k\mathcal{O})^{st}}$
$\mathcal{O}$	1
$(Q\bar{Q}\mathcal{O})^{++}$	$\frac{4(\Delta+\ell)(\Delta+\bar{\ell})(2\Delta+\ell+\bar{\ell}+2)}{(\bar{\ell}+1)^2(\ell+1)^2(2\Delta+\ell+\bar{\ell})}$
$(Q\bar{Q}\mathcal{O})^{+-}$	$\frac{4(\bar{\ell}+1)(\Delta-\bar{\ell}-2)(\Delta+\ell)(2\Delta+\ell-\bar{\ell})}{\bar{\ell}(\ell+1)^2(2\Delta+\ell-\bar{\ell}-2)}$
$(Q\bar{Q}\mathcal{O})^{-+}$	$\frac{4(\ell+1)(\Delta-\ell-2)(\Delta+\bar{\ell})(2\Delta+\bar{\ell}-\ell)}{\ell(\bar{\ell}+1)^2(2\Delta+\bar{\ell}-\ell-2)}$
$(Q\bar{Q}\mathcal{O})^{--}$	$\frac{4(\ell+1)(\bar{\ell}+1)(\Delta-\ell-2)(\Delta-\bar{\ell}-2)(2\Delta-\ell-\bar{\ell}-2)}{\ell\bar{\ell}(2\Delta-\ell-\bar{\ell}-4)}$
$(Q^2\bar{Q}^2\mathcal{O})$	$2^8\left(\frac{(\Delta+\ell)(\Delta-\ell-2)(2\Delta-\ell+\bar{\ell})}{2\Delta-\ell+\bar{\ell}-2}\times(\ell\leftrightarrow\bar{\ell})\right)$ $\times\frac{(2\Delta-\ell-\bar{\ell}-2)(2\Delta+\ell+\bar{\ell}+2)}{(2\Delta-\ell-\bar{\ell}-4)(2\Delta+\ell+\bar{\ell})}$

Table 3.2. Normalization coefficient $n_{(Q^k\bar{Q}^k\mathcal{O})^{st}}$ for the superdescendants generated by the differential operators $\mathcal{D}^{st}_{Q^k\bar{Q}^k}$.

will call "t space" in this paragraph. The notation of the package is as follows

$$\texttt{x3}\to X_3\,,\qquad \texttt{θ3}\to\Theta_3\,,\qquad \texttt{θb3}\to\overline{\Theta}_3\,.$$

The structures are generated by calling internally the package `CFTs4D` and then following the procedure explained in Section 2.2.1. Naturally one can work in t space also for non-supersymmetric three-point functions using the non nilpotent supersymmetric structures as a basis. Expanding any function of $X_3, \eta_i, \bar{\eta}_i$ in such a basis does not require much computational effort. It may however be useful to make contact with more familiar bases. In Table 3.3 we show the mapping between the $\Theta, \overline{\Theta}\to 0$ limit of the non-nilpotent structures in t space and the embedding space structures in `CFTs4D` [91]. Every tensor structure can be constructed as a monomial over the building blocks listed in Table 3.3. In order to pass from t space to embedding one has to multiply both sides by the appropriate prefactors. The structures in t space need to be multiplied by $\mathcal{K}_{\mathcal{O}_1\mathcal{O}_2}$ times an overall scaling X_3^a. While the embedding structures should be multiplied by the kinematic prefactor given in `CFTs4D` as `n3KinematicFactor`. In formulas one has

$$\mathcal{K}_{\mathcal{O}_1\mathcal{O}_2}\,X_3^{2(\Delta_3-\Delta_2-\Delta_1)}\big|_0\cdot\big(t\text{ space}\big)=\prod_{\substack{i<j\\k\neq i,j}}^{3}|x_{ij}|^{\kappa_k-\kappa_i-\kappa_j}\cdot\big(\text{embedding}\big)\,,\tag{3.62}$$

with

$$\Delta_i\equiv q_i+\bar{q}_i\,,\qquad \kappa_i\equiv\Delta_i+\tfrac{1}{2}(j_i+\bar{\jmath}_i)\,.$$

t space	Embedding
$\frac{\eta_1 X_3 \bar{\eta}_2}{(X_3^2)^{1/2}}$	$\hat{\mathbb{I}}^{1,2}$
$-\frac{\eta_2 X_3 \bar{\eta}_1}{(X_3^2)^{1/2}}$	$\hat{\mathbb{I}}^{2,1}$
$\eta_2 \eta_3$	$\hat{\mathbb{I}}^{2,3}$
$-\bar{\eta}_2 \bar{\eta}_3$	$\hat{\mathbb{I}}^{3,2}$
$\eta_1 \eta_3$	$\hat{\mathbb{I}}^{1,3}$
$-\bar{\eta}_1 \bar{\eta}_3$	$\hat{\mathbb{I}}^{3,1}$
$\vdots$	$\vdots$

t space	Embedding
$\vdots$	$\vdots$
$-\frac{\eta_3 X_3 \bar{\eta}_2}{(X_3^2)^{1/2}}$	$\hat{\mathbb{K}}_1^{2,3}$
$-\frac{\eta_3 X_3 \bar{\eta}_1}{(X_3^2)^{1/2}}$	$\hat{\mathbb{K}}_2^{3,1}$
$-\frac{\eta_2 X_3 \bar{\eta}_3}{(X_3^2)^{1/2}}$	$\hat{\bar{\mathbb{K}}}_1^{2,3}$
$-\frac{\eta_1 X_3 \bar{\eta}_3}{(X_3^2)^{1/2}}$	$\hat{\bar{\mathbb{K}}}_2^{3,1}$
$\vdots$	$\vdots$

t space	Embedding
$\vdots$	$\vdots$
$\bar{\eta}_1 \bar{\eta}_2$	$\hat{\mathbb{K}}_3^{1,2}$
$\eta_1 \eta_2$	$\hat{\bar{\mathbb{K}}}_3^{1,2}$
$-\frac{\eta_1 X_3 \bar{\eta}_1}{(X_3^2)^{1/2}}$	$\hat{\mathbb{J}}_{2,3}^1$
$-\frac{\eta_2 X_3 \bar{\eta}_2}{(X_3^2)^{1/2}}$	$\hat{\mathbb{J}}_{3,1}^2$
$-\frac{\eta_3 X_3 \bar{\eta}_3}{(X_3^2)^{1/2}}$	$\hat{\mathbb{J}}_{1,2}^3$

Table 3.3. Mapping between the $\Theta, \overline{\Theta} \to 0$ limit of the non nilpotent three-point tensor structures in t space and the embedding formalism structures in [91]. The equality between neighboring columns holds after we apply the appropriate $\mathcal{K}_{\mathcal{O}_1\mathcal{O}_2} X_3^a$ prefactor to the t space structures and the kinematic prefactor to the embedding structures. See (3.62).

Part II Constraints from locality

4 Averaged Null Energy Condition

This chapter describes the implications of the averaged null energy condition on superconformal field theories. It is entirely based on **Paper II**.

4.1 Conformal collider bounds

Any local conformal field theory must satisfy a positivity constraint called the averaged null energy condition. It states that the following integrated expectation value of the stress tensor must be non-negative

$$\mathcal{E}_\psi := \int_{-\infty}^{\infty} \mathrm{d}\lambda \, \langle\psi|T_{\mu\nu}|\psi\rangle \, u^\mu u^\nu \geqslant 0 \,. \tag{4.1}$$

Here $|\psi\rangle$ is any state and u^μ is the four-velocity of a null geodesic parametrized by λ (i.e. $u^\mu u_\mu = 0$). Since we can choose ψ to be any operator acting on the vacuum, the condition in (4.1) represents an infinite set of linear inequalities on the OPE coefficients $\lambda_{OT\bar{O}}$. Furthermore, as it was discovered in [141], for some conformal dimensions the inequalities have no solutions and thus they imply lower bounds typically stronger than unitarity (1.7). This means that, schematically, we get a system of constraints as follows

$$\Delta_O > \Delta_{\mathrm{ANEC}}(j,\bar{j}) \quad \text{and} \quad M(\lambda_{\bar{O}TO}, \Delta_O) \succeq 0 \,, \tag{4.2}$$

where M is a matrix that can be computed by evaluating the integral (4.1). The goal of this chapter is to extend the results of [141] to superconformal theories. Supersymmetry can potentially give more stringent constraints. The reason is that a generic state may be taken to be

$$|\psi\rangle \sim \left(O + \alpha\, QO + \beta\, \bar{Q}O + \ldots\right)|0\rangle \,. \tag{4.3}$$

And, by varying arbitrarily the coefficients α, β, etc., we get several different inequalities on the same set of OPE coefficients. Indeed $\lambda_{(Q\bar{O})T(\bar{Q}O)}, \lambda_{(\bar{Q}\bar{O})T(QO)}$, etc. are all related to $\lambda_{\bar{O}TO}$. The problem of finding these relations has been addressed extensively in

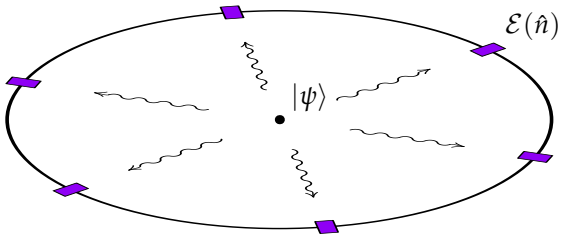


Figure 4.1. The state $|\psi\rangle$ creates a localized excitation and the decay products are measured by calorimeters placed far away.

Chapter 3.

But before diving into the computations let us review the intuition behind (4.1) and mention the proofs available in the literature. The intuitive explanation of the ANEC originates from a thought experiment called conformal collider. It was described in [136] by Hofman and Maldacena and was immediately used to get upper and lower bounds on the OPE coefficients $\lambda_{TTT}^{(i=1,2)}$, which can be related to the conformal anomalies a and c. Later these bounds were also proven rigorously [140]. Consider the setup in Figure 4.1, in which there is a local excitation at the origin created by the state $|\psi\rangle$ and a series of detectors (or calorimeters) placed far away. The energy measured by the detector placed at the direction $\hat{n}$ is computed as

$$\mathcal{E}_\psi(\hat{n}) \coloneqq \lim_{r\to\infty} r^2 \int_{-\infty}^{\infty} \mathrm{d}t\, \hat{n}^i \,\langle\psi|T_{0i}(t, r\,\hat{n}^i)|\psi\rangle\,. \tag{4.4}$$

An energy integrated over all times in quantum field theory must be a non-negative quantity, therefore we must conclude $\mathcal{E}_\psi(\hat{n}) \geqslant 0$. This is the origin of the ANEC. So far this applies to all theories, but for CFTs one can make a conformal transformation and turn this integral into an integral over a null geodesic, making it equivalent to (4.1). Indeed, by looking at Figure 4.2, one can see that the limit at $r \to \infty$ makes the integration run over Minkowski null infinity. Then, by means of a special conformal transformation, we can transform the integration path in any null geodesic we want.

There are rigorous proofs of the ANEC that hold in any conformal field theory and there is also an holographic proof. We will not review them here but only cite the papers in which they appeared. The proof in [139] relies on the principle of causality and on the fact that in the OPE of two scalars one can always single out the stress tensor by taking an appropriate limit, namely the lightcone limit. The holographic proof [137] shows that a violation of the ANEC leads to superluminal propagation of signals in the bulk. There exists also another proof that adopts a completely different strategy and it originates from information theory [138]. Furthermore, the ANEC operator may be seen as a particular case of a so-called light-ray operator, which were introduced in [157]. With the formalism introduced there it is also possible to prove the ANEC by using Rindler positivity, with the only additional assumption that there exists a scalar operator

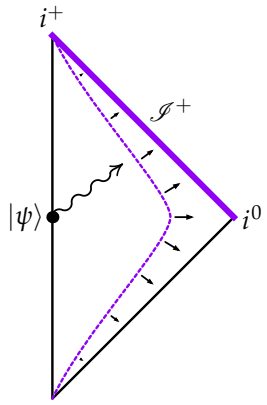


Figure 4.2. The time integral of the energy measured by the calorimeter at large r becomes an integral over Minkowski future null infinity. There is no contribution from past infinity since the excitation is turned off at $t = -\infty$.

of dimension $\Delta_\phi \leqslant \frac{d+2}{2}$.

It is worth to also mention that the ANEC is a special case of a series of constraints which go under the name of "higher spin ANEC" or "deep inelastic scattering bounds" [139, 140, 158, 159]. They consist in similar positivity conditions on the operator $\mathcal{O}_{\mu_1\cdots\mu_s}$ that has minimal twist[1] over all operators of spin s. The statement is that the following expectation value

$$\mathcal{E}_\psi^{(s)} := \int_{-\infty}^{\infty} \mathrm{d}\lambda \, \langle\psi|\mathcal{O}_{\mu_1\cdots\mu_s}|\psi\rangle \, u^{\mu_1}\cdots u^{\mu_s} \,, \tag{4.5}$$

is non-negative. We will not explore this direction in the present work.

4.2 Applying the ANEC to superconformal theories

4.2.1 Strategy

We focus on superconformal multiplets $\mathcal{O}(x,\theta,\bar\theta)$ for which the lowest component field O transforms in the $(j,0)$ irreducible representation of the Lorentz group. Our first goal is to determine the most general form of the three-point function in superspace among $\mathcal{O}$, its complex conjugate superfield, and the Ferrara-Zumino multiplet J, which contains the stress-energy tensor:[2]

$$\langle\overline{\mathcal{O}}(\mathbf{z}_1)J(\mathbf{z}_2)\mathcal{O}(\mathbf{z}_3)\rangle \,. \tag{4.6}$$

In order to determine (4.6), in Section 4.3 we write the most general three-point function consistent with $\mathcal{N} = 1$ superconformal invariance, complex conjugation, and conservation of the Ferrara-Zumino multiplet. Next, we fix certain combinations of the

[1] The twist is defined as the difference between the conformal dimension and the spin $\tau = \Delta - s$. The stress tensor, being conserved, has necessarily minimal twist.

[2] In this section we only present schematic formulas. Details are given in the next sections.

three-point function coefficients entering (4.6) by imposing the Ward identities generated by the conserved currents $J_\mu^{(R)}$, $T_{\mu\nu}$ and S_μ^α in J. Although in principle it should be possible to obtain a superspace version of the Ward identities, along the lines of [115], in this work we impose the constraints at the level of the individual primaries and superdescendants. More specifically, we find that once the $J_\mu^{(R)}$ and $T_{\mu\nu}$ Ward identities are imposed in the three-point function involving the superprimary O, all other ones we checked follow.[3]

As a final step, we need to decompose the superspace three-point function in the various θ components and extract the non -supersymmetric three-point functions of the superprimary O and various primary superdescendants. This task is performed in Section 4.5 and summarized in the Tables in Appendix C.3.2. We only pushed to the fourth order in θ_i or $\bar{\theta}_i$ and computed three-point functions involving at most $T_{\mu\nu}$ and superdescendants $QO^\pm$ and $\bar{Q}O$.

After all these preparatory steps, we can impose the ANEC (4.1) on a general state of the form of (4.3). Due to R-charge conservation, only a few three-point functions are non vanishing. In the end we impose that[4]

$$\begin{aligned} &\langle O|\mathcal{E}|O\rangle \geqslant 0\,, \quad \langle(\bar{Q}O)|\mathcal{E}|(\bar{Q}O)\rangle \geqslant 0\,, \\ &\begin{pmatrix} \langle(QO^+)|\mathcal{E}|(QO^+)\rangle & \langle(QO^+)|\mathcal{E}|(QO^-)\rangle \\ \langle(QO^-)|\mathcal{E}|(QO^+)\rangle & \langle(QO^-)|\mathcal{E}|(QO^-)\rangle \end{pmatrix} \succeq 0\,. \end{aligned} \tag{4.7}$$

We should stress that the above conditions are a subset of all conditions one can impose, since they do not include superdescendants of the form Q^2O or $Q\bar{Q}O$ for example. Nevertheless, we find that in any unitary and local SCFT superprimaries that transform in the $(j,0)$ representation and satisfy the usual unitarity bounds do not necessarily satisfy the conditions (4.7).

In Section 4.6 we obtain closed-form expressions for all the correlators appearing in (4.7) as rational functions of the spin j and dimension Δ. Such formulas allow us to easily compute bounds up to large values of j and in some cases rigorously prove bounds for any j.

Finally, we explore the consequences of our analysis for theories with extended supersymmetry. In Section 4.7 we consider special $\mathcal{N}=2$ and $\mathcal{N}=4$ supermultiplets and decompose them with respect to an $\mathcal{N}=1$ subalgebra. The ANEC constraints presented in the next subsection are then recast as bounds on the $\mathcal{N}=2,4$ superprimaries.

[3] More specifically, we checked the Ward identities for $\langle(\bar{Q}\bar{O})J^{(R)}(QO)\rangle$, $\langle(\bar{Q}\bar{O})T(QO)\rangle$ and $\langle(\bar{Q}\bar{O})SO\rangle$. In principle there could be extra relations that we did not take into account.

[4] For certain short supermultiplets some of these three-point functions vanish.

4.2.2 Summary of results

Let us first mention the results of our analysis for non-supersymmetric CFTs, in the case of a conformal primary with dimension Δ, transforming in the $(j, \bar{j})$ representation, with $\bar{j} = 0, 1$. In Section 4.6.5 we show convincing evidence that the ANEC requires

$$\Delta \geqslant \min\left(j, \tfrac{1}{15}(13j + 42)\right). \tag{4.8}$$

For $\bar{j} = 0, 1$ the above expression is stronger than the corresponding unitarity bound for $j > 2, 6$, respectively. Although we do not have an analytic proof, we checked (4.8) up to $j = 10^3$.

Next, we summarize the results of applying the ANEC inequality to superconformal multiplets $\mathcal{O}^{(j,0)}$. We present them as bounds on the quantum numbers $q, \bar{q}$, which are related to the dimension and the R-charge of a given operator by the simple relations in equation (2.5). We considered all possible shortening conditions. They were reviewed in Section 2.1.

All cases for $j = 0$ In this case the ANEC requires only $q \geqslant 0$ and $\bar{q} \geqslant 0$. Therefore, it is never stronger than the unitarity bound.

$A_1\bar{B}$ for $j = 1$ For these operators there are no free three-point function coefficients and the dimension and R-charge are fixed. It can be easily verified that the ANEC holds.

$A_1\bar{B}$ for $j \geqslant 2$ As shown in Table 4.1, these operators do not admit a three-point function with the Ferrara-Zumino multiplet consistent with all conditions. They are therefore absent in any local SCFT.[5]

$L\bar{B}$ for $j \geqslant 1$ With this shortening condition, corresponding to chiral operators, there are no free three-point function coefficients. Therefore the ANEC for any given j is simply a system of inequalities on q that can be solved algebraically. The result is

$$\Delta = q \geqslant \tfrac{3}{2} j\,. \tag{4.9}$$

This is equivalent to the unitarity bound for $j = 1$ and it is stronger for all $j > 1$. This result is not in contradiction with already known Lagrangian constructions, which so far have only provided examples for $j = 1$ [160, 161]. Also note that the bound is saturated

[5] This conclusion does not require the ANEC.

by j copies of a free $j = 1$ superconformal chiral primary ψ_i^α

$$\Psi^{\alpha_1 \dots \alpha_j} = : \psi_1^{(\alpha_1} \cdots \psi_j^{\alpha_j)} : \,. \tag{4.10}$$

In $\mathcal{N} = 2$ theories, the bound in (4.9) implies a constraint on the dimension of the so called "exotic chiral primaries" discussed in [132]. In Section 4.7.2 we show that

$$\Delta_{\text{exotic}} \geqslant \tfrac{3}{2} j + 1 \,. \tag{4.11}$$

Then in Chapter 5 we apply the full set of constraints stemming from $\mathcal{N} = 2$ Ward identities to show that the exotic operator are actually not allowed at all.

***L* $\bar{L}$ for $j \geqslant 1$** In this case there are two free parameters q and $\bar{q}$ and two undetermined three-point function coefficients (one for $j = 1$). For every value of j we fixed $\bar{q}$ and ran a bisection algorithm on q. The results are in Figure 4.3. See also Figure 4.4 for a plot in the (r, Δ) plane.

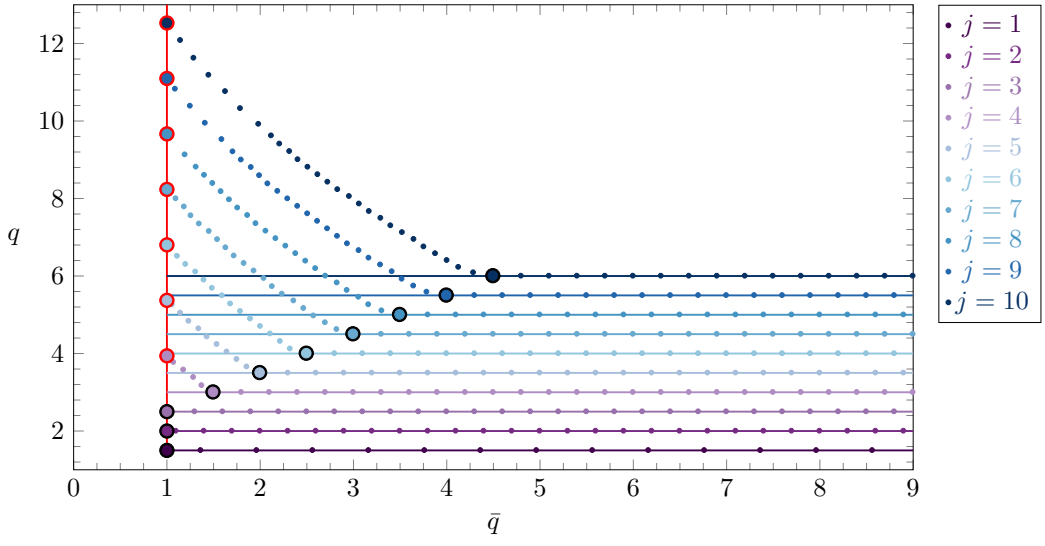


Figure 4.3. Lower bounds on the conformal dimension as a result of the ANEC for long multiples. Each point is the result of a bisection algorithm done with `sdpb` [162] (see Section 4.6.4). The solid lines are the unitarity bound: the red line is the bound on $\bar{q}$ and the colored lines are the j-dependent bounds on q. The larger dots correspond to the points with shortening conditions $L\,\bar{A}_2$ (for the red circled dots) and $A_1\,\bar{L}$ (for the black circled dots).

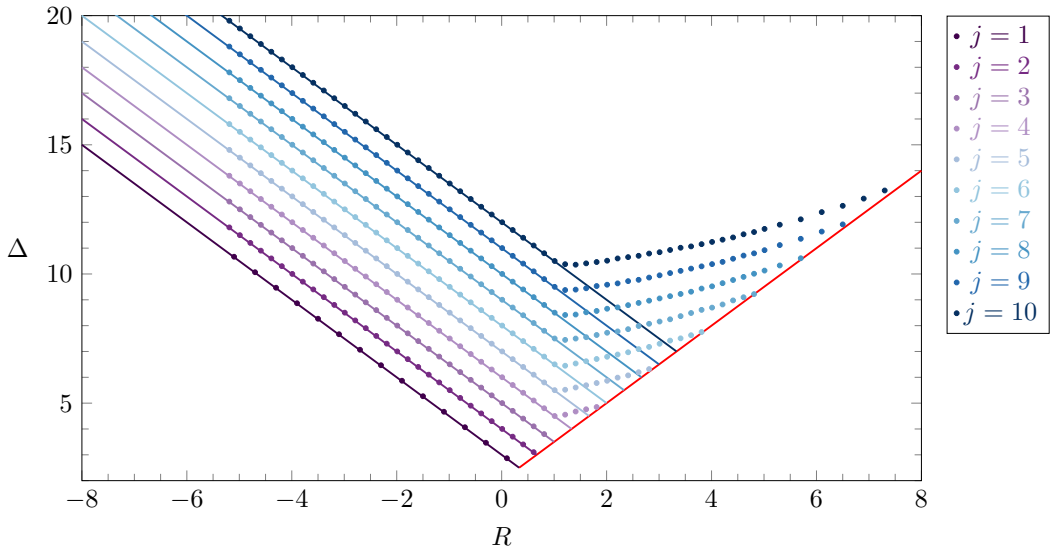


Figure 4.4. Plot of the results in Figure 4.3 in the (r, Δ) plane.

$L\,\bar{A}_2$ for $j \geqslant 1$ For this shortening condition the constraints are equivalent to $[L, \bar{L}]$ for $\bar{q} = 1$. The results are given in Figure 4.5 and correspond to the red circled dots on Figure 4.3. The operators at the unitarity bound, which satisfy the $[A_1, \bar{A}_2]$ shortening, are not allowed for $j > 3$ (see below). Therefore, for $j > 3$ the ANEC provides a constraint strictly stronger than unitarity.

$A_1\,\bar{L}$ for $j \geqslant 1$ Since for this case there is only one free three-point function coefficient and one parameter, $\bar{q}$, the system of inequalities is considerably simpler to solve. The results are given in Figure 4.6 and correspond to the black circled dots on Figure 4.3. As before, for $j > 3$, the ANEC is strictly stronger than unitarity.

$A_1\,\bar{A}_2$ for $j \geqslant 1$ This condition admits solutions only for $j \leqslant 3$. In the edge case $j = 3$ the ANEC inequalities fix the only independent three-point function coefficient to

$$\mathcal{C}_6 = -\frac{16}{\pi^2}\,. \tag{4.12}$$

For all $j > 3$ the ANEC admits no solution and thus such operators must be absent in any local SCFT.

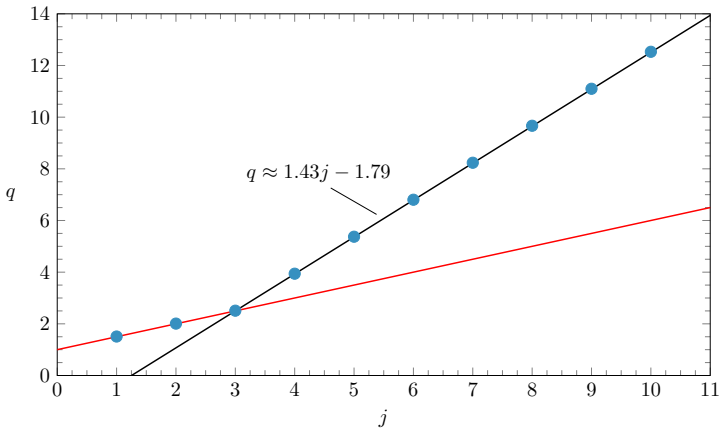


Figure 4.5. Lower bounds on the conformal dimension as a result of the ANEC for $L\,\bar{A}_2$ multiplets. Each point is the result of a bisection algorithm done with `sdpb` [162] (see Section 4.6.4). The red line is the unitarity bound $q = \frac{1}{2}j + 1$. The operators for $j \leqslant 3$ that lie on the red line satisfy $A_1\,\bar{A}_2$.

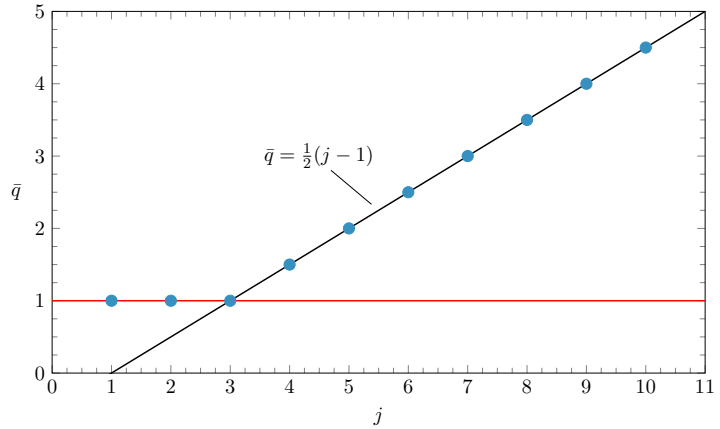


Figure 4.6. Lower bounds on the conformal dimension as a result of the ANEC for $A_1\,\bar{L}$ multiplets. Each point is the result of a bisection algorithm done with *Mathematica*. The operators for $j \leqslant 3$ that lie on the red line satisfy $A_1\,\bar{A}_2$.

4.3 Setup

We can adapt the general formula in Section 2.2.1 to our case: the correlator of an operator $\mathcal{O}$, its conjugate and J. Renaming $t_{\mathcal{O}}^{\bar{\mathcal{O}}J} \equiv t$ the three-point function reads

$$\langle \bar{\mathcal{O}}(\mathbf{z}_1) J(\mathbf{z}_2) \mathcal{O}(\mathbf{z}_3) \rangle = \frac{(\partial_{\chi_1} x_{3\bar{1}} \bar{\eta}_1)^j \, \eta_2 x_{2\bar{3}} \partial_{\bar{\chi}_2} \, \partial_{\chi_2} x_{3\bar{2}} \bar{\eta}_2}{x_{\bar{1}3}^{2q+j} \, x_{3\bar{1}}^{2\bar{q}} \, x_{3\bar{2}}^4 \, x_{2\bar{3}}^4} \, t(Z_3; \chi_1, \chi_2, \bar{\chi}_2, \eta_3) \,, \tag{4.13}$$

where all the definitions are given in and Appendix A.3.2. We can then form fully contracted monomials of the quantities defined above to obtain the building blocks of

the tensor structures in t. We give a complete list in a condensed notation

$$[i\bar{j}] = \frac{\eta_i \mathrm{U} \bar{\eta}_j}{|U|}, \quad [\Theta\bar{\Theta}] = \frac{\Theta \mathrm{U} \bar{\Theta}}{U^2}, \quad [ij] = \eta_i \eta_j, \quad [\bar{i}\bar{j}] = \bar{\eta}_i \bar{\eta}_j, \quad [\Theta^2] = \frac{\Theta^2}{|U|},$$
$$[\bar{\Theta}^2] = \frac{\bar{\Theta}^2}{|U|}, \quad [\Theta j] = \frac{\Theta \eta_j}{|U|^{1/2}}, \quad [\bar{\Theta}\bar{j}] = \frac{\bar{\Theta}\bar{\eta}_j}{|U|^{1/2}}, \quad [j\bar{\Theta}] = \frac{\eta_i \mathrm{U} \bar{\Theta}}{|U|^{3/2}}, \quad [\Theta\bar{j}] = \frac{\Theta \mathrm{U} \bar{\eta}_j}{|U|^{3/2}}. \tag{4.14}$$

The correlator must satisfy the constraints stemming from the conservation of J and the invariance under simultaneous permutation $1 \leftrightarrow 3$ and complex conjugation $\mathcal{O}^* = \bar{\mathcal{O}}$. Moreover, the function t must satisfy the homogeneity properties of (B.3). in this case they read

$$t(\lambda\bar{\lambda}X, \lambda\Theta, \bar{\lambda}\bar{\Theta}; \kappa\eta_1, \mu\eta_2, \bar{\mu}\bar{\eta}_2, \bar{\kappa}\eta_3) = (\lambda\bar{\lambda})^{-3} (\kappa\bar{\kappa})^j \mu\bar{\mu}\, t(X, \Theta, \bar{\Theta}; \eta_i, \bar{\eta}_i). \tag{4.15}$$

All possible tensor structures are built out of the blocks in (4.14) times a factor U^{-3} to take care of the $\lambda\bar{\lambda}$ scaling. By the methods explained in Section 2.2 and 2.3 we can show that t may be written as a linear combination of the following ten tensor structures

$$t(Z; \eta_1, \eta_2, \bar{\eta}_2, \eta_3) = \frac{1}{U^3} \sum_{k=1}^{10} \mathcal{C}_k \, \mathbb{T}_k(Z; \eta_1, \eta_2, \bar{\eta}_2, \eta_3), \tag{4.16}$$

where the explicit expressions for the $\mathbb{T}_k$'s are

$$\begin{aligned}
\mathbb{T}_1 &= i\,[2\bar{2}]\,[13]^j & \mathbb{T}_6 &= [12]\,[1\bar{2}]\,[\Theta 3]\,[3\bar{\Theta}]\,[13]^{j-2} \\
\mathbb{T}_2 &= i\,[12]\,[3\bar{2}]\,[13]^{j-1} & \mathbb{T}_7 &= [12]\,[\bar{\Theta}\bar{2}]\,[\Theta 3]\,[13]^{j-1} \\
\mathbb{T}_3 &= [3\bar{\Theta}]\,[\Theta 2]\,[1\bar{2}]\,[13]^{j-1} & \mathbb{T}_8 &= [12]\,[3\bar{2}]\,[\Theta\bar{\Theta}]\,[13]^{j-1} \\
\mathbb{T}_4 &= [\Theta 2]\,[\bar{\Theta}\bar{2}]\,[13]^j & \mathbb{T}_9 &= i\,[\Theta^2]\,[\bar{\Theta}^2]\,[2\bar{2}]\,[13]^j \\
\mathbb{T}_5 &= [2\bar{2}]\,[\Theta\bar{\Theta}]\,[13]^j & \mathbb{T}_{10} &= i\,[\Theta^2]\,[\bar{\Theta}^2]\,[12]\,[3\bar{2}]\,[13]^{j-1}.
\end{aligned} \tag{4.17}$$

The factors of i are introduced for later convenience. If $j = 1$ then $\mathbb{T}_6$ is not present and if $j = 0$ then $\mathbb{T}_{2,3,6,7,8,10}$ are not present.

4.4 Constraints on the supersymmetric three-point correlator

4.4.1 Conservation

The superconformal multiplet $J(\mathbf{z})$ satisfies a shortening condition as explained in Subsection 2.1.3. In this subsection we will explore the consequences of this constraint on the correlator at separated points. In Section 4.4.3 we will study the contact terms instead. At separated points the prefactor in (4.13) commutes with the conservation

differential operators acting on $\mathbf{z}_2$,[6] thus we can express the conservation condition as an equation involving only t and the variable Z:

$$\partial_{\eta_2}\mathcal{D}\,t(Z;\eta_i,\bar{\eta}_i)=\partial_{\bar{\eta}_2}\overline{\mathcal{D}}\,t(Z;\eta_i,\bar{\eta}_i)=0\,, \tag{4.18}$$

where $\mathcal{D}$ and $\overline{\mathcal{D}}$ have been defined in Appendix A.3.2. Equation (4.18) imposes the following linear constraints for general $j>1$:

$$\mathcal{C}_5=-\mathcal{C}_3-2\mathcal{C}_4\,,\quad \mathcal{C}_7=2\mathcal{C}_2-\mathcal{C}_3-\mathcal{C}_6\,,\quad \mathcal{C}_8=-4\mathcal{C}_2+2\mathcal{C}_3+\mathcal{C}_6\,,\quad \mathcal{C}_9=\mathcal{C}_{10}=0\,. \tag{4.19}$$

When $j=1$ it suffices to set $\mathcal{C}_6$ to zero and when $j=0$ one simply has

$$\mathcal{C}_5=-2\mathcal{C}_4\,,\qquad \mathcal{C}_9=0\,. \tag{4.20}$$

4.4.2 Reality

Since $\mathcal{O}$ and $\overline{\mathcal{O}}=\mathcal{O}^*$ are conjugated to each other and J is hermitian, the correlator under study must be real. Concretely, we want to impose that

$$\left\langle\overline{\mathcal{O}}(\mathbf{z}_1)J(\mathbf{z}_2)\mathcal{O}(\mathbf{z}_3)\right\rangle^*=\left\langle\overline{\mathcal{O}}(\mathbf{z}_3)J(\mathbf{z}_2)\mathcal{O}(\mathbf{z}_1)\right\rangle\,, \tag{4.21}$$

namely that taking the complex conjugation is the same as swapping points 1 and 3. The prefactor in (4.13) is not invariant under this transformation, moreover the exchange $1\leftrightarrow 3$ does not act nicely on Z_3. This means that we cannot translate the reality condition into a constraint for t right away.[7] This obstacle can be overcome by using the results of Appendix B.1, which lead to[8]

$$(-1)^j\bar{X}^{-2j}(t^*)(Z;\bar{X}\eta_1,\eta_2,\bar{\eta}_2,\bar{X}\eta_3)=t(\bar{Z};\eta_3,\eta_2,\bar{\eta}_2,\eta_1)\,. \tag{4.22}$$

We can then solve this equation much more easily. In doing so we find the following linear constraints for even $j>1$:

$$\begin{aligned}
&\mathcal{C}_1{}^*=\mathcal{C}_1\,,\quad \mathcal{C}_2{}^*=\mathcal{C}_2\,,\quad \mathcal{C}_3{}^*=2\mathcal{C}_2-\mathcal{C}_6-\mathcal{C}_7\,,\quad \mathcal{C}_4{}^*=-2\mathcal{C}_2+\mathcal{C}_3+\mathcal{C}_4+\mathcal{C}_6+\mathcal{C}_7\,,\\
&\mathcal{C}_5{}^*=\mathcal{C}_5\,,\quad \mathcal{C}_6{}^*=\mathcal{C}_6\,,\quad \mathcal{C}_7{}^*=2\mathcal{C}_2-\mathcal{C}_3-\mathcal{C}_6\,,\quad \mathcal{C}_8{}^*=\mathcal{C}_8\,,\\
&\mathcal{C}_9{}^*=\mathcal{C}_2-\tfrac{1}{2}(\mathcal{C}_3+\mathcal{C}_6+\mathcal{C}_7)+\mathcal{C}_9\,,\quad \mathcal{C}_{10}{}^*=-2\mathcal{C}_2+\mathcal{C}_3+\mathcal{C}_6+\mathcal{C}_7+\mathcal{C}_{10}\,.
\end{aligned} \tag{4.23}$$

[6] Due to $D_2^\alpha(\mathrm{x}_{2\bar{3}})_{\alpha\dot{\alpha}}/x_{\bar{3}2}^4=\bar{D}_2^{\dot{\alpha}}(\mathrm{x}_{3\bar{2}})_{\alpha\dot{\alpha}}/x_{\bar{2}3}^4=0$ when $x_{23}\neq 0$.

[7] This is obviously a consequence of our parametrization. In the ordering $\langle\mathcal{O}\overline{\mathcal{O}}J\rangle$ the reality condition can be solved easily. On the other hand we would lose the fact that the conservation operator commutes with the prefactor thus making conservation much harder to impose.

[8] By $(t^*)(Z;\dots)$ we mean: first apply the complex conjugation to $t(Z;\eta_1,\eta_2,\bar{\eta}_2,\eta_3)$, then replace $\bar{\eta}_{1(3)}$ with $\bar{X}\eta_{1(3)}$. We also defined $\bar{Z}=(-\bar{X},-\Theta,-\bar{\Theta})$.

If j is odd the equations are obtained by adding an overall minus sign on the right hand side. If $j = 1$ it is sufficient to set $\mathcal{C}_6 = {\mathcal{C}_6}^* = 0$. For $j = 0$ instead one has simply

$$ {\mathcal{C}_1}^* = \mathcal{C}_1 \,, \qquad {\mathcal{C}_4}^* = \mathcal{C}_4 \,, \qquad {\mathcal{C}_5}^* = \mathcal{C}_5 \,, \qquad {\mathcal{C}_9}^* = \mathcal{C}_9 \,. \tag{4.24} $$

Combined with conservation (4.19), these equations imply that the remaining $\mathcal{C}_k$ are purely real (resp. imaginary) if j is even (resp. odd).

4.4.3 Ward identities

There are in principle two ways to impose the Ward identities: one could apply them directly in superspace with the formalism of [115], or alternatively one could expand the correlator in components and apply the non-supersymmetric Ward identity to each triplet of superdescendants. Since we already need the three-point function in components to make contact with the ANEC and since non-supersymmetric Ward identities are much easier to compute, we opted for the second approach. We did not explore all possible combinations of superdescendants but we observed that after imposing the identities for $\langle \bar{O} J^{(R)} O \rangle$ and $\langle \bar{O} T O \rangle$, all other choices of superdescendants that we investigated were not yielding any new constraints.

In Section 1.4 we reviewed the Ward identities for the R-current $J^{(R)}$ and the stress tensor. The results for the former and an operator O of spin $(j, 0)$ are summarized in Table C.1 and those for O of spin $(j, 1)$ in Table C.2. For the latter, we need to first motivate that it is enough to consider only the Killing vectors associated to translations and dilatations. As we argued previously, the number of independent equations equals the number of singlets in the tensor product

$$ \rho_O \otimes \rho_{\bar{O}}^* \otimes (\bullet \oplus (2,0) \oplus (0,2)) \,, \tag{4.25} $$

ρ_O representing the Lorentz representation of O and $\bullet$ the singlet. For $\rho_O = (j, 0)$ the tensor product contains two singlets (one if $j = 0$) and for $\rho = (j, 1)$ it contains three singlets (two if $j = 0$). The equations (1.45) yield the exact same number of independent constraints. When the operator O in (1.45) is the superconformal primary of spin $(j, 0)$, the result is summarized in Table C.3. If instead $O \to \bar{Q} O$ of spin $(j, 1)$ the results are in Table C.4. Finally one could also consider $O \to Q O$; the result is obtained by a simple rescaling of the coefficients in Table C.3 and a replacement $j \to j \pm 1$. For the reader's convenience we report here the relative normalizations for the operators in the $\mathcal{O}$ multiplet as derived in [156]:

$$ \frac{c_{(QO)^+}}{c_O} = 2\,\frac{j + 2q}{(j+1)^2} \,, \quad \frac{c_{(QO)^-}}{c_O} = 2\,\frac{(j+1)(2q - j - 2)}{j} \,, \quad \frac{c_{(\bar{Q}O)}}{c_O} = 4\,\bar{q} \,. \tag{4.26} $$

The other set of identities that we will need are those associated to the generators of

supersymmetries $S_{\mu\alpha}, \bar{S}_{\mu\dot{\alpha}}$. Let us fix the third operator to be O. We then have three choices: $\mathsf{t}_{(Q\bar{O})\bar{S}O}$ and $\mathsf{t}_{(\bar{Q}\bar{O})^{\pm}SO}$. The topological operator obtained by integrating S or $\bar{S}$ over Σ is precisely the supercharge Q or $\bar{Q}$ respectively. We thus readily obtain the following identities

$$\begin{aligned}
\frac{i}{2}\int_{\Sigma} \mathrm{d}\Omega(x_{23})\, x_{23}^2\, \partial_{\eta_2} \mathsf{x}_{23} \partial_{\bar{\eta}_2}\, \mathsf{t}_{(Q\bar{O})\bar{S}O}(x_i;\eta_i,\bar{\eta}_i) &= 2\,\bar{\eta}_2 \partial_{\bar{\eta}_3}\, \mathsf{n}_{(Q\bar{O})(\bar{Q}O)}(x_{13},\eta_{1,3},\bar{\eta}_{1,3})\,,\\
\frac{i}{2}\int_{\Sigma} \mathrm{d}\Omega(x_{23})\, x_{23}^2\, \partial_{\eta_2} \mathsf{x}_{23} \partial_{\bar{\eta}_2}\, \mathsf{t}_{(\bar{Q}\bar{O})^{+}SO}(x_i;\eta_i,\bar{\eta}_i) &= 2\,\eta_2 \partial_{\eta_3}\, \mathsf{n}_{(\bar{Q}\bar{O})^{+}(QO)^{+}}(x_{13},\eta_{1,3},\bar{\eta}_{1,3})\,,\\
\frac{i}{2}\int_{\Sigma} \mathrm{d}\Omega(x_{23})\, x_{23}^2\, \partial_{\eta_2} \mathsf{x}_{23} \partial_{\bar{\eta}_2}\, \mathsf{t}_{(\bar{Q}\bar{O})^{-}SO}(x_i;\eta_i,\bar{\eta}_i) &= \frac{2j}{j+1}\eta_2\eta_3\, \mathsf{n}_{(\bar{Q}\bar{O})^{-}(QO)^{-}}(x_{13},\eta_{1,3},\bar{\eta}_{1,3})\,.
\end{aligned} \tag{4.27}$$

The two-point functions must be normalized according to (4.26). All the results are summarized in Tables C.5, C.6 and C.7.

4.4.4 Shortening conditions

The possible shortening conditions have been reviewed in Section 2.1. For simplicity we will refer to $L\,\bar{B}$ as *chirality*. Furthermore the conditions $\mathcal{X}_L\,\bar{A}_1$ are absent because we are considering the case $\bar{\jmath}=0$. Since $\overline{\mathcal{O}}=\mathcal{O}^*$, $\overline{\mathcal{O}}$ will satisfy the conjugate shortening $\mathcal{X}_R\,\overline{\mathcal{X}}_L$. However, after imposing reality, either one of the two conditions is sufficient.

We can impose the shortening conditions directly on the t thanks to (2.24) and its conjugate. This leads to

$$B_1 \text{ or } \bar{B}_1\colon \qquad \bar{\eta}_1\overline{\mathcal{D}}\, t(Z;\eta_i,\bar{\eta}_i)=0\,, \qquad \eta_1\mathcal{D}\, t(Z;\eta_i,\bar{\eta}_i)=0\,, \tag{4.28a}$$

$$A_1\colon \qquad \partial_{\eta_1}\mathcal{D}\, t(Z;\eta_i,\bar{\eta}_i)=0\,, \tag{4.28b}$$

$$A_2 \text{ or } \bar{A}_2\colon \qquad \mathcal{D}^2\, t(Z;\eta_i,\bar{\eta}_i)=0\,, \qquad \overline{\mathcal{D}}^2\, t(Z;\eta_i,\bar{\eta}_i)\,. \tag{4.28c}$$

In Table 4.2 we summarize all the constraints arising from (4.28a), (4.28b) and (4.28c). All shortening conditions can be easily obtained by combining them. Table 4.1 instead shows how many independent coefficients are left in the superspace correlator as we choose different shortening conditions and impose all other constraints obtained before.

4.5 Expansion of the superspace correlator

In order to apply the various constraints originating from the ANEC to our three-point function in superspace we need to express its components in a basis of non-supersymmetric three-point functions. This will be achieved by Taylor expanding in the Grassmann coordinates $\theta_i, \bar{\theta}_i$. The techniques that can be used to perform such a task have been discussed at length in Chapter 3.

	$\bar{L}$	$\bar{A}_2$	$\bar{B}$
L	2	2	0
A_1	1	1	/

(a) $j > 1$

	$\bar{L}$	$\bar{A}_2$	$\bar{B}$
L	1	1	0
A_1	1	1	0

(b) $j = 1$

	$\bar{L}$	$\bar{A}_2$	$\bar{B}$
L	0	0	0
A_2	0	0	0
B	0	0	**0**

(c) $j = 0$

Table 4.1. Number of independent coefficients $\mathcal{C}_k$ of the superspace correlator as different shortening conditions are chosen. The slash means that there is no consistent three-point function. The boldface zero means that the three-point function is identically zero. Other zeros imply that the three-point function is completely fixed in terms of q, $\bar{q}$ and j. In all cases these numbers refer to *real* degrees of freedom as the $\mathcal{C}_k$ are either all real or all purely imaginary.

	Constraints	Conditions
A_1	$\mathcal{C}_6 = (j-1)\mathcal{C}_3 + \frac{j(j-1)}{j+1}(\mathcal{C}_5 - 4\mathcal{C}_1)$, $\mathcal{C}_7 = -2\mathcal{C}_2 + \mathcal{C}_3 + j\mathcal{C}_4 + \frac{2j}{j+1}(\mathcal{C}_5 + (j-3)\mathcal{C}_1)$, $\mathcal{C}_8 = 4\mathcal{C}_2 - \mathcal{C}_3 + \frac{2j}{1+j}(4\mathcal{C}_1 - \mathcal{C}_5)$, $\mathcal{C}_{10} = j\mathcal{C}_9 = j\mathcal{C}_4 + \frac{j}{2}(\mathcal{C}_3 + \mathcal{C}_5)$.	$j \geqslant 1, q = \frac{1}{2}j + 1$
A_2	$\mathcal{C}_9 = \mathcal{C}_4 + \frac{1}{2}\mathcal{C}_5$.	$j = 0, q = 1$
$\bar{A}_2$	$\mathcal{C}_9 = -\frac{1}{2}(\mathcal{C}_3 + \mathcal{C}_5) - \mathcal{C}_4$, $\mathcal{C}_{10} = -\frac{1}{2}(\mathcal{C}_6 + \mathcal{C}_8) - \mathcal{C}_7$.	$\bar{q} = 1$
B	$\mathcal{C}_4 = -2\mathcal{C}_1$, $\mathcal{C}_5 = 4\mathcal{C}_1$, $\mathcal{C}_9 = 0$.	$j = 0, q = 0$
$\bar{B}$	$\mathcal{C}_4 = 2\mathcal{C}_1$, $\mathcal{C}_5 = -4\mathcal{C}_1$, $\mathcal{C}_7 = 2\mathcal{C}_2$, $\mathcal{C}_8 = -4\mathcal{C}_2$, $\mathcal{C}_3 = \mathcal{C}_6 = \mathcal{C}_9 = \mathcal{C}_{10} = 0$.	$\bar{q} = 0$

Table 4.2. Constraints on the coefficients $\mathcal{C}_k$ following from the various shortening conditions on the multiplet $\mathcal{O}$ (here $\bar{\jmath} = 0$ is implicit). Case A_1 for $j = 1$ and cases $\bar{A}_2$ and $\bar{B}$ for $j = 0, 1$ can be obtained by setting to zero the absent coefficients ($\mathcal{C}_6$ for $j = 1$ and $\mathcal{C}_{2,3,6,7,8,10}$ for $j = 0$).

4.5.1 Lowest order

At this order we simply have $J^{(R)}$. Consistently with the previous sections we denote the three-point function coefficients by

$$\mathsf{t}_{\bar{O}JO} \longrightarrow C_k \,. \tag{4.29}$$

The results, without assuming the reality condition and conservation, are shown in Table C.8.

4.5.2 Three-point function $\langle \bar{O}TO \rangle$

At order $\theta_2\bar{\theta}_2$ we have the stress-energy tensor. Consistently with the previous sections we denote the three-point function coefficients by

$$\mathsf{t}_{\bar{O}TO} \longrightarrow D_k \,. \tag{4.30}$$

The results are shown in Table C.9. The conservation of J and the reality condition are not assumed there. In principle the expansion also contains superdescendants of J of spin $(0,0)$, $(0,2)$ and $(2,0)$. We checked that those contributions vanish after imposing conservation and we will not report those results here.

4.5.3 Three-point functions $\langle (\bar{Q}\bar{O})SO \rangle$ and $\langle (Q\bar{O})\bar{S}O \rangle$

At order $\theta_1\bar{\theta}_2$, $\bar{\theta}_1\theta_2$ we have the supersymmetry current with the first superdescendant of $\bar{\mathcal{O}}$. The naming of the coefficients is

$$\mathsf{t}_{(\bar{Q}\bar{O})^+SO} \longrightarrow E_k \,, \qquad \mathsf{t}_{(\bar{Q}\bar{O})^-SO} \longrightarrow F_k \,, \qquad \mathsf{t}_{(Q\bar{O})\bar{S}O} \longrightarrow G_k \,.$$

Also in these cases the results are presented without conservation and reality applied — they can be found in Tables C.10, C.11 and C.12. There are also contributions from superdescendants of spin $(0,1)$ or $(1,0)$. As in the previous subsection we have verified that they vanish after conservation is imposed and we will not report those results.

4.5.4 Three-point functions $\langle (\bar{Q}\bar{O})J(QO) \rangle$ and $\langle (Q\bar{O})J(\bar{Q}O) \rangle$

At order $\theta_1\bar{\theta}_3$, $\bar{\theta}_1\theta_3$ we extract the descendants QO, $\bar{Q}O$ and their conjugates. We need this mainly as a preliminary result for the computation of the next subsection. We named

$$\begin{aligned} &\mathsf{t}_{(\bar{Q}\bar{O})^+J(QO)^+} \longrightarrow N_k \,, \qquad \mathsf{t}_{(\bar{Q}\bar{O})^+J(QO)^-} \longrightarrow O_k \,, \\ &\mathsf{t}_{(\bar{Q}\bar{O})^-J(QO)^+} \longrightarrow P_k \,, \qquad \mathsf{t}_{(\bar{Q}\bar{O})^-J(QO)^-} \longrightarrow Q_k \,, \end{aligned} \tag{4.31}$$

where

$$\mathsf{t}_{(Q\bar{O})J(\bar{Q}O)} \longrightarrow I_k\,. \tag{4.32}$$

In order to make the computation more manageable, this time we applied conservation and reality from the start. The results are in Tables C.13, C.14, C.15, C.16 and C.17.

4.5.5 Three-point functions $\langle(\bar{Q}\bar{O})T(QO)\rangle$ and $\langle(Q\bar{O})T(\bar{Q}O)\rangle$

At order $\theta_1\theta_2\bar{\theta}_2\bar{\theta}_3$, $\bar{\theta}_1\theta_2\bar{\theta}_2\theta_3$ we extract the descendants $QO, \bar{Q}O$ and their conjugates coupled with the stress tensor. These terms are needed in order to impose the ANEC on superconformal descendants inside $\mathcal{O}$. We named

$$\begin{aligned}
&\mathsf{t}_{(\bar{Q}\bar{O})^+T(QO)^+} \longrightarrow J_k\,, \qquad \mathsf{t}_{(\bar{Q}\bar{O})^+T(QO)^-} \longrightarrow K_k\,,\\
&\mathsf{t}_{(\bar{Q}\bar{O})^-T(QO)^+} \longrightarrow L_k\,, \qquad \mathsf{t}_{(\bar{Q}\bar{O})^-T(QO)^-} \longrightarrow M_k\,,
\end{aligned} \tag{4.33}$$

$$\mathsf{t}_{(Q\bar{O})T(\bar{Q}O)} \longrightarrow H_k\,. \tag{4.34}$$

Also this time we applied conservation and reality from the start. The results are in Tables C.18, C.19, C.20, C.21 and C.22.

4.6 The averaged null energy condition

Following [141, 163] we define the state $|\psi\rangle$ of (4.1) by acting with some operator $O(x,\eta,\bar{\eta})$ on the CFT vacuum $|0\rangle$. Then we take the Fourier transform in order to give the state a definite momentum,[9] which can be set to $q^\mu = (1,\mathbf{0})$ without loss of generality. Next we multiply by $(x^+)^2/16$ and send $x^+ \to \infty$ to simplify the computations. Lastly we need to specify a polarization, but using the auxiliary spinors η and $\bar{\eta}$ we can obtain all possible polarizations at once.

The ANEC integral breaks rotation invariance to an SO(2) generated by $\sigma^{12}{}_\alpha{}^\beta$ and $\bar{\sigma}^{12\dot{\alpha}}{}_{\dot{\beta}}$ in the respective representations. Under a φ rotation of this subgroup, fundamental spinors with a lower index transform as follows:

$$\begin{pmatrix} a \\ b \end{pmatrix}_\alpha \longrightarrow \begin{pmatrix} a\,e^{-i\varphi/2} \\ b\,e^{i\varphi/2} \end{pmatrix}_\alpha, \qquad \begin{pmatrix} \bar{a} \\ \bar{b} \end{pmatrix}_{\dot{\alpha}} \longrightarrow \begin{pmatrix} \bar{a}\,e^{i\varphi/2} \\ \bar{b}\,e^{-i\varphi/2} \end{pmatrix}_{\dot{\alpha}}. \tag{4.35}$$

This will help us in the following way: in principle, if there are s choices for the polarization of O and $\bar{O}$ one would have to apply the ANEC integral to each pair of choices, diagonalize an $s \times s$ matrix and require the positivity of each eigenvalue (or equivalently

[9] Due to translational invariance, Fourier transforming in both x_1 and x_3 will lead to a divergent answer. This can be fixed by using Gaussian wavepackets and taking the limit of plane waves in the end. Alternatively we can simply keep the third point fixed and integrate in x_{13} only.

require semidefinite positiveness of an $s \times s$ matrix). This rotational symmetry reduces the matrix to a block diagonal form, making much simpler the study of its positiveness.

4.6.1 Operators of spin $(j, 0)$

Let us focus first on the case where $O(x, \eta, \bar{\eta})$ has spin $(j, 0)$. We can expand the η's in the eigenbasis of the SO(2) spin,

$$\eta_3^\alpha = \begin{pmatrix} m \\ p \end{pmatrix} := m\, \xi_-^\alpha + p\, \xi_+^\alpha\,, \qquad \bar{\eta}_1^{\dot{\alpha}} = \begin{pmatrix} \bar{p} \\ \bar{m} \end{pmatrix} := \bar{p}\, \bar{\xi}_+^{\dot{\alpha}} + \bar{m}\, \bar{\xi}_-^{\dot{\alpha}}\,, \tag{4.36}$$

where the redundancy $\xi_\pm = \bar{\xi}_\mp$ has been introduced for convenience. The stress tensor is instead polarized along the null geodesic u^μ, which is translated to

$$\eta_2^\alpha = \xi_-^\alpha\,, \qquad \bar{\eta}_2^{\dot{\alpha}} = \bar{\xi}_+^{\dot{\alpha}}\,. \tag{4.37}$$

Now we can perform the ANEC integral (4.1) with the prescriptions defined above on an arbitrary three-point function $\mathsf{t}_{\bar{O}TO}$.[10] We define $x_{13} = x$, $x_{23} = y$ and

$$\mathcal{A}[\mathsf{t}_{\bar{O}TO}] := \int_{-\infty}^{\infty} \mathrm{d}y^- \lim_{y^+\to\infty} \frac{(y^+)^2}{16} \int_{\mathbb{R}^4} \mathrm{d}^4x\, e^{-ix^0}\, \mathsf{t}_{\bar{O}TO}(x, y; \bar{\eta}_1, \eta_2, \bar{\eta}_2, \eta_3)\Big|_{\substack{\bar{\eta}_1,\eta_3\to(4.36)\\ \bar{\eta}_2,\eta_2\to(4.37)}}\,. \tag{4.38}$$

In order to enforce the correct ordering, the integral in y^- must be supplemented with the appropriate $i\epsilon$ prescription, namely $y^0 \to y^0 - i\epsilon$ and $x^0 \to x^0 - 2i\epsilon$. The integrals and the limit $y^+ \to \infty$ remove all dependence on the points x, y. The result is therefore a polynomial in the variables $p, m, \bar{p}$ and $\bar{m}$. The same considerations apply for the norm of the state, which is computed by Fourier transforming the two-point function

$$\mathcal{F}[\mathsf{n}_{\bar{O}O}] := \int_{\mathbb{R}^4} \mathrm{d}^4x\, e^{-ix^0}\, \mathsf{n}_{\bar{O}O}(x; \bar{\eta}_1, \eta_3)\Big|_{\bar{\eta}_1,\eta_3\to(4.36)}\,. \tag{4.39}$$

The restrictions imposed by SO(2) invariance imply that only certain terms can appear, i.e.

$$\mathcal{A}[\mathsf{t}_{\bar{O}TO}] = \sum_{s=0}^{j} \mathcal{A}_s[\mathsf{t}_{\bar{O}TO}]\, (p\bar{m})^s (m\bar{p})^{j-s}\,, \qquad \mathcal{F}[\mathsf{n}_{\bar{O}O}] = \sum_{s=0}^{j} \mathcal{F}_s[\mathsf{n}_{\bar{O}O}]\, (p\bar{m})^s (m\bar{p})^{j-s}\,. \tag{4.40}$$

Each coefficient of this polynomial corresponds to a different choice for the polarizations

[10] The conventions are

$$x^+ = x^0 + x^3 = \xi_- \mathrm{x}\, \bar{\xi}_+\,, \qquad x^- = x^0 - x^3 = \xi_+ \mathrm{x}\, \bar{\xi}_-\,, \qquad x^2 = -x^+x^- + \vec{x}_\perp^2\,.$$

of O and $\bar{O}$, therefore the polarization matrix is diagonal and the ANEC states

$$\mathcal{E}[\Delta;(j,0);s] := \frac{\mathcal{A}_s[\mathsf{t}_{\bar{O}TO}]}{\mathcal{F}_s[\mathsf{n}_{\bar{O}O}]} \geqslant 0\,, \qquad \text{for } s=0,\dots,j\,. \tag{4.41}$$

The integrals have been computed explicitly for some values of j in [141]. Here we provide a general formula, whose proof can be found in Appendix C.2:

$$\begin{aligned}\mathcal{E}[\Delta;(j,0);s] = \frac{3\pi(-i)^j}{8}\frac{(\delta-1)(\delta+j)}{(\delta+j-s-1)_3}\Bigg(D_1 + \frac{j-s}{j}\frac{\delta+j-1}{\delta+j-s-2}D_2 + \\ + \frac{(j-s-1)_2}{(j-1)_2}\frac{(\delta-j-2)_2}{(\delta+j-s-3)_2}D_3\Bigg)\,,\end{aligned} \tag{4.42}$$

where $\delta = \Delta - \frac{1}{2}j - 1$ and $(a)_n = \Gamma(a+n)/\Gamma(a)$ is the Pochhammer symbol. See Table C.3 for the meaning of the three-point function coefficients. For the special cases $j=0,1$ it suffices to set to zero the absent coefficient(s). Note that (4.42) is real because the coefficients D_i are purely real (resp. imaginary) if j is even (resp. odd).

4.6.2 ANEC on a superposition of states

In the previous subsection the operator O could have been either the superconformal primary or the first superdescendant $QO^{\pm}$. However, these operators mix with each other, i.e. the three-point function $\langle(\bar{Q}\bar{O})^+T(QO)^-\rangle$ is nonzero. This means that we can impose an even stronger constraint by demanding positivity on the general superposition

$$|\psi\rangle = \frac{v\,(QO)^+|0\rangle}{|\langle(\bar{Q}\bar{O})^+(QO)^+\rangle|^{1/2}} + \frac{w\,(QO)^-|0\rangle}{|\langle(\bar{Q}\bar{O})^-(QO)^-\rangle|^{1/2}}\,. \tag{4.43}$$

A similar approach was used in [164]. Since v and w can be chosen arbitrarily, the ANEC now becomes a semidefinite-positiveness constraint on a $2(j+1)\times 2(j+1)$ matrix. Such a matrix can be decomposed in j blocks of size 2×2 and two 1×1 blocks, resulting in

$$\begin{aligned}\begin{pmatrix}\mathcal{E}[\Delta+\frac{1}{2};(j+1,0);s+1] & \mathcal{E}_{\text{int}}[\Delta+\frac{1}{2};(j\pm1,0);s] \\ \mathcal{E}_{\text{int}}[\Delta+\frac{1}{2};(j\pm1,0);s] & \mathcal{E}[\Delta+\frac{1}{2};(j-1,0);s]\end{pmatrix} \succeq 0 \qquad \text{for } s=0,\dots,j-1\,, \\ \mathcal{E}[\Delta+\tfrac{1}{2};(j+1,0);s] \geqslant 0 \qquad \text{for } s=0,j+1\,.\end{aligned} \tag{4.44}$$

The diagonal entries have the same expression as (4.42) with the substitution $D_i \to J_i$ or $D_i \to M_i$ (see Tables C.18, C.21), together with the appropriate redefinition of δ. The

"interference" terms $\mathcal{E}_{\text{int}}$ are defined as follows:[11]

$$\begin{aligned}&\mathcal{E}_{\text{int}}[\Delta+\tfrac{1}{2};(j+1,0);s]\\&\quad=\mathcal{E}_{\text{int}}[\Delta+\tfrac{1}{2};(j-1,0);s]\equiv\frac{\mathcal{A}_s[\mathsf{t}_{(\bar{Q}\bar{O}^+)T(QO^-)}]}{\left(\mathcal{F}_{s+1}[\mathsf{n}_{(\bar{Q}\bar{O}^+)(QO^+)}]\,\mathcal{F}_s[\mathsf{n}_{(\bar{Q}\bar{O}^-)(QO^-)}]\right)^{1/2}}\,.\end{aligned}\tag{4.45}$$

Following steps similar to the ones illustrated in Appendix C.2 one can prove the general formula

$$\begin{aligned}\mathcal{E}_{\text{int}}[\Delta+\tfrac{1}{2};(j\pm1,0);s]=\frac{3\pi(-i)^{j-1}}{16}\sqrt{\frac{\delta(s+1)(j-s)}{j(j+1)(\delta+j+1)}}\frac{(\delta+j-1)_3}{(\delta+j-s-2)_4}\times\\\times\left(\frac{\delta+j-s-2}{\delta+j-1}K_1+\frac{j-s-1}{j-1}K_2\right),\end{aligned}\tag{4.46}$$

where the coefficients $K_i=L_i$ are defined in Tables C.19, C.20 and $\delta=\Delta_{QO}-\frac{1}{2}j-\frac{3}{2}$. Here $\Delta_{QO}=\Delta+\frac{1}{2}$ is the dimension of the superdescendant. The polarization s takes values from 0 to $j-1$.

4.6.3 Operators of spin $(j,1)$

The only difference when considering more general $\mathrm{SO}(1,3)$ representations is that the polarization matrix will not be diagonal. This means that the ANEC will not be a set of simple inequalities but rather semidefinite positiveness constraints. In the $(j,1)$ case we further have to specify the polarizations η_1 and $\bar{\eta}_3$; thus together with (4.36) and (4.37) one has

$$\eta_1^\alpha=\begin{pmatrix}m'\\p'\end{pmatrix}\equiv m'\,\boldsymbol{\xi}_-^\alpha+p'\,\boldsymbol{\xi}_+^\alpha\,,\qquad\bar{\eta}_3^{\dot{\alpha}}=\begin{pmatrix}\bar{p}'\\\bar{m}'\end{pmatrix}\equiv\bar{p}'\,\bar{\boldsymbol{\xi}}_+^{\dot{\alpha}}+\bar{m}'\,\bar{\boldsymbol{\xi}}_-^{\dot{\alpha}}\,.\tag{4.47}$$

The ANEC integral for an arbitrary operator O of spin $(j,1)$ takes the form

$$\tilde{\mathcal{A}}[\mathsf{t}_{\bar{O}TO}]\equiv\int_{-\infty}^{\infty}\mathrm{d}y^-\lim_{y^+\to\infty}\frac{(y^+)^2}{16}\int_{\mathbb{R}^4}\mathrm{d}^4x\,e^{-ix^0}\,\mathsf{t}_{\bar{O}TO}(x,y;\eta_{1,2,3},\bar{\eta}_{1,2,3})\Big|_{\substack{\bar{\eta}_1,\eta_3\to(4.36)\\\bar{\eta}_3,\eta_1\to(4.47)\\\bar{\eta}_2,\eta_2\to(4.37)}}.\tag{4.48}$$

We also define $\tilde{\mathcal{F}}[\mathsf{n}_{\bar{O}O}]$ in a similar way. The constraints of $\mathrm{SO}(2)$ invariance allow us to express

$$\tilde{\mathcal{A}}[\mathsf{t}_{\bar{O}TO}]=\sum_{s=0}^{j+1}\sum_{a,b=0}^{1}\left(\tilde{\mathcal{A}}_s[\mathsf{t}_{\bar{O}TO}]\right)_{ab}(p\bar{m})^s(m\bar{p})^{j-s}p'\bar{m}'\left(\frac{\bar{p}m'}{\bar{m}p'}\right)^a\left(\frac{m\bar{p}'}{p\bar{m}'}\right)^b,\tag{4.49}$$

[11] The definition of $\mathcal{A}_s$ for the interference correlator is similar to (4.40) with the difference that we pick up the term $\bar{m}\bar{p}(p\bar{m})^s(m\bar{p})^{j-s-1}$ for $\langle(\bar{Q}\bar{O})^+T(QO)^-\rangle$ and $mp(p\bar{m})^s(m\bar{p})^{j-s-1}$ for its conjugate.

and similarly for $\tilde{\mathcal{F}}[\mathsf{n}_{\bar{O}O}]$. The terms for $s = 0$ and $s = j + 1$ are restricted to, respectively, $a = b = 0$ and $a = b = 1$. Thus we can see that the polarization matrix is block diagonal with j blocks of size 2×2 and two blocks of size 1×1. Defining

$$\big(\mathcal{E}[\Delta;(j,1);s]\big)_{ab} \equiv \frac{(\tilde{\mathcal{A}}_s[\mathsf{t}_{\bar{O}TO}])_{ab}}{\big((\tilde{\mathcal{F}}_s[\mathsf{n}_{\bar{O}O}])_{aa}\,\tilde{\mathcal{F}}_s[\mathsf{n}_{\bar{O}O}])_{bb}\big)^{1/2}}\,, \tag{4.50}$$

the positivity constraints are

$$\begin{aligned} &\mathcal{E}[\Delta;(j,1);s] \succeq 0\,, \qquad \text{for } s = 1,\dots,j\,,\\ &\mathcal{E}[\Delta;(j,1);s] \geqslant 0\,, \qquad \text{for } s = 0, j+1\,. \end{aligned} \tag{4.51}$$

In the next subsection we will explain how to implement a numerical study of this system of inequalities. We obtained a general formula for $\mathcal{E}[\Delta;(j,1);s]$ as well—unfortunately, however, the expression is too unwieldy to be reported here. In Appendix C.2 we briefly explain how to obtain it.

4.6.4 The ANEC as a semidefinite programming problem

Imposing semidefinite positiveness on a symmetric matrix is a well known problem for which there exist algorithms that go under the name of *semidefinite programming*. We will make use of the implementation realized by the software `sdpb` [162], which was developed for the numerical bootstrap approach for the study of CFTs [147], but is general purpose enough to work for our problem too. For an introduction to semidefinite programming in the context of the conformal bootstrap see Section 6.2.

In general we need to solve a system of inequalities

$$\mathcal{E}[\Delta;(j,\bar{\jmath});s] \succeq 0\,, \qquad \text{for } s = 0,\dots,j+\bar{\jmath}\,, \tag{4.52}$$

where $\mathcal{E}[\Delta;(j,\bar{\jmath});s]$ is a symmetric $m_s \times m_s$ matrix with $m_s = \min\{j,\,\bar{\jmath},\,s,\,j+\bar{\jmath}-s\}+1$. The matrices $\mathcal{E}$ will depend on N arbitrary three-point function coefficients (given by Table 4.1) plus an inhomogeneous part which is fixed by the Ward identities. Dropping the Δ and $(j,\bar{\jmath})$ labels for brevity one has

$$\mathcal{E}[s] = \mathcal{E}^{(0)}[s] + \sum_{n=1}^{N} \lambda_n\,\mathcal{E}^{(n)}[s] \succeq 0\,, \qquad \text{for } s = 0,\dots,j+\bar{\jmath}\,. \tag{4.53}$$

This is known as the *dual* formulation of a semidefinite problem. We are interested in studying the feasibility of (4.53). The algorithm we used only terminates when either a solution λ_n is found, or when a numerical threshold for the internal computations[12] is exceeded. For our purposes, a problem that terminates for the latter condition is

[12] Called `-maxComplementarity`.

considered to have no solution. This means that our ANEC-disallowed points are not disallowed in a mathematically rigorous way. We expect this to not have any practical consequences.[13]

4.6.5 Details on ANEC bounds: non-supersymmetric case

Let us briefly review the results obtained in [141] and prove a few results for generic values of j. First let us consider conformal primaries in the $(j,0)$ Lorentz representation. The ANEC condition is expressed by the formula (4.42), where the coefficients D_i are given in Table C.3. In particular, one can take $\hat{D}_1 = -i^j D_1$ to be the only independent real coefficient. By choosing the value $s = 0$ and $s = j$ in (4.42) and restricting to the case $j > 2$ for simplicity we obtain

$$(\delta - 1)\Big((\pi^2\hat{D}_1 - 4)\delta + j(\pi^2\hat{D}_1 + 2\delta - 6) + 2j^2 + 4\Big) \geqslant 0\,, \qquad \hat{D}_1 \geqslant 0\,, \tag{4.54}$$

where $\delta \geqslant 0$ represents the distance from the unitarity bound. It is straightforward to verify that the above conditions cannot be simultaneously satisfied unless $\delta \geqslant 1$.

By considering all polarizations we can obtain stronger bounds at the price of fixing the value of j, for instance by using the function `Reduce` of *Mathematica*. We show our results in Figure 4.7 up to $j = 10^3$. Although the bound initially agrees with the conjecture of [141], it departs from it for $j \geqslant 21$ and follows a different pattern which is well fitted by the expression $\Delta = \frac{1}{2}j + 1 + \delta \geqslant \frac{1}{15}(13j + 42)$. It would be tempting to assign a meaning to the kink at $j \sim 21$, but the explanation might simply reside in the fact that, going to large values, the integer nature of j becomes less and less important and new solutions for $\hat{D}_1$ become available.

Let us now move to the case of conformal primaries in the $(j,1)$ representation. The procedure to obtain the general formula is described in Appendix C.2.2. After imposing the Ward identities, whose solution is reported in Table C.4, one is left with four independent three-point function coefficients H_i. In order to systematically address the feasibility of the ANEC we translated the linear matrix inequality into a semidefinite problem as discussed in the previous subsection. We found agreement with the results of [141] for $j \leqslant 7$ and extended the bounds up to $j = 50$. A lower bound on Δ as a function of j is shown in Figure 4.8: again we observe that for $j > 21$ the bounds departs from the conjecture $\Delta \geqslant j$ of [141] and closely follows the bound $\Delta \geqslant \frac{1}{15}(13j + 42)$ instead.

In the case of conserved operators the problem simplifies considerably: only two coeffi-

[13] In principle there is also a way to mathematically prove that no solutions exist by providing a certificate of infeasibility [165]. By using [166–168] this amounts to finding a solution of another (larger) semidefinite problem.

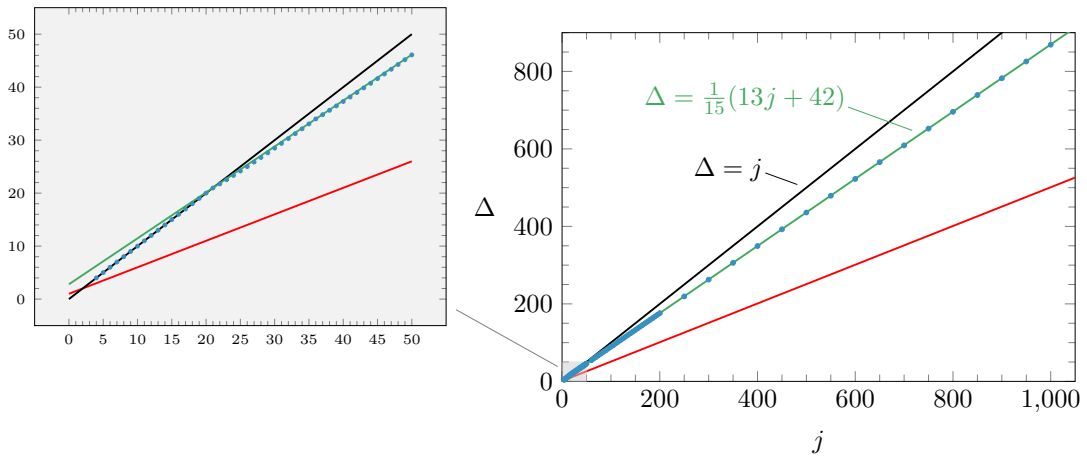


Figure 4.7. Lower bounds on the conformal dimension Δ as a result of the ANEC for primaries transforming in the $(j,0)$ Lorentz representation. Each point is the result of a bisection in Δ. The red line is the unitarity bound, $\Delta = \frac{1}{2}j + 1$. The black line corresponds to the conjecture of [141], $\Delta = j$, and the green line gives an approximate behavior of the bound valid above $j = 20$.

cients remain independent[14] and we can easily prove that conserved currents cannot exist for $j > 5$. For instance, we can take $\hat{H}_{9,10} = -i^{j+1} H_{9,10}$ to be the two independent real coefficients. By considering the eigenvalues of matrices with $s = j-3, \dots, j$ and the condition at $s = j+1$, we obtain the following set of inequalities:

$$\begin{aligned} &\hat{H}_{10} \geqslant 0\,, \qquad 3\hat{H}_9 + \frac{18}{\pi^2}\frac{j-1}{j+1} \leqslant \hat{H}_{10}\frac{2j+1}{j-1}\,, \\ &\hat{H}_9 \leqslant \frac{2}{3}\hat{H}_{10}\,, \qquad 3\hat{H}_9 + \frac{12}{\pi^2} \geqslant 2\hat{H}_{10}\frac{j+1}{j-1}\,. \end{aligned} \tag{4.55}$$

One can immediately check that the above conditions admit a solution only for $j \leqslant 5$, corresponding to the cases when conserved currents can be constructed in free theories. Interestingly, for the boundary case $j = 5$ the solution to the ANEC is unique:

$$\hat{H}_9 = -\frac{4}{\pi^2}\,, \qquad \hat{H}_{10} = 0\,. \tag{4.56}$$

It is easy to construct conserved operators explicitly out of free fields for the $(j,1)$ case when $j \leqslant 5$. Denoting with ϕ a complex boson, ψ a Weyl fermion and F a self-dual field strength one has

$$\begin{aligned} &(1,1): && \bar{\psi}\gamma_\mu\psi\,, && \phi^*\partial_\mu\phi\,, && \Delta = 3\,, \\ &(2,1): && \bar{\psi}_{\dot{\alpha}}F_{\alpha\beta}\,, && \psi_\alpha\partial_{\beta\dot{\alpha}}\phi\,, && \Delta = \tfrac{7}{2}\,, \\ &(3,1): && \psi_\alpha\partial_{\beta\dot{\alpha}}\psi_\gamma\,, && && \Delta = 4\,, \end{aligned}$$

[14] The relation imposed by conservation of the operator O can be easily computed using the package `CFTs4D`.

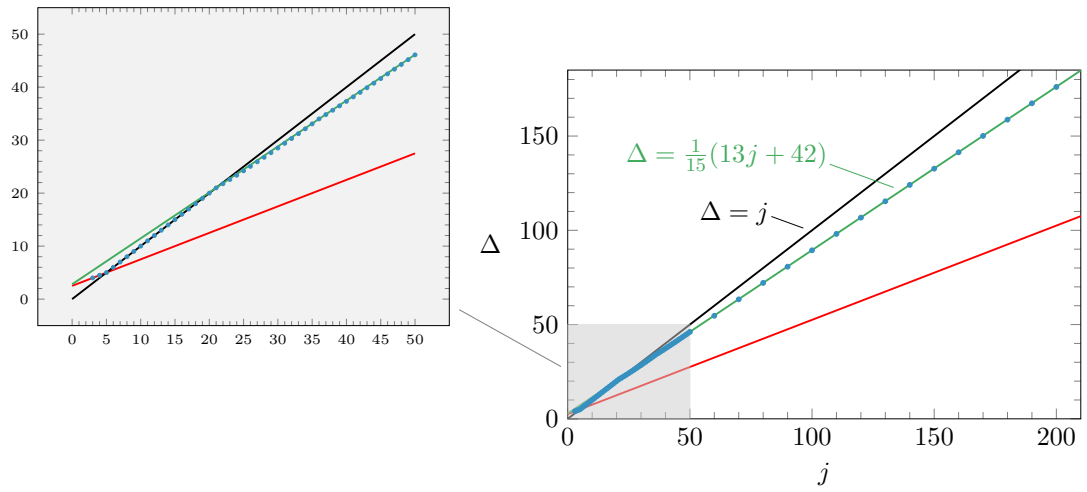


Figure 4.8. Lower bounds on the conformal dimension Δ as a result of the ANEC for primaries transforming in the $(j,1)$ Lorentz representation. Each point is the result of a bisection in Δ. The red line is the unitarity bound, $\Delta = \frac{1}{2}j + \frac{5}{2}$. The operators for $j \leqslant 5$ lie on the red line. The black line corresponds to the conjecture of [141], $\Delta = j$, and the green line gives an approximate behavior of the bound valid above $j = 20$.

$$
\begin{aligned}
&(4,1): && \psi_\alpha \partial_{\beta\dot{\alpha}} F_{\gamma\delta}\,, && \Delta = \tfrac{9}{2}\,,\\
&(5,1): && F_{\alpha\beta}\partial_{\gamma\dot{\alpha}} F_{\delta\varepsilon}\,, && \Delta = 5\,.
\end{aligned}
$$

Similar attempts for constructing conserved operators for $j > 5$ will fail, as one would expect from the ANEC bound.

4.6.6 Details on ANEC bounds: supersymmetric case

In the supersymmetric case the analysis follows the same steps as before, except that now one needs to combine multiple conditions. Let us discuss some of the results presented in the introduction. We first start from a multiplet whose zero component transforms in the $(j,0)$ representation and satisfies the $L\,\bar{B}$ shortening condition. These are the generalizations to $j > 1$ of the usual chiral scalar and gauge-invariant spin-$\frac{1}{2}$ multiplets. In this case $\bar{q} = 0$ and $q = \Delta$. The multiplet contains only four conformal primaries: O, $QO^{\pm}$ and Q^2O. In this work we only consider the first three. As discussed in Section 4.4.4 the superspace three-point function does not have any free parameters. Let us consider, then, the ANEC applied to the superprimary only. The condition is again encoded in (4.42), where now the coefficients D_i are related to the superspace coefficients through the relations in Table C.9, supplemented by the relations in Table 4.2. The analog of $\hat{D}_1 \geqslant 0$ in (4.54) is now simply

$$2q - 3j \geqslant 0\,. \tag{4.57}$$

We explicitly checked that including other constraints does not strengthen the bound. This is expected since one can construct chiral operators with $\Delta = \frac{3}{2}j$ by taking products of free chiral vector multiplets. The bound is therefore optimal.

Let us move to another simple case, namely $A_1\,\bar{A}_2$, corresponding to superprimaries again in the $(j,0)$ representation with $q = \frac{1}{2}j + 1$ and $\bar{q} = 1$. This multiplet contains conserved operators in the $(j+1,1)$ and, due to the results of the previous subsection, we can immediately conclude that $j \leqslant 4$. It turns out, however, that $j = 4$ is excluded since the values $\hat{H}_9$ and $\hat{H}_{10}$ fixed by supersymmetry do not satisfy (4.56). Smaller values of j must be consistent since these operators appear in the decomposition of extended supersymmetry multiplets in the free limit.

All other bounds found in this work were obtained with a numerical approach. For completeness we collect here all the conditions we imposed in the most complicated case $L\,\bar{L}$. In simpler cases some of them do not appear since the corresponding superdescendant is absent. At the same time, the correct three-point function coefficient relations must be imposed. Given an $L\,\bar{L}$ supermultiplet with a superprimary transforming in the $(j,0)$ representation and $q \geqslant j/2+1$, $\bar{q} \geqslant 1$, the ANEC can be satisfied if there exist real coefficients $\hat{\mathcal{C}}_k = i^j \mathcal{C}_k$, $k = 2, 6$, such that

$$
\begin{aligned}
&\langle \bar{O}TO\rangle : \\
&\quad \mathcal{E}[\Delta,(j,0);s] \geqslant 0\,, \qquad \text{for } s = 0,\dots,j\,, \\
&\langle (\bar{Q}\bar{O})T(QO)\rangle : \\
&\begin{pmatrix} \mathcal{E}[\Delta+\frac{1}{2};(j+1,0);s+1] & \mathcal{E}_{\text{int}}[\Delta+\frac{1}{2};(j\pm1,0);s] \\ \mathcal{E}_{\text{int}}[\Delta+\frac{1}{2};(j\pm1,0);s] & \mathcal{E}[\Delta+\frac{1}{2};(j-1,0);s] \end{pmatrix} \succeq 0 \quad \text{for } s = 0,\dots,j-1\,, \\
&\mathcal{E}[\Delta+\tfrac{1}{2};(j+1,0);s] \geqslant 0 \qquad\qquad\qquad\qquad\qquad\qquad\quad \text{for } s = 0, j+1\,, \\
&\langle (Q\bar{O})T(\bar{Q}O)\rangle : \\
&\quad \mathcal{E}[\Delta+\tfrac{1}{2};(j,1);s] \succeq 0\,, \qquad \text{for } s = 0,\dots,j+1\,.
\end{aligned}
\tag{4.58}
$$

As usual we defined $\Delta = q + \bar{q}$. Whenever the above system of conditions does not admit a solution, we conclude that the corresponding supersymmetry multiplet cannot exist in a local unitary SCFT.

4.7 Bounds on extended supersymmetry multiplets

4.7.1 Conventions

The aim of this section is to constrain the superconformal multiplets of theories with $\mathcal{N} > 1$ supersymmetry by decomposing them into $\mathcal{N} = 1$ multiplets. This approach does not make use of the additional linear relations among the three-point function

coefficients and thus may not yield optimal bounds.

We consider for any $\mathcal{N}$ the superalgebra generated by Q^1_α and $\bar{Q}_{1\dot{\alpha}}$. The embedding of the $\mathcal{N}=1$ $\mathfrak{u}(1)$ R-charge in the larger R-symmetry group is[15]

$$\begin{aligned} \mathcal{N}=2: &\qquad r_{\mathcal{N}=1} = -\tfrac{4}{3}R_3 + \tfrac{1}{3}r_{\mathcal{N}=2}\,, \\ \mathcal{N}=4: &\qquad r_{\mathcal{N}=1} = -\tfrac{1}{3}(3H_1+2H_2+H_3)\,, \end{aligned} \tag{4.59}$$

where H_i is the Cartan generator associated to the i-th Dynkin label in $[p_1,p_2,p_3]$. The generator R_3 is the $\mathfrak{su}(2)$ Cartan in units of $\frac{1}{2}$ ($R_3 = -R/2,\dots,R/2$).

4.7.2 $\mathcal{N}=2$

Let us start by considering the so-called "exotic chiral primaries," namely the $L\bar{B}_1[j;0]^{(0,r)}_\Delta$ multiplets, with $\Delta=\frac{1}{2}r$.[16] The bound on chiral multiplets (4.9) for the $\mathcal{N}=1$ subalgebra generated by Q^1_α, applied to the chiral superprimary $Q^2_{(\alpha_1}\mathcal{O}^{\text{exotic}}_{\alpha_2\dots\alpha_{j+1})}$ implies that

$$\Delta+\tfrac{1}{2}\geqslant\tfrac{3}{2}(j+1) \quad\Rightarrow\quad \Delta\geqslant\tfrac{3}{2}j+1\,. \tag{4.60}$$

The unitarity bound is $\Delta\geqslant\frac{1}{2}j+1$, and so we see that the ANEC bound is stronger for $j>0$.

A similar argument can be made on operators with nonzero $\mathfrak{su}(2)$ R-charge $L\bar{B}_1[j;0]^{(R,r)}_\Delta$, where $\Delta=R+\frac{1}{2}r$ and R is in integer units. We considered several values of R and performed the decomposition into $\mathcal{N}=1$ multiplets. Imposing (4.9) on each of the chiral multiplets that appear yields the following pattern (which we conjecture to be true for arbitrary R):

$$r\geqslant 3j+2-2R \qquad\Rightarrow\qquad \Delta\geqslant\tfrac{3}{2}j+1\,. \tag{4.61}$$

This is stronger than unitarity ($r\geqslant j+2$) for $j>R$. As a consequence, short multiplets of the form $A_\ell\bar{B}_1[j;0]^{(R,r)}_\Delta$ are only allowed for $j\leqslant R$.

The multiplets $A_1\bar{B}_1[j;0]^{(1,j+2)}_\Delta$ and $A_1\bar{A}_2[j;0]^{(0,j)}_\Delta$ with $\Delta=\frac{1}{2}j+2$ are absent from any local SCFT for $j>2$. This is a consequence of the presence of an $A_1\bar{A}_2[j+1;0]$ multiplet in their $\mathcal{N}=1$ decomposition, which we have shown to be forbidden by the ANEC when $j+1>3$.

We also considered long multiplets $L\bar{L}[j;0]^{(R,r)}_\Delta$ for some values of R. Calling δ the

[15] For $\mathcal{N}=2$ this embedding is the same as the one explained in Subsection 2.1.2

[16] Denoted $\bar{\mathcal{E}}_{\frac{r}{2}(j,0)}$ in [13].

difference of their dimension and their unitarity bound,

$$\delta = \Delta - 2 - j - R + \tfrac{1}{2} r \,, \tag{4.62}$$

and calling $f(R, j)$ the separation between the unitarity and the ANEC bound in Figure 4.4, we find the following pattern

$$\delta \geqslant f\big(\tfrac{1}{3}(r+1), j+1\big) - R \,. \tag{4.63}$$

4.7.3 $\mathcal{N} = 4$

We considered a few short multiplets and found no constraints from the ANEC. Interestingly, $B_1\bar{B}_1[0;0]_2^{(1,0,1)}$ contains a chiral multiplet that saturates (4.9), namely

$$B_1\bar{B}_1[0;0]_2^{(1,0,1)} \supset L\bar{B}_1[2;0]_3^{(2)} \,. \tag{4.64}$$

The simplest long multiplet is the Konishi multiplet $L\bar{L}[0;0]^{(0,0,0)}$. In its $\mathcal{N} = 1$ decomposition we find a long multiplet of spin $(3,0)$ and R-charge 1 with dimension $\Delta_{\text{Konishi}} + \frac{3}{2}$. In terms of the Q^1_α subalgebra, calling ϕ the Konishi operator, one has

$$\mathcal{O}_{\alpha_1\alpha_2\alpha_3} = \varepsilon_{1IJK} Q^I_{(\alpha_1} Q^J_{\alpha_2} Q^K_{\alpha_3)} \phi \,. \tag{4.65}$$

Since in perturbation theory one can compute $\Delta_{\text{Konishi}} = 2 + O(g^2)$, we see that the ANEC and the unitarity bound for $\mathcal{N} = 1$ long multiplets of spin $(3,0)$ are saturated.

More generally, we checked some cases of long multiplets $L\bar{L}[j;0]^{(p_1,p_2,p_3)}$, namely those with Dynkin labels $[p_1, p_2, p_3] = [0,0,0]$, $[0,2,0]$ and $[1,0,1]$. Calling δ the difference of their dimension and their unitarity bound,

$$\delta = \Delta - 2 - j - \tfrac{1}{2}(3p_1 + 2p_2 + p_3) \,, \tag{4.66}$$

and calling $f(R, j)$ the separation between the unitarity and the ANEC bound in Figure 4.4, we find

$$\begin{aligned} [0,0,0] &: \quad \delta \geqslant f\big(\tfrac{4}{3}, j+2\big) - 2 \,, \\ [0,2,0] &: \quad \delta \geqslant f\big(\tfrac{7}{3}, j+3\big) - 4 \,, \\ [1,0,1] &: \quad \delta \geqslant f\big(\tfrac{7}{3}, j+3\big) - 4 \,. \end{aligned} \tag{4.67}$$

5 Spinning chiral primaries

This brief chapter discusses the fate of a class of superconformal chiral primaries in $\mathcal{N}=2$ SCFTs which have been dubbed "exotic" primaries. It is based on Section 5 of **Paper III**.

5.1 Introduction

The presence of a conserved stress tensor gives non trivial constraints on the operator spectrum. We already saw in Chapter 4 the consequences of the ANEC. Sometimes even simply applying the Ward identities could lead to strong results. Here we will consider a class of operators called "exotic chiral primaries." They are consistent with the representation theory of four dimensional $\mathcal{N}=2$ superconformal symmetry but are forbidden from any local SCFT due to the stress tensor Ward identities. This result requires the use of the differential operators defined in Chapter 3.[1]

The chiral operators are $\mathcal{N}=2$ superconformal multiplets that satisfy an $L\bar{B}_1$ shortening condition.[2] That means that they are annihilated by the $\bar{Q}$ supercharges

$$[\bar{Q}_{I\dot{\alpha}}\,,\,\mathcal{X}(\mathbf{z})\}=0\,,\qquad I=1,2\,. \tag{5.1}$$

This dictates that they must have spin $(j,0)$, with j any non-negative integer, and their conformal dimension must be half their $\mathfrak{u}(1)$ R-charge r. In terms of the q, $\bar{q}$ charges in (2.5) one has $q=r/2$ and $\bar{q}=0$. The exotic primaries $\mathcal{X}$ are defined as those chiral operators with $j>0$ that are $\mathfrak{su}(2)$ singlets. If $j=0$ the chiral operators are often called Coulomb branch operators. There exist also chiral operators with non-vanishing $\mathfrak{su}(2)$ R-charge but we will not consider those here.

[1] We should point out that the differential operator needed for the proof, while being a particular case of the ones defined in this paper, was already known from [117].

[2] In the notation of [13] this shortening condition is denoted as $\bar{\mathcal{E}}$.

The name "exotic" reflects the fact that such operators were proven to be absent from a very large class of theories. Arguments that excluded them from certain theories of class $\mathcal{S}$ appeared in [169].[3] The more general result of [132] instead excludes them from:

i) theories with a Lagrangian description,
ii) theories related to Lagrangian ones via a generalized Argyres-Seiberg-Gaiotto duality [28, 170],
iii) theories that flow to an IR Lagrangian theory via an $\mathcal{N} = 2$–preserving deformation.

The proof that will be shown in the next section is logically independent from these earlier results. Therefore we will not review them.

5.2 Absence of spinning chiral primaries in $\mathcal{N} = 2$ SCFTs

We start by constructing a three-point function of $\mathcal{X}$, its conjugate[4] $\overline{\mathcal{X}}$ and the stress tensor multiplet $\mathcal{J}$. Let us choose the following parametrization

$$\langle \overline{\mathcal{X}}(\mathbf{z}_1)\, \mathcal{J}(z_2)\, \mathcal{X}(\mathbf{z}_3)\rangle = \mathcal{K}_{\overline{\mathcal{X}}\mathcal{J}}\, t_{\mathcal{X}}^{\overline{\mathcal{X}}\mathcal{J}}(Z_3, \chi_1, \eta_3)\,. \tag{5.2}$$

The function $t_{\mathcal{X}}^{\overline{\mathcal{X}}\mathcal{J}}$ must be chiral at point z_1, for example. This implies

$$\overline{\mathcal{D}}^I_{\dot{\alpha}}\, t_{\mathcal{X}}^{\overline{\mathcal{X}}\mathcal{J}}(Z_3, \chi_1, \eta_3) = 0\,, \qquad I = 1, 2\,. \tag{5.3}$$

By using the representation (A.32a) of the differential operators one can see that $t_{\mathcal{X}}^{\overline{\mathcal{X}}\mathcal{J}}$ may only depend on X_3 and Θ_3^I. However, since the sum of R-charges is zero there cannot be an isolated Θ_3^I and thus $t_{\mathcal{X}}^{\overline{\mathcal{X}}\mathcal{J}}$ is a function of X_3, which can be fixed by scaling (B.3) up to an overall constant.

$$t_{\mathcal{X}}^{\overline{\mathcal{X}}\mathcal{J}}(Z_3, \eta_1, \eta_3) = \mathcal{A}\, \frac{(\eta_1\eta_3)^j}{{X_3}^2}\,. \tag{5.4}$$

It is also easy to verify that the conservation of $\mathcal{J}$ is satisfied. The conservation operators are $(\mathcal{Q}_I)^2$ and $(\overline{\mathcal{Q}}^I)^2$. If we rewrite (5.4) as a function of $\overline{X}_3$, Θ_{3I}, $\overline{\Theta}_3^I$ we see that there are no terms of order $\Theta_3^2\overline{\Theta}_3^2$. This means that we can use the representation (A.33a) for $(\overline{\mathcal{Q}}^I)^2$ and the representation (A.33b) for $(\mathcal{Q}_I)^2$ and get trivially zero in both cases.

The proof now consists in showing that (5.4) does not satisfy the stress tensor Ward identities unless $j = 0$. In order to obtain the Ward identities we have to expand

[3] Theories of class $\mathcal{S}$ have been introduced in [28, 29]. They are obtained by compactifying a $6d$ SCFT on a two dimensional puntured Riemann surface. Most of them do not admit a Lagrangian.

[4] The conjugate has R-charge $-r$, spin $(0, j)$ and charges $q = 0$, $\bar{q} = r/2$. It satisfies a $B_1\bar{L}$ shortening.

this function in components and extract the contribution from $T(\mathbf{x}_2)$, the stress tensor, and $J(\mathbf{x}_2)$, the R-symmetry current. Then impose the equalities in (1.45) and (1.44). Fortunately, however, half of the work has already been done. It suffices to extract the contribution from the Ferrara-Zumino multiplet $J(\mathbf{z}_2)$ inside of $\mathcal{J}(\mathbf{z}_2)$. Then we can use the results of Chapter 4 to apply the Ward identities. More precisely, we need the tables C.1, C.3, C.8 and C.9 subject to the constraints in the last line of Table 4.2.

We will keep denoting as $\mathcal{X}$ the $\mathcal{N}=1$ primary appearing as the lowest order in the exotic operator multiplet. Extracting the contribution of the Ferrara-Zumino gives

$$\langle\overline{\mathcal{X}}(\mathbf{z}_1)\,J(\mathbf{z}_2)\,\mathcal{X}(\mathbf{z}_3)\rangle = \mathcal{K}_{\overline{\mathcal{X}}(\mathcal{Q}^2\bar{\mathcal{Q}}_{\underline{2}}\mathcal{J})}\,\mathfrak{Q}^{++}_{\mathcal{Q}\bar{\mathcal{Q}}}\,t^{\overline{\mathcal{X}}\mathcal{J}}_{\mathcal{X}}(Z_3,\chi_1,\eta_3)\big|_0\,, \tag{5.5}$$

with the definition $\mathfrak{Q}^{++}_{\mathcal{Q}\bar{\mathcal{Q}}} = \mathfrak{D}^{++}_{\mathcal{Q}\bar{\mathcal{Q}}}\big|_{\mathcal{D}\to\mathcal{Q}}$, $\mathfrak{D}$ defined in (3.52) and $\mathcal{Q}$ defined in (A.33). The result is

$$\mathfrak{Q}^{++}_{\mathcal{Q}\bar{\mathcal{Q}}}\,t^{\overline{\mathcal{X}}\mathcal{J}}_{\mathcal{X}}(Z_3,\eta_1,\eta_3)\big|_0 = -\frac{4i}{3}\mathcal{A}\,(\eta_1\eta_3)^j\,\frac{\eta_2 X_3\bar{\eta}_2}{{X_3}^4}\,, \tag{5.6}$$

where now X_3 follows the $\mathcal{N}=1$ definition. A three-point function of an $\mathcal{N}=1$ chiral operator instead reads[5]

$$t^{\overline{\mathcal{X}}J}_{\mathcal{X}}(Z_3,\eta_1,\eta_3) = i\,A_1\,(\eta_1\eta_3)^j\,\frac{\eta_2 X_3\bar{\eta}_2}{{X_3}^4} + i\,A_2\,(\eta_1\eta_3)^{j-1}\,\frac{\eta_1 X_3\bar{\eta}_2\,\eta_2\eta_3}{{X_3}^4}\,. \tag{5.7}$$

The comparison between (5.6) and (5.7) is straightforward and yields

$$A_1 = -\frac{4}{3}\mathcal{A}\,,\qquad A_2 = 0\,, \tag{5.8}$$

while the Ward identities for chiral operators require

$$A_1 = i^j\,\frac{r-3j}{3\pi^2}\,,\qquad A_2 = i^j\,\frac{2j}{\pi^2}\,. \tag{5.9}$$

This immediately implies that there are no solutions for $j>0$ and thus the exotic primaries cannot couple consistently with the stress tensor and must be absent from any local theory. It also tells us that if $j=0$ then $\mathcal{A}$ is fixed to be

$$\mathcal{A} = -\frac{r}{4\pi^2}\,. \tag{5.10}$$

As a check of our formalism we expanded (5.4) to higher orders in the supercharges and extracted also the contributions from $\mathcal{Q}^{\underline{2}}\mathcal{X}$ and $\mathcal{Q}^{\underline{2}2}\mathcal{X}$, which are all $\mathcal{N}=1$ $L\bar{B}_1$ chiral multiplets. We verified that for $j=0$ all the components satisfy the Ward identities when $\mathcal{A}$ takes the value in (5.10).

[5] In the basis of Chapter 4 the coefficients $A_{1,2}$ are translated into the coefficients $\mathcal{C}_{1,\dots,10}$ as follows:

$$\mathcal{C}_1 = \tfrac{1}{2}\mathcal{C}_4 = -\tfrac{1}{4}\mathcal{C}_5 = A_1 + A_2,\qquad \mathcal{C}_2 = \tfrac{1}{2}\mathcal{C}_7 = -\tfrac{1}{4}\mathcal{C}_8 = -A_2\,,\qquad \mathcal{C}_{3,6,9,10} = 0\,.$$

Abelian currents Part III

6 Conformal bootstrap

In the previous chapters we described all the elements for performing a thorough analysis of the kinematics of a conformal or superconformal field theory. This has already produced the results that we showed in Chapter 4 and 5. However, the main motivation behind this thesis is different: we want to perform a bootstrap study of (S)CFTs in four dimensions. The work that we did so far will be fundamental to this aim. However, before diving in our case study, namely that of four abelian currents, we would like to introduce to the reader the main ideas behind the numerical bootstrap.

The notions explained in the first two sections are very general and can be found in any review of the conformal bootstrap (e.g. [144–147]). The rest of the chapter instead focuses mainly on four dimensions.

6.1 Crossing equations

As we discussed in Chapter 1, operators in a CFT always admit an operator product expansion, or OPE. In a four-point function there is more than one way to do it and they must all agree. This gives a set of nontrivial constraints which go under the name of crossing equations. We denote the OPE between $\mathcal{O}_1$ and $\mathcal{O}_2$ as a contraction

$$\overbrace{\mathcal{O}_1(\mathbf{x}_1)\mathcal{O}_2(\mathbf{x}_2)} := \sum_{\rho}\sum_{a=1}^{n_{12\rho}} \frac{\lambda^{(a)}_{\mathcal{O}_1\mathcal{O}_2\mathcal{O}_\rho}}{|x_{12}|^{\Delta_1+\Delta_2-\Delta}} C_\rho^{(a)}(\partial_\mu, x_{12}, \partial_\eta)\mathcal{O}_{\rho^*}(0,\eta)\,, \tag{6.1}$$

where, $\mathbf{x}_i = (x_i, \eta_i)$,[1] η is a polarization that contracts the indices of $\mathcal{O}_{\rho^*}$ and a is an index that runs over all three-point tensor structures. $\rho = (\Delta, \ell, \dots)$ is a label that contains the quantum numbers of the conformal group (dimension and spin) and eventual other global symmetries that might appear in the theory under consideration, whereas ρ^* stands for the conjugate representation. We also abbreviated $n(\mathcal{O}_1 \cdots \mathcal{O}_k)$ defined in

[1] We will suppress the dependence on η_i in C_ρ for brevity.

(1.33) as $n_{1\dots k}$. As explained before, $C_\rho(\partial_\mu, x, \partial_\eta)$ is a function that can be completely fixed by conformal symmetry but its precise form will not be important to us. With this notation in mind, any four-point function should satisfy the following constraints

$$\langle \overbrace{\mathcal{O}_1 \mathcal{O}_2} \overbrace{\mathcal{O}_3 \mathcal{O}_4} \rangle = \langle \overbrace{\mathcal{O}_1 \underbrace{\mathcal{O}_2 \mathcal{O}_3}} \mathcal{O}_4 \rangle = \langle \overbrace{\mathcal{O}_1 \underbrace{\mathcal{O}_2} \mathcal{O}_3} \mathcal{O}_4 \rangle \,. \tag{6.2}$$

These three different choices for the OPE lead to different expansions of the four-point function. Indeed, after we apply the definition in (6.1), we get the following expressions

$$\langle \overbrace{\mathcal{O}_i \mathcal{O}_j} \overbrace{\mathcal{O}_k \mathcal{O}_l} \rangle = \sum_{\rho,a,b} \lambda^{(a)}_{\mathcal{O}_i \mathcal{O}_j \mathcal{O}_\rho} \lambda^{(b)}_{\mathcal{O}_{\rho^*} \mathcal{O}_k \mathcal{O}_l} \frac{C^{(a)}_\rho(\partial_\mu, x_{ij}, \partial_\eta)}{|x_{ij}|^{\Delta_i+\Delta_j-\Delta}} \frac{I_\rho(x_{jl}, \eta, \eta')}{|x_{jl}|^{2\Delta}} \frac{C^{(b)}_{\rho^*}(\bar{\partial}_\mu, x_{kl}, \bar{\partial}'_\eta)}{|x_{kl}|^{\Delta_k+\Delta_l-\Delta}} \,, \tag{6.3}$$

where the quantity $I(x, \eta, \eta')$ is the two-point function tensor structure. There is only one sum over ρ because the two-point function $\langle \mathcal{O}_\rho \mathcal{O}_{\rho'} \rangle$ is proportional to $\delta_{\rho' \rho^*}$. The expression that multiplies the OPE coefficient is called conformal partial wave, which in turn can be written as a sum of conformal blocks once a basis of four-point tensor structures is fixed. In practice, however, the conformal blocks are computed by other, more efficient, methods that do not require the explicit form of C_ρ. We will discuss them in Section 6.4.

Consider now a basis of tensor structures $\mathbb{T}_i$ so that the four-point function reads

$$\langle \mathcal{O}_1(\mathbf{x}_1) \mathcal{O}_2(\mathbf{x}_2) \mathcal{O}_3(\mathbf{x}_3) \mathcal{O}_4(\mathbf{x}_4) \rangle = \sum_{i=1}^{n_{1234}} \mathbb{T}_i(\mathbf{x}_1, \dots, \mathbf{x}_4) \, f_i(u, v) \,, \tag{6.4}$$

where u and v have been defined in (1.17). An explicit definition of the $\mathbb{T}_i$ in the case of four dimensions will be given in Section 6.3. In the OPE $\langle \overbrace{\mathcal{O}_1 \mathcal{O}_2} \overbrace{\mathcal{O}_3 \mathcal{O}_4} \rangle$ the functions f_i can be expanded as

$$f_i(u, v) = \sum_\rho \sum_{a=1}^{n_{12\rho}} \sum_{b=1}^{n_{\rho^* 34}} \lambda^{(a)}_{\mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_\rho} \lambda^{(b)}_{\mathcal{O}_{\rho^*} \mathcal{O}_3 \mathcal{O}_4} \, G^{a,b}_{i,\rho}(u, v) \,, \tag{6.5}$$

and $G^{a,b}_{i,\rho}(u, v)$ is what we call a conformal block. The simplest way to impose the crossing equations is to always consider the OPE among the first two and the last two operators and then permute the order of the points. The permutations that leave the OPE invariant are symmetries of either the cross ratios or the blocks themselves. More precisely, the group S_4 is generated by[2] π_{12}, π_{13} and π_{34}. The permutations

$$\pi_{12}\pi_{34} \,, \qquad \pi_{13}\pi_{24} \,, \qquad \pi_{14}\pi_{23} \,, \tag{6.6}$$

leave the OPE untouched and u and v are mapped to themselves (note that the last one

[2] We use the notation: π_{ij} to indicate the permutation that swaps $i \leftrightarrow j$. Then the product $\pi_{ij}\pi_{kl} \cdots$ is just the composition where the leftmost element acts last.

is, in fact, the composition of the first two). These are called kinematic permutations. On the other hand π_{12} and π_{34} do change u and v, but the blocks are mapped to a multiple of themselves. For the conformal blocks of four identical scalars for example one has [96][3]

$$G_{\Delta,\ell}(u,v) \xrightarrow{\pi_{12} \text{ or } \pi_{34}} G_{\Delta,\ell}\left(\tfrac{u}{v},\tfrac{1}{v}\right) = (-1)^{\ell} G_{\Delta,\ell}(u,v)\,. \tag{6.7}$$

This leaves us with only π_{13} as a nontrivial permutation. Its action on u and v is rather simple, it simply swaps them: $u \leftrightarrow v$. Let us first see how it acts on the tensor structures. The permutation applied to any of the $\mathbb{T}_i$ in general will be a linear combination of the structures of the permuted four-point function. Thus letting

$$\langle \mathcal{O}_3(\mathbf{x}_1)\mathcal{O}_2(\mathbf{x}_2)\mathcal{O}_1(\mathbf{x}_3)\mathcal{O}_4(\mathbf{x}_4)\rangle = \sum_{i=1}^{n_{1234}} \widetilde{\mathbb{T}}_i(\mathbf{x}_1,\dots,\mathbf{x}_4)\,\tilde{f}_i(u,v)\,, \tag{6.8}$$

one has

$$\widetilde{\mathbb{T}}_i\big|_{1\leftrightarrow 3} = \sum_{j=1}^{n_{1234}} R_{ij}\,\mathbb{T}_j\,, \tag{6.9}$$

where R is a matrix satisfying $R^2 = \mathbb{1}$. Since $\langle \mathcal{O}_3\mathcal{O}_2\mathcal{O}_1\mathcal{O}_4\rangle|_{\mathbf{x}_1\leftrightarrow\mathbf{x}_3} = \langle \mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\mathcal{O}_4\rangle$, one has a condition like (6.9) on the functions f_i as well. We can always choose a basis of eigenvectors for R so that the first n_o functions are mapped to minus themselves and the remaining ones are mapped to themselves. With this choice of basis we obtain the following list of equations

$$\sum_{\rho,a,b} \lambda^{(a)}_{\mathcal{O}_1\mathcal{O}_2\mathcal{O}_\rho}\lambda^{(b)}_{\mathcal{O}_{\rho^*}\mathcal{O}_3\mathcal{O}_4}\,G^{a,b}_{i,\rho}(u,v) = \sum_{\tilde\rho,a,b} \lambda^{(a)}_{\mathcal{O}_3\mathcal{O}_2\mathcal{O}_{\tilde\rho}}\lambda^{(b)}_{\mathcal{O}_{\tilde\rho^*}\mathcal{O}_1\mathcal{O}_4}\,\widetilde{G}^{a,b}_{i,\tilde\rho}(v,u)\cdot\begin{cases}1 & i\leqslant n_o\\ -1 & i>n_o\end{cases}\,, \tag{6.10}$$

where we called $\widetilde{G}$ the conformal block of the permuted four-point function. On its own this constraint is not very powerful because we have different sets of OPE coefficients on either side. That is why in the case of non identical operators one has to take into account all possible correlators that can be made out of the operators considered. Then the equality above will be just a part of the entire system crossing equations. Since our case will consist in four identical operators, let us from now on consider $\mathcal{O}_i = \mathcal{O}$ for all i. For convenience let us group all OPE coefficients $\lambda^{(a)}_{\mathcal{O}\mathcal{O}\mathcal{O}_\rho}$ in a vector $\vec{\lambda}_\rho$. Then we can write the crossing equations in the form

$$\begin{aligned}
&\sum_{\rho} \vec{\lambda}_\rho^{\,T}\cdot\mathcal{V}_{i,\rho}\cdot\vec{\lambda}_\rho = 0\,,\\
&\big(\mathcal{V}_{i,\rho}(u,v)\big)_{ab} = F^{a,b}_{-;i,\rho}(u,v) \coloneqq G^{a,b}_{i,\rho}(u,v) - G^{a,b}_{i,\rho}(v,u)\,, \qquad i\leqslant n_o\,,\\
&\big(\mathcal{V}_{i,\rho}(u,v)\big)_{ab} = F^{a,b}_{+;i,\rho}(u,v) \coloneqq G^{a,b}_{i,\rho}(u,v) + G^{a,b}_{i,\rho}(v,u)\,, \qquad i> n_o\,,
\end{aligned} \tag{6.11}$$

[3]This does not mean that the crossing equations for π_{12} or π_{34} are useless. In this particular case, for instance, they require all OPE coefficients of the operators of odd spin to vanish.

where we defined $F_\pm$ as the even and odd combinations of the conformal blocks. We can always isolate one operator from the sum, namely the identity $\rho = (0, 0, \dots)$. The number of three-point structures $n(\mathcal{O}\mathcal{O}\mathbb{1})$ is of course 1 and the OPE coefficients are unit normalized. So the vector of matrices degenerates to an ordinary vector and (6.11) can be rewritten as

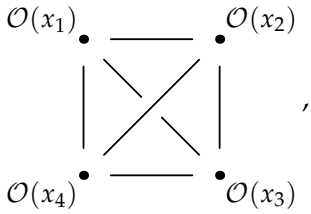
$$V_{i,\mathbb{1}} + \sum_{\rho \neq 0} \vec{\lambda}_\rho^T \cdot \mathcal{V}_{i,\rho} \cdot \vec{\lambda}_\rho = 0\,, \qquad V_{i,\mathbb{1}} \coloneqq \left(\mathcal{V}_{i,(0,0,\dots)}\right)_{1,1}\,. \tag{6.12}$$

This form of the crossing equations is very useful. It is part of a very general class of problems that go under the name of convex optimization. More precisely, it is a linear programming problem if the matrices $F_\pm$ are all one by one and is a semidefinite programming problem otherwise. Finding exact or approximate solutions to (6.11) is often out of the question, however there are several techniques to extract useful information from the CFT and they will all be addressed in the following section.

Before concluding this section let us introduce a convenient set of variables that replace u and v:

$$u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2} = z\bar{z}\,, \qquad v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2} = (1-z)(1-\bar{z})\,. \tag{6.13}$$

In these variables the permutation π_{13} sends $z, \bar{z} \to 1-z, 1-\bar{z}$. The point $z = \bar{z} = \frac{1}{2}$ is special because it corresponds to a crossing symmetric configuration



$$, \qquad \begin{aligned} x_{12}^2 = x_{23}^2 = x_{34}^2 = x_{41}^2 = 1\,, \\ x_{13}^2 = x_{24}^2 = 2\,. \end{aligned} \tag{6.14}$$

It also provides the best compromise for the convergence of both OPE channels, namely $\langle \overbrace{\mathcal{O}\mathcal{O}}\overbrace{\mathcal{O}\mathcal{O}}\rangle$ and $\langle \overbrace{\mathcal{O}\underbrace{\mathcal{O}\mathcal{O}}\mathcal{O}}\rangle$. Indeed the former (the "s" channel) converges well for $z, \bar{z} \sim 0$ and the latter (the "t" channel) converges well for $z, \bar{z} \sim 1$.

We warn the reader that often we will switch from one set of variables to the other. When writing, for instance, $G_{i,\rho}^{a,b}(z, \bar{z})$ we implicitly mean $G_{i,\rho}^{a,b}(u(z, \bar{z}), v(z, \bar{z}))$.

6.2 Semidefinite programming

The first step to address (6.11) is to discretize the equations. This is done by Taylor expanding the functions $F_\pm(z, \bar{z})$ that appear in the crossing equations around a fixed point, retaining only a finite number of coefficients. As discussed before, the best choice is the point $z = \bar{z} = 1/2$. This still leaves an infinite sum over Δ and ℓ. The sum is thus truncated to a maximal value of the spin and the Taylor coefficients $(\partial_z^n \partial_{\bar{z}}^m F(z, \bar{z}))|_{z=\bar{z}=\frac{1}{2}}$

are approximated by rational functions of Δ. After the truncation we end up with a system of equation of the form

$$\sum_{\rho}\sum_{a,b}(\lambda_\rho)_a\,(\mathcal{V}_{I,\rho})_{a,b}\,(\lambda_\rho)_b = -V_{I,\mathbb{1}}\,, \qquad \forall\, I\,. \tag{6.15}$$

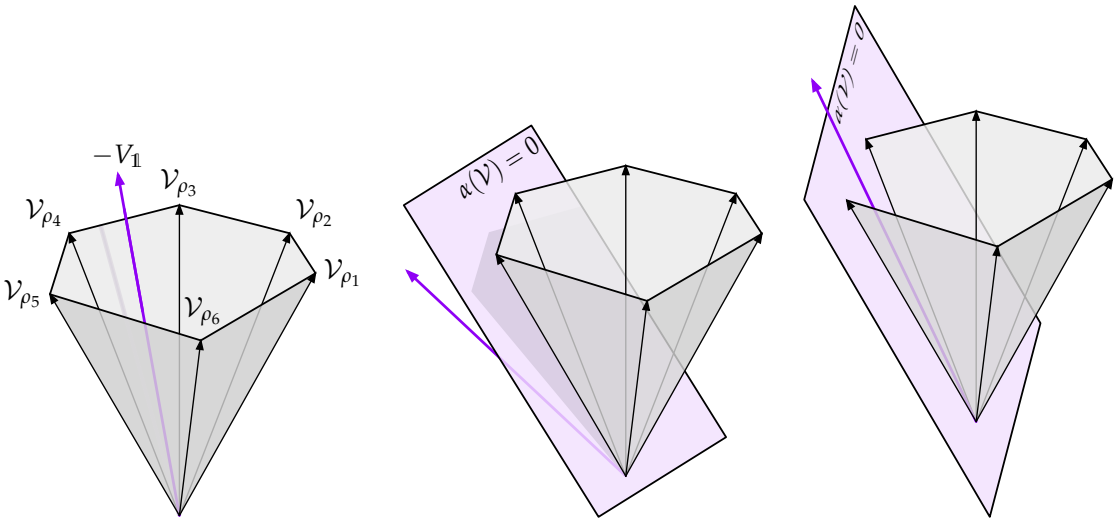
The index I runs over all independent four-point structures times the number of Taylor coefficients kept in the discretization of $F_\pm(z,\bar{z})$. The right hand side of the equation is obtained from the vector $V_{i,\mathbb{1}}$ of the identity (6.12). The strategy of the conformal bootstrap is to try and rule out possible candidate solutions to (6.11) by showing that they lead to a contradiction. One typically starts from a set of assumptions that depend on a small number of parameters and tries to "carve out" regions in parameter space. The contradictions may be found with the following general strategy: suppose that there exists a linear functional α_I such that

$$\begin{aligned}&\sum_I \alpha_I\,\mathcal{V}_{I,\rho} \succeq 0\,, \qquad \forall\,\rho\in\{\text{assumptions}\}\,,\\ &\sum_I \alpha_I\,V_{I,\mathbb{1}} = 1\,.\end{aligned} \tag{6.16}$$

This looks like an infinite set of constraints because we have not restricted the conformal dimensions Δ to a finite set, therefore ρ may assume a continuum of values. Recall, however, that the functions appearing in $\mathcal{V}$ are rational and their denominator is known. Thus one simply has to impose positivity on a polynomial in Δ over some interval, typically of the form $[\Delta_0,\infty)$. This can be done rigorously with a computer since the polynomials can be implemented as matrices. With this approach, even the case with a single correlator requires a semidefinite programming setup. If there exists such an α, then, by contracting α_I with (6.15), one would obtain $0 \leqslant \sum_\rho \lambda_\rho^T(\alpha\cdot\mathcal{V}_\rho)\lambda_\rho = -1$, namely a contradiction. There is a nice geometrical interpretation of this fact. We are simply asking whether the vector $-V_{\mathbb{1}}$ lies or not in the convex hull formed by the non-negative linear combinations of the vectors $\mathcal{V}_\rho$.[4] If the answer is no, then we will never be able to satisfy (6.12) and the assumptions we made are inconsistent. This also means that there is a plane that separates the vector $-V_{\mathbb{1}}$ from all other vectors $\mathcal{V}_\rho$. The normal to this plane will have a non-negative scalar product with all $\mathcal{V}_\rho$'s and a negative scalar product with $-V_{\mathbb{1}}$. Therefore the plane defines the functional α. This is shown in Figure 6.1.

There is an important detail that we have glossed over: until now, the formulas we presented implicitly assumed that the operators $\mathcal{O}_\rho$ were non degenerate. Meaning that there is only one operator per representation $\rho = (\Delta,\ell,\ldots)$. This is generically true, but we do not want to rely on this fact. For instance, we want to be agnostic about possible additional global symmetries that have not been accounted for in the bootstrap equations. In order to fix this issue, let us notice that the operators that share

[4] For this geometrical picture, let us assume that $\mathcal{V}_\rho$ is a vector of one by one matrices and thus the sum is of the form $\sum\lambda_\rho^2\mathcal{V}_\rho$ with $\lambda_\rho^2 \geqslant 0$.



(a) The identity block is inside the convex hull of the conformal blocks.

(b) The identity block is outside the convex hull.

(c) The identity block lies on the boundary of the convex hull.

Figure 6.1. Different scenarios for the bootstrap equation. In case **(b)** there is a plane that separates the identity from the other blocks, which defines a functional α. In case **(c)** some blocks lie on the plane and thus the functional has zeros.

the same quantum numbers ρ enter in the sum (6.5) with the same conformal blocks. This motivates defining the following quantity,[5]

$$P_\rho^{b,a} := \sum_{\mathcal{O}_\rho} \lambda^{(a)}_{\mathcal{O}\mathcal{O}\mathcal{O}_\rho} \lambda^{(b)}_{\mathcal{O}_{\rho^*}\mathcal{O}\mathcal{O}}\,, \tag{6.17}$$

where we sum over all operators in the same representation ρ. It is always possible to define the three-point function basis so that $\lambda^{(a)*}_{\mathcal{O}\mathcal{O}\mathcal{O}_\rho} = \lambda^{(a)}_{\mathcal{O}_{\rho^*}\mathcal{O}\mathcal{O}}$, which in turn implies that

$$P_\rho \succeq 0\,. \tag{6.18}$$

Namely the P's are positive semidefinite hermitian matrices. This is fundamental for obtaining a semidefinite programming problem. The crossing equation (6.11) now reads

$$\sum_\rho \mathrm{tr}\left(P_\rho\, \mathcal{V}_{i,\rho}\right) = -V_{i,\mathbb{1}}\,. \tag{6.19}$$

We can now use the same strategy as before. We look for a functional that satisfies (6.16)

[5]Recall that we are assuming all external operators to be the same for simplicity. At the end, we will only need to consider this case.

and if we find one we obtain the following contradiction[6]

$$-1 = \sum_{\rho} \text{tr}\left(P_\rho\, \alpha[\mathcal{V}_{i,\rho}]\right) \geqslant 0\,. \tag{6.20}$$

The equality holds by the linearity of α and the inequality follows from the fact that the trace of a product of two semidefinite positive matrices is non-negative.[7]

The problem now consists in implementing a numerical algorithm that searches for a functional α with the property described in (6.16). To this end, we use the program `sdpb` [162, 171] which is a semidefinite program solver optimized for the conformal bootstrap.

According to the kind of assumptions that we make, the bootstrap problems that we need to solve can be quite different. We will now explain the most common approaches. First we start with the assumptions of the type

All operators in a representation (Δ, ℓ) have conformal dimension $\Delta \geqslant \Delta_\ell$,

where Δ_ℓ is a function that we can choose. In the most common case it is

$$\begin{aligned} \Delta_{\ell^\star} &= \Delta^\star\,, \\ \Delta_\ell &= \Delta_\ell^{\text{unitarity}}\,, \qquad \forall\ \ell \neq \ell^\star\,, \end{aligned} \tag{6.21}$$

for a chosen representation $\ell^\star$ and a real number $\Delta^\star$. We defined $\Delta_\ell^{\text{unitarity}}$ as the unitarity bound (1.7) for the representation with spin ℓ.[8] The goal in this kind of problem is to find an upper bound on Δ_ℓ. This can be done by a simple binary search. Namely one fixes two values Δ^{max} and Δ^{min}, which are respectively disallowed and allowed. Then, iteratively, the interval is divided in half and either Δ^{max} or Δ^{min} is updated according to whether the middle point results allowed or disallowed by the bootstrap.

Another setup that can be used consists in assuming a gap for a certain class of operators, just as in the previous case, and further assuming the following:

The lightest operator $\mathcal{O}$ with spin $\ell^\star$ has conformal dimension $\Delta^\star$. Meanwhile all other operators in the same representation have $\Delta > \Delta' > \Delta^\star$.

[6] We denote as $\alpha[\mathcal{V}_{\Delta,\ell}]$ the action of the functional in the infinite dimensional space of functions. It reduces to $\sum_I \alpha_I \mathcal{V}_{I,\rho}$ when we discretize and truncate the equations.

[7] This can be seen by applying a Cholesky decomposition to both matrices. Namely, if $A, B \succeq 0$ then there exist L, M so that $A = L^\dagger L$ and $B = M^\dagger M$. Therefore

$$\text{tr}(AB) = \text{tr}(LL^\dagger M^\dagger M) = \text{tr}((ML)(ML)^\dagger) \geqslant 0\,.$$

[8] Since we only consider unitary theories, the assumption $\Delta \geqslant \Delta^{\text{unitarity}}$ is always the default one.

In this case we cannot rely on a binary search and we need to scan over all values of $\Delta^\star$ between unitarity and Δ'. Moreover, if the representation under consideration has more than one three-point structure, we can get more stringent bound by considering a variation of the bootstrap problem. Normally we should impose

$$\alpha[\mathcal{V}_{\Delta^\star,\ell^\star}] \succeq 0\,, \qquad \alpha[\mathcal{V}_{\Delta,\ell^\star}] \succeq 0\,, \quad \forall\, \Delta > \Delta'\,. \tag{6.22}$$

This is a sufficient condition for getting a contradiction, but it is not necessary. We can contract the matrix $\vec{\mathcal{V}}$ with unit vectors directed in an arbitrary direction[9] $\hat{n}_\theta = (\sin\theta, \cos\theta)$ and search for α_θ that makes the result positive. Then, if we chose a sufficiently dense set of $\theta \in [0, \pi)$ and found a functional for all those values, we can conclude that the point is disallowed. We thus use the stronger formulation of the problem

$$\alpha_\theta[\hat{n}_\theta^T \cdot \mathcal{V}_{\Delta^\star,\ell^\star} \cdot \hat{n}_\theta] \geqslant 0\,, \qquad \forall\, \theta \in [0, \pi)\,. \tag{6.23}$$

In this particular formulation, however, we are also imposing that the operator is non-degenerate.

Next, we would like to discuss a slightly different setup that allows us to find upper and lower bounds on OPE coefficients without having to make multiple runs of the semidefinite solver. Indeed `sdpb` can also maximize a given objective vector, subject to certain semidefinite positiveness conditions. Suppose we are interested in the OPE coefficients of a certain operator in the representation[10] $\rho^\star = (\Delta^\star, \ell^\star)$. The crossing equations can be rewritten to isolate the contribution of that operator

$$|\lambda_{\rho^\star}|^2\, \hat{n}_\theta^T \cdot \mathcal{V}_{\rho^\star} \cdot \hat{n}_\theta + \sum_{\rho \neq \rho^\star} \lambda_\rho^T \cdot \mathcal{V}_\rho \cdot \lambda_\rho = -V_{\mathbb{1}}\,. \tag{6.24}$$

If there is only one tensor structure the unit vectors are trivial (i.e. $\hat{n} = 1$). Then we search for functionals α satisfying

$$\begin{aligned} \alpha[\hat{n}_\theta^T \cdot \mathcal{V}_{\rho^\star} \cdot \hat{n}_\theta] &= \pm 1 =: s\,, \\ \alpha[\mathcal{V}_\rho] &\succeq 0\,, \qquad \forall\, \rho \neq \rho^\star\,, \\ \alpha[V_{\mathbb{1}}] &= -B\,, \end{aligned} \tag{6.25}$$

and minimize the value of B. This yields an upper or lower bound on $|\lambda_{\rho^\star}|^2$, namely

$$\begin{aligned} |\lambda_{\rho^\star}|^2 &\leqslant B\,, \qquad \text{if } s = 1\,, \\ |\lambda_{\rho^\star}|^2 &\geqslant -B\,, \qquad \text{if } s = -1\,. \end{aligned} \tag{6.26}$$

[9] We restrict to the simple case where the matrices are at most two by two. In the more general case clearly one would have to consider a vector in the unit n-sphere and thus introduce more angles.

[10] $\rho^\star$ is an arbitrarily chosen representation. It should not be confused with ρ^*, which is the conjugate representation of ρ.

Finally we briefly introduce the extremal functional method. It is a technique that can be used to extract an approximation of the spectrum (conformal dimensions and OPE coefficients) of the theory that lives at the boundary of an allowed region in parameter space. First notice that the functional α takes its normalization from the condition $\alpha[V_{\text{short}}] = 1$. We can relax this to $\alpha[V_{\text{short}}] > 0$ while still getting a contradiction when α is found. The boundary of this region in functional space is given by all α_∂ that satisfy

$$\alpha_\partial[\mathcal{V}_\rho] \succeq 0 \quad \forall\, \rho\,, \qquad \alpha_\partial[V_{\mathbb{1}}] = 0\,. \tag{6.27}$$

This implies that the only terms that can contribute to the sum of a consistent theory living on the boundary must satisfy

$$\alpha_\partial[\hat{n}_\theta^T \cdot \mathcal{V}_\rho \cdot \hat{n}_\theta] = 0\,. \tag{6.28}$$

In the geometric interpretation we explained earlier, this situation would correspond to Figure 6.1c. For simplicity, let us consider only the simpler case where all $\mathcal{V}_\rho$ are one by one matrices. The equality above together with the positivity constraint implies that the function f_ℓ defined by

$$f_\ell(\Delta) \coloneqq \alpha_\partial[\mathcal{V}_{\Delta,\ell}]\,, \tag{6.29}$$

has even order zeros (typically double zeros) only on those values of Δ that belong to the physical spectrum of operators with spin ℓ. Naturally in a numerical computation we will only find a finite number of such zeros. Then, in order to find the OPE coefficients, we can truncate the crossing equation to that finite number of operators and solve the linear system of equations

$$\sum_{\Delta,\ell \in \{f_\ell(\Delta)=0\}} \mathcal{V}_{(\Delta,\ell)} a_{(\Delta,\ell)} = -V_{\mathbb{1}}\,, \tag{6.30}$$

with $a_\rho = \lambda_\rho^2$. If we find a forbidden point in parameter space very close to the allowed region,[11] the functional α that excludes it will be a very good approximation of α_∂. We can therefore use it to define the function $f_\ell(\Delta)$ and extract the spectrum. For the results to be meaningful one needs to both be very close to the boundary and to observe that the position of the boundary remains stable if the size of the numerics is increased.

6.3 Four-point functions

6.3.1 Conformal frame basis

In (6.4) we introduced the four-point tensor structures $\mathbb{T}_i$. In this section we will construct them explicitly for four dimensional CFTs. There are two possible approaches,

[11] This can be done by either running a binary search to very high accuracy or by extremizing an OPE coefficient with the methods explained before.

one is to use the embedding formalism as explained at the end of Subsection 1.2.2, the other is to fix the coordinates to the conformal frame introduced in Section 1.3. The first approach is convenient if we want to act on the tensor structures with differential operators because it is manifestly covariant. The ability to do so will be essential for the computation of conformal blocks, as we will explain in Subsection 6.4.2. The drawback is that, as we explained before, it is very hard to obtain a list of linearly independent tensor structures due to the complexity of the identities that arise in the in embedding space for four points.

The situation is opposite in the conformal frame approach. Indeed, once the frame is fixed, it suffices to list structures that are singlets under the stability group H (1.32). In the case of four points in four dimensions this group is $\mathrm{SO}(2)$, which is very easy to deal with. Furthermore, conformal frame tensor structures are manifestly linearly independent and transform simply under permutation and crossing. On the other hand, covariance is lost and the action of differential operators on a fixed frame is much harder to obtain.

Let us then adopt the conformal frame approach and then explain how to translate between the two. We will follow [91]. In four dimension the conformal frame reads

$$\begin{aligned} x_1^\mu &= (0,0,0,0)\,, \\ x_2^\mu &= \left(\tfrac{1}{2}(z-\bar{z}),0,0,\tfrac{1}{2}(z+\bar{z})\right)\,, \\ x_3^\mu &= (0,0,0,1)\,, \\ x_4^\mu &= (0,0,0,\infty)\,. \end{aligned} \tag{6.31}$$

The last point being set to infinity means that we take $x_4 = (0,0,0,L)$, rescale $\mathcal{O}_4 \to L^{2\Delta_4}\mathcal{O}_4$ and that let $L \to \infty$. With this choice the stability group $H = \mathrm{SO}(2) = \mathrm{U}(1)$ is generated by M^{12}. This situation is analogous to the one encountered in Section 4.6. We can write the polarizations as

$$\eta_i^\alpha = \begin{pmatrix} m_i \\ p_i \end{pmatrix}\,, \qquad \bar{\eta}_i^{\dot{\alpha}} = \begin{pmatrix} \bar{p}_i \\ \bar{m}_i \end{pmatrix}\,. \tag{6.32}$$

The $\mathrm{SO}(2)$ charge of the complex numbers $m, \bar{m}, p, \bar{p}$ is given by

$$Q[p] = Q[\bar{p}] = 1\,, \qquad Q[m] = Q[\bar{m}] = -1\,. \tag{6.33}$$

The most generic contribution to a four-point function in conformal frame will be a function of z and $\bar{z}$, which we called f_i in (6.4), times a function of the polarizations. This means that the tensor structures are monomials in $p, \bar{p}, m, \bar{m}$ in such a way that their

total Q charge is zero, namely they are H singlets. We can represent them as follows[12]

$$\mathbb{T}_i = \begin{bmatrix} q_1 & q_2 & q_3 & q_4 \\ \bar{q}_1 & \bar{q}_2 & \bar{q}_3 & \bar{q}_4 \end{bmatrix} := \prod_{i=1}^{4} p_i^{-q_i+\ell_i/2}\, m_i^{q_i+\ell_i/2}\, \bar{m}_i^{-\bar{q}_i+\bar{\ell}_i/2}\, \bar{p}_i^{\bar{q}_i+\bar{\ell}_i/2}\,, \tag{6.34}$$

where the charges q_i and $\bar{q}_i$ range in

$$q_i \in \left\{-\frac{\ell_i}{2},\dots,\frac{\ell_i}{2}\right\}, \qquad \bar{q}_i \in \left\{-\frac{\bar{\ell}_i}{2},\dots,\frac{\bar{\ell}_i}{2}\right\}, \qquad \sum_{i=1}^{4}(q_i-\bar{q}_i)=0\,. \tag{6.35}$$

By construction these tensor structures are all linearly independent and it is trivial to enumerate them. Going from embedding to conformal frame is also very straightforward. It suffices to perform the Poincaré projection defined in (1.21) and (1.25) followed by setting the coordinates as in (6.31). The only bit of computation needed is the limit $L \to \infty$ for the fourth point. This is however easily done thanks to the scaling shown in (1.24). The rescaled operator $\mathcal{O}_4$ reads

$$L^{2\Delta_4}\mathcal{O}_4(X,S,\bar{S}) = (L^2)^{\Delta_4+\frac{1}{2}(\ell+\bar{\ell})}L^{-\ell}L^{-\bar{\ell}}\mathcal{O}_4(X,S,\bar{S}) = \mathcal{O}_4(X/L^2,S/L,\bar{S}/L)\,. \tag{6.36}$$

This means that we simply have to make the replacements

$$X_4 \to \lim_{L\to\infty} X_4/L^2\,, \qquad S_4 \to \lim_{L\to\infty} S_4/L\,, \qquad \bar{S}_4 \to \lim_{L\to\infty} \bar{S}_4/L\,, \tag{6.37}$$

which are easily computed.

Translating the other way around is certainly more challenging as the procedure is bound to be ambiguous: in embedding space there are non trivial identities between structures. However, for any specific case, once a basis is fixed one can easily write a dictionary between the two formalisms.

We should point out that going to embedding space is not necessary, even if we want to act with derivatives on a four point function. It is perfectly possible to do it in conformal frame, even though it is far more involved. Due to conformal invariance it is always true that the sum of the L_{MN} generators annihilates the four-point function

$$\left(\sum_{i=1}^{4} L_{i,MN}\right)\langle \mathcal{O}_1(\mathbf{x}_1)\mathcal{O}_2(\mathbf{x}_2)\mathcal{O}_3(\mathbf{x}_3)\mathcal{O}_4(\mathbf{x}_4)\rangle = 0\,, \tag{6.38}$$

$L_{i,MN}$ being the conformal group generator at point i. These are 15 differential equations that relate the various $\partial/\partial x_i^\mu$ derivatives among each other. One of them imposes the H invariance and so does not contain x_i^μ derivatives in conformal frame. On the other hand, the other 14 are sufficient to express the derivatives with respect to $\partial/\partial x_i^\mu$ in

[12] By convention, in conformal frame we include the conformally covariant prefactor $\mathcal{K}_4$ in the definition of the f_i. E.g. $(x_{12}^2 x_{34}^2)^{-\Delta_\phi}$ would result in a contribution $(z\bar{z})^{-\Delta_\phi}$ in the functions f_i. It is important to keep this in mind when translating to embedding or position space.

terms of only $\partial/\partial x_2^0$, $\partial/\partial x_2^3$. Therefore we can use (6.38) to convert any derivative into derivatives with respect to z and $\bar{z}$. The spinor derivatives do not pose any problem as the polarizations are not affected by going to conformal frame. Clearly this procedure is very cumbersome but it can be automated with, for instance, Mathematica.

6.3.2 Enhanced symmetry

This discussion is based on Appendix D of [82]. The stability group of the conformal frame H can be enhanced if the points are put in a special configuration. To see this, imagine that in Figure 1.2 three points are chosen to be collinear. Then the plane that they span collapses to a line and the stability group is enlarged by the rotations that keep that line fixed. In the case at hand, namely four points in four dimensions, the special configuration is the one that gives $z = \bar{z}$, for which H is enhanced to $\mathrm{SO}(3)$.[13] Let us denote the enhanced group as $\widehat{H}$.

The representations of $\widehat{H}$ appearing in the correlator are given by

$$\mathrm{Res}_{\mathrm{SO}(3)}^{\mathrm{SO}(4)} \bigotimes_{k=1}^{4} \rho_k \,. \tag{6.39}$$

And, when $z = \bar{z}$, only the singlets survive. This means that the functions f_i that are not associated to singlets will satisfy the property

$$\lim_{\bar{z}\to z} f_i(z,\bar{z}) = 0 \quad \Longleftrightarrow \quad \mathbb{T}_i \notin \mathrm{SO}(3) \text{ singlet} \,. \tag{6.40}$$

But we can say more about it: we can organize the tensor structures based on how fast they go to zero as $\bar{z} \to z$. The group $\widehat{H}$ acts on the first three coordinates of x_i^μ. Therefore, by looking at the definition of x_2^μ in (6.31), one can see that $y^a = (\frac{1}{2}(z-\bar{z}), 0, 0)$ is an $\mathrm{SO}(3)$ vector and $\frac{1}{2}(z+\bar{z})$ is a scalar. This implies that we can expand any function f_i as

$$f_i(z,\bar{z}) = \sum_{J=0}^{J_{\max}} f_i(z+\bar{z})^{a_1\ldots a_J} y_{a_1} \cdots y_{a_J} \,, \tag{6.41}$$

where J is the $\mathrm{SO}(3)$ spin and $J_{\max}$ is the maximal spin appearing in (6.39). This expansion does not tell us much about f_i in general, but it becomes very useful if the f_i's transform in a definite way under $\widehat{H}$. The basis that achieves this can be found by acting with the $\mathrm{SO}(3)$ Casimir operator

$$\mathcal{C}_{\mathrm{SO}(3)} = (L_{01})^2 + (L_{02})^2 - (L_{12})^2 \,, \tag{6.42}$$

on the structures f_i, and then diagonalizing the resulting matrix. The eigenfunctions

[13] As we remarked in footnote 6, the actual stability group would be $\mathrm{O}(2)$, which is then enhanced to $\mathrm{O}(3)$. For the purpose of this discussion we will ignore this detail.

of the Casimir will have definite spin J_i and thus they can be written as a symmetric traceless tensor of rank J_i contracted with the vectors y^a. More concretely, if one has

$$\mathcal{C}_{\mathrm{SO}(3)} f_i = C_i^{\ k} f_k \,, \qquad M \cdot C \cdot M^{-1} = \mathrm{diag}(J_i) \,, \tag{6.43}$$

then the new basis is $\hat{f}_i = M_i^{\ k} f_k$. It follows that $\hat{f}_i$ has spin J_i and we can write it as

$$\hat{f}_i(z,\bar{z}) = \hat{f}_i(z+\bar{z})^{i_1 \ldots i_J} y_{i_1} \cdots y_{i_J} \,, \tag{6.44}$$

which in turn implies

$$\hat{f}_i(z,\bar{z}) \ \propto \ (z-\bar{z})^{J_i} \,, \qquad \hat{f}(\bar{z},z) = (-1)^{J_i} \hat{f}(z,\bar{z}) \,. \tag{6.45}$$

With this knowledge one can obtain a series of smoothness conditions on the functions f_i at $z = \bar{z}$. It is important to include in the bootstrap problem also this class of constraints.

6.4 Conformal blocks

The most important elements needed to study the crossing equations (6.11) are obviously the conformal blocks $G^{a,b}_{i,\rho}$. They are completely determined by the conformal group, but they can be very difficult to compute. The main idea of the method is to define a set of conformally covariant differential operators that add spinning indices to a correlation function. These operators applied to a conformal block will increase the spin of its external operators. Therefore a small set of "seed" blocks are sufficient for knowing all conformal blocks of any external spin. We will present this concept more precisely along the way. For now, let us discuss the simplest possible seed block: the conformal block of four external scalars exchanging a symmetric traceless primary of spin ℓ.

6.4.1 Conformal blocks of external scalars

The definition of conformal block given in (6.3) is rather cumbersome even for the simplest case of external scalar operators. One would need to compute the functions $C_\rho(\partial_\mu, x_{ij}, \partial_\eta)$ by matching them with three-point function and then evaluate the product of two C_ρ's and the two point function $\langle \mathcal{O}_{\rho^*} \mathcal{O}_\rho \rangle$. Nevertheless, this is the way the blocks were first computed [94]. Later the same authors found a simpler method that requires solving a second order differential equation: the Casimir equation [95]. Conformal blocks can be schematically represented as a sum over all descendants of $\mathcal{O}_\rho$

$$\mathcal{K}_4 \, G_\rho \sim \sum_{A,B \in \mathcal{O}_\rho} \langle \phi_1 \phi_2 A \rangle \, g^{AB} \langle B \phi_3 \phi_4 \rangle \,, \tag{6.46}$$

where $\mathcal{K}_4$ is the prefactor defined in (1.18) and g^{AB} is the inverse of the matrix of two-point functions $g_{AB} = \langle AB \rangle$. Each state in a representation $\rho = (\Delta, \ell)$ is an eigenvector

of the Casimir differential operator. In embedding coordinates this reads

$$L^2|O\rangle := \tfrac{1}{2} L_{MN} L^{MN} |O\rangle = \mathcal{C}_{\Delta,\ell}|O\rangle\,,$$

with

$$L_{MN} := i\left(X_M \frac{\partial}{\partial X^N} - X_N \frac{\partial}{\partial X^M}\right), \qquad \mathcal{C}_{\Delta,\ell} = \Delta(\Delta - d) + \ell(\ell + d - 2)\,. \tag{6.47}$$

Naturally, since the correlators are conformally invariant, $\sum_{i=1}^{n} L_i$ applied on a n-point function gives zero, as we noted in (6.38). Therefore the Casimir at points 1 and 2 applied on (6.46) gives precisely $\mathcal{C}_{\Delta,\ell}$. We thus obtain the Casimir equation

$$(L_1 + L_2)^2 (\mathcal{K}_4\, G_\rho) = \mathcal{C}_{\Delta,\ell}\, \mathcal{K}_4\, G_\rho\,. \tag{6.48}$$

After rewriting it in terms of the $z, \bar{z}$ variables we are left dealing with a second order partial differential equation. With some redefinitions, (6.48) can be seen to factorize when d is an even integer. In those cases it is possible to find a closed form solution in terms of ${}_2F_1$ hypergeometric functions. Whereas if d is odd no closed form solution exists. Without dwelling to much on the details, we will present the solution for $d = 4$. It is expressed in terms of a function κ_β

$$\begin{aligned} \kappa_\beta(x) &:= x^\beta\, {}_2F_1\left(\beta - \tfrac{\Delta_{12}}{2}, \beta + \tfrac{\Delta_{34}}{2}; 2\beta; x\right), \\ G_\rho(z,\bar{z}) &= \frac{1}{(-2)^\ell} \frac{z\bar{z}}{z - \bar{z}} \left(\kappa_{\frac{\Delta+\ell}{2}}(z)\, \kappa_{\frac{\Delta-\ell-2}{2}}(\bar{z}) - (z \leftrightarrow \bar{z})\right), \end{aligned} \tag{6.49}$$

where $\Delta_{ij} = \Delta_i - \Delta_j$ is the difference of the conformal dimensions of the external fields.

For numerical applications we need to find an efficient and precise approximation of the derivatives of the conformal blocks at $z = \bar{z} = 1/2$ that consists in rational functions in Δ. The Casimir equation is particularly useful in that regard because it allows us to know all derivatives of the conformal block at a given point once the function and its first derivative are known at that point. However, using the expression (6.49) for evaluating and approximating the blocks is not the most convenient approach. Other faster methods, which we will not review, are available in the literature. The interested reader may consult the reviews cited at the beginning of this chapter or the references at the end of Subsection 1.2.1. In what follows we will only worry about "spinning up" the scalar blocks, assuming that we already have an efficient method for evaluating and approximating them.

6.4.2 Differential operators in embedding space

In this subsection we will briefly review the work done in [90]. We are thus focusing in four dimensions and making use of the embedding space formalism (which we

introduced in Section 1.2.2). Unlike the previous chapters, we will from now on denote spin labels as $(\ell, \bar{\ell})$ instead of $(j, \bar{j})$.

Before diving into the definition of the differential operators, let us understand from the representation theoretic point of view what can we gain from them. Let us define for convenience the conformal partial waves $W_\rho^{a,b}(\mathbf{x}_1, \dots, \mathbf{x}_4)$ as

$$W_\rho^{a,b}(\mathbf{x}_1, \dots, \mathbf{x}_4) \coloneqq \sum_{i=1}^{n_{1234}} \mathbb{T}_i(\mathbf{x}_1, \dots, \mathbf{x}_4)\, G_{i,\rho}^{a,b}(u, v)\,, \tag{6.50}$$

where we also used a shorthand $\mathbf{x}_i = (x_i, \eta_i, \bar{\eta}_i)$. Conformal partial waves can be written schematically in the same fashion as (6.46). More precisely, it is possible to express them as a certain pairing of three-point tensor structures [103, 105], which is sometimes denoted by

$$W^{a,b}_{\rho,\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\mathcal{O}_4}(\mathbf{x}_i) = \mathrm{t}^a_{\mathcal{O}_1\mathcal{O}_2\mathcal{O}_\rho}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_0) \bowtie \mathrm{t}^b_{\mathcal{O}_{\rho^*}\mathcal{O}_3\mathcal{O}_4}(\mathbf{x}_0, \mathbf{x}_3, \mathbf{x}_4). \tag{6.51}$$

One may think of the operation $\bowtie$ roughly as performing an integral over the coordinates and summing over polarizations of $\mathbf{x}_0$.[14] Its precise definition will not be important to us.

So, suppose we have a differential operator $\mathbf{D}_{12}$ that takes $\langle\phi_1\phi_2\mathcal{O}\rangle$ to $\langle\mathcal{O}_1\mathcal{O}_2\mathcal{O}\rangle$, where $\mathcal{O}_1$ and $\mathcal{O}_2$ are symmetric traceless tensors with nonzero spin.[15] We can use it to write

$$W^{a,b}_{\rho,\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\mathcal{O}_4}(\mathbf{x}_1, \dots, \mathbf{x}_4) = \mathbf{D}_{12}^{(a)}\mathbf{D}_{34}^{(b)} W^{1,1}_{\rho,\phi_1\phi_2\phi_3\phi_4}(\mathbf{x}_1, \dots, \mathbf{x}_4)\,, \tag{6.52}$$

where we used the fact that there is only one tensor structure for $\langle\phi\phi\mathcal{O}\rangle$. The indices a, b on $\mathbf{D}_{ij}$ will be made clear later. One might naîvely say that, due to this, the only seed block needed is the scalar one. After all we can generate all partial waves starting from the one with external scalars. This would be wrong because the differential operators cannot act on the exchanged operator[16] and therefore we can only do the trick when the three-point function $\langle\phi_1\phi_2\mathcal{O}\rangle$ is nonzero. This is the case only for symmetric traceless operators. The correlator with the lowest possible spin that contains an operator of spin $(\ell, \bar{\ell})$ with $p = |\ell - \bar{\ell}|$ is

$$\langle\phi F^{(p)}\mathcal{O}\rangle \qquad \text{or} \qquad \langle\phi \bar{F}^{(p)}\mathcal{O}\rangle\,, \tag{6.53}$$

[14] Note that the $\bowtie$ notation obscures the fact that the normalization of conformal blocks also implicitly depends on the normalization of the two-point functions.

[15] The operators $\mathcal{O}_1$ and $\mathcal{O}_2$ should not be thought of as real operators of the theory. The sense in which $\mathbf{D}_{12}\langle\phi_1\phi_2\mathcal{O}\rangle = \langle\mathcal{O}_1\mathcal{O}_2\mathcal{O}\rangle$ holds is that the left hand side has the functional form of a conformal correlator with the quantum numbers of the operators in the right hand side. Equivalently, it may be seen as a relation between tensor structures, without any reference to physical correlators.

[16] Actually this is not correct: it is possible to act on the exchanged primary with the aid of the so-called weight shifting operators [103] by using the crossing equations that they satisfy. We will not use this method in the thesis and so we will not discuss it any further.

where $F^{(p)}$ is an operator of spin $(p,0)$ and $\bar{F}^{(p)}$ of spin $(0,p)$. Furthermore, for these operators the three-point structure is unique, so we do not have a,b labels. Then, for every p, we define the partial waves[17]

$$W_\rho^{\text{seed}} := W^{1,1}_{\rho,\,\phi_1 F_2^{(p)}\,\phi_3\,\bar{F}_4^{(p)}}\,. \tag{6.54}$$

These together with the block of four external scalars complete the list of all needed seeds. We could have equivalently chosen seeds with two $F^{(p)}$ and two scalars or two $\bar{F}^{(p)}$ and two scalars in place of (6.54). We will now address the problem of obtaining general partial waves from the seed ones. In the next subsection we will show how to compute the seeds. The final step would be to go from partial waves to conformal blocks, but this is a relatively trivial task as it only requires to expand W_ρ in the basis of the $\mathbb{T}_i$.

Now that we have our objective clear we can start writing down the differential operators $\mathbf{D}_{ij}$. They will be constructed by simpler building blocks which modify the spin by the smallest possible amount. We will define them for the first two point. The other operators are obtained with the obvious replacement $1 \to 3$, $2 \to 4$. Let us list these building blocks. The simplest ones that we can construct are

$$\begin{aligned}
\sqrt{X_{12}}\,, &\qquad (-\tfrac{1}{2};-\tfrac{1}{2}|0,0;0,0)\,,\\
I^{12}\,, &\qquad (-\tfrac{1}{2};-\tfrac{1}{2}|0,1;1,0)\,,\\
I^{21}\,, &\qquad (-\tfrac{1}{2};-\tfrac{1}{2}|1,0;0,1)\,.
\end{aligned}$$

These are just multiplicative operators. Next to them we wrote the shifts that they make on the conformal dimensions and spins $\delta := (\delta\Delta_1;\delta\Delta_2|\delta\ell_1,\delta\bar{\ell}_1;\delta\ell_2,\delta\bar{\ell}_2)$ (for example, $\delta\ell_1$ means $\ell_1^{\text{new}} - \ell_1^{\text{old}}$). The same notation will be used for what follows. At first order in the number of derivatives we have

$$\begin{aligned}
D_{12} &= \frac{1}{2}\bar{S}_1\Sigma^M\bar{\Sigma}^N S_1\left(X_{2M}\frac{\partial}{\partial X_1^N} - X_{2N}\frac{\partial}{\partial X_1^M}\right), && (0;-1|1,1;0,0)\,,\\
D_{21} &= D_{12}|_{1\leftrightarrow 2}\,, && (-1;0|0,0;1,1)\,,\\
\widetilde{D}_{12} &= \bar{S}_1\mathbf{X}_2\bar{\Sigma}^N S_1\frac{\partial}{\partial X_2^N} + 2I^{12}S_{1a}\frac{\partial}{\partial S_{2a}} - 2I^{21}\bar{S}_1^a\frac{\partial}{\partial \bar{S}_2^a}\,, && (-1;0|1,1;0,0)\,,\\
\widetilde{D}_{21} &= \widetilde{D}_{12}|_{1\leftrightarrow 2}\,, && (0;-1|0,0;1,1)\,.
\end{aligned} \tag{6.55}$$

The operators listed so far do not change the overall difference between ℓ and $\bar{\ell}$. Here

[17] The dependence on p of the left hand side is in $\rho = (\Delta,\ell,\bar{\ell})$ with $p = |\ell - \bar{\ell}|$. The dependence on the external dimensions is kept implicit for brevity.

are some operators that change it by two:

$$\begin{aligned} d_{12} &= S_2 \bar{\mathbf{X}}_1 \frac{\partial}{\partial \bar{S}_1}\,, && (-\tfrac{1}{2};-\tfrac{1}{2}|0,-1;1,0)\,,\\ d_{21} &= d_{12}|_{1\leftrightarrow 2}\,, && (-\tfrac{1}{2};-\tfrac{1}{2}|1,0;0,-1)\,,\\ \bar{d}_{12} &= \bar{S}_2 \mathbf{X}_1 \frac{\partial}{\partial S_1}\,, && (-\tfrac{1}{2};-\tfrac{1}{2}|-1,0;0,1)\,,\\ \bar{d}_{21} &= \bar{d}_{12}|_{1\leftrightarrow 2}\,, && (-\tfrac{1}{2};-\tfrac{1}{2}|0,1;-1,0)\,. \end{aligned} \tag{6.56}$$

There are also other operators that change the spins in a similar way, but they can be shown to be redundant. Lastly, we need an operator to decrease the spin and change the difference $\ell_1 - \bar{\ell}_1 - (\ell_2 - \bar{\ell}_2)$. This is necessary in order to connect three-point functions with one operator of spin $(\ell, \bar{\ell})$ to those with an operator of spin $(\ell \pm 1, \bar{\ell} \mp 1)$, while keeping the other operators unchanged. It is sufficient to introduce the following second order differential operators

$$\begin{aligned} \nabla_{12} &= (\bar{\mathbf{X}}_1 \mathbf{X}_2)^a_b \frac{\partial^2}{\partial \bar{S}_1^a \partial S_{2b}}\,, && (-\tfrac{1}{2};-\tfrac{1}{2}|0,-1;-1,0)\,,\\ \nabla_{21} &= \nabla_{12}|_{1\leftrightarrow 2}\,, && (-\tfrac{1}{2};-\tfrac{1}{2}|-1,0;0,-1)\,. \end{aligned} \tag{6.57}$$

One of the crucial aspects of these operators is that there is generally more than on way to connect two three-point functions of different spins. These different ways can be used to create a many-to-one mapping between the space of differential operators $\mathbf{D}_{12}^{(a)}$ and the three-point tensor structures $\mathrm{t}^a_{\mathcal{O}_1\mathcal{O}_2\mathcal{O}}$. That means that we can define a basis such that

$$\mathbf{D}_{12}^{(a)}\, \mathrm{t}_{\phi F^{(p)}\mathcal{O}}(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3) = \mathrm{t}^a_{\mathcal{O}_1\mathcal{O}_2\mathcal{O}}(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3)\,. \tag{6.58}$$

Typically the most natural basis from the differential operators perspective is not a very convenient one in terms of embedding space structures, and vice versa. This is not important as one can always work in the best basis for the purpose at hand and then translate from one to the other when needed. The most general differential operator reads

$$\mathbf{D}_{12}^{(a)} = (I^{12})^{m_{12}} (I^{21})^{m_{21}} D_{12}^{m_{13}}\, D_{21}^{m_{23}}\, \widetilde{D}_{12}^{j_1}\, \widetilde{D}_{21}^{j_2}\, d_{12}^{k_1}\, d_{21}^{k_2}\, \bar{d}_{12}^{\bar{k}_1}\, \bar{d}_{21}^{\bar{k}_2}\, \nabla_{12}^{r}\,. \tag{6.59}$$

This differential operator should be applied on a three-point structure of a scalar, an operator of spin $(p = |\ell - \bar{\ell}|, 0)$ and an operator of spin $(\ell, \bar{\ell})$ with conformal dimensions given by Δ_1', Δ_2' and Δ.[18] The result is a structure with spins $(\ell_1, \bar{\ell}_1)$,$(\ell_2, \bar{\ell}_2)$,$(\ell, \bar{\ell})$ and

[18] If we want to act on a correlator with an operator of spin $(0, p)$ at the second point it suffices to replace ∇_{12} by ∇_{21}.

conformal dimensions Δ_1, Δ_2 and Δ given as follows

$$\begin{cases} \Delta_1 = \Delta_1' - m_{23} - j_1 - \frac{1}{2}(m_{12} + m_{21} + k_1 + k_2 + \bar{k}_1 + \bar{k}_2 + r)\,, \\ \Delta_2 = \Delta_2' - m_{13} - j_2 - \frac{1}{2}(m_{12} + m_{21} + k_1 + k_2 + \bar{k}_1 + \bar{k}_2 + r)\,, \\ \ell_1 = j_1 + k_2 - \bar{k}_1 + m_{13} + m_{21}\,, \\ \bar{\ell}_1 = j_1 + \bar{k}_2 - k_1 + m_{13} + m_{12} - r\,, \\ \ell_2 = j_2 + k_1 - \bar{k}_2 + m_{23} + m_{12} - r + p\,, \\ \bar{\ell}_2 = j_2 + \bar{k}_1 - k_2 + m_{23} + m_{21}\,. \end{cases} \tag{6.60}$$

The basis $\mathbf{D}_{12}^{(a)}\, \mathrm{t}_{\phi F^{(p)}\mathcal{O}}$ is often referred to as the differential basis, while the basis given in (1.28) is called the OPE basis. Mind that we are not claiming that the solutions to (6.60) are in one-to-one correspondence with the elements in the OPE basis. There might still be some redundancies. We will be content with defining a possibly overcomplete basis here. A general prescription for when the first two operators are symmetric traceless is given in [90].[19]

We end this subsection with a technical remark: often it is necessary to replace $\widetilde{D}$ with $D + \widetilde{D}$ in the definition (6.59). This yields a different but still perfectly valid basis. The reason for doing so is that in the former case the matrix that translates from the differential basis to the OPE basis will contain elements with poles for unitary values of Δ, ℓ and $\bar{\ell}$. These are unphysical poles which should be removed.

6.4.3 Seed partial waves

We have shown in the last subsection that a general partial wave for the exchange of $\rho = (\Delta, \ell, \bar{\ell})$ in the correlator $\langle \mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3 \mathcal{O}_4 \rangle$ can be obtained by acting on the "seed" partial wave with some conformal differential operators

$$W^{a,b}_{\rho,\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\mathcal{O}_4} = \mathbf{D}_{12}^{(a)} \mathbf{D}_{34}^{(b)}\, W^{\text{seed}}_{\rho}\,, \tag{6.61}$$

Before giving the expressions for the seeds, we need to write down their four point-function. The partial wave decomposition reads

$$\begin{aligned} &\langle \phi_1(x_1) F_2^{(p)}(x_2, \eta_2) \phi_3(x_3) \bar{F}^{(p)}(x_4, \bar{\eta}_4) \rangle \\ &\qquad = \sum_{q=p,p-2,\ldots} \sum_{\rho=(\Delta,\ell,\ell+q)} \left(\lambda_\rho^2\, W^{1,1}_{\rho,\phi_1 F_2^{(p)} \phi_3 \bar{F}_4^{(p)}} + (\rho \leftrightarrow \rho^*) \right). \end{aligned} \tag{6.62}$$

[19] It should be stressed that the prescription in [90] is defined so that it works in general (when the first two operators are symmetric traceless) but it is not necessarily optimized for minimizing the number of terms or the order of the differential operators. Such an optimization is not necessary from a theoretical perspective but it may make a significant difference in the numerical computation of the blocks.

We are interested in the waves for $q = p$. Next we want to decompose the partial waves in the tensor structures $\mathbb{T}_i$. For these minimal four-point functions, listing the tensor structures is not a difficult problem. Furthermore we want to be able to act with differential operators on them. As per the discussion at the beginning of Section 6.3, it is more convenient to use the embedding formalism. There are in total $p+1$ structures given by

$$\begin{aligned} W^{\text{seed}}_{\Delta,(\ell,\ell+p)} &= \mathcal{K}_4 \sum_{e=0}^{p} G^{(p)}_{\Delta,\ell;e}(z,\bar{z})\,(I^{42})^e\,(I^{42}_{31})^{p-e}\,, \\ W^{\text{seed}}_{\Delta,(\ell+p,\ell)} &= \mathcal{K}_4 \sum_{e=0}^{p} \bar{G}^{(p)}_{\Delta,\ell;e}(z,\bar{z})\,(I^{42})^e\,(I^{42}_{31})^{p-e}\,, \end{aligned} \tag{6.63}$$

where $\mathcal{K}_4$ is the prefactor

$$\mathcal{K}_4 = X_{12}^{-\frac{p}{4}-\frac{1}{2}(\Delta_1+\Delta_2)} X_{34}^{-\frac{p}{4}-\frac{1}{2}(\Delta_3+\Delta_4)} \left(\frac{X_{24}}{X_{14}}\right)^{-\frac{p}{4}+\frac{1}{2}\Delta_{12}} \left(\frac{X_{14}}{X_{13}}\right)^{-\frac{p}{4}+\frac{1}{2}\Delta_{34}}. \tag{6.64}$$

The functions $G^{(p)}_{\Delta,\ell;e}$, $\bar{G}^{(p)}_{\Delta,\ell;e}$ are the seed conformal blocks and they will be the object of study of this subsection. In what follows we will suppress their dependence on Δ and ℓ for brevity. We will review the computation of [172] which uses the method of the Casimir equation. We already explained the idea of the Casimir equation in Subsection 6.4.1. The complications that arise in the spinning case are two: the equation will actually be a system of partial differential equations relating $G^{(p)}_e$ for different values of e. Moreover, now the operator L_{MN} has also a non-vanishing spin part

$$L_{i,MN} = i\left(X_{iM}\frac{\partial}{\partial X_{iN}} - X_{iN}\frac{\partial}{\partial X_{iM}} + S_i\bar{\Sigma}_{MN}\frac{\partial}{\partial S_i} + \bar{S}_i\Sigma_{MN}\frac{\partial}{\partial \bar{S}_i}\right). \tag{6.65}$$

The eigenvalue for a representation $(\ell,\bar{\ell})$ is given by

$$\mathcal{C}_{\Delta,\ell,\bar{\ell}} = \Delta(\Delta-4) + \tfrac{1}{2}\left(\ell(\ell+2) + \bar{\ell}(\bar{\ell}+2)\right). \tag{6.66}$$

After applying this operator on (6.63) and requiring that the coefficient of each tensor structure vanishes[20] we obtain this system of second order partial differential equations

$$\left(\Delta^{(a_e,b_e;c_e)}_{2+p} - \tfrac{1}{2}(\mathcal{C}_{\Delta,\ell+p,\ell} - \varepsilon^p_e)\right) G^{(p)}_e + A^p_e\, z\bar{z}\, L(a_{e-1})\, G(p)_{e-1} + B_e\, L(b_{e+1})\, G^{(e)}_{e+1} = 0\,, \tag{6.67}$$

with the definitions

$$a_e := \frac{\Delta_2-\Delta_1}{2} + \frac{p}{4}\,, \quad b_e := \frac{\Delta_3-\Delta_4}{2} - \frac{p}{4} + p - e\,, \quad c_e := p-e\,, \tag{6.68a}$$

$$\varepsilon^p_e := \tfrac{3}{4}p^2 - (1+2e)p + 2e(2+e)\,, \quad A^p_e := 2(p-e+1)\,, \quad B_e := \frac{e+1}{2}\,, \tag{6.68b}$$

[20] This is not at all a straightforward step because the action of the Casimir operator on the $\mathbb{T}_i$ generates many invariants which are linearly dependent. It is thus necessary to apply the various relations in embedding space to eliminate them.

$$L(\mu) \coloneqq -\frac{1}{z-\bar{z}}\left(z(1-z)\partial_z - \bar{z}(1-\bar{z})\partial_{\bar{z}}\right) + \mu\,, \tag{6.68c}$$

$$D_z^{(a,b;c)} \coloneqq z^2(1-z)\partial_z^2 - \left((a+b+1)z^2 - cz\right)\partial_z - abz\,, \tag{6.68d}$$

$$\Delta_\epsilon^{(a,b;c)} \coloneqq D_z^{(a,b;c)} + D_{\bar{z}}^{(a,b;c)} + \epsilon\frac{z\bar{z}}{z-zb}\left((1-z)\partial_z - (1-\bar{z})\partial_{\bar{z}}\right)\,. \tag{6.68e}$$

In (6.67) it is understood that $G_{-1}^{(p)} = G_{p+1}^{(p)} = 0$. The equations for $\bar{G}_e^{(p)}$ are analogous. It is interesting to notice a nearest-neighbor structure in the Casimir equation. Namely every block $G_e^{(p)}$ only couples to its neighbors $G_{e\pm1}^{(p)}$.

In order to attempt solving this system we need some guidance. In particular, we can benefit from knowing the asymptotic behavior of the solution as $z, \bar{z} \to 0$. This is not easy to obtain from the equations themselves, but there is an alternative way to obtain the blocks which is viable for small values of p, namely the shadow formalism [104, 105]. By studying the asymptotic behavior of the solutions obtained with this latter method we can extrapolate a pattern for all p and then we can use that knowledge to come up with an ansatz. We will not show here the shadow formalism solution nor the general asymptotic behavior and refer the reader to [172].

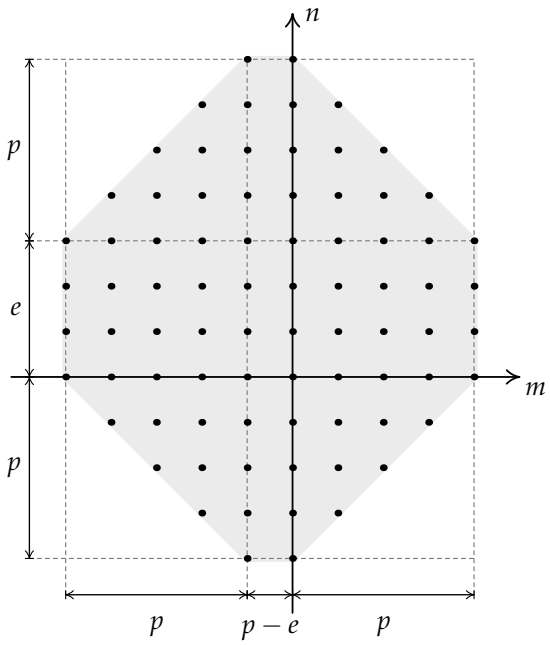


Figure 6.2. Octagon $\mathrm{Oct}_e^{(p)}$ for $p = 4$ and $e = 3$ representing the values of m, n for which $c_{m,n}^e$ is nonzero.

The solution can be parametrized by a matrix of coefficients $c_{m,n}^e$ for $G_e^{(p)}$ and $\bar{c}_{m,n}^e$ for $\bar{G}_e^{(p)}$ that are nonzero only for m and n belonging to an octagon region on the plane $\mathrm{Oct}_e^{(p)} \subset \mathbb{Z}^2$ that we will describe shortly. In terms of these coefficients the solution

reads

$$G_e^{(p)}(z,\bar{z}) = \left(\frac{z\bar{z}}{z-\bar{z}}\right)^{2p+1} \sum_{m,n\in \mathrm{Oct}_e^{(p)}} c_{m,n}^e \, \mathcal{F}^{(a_e,b_e;c_e)}_{\beta_1+m,\beta_2+n}(z,\bar{z})\,, \tag{6.69a}$$

$$\bar{G}_e^{(p)}(z,\bar{z}) = \left(\frac{z\bar{z}}{z-\bar{z}}\right)^{2p+1} \sum_{m,n\in \mathrm{Oct}_{p-e}^{(p)}} \bar{c}_{m,n}^e \, \mathcal{F}^{(a_e,b_e;c_e)}_{\bar{\beta}_1+e+m,\bar{\beta}_2+e+n}(z,\bar{z})\,, \tag{6.69b}$$

where we defined

$$\begin{aligned}
\beta_1 &:= \frac{\Delta+\ell}{2} + \frac{p}{4}\,, \qquad \beta_2 := \frac{\Delta-\ell}{2} + \frac{p}{4}\,,\\
\bar{\beta}_1 &:= \frac{\Delta+\ell}{2} - \frac{p}{4}\,, \qquad \bar{\beta}_2 := \frac{\Delta-\ell}{2} - \frac{5p}{4} - 1\,,\\
\kappa_\beta^{(a,b;c)}(z) &:= z^\beta \, {}_2F_1(a+\beta, b+\beta; c+2\beta; z)\,,\\
\mathcal{F}^{(a,b;c)}_{\beta_1,\beta_2}(z,\bar{z}) &:= \kappa_{\beta_1}^{(a,b;c)}(z)\,\kappa_{\beta_2}^{(a,b;c)}(\bar{z}) - (z \leftrightarrow \bar{z})\,.
\end{aligned} \tag{6.70}$$

We have already encountered κ_β in (6.49) for the case of $c = 0$. The octagon $\mathrm{Oct}_e^{(p)}$ on which the coefficients $c_{m,n}^e$ lie is defined by the boundaries

$$\begin{aligned}
-p \leqslant n \leqslant e+p\,, \qquad & e-2p \leqslant m \leqslant p\,,\\
e-2p \leqslant m+n \leqslant p+e\,, \qquad & -2p \leqslant m-n \leqslant p\,.
\end{aligned} \tag{6.71}$$

and it contains $N_e^{(p)} = p(4p+3) - e^2 + ep + 1$ points, which is easily seen to be invariant under $e \to p - e$. The shape of the octagon can be seen in Figure 6.2. The coefficients $c_{m,n}^e$ and $\bar{c}_{m,n}^e$ may be obtained by means of a recursion relation, which we will not reproduce here. The growth in the number of coefficients together with the complexity of the recursion relation make this task very computationally demanding. Luckily, however, the coefficients for p up to four have been already computed and can be found in this repository: `gitlab.com/bootstrapcollaboration/CFTs4D`. Those are all we need for bootstrapping four currents. Indeed, the exchanged operators in the OPE $J \times J$ can have $p = 0, 2$ or 4.

7 Conformal blocks

In this chapter we describe all the steps needed to prepare the bootstrap problem of four abelian currents in four dimensions. It is based on **Paper I** and **Paper IV**. First we do the three-point function analysis, then we study the four-point function $\langle JJJJ \rangle$, and finally we show the computation of the conformal and superconformal blocks.

7.1 Non-supersymmetric operator product expansion

An abelian current is a primary operator of spin $(1,1)$ with conformal dimension $\Delta_J = 3$. It therefore satisfies a shortening condition

$$\partial_\eta \partial_x \partial_{\bar{\eta}} J(\mathbf{x}) = 0\,, \qquad J(\mathbf{x}) \coloneqq \eta^\alpha \bar{\eta}^{\dot{\alpha}} J_{\alpha\dot{\alpha}}(x)\,. \tag{7.1}$$

It is called abelian because it is associated to a $\mathrm{U}(1)$ global symmetry. If we are considering a superconformal theory then this is precisely the R-symmetry and J represents the superprimary of the Ferrara-Zumino multiplet. This multiplet was introduced in Subsection 2.1.3 and it satisfies the shortening condition (2.13a).

The four-point function under study is

$$\langle J(\mathbf{x}_1) J(\mathbf{x}_2) J(\mathbf{x}_3) J(\mathbf{x}_4) \rangle\,. \tag{7.2}$$

Associativity of the OPE requires that the crossing equations (6.2) hold. The two relations are, in fact, not independent. It suffices therefore to impose

$$\langle \overbrace{J(\mathbf{x}_1) J(\mathbf{x}_2)} \overbrace{J(\mathbf{x}_3) J(\mathbf{x}_4)} \rangle = \langle \overbrace{J(\mathbf{x}_1) \underbrace{J(\mathbf{x}_2) J(\mathbf{x}_3)} J(\mathbf{x}_4)} \rangle\,. \tag{7.3}$$

The contractions represent the OPE as in (6.1), which, for the case at hand, reads

$$\overbrace{J(\mathbf{x}_1)J}(\mathbf{x}_2) \coloneqq \sum_{\rho}\sum_{a=1}^{n_{JJ\rho}} \frac{\lambda^{(a)}_{JJ\mathcal{O}_\rho}}{|x_{12}|^{6+\Delta}} C^{(a)}_{\rho}(\partial_\mu, x_{12}, \partial_\eta, \partial_{\bar\eta})\mathcal{O}_{\rho^*}(0,\eta,\bar\eta)\,, \tag{7.4}$$

where $\mathcal{O}_{\rho^*} = \overline{\mathcal{O}}_\rho$ is the hermitian conjugate operator (or, equivalently, the CPT conjugate). The allowed values for $\rho = (\Delta, \ell, \bar\ell)$ in the sum are

$$\Delta = \ell = \bar\ell = 0 \quad \text{or} \quad \Delta \geqslant \Delta_{\text{unitarity}}(\ell,\bar\ell) \coloneqq \begin{cases} 2 + \frac{1}{2}(\ell+\bar\ell) & \ell\bar\ell \neq 0\,, \\ 1 + \frac{1}{2}(\ell+\bar\ell) & \ell\bar\ell = 0\,, \end{cases} \tag{7.5}$$
$$(\ell,\bar\ell) \in \{(\ell,\ell),(\ell,\ell+2),(\ell+2,\ell),(\ell,\ell+4),(\ell+4,\ell) :\ \ell \in \mathbb{N}\}\,.$$

where $\Delta_{\text{unitarity}}$ is given in (1.7) but was reproduced here for convenience. We also introduce the parameter p as

$$p \coloneqq |\ell - \bar\ell|\,, \tag{7.6}$$

as we did in Subsection 6.4.2. The allowed values of p are therefore 0, 2 and 4. For $p > 0$ there are two types of operators related by conjugation. We will refer to $(\ell, \ell+p)$ as the *primal* and to $(\ell+p, \ell)$ as the *dual*. In this section we will consider all operators $\mathcal{O}_\rho$ that may appear in the OPE of two currents and study the three-point functions with two J's and one $\mathcal{O}$ and the two-point functions of $\mathcal{O}_\rho$ and $\mathcal{O}_{\rho^*}$. In doing so we will make use of the embedding formalism described in Subsection 1.2.2.

7.1.1 Two-point functions

We start by choosing the basis of local operators which appear in the OPE (7.4). First we name all the operators with $\ell \geqslant \bar\ell$ as[1]

$$\mathcal{O}^{(\ell,\ell)}_\Delta\,, \qquad \mathcal{O}^{(\ell+2,\ell)}_\Delta\,, \qquad \mathcal{O}^{(\ell+4,\ell)}_\Delta\,. \tag{7.7}$$

All the operators with $\ell \leqslant \bar\ell$ are obtained by hermitian conjugation of the ones in (7.7). They read

$$\overline{\mathcal{O}}^{(\ell,\ell)}_\Delta\,, \qquad \overline{\mathcal{O}}^{(\ell+2,\ell)}_\Delta\,, \qquad \overline{\mathcal{O}}^{(\ell+4,\ell)}_\Delta\,. \tag{7.8}$$

In the traceless symmetric ($p = 0$) case the local operators are chosen to be hermitian or in other words $\mathcal{O}^{(\ell,\ell)}_\Delta = \overline{\mathcal{O}}^{(\ell,\ell)}_\Delta$. Secondly, we can assume that the two-point functions are diagonal in this basis. This means that the only non-vanishing two-point functions are

[1] In principle there can be operators which cannot be fully specified by dimension and spin alone. For example, there could be degeneracies given by additional global symmetries that were not taken as an assumptions. We can ignore this detail as it is irrelevant for our purposes.

those of an operator and its conjugate

$$\langle \overline{\mathcal{O}}_{\Delta}^{(\ell,\bar{\ell})}(\mathbf{x}_1)\overline{\mathcal{O}}_{\Delta}^{(\ell,\bar{\ell})}(\mathbf{x}_2)\rangle = \frac{i^{\ell-\bar{\ell}}}{(x_{12}^2)^{\Delta+\frac{1}{2}(\ell+\bar{\ell})}}(\mathbb{I}^{12})^{\ell}(\mathbb{I}^{21})^{\bar{\ell}}\,. \tag{7.9}$$

See (A.22) for the definition of the invariants $\mathbb{I}^{ij}$. The choice of normalization implicitly made in (7.9) is the standard one for generic operators. On the other hand, for conserved currents we will use a different normalization, as we are going to discuss now.

Operators with $\ell\bar{\ell} \neq 0$ saturating the unitarity bound $\Delta_{\text{unitarity}}$ defined in (7.5) are necessarily conserved currents.[2] The cases with spin $(1,1)$ and $(2,2)$ are the well known Noether current and stress tensor

$$J \coloneqq \mathcal{O}_3^{(1,1)}\,, \qquad T \coloneqq \mathcal{O}_4^{(2,2)}\,. \tag{7.10}$$

A natural normalization for J and T is one that makes their Ward identities follow the definitions in Section 1.4. This choice is not necessarily the one that makes their two-point functions unit normalized. Thus, instead of (7.9) one has to write

$$\langle J(\mathbf{x}_1)J(\mathbf{x}_2)\rangle = \frac{C_J}{(x_{12}^2)^4}\,\mathbb{I}^{12}\mathbb{I}^{21}\,, \qquad \langle T(\mathbf{x}_1)T(\mathbf{x}_2)\rangle = \frac{C_T}{(x_{12}^2)^6}\,(\mathbb{I}^{12}\mathbb{I}^{21})^2\,, \tag{7.11}$$

for some numbers C_J and C_T which are called the central charges. When we are considering the supersymmetric setup, J and T live in the same multiplet. As a consequence C_T and C_J are related [156]. The precise relation can also be inferred from $n_{(Q\bar{Q}J)^{++}}$ in Table 3.2.

$$C_T^{\text{susy}} = 5C_J^{\text{susy}}\,. \tag{7.12}$$

7.1.2 Three-point functions

We can study the OPE (7.4) by looking at three-point functions[3]

$$\langle J(\mathbf{x}_1)J(\mathbf{x}_2)\mathcal{O}_{\Delta}^{(\ell,\bar{\ell})}(\mathbf{x}_3)\rangle. \tag{7.13}$$

The allowed values for ℓ and $\bar{\ell}$ are given in (7.5). The three-point function satisfies the permutation constraint

$$\langle J(\mathbf{x}_1)J(\mathbf{x}_2)\mathcal{O}_{\Delta}^{(\ell,\bar{\ell})}(\mathbf{x}_3)\rangle = \pi_{12}\langle J(\mathbf{x}_1)J(\mathbf{x}_2)\mathcal{O}_{\Delta}^{(\ell,\bar{\ell})}(\mathbf{x}_3)\rangle\,, \tag{7.14}$$

[2] Operators saturating the unitarity bound for $\ell\bar{\ell} = 0$ can only appear in free theories [173].

[3] Indeed, by multiplying (7.4) by the conjugated operator $\mathcal{O}_\rho$ and taking the vacuum expectation value, we obtain an equality between the three-point function and the function C_ρ.

due to the presence of the identical operators J. Here the permutation operation π_{ij} is defined by

$$\pi_{ij}: \quad \mathbf{x}_i \leftrightarrow \mathbf{x}_j\,. \tag{7.15}$$

The correlator also satisfies the conservation constraints

$$\partial_i \cdot \langle J(\mathbf{x}_1)J(\mathbf{x}_2)\mathcal{O}_\Delta^{(\ell,\bar{\ell})}(\mathbf{x}_3)\rangle = 0\,, \qquad i=1,2\,. \tag{7.16}$$

Both (7.15) and (7.16) are constraints that must be imposed the three-point function. In order to do that, we first consider a more general correlator, by assuming only conformal invariance

$$\langle V_1(\mathbf{x}_1)V_2(\mathbf{x}_2)\mathcal{O}_\Delta^{(\ell,\bar{\ell})}(\mathbf{x}_3)\rangle\,, \qquad V_i(\mathbf{x}) \coloneqq \mathcal{O}_\Delta^{(1,1)}(\mathbf{x})\,. \tag{7.17}$$

Here V_i is a generic vector operator. We then impose the constraints (7.15) and (7.16) to obtain the final form of (7.13) at the end. The three-point function under study can be expanded as (1.16)

$$\langle V_1(\mathbf{x}_1)V_2(\mathbf{x}_2)\mathcal{O}_\Delta^{(\ell,\bar{\ell})}(\mathbf{x}_3)\rangle = \sum_{a=1}^{n(\ell,\bar{\ell})} \lambda^{(a)}_{V_1V_2\mathcal{O}_\Delta^{(\ell,\bar{\ell})}}\, \mathrm{t}^a_{V_1V_2\mathcal{O}_\Delta^{(\ell,\bar{\ell})}}(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3)\,, \tag{7.18}$$

where $n(\ell,\bar{\ell})$ is a shorthand for $n(V_1V_2\mathcal{O}_\Delta^{(\ell,\bar{\ell})})$ defined in (1.40) and it counts the number of tensor structures. For these correlators the values are given by

$$n(\ell,\ell) = \begin{cases} 5^+ + 1^- & \ell \geqslant 2 \\ 4^+ + 1^- & \ell = 1 \\ 2^+ & \ell = 0 \end{cases} \qquad n(\ell+2,\ell) = \begin{cases} 4 & \ell \geqslant 1 \\ 3 & \ell = 0 \end{cases} \qquad n(\ell+4,\ell) = 1\,, \tag{7.19}$$

with the superscripts plus and minus denoting the eigenvalue of the structure under a P parity transformation. The tensor structures $\mathrm{t}^a_{V_1V_2\mathcal{O}}$ depend on the scaling dimensions through the kinematic factor $\mathcal{K}_3$ defined in (1.27). In this case it reads

$$\mathcal{K}_3^{-1} = x_{12}^{\Delta_1+\Delta_2-\Delta-\ell-\frac{p}{2}+2}\, x_{13}^{\Delta_1-\Delta_2+\Delta+\ell+\frac{p}{2}}\, x_{23}^{\Delta_2-\Delta_1+\Delta+\ell+\frac{p}{2}}\,. \tag{7.20}$$

Δ_i being the dimension of V_i, which will be set to three at the end.

Generic constraints on (7.18) come from parity symmetry, time-reversal and complex conjugation. We do not require explicitly neither parity nor time-reversal. Besides the attempt of being more general, the requirement of these symmetries does not give any sizable constraints on the setup. We deduce the constraints from complex conjugation by applying it to (7.18). We get the relation[4]

$$\langle V_1(\mathbf{x}_1)V_2(\mathbf{x}_2)\mathcal{O}_\Delta^{(\ell,\bar{\ell})}(\mathbf{x}_3)\rangle^* = \langle V_1(\mathbf{x}_1)V_2(\mathbf{x}_2)\mathcal{O}_\Delta^{(\bar{\ell},\ell)}(\mathbf{x}_3)\rangle\,. \tag{7.21}$$

[4]We work in equal time quantization in Lorentzian signature, and hence for a generic correlator $\langle\mathcal{O}_1(\mathbf{x}_1)\mathcal{O}_2(\mathbf{x}_2)\mathcal{O}_3(\mathbf{x}_3)\rangle^* = \langle\overline{\mathcal{O}}_3(\mathbf{x}_3)\overline{\mathcal{O}}_2(\mathbf{x}_2)\overline{\mathcal{O}}_1(\mathbf{x}_1)\rangle$.

The tensor structures $\mathfrak{t}^a_{V_1V_2\mathcal{O}}$ have well defined transformation properties under complex conjugation. The condition (7.21) then translates into relations between the OPE coefficients. In what follows we will define the basis of tensor structures for the operators with $\ell \geqslant \bar{\ell}$ only. The basis for the conjugate operators with $\ell < \bar{\ell}$ is chosen in such a way that the associated OPE coefficients are related through (7.21) to the ones with $\ell \geqslant \bar{\ell}$ in the following simple way

$$\lambda^{(a)*}_{V_1V_2\mathcal{O}^{(\ell+p,\ell)}_\Delta} = \lambda^{(a)}_{V_1V_2\mathcal{O}^{(\ell,\ell+p)}_\Delta}\,. \tag{7.22}$$

Since the $p = 0$ operators are hermitian and the three-point function is mapped to itself, the relation above implies that all $p = 0$ OPE coefficients are real.

In order to obtain the conformal partial waves we also need to define the three-point functions related to (7.18) by a π_{13} permutation

$$\langle \mathcal{O}^{(\ell,\bar{\ell})}_\Delta(\mathbf{x}_1)V_2(\mathbf{x}_2)V_1(\mathbf{x}_3)\rangle = \sum_{a=1}^{n(\ell,\bar{\ell})} \lambda^{(a)}_{\mathcal{O}^{(\ell,\bar{\ell})}_\Delta V_2V_1}\,\mathfrak{t}^a_{\mathcal{O}^{(\ell,\bar{\ell})}_\Delta V_2V_1}(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3)\,. \tag{7.23}$$

We choose their basis in such a way that

$$\lambda^{(a)}_{\mathcal{O}^{(\ell,\bar{\ell})}_\Delta V_2V_1} = \lambda^{(a)}_{V_1V_2\mathcal{O}^{(\ell,\bar{\ell})}_\Delta}\,. \tag{7.24}$$

Together with (7.22) this implies that there is the following relation between the OPE coefficients

$$\lambda^{(a)}_{\mathcal{O}^{(\ell,\bar{\ell})}_\Delta V_2V_1} = \lambda^{(a)*}_{V_1V_2\mathcal{O}^{(\bar{\ell},\ell)}_\Delta}\,. \tag{7.25}$$

This relation will be crucial for setting up the semidefinite problem.

Now we impose conservation (7.16) and permutation symmetry (7.14). These two requirements lead to a system of linear equations on the OPE coefficients entering (7.18). We solve them in terms of a smaller set of independent OPE coefficients $\hat{\lambda}$ and plug the solution back in (7.18). This defines in turn a basis of conserved and π_{12}-symmetric tensor structures that we denote as $\hat{\mathfrak{t}}^a_{JJ\mathcal{O}}$. As a result we get

$$\langle J(\mathbf{x}_1)J(\mathbf{x}_2)\mathcal{O}^{(\ell,\bar{\ell})}_\Delta(\mathbf{x}_3)\rangle = \sum_{a=1}^{\hat{n}(\ell,\bar{\ell})} \hat{\lambda}^{(a)}_{JJ\mathcal{O}^{(\ell,\bar{\ell})}_\Delta}\,\hat{\mathfrak{t}}^{\,a}_{JJ\mathcal{O}^{(\ell,\bar{\ell})}_\Delta}(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3)\,, \tag{7.26}$$

where $\hat{n}(\ell,\bar{\ell})$ is the new number of independent OPE coefficients $\hat{\lambda}$. The values of $\hat{n}(\ell,\bar{\ell})$ are given by

$$\hat{n}(\ell,\ell) = \begin{cases} 2 & \ell \geqslant 2 \text{ even} \\ 1 & \text{else} \end{cases} \qquad \hat{n}(\ell+2,\ell) = \begin{cases} 0 & \substack{\ell=0 \text{ and} \\ \Delta\neq 2} \\ 1 & \text{else} \end{cases} \qquad \hat{n}(\ell+4,\ell) = \begin{cases} 0 & \ell \text{ odd} \\ 1 & \ell \text{ even} \end{cases} \tag{7.27}$$

The associated conserved and π_{12}-symmetric tensor structures can be related to the old ones through a rectangular matrix. More specifically we can define the $\hat{n} \times n$ matrix M as follows

$$\hat{\mathrm{t}}^{a}_{JJ\mathcal{O}_\Delta^{(\ell+p,\ell)}}(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3) = \sum_{a=1}^{n(\ell+p,\ell)} (M_{p,\ell})^{ab}\, \mathrm{t}^{b}_{V_1V_2\mathcal{O}_\Delta^{(\ell+p,\ell)}}(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3)\,. \tag{7.28}$$

It is important to check that these matrices are always non-singular for any unitary values of Δ. In what follows we will precisely define the basis of tensor structures in (7.18) and specify the matrices M for $p = 0, 2$ and 4.

7.1.3 Basis for $p = 0$

In the $p = 0$ case there are six independent tensor structures which we choose to be

$$\mathrm{t}^{1}_{V_1V_2\mathcal{O}_\Delta^{(\ell,\ell)}}(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3) \coloneqq \mathbb{I}^{12}\mathbb{I}^{21}(\mathbb{J}^{3}_{12})^{\ell}\,, \tag{7.29a}$$

$$\mathrm{t}^{2}_{V_1V_2\mathcal{O}_\Delta^{(\ell,\ell)}}(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3) \coloneqq \mathbb{J}^{1}_{23}\mathbb{J}^{2}_{13}(\mathbb{J}^{3}_{12})^{\ell}\,, \tag{7.29b}$$

$$\mathrm{t}^{3}_{V_1V_2\mathcal{O}_\Delta^{(\ell,\ell)}}(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3) \coloneqq \mathbb{I}^{13}\mathbb{I}^{31}\mathbb{J}^{2}_{13}(\mathbb{J}^{3}_{12})^{\ell-1}\,, \tag{7.29c}$$

$$\mathrm{t}^{4}_{V_1V_2\mathcal{O}_\Delta^{(\ell,\ell)}}(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3) \coloneqq \mathbb{I}^{23}\mathbb{I}^{32}\mathbb{J}^{1}_{23}(\mathbb{J}^{3}_{12})^{\ell-1}\,, \tag{7.29d}$$

$$\mathrm{t}^{5}_{V_1V_2\mathcal{O}_\Delta^{(\ell,\ell)}}(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3) \coloneqq \mathbb{I}^{13}\mathbb{I}^{31}\mathbb{I}^{23}\mathbb{I}^{32}(\mathbb{J}^{3}_{12})^{\ell-2}\,, \tag{7.29e}$$

$$\mathrm{t}^{6}_{V_1V_2\mathcal{O}_\Delta^{(\ell,\ell)}}(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3) \coloneqq i\left(\mathbb{I}^{12}\mathbb{I}^{23}\mathbb{I}^{31} + \mathbb{I}^{21}\mathbb{I}^{13}\mathbb{I}^{32}\right)(\mathbb{J}^{3}_{12})^{\ell-1}\,. \tag{7.29f}$$

The tensor structures (7.29) have simple transformation properties under parity. More precisely, the first five structures (7.29a)–(7.29e) are parity even and the last one (7.29f) is parity odd. As we already mentioned, we do not require parity symmetry in our setup.

The $\ell = 0$ and $\ell = 1$ are special cases since not all the six structures exist. In particular (7.29d) is absent for $\ell \leqslant 1$ and (7.29c)–(7.29f) are absent for $\ell = 0$.

We will now provide the matrices M appearing in (7.28). There are three different cases to be considered. When $\ell = 0$ there is a single structure therefore the matrix is 1×2 and it reads

$$M_{0,0} = \begin{pmatrix} 2(3-\Delta) & \Delta \end{pmatrix}. \tag{7.30}$$

When $\ell \geqslant 1$ and is odd again we have a single structure and the matrix can be written as

$$M_{0,\ell\ \mathrm{odd}} = \begin{pmatrix} 0_n & 1 \end{pmatrix}, \tag{7.31}$$

where 0_n is a row of $n(\ell,\ell)$ zeros, which means a row of length four for $\ell = 1$ and of length five for $\ell \geqslant 3$. In the very special case of the conserved current J when $\ell = 1$ and

$\Delta = 3$ all the additional constraints are satisfied automatically. We can however use in addition the Ward identities to relate the OPE coefficient to the two-point normalization of the conserved current J given in (7.11). Since J has no $\mathrm{U}(1)$ charge and its only three-point tensor structure is parity odd while its two-point function is parity even, both sides of the the Ward identity vanish and therefore there are no further constraints.

The case for even $\ell \geqslant 2$ is the only one with two independent structures. The associated matrix M is therefore 2×6 and it reads

$$M_{0,\ell \text{ even}} = \begin{pmatrix} 2(4+\ell-\Delta)(\Delta-3) & -(4+\ell-\Delta)(\Delta+\ell) & 0 & 0 & 4\ell(\Delta-3) & 0 \\ 4(\Delta-2)(\Delta-3) & 2\ell(\ell+8)-2\Delta(\Delta+2\ell-2) & -4\ell(\Delta-3) & 4\ell(\Delta-3) & 0 & 0 \end{pmatrix}. \tag{7.32}$$

A very special situation is given by the stress tensor T when $\ell = 2$ and $\Delta = 4$. No further constraints appear on (7.28). However, due to the Ward identities, we can relate the OPE coefficients to the two-point function normalization of two currents J given in (7.11) as

$$\hat{\lambda}^{(2)}_{\langle JJT\rangle} = \frac{1}{2}\hat{\lambda}^{(1)}_{\langle JJT\rangle} - \frac{C_J}{8\pi^2}\,. \tag{7.33}$$

This computation follows from the discussion in Section 1.4. However, in this chapter we have made a minor change in the conventions. Namely we do not include the factor of two in the right hand sides of (1.45) and, of course, due to the normalization of J, the two-point functions n_{JJ} take an overall C_J factor.

The Hofman-Maldacena bounds [136] impose an inequality on a combination of these coefficients. Following [84] we can define a parameter γ as follows

$$\hat{\lambda}^{(1)}_{\langle JJT\rangle} = \frac{C_J(1-32\gamma)}{36\pi^2}\,, \qquad \hat{\lambda}^{(2)}_{\langle JJT\rangle} = -\frac{C_J(1+4\gamma)}{9\pi^2}\,. \tag{7.34}$$

Then γ needs to satisfy the inequality

$$-\frac{1}{16} \leqslant \gamma \leqslant \frac{1}{32}\,. \tag{7.35}$$

The two extremes of this window are associated to free theories. More specifically $\gamma = -1/16$ corresponds to the free complex boson and $\gamma = 1/32$ corresponds to the free fermion.

7.1.4 Basis for $p = 2$

The second class of three-point functions is given by $p = 2$ operators. The number of independent structures is summarized in (7.19). For general $\ell \geqslant 1$ the basis is taken to

be

$$\mathsf{t}^1_{V_1V_2\mathcal{O}_\Delta^{(\ell+2,\ell)}}(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3) \coloneqq \mathbb{I}^{12}\mathbb{I}^{23}\mathbb{K}_2^{13}(\mathbb{J}_{12}^3)^\ell\,, \tag{7.36a}$$

$$\mathsf{t}^2_{V_1V_2\mathcal{O}_\Delta^{(\ell+2,\ell)}}(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3) \coloneqq \mathbb{I}^{13}\mathbb{I}^{21}\mathbb{K}_1^{23}(\mathbb{J}_{12}^3)^\ell\,, \tag{7.36b}$$

$$\mathsf{t}^3_{V_1V_2\mathcal{O}_\Delta^{(\ell+2,\ell)}}(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3) \coloneqq \mathbb{I}^{13}\mathbb{I}^{31}\mathbb{I}^{23}\mathbb{K}_1^{23}(\mathbb{J}_{12}^3)^{\ell-1}\,, \tag{7.36c}$$

$$\mathsf{t}^4_{V_1V_2\mathcal{O}_\Delta^{(\ell+2,\ell)}}(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3) \coloneqq \mathbb{I}^{23}\mathbb{I}^{32}\mathbb{I}^{13}\mathbb{K}_2^{13}(\mathbb{J}_{12}^3)^{\ell-1}\,. \tag{7.36d}$$

In the special case for $\ell = 0$ we can only write three independent structures, which are given by

$$\mathsf{t}^1_{V_1V_2\mathcal{O}_\Delta^{(2,0)}}(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3) \coloneqq \mathbb{I}^{12}\mathbb{I}^{23}\mathbb{K}_2^{13}\,, \tag{7.37a}$$

$$\mathsf{t}^2_{V_1V_2\mathcal{O}_\Delta^{(2,0)}}(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3) \coloneqq \mathbb{I}^{13}\mathbb{I}^{21}\mathbb{K}_1^{23}\,, \tag{7.37b}$$

$$\mathsf{t}^3_{V_1V_2\mathcal{O}_\Delta^{(2,0)}}(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3) \coloneqq \mathbb{I}^{13}\mathbb{I}^{23}\mathbb{K}_3^{12}\,. \tag{7.37c}$$

The conservation and permutation conditions are very restrictive for $\ell = 0$ and happen to not only fix the OPE coefficients but the dimension Δ as well. We have

$$\Delta = 2\,, \qquad M_{2,0} = \begin{pmatrix} 1 & 1 & 0 \end{pmatrix}. \tag{7.38}$$

For this reason, only the operators saturating the unitarity bound $\Delta_{\text{unitarity}}$ in (7.5) are allowed. According to [173] such operators can only belong to a decoupled free subsector of the theory. The conservation condition for $\ell \geqslant 1$, on the other hand, has a nontrivial solution. We obtain a 1×4 matrix which, if ℓ is even, reads

$$M_{2,\ell\,\text{even}} = \begin{pmatrix} x & x & -z & z \end{pmatrix}, \qquad x \coloneqq \ell + 6 - \Delta\,, \quad z \coloneqq 2(\Delta - 2)\,, \tag{7.39}$$

whereas if ℓ is odd it reads

$$M_{2,\ell\,\text{even}} = \begin{pmatrix} -x & x & y & y \end{pmatrix}, \qquad y \coloneqq 2(\ell + 2)\,. \tag{7.40}$$

7.1.5 Basis for $p = 4$

The last case to be considered is that of $p = 4$. Luckily there are no special cases to be treated separately. We only have one allowed structure which reads

$$\mathsf{t}_{V_1V_2\mathcal{O}_\Delta^{(\ell+4,\ell)}}(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3) \coloneqq \mathbb{I}^{13}\mathbb{I}^{23}\mathbb{K}_2^{13}\mathbb{K}_1^{23}(\mathbb{J}_{12}^3)^\ell\,. \tag{7.41}$$

This structure is automatically conserved and π_{12}-permutation symmetry forces ℓ to be even. Thus M is a very simple 1×1 matrix that reads

$$M_{4,\ell\,\text{even}} = (1)\,, \qquad M_{4,\ell\,\text{odd}} = (0)\,. \tag{7.42}$$

7.2 Four-point tensor structures

We now perform a detailed analysis of the four-point function (7.2) and the kinematic constraints that it must satisfy. Then in Section 8.1 we will analyze the constraints due to crossing. First, we need to define a basis of tensor structures for the four-point function of four generic vectors

$$\langle V_1(\mathbf{x}_1)V_2(\mathbf{x}_2)V_3(\mathbf{x}_3)V_4(\mathbf{x}_4)\rangle\,. \tag{7.43}$$

Then we proceed by studying its properties under complex conjugation, permutation and conservation and its analytic properties. Contrary to Section 7.1 we will work in conformal frame. This formalism has been introduced in Section 1.3 and then specialized for four-point functions in Section 6.3.

The four point function (7.43) can be expanded in a basis of 70 structures, before imposing any kinematic constraint

$$\langle V_1(\mathbf{x}_1)V_2(\mathbf{x}_2)V_3(\mathbf{x}_3)V_4(\mathbf{x}_4)\rangle = \sum_{i=1}^{70} \mathbb{T}_i^0\, f_i^0(z,\bar{z})\,. \tag{7.44}$$

This is going to be just an interim basis, hence the superscript "0". The explicit definition of all 70 structures can be found in Appendix D.2 and it uses the conformal frame notation given in (6.34). The cross ratios $z, \bar{z}$ have been defined in (6.13). Analogously to the $p = 0$ three-point structures in (7.29), the four-point structures $\mathbb{T}_i^0$ have well defined transformation properties under P parity. It could be convenient to label them according to these properties even if we do not require parity symmetry in our setup. However here we decided not to do so. The 70 structures split into

$$n(V_1V_2V_3V_4) = 70 = 43^+ + 27^-\,, \tag{7.45}$$

where $+$ stands for parity even and $-$ for parity odd. Using complex conjugation[5,6] one can write the following identity

$$\langle V_1(\mathbf{x}_1)V_2(\mathbf{x}_2)V_3(\mathbf{x}_3)V_4(\mathbf{x}_4)\rangle^* = \langle V_4(\mathbf{x}_4)V_3(\mathbf{x}_3)V_2(\mathbf{x}_2)V_1(\mathbf{x}_1)\rangle\,. \tag{7.46}$$

The first 16 functions f_i^0 are real and the remaining ones come in complex conjugate

[5]Similar to the three-point unction case, see 4, for a generic four-point function we have $\langle \mathcal{O}_1(\mathbf{x}_1)\mathcal{O}_2(\mathbf{x}_2)\mathcal{O}_3(\mathbf{x}_3)\mathcal{O}_4(\mathbf{x}_4)\rangle^* = \langle \overline{\mathcal{O}}_4(\mathbf{x}_4)\overline{\mathcal{O}}_3(\mathbf{x}_3)\overline{\mathcal{O}}_2(\mathbf{x}_2)\overline{\mathcal{O}}_1(\mathbf{x}_1)\rangle$

[6]We work in Lorentzian signature. In particular, Hermitian conjugation does not act on the coordinates to the local operators and the cross ratios $z, \bar{z}$ are real.

pairs. In the basis that we will define later the associated functions will be all real.

7.2.1 Kinematic permutations

Let us now discuss the constraints due to permutation symmetry. As we said earlier, we are going to discuss crossing symmetry in Section 8.1. Here we are only interested in the permutations that map the cross ratios to themselves, so that they imply constraints for the functions f_i^0 that hold for each $z, \bar{z}$, instead of relating the functions at different points. The permutations that leave $z, \bar{z}$ invariant are called kinematic and they have been introduced in (6.6). We find that by imposing

$$\langle V(\mathbf{x}_1)V(\mathbf{x}_2)V(\mathbf{x}_3)V(\mathbf{x}_4)\rangle = \pi\langle V(\mathbf{x}_1)V(\mathbf{x}_2)V(\mathbf{x}_3)V(\mathbf{x}_4)\rangle\,, \qquad \pi \in (6.6)\,, \tag{7.47}$$

the number of independent structures is reduced to

$$n(VVVV) = 22 = 19^+ + 3^-\,. \tag{7.48}$$

Furthermore we want to make sure that the functions have definite properties under the exchange $\zeta\colon\ z \leftrightarrow \bar{z}$. This can be achieved by studying a rotation in the plane 0–2, as one can see from (6.31). On the conformal frame structures the transformation amounts to

$$\zeta\colon\ \begin{bmatrix} q_1 & q_2 & q_3 & q_4 \\ \bar{q}_1 & \bar{q}_2 & \bar{q}_3 & \bar{q}_4 \end{bmatrix} f(z,\bar{z}) \ \to\ i^{\sum_i \ell_i - \bar{\ell}_i} \begin{bmatrix} -q_1 & -q_2 & -q_3 & -q_4 \\ -\bar{q}_1 & -\bar{q}_2 & -\bar{q}_3 & -\bar{q}_4 \end{bmatrix} f(\bar{z},z)\,. \tag{7.49}$$

After taking the appropriate linear combinations of structures that diagonalize ζ we can write

$$\langle V(\mathbf{x}_1)V(\mathbf{x}_2)V(\mathbf{x}_3)V(\mathbf{x}_4)\rangle = \sum_{i=1}^{22} \mathbb{T}_i\, f_i(z,\bar{z})\,. \tag{7.50}$$

These new structures are defined in Appendix D.2 in terms of the structures $\mathbb{T}_i^0$ defined in the previous subsection. This result can be obtained by acting with the permutations π in (6.6) and with ζ in (7.49) on the conformal frame structures. The factors of $z, \bar{z}$ appear because the permutations do not leave the conformal frame unchanged. Therefore one has to act with a transformation r_π to restore the points to the original frame. Since the four operators lie on a plane, the transformation r_π can be taken to be an element of the two dimensional conformal group $\mathrm{SL}(2,\mathbb{C})$. More details can be found in Appendix B of [93].

The functions f_i are all real and the structures $\mathbb{T}_i$ are CPT invariant. They also have definite properties under P parity and $z \leftrightarrow \bar{z}$ parity: the first 14 are P and $z \leftrightarrow \bar{z}$ even, the structures $\mathbb{T}_{15,16,17}$ are P odd and $z \leftrightarrow \bar{z}$ even and the remaining ones are P even and $z \leftrightarrow \bar{z}$ odd.

7.2.2 Smoothness at $z = \bar{z}$

As discussed in Subsection 6.3.2, the point $z = \bar{z}$ is a point of enhanced symmetry and this implies that the functions $f_i(z, \bar{z})$ must satisfy some smoothness condition as $z \to \bar{z}$. By diagonalizing the action of the $\mathrm{O}(3)$ Casimir operator on the f_i's we can define a new basis of functions, where the smoothness properties become manifest

$$f^{\mathrm{O}(3)}_{J,i}(z, \bar{z}) \,, \qquad f^{\mathrm{O}(3)}_{J,i}(z, \bar{z}) \propto (z - \bar{z})^J \,. \tag{7.51}$$

We can use the formula (6.39) to predict the number of functions in each spin sector.[7] However, now we need to restrict the counting to only those structures that are permutation symmetric. In order to do that, it suffices to take the singlets under the group of kinematic permutations $\mathbb{Z}_2 \times \mathbb{Z}_2$. All in all we get

$$\left(\mathrm{Res}^{\mathrm{O}(4)}_{\mathrm{O}(3)} \bigotimes_{k=1}^{4} (1,1) \right)^{\mathbb{Z}_2 \times \mathbb{Z}_2} = 7 \cdot \mathbf{0}^+ \oplus \mathbf{0}^- \oplus 4 \cdot \mathbf{1}^+ \oplus 6 \cdot \mathbf{2}^+ \oplus 8 \cdot \mathbf{2}^- \oplus \mathbf{3}^+ \oplus \mathbf{4}^+ \,, \tag{7.52}$$

where we indicated the $\mathrm{O}(3)$ representation of spin J and parity π by $\mathbf{J}^\pi$. In particular, the number of independent functions $f^{\mathrm{O}(3)}_{0,i}(z, \bar{z})$ is eight and the non singlets are fourteen.

We will not reproduce here explicitly the Casimir operator in the basis (D.11), nor the definition of the functions (7.51) in terms of the f_i. Before studying the smoothness conditions we introduce a modification of our basis f_i so that all functions are even under $z \leftrightarrow \bar{z}$. As a consequence, the constraints of spin **1** will be automatically satisfied, since the formerly odd functions now vanish as $(z - \bar{z})$. The only orders on which the regularity condition is not trivial are the order $(z - \bar{z})^2$ at spin **2**, **4**, order $(z - \bar{z})^3$ at spin **3** and order $(z - \bar{z})^4$ at spin **4**. At the end it will turn out that all these constraints are redundant when combined with conservation. This fact is nontrivial and we checked it explicitly by doing a Taylor expansion of the functions. We will discuss this in the next subsection.

7.2.3 Conservation

Finally we address the most important issue of this analysis: conservation. We have to impose in the four-point function basis that the current J is a short multiplet, satisfying $\partial_\mu J^\mu = 0$. Unlike the case of three-point functions, where this constraint simply results in a system of linear relations, now we are dealing with a system of linear differential equations. We do not need to explicitly solve this system, but we want to study it in order to understand which functions are left unconstrained and which can be obtained by integrating a suitable set of initial data.

[7] In this subsection we also take into account the parity of the structures for completeness. Therefore in the formula SO is replaced by O.

In the literature there have been two different approaches. In [84] the authors considered an evolution system with initial data on the line $u = v$. This is precisely the crossing symmetric point. Therefore the aim is to start from crossing symmetric data on the line, and evolve them to a fully crossing symmetric function on the whole plane. More concretely, one chooses a "time" direction t and an orthogonal direction y as

$$t = u - v\,, \qquad y = u + v - \tfrac{1}{2}\,, \tag{7.53}$$

and then writes the conservation condition in a matrix form

$$\sum_{i=1}^{22} A_{Ii}\,\partial_t f_i(u,v) + B_{Ii}\,\partial_y f_i(u,v) + C_{Ii}\,f_i(u,v) = 0\,, \qquad I = 1,\dots,20\,. \tag{7.54}$$

Here A, B, C are matrices with t, y dependent entries and 20 is the number of tensor structures of a scalar and three vectors. The functions that live in the kernel of A are not affected by the evolution equation, so we have no additional information on them and we have to impose crossing in the whole plane u, v. We call these "bulk" degrees of freedom. The functions in the image of A instead can be evolved from the data at $u = v$. Now crossing may either act trivially on those functions, thus giving no extra constraints, or it might set them to zero. Those among the latter which are also in the kernel of B wil be called "line" degrees of freedom. On the other hand, the functions that are set to zero on the line and are in the image of B can be evolved from $u = v = 1/4$. Then we might have a condition that sets the function to zero at that point. These last functions will be called "point" degrees of freedom.

Unfortunately, with this method the constraints stemming from smoothness at $z = \bar{z}$ are not easy to impose. We will therefore describe a different approach where these conditions can be introduced much more naturally. It was adopted in [83]. The idea is to consider as initial condition the line $z = \bar{z}$. The time and the orthogonal direction can be then defined as

$$t = \frac{z - \bar{z}}{2}\,, \qquad y = \frac{z + \bar{z} - 1}{2}\,. \tag{7.55}$$

The differential equation looks exactly like (7.54), but obviously the matrices A, B and C will be different. Now the logic changes slightly. It is still true that the functions in the kernel of A constitute the bulk degrees of freedom. Now the line degrees of freedom are strongly constrained by the smoothness condition. Out of the functions in the image of A, only those that belong to $\ker \mathcal{C}_{\mathrm{SO}(3)}$ survive. The other vanish identically in the whole plane as they are the evolution of vanishing initial data. Then, as before, we can use the equations in the image of B to evolve the line degrees of freedom starting from the point $z = \bar{z} = 1/2$. Whether or not these point degrees of freedom will survive depends on the crossing equations.

In this work, we will adopt the second method, namely the one that evolves from the line $z = \bar{z}$. The solution of (7.54) may be expressed as a power series in the time variable

t. We define

$$A_{Ii} = \sum_{n=0}^{\infty} A_{Ii}^{(n)} t^n \,, \qquad B_{Ii} = \sum_{n=0}^{\infty} B_{Ii}^{(n)} t^n \,, \qquad C_{Ii} = \sum_{n=0}^{\infty} C_{Ii}^{(n)} t^n \,, \tag{7.56a}$$

$$f_i(t,y) = \sum_{n=0}^{\infty} f_i^{(n)}(y)\, t^n \,. \tag{7.56b}$$

At order t^{n-1} we have the following ordinary differential equation in y

$$\left(n A^{(0)} + C^{(0)}\right) \cdot f^{(n)}(x) = -\sum_{k=0}^{n-1} \left(k A^{(n-k)} + B^{(n-k-1)} \partial_y + C^{(n-k)}\right) \cdot f^{(k)}(x) \,. \tag{7.57}$$

If $f_i^{(n)}$ is not in the kernel of $n A^{(0)} + C^{(0)}$, then we can recursively determine it from the Taylor coefficients of lower order $f_i^{(k)}$, $k < n$. We thus study the rank of the following matrix

$$E_n \coloneqq n A^{(0)} + C^{(0)} \,, \tag{7.58}$$

as n varies. It turns out that for all n bigger than zero $\operatorname{rk} E_n = 15$, while $\operatorname{rk} E_0 = 14$. This means that we can determine all functions in terms of seven bulk degrees of freedom and possibly a line degree of freedom. To complete the analysis we need to check if some other conditions apply to the line function. Indeed one can define a matrix N such that $N \cdot A = 0$. After multiplying by N on the left we obtain the system

$$\left(N \cdot B\, \partial_y + N \cdot C\right) f_i = 0 \,. \tag{7.59}$$

It is possible to choose the function on the line such that $N \cdot B$ is nonzero when restricted to it. Therefore we can integrate the above equation and end up with just one integration constant. So, to summarize, we have seven bulk degrees of freedom and one point degree of freedom.

We now make a uniform choice of all the bulk functions and the point function. The former are defined by taking the structures $\mathbb{T}$ that have the following property: for every $i = 1,2,3,4$, at least q_i or $\bar{q}_i$ is equal to $+1/2$ and the total charge $|\sum_i q_i| = |\sum_i \bar{q}_i|$ is even. This choice is inspired by studying the kernel of the evolution operator A in the time $t = \bar{z}$.[8] The bulk structures are all parity even. The point degree of freedom instead is parity odd. We have three structures to choose from, one of them is not in the kernel of E_0, so we can choose any of the other two. To summarize, in terms of the basis defined in (D.11), we have

$$\begin{aligned} &\text{Bulk d.o.f:} && f_i(t,y) \,, && i \in \{1,3,4,5,9,10,11\} \,, \\ &\text{Point d.o.f:} && f_{15}(0,0) \,. \end{aligned} \tag{7.60}$$

[8] It is not obvious that a basis that works for a certain choice of time t will be suitable for other choices too. In this case, it happens to work.

We note that all these functions are even under $z \leftrightarrow \bar{z}$.

As a final check we expanded all quantities in both t and y, in a similar fashion as (7.56)

$$f_i(t,y) = \sum_{n,m=0}^{\Lambda} f_i^{(m,n)}\, t^n y^m\,, \qquad \text{etc.} \tag{7.61}$$

Then we plugged this expansion in the conservation equation (7.54) and we imposed the regularity conditions discussed in Subsection 7.2.2. We find that the system obtained by combining all these equations has full rank when restricted to the choice of degrees of freedom shown in (7.60). This means that they are truly unconstrained and they can be used to determine all other functions. Moreover, we observe that the regularity constraints do not add anything on top of conservation.

7.3 Conformal blocks

7.3.1 Decomposition into conformal blocks

By using the OPE (7.4) we can express the four-point function (7.2) as a sum over the exchanged primary operators. In this way we can express the functions f_i defined in Section 7.2 in terms of the CFT data. In what follows we discuss this decomposition in details.

We apply the OPE (7.4) to the first two and the last two operators. As a result, each function f_i may be expanded in conformal blocks in the same way as in (6.5). Namely we have

$$f_i(z,\bar{z}) = \sum_{\Delta,\ell,\bar{\ell}} \sum_{a=1}^{n_{12\rho}} \sum_{b=1}^{n_{\rho^*34}} \hat{\lambda}^{(a)}_{JJ\mathcal{O}^{(\ell,\bar{\ell})}_{\Delta}} \hat{\lambda}^{(b)}_{\mathcal{O}^{(\bar{\ell},\ell)}_{\Delta}JJ}\, G^{a,b}_{i,\Delta,(\ell,\bar{\ell})}(z,\bar{z})\,. \tag{7.62}$$

Notice that the blocks are not automatically $\pi_{13}\pi_{24}$ or $\pi_{14}\pi_{23}$–symmetric. Since the conformal block $G^{a,b}_{i,\Delta,(\ell,\bar{\ell})}$ is built out of the conserved and π_{12}, π_{34}–symmetric structures, it automatically inherits their properties. Also, since all the structures are conserved, the topology of blocks is simple (all the blocks are disconnected) and thus there is no "fake primary effect" in this case. See Section 5 of [82].

The conformal blocks follow the normalization of two-point functions discussed previously. Namely for most operators we use (7.9) and for the conserved currents J and the stress tensor T we use (7.11) instead. As a consequence, the associated conformal blocks that we compute should be rescaled as[9]

$$G^{ab}_J \longrightarrow \frac{1}{C_J} G^{ab}_J\,, \quad G^{ab}_T \longrightarrow \frac{1}{C_T} G^{ab}_T\,. \tag{7.63}$$

[9]Notice that the OPE functions C get rescaled as well but in an opposite way to two-point functions. See footnote 3.

Following the discussion surrounding (6.17), we will also define the matrices

$$P^{ba}_{\Delta,(\ell,\bar{\ell})} \coloneqq \sum_{\mathcal{O}^{(\ell,\bar{\ell})}_{\Delta}} \hat{\lambda}^{(a)}_{JJ\mathcal{O}^{(\ell,\bar{\ell})}_{\Delta}} \hat{\lambda}^{(b)}_{\mathcal{O}^{(\ell,\bar{\ell})}_{\Delta}JJ}\,, \tag{7.64}$$

to allow for possible degeneracies in the spectrum. According to the discussion in Section 7.1, the operators contributing to JJ OPE come in five different families. Thus there are also five families of matrices P:

$$P^{ab}_{\Delta,(\ell,\ell)}\,, \quad P_{\Delta,(\ell+2,\ell)}\,, \quad P_{\Delta,(\ell,\ell+2)}\,, \quad P_{\Delta,(\ell+4,\ell)}\,, \quad P_{\Delta,(\ell,\ell+4)}\,. \tag{7.65}$$

The first matrix in (7.65) is 2×2 for $\ell \geqslant 2$ even and 1×1 for $\ell = 0$ and $\ell \geqslant 1$ odd. The rest are the 1×1 matrices or simply real non-negative numbers. The very last matrix is non-vanishing only for ℓ even. Due to (7.22) we have the following relations

$$P_{\Delta,(\ell,\ell+2)} = P_{\Delta,(\ell+2,\ell)}\,, \qquad P_{\Delta,(\ell,\ell+4)} = P_{\Delta,(\ell+4,\ell)}\,.$$

There is a corresponding relation between the non traceless symmetric partial waves[10] which reads

$$W_{\Delta,(\ell,\ell+2)}(\mathbf{x}_i) = \pi_{13}\pi_{24} W_{\Delta,(\ell+2,\ell)}(\mathbf{x}_i)\,, \tag{7.66a}$$

$$W_{\Delta,(\ell,\ell+4)}(\mathbf{x}_i) = \pi_{13}\pi_{24} W_{\Delta,(\ell+4,\ell)}(\mathbf{x}_i)\,. \tag{7.66b}$$

One can then define $\pi_{13}\pi_{24}$–symmetric blocks as

$$W^{\pi_{13}\pi_{24}}_{\Delta,(\ell+2,\ell)}(\mathbf{x}_i) \coloneqq W_{\Delta,(\ell+2,\ell)}(\mathbf{x}_i) + \pi_{13}\pi_{24} W_{\Delta,(\ell+2,\ell)}(\mathbf{x}_i)\,, \tag{7.67a}$$

$$W^{\pi_{13}\pi_{24}}_{\Delta,(\ell+4,\ell)}(\mathbf{x}_i) \coloneqq W_{\Delta,(\ell+4,\ell)}(\mathbf{x}_i) + \pi_{13}\pi_{24} W_{\Delta,(\ell+4,\ell)}(\mathbf{x}_i)\,. \tag{7.67b}$$

Using these and the relations (7.66) one can rewrite (7.62) in the following way

$$\begin{aligned} f_i(z,\bar{z}) = &\sum_{\Delta,\ell} \mathrm{tr}\left(P_{\Delta,(\ell,\ell)}\, G_{i,\Delta,(\ell,\ell)}(z,\bar{z})\right) \\ &+ \sum_{\Delta,\ell} P_{\Delta,(\ell+2,\ell)}\, G^{\pi_{13}\pi_{24}}_{i,\Delta,(\ell+2,\ell)}(z,\bar{z}) + \sum_{\Delta,\ell} P_{\Delta,(\ell+4,\ell)}\, G^{\pi_{13}\pi_{24}}_{i,\Delta,(\ell+4,\ell)}(z,\bar{z})\,, \end{aligned} \tag{7.68}$$

where $G^{\pi_{13}\pi_{24}}_{i,\Delta,(\ell+p,\ell)}$ are obtained by expanding in the basis of four point tensor structures the partial waves in (7.67).

7.3.2 Computation of the conformal blocks

We are now ready to write down the conformal blocks. We will follow the strategy described in Section 6.4. Namely we will provide explicitly the differential operators

[10] Recall the relation between a partial wave and a conformal block (6.50).

that must be applied on the seed partial waves. The latter have been already computed for p up to four, as we reviewed in Subsection 6.4.3.

We allow ourselves to make a small abuse of notation in order to improve readability. The differential operators $\mathbf{D}_{ij} = \{I^{ij}, D_{ij}, \widetilde{D}_{ij}, d_{ij}, \bar{d}_{ij}, \nabla_{ij}\}$ change the conformal dimensions at the points on which they act. This is summarized in equation (6.60). In what follows, whenever we write one of those differential operators, we imagine to compose it with a suitable shift in the conformal dimensions so that their action on the seed leaves the dimensions invariant. In the notation of [91] that means that we rename

$$\mathbf{D}_{ij} \longrightarrow \Xi[\mathbf{D}_{ij}]\,, \tag{7.69}$$

Ξ being the formal dimension shifting operator.

Exchanged operators with $p = 0$ When the exchanged operator is symmetric traceless we can write the left and right three-point structures as

$$\mathrm{t}^{a}_{V_1 V_2 \mathcal{O}_\Delta^{(\ell,\ell)}}(\mathbf{x}_i) = \sum_{b=1}^{6} N^{ab}_{p=0}\, \mathbf{D}^{(b)}_{12,\,p=0}\, \mathrm{t}^{1}_{\phi\phi\mathcal{O}_\Delta^{(\ell,\ell)}}(\mathbf{x}_i)\,, \tag{7.70a}$$

$$\mathrm{t}^{a}_{\mathcal{O}_\Delta^{(\ell,\ell)} V_3 V_4}(\mathbf{x}_i) = \sum_{b=1}^{6} N'^{ab}_{p=0}\, \mathbf{D}^{(b)}_{34,\,p=0}\, \mathrm{t}^{1}_{\mathcal{O}_\Delta^{(\ell,\ell)}\phi\phi}(\mathbf{x}_i)\,, \tag{7.70b}$$

where we abbreviated the scalar operator $\mathcal{O}_3^{(0,0)}$ by ϕ. The differential operators $\mathbf{D}^{(b)}_{ij,\,p=0}$ are given by

$$\begin{aligned}
\mathbf{D}^{(1)}_{ij,\,p=0} &\coloneqq (x_{ij}^2)^{-1/2}\, \mathbb{I}^{ij}\, \mathbb{I}^{ji}\,, & \mathbf{D}^{(4)}_{ij,\,p=0} &\coloneqq D_{ij}\, \widetilde{D}_{ji}\,,\\
\mathbf{D}^{(2)}_{ij,\,p=0} &\coloneqq D_{ij}\, D_{ji}\,, & \mathbf{D}^{(5)}_{ij,\,p=0} &\coloneqq \widetilde{D}_{ij}\, \widetilde{D}_{ji}\,,\\
\mathbf{D}^{(3)}_{ij,\,p=0} &\coloneqq \widetilde{D}_{ij}\, D_{ji}\,, & \mathbf{D}^{(6)}_{ij,\,p=0} &\coloneqq \left(\bar{d}_{ji}\, d_{ij} - d_{ji}\, \bar{d}_{ij}\right)\, D_{ij}\, D_{ji}\,.
\end{aligned} \tag{7.71}$$

The matrices $N^{ab}_{p=0}$ and $N'^{ab}_{p=0}$ are given in Appendix D.1. Using these definition we can write the final expression for the $p = 0$ conformal blocks of conserved currents[11]

$$W^{\hat{a}\hat{b}}_{\Delta,(\ell,\ell)}(\mathbf{x}_i) = \sum_{a,b,c,d=1}^{6} M^{\hat{a}a}_{0,\ell}\, M^{\hat{b}b}_{0,\ell}\, N^{ac}_{p=0}\, N'^{bd}_{p=0}\, \mathbf{D}^{(c)}_{12,\,p=0}\, \mathbf{D}^{(d)}_{34,\,p=0}\, W^{\text{seed}}_{\Delta,(\ell,\ell)}(\mathbf{x}_i)\,, \tag{7.72}$$

where W^{seed} is the partial wave of a symmetric traceless exchange in the four-point function of four scalars, defined in (6.54).

[11] In the $p = 0$ case we do not need to distinguish between primal or dual seed conformal blocks.

Exchanged operators with $p = 2$ For $p = 2$ operators we can write the left and right three-point structures as

$$\mathsf{t}^{a}_{V_1 V_2 \mathcal{O}_\Delta^{(\ell+2,\ell)}}(\mathbf{x}_i) = \sum_{b=1}^{4} N^{ab}_{p=2}\, \mathbf{D}^{\text{left}\,(b)}_{12,\,p=2}\, \mathsf{t}^{1}_{\phi F^{(2)} \mathcal{O}_\Delta^{(\ell+2,\ell)}}(\mathbf{x}_i)\,, \tag{7.73a}$$

$$\mathsf{t}^{a}_{\mathcal{O}_\Delta^{(\ell,\ell+2)} V_3 V_4}(\mathbf{x}_i) = \sum_{b=1}^{4} N'^{ab}_{p=2}\, \mathbf{D}^{\text{right}\,(b)}_{34,\,p=2}\, \mathsf{t}^{1}_{\mathcal{O}_\Delta^{(\ell,\ell+2)} \phi \bar{F}^{(2)}}(\mathbf{x}_i)\,, \tag{7.73b}$$

where we defined the scalar ϕ as before and $F^{(2)}$ as $\mathcal{O}_3^{(2,0)}$ and $\bar{F}^{(2)}$ as its conjugate. The left differential operators are defined as

$$\begin{aligned} \mathbf{D}^{\text{left}\,(1)}_{ij,\,p=2} &\coloneqq \nabla_{ji} D_{ji} \bar{d}_{ji} D_{ij}\,, & \mathbf{D}^{\text{left}\,(3)}_{ij,\,p=2} &\coloneqq \nabla_{ji} D_{ij} \bar{d}_{ji} D_{ji}\,, \\ \mathbf{D}^{\text{left}\,(2)}_{ij,\,p=2} &\coloneqq \nabla_{ji} \widetilde{D}_{ji} \bar{d}_{ji} \widetilde{D}_{ij}\,, & \mathbf{D}^{\text{left}\,(4)}_{ij,\,p=2} &\coloneqq \nabla_{ji} \widetilde{D}_{ij} \bar{d}_{ji} \widetilde{D}_{ji}\,. \end{aligned} \tag{7.74}$$

The right differential operators are given by this simple substitution

$$\mathbf{D}^{\text{right}\,(p)}_{ij,\,p=2} \coloneqq \mathbf{D}^{\text{left}\,(p)}_{ij,\,p=2}\big|_{\nabla_{ji} \to \nabla_{ij}}\,. \tag{7.75}$$

The matrices $N^{ab}_{p=2}$ and $N'^{ab}_{p=2}$ are defined in Appendix D.1. Now we can write the final expression for the $p = 2$ conformal partial waves

$$W_{\Delta,(\ell+2,\ell)}(\mathbf{x}_i) = \sum_{a,b,c,d=1}^{4} M^{a}_{2,\ell} M^{b}_{2,\ell} N^{ac}_{p=2} N'^{bd}_{p=2}\, \mathbf{D}^{\text{left}\,(c)}_{12,\,p=2}\, \mathbf{D}^{\text{right}\,(d)}_{34,\,p=2}\, W^{\text{seed}}_{\Delta,(\ell+2,\ell)}(\mathbf{x}_i)\,. \tag{7.76}$$

Exchanged operators with $p = 4$ The last case we need to consider is that of $p = 4$. The left and right three-point structures read

$$\mathsf{t}^{a}_{V_1 V_2 \mathcal{O}_\Delta^{(\ell+4,\ell)}}(\mathbf{x}_i) = N_{p=4}\, \mathbf{D}^{\text{left}}_{12,\,p=4}\, \mathsf{t}^{1}_{\phi F^{(4)} \mathcal{O}_\Delta^{(\ell+4,\ell)}}(\mathbf{x}_i)\,, \tag{7.77a}$$

$$\mathsf{t}^{a}_{\mathcal{O}_\Delta^{(\ell,\ell+4)} V_3 V_4}(\mathbf{x}_i) = N_{p=4}\, \mathbf{D}^{\text{right}}_{34,\,p=4}\, \mathsf{t}^{1}_{\mathcal{O}_\Delta^{(\ell,\ell+4)} \phi \bar{F}^{(4)}}(\mathbf{x}_i)\,, \tag{7.77b}$$

where again $F^{(4)}$ is a shorthand for $\mathcal{O}_3^{(\ell+4,\ell)}$ and $\bar{F}^{(4)}$ is its conjugate. The left and right differential operators are defined as

$$\begin{aligned} \mathbf{D}^{\text{left}}_{12,\,p=4} &\coloneqq \nabla_{ji} \nabla_{ji} D_{ji} \bar{d}_{ji} \bar{d}_{ji} D_{ij}\,, \\ \mathbf{D}^{\text{right}}_{12,\,p=4} &\coloneqq \nabla_{ij} \nabla_{ij} D_{ji} d_{ji} d_{ji} D_{ij}\,. \end{aligned} \tag{7.78}$$

The coefficient $N_{p=4}$ is given in Appendix D.1. As a result the the final expression for the $p = 4$ conformal partial wave reads

$$W_{\Delta,(\ell+4,\ell)}(\mathbf{x}_i) = (N_{p=4})^2\, \mathbf{D}^{\text{left}}_{12,\,p=4}\, \mathbf{D}^{\text{right}}_{34,\,p=4}\, W^{\text{seed}}_{\Delta,(\ell+4,\ell)}(\mathbf{x}_i)\,. \tag{7.79}$$

7.4 Supersymmetric three-point functions

Now it is time to introduce supersymmetry in our setup. In principle we should distinguish two cases: one where the superconformal primary itself contributes to the three-point function, and one where only its superdescendants do. For example, since the operator J is real, one has contributions of superprimaries with vanishing R-charge. But in principle one can expect that a superprimary with R-charge 1 can appear at order $\bar{Q}$. We will now argue that the latter situation never happens. More precisely: all three-point functions in superspace that consist of only nilpotent structures do not have a non zero solution to the conservation equations (2.13a). For this reason, in what follows all operators $\mathcal{O}$ will have vanishing R-charge.

In order to obtain this result we need to count the π_{12}-symmetric and conserved supersymmetric tensor structures by means of the conformal frame formula introduced in Section 2.3. We will find that the number of conserved structures vanishes when the R-charge is not zero, and furthermore we will derive (7.91).

7.4.1 Counting superconformal three-point functions

In Section 2.3 we have derived a formula to count the number of independent tensor structures in superspace for a triplet of superconformal primaries. The formula specialized to the case of four dimensions can be found in (2.48). However, as we remarked previously, the formula (2.48) does not account for the kinematic constraints of π_{12} permutation and conservation. We address now this issue.

The constraints we need to impose are (see (3.8) for the definition of $D^{\pm}_{Q}, D^{\pm}_{\bar{Q}}$)

$$\langle D^{-}_{Q} J(\mathbf{z}_1)\, J(\mathbf{z}_2)\, \mathcal{O}^{(\ell,\bar{\ell})}_{\Delta}(\mathbf{z}_3)\rangle = 0\,, \tag{7.80a}$$

$$\langle D^{-}_{\bar{Q}} J(\mathbf{z}_1)\, J(\mathbf{z}_2)\, \mathcal{O}^{(\ell,\bar{\ell})}_{\Delta}(\mathbf{z}_3)\rangle = 0\,. \tag{7.80b}$$

These conditions are not independent. First we can observe that taking the derivative $D_{\bar{Q}}$ at the second point of (7.80a) and the derivative D_{Q} at the second point of (7.80b) give the same result, modulo permuting the first two operators,

$$\langle D^{-}_{Q} J(\mathbf{z}_1)\, D^{-}_{\bar{Q}} J(\mathbf{z}_2)\, \mathcal{O}^{(\ell,\bar{\ell})}_{\Delta}(\mathbf{z}_3)\rangle = \pi_{12}\langle D^{-}_{\bar{Q}} J(\mathbf{z}_1)\, D^{-}_{Q} J(\mathbf{z}_2)\, \mathcal{O}^{(\ell,\bar{\ell})}_{\Delta}(\mathbf{z}_3)\rangle\,. \tag{7.81}$$

Moreover, by taking D_Q of (7.80a) and permuting points z_1 and z_2 we obtain identically zero. The same holds if we take $D_{\bar{Q}}$ of (7.80b). The prescription to count the number of conserved tensor structures [93] is to take the number of non-conserved tensor structures, subtract all degrees of freedom contained in the equations (7.80) and add back all linear relations between such equations. The complication with supersymmetry is that a superspace equation decomposes into a certain number of ordinary bosonic equations

by projecting on the various terms in (2.46). This depends on the R-charge of $\mathcal{O}$. Let us start assuming that $\mathcal{O}$ is real. The conservation conditions impose a number of constraints equal to the number of tensor structures present in (7.80). This number is given by[12]

$$n_{\bar{\Theta}}(DJ, J, \mathcal{O}) + n_{\Theta\bar{\Theta}^2}(DJ, J, \mathcal{O}) + n_{\Theta}(\bar{D}J, J, \mathcal{O}) + n_{\Theta^2\bar{\Theta}}(\bar{D}J, J, \mathcal{O})\,. \tag{7.83}$$

Even though $n_{\bar{\Theta}} = n_{\Theta^2\bar{\Theta}}$, etc., we keep them distinct to track down the various contributions. As anticipated, however, not all the tensor structures in (7.80) give a non trivial constraint. This is a consequence of the fact that the three-point functions D_Q(7.80a) and $D_{\bar{Q}}$(7.80b) are made of identical operators. To take this into account one must subtract from (7.83) the numbers

$$n_{\bar{\Theta}^2}(DJ, DJ, \mathcal{O})\,, \tag{7.84a}$$

$$n_{\Theta^2}(\bar{D}J, \bar{D}J, \mathcal{O})\,. \tag{7.84b}$$

Similarly, given the relation $D_{\bar{Q}}$(7.80a) $\sim$ D_Q(7.80b), we should naîvely subtract from (7.83) the number

$$n_1(DJ, \bar{D}J, \mathcal{O}) + n_{\Theta\bar{\Theta}}(DJ, \bar{D}J, \mathcal{O}) + n_{\Theta^2\bar{\Theta}^2}(DJ, \bar{D}J, \mathcal{O})\,. \tag{7.85}$$

However, the above expression would give rise to an over-counting: the conditions given by $n_{\Theta^2\bar{\Theta}^2}(DJ, \bar{D}J, \mathcal{O})$ and by $n_1(DJ, \bar{D}J, \mathcal{O})$ are dependent. Indeed, by using a suitable representation of the differential operators,[13] one can show that the terms $\Theta^2\bar{\Theta}^2$ cannot be generated by applying D_Q and $D_{\bar{Q}}$ on $\langle JJ\mathcal{O}\rangle$. Consistently with Section 7.1, we denote with a hat the number of structures after the kinematic constraints have been applied. The correct counting is

$$\begin{aligned}\hat{\mathfrak{n}}(J, J, \mathcal{O}; 0) = {}& \mathfrak{n}(J, J, \mathcal{O}; 0) - \mathfrak{n}(DJ, J, \mathcal{O}; 1) - \mathfrak{n}(\bar{D}J, J, \mathcal{O}; -1) \\ & + \mathfrak{n}(DJ, DJ, \mathcal{O}; 2) + \mathfrak{n}(\bar{D}J, \bar{D}J, \mathcal{O}; -2) \\ & + n_1(DJ, \bar{D}J, \mathcal{O}) + n_{\Theta\bar{\Theta}}(DJ, \bar{D}J, \mathcal{O})\,,\end{aligned} \tag{7.86}$$

where we added an extra argument to $\mathfrak{n}$, namely $\mathfrak{n}(\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3; \delta)$, in order to emphasize the value of the R-charges.[14] In addition, since the currents J are identical, we need to take into account the permutation symmetry as we explained in the previous section by replacing the product $\rho_1 \otimes \rho_2$ by either $S^2\rho_1$ or $\wedge^2\rho_1$. There is a subtlety in the (anti)symmetrization of two DJ's or two $\bar{D}J$'s: these operators get an extra minus due

[12] We denote the various representations in $n_{\mathcal{X}}(\ldots)$ in the following way:

$$J = (1,1)\,, \quad DJ = (0,1)\,, \quad \bar{D}J = (1,0)\,, \quad \mathcal{O} = (\ell, \bar{\ell})\,. \tag{7.82}$$

[13] See Appendix A.3.2, specifically equation (A.32a).

[14] If $\mathcal{O}_i$ has R-charge r_i then $\delta = r_1 + r_2 + r_3$.

to their fermionic nature. Thus for ℓ even (odd) we must take the S^2 ($\wedge^2$) product in $\mathfrak{n}(DJ, DJ, \mathcal{O}; -2)$ and $\mathfrak{n}(\bar{D}J, \bar{D}J, \mathcal{O}; 2)$. Explicitly, for ℓ large enough, this formula yields

$$\begin{aligned} \hat{\mathfrak{n}}(\ell,\ell)^{(\ell\text{ even})} &= 2\,, & \hat{\mathfrak{n}}(\ell,\ell)^{(\ell\text{ odd})} &= 2\,, \\ \hat{\mathfrak{n}}(\ell+2,\ell)^{(\ell\text{ even})} &= 1\,, & \hat{\mathfrak{n}}(\ell+2,\ell)^{(\ell\text{ odd})} &= 2\,, \\ \hat{\mathfrak{n}}(\ell+4,\ell)^{(\ell\text{ even})} &= 1\,, & \hat{\mathfrak{n}}(\ell+4,\ell)^{(\ell\text{ odd})} &= 1\,, \end{aligned} \tag{7.87}$$

where, as before, $\hat{\mathfrak{n}}(\ell,\bar{\ell})$ is a shorthand for $\hat{\mathfrak{n}}(J, J, \mathcal{O}^{(\ell,\bar{\ell})}; 0)$. This result, and the special cases for smaller values of ℓ, are summarized in (7.91).

We can similarly obtain the respective formulas when $\mathcal{O}$ has non-zero R-charge. Without loss of generality we take the R-charge to be negative.[15] Skipping the details of the derivation we show the answer for $R = -1$,

$$\begin{aligned} \hat{\mathfrak{n}}(J, J, \mathcal{O}; -1) = \mathfrak{n}(J, J, \mathcal{O}; 1) - \mathfrak{n}(\bar{D}J, J, \mathcal{O}; -2) - n_1(DJ, J, \mathcal{O}) \\ - n_{\Theta\bar{\Theta}}(DJ, J, \mathcal{O}) + n_{\Theta}(DJ, DJ, \mathcal{O}) + n_{\bar{\Theta}}(DJ, \bar{D}J, \mathcal{O})\,, \end{aligned} \tag{7.88}$$

and for $R = -2$,

$$\hat{\mathfrak{n}}(J, J, \mathcal{O}; -2) = \mathfrak{n}(J, J, \mathcal{O}; -2) - n_{\bar{\Theta}}(DJ, J, \mathcal{O}) + n_1(DJ, DJ, \mathcal{O})\,. \tag{7.89}$$

In all cases with non-zero R-charge (7.88) and (7.89) yield non-positive results. Therefore we conclude that there are no structures allowed after conservation, as mentioned at the beginning of the section. We can thus proceed with the analysis considering only operators with $R = 0$ that are superconformal primaries.

7.4.2 Generalities

We start by defining a list of supersymmetric tensor structures for the correlators under consideration

$$t^{V_1V_2}_{\mathcal{O}^{(\ell,\bar{\ell})}_\Delta}(Z, \eta_i, \bar{\eta}_i) = \sum_{a=1}^{\mathfrak{n}(\ell,\bar{\ell})} \mathcal{C}^{(a)}_{\langle V_1V_2\mathcal{O}^{(\ell,\bar{\ell})}_\Delta\rangle}\, t^{V_1V_2}_{\mathcal{O}^{(\ell,\bar{\ell})}}(Z, \eta_i, \bar{\eta}_i)^{(a)}\,. \tag{7.90}$$

The number of independent OPE coefficients in superspace $\mathfrak{n}(\ell,\bar{\ell})$ is significantly bigger than its non-supersymmetric analog $n(\ell,\bar{\ell})$. However, as we have seen in the previous subsection, after applying the constraints of conservation and π_{12} permutation symmetry, the number of independent structures is greatly reduced. Let us summarize the number

[15] The other case can be obtained by complex conjugation. In order to prove the same formula for $R = 1$ we would need a representation of the differential operators where $D_{\bar{Q}} \to \partial/\partial\Theta^\alpha$, namely (A.32b).

of conserved structures for each representation

$$\hat{\mathfrak{n}}(\ell,\ell)=\begin{cases}2 & \ell\geqslant 2 \text{ even}\\ 2 & \ell\geqslant 3 \text{ odd}\\ 1 & \ell=0\\ 2 & \ell=1 \text{ and } \Delta=3\\ 1 & \ell=1 \text{ and } \Delta\neq 3\end{cases}\qquad \hat{\mathfrak{n}}(\ell+2,\ell)=\begin{cases}1 & \ell\geqslant 2 \text{ even}\\ 2 & \ell\geqslant 3 \text{ odd}\\ 1 & \ell=1\\ 0 & \ell=0\end{cases}\qquad \hat{\mathfrak{n}}(\ell+4,\ell)=1\,, \tag{7.91}$$

By comparing with the non-supersymmetric result we see that the only cases where $\hat{\mathfrak{n}} > \hat{n}$ are for $\bar{\ell}=\ell\geqslant 3$ odd, for $\bar{\ell}=\ell+2\geqslant 3$ odd, for $\bar{\ell}=\ell+4$ odd and for $\ell=1,\Delta=3$. The latter corresponds to the third operator being the Ferrara-Zumino multiplet itself. The additional structures that appear in those cases are called nilpotent because they are proportional to the Grassmann variables $\Theta,\overline{\Theta}$. We choose a basis of conserved supersymmetric tensor structures in such a way that the solution to

$$\langle J(\mathbf{z}_1)J(\mathbf{z}_2)\mathcal{O}_\Delta^{(\ell,\bar{\ell})}(\mathbf{z}_3)\rangle\big|_{\theta_i,\bar{\theta}_i=0}=\langle J(\mathbf{x}_1)J(\mathbf{x}_2)\mathcal{O}_\Delta^{(\ell,\bar{\ell})}(\mathbf{x}_3)\rangle\,, \tag{7.92}$$

is simply given by

$$\hat{\lambda}^{(a)}_{\langle JJ\mathcal{O}_\Delta^{(\ell,\bar{\ell})}\rangle}=\mathcal{C}^{(a)}_{\langle JJ\mathcal{O}_\Delta^{(\ell,\bar{\ell})}\rangle}\,,\qquad a=1,\dots,\hat{n}(\ell,\bar{\ell})\,. \tag{7.93}$$

Namely we chose the first $\hat{n}(\ell,\bar{\ell})$ structures to be non nilpotent and to have coefficients exactly equal to the non-supersymmetric ones. There will be some exception to this choice that we will emphasize later.

Since we want to construct the superconformal blocks, we will also need the other permutation of the three-point function. Namely $t_{V_1}^{\mathcal{O}V_2}$. This is not entirely straightforward because $\mathcal{K}_{\mathcal{O}_1\mathcal{O}_2}$ is not invariant under cyclic permutations. This problem is addressed in Appendix B.1. We will keep adopting the convention that the coefficients multiplying the permuted three-point structures are related to the original ones by

$$\mathcal{C}^{(a)}_{\langle\mathcal{O}_\Delta^{(\ell,\bar{\ell})}JJ\rangle}=\mathcal{C}^{(a)}_{\langle JJ\mathcal{O}_\Delta^{(\ell,\bar{\ell})}\rangle}\,. \tag{7.94}$$

We will now report the explicit expressions for the non-conserved tensor structures $t_{\mathcal{O}}^{V_1V_2}(Z,\eta_i,\bar{\eta}_i)^{(a)}$ when $\mathcal{O}$ has spin $(\ell+p,\ell)$. In what follows we assume a generic value of the dimension Δ. The basis of conserved structures will not be explicitly reported here because the results are a bit too unwieldy. However we will give enough information to make our conventions unambiguous. For the cases where $\hat{\mathfrak{n}}=\hat{n}$ the choice of basis is entirely fixed by (7.93). For the other cases we will write down only the necessary information that completely fixes the nilpotent contributions. The permutation symmetric and conserved basis is denoted as $\hat{t}_{\mathcal{O}}^{JJ}(Z,\eta_i,\bar{\eta}_i)^{(a)}$.

In order to make the expressions more compact, we use the following shorthand

$$\mathcal{I}_{ij} = \frac{\eta_i \mathrm{U} \bar{\eta}_j}{|U|}, \qquad \mathcal{I}_{i\bar{\Theta}} = \frac{\eta_i \mathrm{U} \bar{\Theta}}{|U|^{3/2}}, \qquad \mathcal{I}_{\Theta j} = \frac{\Theta \mathrm{U} \bar{\eta}_j}{|U|^{3/2}}, \qquad \mathcal{I}_{\Theta\bar{\Theta}} = \frac{\bar{\Theta}\mathrm{U}\Theta}{U^2},$$
$$\mathcal{K}_{ij} = \eta_i \eta_j, \qquad \mathcal{K}_{i\Theta} = \frac{\eta_i \Theta}{|U|^{1/2}}, \qquad \bar{\mathcal{K}}_{ij} = \bar{\eta}_i \bar{\eta}_j, \qquad \bar{\mathcal{K}}_{i\bar{\Theta}} = \frac{\bar{\eta}_i \bar{\Theta}}{|U|^{1/2}},$$
$$\xi = \frac{\Theta^2 \bar{\Theta}^2}{U^2}.$$

With these definitions, we can now list the tensor structures for all values of p. We use the shorthand $\mathcal{C}_a \equiv \mathcal{C}^{(a)}_{\langle V_1 V_2 \mathcal{O}\rangle}$ for the coefficients of the structure before conservation and permutation symmetry and $\widehat{\mathcal{C}}_a \equiv \widehat{\mathcal{C}}^{(a)}_{\langle JJ\mathcal{O}\rangle}$ for the coefficients after such constraints have been imposed (the operator $\mathcal{O}$ to which it refers will be clear from the context).

7.4.3 Basis for $p = 0$

The basis for $p = 0$ reads

$$\begin{aligned}
&t^{V_1 V_2}_{\mathcal{O}^{(\ell,\ell)}_\Delta}(Z, \eta_i, \bar{\eta}_i) = \\
&\quad U^{\Delta-6}\Big((\mathcal{C}_1 + \mathcal{C}_{27}\,\xi)\, \mathcal{I}_{11}\mathcal{I}_{23}\mathcal{I}_{32}(\mathcal{I}_{33})^{\ell-1} + (\mathcal{C}_2 + \mathcal{C}_{28}\,\xi)\, \mathcal{I}_{13}\mathcal{I}_{22}\mathcal{I}_{31}(\mathcal{I}_{33})^{\ell-1} \\
&\qquad + (\mathcal{C}_3 + \mathcal{C}_{29}\,\xi)\, \mathcal{I}_{13}\mathcal{I}_{23}\mathcal{I}_{31}\mathcal{I}_{32}(\mathcal{I}_{33})^{\ell-2} + (\mathcal{C}_4 + \mathcal{C}_{30}\,\xi)\, \mathcal{I}_{12}\mathcal{I}_{23}\mathcal{I}_{31}(\mathcal{I}_{33})^{\ell-1} \\
&\qquad + (\mathcal{C}_5 + \mathcal{C}_{31}\,\xi)\, \mathcal{I}_{13}\mathcal{I}_{21}\mathcal{I}_{32}(\mathcal{I}_{33})^{\ell-1} + (\mathcal{C}_6 + \mathcal{C}_{32}\,\xi)\, \mathcal{I}_{12}\mathcal{I}_{21}(\mathcal{I}_{33})^{\ell} \\
&\qquad + \mathcal{C}_7\, \mathcal{I}_{11}\mathcal{I}_{23}\mathcal{I}_{32}(\mathcal{I}_{33})^{\ell-1}\mathcal{I}_{\Theta\bar{\Theta}} + \mathcal{C}_8\, \mathcal{I}_{13}\mathcal{I}_{22}\mathcal{I}_{31}(\mathcal{I}_{33})^{\ell-1}\mathcal{I}_{\Theta\bar{\Theta}} \\
&\qquad + \mathcal{C}_9\, \mathcal{I}_{13}\mathcal{I}_{23}\mathcal{I}_{31}\mathcal{I}_{32}(\mathcal{I}_{33})^{\ell-2}\mathcal{I}_{\Theta\bar{\Theta}} + \mathcal{C}_{10}\, \mathcal{I}_{12}\mathcal{I}_{23}\mathcal{I}_{31}(\mathcal{I}_{33})^{\ell-1}\mathcal{I}_{\Theta\bar{\Theta}} \\
&\qquad + \mathcal{C}_{11}\, \mathcal{I}_{13}\mathcal{I}_{21}\mathcal{I}_{32}(\mathcal{I}_{33})^{\ell-1}\mathcal{I}_{\Theta\bar{\Theta}} + \mathcal{C}_{12}\, \mathcal{I}_{12}\mathcal{I}_{21}(\mathcal{I}_{33})^{\ell}\mathcal{I}_{\Theta\bar{\Theta}} \\
&\qquad + \mathcal{C}_{13}\, \mathcal{K}_{3\Theta}\bar{\mathcal{K}}_{1\bar{\Theta}}\mathcal{I}_{13}\mathcal{I}_{23}\mathcal{I}_{32}(\mathcal{I}_{33})^{\ell-2} + \mathcal{C}_{14}\, \mathcal{K}_{3\Theta}\bar{\mathcal{K}}_{2\bar{\Theta}}\mathcal{I}_{13}\mathcal{I}_{23}\mathcal{I}_{31}(\mathcal{I}_{33})^{\ell-2} \\
&\qquad + \mathcal{C}_{15}\, \mathcal{K}_{3\Theta}\bar{\mathcal{K}}_{2\bar{\Theta}}\mathcal{I}_{13}\mathcal{I}_{21}(\mathcal{I}_{33})^{\ell-1} + \mathcal{C}_{16}\, \mathcal{K}_{3\Theta}\bar{\mathcal{K}}_{1\bar{\Theta}}\mathcal{I}_{12}\mathcal{I}_{23}(\mathcal{I}_{33})^{\ell-1} \\
&\qquad + \mathcal{C}_{17}\, \bar{\mathcal{K}}_{3\bar{\Theta}}\mathcal{K}_{1\Theta}\mathcal{I}_{23}\mathcal{I}_{31}\mathcal{I}_{32}(\mathcal{I}_{33})^{\ell-2} + \mathcal{C}_{18}\, \bar{\mathcal{K}}_{3\bar{\Theta}}\mathcal{K}_{2\Theta}\mathcal{I}_{13}\mathcal{I}_{31}\mathcal{I}_{32}(\mathcal{I}_{33})^{\ell-2} \\
&\qquad + \mathcal{C}_{19}\, \bar{\mathcal{K}}_{3\bar{\Theta}}\mathcal{K}_{2\Theta}\mathcal{I}_{12}\mathcal{I}_{31}(\mathcal{I}_{33})^{\ell-1} + \mathcal{C}_{20}\, \bar{\mathcal{K}}_{3\bar{\Theta}}\mathcal{K}_{1\Theta}\mathcal{I}_{21}\mathcal{I}_{32}(\mathcal{I}_{33})^{\ell-1} \\
&\qquad + \mathcal{C}_{21}\, \mathcal{K}_{3\Theta}\bar{\mathcal{K}}_{3\bar{\Theta}}\mathcal{I}_{11}\mathcal{I}_{23}\mathcal{I}_{32}(\mathcal{I}_{33})^{\ell-2} + \mathcal{C}_{22}\, \mathcal{K}_{3\Theta}\bar{\mathcal{K}}_{3\bar{\Theta}}\mathcal{I}_{13}\mathcal{I}_{22}\mathcal{I}_{31}(\mathcal{I}_{33})^{\ell-2} \\
&\qquad + \mathcal{C}_{23}\, \mathcal{K}_{3\Theta}\bar{\mathcal{K}}_{3\bar{\Theta}}\mathcal{I}_{13}\mathcal{I}_{23}\mathcal{I}_{31}\mathcal{I}_{32}(\mathcal{I}_{33})^{\ell-3} + \mathcal{C}_{24}\, \mathcal{K}_{3\Theta}\bar{\mathcal{K}}_{3\bar{\Theta}}\mathcal{I}_{12}\mathcal{I}_{23}\mathcal{I}_{31}(\mathcal{I}_{33})^{\ell-2} \\
&\qquad + \mathcal{C}_{25}\, \mathcal{K}_{3\Theta}\bar{\mathcal{K}}_{3\bar{\Theta}}\mathcal{I}_{13}\mathcal{I}_{21}\mathcal{I}_{32}(\mathcal{I}_{33})^{\ell-2} + \mathcal{C}_{26}\, \mathcal{K}_{3\Theta}\bar{\mathcal{K}}_{3\bar{\Theta}}\mathcal{I}_{12}\mathcal{I}_{21}(\mathcal{I}_{33})^{\ell-1}\Big).
\end{aligned} \tag{7.95}$$

If $\ell \geqslant 2$ and is even, conservation and permutation symmetry reduces the number of independent coefficients to two. The nilpotent structures are thus uniquely determined

from the non-nilpotent ones, which are in turn fixed by (7.93). Similarly, if $\ell = 0$ or 1 the coefficients are reduced down to a single one, whose definition is again fixed by (7.93).

By contrast, if $\ell \geqslant 3$ and is odd we get two independent coefficients after conservation and permutation. We define $\widehat{\mathcal{C}}_1$ to be the non-nilpotent one. Then $\widehat{\mathcal{C}}_2$ is defined such that

$$\mathcal{C}_8 = \frac{(\Delta+\ell)(\Delta-\ell-6)}{5\Delta^2-20\Delta-3\ell^2-6\ell+24}\left((\Delta-\ell-2)\,\widehat{\mathcal{C}}_1+\frac{(\Delta-\ell-4)}{4(\Delta-2)}\,\widehat{\mathcal{C}}_2\right). \tag{7.96}$$

All other coefficients can be obtained as a consequence.

The case $\ell = 1$ and $\Delta = 3$ is special as it does not follow the pattern for $\Delta \neq 3$. Furthermore it corresponds to the case where the operator at the third point is J itself. This case was already analyzed long ago in [115]. Since the first superdescendant is the stress tensor, it is convenient to define the conserved coefficients in the same way as the non-supersymmetric ones for the stress tensor. Thus we define

$$\widehat{\mathcal{C}}_1 = \hat{\lambda}^{(1)}_{\langle JJT\rangle}\,, \qquad \widehat{\mathcal{C}}_2 = \hat{\lambda}^{(2)}_{\langle JJT\rangle}\,. \tag{7.97}$$

Note that this is in contrast with the general rule stated in (7.93), which says that the non-nilpotent structures have the same coefficient as the lowest component of the multiplet. For the reader's convenience, here is the relation between the non-nilpotent coefficient and the three-point function of J

$$\hat{\lambda}^{(1)}_{\langle JJJ\rangle} = 16\widehat{\mathcal{C}}_1 + 8\widehat{\mathcal{C}}_2. \tag{7.98}$$

If we make further use of supersymmetry we can relate these two coefficients with the OPE coefficients of $\langle TTT\rangle$. In particular, we can express them in terms of the anomaly coefficients a and c. The relation between $\widehat{\mathcal{C}}_a$ and a, c reads

$$\widehat{\mathcal{C}}_1 = -\frac{2\,(2a-3c)}{9\pi^6}\,, \qquad \widehat{\mathcal{C}}_2 = -\frac{2\,(2a+3c)}{9\pi^6}\,. \tag{7.99}$$

This result can be obtained using [115, Eq. (11.7)], or, equivalently, following [140, Appendix C] and using the relation between C_J and C_T stemming from supersymmetry (7.12). After using (7.33) we could also express the c anomaly in terms of C_T, namely

$$c = \frac{\pi^4}{40}\,C_T\,. \tag{7.100}$$

The Hofman-Maldacena bounds [136] restrict the ratio of a and c in any $\mathcal{N} = 1$ SCFT to lie between the following values

$$\frac{1}{2} \leqslant \frac{a}{c} \leqslant \frac{3}{2}\,. \tag{7.101}$$

7.4.4 Basis for $p = 2$

For $p = 2$ we choose the following basis

$$
\begin{aligned}
&t^{V_1V_2}_{\mathcal{O}_\Delta^{(\ell+2,\ell)}}(Z,\eta_i,\bar\eta_i) = \\
&\quad U^{\Delta-6}\Big((\mathcal{C}_1+\mathcal{C}_{20}\,\xi)\,\mathcal{I}_{23}\mathcal{I}_{31}\mathcal{I}_{32}\mathcal{K}_{13}\,(\mathcal{I}_{33})^{\ell-1} \\
&\qquad + (\mathcal{C}_2+\mathcal{C}_{21}\,\xi)\,\mathcal{I}_{13}\mathcal{I}_{31}\mathcal{I}_{32}\mathcal{K}_{23}\,(\mathcal{I}_{33})^{\ell-1} + (\mathcal{C}_3+\mathcal{C}_{22}\,\xi)\,\mathcal{I}_{12}\mathcal{I}_{31}\mathcal{K}_{23}\,(\mathcal{I}_{33})^{\ell} \\
&\qquad + (\mathcal{C}_4+\mathcal{C}_{23}\,\xi)\,\mathcal{I}_{21}\mathcal{I}_{32}\mathcal{K}_{13}\,(\mathcal{I}_{33})^{\ell} + \mathcal{C}_5\,\mathcal{I}_{23}\mathcal{I}_{31}\mathcal{I}_{32}\mathcal{K}_{13}\mathcal{I}_{\Theta\bar\Theta}\,(\mathcal{I}_{33})^{\ell-1} \\
&\qquad + \mathcal{C}_6\,\mathcal{I}_{13}\mathcal{I}_{31}\mathcal{I}_{32}\mathcal{K}_{23}\mathcal{I}_{\Theta\bar\Theta}\,(\mathcal{I}_{33})^{\ell-1} + \mathcal{C}_7\,\mathcal{I}_{12}\mathcal{I}_{31}\,(\mathcal{I}_{33})^{\ell}\mathcal{I}_{\Theta\bar\Theta}\mathcal{K}_{23} \\
&\qquad + \mathcal{C}_8\,\mathcal{I}_{21}\mathcal{I}_{32}\mathcal{K}_{13}\mathcal{I}_{\Theta\bar\Theta}\,(\mathcal{I}_{33})^{\ell} + \mathcal{C}_9\,\mathcal{K}_{3\Theta}\mathcal{I}_{2\bar\Theta}\mathcal{I}_{11}\mathcal{I}_{32}\,(\mathcal{I}_{33})^{\ell} \\
&\qquad + \mathcal{C}_{10}\,\mathcal{K}_{3\Theta}\mathcal{I}_{1\bar\Theta}\mathcal{I}_{22}\mathcal{I}_{31}\,(\mathcal{I}_{33})^{\ell} + \mathcal{C}_{11}\,\mathcal{K}_{3\Theta}\mathcal{I}_{1\bar\Theta}\mathcal{I}_{23}\mathcal{I}_{31}\mathcal{I}_{32}\,(\mathcal{I}_{33})^{\ell-1} \\
&\qquad + \mathcal{C}_{12}\,\mathcal{K}_{3\Theta}\mathcal{I}_{2\bar\Theta}\mathcal{I}_{12}\mathcal{I}_{31}\,(\mathcal{I}_{33})^{\ell} + \mathcal{C}_{13}\,\mathcal{K}_{3\Theta}\mathcal{I}_{1\bar\Theta}\mathcal{I}_{21}\mathcal{I}_{32}\,(\mathcal{I}_{33})^{\ell} \\
&\qquad + \mathcal{C}_{14}\,\mathcal{K}_{3\Theta}\mathcal{I}_{3\bar\Theta}\mathcal{I}_{12}\mathcal{I}_{21}\,(\mathcal{I}_{33})^{\ell} + \mathcal{C}_{15}\,\bar{\mathcal{K}}_{3\bar\Theta}\mathcal{K}_{1\Theta}\mathcal{I}_{31}\mathcal{I}_{32}\mathcal{K}_{23}\,(\mathcal{I}_{33})^{\ell-1} \\
&\qquad + \mathcal{C}_{16}\,\mathcal{K}_{3\Theta}\bar{\mathcal{K}}_{3\bar\Theta}\mathcal{I}_{23}\mathcal{I}_{31}\mathcal{I}_{32}\mathcal{K}_{13}\,(\mathcal{I}_{33})^{\ell-2} + \mathcal{C}_{17}\,\mathcal{K}_{3\Theta}\bar{\mathcal{K}}_{3\bar\Theta}\mathcal{I}_{13}\mathcal{I}_{31}\mathcal{I}_{32}\mathcal{K}_{23}\,(\mathcal{I}_{33})^{\ell-2} \\
&\qquad + \mathcal{C}_{18}\,\mathcal{K}_{3\Theta}\bar{\mathcal{K}}_{3\bar\Theta}\mathcal{I}_{12}\mathcal{I}_{31}\mathcal{K}_{23}\,(\mathcal{I}_{33})^{\ell-1} + \mathcal{C}_{19}\,\mathcal{K}_{3\Theta}\bar{\mathcal{K}}_{3\bar\Theta}\mathcal{I}_{21}\mathcal{I}_{32}\mathcal{K}_{13}\,(\mathcal{I}_{33})^{\ell-1}\Big)\,. \qquad (7.102)
\end{aligned}
$$

When ℓ is even or $\ell = 1$ we have the same number of structures as the non-supersymmetric case after conservation and permutation. Therefore the coefficients are fixed by (7.93) as before. The case $\ell = 0$ in particular is trivial because the three-point function is simply vanishing.

The case of odd spin with $\ell \geqslant 3$, on the other hand, contains one nilpotent coefficient, which we define to be $\widehat{\mathcal{C}}_2$ as before. Its definition is can be inferred from

$$
\mathcal{C}_{16} = \frac{4i\ell(\ell+2)(2\Delta-\ell^2-2\Delta\ell+15\ell+16)}{3(\ell+4)(\ell+5)}\,\widehat{\mathcal{C}}_1 - \frac{4i}{3}(\Delta-\ell-4)\,\widehat{\mathcal{C}}_2\,. \qquad (7.103)
$$

This choice of $\widehat{\mathcal{C}}_2$ ensures that the correlator is conserved at the third point when the superprimary hits the unitarity bound $\Delta = \ell + 4$.

7.4.5 Basis for $p = 4$

Finally, the basis for $p = 4$ is given by

$$
\begin{aligned}
&t^{V_1V_2}_{\mathcal{O}_\Delta^{(\ell+4,\ell)}}(Z,\eta_i,\bar\eta_i) = \\
&\quad U^{\Delta-6}\Big((\mathcal{C}_1+\mathcal{C}_8\,\xi)\,\mathcal{I}_{31}\mathcal{I}_{32}\mathcal{K}_{13}\mathcal{K}_{23}\,(\mathcal{I}_{33})^{\ell}
\end{aligned}
$$

$$
\begin{aligned}
&+ \mathcal{C}_2\, \mathcal{I}_{31}\mathcal{I}_{32}\mathcal{K}_{13}\mathcal{K}_{23}\mathcal{I}_{\Theta\bar{\Theta}}\,(\mathcal{I}_{33})^{\ell} + \mathcal{C}_3\, \mathcal{K}_{3\Theta}\mathcal{I}_{2\bar{\Theta}}\mathcal{I}_{31}\mathcal{I}_{32}\mathcal{K}_{13}\,(\mathcal{I}_{33})^{\ell} \\
&+ \mathcal{C}_4\, \mathcal{K}_{3\Theta}\mathcal{I}_{1\bar{\Theta}}\mathcal{I}_{31}\mathcal{I}_{32}\mathcal{K}_{23}\,(\mathcal{I}_{33})^{\ell} + \mathcal{C}_5\, \mathcal{K}_{3\Theta}\mathcal{I}_{3\bar{\Theta}}\mathcal{I}_{12}\mathcal{I}_{31}\mathcal{K}_{23}\,(\mathcal{I}_{33})^{\ell} \\
&+ \mathcal{C}_6\, \mathcal{K}_{3\Theta}\mathcal{I}_{3\bar{\Theta}}\mathcal{I}_{21}\mathcal{I}_{32}\mathcal{K}_{13\,(\mathcal{I}_{33})^{\ell}} + \mathcal{C}_7\, \mathcal{K}_{3\Theta}\bar{\mathcal{K}}_{3\bar{\Theta}}\mathcal{I}_{31}\mathcal{I}_{32}\mathcal{K}_{13}\mathcal{K}_{23}\,(\mathcal{I}_{33})^{\ell-1}\Big)\,. \qquad (7.104)
\end{aligned}
$$

Here after conservation and permutation we end up with just one coefficient for every ℓ. The non-supersymmetric case for ℓ even also has one coefficient, however $\widehat{\mathcal{C}}_1$ is *not* defined by (7.93). We added a factor to enforce the conservation at the third point when the superprimary hits the unitarity bound $\Delta = \ell + 6$. The definition of $\widehat{\mathcal{C}}_1$ is

$$
\hat{\lambda}_{\langle JJ\mathcal{O}_{\Delta}^{(\ell+4,\ell)}\rangle} = (\Delta - \ell - 6)\,\widehat{\mathcal{C}}_1\,. \qquad (7.105)
$$

If instead ℓ is odd we define $\widehat{\mathcal{C}}_1$ by

$$
\mathcal{C}_4 = i\,(\Delta - \ell - 6)\,\widehat{\mathcal{C}}_1\,, \qquad (7.106)
$$

while all other coefficients follow by imposing permutation symmetry and conservation.

7.4.6 Applying the differential operators

As we discussed in the previous subsection, we will only consider superprimaries with vanishing R-charge. This means that the only differential operators needed are $\mathcal{D}_{Q^k\bar{Q}^k}$ for $k = \bar{k}$.

Let us start with the three-point functions $t_J^{\mathcal{O}J}$ since those are the ones where the operators $\mathcal{D}$ and $\bar{\mathcal{D}}$ act simply. Given a product of supercharges $(Q^k\bar{Q}^k\mathcal{O})^{st}$ with $k = 0, 1, 2$ and $s, t = \pm$, one has the following shifts in the quantum numbers

$$
\Delta' = \Delta + k\,, \qquad \ell' = \ell + s\,, \qquad \bar{\ell}' = \bar{\ell} + t\,. \qquad (7.107)
$$

With this notation in mind, we want to solve the linear equation given by

$$
\mathcal{D}^{st}_{Q^k\bar{Q}^k}\,\hat{t}_J^{\mathcal{O}_{\Delta}^{(\ell,\bar{\ell})}J}(Z,\eta_i,\bar{\eta}_i)\big|_{\Theta,\bar{\Theta}=0} = \sum_{a=1}^{\hat{n}(\ell',\bar{\ell}')} \mathcal{E}^{k,s,t|\,a}_{\mathcal{O}_{\Delta}^{(\ell,\bar{\ell})}}\,\hat{t}_J^{\mathcal{O}_{\Delta'}^{(\ell',\bar{\ell}')}J}(X,\eta_i,\bar{\eta}_i)^{(a)}\,. \qquad (7.108)
$$

As for the other ordering, namely $\langle JJ\mathcal{O}\rangle$, we can again use the results of Appendix B.1. We do not need to recompute the derivatives. It suffices to use the one-to-one mapping that relates a structure $\hat{t}^{JJ}_{\mathcal{O}}(Z,\eta_i,\bar{\eta}_i)^{(a)}$ to a sum of $\hat{t}_J^{\mathcal{O}J}(Z,\eta_i,\bar{\eta}_i)^{(b)}$. We then use the coefficients $\mathcal{E}^{k,s,t,|b}_{\mathcal{O}}$ computed before, and obtain the wanted result by using the inverse

map from $\hat{t}^{\mathcal{O}J}_J$ to $\hat{t}^{JJ}_{\mathcal{O}}$. All in all this results in

$$D^{st}_{3,\,Q^k\bar{Q}^k}\,\mathcal{K}_{JJ}\,\hat{t}^{JJ}_{\mathcal{O}^{(\ell,\bar{\ell})}_\Delta}(Z,\eta_i,\bar{\eta}_i)\big|_{\Theta,\bar{\Theta}=0}=\mathcal{K}_{JJ}\sum_{a=1}^{\hat{n}(\ell',\bar{\ell}')}\mathcal{F}^{k,s,t|\,a}_{\mathcal{O}^{(\ell,\bar{\ell})}_\Delta}\,\hat{t}^{JJ}_{\mathcal{O}^{(\ell',\bar{\ell}')}_{\Delta'}}(X,\eta_i,\bar{\eta}_i)^{(a)}\,, \tag{7.109}$$

where the subscript "3" in D signifies that the derivative acts on the third point. The products of coefficients $\mathcal{F}^{k,s,t|\hat{a}}_{\mathcal{O}}\mathcal{E}^{k,s,t|\hat{b}}_{\mathcal{O}}$ are precisely those that appear in the linear combinations that make up the superconformal blocks. We will see this in the next subsection.

7.5 Superconformal blocks

Before writing the conformal blocks we need to properly normalize the superdescendants. The differential operators $\mathcal{D}_{Q^k\bar{Q}^k}$ are defined so that $(Q^k\bar{Q}^k\mathcal{O})$ is normalized as in [156]. Namely we have

$$\langle(Q^k\bar{Q}^k\overline{\mathcal{O}})(\mathbf{x}_1)(Q^k\bar{Q}^k\mathcal{O})(\mathbf{x}_2)\rangle=i^{\ell-\bar{\ell}}\,n_{(Q^k\bar{Q}^k\mathcal{O})}\frac{(\eta_1\mathrm{x}_{1\bar{2}}\bar{\eta}_2)^\ell(\eta_2\mathrm{x}_{2\bar{1}}\bar{\eta}_1)^{\bar{\ell}}}{x_{1\bar{2}}{}^{\Delta+\ell}x_{\bar{1}2}{}^{\Delta+\bar{\ell}}}\,, \tag{7.110}$$

where $n_{Q^k\bar{Q}^k\mathcal{O}}$ is given in Table 3.2. As a first step, the four-point function of four J's can be expanded in a basis of tensor structures as before

$$\langle J(\mathbf{x}_1)J(\mathbf{x}_2)J(\mathbf{x}_3)J(\mathbf{x}_4)\rangle=\sum_{\mathcal{O}^{(\ell,\bar{\ell})}_\Delta}\sum_{i=1}^{n_{JJJJ}}\mathbb{T}_i(\mathbf{x}_1,\ldots,\mathbf{x}_4)\,\mathcal{G}_{i,\Delta,(\ell,\bar{\ell})}(z,\bar{z})\,. \tag{7.111}$$

The first sum runs over only the superconformal primaries and the function $\mathcal{G}_{i,\Delta,(\ell,\bar{\ell})}$ is defined as

$$\mathcal{G}_{i,\Delta,(\ell,\bar{\ell})}(z,\bar{z})=\sum_{\substack{k=0,1,2\\ s,t=\pm}}\frac{1}{n_{(Q^k\bar{Q}^k\mathcal{O})^{st}}}\sum_{a,b=1}^{\hat{n}(\ell',\bar{\ell}')}\mathcal{F}^{k,t,s|a}_{\overline{\mathcal{O}}^{(\bar{\ell},\ell)}_\Delta}\,\mathcal{E}^{k,s,t|b}_{\mathcal{O}^{(\ell,\bar{\ell})}_\Delta}\,G^{ab}_{i,\Delta',(\ell',\bar{\ell}')}(z,\bar{z})\,, \tag{7.112}$$

where the primes on Δ, ℓ and $\bar{\ell}$ refer to the notation introduced in (7.107) and the functions $G^{ab}_{i,\Delta,(\ell,\bar{\ell})}(z,\bar{z})$ are the non-supersymmetric conformal blocks computed in the previous sections. The quantities $\mathcal{F}_{\mathcal{O}}$ and $\mathcal{E}_{\mathcal{O}}$ are linear combinations of the supersymmetric OPE coefficients. We can therefore obtain the final results, namely the superconformal blocks $\mathcal{G}^{ab}_{i,\Delta,(\ell,\bar{\ell})}(z,\bar{z})$, as follows

$$\mathcal{G}_{i,\Delta,(\ell,\bar{\ell})}(z,\bar{z})=\sum_{a,b=1}^{\hat{\mathfrak{n}}(\ell,\bar{\ell})}\widehat{\mathcal{C}}^{(a)}_{\langle JJ\bar{\mathcal{O}}^{(\bar{\ell},\ell)}_\Delta\rangle}\widehat{\mathcal{C}}^{(b)}_{\langle\mathcal{O}^{(\ell,\bar{\ell})}_\Delta JJ\rangle}\,\mathcal{G}^{ab}_{i,\Delta,(\ell,\bar{\ell})}(z,\bar{z})\,. \tag{7.113}$$

Note that now the sum over a and b runs from 1 to $\hat{\mathfrak{n}}(\ell,\bar{\ell})$. The relation between the superconformal blocks and the ordinary conformal blocks can be summarized in a

rectangular matrix $\mathcal{S}$ defined by

$$\mathcal{G}^{ab}_{i,\Delta,(\ell,\bar{\ell})}(z,\bar{z}) = \sum_{\substack{k=0,1,2\\ s,t=\pm}} \sum_{c,d=1}^{\hat{n}(\ell,\bar{\ell})} \mathcal{S}^{ab}_{k,s,t|cd}(\Delta,\ell,\bar{\ell})\; G^{cd}_{i,\Delta',(\ell',\bar{\ell}')}\,, \tag{7.114}$$

where, again, the primes refer to (7.107). In Appendix D.3 we report the nonzero entries of $\mathcal{S}^{ab}_{k,s,t|cd}$ for all cases with $\ell \geqslant \bar{\ell}$. Naturally, the coefficients are symmetric in a, b and c, d. We will thus avoid listing the repeated entries.

The block of the Ferrara-Zumino multiplet itself is special because it is a short multiplet. We can write here its expression since it is very simple compared to the others. It only contains two blocks: the R-current and the stress tensor.[16] Due to the choice made in (7.97) of parametrizing the OPE coefficients as $\lambda^{(a)}_{\langle JJT\rangle}$ and due to the relation between the normalizations of J and T (7.12), we expect the block of T to simply appear with an overall factor of $1/5$. Indeed this is the result

$$\begin{aligned}
\mathcal{G}^{11}_{i,3,(1,1)}(z,\bar{z}) &= 256\, G^{11}_{i,3,(1,1)}(z,\bar{z}) + \frac{1}{5}\, G^{11}_{i,4,(2,2)}(z,\bar{z})\,,\\
\mathcal{G}^{12}_{i,3,(1,1)}(z,\bar{z}) &= 128\, G^{11}_{i,3,(1,1)}(z,\bar{z}) + \frac{1}{10}\,\left(G^{12}_{i,4,(2,2)}(z,\bar{z}) + G^{21}_{i,4,(2,2)}(z,\bar{z})\right)\,,\\
\mathcal{G}^{22}_{i,3,(1,1)}(z,\bar{z}) &= 64\, G^{11}_{i,3,(1,1)}(z,\bar{z}) + \frac{1}{5}\, G^{22}_{i,4,(2,2)}(z,\bar{z})\,.
\end{aligned} \tag{7.115}$$

In terms of the $\mathcal{S}$ matrix defined above, we have

$$\begin{aligned}
\mathcal{S}^{11}_{0|11} = 256\,, \qquad \mathcal{S}^{12}_{0|11} = 128\,, \qquad \mathcal{S}^{22}_{0|11} = 64\,,\\
\mathcal{S}^{11}_{1,+,+|11} = 2\mathcal{S}^{12}_{1,+,+|12} = \mathcal{S}^{22}_{1,+,+|22} = \frac{1}{5}\,.
\end{aligned} \tag{7.116}$$

[16] Naturally the supersymmetry currents are not exchanged in this four-point function given that they have spin $(1,2)$ and $(2,1)$.

8 Numerical studies

In this chapter we present the crossing equations and propose possible directions for the numerical studies. The actual numerical investigations and the subsequent discussions will be left to the forthcoming **Paper IV**.

8.1 Crossing equations

As discussed in Subsection 7.2.3, we have in total 7 bulk degrees of freedom and potentially one point degree of freedom. For convenience we will repeat them here

$$\begin{aligned} &\text{Bulk:} && f_i(z,\bar{z})\,, && i \in \{1,3,4,5,9,10,11\}\,, \\ &\text{Point:} && f_{15}(\tfrac{1}{2},\tfrac{1}{2})\,. \end{aligned} \tag{8.1}$$

The crossing conditions may be obtained by acting with the permutation π_{13} on the four-point tensor structures $\mathbb{T}_i$ defined in (D.11). The equation for f_{15} is odd, so the solution is $f_{15}(z,\bar{z}) = 0$ identically. This is because the crossing equations require that it vanishes on $z = \bar{z} = \frac{1}{2}$ while conservation implies that it vanishes everywhere. For the other functions we have

$$\begin{aligned} f_1(z,\bar{z}) &= f_1(1-z,1-\bar{z})\,, \\ f_3(z,\bar{z}) &= f_5(1-z,1-\bar{z})\,, \\ f_4(z,\bar{z}) &= f_4(1-z,1-\bar{z})\,, \\ f_9(z,\bar{z}) &= f_{11}(1-z,1-\bar{z})\,, \\ f_{10}(z,\bar{z}) &= f_{10}(1-z,1-\bar{z})\,. \end{aligned} \tag{8.2}$$

We would like to discuss the fate of the function f_{15}. If one were to redo the analysis of the four-point correlator by assuming P parity from the start, then f_{15} would have been set to zero automatically. All the other bulk functions instead would have survived as they are parity even. Therefore it seems that this setup wants to be parity preserving, even if we do not require it. The explanation lies in the three-point function analysis.

The reader can notice that the conservation matrices $M_{0,\ell}$ are such that parity odd structures are coupled with parity odd ones, while parity even structures are coupled with parity even ones.[1] This results in the exchanged partial waves to be effectively parity preserving, even though that is just the result of conservation at the level of three-point functions.

This incidentally leads to a curious puzzle: at the level of four-point functions, conservation alone is not sufficient to set to zero the parity odd functions f_{15} — the assumption of crossing is needed. However the previous argument about three-point functions implies that all conformal blocks are parity even and therefore none of them can contribute to f_{15}. Therefore f_{15} is a function which is perfectly consistent with conservation and yet cannot be expanded in conformal blocks. This is not a paradox because all theories satisfy crossing anyway, although it is an interesting fact. A possible explanation is that f_{15} can be expanded in a basis of sporadic solutions to the Casimir equation that do not correspond to any physical operator being exchanged. We leave a more careful treatment of this issue to future studies.

The equations (8.2) can be expanded in conformal blocks with (7.68) and they can be studied with the numerical techniques discussed in Section 6.2. The assumption of supersymmetry modifies the problem minimally: instead of using the blocks $G^{ab}_{i,\rho}$ one has to use the superconformal blocks $\mathcal{G}^{ab}_{i,\rho}$ in (7.114). In addition, the unitarity bounds for operators with $\ell\bar{\ell} = 0$ are different (cfr. (2.6) and (1.7) where $j, \bar{\jmath}$ are $\ell, \bar{\ell}$).

8.2 Possible directions

The goal of this bootstrap study is twofold. On one hand we want to explore the space of solutions to crossing in the hope of gaining evidence for a new CFT — so far, in four dimensions, we have not observed any island, so finding one would be a remarkable achievement. On the other hand we also want to focus on some specific theories. In particular, we want to direct our attention to the putative theory at the kink in the $\mathcal{N} = 1$ setup, called minimal SCFT [25, 26].

We can certainly draw inspiration from a previous study that considered a similar setup in three dimensions [84] and also from the three-dimensional stress tensor bootstrap [83]. An important result of [84] was that, without any assumptions other than the $\mathrm{U}(1)$ global symmetry, it is possible to show numerically that the dimensions of the first parity odd scalar and the first parity even scalar must lie within a bounded two-dimensional region. This is a completely general result. It would be nice to see whether even in four dimensions one can obtain a similar bound. In our setup we do not have parity odd

[1] Since for $p > 0$ parity does not map an operator to itself, the corresponding three-point structures have no notion of intrinsic parity. Therefore the parity invariance of the $p > 0$ partial waves is simply a consequence of the permutation symmetry of the four point-function.

scalars, thus the situation is not exactly the same. We could still try to use another pair of representations. Other than the scalar one has spin $(3,1)$[2] and spin $(4,0)$ as candidates.

Alternatively one could consider the dimensions of the lightest scalar and the lightest spin two operator after the stress tensor. The situation would be identical to the one in the plot already made in [84] since also here we can parametrize $\lambda^{(a)}_{JJT}$ with the central charge and an extra free parameter γ (see Subsection 7.1.3). Furthermore, one can make a supersymmetric version of this plot in two different ways: one would be to look at the first vector after the Ferrara-Zumino multiplet, the other would be to look at the first spin two superprimary.[3] Looking for bounds on the second operator in a given representation has been proven useful in recent studies [58].

The bounds on the central charge have been studied extensively in [84]. The central charge C_T can be thought of as a measure of the number of degrees of freedom of a theory. If we are looking at, for instance, a gauge theory of a group $G = \mathrm{SU}(N)$ with N very large, the central charge will scale with N^2. Similarly, a flavor group of large rank will also make the central charge grow. As a consequence, a CFT with C_T smaller than the free theory value is normally expected to have the smallest possible amount of symmetry. In this case it would be the $\mathrm{U}(1)$ that we have built in. Observing that the lower bound on C_T stays below its free theory value might be an indication of some minimal CFT without any gauge or flavor symmetry (other than $\mathrm{U}(1)$). This would be a very exciting discovery as it would defy the paradigm of the CBZ fixed points [16,17], which are the only candidates for a non-supersymmetric four dimensional CFT known so far. Another motivation for studying lower bounds on C_T is to compare it with the known Hofman-Maldacena bounds [136]. The expected result, at infinitely high numerical accuracy, is a bound that goes to the free theory value at the edges of the allowed window of the Hofman-Maldacena bounds, and to infinity outside of the window. This is because theories that do not satisfy those bounds cannot have a stress tensor (due to the ANEC) and the theories at the edges have been proven to be free. Obtaining a plot close to the expected result will give us confidence that the numerical power of this setup is sufficiently high. Concretely, the lower bound on the central charge can be obtained by normalizing to one the functional acting on the block of the stress tensor and maximizing the functional acting on the identity operator. More

[2] Recall that, from (7.27), the operators of spin $(2,0)$ decouple as a consequence of conservation at the level of three-point functions.

[3] In interacting $\mathcal{N} = 1$ SCFTs the lightest multiplet in the spin two sector cannot be protected because it contains higher spin currents [14]. Furthermore, the amount of OPE coefficients of λ_{JJJ} and $\lambda_{JJ\mathcal{O}^{(2,2)}_{\Delta}}$ is the same (i.e. 2).

precisely, using the notation of Section 6.2, we need to impose

$$\begin{aligned} \alpha\big[\tfrac{1}{C_J^2}\,\lambda^T_{4,(2,2)}\cdot\mathcal{V}_{4,(2,2)}\cdot\lambda_{4,(2,2)}\big] &= 1 \\ \alpha[\mathcal{V}_\rho] &\succeq 0\,, \qquad \rho\neq(4,(2,2))\,, \\ \alpha[V_{\mathbb{1}}] &= B\,, \end{aligned} \tag{8.3}$$

and maximize B. At first is may sound confusing that the result of maximizing the OPE coefficient λ_{JJT} gives us a bound on C_T even if C_T does not appear in (7.34). The explanation lies on our normalization of the operators J and T. In particular, the identity operator and the block of T will contribute to the crossing equations as

$$\langle JJJJ\rangle \sim C_J^2\,G_{0,0} + \frac{1}{C_T}\hat{\lambda}^{(a)}_{\langle JJT\rangle}\,\hat{\lambda}^{(b)}_{\langle JJT\rangle}\,G^{a,b}_{4,2} + \cdots\,. \tag{8.4}$$

Thus the conditions imposed in (8.3) require

$$C_J^2\,B + \frac{C_J^2}{C_T} \leqslant 0\,.$$

Upper bounds on the central charge are also important, but for a more practical reason: there are always solutions to crossing given by generalized free theories.[4] The operators in a genralized free theory tend to have large gaps, and thus drive the bounds on the scalar dimensions, potentially hiding some interesting physical theories underneath. Assuming an upper bound on C_T has the purpose of making it finite. An infinite central charge implies that the stress tensor has a zero two-point function, therefore it corresponds to a non-local theory. Just like the case of three dimensions, we expect also here the presence of generalized free theories driving the bounds. Therefore it might be beneficial to study the effects of bounding C_T from above in order to potentially reveal interesting features. We want to emphasize that, unlike the previous case, here we are not searching for a bound, we are rather imposing it by hand. The way this is done in practice is a little different from before. We have to look for functionals satisfying[5]

$$\begin{aligned} \alpha\left[V_{\mathbb{1}} + \tfrac{1}{C_T^{\max}}\lambda^T_{4,(2,2)}\cdot\mathcal{V}_{4,(2,2)}\cdot\lambda_{4,(2,2)}\right] &= 1 \\ \alpha[\mathcal{V}_\rho] &\succeq 0\,, \qquad \rho\neq(4,(2,2))\,. \end{aligned} \tag{8.5}$$

This isolates a stress tensor contribution in the crossing equation with coefficient

[4] The generalized free theory of an operator $\mathcal{O}$ is a nonlocal theory defined as follows: the operator spectrum consists in all normal ordered composite operators of the form $:\partial^{n_1}\mathcal{O}\partial^{n_2}\mathcal{O}\cdots\partial^{n_k}\mathcal{O}:$ and the correlators are obtained by performing Wick contractions. The conformal dimension of $\mathcal{O}$ differs from the free theory value (otherwise the theory would be just an ordinary free theory and the operators would satisfy the equations of motion).

[5] For simplicity, let us consider all two-point functions to be unit normalized, so that the dependence on C_T does not appear on the blocks but rather it appears in the OPE coefficients only.

$1/C_T - 1/C_T^{\max}$. Since we implicitly assume all contributions to be positive it follows that this functional will only exclude theories for which $C_T \leqslant C_T^{\max}$.

Another interesting quantity to focus on is λ_{JJJ}. It is related to a 't Hooft anomaly of the global symmetry to which J_μ is associated. Such an anomaly can be diagnosed by coupling the current to an external background gauge field. The symmetry is 't Hooft anomalous if the partition function changes under a gauge variation the background field. More precisely, if J_μ is coupled to an external gauge field A_μ, then the partition function $Z[A]$ will transform as

$$Z[A + \mathrm{d}\lambda] = \exp\left(-\frac{i}{24\pi^2}\mathcal{A}\int \lambda F \wedge F\right) Z[A]\,, \tag{8.6}$$

where $F = \mathrm{d}A$ is the field strength. Then the 't Hooft anomaly is given by the number $\mathcal{A}$. A nonzero $\mathcal{A}$ represents an obstruction to gauging a global symmetry, as the partition function is ill defined and one cannot make the field A dynamical. The symmetry is however perfectly fine as a global symmetry and its associated Noether current satisfies the usual conservation law and Ward identities. In our setup we have access to $\mathcal{A}$ thanks to the relation

$$\mathcal{A} = \frac{\lambda_{JJJ}}{C_J^{3/2}}\,. \tag{8.7}$$

In $\mathcal{N} = 1$ theories the 't Hooft anomaly of the R-symmetry is particularly interesting because it has been proven to imply also a 't Hooft anomaly for global supersymmetry [174, 175]. This means that a theory with an anomalous R-symmetry cannot be coupled to supergravity.

Now that we have laid out some ideas for an explorative approach, let us discuss a more targeted study: that of the minimal $\mathcal{N} = 1$ SCFT. Such a theory, if it exists, has a chiral primary of dimension $\Delta_\phi \simeq 1.407$ and a gap in the singlet scalar sector of approximately $\Delta^\star_{\bar\phi\phi} \simeq 3.2$. This last feature also implies that the flavor current multiplets (scalars of dimension 2) are excluded, therefore the only global symmetry is the R-symmetry. The chiral multiplets are charged under $\mathrm{U}(1)_R$ thus ϕ will not appear in the $J \times J$ OPE. However $\bar\phi\phi$ will appear, so we can try to see if there are some noticeable features around the predicted value of $\Delta^\star_{\bar\phi\phi}$. In [26] we can find estimates for the central charge c which is very small: $c/c_{\text{free}} \simeq 8/3$, where $c_{\text{free}} = 1/24$ is the central charge of a free chiral multiplet. One thus may hope to find this theory close to the bounds on the central charge that we discussed before. Moreover, using our setup we can find additional data, such as the a coefficient (7.99). The knowledge of a can also be helpful for learning more about the UV realization of the theory. This is because, due to the a-theorem [176, 177], a_{UV} is always bigger than a_{IR}. Therefore, if we find numerically $a \simeq a^\star$, all theories with $a \leqslant a^\star$ cannot flow to the minimal SCFT in the IR under any deformation and are therefore ruled out as microscopic realizations of the latter. Moreover, for some theories the exact values of a and c are known [178]. This allows us to select a specific theory and

use the bootstrap to study its properties.

It should also be noted that the papers addressing the minimal SCFT predate the discovery of the new "cutting surface" algorithm [57]. It is an algorithm designed to efficiently carve out portions of OPE data. A possible strategy could be to first learn more about the spectrum of the minimal SCFT, specifically to get an estimate on the first vector after J. Then one could impose a safe gap after J, vary the gap on the scalar and apply the surface cutting algorithm on the coefficients a and c. Without going too much in the details, let us briefly mention how does the cutting surface algorithm work. Suppose we are scanning over the values of the angles of some OPE coefficient and searching for functionals

$$\alpha_{\theta_1,\dots,\theta_n}[\hat{n}^T_{\theta_1,\dots,\theta_n}\cdot\mathcal{V}_{\rho^\star}\cdot\hat{n}_{\theta_1,\dots,\theta_n}]\geqslant 0\,,\qquad \forall\,\theta_i\in\mathbb{RP}^{n-1}\,. \tag{8.8}$$

If we do not find a functional, then the point is allowed and we stop the algorithm. However, if we find a functional for some values of the angles $\bar{\theta}_i$, then we might expect the same functional to be good for excluding nearby values of $\bar{\theta}_i$ as well. The angles excluded by the functional can be found by solving a quadratic equation. As it turns out, most of the times this operation reduces the size of the space of angles by roughly a half. Then the algorithm continues by choosing new values of θ_i that are outside of the region excluded so far. This is the hardest step and it involves solving a so-called quadratically constrained quadratic program. The termination occurs when either the whole space has been ruled out or when a set of angles does not admit a functional α.

Finally we would like to mention a possible problem that may arise in the numerics. It goes under the name of "fake primary effect" and it was first understood in [82]. We mentioned it en-passant in Subsection 7.3.1. Due to this effect, an operator at the unitarity bound can "simulate" another operator of lower spin. This happens because the conformal blocks have poles at the locations of the zero norm states (in this case, the null state that arises at unitarity), and the residue of that pole is a conformal block itself. This is problematic because the bootstrap setup does not care about the overall normalization, so the residue will look like any other conformal block. This in turn may reintroduce in the spectrum operators that have been assumed to be absent by putting gaps in the usual way. The solution is to put small gaps in all dangerous channels so that the unitarity values are never reached. In our setup, some channels are safe from the fake primary effect. For instance, the $p=4$ operators appear only for ℓ even, thus there are no blocks at spin $\ell-1$ that can simulate a fake primary. Moreover, the symmetric traceless operators are such that their three-point functions are automatically conserved at unitarity. This implies that the residues are actually zero and no fake primary effect occurs. The only problematic channel is that of $p=2$. It is thus necessary to impose some small gaps accordingly where needed. Similar phenomena could also arise at the supersymmetric unitarity bound threshold.

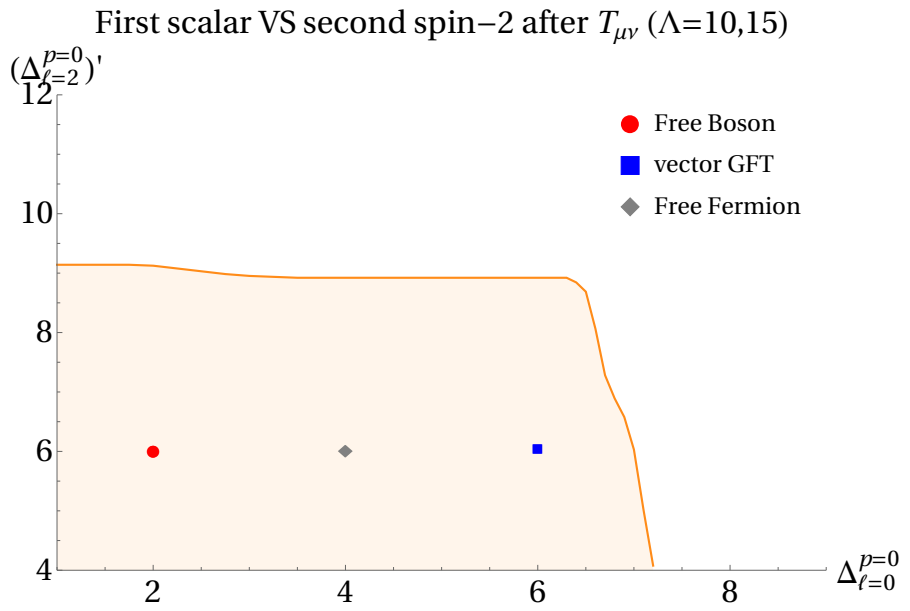


Figure 8.1. Upper bound on the second spin $(2,2)$ operator after the stress tensor as a function on the gap in the scalar sector.

Of course one can set goals and make proposals for various plots or specific investigations, but ultimately the numerics will guide us. We hope to be able to put into practice all the results of this thesis in the forthcoming publication **Paper IV** and to obtain new interesting results for CFTs and SCFTs in four dimensions.

8.3 Preliminary results

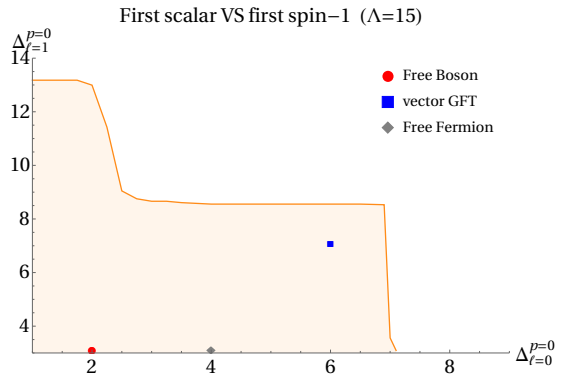
In this section we show some preliminary results for the non-supersymmetric case. As we discussed in Section 6.2, the crossing equations are discretized by turning the conformal blocks into a vector of Taylor coefficients. The number of derivatives kept is parametrized by an integer Λ. More precisely, we take derivatives with respect to y and t^2, defined in (7.55), around $y = t = 0$. Then we construct the vectors as follows

$$(\mathcal{V}_{I,\rho})_{I=1,\dots,d(\Lambda)} = \left\{ (\partial_y^n \partial_{t^2}^m F_\rho(y,t^2))|_{y=t=0} \quad \text{with} \quad m \leqslant \lfloor \tfrac{\Lambda-n}{2} \rfloor\,,\ n \leqslant \Lambda \right\}, \tag{8.9}$$

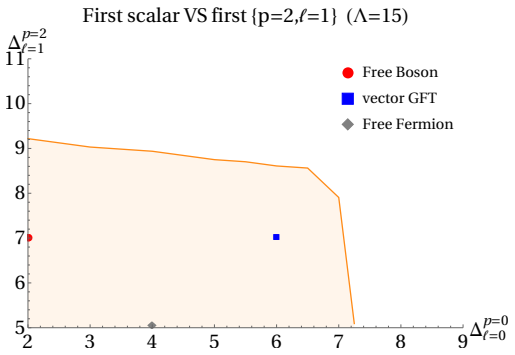
where the dimension of the vector $d(\Lambda)$ is

$$d(\Lambda) = \begin{cases} \frac{1}{4}(\Lambda+2)^2 & \Lambda \text{ even}\,, \\ \frac{1}{4}(\Lambda+1)(\Lambda+3) & \Lambda \text{ odd}\,. \end{cases} \tag{8.10}$$

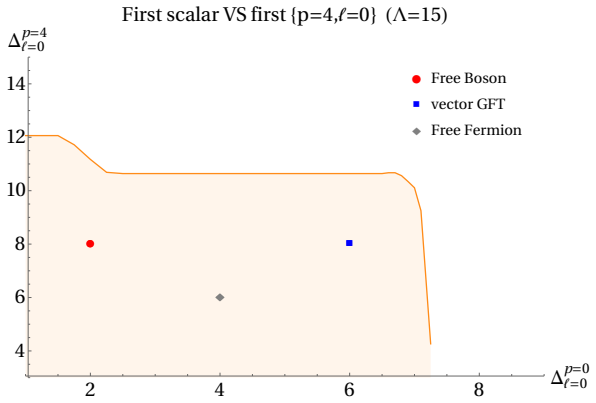
In Figure 8.1 we show the plot of the upper bound on the second spin $(2,2)$ operator after the stress tensor as a function of the gap in the scalar sector. The semidefinite



(a) Upper bound on the first spin $(1,1)$ as a function of the gap on the scalar sector



(b) Upper bound on the first spin $(3,1)$ as a function of the gap on the scalar sector



(c) Upper bound on the first spin $(4,0)$ as a function of the gap on the scalar sector

Figure 8.2. Upper bounds on the gap of other representations.

problem simply requires positivity on the scalar block above a certain fixed gap, on the the stress tensor block, on the block of spin $(2,2)$ above a certain gap and on all other representations above unitarity. The value of the maximal gap on spin 2 is obtained by a bisection algorithm. On the picture we also placed some known free theory values for the conformal dimensions, which are of course in the allowed region. The remarkable feature of this plot is that the allowed region is bounded. This means that we have a universal upper bound on the gap of spin 0 and spin 2 which does not rely on any assumption other than the existence of a $\mathrm{U}(1)$ global symmetry. The corner of the plot might also indicate that a theory lives there, but at this value of Λ we have not observed the convergence of the bounds, so that corner might in principle retract to the vector generalized free theory point.

Similar plots can also be made for other representations. In Figure 8.2 we show the upper bounds for the gap on the sectors of spin $(1,1)$, $(3,1)$[6] and $(4,0)$. In this case

[6]Recall that non-conserved $(2,0)$ operators do not couple to J.

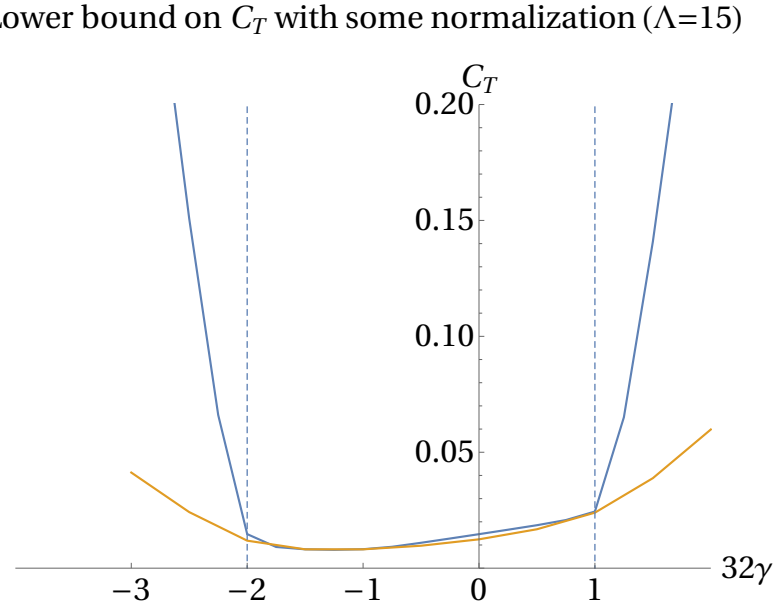


Figure 8.3. Lower bound on the central charge C_T as a function of the parameter γ. The scale of C_T is on an arbitrary normalization. The vertical dashed lines indicate the Hofman-Maldacena bound (7.35). The blue curve assumes a gap on the next spin-two operator while the yellow one does not.

we do not require the presence of an operator at unitarity, thus, in particular, the first plot assumes $\lambda_{JJJ} = 0$. This is not always true of course: λ_{JJJ} might be non-zero if there is a 't Hooft anomalies for example. The plot in Figure 8.2a is particularly interesting due to its two features. The rightmost corner is probably indicating the presence of the vector generalized free theory. But the sudden drop around $\Delta_{\ell=0}^{p=0}$ is an interesting feature worth exploring. At the moment we have no candidates for an explanation of the latter. A similar drop can also be observed at a similar value of $\Delta_{\ell=0}^{p=0}$ in Figure 8.2c.

Finally, in Figure 8.3 we show the lower bound on the central charge C_T as a function of the parameter γ defined in (7.34). The blue curve assumes a gap for the second spin two operator — thus isolating the stress tensor — while the yellow curve does not assume anything. The normalization of C_T is arbitrary, so the actual value of the bound does not give us any information. It is however interesting to see that the blue grows rapidly outside the region allowed by the Hofman-Maldacena ANEC bound (7.35). Presumably the lower bound on C_T will go to infinity as we increase $\Lambda \to \infty$. The yellow curve, on the other hand, does not show this behavior. The explanation is that without assuming $T_{\mu\nu}$ to be isolated, operators of a nearby dimension can "simulate" it, thus invalidating the bound. The fact that we observe the ANEC bound so clearly on the plot is a confirmation of the correctness of our setup.

Appendices Part IV

A Notation and conventions

A.1 Notation and conventions for four dimensions

We adopt the mostly plus signature for the metric

$$g_{\mu\nu} = \text{diag}(-1, 1, \ldots, 1)\,. \tag{A.1}$$

The Levi-Civita tensors in two and four dimensions are defined as

$$\epsilon^{12} = -\epsilon_{12} = 1\,, \qquad \varepsilon^{0123} = -\varepsilon_{0123} = 1\,. \tag{A.2}$$

The Pauli matrices follow the conventions of [155]

$$\begin{aligned} \sigma^0_{\alpha\dot{\alpha}} &= \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}_{\alpha\dot{\alpha}}\,, \quad \sigma^1_{\alpha\dot{\alpha}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}_{\alpha\dot{\alpha}}\,, \\ \sigma^2_{\alpha\dot{\alpha}} &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}_{\alpha\dot{\alpha}}\,, \quad \sigma^3_{\alpha\dot{\alpha}} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}_{\alpha\dot{\alpha}}\,, \end{aligned} \tag{A.3}$$

with

$$\bar{\sigma}^{\mu\dot{\alpha}\alpha} = \epsilon^{\alpha\beta}\epsilon^{\dot{\alpha}\dot{\beta}}\sigma^{\mu}_{\beta\dot{\beta}}\,. \tag{A.4}$$

The spin generators in the irreducible representations $(1,0)$ and $(0,1)$ are, respectively,

$$\sigma^{\mu\nu}{}_{\alpha}{}^{\beta} = \frac{1}{4}\left(\sigma^{\mu}{}_{\alpha\dot{\alpha}}\bar{\sigma}^{\nu\dot{\alpha}\beta} - \sigma^{\nu}{}_{\alpha\dot{\alpha}}\bar{\sigma}^{\mu\dot{\alpha}\beta}\right)\,, \tag{A.5}$$

$$\bar{\sigma}^{\mu\nu\dot{\alpha}}{}_{\dot{\beta}} = \frac{1}{4}\left(\bar{\sigma}^{\mu\dot{\alpha}\alpha}\sigma^{\nu}{}_{\alpha\dot{\beta}} - \bar{\sigma}^{\nu\dot{\alpha}\alpha}\sigma^{\mu}{}_{\alpha\dot{\beta}}\right)\,. \tag{A.6}$$

Lorentz vectors can be written as matrices by contracting them with σ^{μ}. Namely

$$\mathrm{x}_{\alpha\dot{\alpha}} = \sigma^{\mu}_{\alpha\dot{\alpha}}x_{\mu}\,, \qquad x^{\mu} = -\tfrac{1}{2}\mathrm{x}_{\alpha\dot{\alpha}}\bar{\sigma}^{\mu\dot{\alpha}\alpha}\,, \qquad \det(\mathrm{x}) = -x^2\,. \tag{A.7}$$

A convenient way to deal with complicated expressions involving such matrices is to get rid of all the indices by contracting them with bosonic polarization spinors $\eta_\alpha, \bar{\eta}_{\dot{\alpha}}$. Any irreducible representation of the Lorentz group in four dimensions is given by a tensor with j "undotted" indices and $\bar{\jmath}$ "dotted" indices, separately symmetrized.[1] We can therefore contract all the symmetrized indices with the same polarization without losing any information. Thus for $\mathcal{O} \in (j, \bar{\jmath})$ one has

$$\mathcal{O}(\eta, \bar{\eta}) \coloneqq \frac{1}{j!\,\bar{\jmath}!}\, \eta^{\alpha_1} \cdots \eta^{\alpha_j} \bar{\eta}^{\dot{\alpha}_1} \cdots \bar{\eta}^{\dot{\alpha}_{\bar{\jmath}}}\, \mathcal{O}_{(\alpha_1 \cdots \alpha_j),(\,\dot{\alpha}_1 \cdots \dot{\alpha}_{\bar{\jmath}})}\,. \tag{A.8}$$

If $\mathcal{O}$ is an operator we will often use the shorthand $\mathbf{x}$ to denote $x, \eta, \bar{\eta}$. We use the Dynkin label notation for the representations. This means that, for example, the vector belongs to the representation $(1,1)$. The spinor indices are raised and lowered with the aid of the ϵ tensor, namely

$$\eta_\alpha = \epsilon_{\alpha\beta}\eta^\beta\,, \qquad \bar{\eta}_{\dot{\alpha}} = \epsilon_{\dot{\alpha}\dot{\beta}}\bar{\eta}^{\dot{\beta}}\,. \tag{A.9}$$

Since the polarization are taken to be bosonic, this implies that they square to zero

$$\eta^2 \coloneqq \eta^\alpha \eta_\alpha = 0\,, \qquad \bar{\eta}^2 \coloneqq \bar{\eta}_{\dot{\alpha}}\bar{\eta}^{\dot{\alpha}} = 0\,. \tag{A.10}$$

Note the convention for contracting indices: descending undotted indices and ascending dotted ones. In principle any tensor can be recovered from its contracted form by applying spinor derivatives $\partial/\partial\eta^\alpha$ and $\partial/\partial\bar{\eta}^{\dot{\alpha}}$. These derivatives are also useful for performing index contractions. For instance

$$x \cdot y = -\tfrac{1}{2}\,\partial_\eta \mathrm{x} \partial_{\bar{\eta}}\; \eta \mathrm{y} \bar{\eta}\,, \tag{A.11}$$

where we abbreviated $\partial/\partial\eta$ with ∂_η and we suppressed all contracted indices, implying the usual rules for matrix multiplication. From the above example this formalism might look somewhat cumbersome, but it will turn out to be very convenient. The main reason is that it is possible to deal with all representations in the same way. Furthermore, index contractions now can be regarded as an operator acting on some expression. This makes it possible to neatly organize rather complicated computations.

Complex conjugation swaps barred and the unbarred quantities and then acts with the conjugation matrix $i\sigma^2 = \epsilon$. Due to (A.4) and (A.9) this results in

$$(\sigma^\mu_{\alpha\dot{\beta}})^* = \sigma^\mu_{\beta\dot{\alpha}}\,, \qquad (\bar{\sigma}^{\mu\dot{\alpha}\beta})^* = \bar{\sigma}^{\mu\dot{\beta}\alpha}\,, \qquad \bar{\eta}^{\dot{\alpha}} = \epsilon^{\dot{\alpha}\dot{\beta}}(\eta_\beta)^*\,. \tag{A.12}$$

Finally we would like to mention a class of identities that set to zero expressions which

[1] Here we clarify a minor clash in notations in this manuscript. We use $j, \bar{\jmath}$ to denote Lorentz Dynkin labels and $\ell, \bar{\ell}$ to denote the level of the supercharges in Parts I and II. Then for Part III we rename $\ell = \bar{\ell} \to k$ (we only need operators with the same number of Q and $\bar{Q}$.) and, in order to agree with the most commonly used notation, we rename $j, \bar{\jmath} \to \ell, \bar{\ell}$.

are not manifestly vanishing. These are called Schouten identities and they all stem from

$$\epsilon^{\alpha\beta}\epsilon^{\gamma\delta} + \epsilon^{\gamma\alpha}\epsilon^{\beta\delta} + \epsilon^{\beta\gamma}\epsilon^{\alpha\delta} = 0\,, \tag{A.13}$$

and the corresponding one with dotted indices. The above relation is a trivial consequence of the fact that there are three antisymmetrized indices (α, β and γ) that can only take two values. Some examples of identities that can be derived from that relation are

$$\begin{aligned} \eta_1\eta_2\ \eta_3\eta_4 &= \eta_1\eta_3\ \eta_2\eta_4 + \eta_1\eta_4\ \eta_3\eta_2\,, \\ \eta_1 \mathrm{x} \bar{\eta}_1\ \eta_2 \mathrm{x} \bar{\eta}_2 &= \eta_2 \mathrm{x} \bar{\eta}_1\ \eta_1 \mathrm{x} \bar{\eta}_2 + \eta_1\eta_2\ \bar{\eta}_1\bar{\eta}_2\ x^2\,, \end{aligned} \tag{A.14}$$

where the subscripts now represent labels and not indices.

Even though we will not use it, it might be convenient to be able to translate between spinor polarizations $\eta, \bar{\eta}$ and more familiar vector polarizations h that apply to any symmetric traceless tensor. Namely

$$\mathcal{O}(h) := \frac{1}{\ell!}\mathcal{O}_{\mu_1\cdots\mu_\ell}h^{\mu_1}\cdots h^{\mu_\ell}\,, \qquad h^\mu h_\mu = 0\,. \tag{A.15}$$

The dictionary between $\mathcal{O}(\eta, \bar{\eta})$ and $\mathcal{O}(h)$ is given by

$$\mathcal{O}(h) = \left(-\tfrac{1}{2}\right)^\ell \frac{1}{\ell!}(\partial_\eta \mathrm{h} \partial_{\bar{\eta}})^\ell\, \mathcal{O}(\eta, \bar{\eta})\,, \qquad \mathrm{h}_{\alpha\dot{\alpha}} = \sigma^\mu_{\alpha\dot{\alpha}} h_\mu\,. \tag{A.16}$$

A.2 Notation and conventions for six dimensions

The six dimensional metric G is taken as

$$G_{\mu\nu} = g_{\mu\nu}\,, \qquad G_{+-} = \tfrac{1}{2}\,, \qquad G^{+-} = 2\,. \tag{A.17}$$

The six dimensional Pauli matrices are defined as

$$\begin{aligned} \{\Sigma^\mu_{ab}, \Sigma^+_{ab}, \Sigma^-_{ab}\} &= \left\{ \begin{pmatrix} 0 & \sigma^\mu_{\alpha\dot{\gamma}}\epsilon^{\dot{\beta}\dot{\gamma}} \\ -\bar{\sigma}^{\mu\dot{\alpha}\gamma}\epsilon_{\beta\gamma} & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 2\epsilon^{\dot{\alpha}\dot{\beta}} \end{pmatrix}, \begin{pmatrix} -2\epsilon_{\alpha\beta} & 0 \\ 0 & 0 \end{pmatrix} \right\}, \\ \{\bar{\Sigma}^{\mu ab}, \bar{\Sigma}^{+ab}, \bar{\Sigma}^{-ab}\} &= \left\{ \begin{pmatrix} 0 & -\epsilon^{\alpha\gamma}\sigma^\mu_{\gamma\dot{\beta}} \\ \epsilon_{\dot{\alpha}\dot{\gamma}}\bar{\sigma}^{\mu\dot{\gamma}\beta} & 0 \end{pmatrix}, \begin{pmatrix} -2\epsilon^{\alpha\beta} & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 2\epsilon_{\dot{\alpha}\dot{\beta}} \end{pmatrix} \right\}, \end{aligned} \tag{A.18}$$

where $\alpha = a$, $\beta = b$, $\dot{\alpha} = a - 2$ and $\dot{\beta} = b - 2$. The indices a, b are fundamental $\mathrm{SU}(2,2)$ indices, namely they belong to the spinor representation of $\mathrm{Spin}(4,2)$. With these matrices one can define the twistor-space coordinates as

$$\mathbf{X}_{ab} := X_M \Sigma^M_{ab} = -\mathbf{X}_{ba}\,, \qquad \bar{\mathbf{X}}^{ab} := X_M \bar{\Sigma}^{Mab} = -\bar{\mathbf{X}}^{ba}\,. \tag{A.19}$$

From the Clifford algebra the following relation follows

$$(\mathbf{X}_i\bar{\mathbf{X}}_j)_a{}^b + (\mathbf{X}_j\bar{\mathbf{X}}_i)_a{}^b = 2(X_i \cdot X_j)\delta_b^a\,. \tag{A.20}$$

The six dimensional polarizations transform in the (anti)fundamental of $\mathrm{SU}(2,2)$ and contract the respective indices of a six dimensional tensor

$$O(S,\bar{S}) := S_{a_1}\cdots S_{a_{\bar{j}}}\,\bar{S}^{b_1}\cdots\bar{S}^{b_j}\,O^{a_1\cdots a_{\bar{j}}}_{b_1\cdots b_j}\,. \tag{A.21}$$

Under the projection defined in (1.21) and (1.25) we have the following mapping of three-point building blocks

$$I^{ij}\big|_{\text{Poinc.}} = \mathbb{I}^{ij} := \eta_j \mathrm{x}_{ij}\bar{\eta}_i\,, \tag{A.22a}$$

$$J^i_{jk}\big|_{\text{Poinc.}} = \mathbb{J}^i_{jk} := \frac{x_{ik}^2 x_{jk}^2}{x_{ij}^2}\left(\frac{\eta_k \mathrm{x}_{ik}\bar{\eta}_k}{x_{ik}^2} - \frac{\eta_k \mathrm{x}_{jk}\bar{\eta}_k}{x_{jk}^2}\right)\,, \tag{A.22b}$$

$$K^{ij}_k\big|_{\text{Poinc.}} = \mathbb{K}^{ij}_k := \frac{1}{2}\frac{|x_{ij}|}{|x_{ik}||x_{jk}|}\left((x_{ik}^2 + x_{jk}^2 - x_{ij}^2)\eta_i\eta_j - 4x_{ik}^\mu x_{jk}^\nu\,\eta_i\sigma_{\mu\nu}\eta_j\right)\,, \tag{A.22c}$$

$$\bar{K}^{ij}_k\big|_{\text{Poinc.}} = \bar{\mathbb{K}}^{ij}_k := \frac{1}{2}\frac{|x_{ij}|}{|x_{ik}||x_{jk}|}\left((x_{ik}^2 + x_{jk}^2 - x_{ij}^2)\bar{\eta}_i\bar{\eta}_j - 4x_{ik}^\mu x_{jk}^\nu\,\bar{\eta}_i\bar{\sigma}_{\mu\nu}\bar{\eta}_j\right)\,. \tag{A.22d}$$

Whereas for four-point building blocks the projections read as follows

$$\begin{aligned} I^{ij}_{kl}\big|_{\text{Poinc.}} = \mathbb{I}^{ij}_{kl} := \frac{1}{2x_{kl}^2}\big((x_{ik}^2\,\eta_j\mathrm{x}_{jl}\bar{\eta}_i - x_{il}^2\,\eta_j\mathrm{x}_{jk}\bar{\eta}_i) + (x_{jk}^2\,\eta_j\mathrm{x}_{il}\bar{\eta}_i - x_{jl}^2\,\eta_j\mathrm{x}_{ik}\bar{\eta}_i) \\ - x_{ij}^2\,\eta_j\mathrm{x}_{kl}\bar{\eta}_i - x_{kl}^2\,\eta_j\mathrm{x}_{ij}\bar{\eta}_i - 2i\epsilon^{\mu\nu\rho\lambda}x_{ik\,\nu}x_{ij\,\rho}x_{lk\,\lambda}\;\eta_j\sigma_\mu\bar{\eta}_i\big)\,, \end{aligned} \tag{A.23a}$$

$$L^i_{jkl}\big|_{\text{Poinc.}} = \mathbb{L}^i_{jkl} := \frac{2}{|x_{jk}||x_{kl}||x_{lj}|}\left(x_{ij}^2 x_{kl}^\mu x_{il}^\nu + x_{ik}^2 x_{lj}^\mu x_{ij}^\nu + x_{il}^2 x_{jk}^\mu x_{ik}^\nu\right)\eta_i\sigma_{\mu\nu}\eta_i\,, \tag{A.23b}$$

$$\bar{L}^i_{jkl}\big|_{\text{Poinc.}} = \bar{\mathbb{L}}^i_{jkl} := \frac{2}{|x_{jk}||x_{kl}||x_{lj}|}\left(x_{ij}^2 x_{kl}^\mu x_{il}^\nu + x_{ik}^2 x_{lj}^\mu x_{ij}^\nu + x_{il}^2 x_{jk}^\mu x_{ik}^\nu\right)\bar{\eta}_i\bar{\sigma}_{\mu\nu}\bar{\eta}_i\,. \tag{A.23c}$$

We refer the reader to Appendix B of [91] for more details on the formalism.

A.3 Conventions for supersymmetry

A.3.1 Superconformal algebras

Superconformal algebras can be classified according to the number of spacetime dimensions d and the number of supercharges in unit of a minimal spinor $\mathcal{N}$. In Table A.1 there is a complete list. There exist also superconformal algebras for greater $\mathcal{N}$ than the ones listed, however for $d = 3$ they lead to free superconformal field theories and for

$d = 4, 5, 6$ they lead to nonlocal theories. Superconformal algebras for $d > 6$, however, simply do not exist.

We list below the (anti)commutators that need to be added to (1.4), for the case $4d$, $\mathcal{N} = 2$. The indices I, J are $\mathfrak{su}(2)$ indices. The case $\mathcal{N} = 1$ may be obtained by dropping the $\mathfrak{su}(2)$ indices and the generators $R^I{}_J$. We follow the conventions of [13] except for the R-symmetry generators.[2]

$$
\begin{aligned}
\{Q^I_\alpha, \bar{Q}_{J\dot{\alpha}}\} &= 2\delta^I_J \mathrm{P}_{\alpha\dot{\alpha}}\,, & \{\bar{S}^{I\dot{\alpha}}, S^\alpha_J\} &= 2\delta^I_J \tilde{\mathrm{K}}^{\dot{\alpha}\alpha}\,,\\
\{Q^I_\alpha, S^\beta_J\} &= 4\left(\delta^I_J (M_\alpha{}^\beta - \tfrac{i}{2}\delta^\beta_\alpha D) - \delta^\beta_\alpha (R^I{}_J - \tfrac{1}{4}\delta^I_J r)\right)\,,\\
\{\bar{S}^{I\dot{\alpha}}, \bar{Q}_{J\dot{\beta}}\} &= 4\left(\delta^I_J (\bar{M}^{\dot{\alpha}}{}_{\dot{\beta}} + \tfrac{i}{2}\delta^{\dot{\beta}}_{\dot{\alpha}} D) - \delta^{\dot{\beta}}_{\dot{\alpha}} (R^I{}_J - \tfrac{1}{4}\delta^I_J r)\right)\,,\\
[M_\alpha{}^\beta, Q^I_\gamma] &= \delta^\beta_\gamma Q^I_\alpha - \tfrac{1}{2}\delta^\alpha_\beta Q^I_\gamma\,, & [M_\alpha{}^\beta, S^\gamma_I] &= -\delta^\gamma_\alpha S^\beta_I + \tfrac{1}{2}\delta^\beta_\alpha S^\gamma_I\,,\\
[\bar{M}^{\dot{\alpha}}{}_{\dot{\beta}}, \bar{Q}_{I\dot{\gamma}}] &= -\delta^{\dot{\alpha}}_{\dot{\gamma}} \bar{Q}_{I\dot{\beta}} + \tfrac{1}{2}\delta^{\dot{\alpha}}_{\dot{\beta}} \bar{Q}_{I\dot{\gamma}}\,, & [\bar{M}^{\alpha}{}_{\dot{\beta}}, \bar{S}^{I\dot{\gamma}}] &= \delta^{\dot{\gamma}}_{\dot{\beta}} \bar{S}^{I\dot{\alpha}} - \tfrac{1}{2}\delta^{\dot{\beta}}_{\dot{\alpha}} \bar{S}^{I\dot{\gamma}}\,,\\
[D, Q^I_\alpha] &= \tfrac{i}{2} Q^I_\alpha\,, & [D, \bar{Q}^{I\dot{\alpha}}] &= \tfrac{i}{2} \bar{Q}^{I\dot{\alpha}}\,,\\
[D, S^\alpha_I] &= -\tfrac{i}{2} S^\alpha_I\,, & [D, \bar{S}^{I\dot{\alpha}}] &= -\tfrac{i}{2} \bar{S}^{I\dot{\alpha}}\,,\\
[K_\mu, Q^I_\alpha] &= -\sigma_{\mu\alpha\dot{\alpha}} \bar{S}^{I\dot{\alpha}}\,, & [K_\mu, \bar{Q}_{I\dot{\alpha}}] &= S^\alpha_I \sigma_{\mu\alpha\dot{\alpha}}\,,\\
[P_\mu, S^\alpha_I] &= \bar{Q}_{I\dot{\alpha}} \bar{\sigma}^{\dot{\alpha}\alpha}_\mu\,, & [P_\mu, \bar{S}^{I\dot{\alpha}}] &= -\sigma^{\dot{\alpha}\alpha}_\mu Q^I_\alpha\,,\\
[R^I{}_J, Q^K_\alpha] &= \delta^K_J Q^I_\alpha - \tfrac{1}{2}\delta^I_J Q^K_\alpha\,, & [R^I{}_J, \bar{Q}_{K\dot{\alpha}}] &= -\delta^I_K \bar{Q}_{J\dot{\alpha}} + \tfrac{1}{2}\delta^I_J \bar{Q}_{K\dot{\alpha}}\,,\\
[R^I{}_J, S^\alpha_K] &= -\delta^I_K S^\alpha_J + \tfrac{1}{2}\delta^I_J S^\alpha_K\,, & [R^I{}_J, \bar{S}^{K\dot{\alpha}}] &= \delta^K_J \bar{S}^{I\dot{\alpha}} - \tfrac{1}{2}\delta^I_J \bar{S}^{K\dot{\alpha}}\,,\\
[r, Q^I_\alpha] &= -Q^I_\alpha\,, & [r, \bar{Q}_{I\dot{\alpha}}] &= \bar{Q}_{I\dot{\alpha}}\,,\\
[r, S^\alpha_I] &= S^\alpha_I\,, & [r, \bar{S}^{I\dot{\alpha}}] &= -\bar{S}^{I\dot{\alpha}}\,,\\
[R^I{}_J, R^K{}_L] &= \delta^K_J R^I{}_L - \delta^I_L R^K{}_J\,.
\end{aligned}
\tag{A.24}
$$

where we introduced

$$
\begin{aligned}
\mathrm{P}_{\alpha\dot{\alpha}} &= \sigma^\mu_{\alpha\dot{\alpha}} P_\mu\,, & \tilde{\mathrm{K}}^{\dot{\alpha}\alpha} &= \bar{\sigma}^{\mu\dot{\alpha}\alpha} K_\mu\,,\\
M_\alpha{}^\beta &= -i\sigma^{\mu\nu}{}_\alpha{}^\beta M_{\mu\nu}\,, & \bar{M}^{\dot{\alpha}}{}_\beta &= -i\bar{\sigma}^{\mu\nu\dot{\alpha}}{}_{\dot{\beta}} M_{\mu\nu}\,.
\end{aligned}
\tag{A.25}
$$

A.3.2 Superspace

We use lowercase greek letters $\alpha, \dot{\alpha}$ for spinor indices, lowercase latin letters i, j for operator labels and uppercase latin letters I, J for $\mathfrak{su}(2)_R$ indices (only for $\mathcal{N} = 2$). The

[2] Namely $(R^I{}_J)_{\mathrm{DO}} = (R^I{}_J)_{\mathrm{ours}} - \frac{1}{4} r \delta^I_J$ and $r = -2(R^I{}_I)_{\mathrm{DO}}$.

d	$\mathcal{N}$	Spin(d)	R-symmetry	Representation of Q
3	1	$\mathfrak{su}(2)$	/	$[1]$
3	$\leqslant 8$	"	$\mathfrak{so}(\mathcal{N})$	$[1]^{\mathcal{N}}$
4	1	$\mathfrak{su}(2) \oplus \mathfrak{su}(2)$	$\mathfrak{u}(1)$	$[1;0]^{(-1)} \oplus [0;1]^{(1)}$
4	2	"	$\mathfrak{su}(2) \oplus \mathfrak{u}(1)$	$[1;0]^{(1;-1)} \oplus [0;1]^{(1;1)}$
4	3	"	$\mathfrak{su}(3) \oplus \mathfrak{u}(1)$	$[1;0]^{(1,0;-1)} \oplus [0;1]^{(0,1;1)}$
4	4	"	$\mathfrak{su}(4)$	$[1;0]^{(1,0,0)} \oplus [0;1]^{(0,0,1)}$
5	1	$\mathfrak{sp}(4)$	$\mathfrak{su}(2)$	$[1,0]^{(1)}$
6	$(1,0)$	$\mathfrak{su}(4)$	$\mathfrak{su}(2)$	$[1,0,0]^{(1)}$
6	$(2,0)$	"	$\mathfrak{sp}(4)$	$[1,0,0]^{(1,0)}$

Table A.1. Superconformal algebras in any dimension d for any amount of supercharges $\mathcal{N}$. The notation $[j_1, j_2, \ldots]^{(R_1,R_2,\ldots)}$ denotes the Dynkin labels j_i of the Spin(d) group and the Dynkin labels R_i of the R-symmetry group. In $d = 3$ instead we call $\boldsymbol{\mathcal{N}}$ the $\mathcal{N}$-dimensional vector representation. When the group is not simple, the factors are separated by a semicolon.

bosonic supersymmetric interval is defined as[3]

$$(\mathrm{x}_{i\bar{\jmath}})_{\alpha\dot{\alpha}} = (\mathrm{x}_{ij})_{\alpha\dot{\alpha}} - 2i\theta_{Ii\alpha}\bar{\theta}^I_{i\dot{\alpha}} - 2i\theta_{Ij\alpha}\bar{\theta}^I_{j\dot{\alpha}} + 4i\theta_{Ii\alpha}\bar{\theta}^I_{j\dot{\alpha}}\,, \tag{A.26}$$

with x_{ij} being a shorthand for $x_i - x_j$ and $\mathrm{x}_{\alpha\dot{\alpha}}$ denoting the matrix $\sigma^\mu_{\alpha\dot{\alpha}} x_\mu$. We can also define the matrix with upper indices and the Lorentz square as follows

$$(\tilde{\mathrm{x}}_{\bar{\jmath}i})^{\dot{\alpha}\alpha} = -\epsilon^{\alpha\beta}\epsilon^{\dot{\alpha}\dot{\beta}}(\mathrm{x}_{i\bar{\jmath}})_{\beta\dot{\beta}}\,, \qquad x_{\bar{\jmath}i}{}^2 = \tfrac{1}{2}(\mathrm{x}_{i\bar{\jmath}})_{\alpha\dot{\alpha}}\,(\tilde{\mathrm{x}}_{\bar{\jmath}i})^{\dot{\alpha}\alpha}\,. \tag{A.27}$$

The fermionic supersymmetric intervals are defined as

$$\theta^\alpha_{I\,ij} = \theta^\alpha_{Ii} - \theta^\alpha_{Ij}\,, \qquad \bar{\theta}^I_{\dot{\alpha}\,ij} = \bar{\theta}^I_{\dot{\alpha}i} - \bar{\theta}^I_{\dot{\alpha}j}\,. \tag{A.28}$$

The chiral derivatives in superspace are defined in the standard way

$$D^I_{i\,\alpha} = \frac{\partial}{\partial\theta^\alpha_{I\,i}} + i\sigma^\mu_{\alpha\dot{\alpha}}\,\bar{\theta}^{I\,\dot{\alpha}}_i\frac{\partial}{\partial x^\mu_i}\,, \qquad \bar{D}_{I\,i\,\dot{\alpha}} = -\frac{\partial}{\partial\bar{\theta}^{I\,\dot{\alpha}}_i} - i\theta^\alpha_{I\,i}\sigma^\mu_{\alpha\dot{\alpha}}\frac{\partial}{\partial x^\mu_i}\,. \tag{A.29}$$

The derivative D_i yields zero when acting on $\mathrm{x}_{k\bar{\jmath}}$, $k \neq i$ and similarly for $\bar{D}_i$ when acting on $\mathrm{x}_{j\bar{k}}$, $k \neq i$. For three-point functions in $\mathcal{N} = 1, 2$ superspace we encountered in Section 2.2 the superconformally covariant variables X_3, Θ^I_3 and $\bar{\Theta}_{3\,I}$. Here is their definition

$$\begin{aligned} &\mathrm{X}_3 = \frac{\mathrm{x}_{3\bar{1}}\tilde{\mathrm{x}}_{\bar{1}2}\mathrm{x}_{2\bar{3}}}{x_{\bar{1}3}{}^2 x_{\bar{3}2}{}^2}\,, \quad \bar{\mathrm{X}}_3 = -\frac{\mathrm{x}_{3\bar{2}}\tilde{\mathrm{x}}_{\bar{2}1}\mathrm{x}_{1\bar{3}}}{x_{\bar{3}1}{}^2 x_{\bar{2}3}{}^2} = \mathrm{X}_3^\dagger\,, \\ &\Theta^I_3 = i\left(\frac{\mathrm{x}_{3\bar{1}}\bar{\theta}^I_{31}}{x_{\bar{1}3}{}^2} - \frac{\mathrm{x}_{3\bar{2}}\bar{\theta}^I_{32}}{x_{\bar{2}3}{}^2}\right)\,, \quad \bar{\Theta}_{3\,I} = i\left(\frac{\theta_{31\,I}\,\mathrm{x}_{1\bar{3}}}{x_{\bar{3}1}{}^2} - \frac{\theta_{32\,I}\,\mathrm{x}_{2\bar{3}}}{x_{\bar{3}2}{}^2}\right) = (\Theta^I_3)^\dagger\,. \end{aligned} \tag{A.30}$$

[3]Here and in the following equations the expressions for $\mathcal{N} = 1$ are obtained by dropping the I indices.

Similar objects X_i, Θ_i^I, $\overline{\Theta}_{i\,I}$, $i = 1, 2$, can be defined by a cyclic permutation of the points. We will further define

$$\mathrm{U}_3 = \tfrac{1}{2}(\mathrm{X}_3 + \bar{\mathrm{X}}_3)\,. \tag{A.31}$$

Also, note that $\mathrm{X}_3 - \bar{\mathrm{X}}_3 = 4i\Theta_3^I\overline{\Theta}_{3\,I}$. We can define chiral derivatives in the space of the $X_i, \Theta_i, \overline{\Theta}_i$ variables

$$\mathcal{D}_{\alpha\,I} = \frac{\partial}{\partial\Theta^{\alpha\,I}} - 2i\sigma^{\mu}_{\alpha\dot{\alpha}}\,\overline{\Theta}_I^{\dot{\alpha}}\frac{\partial}{\partial X^{\mu}}\,, \qquad \overline{\mathcal{D}}^I_{\dot{\alpha}} = -\frac{\partial}{\partial\overline{\Theta}^{\dot{\alpha}}_I}\,, \tag{A.32a}$$

$$\mathcal{D}_{\alpha\,I} = \frac{\partial}{\partial\Theta^{\alpha\,I}}\,, \qquad \overline{\mathcal{D}}^I_{\dot{\alpha}} = -\frac{\partial}{\partial\overline{\Theta}^{\dot{\alpha}}_I} + 2i\Theta^{\alpha\,I}\sigma^{\mu}_{\alpha\dot{\alpha}}\frac{\partial}{\partial\bar{X}^{\mu}}\,, \tag{A.32b}$$

$$\mathcal{D}_{\alpha\,I} = \frac{\partial}{\partial\Theta^{\alpha\,I}} - i\sigma^{\mu}_{\alpha\dot{\alpha}}\,\overline{\Theta}_I^{\dot{\alpha}}\frac{\partial}{\partial U^{\mu}}\,, \qquad \overline{\mathcal{D}}^I_{\dot{\alpha}} = -\frac{\partial}{\partial\overline{\Theta}^{\dot{\alpha}}_I} + i\Theta^{\alpha\,I}\sigma^{\mu}_{\alpha\dot{\alpha}}\frac{\partial}{\partial U^{\mu}}\,. \tag{A.32c}$$

All three representations are equivalent, provided one regards as independent the variables that appear therein. Furthermore we can give a list of dual derivatives $\mathcal{Q}_{\alpha}$, $\overline{\mathcal{Q}}_{\dot{\alpha}}$ satisfying $[\mathcal{D}, \mathcal{Q}\} = 0$.

$$\mathcal{Q}_{\alpha\,I} = \frac{\partial}{\partial\Theta^{\alpha\,I}} + 2i\sigma^{\mu}_{\alpha\dot{\alpha}}\,\overline{\Theta}_I^{\dot{\alpha}}\frac{\partial}{\partial X^{\mu}}\,, \qquad \overline{\mathcal{Q}}^I_{\dot{\alpha}} = -\frac{\partial}{\partial\overline{\Theta}^{\dot{\alpha}}_I}\,, \tag{A.33a}$$

$$\mathcal{Q}_{\alpha\,I} = \frac{\partial}{\partial\Theta^{\alpha\,I}}\,, \qquad \overline{\mathcal{Q}}^I_{\dot{\alpha}} = -\frac{\partial}{\partial\overline{\Theta}^{\dot{\alpha}}_I} - 2i\Theta^{\alpha\,I}\sigma^{\mu}_{\alpha\dot{\alpha}}\frac{\partial}{\partial\bar{X}^{\mu}}\,, \tag{A.33b}$$

$$\mathcal{Q}_{\alpha\,I} = \frac{\partial}{\partial\Theta^{\alpha\,I}} + i\sigma^{\mu}_{\alpha\dot{\alpha}}\,\overline{\Theta}_I^{\dot{\alpha}}\frac{\partial}{\partial U^{\mu}}\,, \qquad \overline{\mathcal{Q}}^I_{\dot{\alpha}} = -\frac{\partial}{\partial\overline{\Theta}^{\dot{\alpha}}_I} - i\Theta^{\alpha\,I}\sigma^{\mu}_{\alpha\dot{\alpha}}\frac{\partial}{\partial U^{\mu}}\,. \tag{A.33c}$$

These operators are obtained from the former by letting $(X, \Theta, \overline{\Theta}) \leftrightarrow (-\bar{X}, -\Theta, -\overline{\Theta})$, which is also the result of swapping $1 \leftrightarrow 2$ for $(X_3, \Theta_3, \overline{\Theta}_3)$, and by introducing an overall minus sign by convention. It is possible to derive some very useful identities that arise from acting with D or $\overline{D}$ on a function of Z_3. For $\mathcal{N} = 1$ we have [115]

$$\begin{aligned} D_{1\alpha}\,f(Z_3) &= -i\frac{(\mathrm{x}_{1\bar{3}})_{\alpha\dot{\alpha}}}{{x_{\bar{1}3}}^2}\,\epsilon^{\dot{\alpha}\dot{\beta}}\,\overline{\mathcal{D}}_{\dot{\beta}}\,f(Z_3)\,, \\ \overline{D}_{1\dot{\alpha}}\,f(Z_3) &= -i\frac{(\mathrm{x}_{3\bar{1}})_{\alpha\dot{\alpha}}}{{x_{\bar{3}1}}^2}\,\epsilon^{\alpha\beta}\,\mathcal{D}_{\beta}\,f(Z_3)\,. \end{aligned} \tag{A.34}$$

Similarly, for the second point we have

$$\begin{aligned} D_{2\alpha}\,f(Z_3) &= i\frac{(\mathrm{x}_{2\bar{3}})_{\alpha\dot{\alpha}}}{{x_{\bar{2}3}}^2}\,\epsilon^{\dot{\alpha}\dot{\beta}}\,\overline{\mathcal{Q}}_{\dot{\beta}}\,f(Z_3)\,, \\ \overline{D}_{2\dot{\alpha}}\,f(Z_3) &= i\frac{(\mathrm{x}_{3\bar{2}})_{\alpha\dot{\alpha}}}{{x_{\bar{3}2}}^2}\,\epsilon^{\alpha\beta}\,\mathcal{Q}_{\beta}\,f(Z_3)\,. \end{aligned} \tag{A.35}$$

And for $\mathcal{N}=2$ we have [117]

$$\begin{aligned} D^I_{1\,\alpha}\, f(Z_3) &= -i\frac{(\mathrm{x}_{1\bar{3}})_{\alpha\dot{\alpha}}}{(x_{\bar{1}3}{}^2 x_{\bar{3}1}{}^2)^{1/2}}\, \hat{u}_J{}^I(z_{31})\, \epsilon^{\dot{\alpha}\dot{\beta}}\, \overline{\mathcal{D}}_{\dot{\beta}}{}^J\, f(Z_3)\,, \\ \overline{D}_{1\dot{\alpha}\,I}\, f(Z_3) &= -i\frac{(\mathrm{x}_{3\bar{1}})_{\alpha\dot{\alpha}}}{(x_{\bar{1}3}{}^2 x_{\bar{3}1}{}^2)^{1/2}}\, \hat{u}_I{}^J(z_{13})\, \epsilon^{\alpha\beta}\, \mathcal{D}_{\beta\,J}\, f(Z_3)\,, \end{aligned} \tag{A.36}$$

having defined $u_I{}^J$ in (2.25). Similarly, for the second point we have

$$\begin{aligned} D^I_{2\,\alpha}\, f(Z_3) &= i\frac{(\mathrm{x}_{2\bar{3}})_{\alpha\dot{\alpha}}}{(x_{\bar{2}3}{}^2 x_{\bar{3}2}{}^2)^{1/2}}\, \hat{u}_J{}^I(z_{32})\, \epsilon^{\dot{\alpha}\dot{\beta}}\, \overline{\mathcal{Q}}_{\dot{\beta}}{}^J\, f(Z_3)\,, \\ \overline{D}_{2\dot{\alpha}\,I}\, f(Z_3) &= i\frac{(\mathrm{x}_{3\bar{2}})_{\alpha\dot{\alpha}}}{(x_{\bar{2}3}{}^2 x_{\bar{3}2}{}^2)^{1/2}}\, \hat{u}_I{}^J(z_{23})\, \epsilon^{\alpha\beta}\, \mathcal{Q}_{\beta\,J}\, f(Z_3)\,. \end{aligned} \tag{A.37}$$

B Superconformal correlators

B.1 Acting on different points

The formulas shown in Chapter 3 only consider the application of the differential operators at the first point. Due to the symmetry of exchanging the first two points, it is very easy to derive similar formulas for the second one. It suffices to use (A.35) instead of (A.34). The result amounts to simply replacing all $\mathcal{D}$'s to $\mathcal{Q}$'s and $\bar{\mathcal{D}}$'s to $\bar{\mathcal{Q}}$'s and including an extra minus sign for every derivative. The result concerning the commutativity with the prefactor clearly holds as well thanks to the symmetry of $\mathcal{K}_{\mathcal{O}_1\mathcal{O}_2}$ upon exchanging the first two operators.

On the other hand, the point $\mathbf{z}_3$ is treated differently by the parametrization of (2.20). As a consequence it is not possible to apply a differential operator on $\mathcal{O}_3$ using the formulas shown here. Fortunately there is a way to switch between different parametrizations by working on the t only. For simplicity, we will assume all three operators to be $\mathfrak{su}(2)$ singlets. We have

$$\langle \mathcal{O}_1(\mathbf{z}_1)\mathcal{O}_2(\mathbf{z}_2)\mathcal{O}_3(\mathbf{z}_3)\rangle = \mathcal{K}_{\mathcal{O}_1\mathcal{O}_2}(\mathbf{z}_{1,2}, z_3)\, t_{\mathcal{O}_3}^{\mathcal{O}_1\mathcal{O}_2}(Z_3)\,, \tag{B.1}$$

where some arguments of the various objects are not shown for brevity. Recall $\mathbf{z} = z, \eta, \bar{\eta}$, $z = x, \theta, \bar{\theta}$. By cyclically permuting the operators (which is a trivial operation) we get

$$\langle \mathcal{O}_2(\mathbf{z}_2)\mathcal{O}_3(\mathbf{z}_3)\mathcal{O}_1(\mathbf{z}_1)\rangle = \mathcal{K}_{\mathcal{O}_2\mathcal{O}_3}(\mathbf{z}_{2,3}, z_1)\, t_{\mathcal{O}_1}^{\mathcal{O}_2\mathcal{O}_3}(Z_1)\,, \tag{B.2}$$

having used $Z_3|_{\{1,2,3\}\to\{2,3,1\}} = Z_1$. The reduced three-point functions t satisfy a scaling property

$$\begin{aligned} t_{\mathcal{O}_3}^{\mathcal{O}_1\mathcal{O}_2}(\lambda\bar{\lambda}X, \lambda\Theta, \bar{\lambda}\bar{\Theta}, \eta_i, \bar{\eta}_i) &= \lambda^{2a}\bar{\lambda}^{2\bar{a}}\, t_{\mathcal{O}_3}^{\mathcal{O}_1\mathcal{O}_2}(X, \Theta, \bar{\Theta}, \eta_i, \bar{\eta}_i)\,, \\ t_{\mathcal{O}_3}^{\mathcal{O}_1\mathcal{O}_2}(Z, \kappa_i\eta_i, \bar{\kappa}_i\bar{\eta}_i) &= \kappa_3^{\ell_3}\bar{\kappa}_3^{\bar{\ell}_3} \prod_{i=1,2} \kappa_i^{\bar{\ell}_i}\bar{\kappa}_i^{\ell_i}\, t_{\mathcal{O}_3}^{\mathcal{O}_1\mathcal{O}_2}(Z, \eta_i, \bar{\eta}_i)\,. \end{aligned} \tag{B.3}$$

with

$$\begin{aligned} a &= \tfrac{1}{3}(q_3 - \bar{q}_1 - \bar{q}_2) + \tfrac{2}{3}(\bar{q}_3 - q_1 - q_2)\,, \\ \bar{a} &= \tfrac{2}{3}(q_3 - \bar{q}_1 - \bar{q}_2) + \tfrac{1}{3}(\bar{q}_3 - q_1 - q_2)\,. \end{aligned} \tag{B.4}$$

Let us be now more precise with the arguments of the t. We will denote it as

$$t^{\mathcal{O}_i\mathcal{O}_j}_{\mathcal{O}_k}(Z;\eta_i,\bar{\eta}_i;\eta_j,\bar{\eta}_j;\eta_k,\bar{\eta}_k)\,, \tag{B.5}$$

implying that the spinors are associated to the operator with the matching label. Following [115] we can then write the following formula that relates the two

$$t^{\mathcal{O}_2\mathcal{O}_3}_{\mathcal{O}_1}(Z;\eta_2,\bar{\eta}_2;\bar{\eta}_3\bar{\mathrm{X}},\mathrm{X}\eta_3;\eta_1,\bar{\eta}_1) = C(X,\bar{X})\, t^{\mathcal{O}_1\mathcal{O}_2}_{\mathcal{O}_3}(Z;\bar{\eta}_1\mathrm{X},\bar{\mathrm{X}}\eta_1;\eta_2,\bar{\eta}_2;\eta_3,\bar{\eta}_3)\,. \tag{B.6}$$

with

$$C(X,\bar{X}) = \frac{(-1)^{j_1+j_2+\bar{j}_2+\bar{j}_3}}{X^{2(a+\bar{q}_2)+\bar{j}_1-j_3}\,\bar{X}^{2(\bar{a}+q_2)+j_1-\bar{j}_3}}\,. \tag{B.7}$$

The matching involves only quantities in the Z space. If one wants to have the $\mathrm{X}\eta$ replacements only in one side, it is possible to consider equation (B.6) with, for instance, $\eta_1 \to \bar{\mathrm{X}}\bar{\eta}_1$ and $\bar{\eta}_1 \to \eta_1\mathrm{X}$ and then use simply

$$\bar{\mathrm{X}}\bar{\mathrm{X}}\bar{\eta}_1 = -\bar{X}^2\bar{\eta}_1\,, \qquad \eta_1\mathrm{X}\mathrm{X} = -X^2\eta_1\,. \tag{B.8}$$

The factors of X^2 and $\bar{X}^2$ can be then taken out using the scaling in $j_1, \bar{j}_1$. Often it is more convenient to express the correlator in terms of U rather than X and $\bar{X}$. The relation is very simple

$$\mathrm{X}_{\alpha\dot{\alpha}} = \mathrm{U}_{\alpha\dot{\alpha}} + 2i\Theta_\alpha\bar{\Theta}_{\dot{\alpha}}\,, \qquad \bar{\mathrm{X}}_{\alpha\dot{\alpha}} = \mathrm{U}_{\alpha\dot{\alpha}} - 2i\Theta_\alpha\bar{\Theta}_{\dot{\alpha}}\,. \tag{B.9}$$

A function that solves (B.6) has been implemented in the Mathematica package of Sec. 3.4 under the name of `permuteCyclic`.

B.2 Superspace expansion

The expansion in $\mathcal{N}=1$ superspace of a general superfield reads

$$\mathcal{O}(\mathbf{z})\big|_{\theta=\bar{\theta}=0} = O(\mathbf{x})\,. \tag{B.10}$$

$$\begin{aligned} \mathcal{O}\big|_{\theta,\bar{\theta}} = \quad & i\theta\partial_\eta\,(QO)^+ + \tfrac{j}{j+1}\,i\theta\eta\,(QO)^- \\ & - i\bar{\theta}\partial_{\bar{\eta}}\,(\bar{Q}O)^+ + \tfrac{\bar{j}}{\bar{j}+1}\,i\bar{\theta}\bar{\eta}\,(\bar{Q}O)^-\,. \end{aligned} \tag{B.11}$$

$$\mathcal{O}\big|_{\theta^2,\bar{\theta}^2} = \tfrac{1}{4}\theta^2(Q^2O) + \tfrac{1}{4}\bar{\theta}^2(\bar{Q}^2O)\,. \tag{B.12}$$

$$\begin{aligned}\mathcal{O}\big|_{\theta\bar\theta} =& -\theta\partial_\eta\,\bar\theta\partial_{\bar\eta}\big((Q\bar Q O)^{++} - ic_1\,\eta\,\partial_{\rm x}\bar\eta\,O\big)\\ &-\tfrac{j}{j+1}\,\theta\eta\,\bar\theta\partial_{\bar\eta}\big((Q\bar Q O)^{-+} - ic_2\,\partial_\eta\partial_{\rm x}\bar\eta\,O\big)\\ &+\tfrac{\bar\jmath}{\bar\jmath+1}\,\theta\partial_\eta\,\bar\theta\bar\eta\big((Q\bar Q O)^{+-} - ic_3\,\eta\,\partial_{\rm x}\partial_{\bar\eta}\,O\big)\\ &+\tfrac{j\bar\jmath}{(j+1)(\bar\jmath+1)}\,\theta\eta\,\bar\theta\bar\eta\big((Q\bar Q O)^{--} - ic_4\,\partial_\eta\partial_{\rm x}\partial_{\bar\eta}\,O\big)\,.\end{aligned} \tag{B.13}$$

$$\begin{aligned}\mathcal{O}\big|_{\theta^2\bar\theta,\bar\theta^2\theta} =& -\tfrac{i}{4}\theta^2\,\bar\theta\partial_{\bar\eta}\left((Q^2\bar Q O)^{+} - ic_5\,\partial_\eta\partial_{\rm x}\bar\eta\,(QO)^{+} - ic_6\,\eta\,\partial_{\rm x}\bar\eta\,(QO)^{-}\right)\\ &+\tfrac{i}{4}\tfrac{\bar\jmath}{\bar\jmath+1}\bar\theta^2\,\bar\theta\bar\eta\left((Q^2\bar Q O)^{-} - ic_7\,\partial_\eta\partial_{\rm x}\partial_{\bar\eta}\,(QO)^{+} - ic_8\,\eta\,\partial_{\rm x}\partial_{\bar\eta}\,(QO)^{-}\right)\\ &+\tfrac{i}{4}\bar\theta^2\,\theta\partial_\eta\left((\bar Q^2 QO)^{+} - i\bar c_5\,\eta\,\partial_{\rm x}\partial_{\bar\eta}\,(\bar QO)^{+} - i\bar c_6\,\eta\,\partial_{\rm x}\bar\eta\,(\bar QO)^{-}\right)\\ &+\tfrac{i}{4}\tfrac{j}{j+1}\bar\theta^2\,\theta\eta\left((\bar Q^2 QO)^{-} - i\bar c_7\,\partial_\eta\partial_{\rm x}\partial_{\bar\eta}\,(\bar QO)^{+} - i\bar c_8\,\partial_\eta\partial_{\rm x}\bar\eta\,(\bar QO)^{-}\right).\end{aligned} \tag{B.14}$$

$$\begin{aligned}\mathcal{O}\big|_{\theta^2\bar\theta^2} =& \tfrac{1}{16}\,\theta^2\bar\theta^2\left((Q^2\bar Q^2 O)\right.\\ &-ic_9\,\partial_\eta\partial_{\rm x}\partial_{\bar\eta}\,(Q\bar QO)^{++} - ic_{10}\,\eta\,\partial_{\rm x}\partial_{\bar\eta}\,(Q\bar QO)^{-+}\\ &-ic_{11}\,\partial_\eta\partial_{\rm x}\bar\eta\,(Q\bar QO)^{+-} - ic_{12}\,\eta\,\partial_{\rm x}\bar\eta\,(Q\bar QO)^{--}\\ &\left.-c_{13}\,\partial_x^2\,O - c_{14}\,\partial_\eta\partial_{\rm x}\partial_{\bar\eta}\,\eta\,\partial_{\rm x}\bar\eta\,O\right).\end{aligned} \tag{B.15}$$

The coefficients c_1 through c_4 agree with [156]. Instead c_5 through c_{12} and c_{14} differ by a simple normalization (cfr. [156, (A.8–10)]). Finally c_{13} is a bit different

$$c_{13}^{\rm here} = -4\,(2+j+\bar\jmath)\big(c_{13}^{\rm there} + 8\,c_{14}^{\rm there}\big)\,. \tag{B.16}$$

The reason is because they define $\partial_x^2 O$ to be

$$-\tfrac{1}{2}\,\partial_\eta\partial_{\rm x}\partial_{\bar\eta}\,\eta\,\partial_{\rm x}\bar\eta\,O + \tfrac{1}{2}\,\eta\,\partial_{\rm x}\bar\eta\,\partial_\eta\partial_{\rm x}\partial_{\bar\eta}\,O = \tfrac{1}{2}(2+j+\bar\jmath)\,\partial_\mu\partial^\mu O\,, \tag{B.17}$$

whereas we define it simply as $\partial_\mu\partial^\mu O$.

B.3 Some identities for the superspace derivatives

Let us denote in this way the following shifts of the quantum numbers in $\mathcal{K}_{\mathcal{O}_1\mathcal{O}_2}$

$$\mathcal{K}_{s_\pm\mathcal{O}_1\mathcal{O}_2} = \mathcal{K}_{\mathcal{O}_1\mathcal{O}_2}\Big|^{q_1\to q_1+1/2\,,}_{j_1\to j_1\pm1}\qquad \mathcal{K}_{\bar s_\pm\mathcal{O}_1\mathcal{O}_2} = \mathcal{K}_{\mathcal{O}_1\mathcal{O}_2}\Big|^{\bar q_1\to \bar q_1+1/2\,.}_{\bar\jmath_1\to \bar\jmath_1\pm1} \tag{B.18}$$

Note that this does not correspond to the shifts that follow from applying Q or $\bar Q$ on $\mathcal{O}_1$. Those have been defined in (3.35). This definition simply happens to be convenient for the formulas that will follow. The factors of j are inserted to cancel the contributions

from the factorials at the denominator of (2.21).

The first set of formulae that we need is the action of the first order derivatives on $\mathcal{K}_{\mathcal{O}_1\mathcal{O}_2}$. The quantum numbers of $\mathcal{O}_1$ will be denoted as $q, \bar{q}, j$ and $\bar{\jmath}$. Since in some cases $\mathcal{O}_1$ might be a superdescendant we will denote as j_0 and $\bar{\jmath}_0$ the spin labels of the superconformal primary. They will show up because they are inside the definition of the differential operators (see (3.8)).

$$D_Q^- \mathcal{K}_{\mathcal{O}_1\mathcal{O}_2} = -2(2q - j - 2)\frac{1}{j_0}\mathcal{K}_{s_-\mathcal{O}_1\mathcal{O}_2}\,\bar{\theta}_1\partial_{\bar{\chi}}\,, \tag{B.19a}$$

$$D_Q^+ \mathcal{K}_{\mathcal{O}_1\mathcal{O}_2} = -2\frac{2q + j}{j_0 + 1}\mathcal{K}_{s_+\mathcal{O}_1\mathcal{O}_2}\,\bar{\chi}\bar{\theta}_1\,, \tag{B.19b}$$

$$D_{\bar{Q}}^- \mathcal{K}_{\mathcal{O}_1\mathcal{O}_2} = -2(2\bar{q} - \bar{\jmath} - 2)\frac{1}{\bar{\jmath}_0}\mathcal{K}_{\bar{s}_-\mathcal{O}_1\mathcal{O}_2}\,\partial_\chi\theta_1\,. \tag{B.19c}$$

$$D_{\bar{Q}}^+ \mathcal{K}_{\mathcal{O}_1\mathcal{O}_2} = -2\frac{2\bar{q} + \bar{\jmath}}{\bar{\jmath}_0 + 1}\mathcal{K}_{\bar{s}_+\mathcal{O}_1\mathcal{O}_2}\,\theta_1\chi\,. \tag{B.19d}$$

Next we need the action on $\mathcal{K}_{\mathcal{O}_1\mathcal{O}_2}$ times a Grassmann variable. Below, the χ and $\bar{\chi}$ that appear on both sides of the equalities can also be replaced by ∂_χ or $\partial_{\bar{\chi}}$ being careful with the signs.[1]

$$D_Q^- \mathcal{K}_{\mathcal{O}_1\mathcal{O}_2}\,\theta_1\chi = (D_Q^- \mathcal{K}_{\mathcal{O}_1\mathcal{O}_2})\,\theta_1\chi - \frac{i}{j_0}\mathcal{K}_{s_-\mathcal{O}_1\mathcal{O}_2}\,\chi \mathrm{x}_{1\bar{3}}\partial_{\bar{\chi}}\,, \tag{B.20a}$$

$$D_Q^+ \mathcal{K}_{\mathcal{O}_1\mathcal{O}_2}\,\theta_1\chi = (D_Q^+ \mathcal{K}_{\mathcal{O}_1\mathcal{O}_2})\,\theta_1\chi + \frac{i}{j_0 + 1}\mathcal{K}_{s_+\mathcal{O}_1\mathcal{O}_2}\,\chi \mathrm{x}_{1\bar{3}}\bar{\chi}\,, \tag{B.20b}$$

$$D_{\bar{Q}}^- \mathcal{K}_{\mathcal{O}_1\mathcal{O}_2}\,\bar{\chi}\bar{\theta}_1 = (D_{\bar{Q}}^- \mathcal{K}_{\mathcal{O}_1\mathcal{O}_2})\,\bar{\chi}\bar{\theta}_1 + \frac{i}{\bar{\jmath}_0}\mathcal{K}_{\bar{s}_-\mathcal{O}_1\mathcal{O}_2}\,\partial_\chi \mathrm{x}_{3\bar{1}}\bar{\chi}\,, \tag{B.20c}$$

$$D_{\bar{Q}}^+ \mathcal{K}_{\mathcal{O}_1\mathcal{O}_2}\,\bar{\chi}\bar{\theta}_1 = (D_{\bar{Q}}^+ \mathcal{K}_{\mathcal{O}_1\mathcal{O}_2})\,\bar{\chi}\theta_1 - \frac{i}{\bar{\jmath}_0 + 1}\mathcal{K}_{\bar{s}_+\mathcal{O}_1\mathcal{O}_2}\,\chi \mathrm{x}_{3\bar{1}}\bar{\chi}\,. \tag{B.20d}$$

Finally we will also need the action of the derivatives on the t. These equations will make use of a different definition for the shifts, which can be found in (3.35). The result is

$$\mathcal{K}_{\mathcal{O}_1\mathcal{O}_2}\,D_Q^\pm\, t_{\mathcal{O}_3}^{\mathcal{O}_1\mathcal{O}_2} = \mathcal{K}_{(Q\mathcal{O}_1)^\pm\mathcal{O}_2}\,\mathcal{D}_Q^\pm\, t_{\mathcal{O}_3}^{\mathcal{O}_1\mathcal{O}_2}\,, \tag{B.21a}$$

$$\mathcal{K}_{\mathcal{O}_1\mathcal{O}_2}\,D_{\bar{Q}}^\pm\, t_{\mathcal{O}_3}^{\mathcal{O}_1\mathcal{O}_2} = \mathcal{K}_{(\bar{Q}\mathcal{O}_1)^\pm\mathcal{O}_2}\,\mathcal{D}_{\bar{Q}}^\pm\, t_{\mathcal{O}_3}^{\mathcal{O}_1\mathcal{O}_2}\,, \tag{B.21b}$$

where we have used the derivatives defined in (3.38).

All these results can be easily proven by first applying the differential operators on simple terms such as $\mathrm{x}_{i\bar{\jmath}}$ and ${x_{\bar{\imath}j}}^2$ and then working our way up to more complicated expressions. The main trick used involves shifting the labels of the prefactor. When the

[1] In our conventions the replacement is $\chi^\alpha \to \partial_{\chi_\alpha}$ and $\chi_\alpha \to -\partial_{\chi^\alpha}$. The same holds for $\bar{\chi}$.

labels are shifted down that means that a derivative acted on η or $\bar{\eta}$, bringing down a $\partial_{\bar{\chi}}$ or $\partial_{\bar{\chi}}$ respectively. When the labels are shifted up that means that we have introduced an extra auxiliary spinor χ or $\bar{\chi}$ on which the added derivative can act and reproduce the needed expression.

C Appendices for the ANEC

C.1 Supersymmetric inversion tensors

Here we list the properties needed to derive equation (4.22). The order in which they appear is roughly the order in which one needs to apply them. First of all, the explicit definition of the tensors is

$$I_{\mu\nu}(x_{1\bar{2}}, x_{\bar{1}2}) = \bar{I}_{\nu\mu}(x_{\bar{2}1}, x_{2\bar{1}}) = \frac{\mathrm{tr}(\sigma_\mu \tilde{\mathrm{x}}_{\bar{1}2}\sigma_\nu \tilde{\mathrm{x}}_{\bar{2}1})}{2\sqrt{x_{\bar{1}2}{}^2 x_{\bar{2}1}{}^2}} = \frac{\mathrm{tr}(\bar{\sigma}_\mu \mathrm{x}_{1\bar{2}}\bar{\sigma}_\nu \tilde{\mathrm{x}}_{2\bar{1}})}{2\sqrt{x_{\bar{1}2}{}^2 x_{\bar{2}1}{}^2}}\,, \tag{C.1a}$$

$$I^{i\bar{\imath}}(x_{1\bar{2}}) = \frac{i^j}{j!}\frac{(\mathrm{x}_{1\bar{2}})_{\alpha_1(\dot{\alpha}_1}\cdots(\mathrm{x}_{1\bar{2}})_{\alpha_j|\dot{\alpha}_j)}}{x_{\bar{2}1}{}^j}\,, \tag{C.1b}$$

$$\bar{I}_{\bar{\imath}i}(x_{\bar{2}1}) = \frac{(-i)^j}{j!}\frac{(\tilde{\mathrm{x}}_{\bar{2}1})^{\dot{\alpha}_1(\alpha_1}\cdots(\tilde{\mathrm{x}}_{\bar{2}1})^{\dot{\alpha}_j|\alpha_j)}}{x_{\bar{2}1}{}^j}\,. \tag{C.1c}$$

The needed properties are

$$I_{\mu\nu}(x,\bar{x})\bar{I}^{\nu\rho}(-x,-\bar{x}) = \delta_\mu^\rho\,, \tag{C.2a}$$

$$I^{i\bar{\imath}}(x)\bar{I}_{\bar{\imath}i'}(-x) = \delta^i_{i'}\,, \tag{C.2b}$$

$$I_{\lambda\rho}(x_{1\bar{3}}, x_{\bar{1}3})\bar{I}^{\rho\nu}(x_{\bar{3}2}, x_{3\bar{2}})I_{\nu\mu}(x_{2\bar{1}}, x_{\bar{2}1}) = I_{\lambda\mu}(-\bar{X}_1, -X_1)\,. \tag{C.2c}$$

The covariance property of the t and its $\lambda\bar{\lambda}$ scaling (4.15) imply

$$\begin{aligned} I^{i_1\bar{\imath}_1}(x_{1\bar{3}})I^{i_4\bar{\imath}_3}(x_{1\bar{3}})\, I_{\lambda\nu}(x_{1\bar{3}}, x_{\bar{1}3})\, t_{\bar{\imath}_1}{}^{\nu}{}_{\bar{\imath}_3}(Z_3) = \\ = X_1{}^3\bar{X}_1{}^3 x_{\bar{1}3}{}^3\, x_{\bar{3}1}{}^3 I^{i_1\bar{\imath}_1}(\bar{X}_1)I^{i_4\bar{\imath}_3}(\bar{X}_1)\, I_{\lambda\nu}(\bar{X}_1, X_1)\, t_{\bar{\imath}_1}{}^{\nu}{}_{\bar{\imath}_3}(Z_1)\,. \end{aligned} \tag{C.3}$$

The last identities that we need are

$$X_1^2 = \frac{x_{\bar{2}3}{}^2}{x_{\bar{2}1}{}^2 x_{\bar{1}3}{}^2}\,, \qquad \bar{X}_1^2 = \frac{x_{\bar{3}2}{}^2}{x_{\bar{3}1}{}^2 x_{\bar{1}2}{}^2}\,. \tag{C.4}$$

C.2 Proof of the general formula

C.2.1 Formula for the $(j,0)$ case

In this section we provide a proof of the formula (4.42) which we reproduce here for convenience:

$$\mathcal{E}[\Delta;(j,0);s]=\frac{\mathcal{A}_s[\mathsf{t}_{\bar{O}TO}]}{\mathcal{F}_s[\mathsf{n}_{\bar{O}O}]}=\frac{3\pi(-i)^j}{8}\frac{(\delta-1)(\delta+j)}{(\delta+j-s-1)_3}\left(D_1+\frac{j-s}{j}\frac{\delta+j-1}{\delta+j-s-2}D_2+\right.$$
$$\left.+\frac{(j-s-1)_2}{(j-1)_2}\frac{(\delta-j-2)_2}{(\delta+j-s-3)_2}D_3\right). \tag{C.5}$$

The first step is to realize that the dependence on j and s is entirely coming from the tensors $(\mathbb{I}^{13})^{\tilde{\jmath}}$ which appear both in $\mathsf{t}_{\bar{O}TO}$ at the numerator (with $\tilde{\jmath}=j,j-1,j-2$) and in $\mathsf{n}_{\bar{O}O}$ at the denominator (with $\tilde{\jmath}=j$). Let us then expand this tensor when the polarizations are replaced as in (4.36),

$$\begin{aligned}(\mathbb{I}^{13})^{\tilde{\jmath}}&=(\eta_3\mathsf{x}\bar{\eta}_1)^{\tilde{\jmath}}=\left(m\bar{p}\,x^+ + p\bar{m}\,x^- + m\bar{m}\,\mathsf{x}_{--} + p\bar{p}\,\mathsf{x}_{++}\right)^{\tilde{\jmath}}\\&=\sum_{s=0}^{\tilde{\jmath}}\sum_{r=0}^{\min(s,\tilde{\jmath}-s)}\binom{\tilde{\jmath}}{2r}\binom{\tilde{\jmath}-2r}{s-r}\binom{2r}{r}(x^-)^{s-r}(x^+)^{\tilde{\jmath}-r-s}(x_\perp^2)^r\,(p\bar{m})^s(m\bar{p})^{\tilde{\jmath}-s}\,.\end{aligned} \tag{C.6}$$

We obtained this result by simply doing a double binomial expansion and using $\mathsf{x}_{--}\mathsf{x}_{++}=x_\perp^2\equiv(x^1)^2+(x^2)^2$. All terms where x_{--} and x_{++} appear with different powers can be thrown away as they are not $\mathrm{SO}(2)$ neutral and there are no other invariants in the tensor structures that can compensate for them.[1] The first sum is precisely the sum over polarizations, and so we can remove it and focus on one s at a time. The second sum, instead, can be extended to $\sum_{r=0}^{\infty}$ since the binomial coefficients are automatically zero when r is out of bounds. This fact will be useful later on.

This expansion completely takes care of the polarizations of $\mathsf{n}_{\bar{O}O}$ and of the structure D_1 of $\mathsf{t}_{\bar{O}TO}$. For the other two structures it is not hard to see that the terms $(p\bar{m})^s(m\bar{p})^{\tilde{\jmath}-s}$ of the $(\mathbb{I}^{13})^{\tilde{\jmath}}$ tensor of each structure all contribute to the same term $(p\bar{m})^s(m\bar{p})^{j-s}$.[2] Concretely we find

$$\begin{aligned}\mathcal{A}_s[\mathsf{t}_{\bar{O}TO}]=-\frac{3i\pi}{4}\int_{\mathbb{R}^4}\mathrm{d}^4x\,e^{-ix^0}\sum_{r=0}(x^-)^{s-r-5}(x^+)^{j-r-s-2}(x_\perp^2)^r(x^2)^{1-\Delta-j/2}\times\\\times\left(I^{(j)}_{r,s}(x^-)^2(x^+)^2D_1-I^{(j-1)}_{r,s}x^-x^+x^2D_2+I^{(j-2)}_{r,s}(x^2)^2D_3\right),\end{aligned} \tag{C.7}$$

[1] This statement holds in the $y^+\to\infty$ limit.

[2] To be more precise there are contributions also to the terms $(p\bar{m})^{s+a}(m\bar{p})^{j-s-a}$ $(a=1,2)$, but it can be verified that in the limit $y^+\to\infty$ they are subleading.

where

$$I^{(j)}_{r,s} = \binom{j}{2r}\binom{j-2r}{s-r}\binom{2r}{r}. \tag{C.8}$$

Similarly, the denominator has the form

$$\mathcal{F}_s[\mathsf{n}_{\bar{O}O}] = i^j \int_{\mathbb{R}^4} \mathrm{d}^4x\, e^{-ix^0} \sum_{r'=0} I^{(j)}_{r',s} (x^-)^{s-r'}(x^+)^{j-r'-s}(x_\perp^2)^{r'}(x^2)^{-\Delta-j/2}. \tag{C.9}$$

The Fourier transforms can be straightforwardly computed using the general formulas

$$\begin{aligned} \int_{\mathbb{R}^2} \mathrm{d}^2x_\perp\, (x^2)^a (x_\perp^2)^b &= \frac{\pi\Gamma(1-a-b)\Gamma(1+b)}{\Gamma(-a)}(-x^-x^+)^{1+a+b}, \\ \int_{\mathbb{R}^2} \mathrm{d}x^+\mathrm{d}x^-\, e^{-i(x^++x^-)/2}\,(x^+)^a(x^-)^b &= \frac{(2\pi)^2(-i)^{a+b}(-2)^{a+b+2}}{\Gamma(-a)\Gamma(-b)}. \end{aligned} \tag{C.10}$$

What remains now is to compute the sums in r and r'. After some simplifications all sums can be reduced to the following general form for some m, n:[3]

$$\Sigma_{m,n} = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} \frac{\Gamma\big(\Delta+\frac{j}{2}-r-m\big)}{\Gamma(1-r+s)\,\Gamma(j-r-s+n)}. \tag{C.11}$$

We stress again that even though the upper limit is ∞, there are actually only a finite number of nonzero terms. After using the property

$$\Gamma(X-r) = (-1)^r \frac{\Gamma(X)}{(1-X)_r} \tag{C.12}$$

of the Γ function, we can rewrite this sum in the form of a ${}_2F_1$ hypergeometric function evaluated at 1, for which the explicit expression is known:

$$\begin{aligned} \Sigma_{m,n} &= \frac{\Gamma\big(\Delta+\frac{j}{2}-m\big)}{\Gamma(1+s)\,\Gamma(j-s+n)}\, {}_2F_1\big(-s, 1-j-n+s; 1-\Delta-\tfrac{j}{2}+m;\, 1\big) \\ &= \frac{\Gamma\big(\Delta+\frac{j}{2}-m\big)}{\Gamma(1+s)\,\Gamma(j-s+n)} \frac{\Gamma\big(1-\Delta-\frac{j}{2}+m\big)\Gamma\big(\frac{j}{2}+m+n-\Delta\big)}{\Gamma\big(s+1-\Delta-\frac{j}{2}+m\big)\Gamma\big(\frac{j}{2}+m+n-\Delta-s\big)}. \end{aligned}$$

The final result will be expressed in terms of ratios $\Sigma_{m,n}/\Sigma_{1,1}$ which are rational functions of Δ, j and s. It is now straightforward to check that it agrees with the general formula (4.42).

C.2.2 Formula for the $(j,1)$ case

In order to obtain a formula for this case we mostly need to follow the same steps as in the previous subsection, with some minor modifications. The main difference is that

[3] (m,n) can be $(1,1)$, $(2,1)$, $(3,0)$ or $(4,-1)$

the invariants $\mathbb{I}^{31}$, $\mathbb{J}^3_{12}$ and $\mathbb{J}^1_{23}$ can yield contributions with SO(2) charge ± 1 in the limit $y^+ \to \infty$. By looking at Table C.22 we see that all tensor structures have at most one of these invariant except for H_8 which contains two. Since that particular structure is zero in our superspace correlator we will not compute a formula for it. As a consequence we need to expand $(\mathbb{I}^{13})^{\tilde{\jmath}}$ keeping also terms of charge ± 1. This is easily done as follows:

$$\begin{aligned}(\mathbb{I}^{13})^{\tilde{\jmath}} &= \left(m\bar{p}\, x^+ + p\bar{m}\, x^- + m\bar{m}\, \mathrm{x}_{--} + p\bar{p}\, \mathrm{x}_{++}\right)^{\tilde{\jmath}} \\ &= \sum_{s=0}^{\tilde{\jmath}} \sum_{r=0}^{\min(s,\tilde{\jmath}-s)} \binom{\tilde{\jmath}}{2r}\binom{\tilde{\jmath}-2r}{s-r}\binom{2r}{r} (x^-)^{s-r}(x^+)^{\tilde{\jmath}-r-s}(x_\perp^2)^r\,(p\bar{m})^s(m\bar{p})^{\tilde{\jmath}-s} \\ &\quad + \sum_{s=0}^{\tilde{\jmath}-1} \sum_{t=0}^{\min(s,\tilde{\jmath}-s-1)} \binom{\tilde{\jmath}}{2t+1}\binom{\tilde{\jmath}-2t-1}{s-t}\binom{2t+1}{t+1} (x^-)^{s-t}(x^+)^{\tilde{\jmath}-t-s-1}(x_\perp^2)^t \times \\ &\qquad \times (m\bar{m}\, \mathrm{x}_{--} + p\bar{p}\, \mathrm{x}_{++})\,(p\bar{m})^s(m\bar{p})^{\tilde{\jmath}-s-1}\,.\end{aligned} \tag{C.13}$$

As before, both sums in r and t can be extended to any range. After taking care of the remaining polarizations and performing the Fourier transform with (C.10) we again end up with sums in the form of (C.11). The result will be a 2×2 matrix whose entries are ratios of Γ functions, which can be reduced to rational functions of Δ, j and s. For the extreme cases $s=0$ and $s=j+1$ one needs to retain only the appropriate entry of this matrix—respectively the upper left and the lower right—and discard the other ones. As an example we show the part of the formula that multiplies the coefficient H_2:

$$\begin{aligned}\mathcal{E}[\Delta;(j,1);s]\Big|_{H_2} &= -\frac{3\pi\,(-i)^{j+1}(\delta+1)(\delta+j+2)}{8\,(\delta+j-s+1)_3}\times \\ &\qquad \begin{pmatrix} \frac{\delta+j-s+3}{\delta+j-s+1} & \sqrt{\frac{s(j-s+1)}{(s+\delta)(\delta+j-s+1)}} \\ \sqrt{\frac{s(j-s+1)}{(s+\delta)(\delta+j-s+1)}} & \frac{(\delta+s-1)(\delta+j-s+1)}{(\delta+s)(\delta+j-s+4)} \end{pmatrix},\end{aligned} \tag{C.14}$$

where now $\delta = \Delta - j/2 - 5/2$ with Δ the dimension of the operator of spin $(j,1)$.

Clearly the same logic can be applied to more general cases $(j,\bar{\jmath})$ with $\bar{\jmath}$ fixed and j arbitrary. It suffices to expand like in (C.13) keeping terms with charge up to $\pm u$ where u is the total number of invariants $\mathbb{I}^{31}$, $\mathbb{J}^3_{12}$ and $\mathbb{J}^1_{23}$ in the tensor structure under consideration. Then all steps follow in the same way, except that one may get sums more complicated than $\Sigma_{m,n}$.

C.3 Tables

C.3.1 Ward identities

C_i	Structure	$j>1$	$j=1$	$j=0$
C_1	$\mathbb{J}^2_{13}(\mathbb{I}^{13})^j$	$-\frac{1}{2}C_2+\frac{2i^{j+1}(q-\bar{q})}{3\pi^2}$	$-\frac{1}{2}C_2-\frac{2(q-\bar{q})}{3\pi^2}$	$\frac{2i(q-\bar{q})}{3\pi^2}$
C_2	$\mathbb{I}^{23}(\mathbb{I}^{13})^{j-1}$	C_2	C_2	$\diagup$

Table C.1. Ward identities of the R-current for the correlator $\langle\bar{O}J^{(R)}O\rangle$ when O has spin $(j,0)$.

I_i	Structure	$j>1$	$j=1$	$j=0$
I_1	$\mathbb{J}^1_{23}\mathbb{I}^{23}\mathbb{I}^{32}(\mathbb{I}^{13})^{j-1}$	I_1		$\diagup$
I_2	$\mathbb{J}^2_{13}\mathbb{I}^{31}(\mathbb{I}^{13})^{j}$	I_2		
I_3	$\mathbb{I}^{21}\mathbb{I}^{32}(\mathbb{I}^{13})^{j}$	$2I_2+I_5-\frac{1}{2}(I_1+I_6+I_4)$ $-\frac{4i^j(3+2(q-\bar{q}))}{3\pi^2}$		
I_4	$\mathbb{J}^1_{23}\mathbb{J}^3_{12}\mathbb{I}^{12}\mathbb{I}^{23}(\mathbb{I}^{13})^{j-2}$	I_4	$\diagup$	$\diagup$
I_5	$\mathbb{I}^{12}\mathbb{I}^{23}\mathbb{I}^{31}(\mathbb{I}^{13})^{j-1}$	I_5		$\diagup$
I_6	$\mathbb{J}^3_{12}\mathbb{I}^{12}\mathbb{I}^{21}(\mathbb{I}^{13})^{j-1}$	I_6		$\diagup$

Table C.2. Ward identities of the R-current for the correlator $\langle\bar{O}'J^{(R)}O'\rangle$ when O' has spin $(j,1)$, R-charge $\frac{2}{3}(q-\bar{q})+1$ and is assumed to be unit normalized. If $O'=\bar{Q}O$ the terms not proportional to I_k must be rescaled by $c_{(\bar{Q}O)}$. The unbarred entries in the $j=1,0$ columns are obtained by setting the absent coefficients to zero.

D_i	Structure	$j>1$	$j=1$	$j=0$
D_1	$(\mathbb{J}^2_{13})^2(\mathbb{I}^{13})^j$	D_1	$\frac{i(2\Delta-3)}{3\pi^2}$	$\frac{2\Delta}{3\pi^2}$
D_2	$\mathbb{I}^{12}\mathbb{I}^{23}\mathbb{J}^2_{13}(\mathbb{I}^{13})^{j-1}$	$-6D_1+\frac{4i^j(\Delta-j)}{\pi^2}$	$\frac{2i}{\pi^2}$	$\diagup$
D_3	$(\mathbb{I}^{12})^2(\mathbb{I}^{23})^2(\mathbb{I}^{13})^{j-2}$	$6D_1-\frac{2i^j(2\Delta-3j)}{\pi^2}$	$\diagup$	$\diagup$

Table C.3. Ward identities of the stress tensor for the correlator $\langle\bar{O}TO\rangle$ when O has spin $(j,0)$. We have defined $\Delta=q+\bar{q}$.

H_i	Structure	$j>1$	$j=1$	$j=0$
H_1	$\mathbb{I}^{23}\mathbb{I}^{32}\mathbb{J}^2_{13}\mathbb{J}^1_{23}(\mathbb{I}^{13})^{j-1}$	H_6	H_6	$\diagup$
H_2	$\mathbb{I}^{31}(\mathbb{J}^2_{13})^2(\mathbb{I}^{13})^j$	$-\frac{1}{9}(3H_5+2H_6+H_{10})$ $-\frac{1}{6}H_9-i^{j+1}\frac{2\Delta-j-2}{3\pi^2}$	$-\frac{2}{3}H_6+\frac{2\Delta-5}{3\pi^2}$	$-i\frac{2(\Delta-1)}{3\pi^2}$
H_3	$\mathbb{I}^{21}\mathbb{I}^{32}\mathbb{J}^2_{13}(\mathbb{I}^{13})^j$	$-\frac{1}{9}(3H_5+8H_6+H_{10})$ $-\frac{1}{3}H_9-\frac{2i^{j+1}(j-3)}{3\pi^2}$	$-\frac{4}{3}H_6-\frac{2}{\pi^2}$	$\frac{2i}{\pi^2}$
H_4	$\mathbb{I}^{12}\mathbb{I}^{32}(\mathbb{I}^{23})^2\mathbb{J}^1_{23}(\mathbb{I}^{13})^{j-2}$	H_{10}	$\diagup$	$\diagup$
H_5	$\mathbb{I}^{12}\mathbb{I}^{31}\mathbb{I}^{23}\mathbb{J}^2_{13}(\mathbb{I}^{13})^{j-1}$	H_5	$\frac{4}{3}H_6+\frac{2}{\pi^2}$	$\diagup$
H_6	$\mathbb{I}^{12}\mathbb{I}^{21}\mathbb{J}^2_{13}\mathbb{J}^3_{12}(\mathbb{I}^{13})^{j-1}$	H_6	H_6	$\diagup$
H_7	$\mathbb{I}^{12}\mathbb{I}^{21}\mathbb{I}^{23}\mathbb{I}^{32}(\mathbb{I}^{13})^{j-1}$	$\frac{2}{3}(H_6-H_{10})$	$\frac{2}{3}H_6$	$\diagup$
H_8	$(\mathbb{I}^{12})^2(\mathbb{I}^{23})^2\mathbb{J}^3_{12}\mathbb{J}^1_{23}$ $\times(\mathbb{I}^{13})^{j-3}$	$-\frac{4}{3}(2H_6+H_{10})$ $+2(H_5+H_9)+\frac{4i^{j+1}j}{\pi^2}$	$\diagup$	$\diagup$
H_9	$(\mathbb{I}^{12})^2(\mathbb{I}^{23})^2\mathbb{I}^{31}(\mathbb{I}^{13})^{j-2}$	H_9	$\diagup$	$\diagup$
H_{10}	$(\mathbb{I}^{12})^2\mathbb{I}^{21}\mathbb{I}^{23}\mathbb{J}^3_{12}(\mathbb{I}^{13})^{j-2}$	H_{10}	$\diagup$	$\diagup$

Table C.4. Ward identities of the stress tensor for the correlator $\langle\bar{O}'TO'\rangle$ when O' has spin $(j,1)$, dimension $\Delta+\frac{1}{2}$ and is assumed to be unit normalized. If $O'=\bar{Q}O$ the terms not proportional to H_k must be rescaled by $c_{(\bar{Q}O)}$.

G_i	Structure	$j>1$	$j=1$	$j=0$
G_1	$\mathbb{J}^1_{23}\mathbb{J}^2_{13}\mathbb{I}^{23}(\mathbb{I}^{13})^{j-1}$	G_1	G_1	$\diagup$
G_2	$\mathbb{I}^{21}\mathbb{J}^2_{13}(\mathbb{I}^{13})^j$	$-\frac{1}{2}(G_1+G_4)-\frac{1}{3}G_3+\frac{8i^j\bar{q}}{3\pi^2}$	$-\frac{1}{2}(G_1+G_4)+\frac{8i\bar{q}}{3\pi^2}$	$\frac{8\bar{q}}{3\pi^2}$
G_3	$\mathbb{I}^{12}(\mathbb{I}^{23})^2\mathbb{J}^1_{23}(\mathbb{I}^{13})^{j-2}$	G_3	$\diagup$	$\diagup$
G_4	$\mathbb{I}^{12}\mathbb{I}^{21}\mathbb{I}^{23}(\mathbb{I}^{13})^{j-1}$	G_4	G_4	$\diagup$

Table C.5. Ward identities of the supersymmetry current for the correlator $\langle(Q\bar{O})\bar{S}O\rangle$ when $Q\bar{O}$ has spin $(1,j)$.

E_i	Structure	$j>1$	$j=1$	$j=0$
E_1	$\mathbb{J}_{13}^2\mathbb{I}^{12}(\mathbb{I}^{13})^j$	$-\frac{2}{3}E_2-\frac{4i^j(2q+j)}{3\pi^2(j+1)}$	$-\frac{2}{3}E_2-\frac{2i(2q+1)}{3\pi^2}$	$-\frac{8q}{3\pi^2}$
E_2	$(\mathbb{I}^{12})^2(\mathbb{I}^{13})^{j-1}$	E_2	E_2	$\diagup$

Table C.6. Ward identities of the supersymmetry current for the correlator $\langle(\bar{Q}\bar{O})SO\rangle$ when $\bar{Q}\bar{O}$ has spin $(0,j+1)$.

F_i	Structure	$j>1$	$j=1$
F_1	$\mathbb{J}_{13}^2\mathbb{K}_1^{23}(\mathbb{I}^{13})^{j-1}$	$-\frac{1}{3}F_2+\frac{4i^j(2q-j-2)}{3\pi^2}$	$\frac{4i(2q-3)}{3\pi^2}$
F_2	$\mathbb{I}^{12}\mathbb{I}^{23}\mathbb{K}_1^{23}(\mathbb{I}^{13})^{j-2}$	F_2	$\diagup$

Table C.7. Ward identities of the supersymmetry current for the correlator $\langle(\bar{Q}\bar{O})SO\rangle$ when $\bar{Q}\bar{O}$ has spin $(0,j-1)$.

C.3.2 Expansion in components

C_i	Structure	$j>0$	$j=0$
C_1	$\mathbb{J}_{13}^2(\mathbb{I}^{13})^j$	$i(\mathcal{C}_1+\mathcal{C}_2)$	$i\mathcal{C}_1$
C_2	$\mathbb{I}^{23}(\mathbb{I}^{13})^{j-1}$	$-i\mathcal{C}_2$	/

Table C.8. Expansion of the supersymmetric correlator in the component $\langle\bar{O}J^{(R)}O\rangle$ when O has spin $(j,0)$.

D_i	Structure	$j>1$	$j=1$	$j=0$
D_1	$(\mathbb{J}_{13}^2)^2(\mathbb{I}^{13})^j$	$-\frac{1}{4}(\mathcal{C}_5+\mathcal{C}_8)$	$-\frac{1}{4}(\mathcal{C}_5+\mathcal{C}_8)$	$-\frac{1}{4}\mathcal{C}_5$
D_2	$\mathbb{I}^{12}\mathbb{I}^{23}\mathbb{J}_{13}^2(\mathbb{I}^{13})^{j-1}$	$\frac{1}{4}(\mathcal{C}_6+\mathcal{C}_8)$	$\frac{1}{4}\mathcal{C}_8$	/
D_3	$(\mathbb{I}^{12})^2(\mathbb{I}^{23})^2(\mathbb{I}^{13})^{j-2}$	$-\frac{1}{4}\mathcal{C}_6$	/	/

Table C.9. Expansion of the supersymmetric correlator in the component $\langle\bar{O}TO\rangle$ when O has spin $(j,0)$.

E_i	Structure	$j>0$	$j=0$
E_1	$\mathbb{J}_{13}^2\mathbb{I}^{12}(\mathbb{I}^{13})^j$	$-\frac{1}{2(1+j)}(4\mathcal{C}_1+4\mathcal{C}_2+\mathcal{C}_3-\mathcal{C}_5+\mathcal{C}_6-\mathcal{C}_8)$	$-2\mathcal{C}_1-\frac{1}{2}\mathcal{C}_5$
E_2	$(\mathbb{I}^{12})^2(\mathbb{I}^{13})^{j-1}$	$\frac{1}{2(1+j)}(4\mathcal{C}_2+\mathcal{C}_3+\mathcal{C}_6-\mathcal{C}_8)$	/

Table C.10. Expansion of the supersymmetric correlator in the component $\langle(\bar{Q}\bar{O})SO\rangle$ when $\bar{Q}\bar{O}$ has spin $(0,j+1)$. The result for $j=1$ is obtained by setting $\mathcal{C}_6=0$.

F_i	Structure	$j>1$	$j=1$
F_1	$\mathbb{J}_{13}^2\mathbb{K}_1^{23}(\mathbb{I}^{13})^{j-1}$	$2(\mathcal{C}_1+\mathcal{C}_2)-\frac{1}{2}(\mathcal{C}_5+\mathcal{C}_8)-\frac{1}{2j}(\mathcal{C}_3+\mathcal{C}_6)$	
F_2	$\mathbb{I}^{12}\mathbb{I}^{23}\mathbb{K}_1^{23}(\mathbb{I}^{13})^{j-2}$	$\frac{1}{j}\mathcal{C}_6-\frac{j-1}{2j}(4\mathcal{C}_2+\mathcal{C}_3-\mathcal{C}_8)$	/

Table C.11. Expansion of the supersymmetric correlator in the component $\langle(\bar{Q}\bar{O})SO\rangle$ when $\bar{Q}\bar{O}$ has spin $(0,j-1)$. The unbarred entry in the $j=1$ column is obtained by setting $\mathcal{C}_6=0$.

G_i	Structure	$j>1$	$j=1$	$j=0$
G_1	$\mathbb{J}^1_{23}\mathbb{J}^2_{13}\mathbb{I}^{23}(\mathbb{I}^{13})^{j-1}$	$\mathcal{C}_2-\frac{1}{2}\mathcal{C}_7$		$\diagup$
G_2	$\mathbb{I}^{21}\mathbb{J}^2_{13}(\mathbb{I}^{13})^{j}$	$-2(\mathcal{C}_1+\mathcal{C}_2)-\frac{1}{2}(\mathcal{C}_5+\mathcal{C}_8)$		$-2\mathcal{C}_1-\frac{1}{2}\mathcal{C}_5$
G_3	$\mathbb{I}^{12}(\mathbb{I}^{23})^2\mathbb{J}^1_{23}(\mathbb{I}^{13})^{j-2}$	$-\frac{1}{2}\mathcal{C}_6$	$\diagup$	$\diagup$
G_4	$\mathbb{I}^{12}\mathbb{I}^{21}\mathbb{I}^{23}(\mathbb{I}^{13})^{j-1}$	$\mathcal{C}_2+\frac{1}{2}(\mathcal{C}_6+\mathcal{C}_7+\mathcal{C}_8)$		$\diagup$

Table C.12. Expansion of the supersymmetric correlator in the component $\langle(Q\bar{O})\bar{S}O\rangle$ when $Q\bar{O}$ has spin $(1,j)$. The unbarred entries in the $j=1$ column can be obtained by setting the $\mathcal{C}_6=0$.

N_i	Structure	$j\geqslant 0$
N_1	$\mathbb{J}^2_{13}(\mathbb{I}^{13})^{j+1}$	$-\frac{2(2q+j-1)}{(j+1)^2}(\mathcal{C}_1+\mathcal{C}_2)-\frac{1}{(j+1)^2}(\mathcal{C}_4+\mathcal{C}_5+\mathcal{C}_7+\mathcal{C}_8)$
N_2	$\mathbb{I}^{12}\mathbb{I}^{23}(\mathbb{I}^{13})^{j}$	$\frac{1}{(j+1)^2}\left(2\mathcal{C}_1+2(2q+j-1)\mathcal{C}_2+\mathcal{C}_4+\mathcal{C}_7+\mathcal{C}_8\right)$

Table C.13. Expansion of the supersymmetric correlator in the component $\langle(\bar{Q}\bar{O})J^{(R)}(QO)\rangle$ when $\bar{Q}\bar{O}$ has spin $(0,j+1)$ and QO has spin $(j+1,0)$. The result for $j=0,1$ can be obtained by setting the absent coefficients to zero (see caption of Table 4.2).

O_i	Structure	$j\geqslant 1$
O_1	$\mathbb{I}^{12}\overline{\mathbb{K}}^{12}_3(\mathbb{I}^{13})^{j-1}$	$\frac{1}{j(j+1)}\left(4(q-1)\mathcal{C}_2+\mathcal{C}_8\right)-\frac{1}{j+1}(2\mathcal{C}_1+\mathcal{C}_4+\mathcal{C}_7)-\frac{1}{j}(\mathcal{C}_3+\mathcal{C}_6)$

Table C.14. Expansion of the supersymmetric correlator in the component $\langle(\bar{Q}\bar{O})J^{(R)}(QO)\rangle$ when $\bar{Q}\bar{O}$ has spin $(0,j+1)$ and QO has spin $(j-1,0)$. The result for $j=1$ can be obtained by setting $\mathcal{C}_6$ to zero.

P_i	Structure	$j\geqslant 1$
P_1	$\mathbb{I}^{23}\mathbb{K}^{23}_1(\mathbb{I}^{13})^{j-1}$	$\frac{1}{j(j+1)}\left(2(2q-j-3)\mathcal{C}_2+\mathcal{C}_8+\mathcal{C}_7\right)-\frac{1}{j+1}(2\mathcal{C}_1+\mathcal{C}_4)$

Table C.15. Expansion of the supersymmetric correlator in the component $\langle(\bar{Q}\bar{O})J^{(R)}(QO)\rangle$ when $\bar{Q}\bar{O}$ has spin $(0,j-1)$ and QO has spin $(j+1,0)$.

Q_i	Structure	$j \geqslant 1$
Q_1	$\mathbb{J}_{13}^2(\mathbb{I}^{13})^{j-1}$	$-\frac{2\Xi_1}{j}\mathcal{C}_1 - \frac{2\Xi_2}{j^2}\mathcal{C}_2 - \frac{1}{j^2}\mathcal{C}_7 + \frac{1}{j}\mathcal{C}_4 + \frac{j+1}{j^2}(\mathcal{C}_3+\mathcal{C}_6+j\mathcal{C}_5) + \frac{j^2+j-1}{j^2}\mathcal{C}_8$
Q_2	$\mathbb{I}^{12}\mathbb{I}^{23}(\mathbb{I}^{13})^{j-2}$	$\frac{2(j-1)\,\Xi_3}{j^2}\mathcal{C}_2 + \frac{j-1}{j}(2\mathcal{C}_1+\mathcal{C}_4) + \frac{j^2-1}{j^2}\mathcal{C}_3 - \frac{2(j+1)}{j^2}\mathcal{C}_6 - \frac{j-1}{j^2}\mathcal{C}_7$ $- \frac{(j-1)(j+2)}{j^2}\mathcal{C}_8$

Table C.16. Expansion of the supersymmetric correlator in the component $\langle(\bar{Q}\bar{O})J^{(R)}(QO)\rangle$ when $\bar{Q}\bar{O}$ has spin $(0, j-1)$ and QO has spin $(j-1, 0)$. The result for $j = 1$ can be obtained by setting $\mathcal{C}_6$ to zero and removing the last row. Furthermore we defined

$$\begin{aligned}\Xi_1 &= j^2 - 2jq + 5j - 2q + 3\,,\\ \Xi_2 &= j^3 - 2j^2q + 5j^2 - 2jq + 3j + 2q - 3\,,\\ \Xi_3 &= j^2 - 2jq + 6j - 4q + 7\,.\end{aligned}$$

I_i	Structure	$j > 1$	$j = 1$	$j = 0$
I_1	$\mathbb{J}_{23}^1\mathbb{I}^{23}\mathbb{I}^{32}(\mathbb{I}^{13})^{j-1}$	$\mathcal{C}_7 - 2\mathcal{C}_2$		$\diagup$
I_2	$\mathbb{J}_{13}^2\mathbb{I}^{31}(\mathbb{I}^{13})^{j}$	$2(2\bar{q}-1)(\mathcal{C}_1+\mathcal{C}_2)$ $-\mathcal{C}_4-\mathcal{C}_5-\mathcal{C}_7-\mathcal{C}_8$		$2(2\bar{q}-1)\mathcal{C}_1 - \mathcal{C}_4 - \mathcal{C}_5$
I_3	$\mathbb{I}^{21}\mathbb{I}^{32}(\mathbb{I}^{13})^{j}$	$2(\mathcal{C}_1+\mathcal{C}_2) - \mathcal{C}_4 - \mathcal{C}_7$		$2\mathcal{C}_1 - \mathcal{C}_4$
I_4	$\mathbb{J}_{23}^1\mathbb{J}_{12}^3\mathbb{I}^{12}\mathbb{I}^{23}(\mathbb{I}^{13})^{j-2}$	$\mathcal{C}_6$	$\diagup$	$\diagup$
I_5	$\mathbb{I}^{12}\mathbb{I}^{23}\mathbb{I}^{31}(\mathbb{I}^{13})^{j-1}$	$-2(2\bar{q}-1)\mathcal{C}_2 - \mathcal{C}_3 + \mathcal{C}_7 + \mathcal{C}_8$		$\diagup$
I_6	$\mathbb{J}_{12}^3\mathbb{I}^{12}\mathbb{I}^{21}(\mathbb{I}^{13})^{j-1}$	$-\mathcal{C}_3 - \mathcal{C}_6$		$\diagup$

Table C.17. Expansion of the supersymmetric correlator in the component $\langle(Q\bar{O})J^{(R)}(\bar{Q}O)\rangle$ when $Q\bar{O}$ has spin $(1, j)$. The unbarred entries in the $j = 1$ column can be obtained by setting $\mathcal{C}_6 = 0$.

J_i	Structure	$j>0$	$j=0$
J_1	$(\mathbb{J}^2_{13})^2(\mathbb{I}^{13})^{j+1}$	$-\frac{i}{2(j+1)^2}\big(4\mathcal{C}_1+(2q+j)(\mathcal{C}_3+\mathcal{C}_6)$ $-(2q+j-2)(4\mathcal{C}_2+2\mathcal{C}_4)\big)$	$-2i\mathcal{C}_1+i(2q-1)\mathcal{C}_4$
J_2	$\mathbb{J}^2_{13}\mathbb{I}^{12}\mathbb{I}^{23}(\mathbb{I}^{13})^j$	$\frac{i}{(j+1)^2}\big(6\mathcal{C}_1+3\mathcal{C}_4-2(2q+j-10)\mathcal{C}_2$ $+(2q+j-1)\mathcal{C}_3+(2q+j)\mathcal{C}_6\big)$	$3i(2\mathcal{C}_1+\mathcal{C}_4)$
J_3	$(\mathbb{I}^{12})^2(\mathbb{I}^{23})^2(\mathbb{I}^{13})^{j-1}$	$-\frac{i}{2(j+1)^2}\big(32\mathcal{C}_2-4\mathcal{C}_3+(2q+j)\mathcal{C}_6\big)$	$\diagup$

Table C.18. Expansion of the supersymmetric correlator in the component $\langle(\bar{Q}\bar{O})T(QO)\rangle$ when $\bar{Q}\bar{O}$ has spin $(0,j+1)$ and QO has spin $(j+1,0)$. The result for $j=1$ can be obtained by setting $\mathcal{C}_6=0$.

K_i	Structure	$j>1$	$j=1$
K_1	$\overline{\mathbb{K}}^{12}_3\mathbb{J}^2_{13}\mathbb{I}^{12}(\mathbb{I}^{13})^{j-1}$	$-\frac{3i}{j+1}(2\mathcal{C}_1+\mathcal{C}_4)+\frac{i}{j(j+1)}\big((2q+j-1)\mathcal{C}_3$ $-(2q+j)\mathcal{C}_6+2(2q+7j-4)\mathcal{C}_2\big)$	
K_2	$\overline{\mathbb{K}}^{12}_3\mathbb{I}^{23}(\mathbb{I}^{12})^2(\mathbb{I}^{13})^{j-2}$	$\frac{i}{j(j+1)}\big(2(j-1)(8\mathcal{C}_2-\mathcal{C}_3)-(2q+j)\mathcal{C}_6\big)$	$\diagup$

Table C.19. Expansion of the supersymmetric correlator in the component $\langle(\bar{Q}\bar{O})T(QO)\rangle$ when $\bar{Q}\bar{O}$ has spin $(0,j+1)$ and QO has spin $(j-1,0)$. The unbarred entry in the $j=1$ column can be obtained by setting $\mathcal{C}_6=0$.

L_i	Structure	$j>1$	$j=1$
L_1	$\mathbb{K}^{23}_1\mathbb{J}^2_{13}\mathbb{I}^{23}(\mathbb{I}^{13})^{j-1}$	$-\frac{3i}{j+1}(2\mathcal{C}_1+\mathcal{C}_4)+\frac{i}{j(j+1)}\big((2q+j-1)\mathcal{C}_3$ $-(2q+j)\mathcal{C}_6-2(2q+7j-4)\mathcal{C}_2\big)$	
L_2	$\mathbb{K}^{23}_1\mathbb{I}^{12}(\mathbb{I}^{23})^2(\mathbb{I}^{13})^{j-2}$	$\frac{i}{j(j+1)}\big(2(j-1)(8\mathcal{C}_2-\mathcal{C}_3)-(2q+j)\mathcal{C}_6\big)$	$\diagup$

Table C.20. Expansion of the supersymmetric correlator in the component $\langle(\bar{Q}\bar{O})T(QO)\rangle$ when $\bar{Q}\bar{O}$ has spin $(0,j-1)$ and QO has spin $(j+1,0)$. The unbarred entry in the $j=1$ column can be obtained by setting $\mathcal{C}_6=0$. Note that this Table is identical to Table C.19.

M_i	Structure	$j>2$	$j=2$	$j=1$
M_1	$(\mathbb{I}^2_{13})^2(\mathbb{I}^{13})^{j-1}$	$-\frac{2i(2j-1)}{j}\mathcal{C}_1+\frac{2i\Xi_4}{j^2}\mathcal{C}_2-\frac{i\Xi_5}{2j^2}\mathcal{C}_3$ $+\frac{i\Xi_6}{j}\mathcal{C}_4-\frac{i(j-1)(\Xi_6-2q+j-1)}{2j^2}\mathcal{C}_6$		
M_2	$\mathbb{I}^2_{13}\mathbb{I}^{12}\mathbb{I}^{23}(\mathbb{I}^{13})^{j-2}$	$\frac{6i(j-1)}{j}\mathcal{C}_1-\frac{2i(j-1)\Xi_7}{j^2}\mathcal{C}_2+\frac{3i(j-1)}{j}\mathcal{C}_4$ $+\frac{i(j-1)(\Xi_7+9j-12)}{j^2}\mathcal{C}_3+\frac{i\Xi_8}{j^2}\mathcal{C}_6$		⁄
M_3	$(\mathbb{I}^{12})^2(\mathbb{I}^{23})^2(\mathbb{I}^{13})^{j-3}$	$\frac{2i(j-1)(j-2)}{j^2}(\mathcal{C}_3-8\mathcal{C}_2)$ $-\frac{i(j-2)(j^2-2jq+j-6q+2)}{2j^2}\mathcal{C}_6$	⁄	⁄

Table C.21. Expansion of the supersymmetric correlator in the component $\langle(\bar{Q}\bar{O})T(QO)\rangle$ when $\bar{Q}\bar{O}$ has spin $(0,j-1)$ and QO has spin $(j-1,0)$. The unbarred entries in the $j=2$ column are identical and the ones in the $j=1$ column are obtained by setting $\mathcal{C}_6=0$. We further defined:

$$\begin{aligned}
\Xi_4 &= j^3-2j^2q-j^2-2jq+5j+2q-4\,,\\
\Xi_5 &= j^3-2j^2q+j^2-2jq+4q-4\,,\\
\Xi_6 &= j^2-2jq+j-2q+3\,,\\
\Xi_7 &= j^2-2jq-8j-4q+18\,,\\
\Xi_8 &= j^3-2j^2q-2jq+8q-3\,.
\end{aligned}$$

H_i	Structure	$j>1$	$j=1$	$j=0$
H_1	$\mathbb{I}^{23}\mathbb{I}^{32}\mathbb{J}^2_{13}\mathbb{J}^1_{23}(\mathbb{I}^{13})^{j-1}$	$\frac{3i}{2}(\mathcal{C}_3+\mathcal{C}_6)$		$\diagup$
H_2	$\mathbb{I}^{31}(\mathbb{J}^2_{13})^2(\mathbb{I}^{13})^j$	$-2i(\mathcal{C}_1+2\bar{q}\mathcal{C}_2)-i(2\bar{q}-1)\mathcal{C}_4$ $+i(\bar{q}-1)(\mathcal{C}_3+\mathcal{C}_6)$		
H_3	$\mathbb{I}^{21}\mathbb{I}^{32}\mathbb{J}^2_{13}(\mathbb{I}^{13})^j$	$-i(6\mathcal{C}_1+2\mathcal{C}_3-3\mathcal{C}_4+2\mathcal{C}_6)$		
H_4	$\mathbb{I}^{12}\mathbb{I}^{32}(\mathbb{I}^{23})^2\mathbb{J}^1_{23}(\mathbb{I}^{13})^{j-2}$	$-\frac{3i}{2}\mathcal{C}_6$	$\diagup$	$\diagup$
H_5	$\mathbb{I}^{12}\mathbb{I}^{31}\mathbb{I}^{23}\mathbb{J}^2_{13}(\mathbb{I}^{13})^{j-1}$	$4i\bar{q}\mathcal{C}_2-2i(\bar{q}-1)\mathcal{C}_3$ $-i(2\bar{q}-3)\mathcal{C}_6$		$\diagup$
H_6	$\mathbb{I}^{12}\mathbb{I}^{21}\mathbb{J}^2_{13}\mathbb{J}^3_{12}(\mathbb{I}^{13})^{j-1}$	$\frac{3i}{2}(\mathcal{C}_3+\mathcal{C}_6)$		$\diagup$
H_7	$\mathbb{I}^{12}\mathbb{I}^{21}\mathbb{I}^{23}\mathbb{I}^{32}(\mathbb{I}^{13})^{j-1}$	$i(\mathcal{C}_3+2\mathcal{C}_6)$		$\diagup$
H_8	$(\mathbb{I}^{12})^2(\mathbb{I}^{23})^2\mathbb{J}^3_{12}\mathbb{J}^1_{23}(\mathbb{I}^{13})^{j-3}$	0	$\diagup$	$\diagup$
H_9	$(\mathbb{I}^{12})^2(\mathbb{I}^{23})^2\mathbb{I}^{31}(\mathbb{I}^{13})^{j-2}$	$i(\bar{q}-2)\mathcal{C}_6$	$\diagup$	$\diagup$
H_{10}	$(\mathbb{I}^{12})^2\mathbb{I}^{21}\mathbb{I}^{23}\mathbb{J}^3_{12}(\mathbb{I}^{13})^{j-2}$	$-\frac{3i}{2}\mathcal{C}_6$	$\diagup$	$\diagup$

Table C.22. Expansion of the supersymmetric correlator in the component $\langle(Q\bar{O})T(\bar{Q}O)\rangle$ when $Q\bar{O}$ has spin $(1,j)$. The unbarred entries in the last two columns can be obtained by setting the absent coefficients to zero (see caption of Table 4.2).

D Appendices for the bootstrap

D.1 Differential basis

In this section we provide the matrices that translate between the OPE or embedding space bases defined in Section 7.1 and the differential operator bases defined in Subsection 7.3.2. More specifically, we provide the explicit values of the matrices N_p^{ab}, $N_p'^{ab}$ that appeared in (7.70), (7.73) and (7.77). We will provide the expressions by assuming different arbitrary values Δ_i for the external dimensions of V_i and the exchanged operator $\mathcal{O}$. Clearly they will be set to the appropriate values at the end, but the for the intermediate steps of the computations we need Δ_i to be arbitrary in order to act with the dimension shifting operators Ξ. For the left three-point functions, which are associated to the matrices N, one has $\Delta_{1,2} \to 3$ and $\Delta_3 \to \Delta$; while for the right tree-point functions, associated to the matrices N', it is $\Delta_{2,3} \to 3$ and $\Delta_1 \to \Delta$.

Matrices for $p = 0$

$$
\begin{aligned}
&N^{11}_{p=0} = 1\,,\\
&N^{12}_{p=0} = N^{13}_{p=0} = N^{14}_{p=0} = N^{15}_{p=0} = N^{16}_{p=0} = N^{26}_{p=0} = N^{36}_{p=0} = N^{46}_{p=0} = N^{56}_{p=0}\\
&\qquad = N^{61}_{p=0} = N^{62}_{p=0} = N^{63}_{p=0} = N^{64}_{p=0} = N^{65}_{p=0} = 0\,,\\
&N^{21}_{p=0} = \frac{2}{\Delta_3}\,,\\
&N^{22}_{p=0} = N^{23}_{p=0} = N^{24}_{p=0} = N^{25}_{p=0} = \frac{1}{4\,(\Delta_3 - 1)\,\Delta_3}\,,\\
&N^{31}_{p=0} = -\frac{\Delta_1 - \Delta_2 + \Delta_3\ell - \ell}{(\Delta_3 - 1)\,\Delta_3\ell}\,,\\
&N^{32}_{p=0} = N^{34}_{p=0} = \frac{\Delta_1 - \Delta_2 - \Delta_3 - \ell}{8\,(\Delta_3 - 1)\,\Delta_3\ell}\,,
\end{aligned}
$$

$$
\begin{aligned}
N^{33}_{p=0} &= N^{35}_{p=0} = \frac{\Delta_1 - \Delta_2 + \Delta_3 - \ell}{8\,(\Delta_3 - 1)\,\Delta_3 \ell}\,, \\
N^{41}_{p=0} &= \frac{\Delta_1 - \Delta_2 + \ell}{\Delta_3 \ell}\,, \\
N^{42}_{p=0} &= N^{43}_{p=0} = \frac{\Delta_1 - \Delta_2 + \Delta_3 + \ell}{8\,(\Delta_3 - 1)\,\Delta_3 \ell}\,, \\
N^{44}_{p=0} &= N^{45}_{p=0} = \frac{\Delta_1 - \Delta_2 - \Delta_3 + \ell}{8\,(\Delta_3 - 1)\,\Delta_3 \ell}\,, \\
N^{51}_{p=0} &= -\frac{(\Delta_1 - \Delta_2 - \Delta_3 + \ell)\,(\Delta_1 - \Delta_2 + \Delta_3 \ell - \ell)}{2\,(\Delta_3 - 1)\,\Delta_3 (\ell - 1)\ell}\,, \\
N^{52}_{p=0} &= \frac{\Delta_1^2 + \Delta_2^2 - \Delta_3^2 - 2\Delta_1\Delta_2 + 4\Delta_3 - \ell^2 - 2\Delta_3 \ell}{16\,(\Delta_3 - 1)\,\Delta_3 (\ell - 1)\ell}\,, \\
N^{53}_{p=0} &= \frac{(\Delta_1 - \Delta_2 + \Delta_3 - \ell)\,(\Delta_1 - \Delta_2 + \Delta_3 + \ell)}{16\,(\Delta_3 - 1)\,\Delta_3 (\ell - 1)\ell}\,, \\
N^{54}_{p=0} &= \frac{(\Delta_1 - \Delta_2 - \Delta_3 + \ell)\,(\Delta_1 - \Delta_2 - \Delta_3 - \ell)}{16\,(\Delta_3 - 1)\,\Delta_3 (\ell - 1)\ell}\,, \\
N^{55}_{p=0} &= \frac{(\Delta_1 - \Delta_2 - \Delta_3 + \ell)\,(\Delta_1 - \Delta_2 + \Delta_3 - \ell)}{16\,(\Delta_3 - 1)\,\Delta_3 (\ell - 1)\ell}\,, \\
N^{66}_{p=0} &= -\frac{1}{4\,(\Delta_3 - 1)\,\ell}\,.
\end{aligned}
\tag{D.1}
$$

For the matrix $N'_{p=0}$ we will only report some entries. The ones that are *not* listed are given by

$$
N'^{ab}_{p=0} = N^{ab}_{p=0}\big|_{\Delta_3 \leftrightarrow \Delta_1}\,. \tag{D.2}
$$

For the other entries we have instead

$$
\begin{aligned}
N'^{31}_{p=0} &= -\frac{\Delta_2 - \Delta_3 + \ell}{\Delta_1 \ell}\,, \\
N'^{33}_{p=0} &= -\frac{\Delta_1 + \Delta_2 - \Delta_3 + \ell}{8\,(\Delta_1 - 1)\,\Delta_1 \ell}\,, \\
N'^{34}_{p=0} &= -\frac{-\Delta_1 + \Delta_2 - \Delta_3 + \ell}{8\,(\Delta_1 - 1)\,\Delta_1 \ell}\,, \\
N'^{41}_{p=0} &= \frac{\Delta_2 - \Delta_3 + \Delta_1 \ell - \ell}{(\Delta_1 - 1)\,\Delta_1 \ell}\,, \\
N'^{43}_{p=0} &= \frac{-\Delta_1 - \Delta_2 + \Delta_3 + \ell}{8\,(\Delta_1 - 1)\,\Delta_1 \ell}\,, \\
N'^{44}_{p=0} &= \frac{\Delta_1 - \Delta_2 + \Delta_3 + \ell}{8\,(\Delta_1 - 1)\,\Delta_1 \ell}\,, \\
N'^{51}_{p=0} &= \frac{(\Delta_1 - \Delta_2 + \Delta_3 - \ell)\,(\Delta_2 - \Delta_3 + \Delta_1 \ell - \ell)}{2\,(\Delta_1 - 1)\,\Delta_1 (\ell - 1)\ell}\,,
\end{aligned}
$$

$$N'^{53}_{p=0} = \frac{(\Delta_1+\Delta_2-\Delta_3+\ell)\,(\Delta_1+\Delta_2-\Delta_3-\ell)}{16\,(\Delta_1-1)\,\Delta_1(\ell-1)\ell}\,,$$
$$N'^{54}_{p=0} = \frac{(\Delta_1-\Delta_2+\Delta_3-\ell)\,(\Delta_1-\Delta_2+\Delta_3+\ell)}{16\,(\Delta_1-1)\,\Delta_1(\ell-1)\ell}\,. \tag{D.3}$$

The special case of spin zero is given by a smaller matrix which reads

$$N_{p=0} = N'_{p=0}\big|_{\Delta_1\to\Delta_3,\,\Delta_3\to\Delta_2,\,\Delta_2\to\Delta_1} = \begin{pmatrix} 1 & 0 \\ -\frac{2}{\Delta_1-\Delta_2-\Delta_3} & -\frac{1}{(\Delta_1-\Delta_2-\Delta_3)(\Delta_1-\Delta_2+\Delta_3)} \end{pmatrix}. \tag{D.4}$$

Matrices for $p = 2$

$$N^{11}_{p=2} = \frac{-\Delta_1^2-\Delta_2^2+\Delta_3^2+10\Delta_1+2\Delta_1\Delta_2-10\Delta_2-6\Delta_3+\ell^2-\Delta_3\ell+6\ell+4}{4\,(\Delta_2-\Delta_1-\Delta_3+\ell+4)\,(\Delta_2-\Delta_1+\Delta_3+\ell)\,(10\Delta_2-10\Delta_2+2\Delta_3+\Delta_3\ell-2\ell-4)}\,,$$
$$N^{12}_{p=2} = -\frac{-\Delta_1^2-\Delta_2^2+\Delta_3^2-10\Delta_1+2\Delta_1\Delta_2+10\Delta_2-2\Delta_3+\ell^2+\Delta_3\ell+2\ell-4}{4\,(\Delta_2-\Delta_1-\Delta_3+\ell+4)\,(\Delta_2-\Delta_1+\Delta_3+\ell)\,(10\Delta_2-10\Delta_2+2\Delta_3+\Delta_3\ell-2\ell-4)}\,,$$
$$N^{13}_{p=2} = -\frac{-\Delta_1^2-\Delta_2^2+\Delta_3^2+2\Delta_1\Delta_2-4\Delta_3+\ell^2+4\ell}{4\,(\Delta_2-\Delta_1-\Delta_3+\ell+4)\,(\Delta_2-\Delta_1+\Delta_3+\ell)\,(10\Delta_2-10\Delta_2+2\Delta_3+\Delta_3\ell-2\ell-4)}\,,$$
$$N^{14}_{p=2} = \frac{-\Delta_1^2-\Delta_2^2+\Delta_3^2+2\Delta_1\Delta_2-4\Delta_3+\ell^2+4\ell}{4\,(\Delta_2-\Delta_1-\Delta_3+\ell+4)\,(\Delta_2-\Delta_1+\Delta_3+\ell)\,(10\Delta_2-10\Delta_2+2\Delta_3+\Delta_3\ell-2\ell-4)}\,,$$
$$N^{21}_{p=2} = N^{24}_{p=2} = \frac{1}{8\,(10\Delta_2-10\Delta_2+2\Delta_3+\Delta_3\ell-2\ell-4)}\,,$$
$$N^{22}_{p=2} = N^{23}_{p=2} = -N^{21}_{p=2}\,,$$
$$N^{31}_{p=2} = -\frac{(-\Delta_1+\Delta_2-\Delta_3+\ell+6)\,(-2\Delta_1+2\Delta_2-2\Delta_3+3\Delta_3\ell+2\ell-4)}{32\Delta_3\ell\,(-\Delta_1+\Delta_2-\Delta_3+\ell+4)\,(10\Delta_2-10\Delta_2+2\Delta_3+\Delta_3\ell-2\ell-4)}\,,$$
$$N^{32}_{p=2} = \frac{(-\Delta_1+\Delta_2-\Delta_3+\ell+6)\,(-2\Delta_1+2\Delta_2-2\Delta_3+3\Delta_3\ell+2\ell-4)}{32\Delta_3\ell\,(-\Delta_1+\Delta_2-\Delta_3+\ell+4)\,(10\Delta_2-10\Delta_2+2\Delta_3+\Delta_3\ell-2\ell-4)}\,,$$

$$N^{33}_{p=2} = \frac{\begin{array}{r}-4\Delta_1^2-4\Delta_2^2+2\Delta_3^2-16\Delta_1+8\Delta_1\Delta_2+16\Delta_2-2\Delta_1\Delta_3+2\Delta_2\Delta_3-4\Delta_3+\Delta_3\ell^2+2\ell^2\\ -\Delta_3^2\ell+2\Delta_1\ell-2\Delta_2\ell-\Delta_1\Delta_3\ell+\Delta_2\Delta_3\ell+6\Delta_3\ell+4\ell-16\end{array}}{16\Delta_3\ell\,(-\Delta_1+\Delta_2-\Delta_3+\ell+4)\,(10\Delta_2-10\Delta_2+2\Delta_3+\Delta_3\ell-2\ell-4)}\,,$$
$$N^{34}_{p=2} = -\frac{(-\Delta_1+\Delta_2-\Delta_3+\ell+6)\,(4\Delta_1-4\Delta_2-2\Delta_3+\Delta_3\ell+2\ell)}{16\Delta_3\ell\,(-\Delta_1+\Delta_2-\Delta_3+\ell+4)\,(10\Delta_2-10\Delta_2+2\Delta_3+\Delta_3\ell-2\ell-4)}\,,$$

$$N^{41}_{p=2} = \frac{\begin{array}{r}-2\Delta_1^3+2\Delta_2^3+2\Delta_3^3+8\Delta_1^2-6\Delta_1\Delta_2^2+8\Delta_2^2+2\Delta_1\Delta_3^2-2\Delta_2\Delta_3^2-8\Delta_3^2+24\Delta_1+6\Delta_1^2\Delta_2\\ -16\Delta_1\Delta_2-24\Delta_2-2\Delta_1^2\Delta_3-2\Delta_2^2\Delta_3+4\Delta_1\Delta_2\Delta_3-24\Delta_3+7\Delta_3\ell^3-2\ell^3-4\Delta_3^2\ell^2\\ +2\Delta_1\ell^2-2\Delta_2\ell^2+40\Delta_3\ell^2-8\ell^2+7\Delta_3^3\ell+2\Delta_1^2\ell+2\Delta_2^2\ell-40\Delta_3^2\ell-4\Delta_1\Delta_2\ell\\ -7\Delta_1^2\Delta_3\ell-7\Delta_2^2\Delta_3\ell+38\Delta_1\Delta_3\ell+14\Delta_1\Delta_2\Delta_3\ell-38\Delta_2\Delta_3\ell+24\Delta_3\ell+24\ell\end{array}}{32\Delta_3\ell\,(\Delta_2-\Delta_1-\Delta_3+\ell+4)\,(\Delta_2-\Delta_1+\Delta_3+\ell)\,(10\Delta_2-10\Delta_2+2\Delta_3+\Delta_3\ell-2\ell-4)}\,,$$

$$
\begin{aligned}
N^{42}_{p=2} = \frac{
&-2\Delta_1^3+2\Delta_2^3+2\Delta_3^3+8\Delta_1^2-6\Delta_1\Delta_2^2+8\Delta_2^2+2\Delta_1\Delta_3^2-2\Delta_2\Delta_3^2-8\Delta_3^2+24\Delta_1+6\Delta_1^2\Delta_2\\
&-16\Delta_1\Delta_2-24\Delta_2-2\Delta_1^2\Delta_3-2\Delta_2^2\Delta_3+4\Delta_1\Delta_2\Delta_3-24\Delta_3+7\Delta_3\ell^3-2\ell^3+4\Delta_3^2\ell^2\\
&+2\Delta_1\ell^2-2\Delta_2\ell^2+24\Delta_3\ell^2-8\ell^2+7\Delta_3^3\ell+2\Delta_1^2\ell+2\Delta_2^2\ell-24\Delta_3^2\ell-4\Delta_1\Delta_2\ell\\
&-7\Delta_1^2\Delta_3\ell-7\Delta_2^2\Delta_3\ell-42\Delta_1\Delta_3\ell+14\Delta_1\Delta_2\Delta_3\ell+42\Delta_2\Delta_3\ell-8\Delta_3\ell+24\ell
}{
32\Delta_3\ell\,(\Delta_2-\Delta_1-\Delta_3+\ell+4)\,(\Delta_1-\Delta_2-\Delta_3-\ell)\,(10\Delta_2-10\Delta_2+2\Delta_3+\Delta_3\ell-2\ell-4)
},
\end{aligned}
$$

$$
\begin{aligned}
N^{43}_{p=2} = \frac{
&2\Delta_1^3-2\Delta_2^3+\Delta_3^3+8\Delta_1^2+6\Delta_1\Delta_2^2+8\Delta_2^2-2\Delta_1\Delta_3^2+2\Delta_2\Delta_3^2-2\Delta_3^2+8\Delta_1-6\Delta_1^2\Delta_2\\
&-16\Delta_1\Delta_2-8\Delta_2-\Delta_1^2\Delta_3-\Delta_2^2\Delta_3-6\Delta_1\Delta_3+2\Delta_1\Delta_2\Delta_3+6\Delta_2\Delta_3-8\Delta_3+2\Delta_3\ell^3\\
&-\ell^3+\Delta_3^2\ell^2-2\Delta_1\ell^2+2\Delta_2\ell^2+6\Delta_3\ell^2-2\ell^2+2\Delta_3^3\ell+\Delta_1^2\ell+\Delta_2^2\ell-6\Delta_3^2\ell+6\Delta_1\ell\\
&-2\Delta_1\Delta_2\ell-6\Delta_2\ell-2\Delta_1^2\Delta_3\ell-2\Delta_2^2\Delta_3\ell-11\Delta_1\Delta_3\ell+4\Delta_1\Delta_2\Delta_3\ell+11\Delta_2\Delta_3\ell-4\Delta_3\ell+8\ell
}{
8\Delta_3\ell\,(\Delta_2-\Delta_1-\Delta_3+\ell+4)\,(\Delta_1-\Delta_2-\Delta_3-\ell)\,(10\Delta_2-10\Delta_2+2\Delta_3+\Delta_3\ell-2\ell-4)
},
\end{aligned}
$$

$$
\begin{aligned}
N^{44}_{p=2} = \frac{
&2\Delta_1^3-2\Delta_2^3+\Delta_3^3-12\Delta_1^2+6\Delta_1\Delta_2^2-12\Delta_2^2-2\Delta_1\Delta_3^2+2\Delta_2\Delta_3^2-6\Delta_3^2-6\Delta_1^2\Delta_2+24\Delta_1\Delta_2\\
&-\Delta_1^2\Delta_3-\Delta_2^2\Delta_3+18\Delta_1\Delta_3+2\Delta_1\Delta_2\Delta_3-18\Delta_2\Delta_3+2\Delta_3\ell^3-\ell^3-\Delta_3^2\ell^2-2\Delta_1\ell^2+2\Delta_2\ell^2\\
&+12\Delta_3\ell^2-6\ell^2+2\Delta_3^3\ell+\Delta_1^2\ell+\Delta_2^2\ell-12\Delta_3^2\ell-18\Delta_1\ell-2\Delta_1\Delta_2\ell+18\Delta_2\ell\\
&-2\Delta_1^2\Delta_3\ell-2\Delta_2^2\Delta_3\ell+11\Delta_1\Delta_3\ell+4\Delta_1\Delta_2\Delta_3\ell-11\Delta_2\Delta_3\ell+12\Delta_3\ell
}{
8\Delta_3\ell\,(\Delta_2-\Delta_1-\Delta_3+\ell+4)\,(\Delta_2-\Delta_1+\Delta_3+\ell)\,(10\Delta_2-10\Delta_2+2\Delta_3+\Delta_3\ell-2\ell-4)
},
\end{aligned}
\tag{D.5}
$$

$$
N'^{11}_{p=2} = N'^{14}_{p=2} = -\frac{1}{8\,(2\Delta_1-10\Delta_2+10\Delta_3+\Delta_1\ell-2\ell-4)},
$$

$$
N'^{12}_{p=2} = N'^{13}_{p=2} = \frac{1}{8\,(2\Delta_1-10\Delta_2+10\Delta_3+\Delta_1\ell-2\ell-4)},
$$

$$
N'^{21}_{p=2} = -\frac{\Delta_1^2-\Delta_2^2-\Delta_3^2-6\Delta_1+10\Delta_2+2\Delta_2\Delta_3-10\Delta_3+\ell^2-\Delta_1\ell+6\ell+4}{4\,(-\Delta_1-\Delta_2+\Delta_3+\ell+4)\,(\Delta_1-\Delta_2+\Delta_3+\ell)\,(2\Delta_1-10\Delta_2+10\Delta_3+\Delta_1\ell-2\ell-4)},
$$

$$
N'^{22}_{p=2} = \frac{\Delta_1^2-\Delta_2^2-\Delta_3^2-2\Delta_1-10\Delta_2+2\Delta_2\Delta_3+10\Delta_3+\ell^2+\Delta_1\ell+2\ell-4}{4\,(-\Delta_1-\Delta_2+\Delta_3+\ell+4)\,(\Delta_1-\Delta_2+\Delta_3+\ell)\,(2\Delta_1-10\Delta_2+10\Delta_3+\Delta_1\ell-2\ell-4)},
$$

$$
N'^{23}_{p=2} = \frac{\Delta_1^2-\Delta_2^2-\Delta_3^2-4\Delta_1+2\Delta_2\Delta_3+\ell^2+4\ell}{4\,(-\Delta_1-\Delta_2+\Delta_3+\ell+4)\,(\Delta_1-\Delta_2+\Delta_3+\ell)\,(2\Delta_1-10\Delta_2+10\Delta_3+\Delta_1\ell-2\ell-4)},
$$

$$
N'^{24}_{p=2} = -\frac{\Delta_1^2-\Delta_2^2-\Delta_3^2-4\Delta_1+2\Delta_2\Delta_3+\ell^2+4\ell}{4\,(-\Delta_1-\Delta_2+\Delta_3+\ell+4)\,(\Delta_1-\Delta_2+\Delta_3+\ell)\,(2\Delta_1-10\Delta_2+10\Delta_3+\Delta_1\ell-2\ell-4)},
$$

$$N'^{31}_{p=2} = \frac{\begin{array}{r}2\Delta_1^3-2\Delta_2^3+2\Delta_3^3-8\Delta_1^2-2\Delta_1\Delta_2^2+8\Delta_2^2-2\Delta_1\Delta_3^2-6\Delta_2\Delta_3^2+8\Delta_3^2-24\Delta_1+2\Delta_1^2\Delta_2\\ +24\Delta_2-2\Delta_1^2\Delta_3+6\Delta_2^2\Delta_3+4\Delta_1\Delta_2\Delta_3-16\Delta_2\Delta_3-24\Delta_3+7\Delta_1\ell^3-2\ell^3-4\Delta_1^2\ell^2\\ +40\Delta_1\ell^2+2\Delta_2\ell^2-2\Delta_3\ell^2-8\ell^2+7\Delta_1^3\ell-40\Delta_1^2\ell-7\Delta_1\Delta_2^2\ell+2\Delta_2^2\ell-7\Delta_1\Delta_3^2\ell\\ +2\Delta_3^2\ell+24\Delta_1\ell+38\Delta_1\Delta_2\ell-38\Delta_1\Delta_3\ell+14\Delta_1\Delta_2\Delta_3\ell-4\Delta_2\Delta_3\ell+24\ell\end{array}}{32\Delta_1\ell\,(\Delta_3-\Delta_1-\Delta_2+\ell+4)\,(\Delta_1-\Delta_2+\Delta_3+\ell)\,(2\Delta_1-10\Delta_2+10\Delta_3+\Delta_1\ell-2\ell-4)},$$

$$N'^{32}_{p=2} = \frac{\begin{array}{r}2\Delta_1^3-2\Delta_2^3+2\Delta_3^3-8\Delta_1^2-2\Delta_1\Delta_2^2+8\Delta_2^2-2\Delta_1\Delta_3^2-6\Delta_2\Delta_3^2+8\Delta_3^2-24\Delta_1+2\Delta_1^2\Delta_2\\ +24\Delta_2-2\Delta_1^2\Delta_3+6\Delta_2^2\Delta_3+4\Delta_1\Delta_2\Delta_3-16\Delta_2\Delta_3-24\Delta_3+7\Delta_1\ell^3-2\ell^3+4\Delta_1^2\ell^2\\ +24\Delta_1\ell^2+2\Delta_2\ell^2-2\Delta_3\ell^2-8\ell^2+7\Delta_1^3\ell-24\Delta_1^2\ell-7\Delta_1\Delta_2^2\ell+2\Delta_2^2\ell-7\Delta_1\Delta_3^2\ell\\ +2\Delta_3^2\ell-8\Delta_1\ell-42\Delta_1\Delta_2\ell+42\Delta_1\Delta_3\ell+14\Delta_1\Delta_2\Delta_3\ell-4\Delta_2\Delta_3\ell+24\ell\end{array}}{32\Delta_1\ell\,(\Delta_1+\Delta_2-\Delta_3-\ell-4)\,(\Delta_1-\Delta_2+\Delta_3+\ell)\,(2\Delta_1-10\Delta_2+10\Delta_3+\Delta_1\ell-2\ell-4)},$$

$$N'^{33}_{p=2} = \frac{\begin{array}{r}\Delta_1^3+2\Delta_2^3-2\Delta_3^3-2\Delta_1^2-\Delta_1\Delta_2^2+8\Delta_2^2-\Delta_1\Delta_3^2+6\Delta_2\Delta_3^2+8\Delta_3^2-8\Delta_1-2\Delta_1^2\Delta_2\\ -6\Delta_1\Delta_2+8\Delta_2+2\Delta_1^2\Delta_3-6\Delta_2^2\Delta_3+6\Delta_1\Delta_3+2\Delta_1\Delta_2\Delta_3-16\Delta_2\Delta_3-8\Delta_3+2\Delta_1\ell^3\\ -\ell^3+\Delta_1^2\ell^2+6\Delta_1\ell^2-2\Delta_2\ell^2+2\Delta_3\ell^2-2\ell^2+2\Delta_1^3\ell-6\Delta_1^2\ell-2\Delta_1\Delta_2^2\ell+\Delta_2^2\ell-2\Delta_1\Delta_3^2\ell\\ +\Delta_3^2\ell-4\Delta_1\ell-11\Delta_1\Delta_2\ell+6\Delta_2\ell+11\Delta_1\Delta_3\ell+4\Delta_1\Delta_2\Delta_3\ell-2\Delta_2\Delta_3\ell-6\Delta_3\ell+8\ell\end{array}}{8\Delta_1\ell\,(\Delta_1+\Delta_2-\Delta_3-\ell-4)\,(\Delta_1-\Delta_2+\Delta_3+\ell)\,(2\Delta_1-10\Delta_2+10\Delta_3+\Delta_1\ell-2\ell-4)},$$

$$N'^{34}_{p=2} = \frac{\begin{array}{r}\Delta_1^3+2\Delta_2^3-2\Delta_3^3-6\Delta_1^2-\Delta_1\Delta_2^2-12\Delta_2^2-\Delta_1\Delta_3^2+6\Delta_2\Delta_3^2-12\Delta_3^2-2\Delta_1^2\Delta_2+18\Delta_1\Delta_2\\ +2\Delta_1^2\Delta_3-6\Delta_2^2\Delta_3-18\Delta_1\Delta_3+2\Delta_1\Delta_2\Delta_3+24\Delta_2\Delta_3+2\Delta_1\ell^3-\ell^3-\Delta_1^2\ell^2+12\Delta_1\ell^2\\ -2\Delta_2\ell^2+2\Delta_3\ell^2-6\ell^2+2\Delta_1^3\ell-12\Delta_1^2\ell-2\Delta_1\Delta_2^2\ell+\Delta_2^2\ell-2\Delta_1\Delta_3^2\ell+\Delta_3^2\ell+12\Delta_1\ell\\ +11\Delta_1\Delta_2\ell-18\Delta_2\ell-11\Delta_1\Delta_3\ell+4\Delta_1\Delta_2\Delta_3\ell-2\Delta_2\Delta_3\ell+18\Delta_3\ell\end{array}}{8\Delta_1\ell\,(\Delta_3-\Delta_1-\Delta_2+\ell+4)\,(\Delta_1-\Delta_2+\Delta_3+\ell)\,(2\Delta_1-10\Delta_2+10\Delta_3+\Delta_1\ell-2\ell-4)},$$

$$N'^{41}_{p=2} = -\frac{(-\Delta_1-\Delta_2+\Delta_3+\ell+6)\,(-2\Delta_1-2\Delta_2+2\Delta_3+3\Delta_1\ell+2\ell-4)}{32\Delta_1\ell\,(-\Delta_1-\Delta_2+\Delta_3+\ell+4)\,(2\Delta_1-10\Delta_2+10\Delta_3+\Delta_1\ell-2\ell-4)},$$

$$N'^{42}_{p=2} = \frac{(-\Delta_1-\Delta_2+\Delta_3+\ell+6)\,(-2\Delta_1-2\Delta_2+2\Delta_3+3\Delta_1\ell+2\ell-4)}{32\Delta_1\ell\,(-\Delta_1-\Delta_2+\Delta_3+\ell+4)\,(2\Delta_1-10\Delta_2+10\Delta_3+\Delta_1\ell-2\ell-4)},$$

$$N'^{43}_{p=2} = \frac{\begin{array}{r}2\Delta_1^2-4\Delta_2^2-4\Delta_3^2-4\Delta_1-2\Delta_1\Delta_2-16\Delta_2+2\Delta_1\Delta_3+8\Delta_2\Delta_3+16\Delta_3+\Delta_1\ell^2+2\ell^2\\ -\Delta_1^2\ell+6\Delta_1\ell-\Delta_1\Delta_2\ell+2\Delta_2\ell+\Delta_1\Delta_3\ell-2\Delta_3\ell+4\ell-16\end{array}}{16\Delta_1\ell\,(-\Delta_1-\Delta_2+\Delta_3+\ell+4)\,(2\Delta_1-10\Delta_2+10\Delta_3+\Delta_1\ell-2\ell-4)},$$

$$N'^{44}_{p=2} = -\frac{(-2\Delta_1+4\Delta_2-4\Delta_3+\Delta_1\ell+2\ell)\,(-\Delta_1-\Delta_2+\Delta_3+\ell+6)}{16\Delta_1\ell\,(-\Delta_1-\Delta_2+\Delta_3+\ell+4)\,(2\Delta_1-10\Delta_2+10\Delta_3+\Delta_1\ell-2\ell-4)}. \tag{D.6}$$

Matrix for $p = 4$

$$N^{-1}_{p=4} \equiv -96(\Delta_1 - \Delta_2 - \Delta_3 + \ell + 6)(\Delta_1 - \Delta_2 + \Delta_3 + \ell)\,. \tag{D.7}$$

D.2 Four-point function basis

In this appendix we provide the explicit choice of basis for the four-point function of currents defined in Section 7.2. We are working in conformal frame so the structures are given in the notation (6.34). In particular, all structures will look like this

$$\mathbb{T}^0_i = \begin{bmatrix} q_1 & q_2 & q_3 & q_4 \\ \bar{q}_1 & \bar{q}_2 & \bar{q}_3 & \bar{q}_4 \end{bmatrix}, \qquad q_i, \bar{q}_i = \pm\frac{1}{2}, \quad \sum_{i=1}^{4}(q_i - \bar{q}_i) = 0\,. \tag{D.8}$$

There are 70 choices of the charges $q_i, \bar{q}_i$ subject to the constraints of (D.8). They can all be represented with a binary number of eight digits by taking $q_i, \bar{q}_i \to q_i + \frac{1}{2}, \bar{q}_i + \frac{1}{2}$ and regarding the charges as binary digits $0, 1$. If we express this number in the decimal decimal basis we obtain a compact representation of all structures (obviously not all numbers from 0 to 255 will appear as not all of them satisfy the constraint of zero total charge). As an example

$$\mathbb{T}^0_{20} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \longrightarrow 11001010_2 = 202\,. \tag{D.9}$$

With this funny notation, we now present our choice of basis

$$\begin{array}{lllll}
\mathbb{T}^0_1 \to 255, & \mathbb{T}^0_2 \to 238, & \mathbb{T}^0_3 \to 221, & \mathbb{T}^0_4 \to 204, & \mathbb{T}^0_5 \to 187, \\
\mathbb{T}^0_6 \to 170, & \mathbb{T}^0_7 \to 153, & \mathbb{T}^0_8 \to 136, & \mathbb{T}^0_9 \to 119, & \mathbb{T}^0_{10} \to 102, \\
\mathbb{T}^0_{11} \to 85, & \mathbb{T}^0_{12} \to 68, & \mathbb{T}^0_{13} \to 51, & \mathbb{T}^0_{14} \to 34, & \mathbb{T}^0_{15} \to 17, \\
\mathbb{T}^0_{16} \to 0, & \mathbb{T}^0_{17} \to 237, & \mathbb{T}^0_{18} \to 235, & \mathbb{T}^0_{19} \to 219, & \mathbb{T}^0_{20} \to 202, \\
\mathbb{T}^0_{21} \to 201, & \mathbb{T}^0_{22} \to 169, & \mathbb{T}^0_{23} \to 231, & \mathbb{T}^0_{24} \to 215, & \mathbb{T}^0_{25} \to 198, \\
\mathbb{T}^0_{26} \to 197, & \mathbb{T}^0_{27} \to 195, & \mathbb{T}^0_{28} \to 183, & \mathbb{T}^0_{29} \to 166, & \mathbb{T}^0_{30} \to 165, \\
\mathbb{T}^0_{31} \to 150, & \mathbb{T}^0_{32} \to 149, & \mathbb{T}^0_{33} \to 132, & \mathbb{T}^0_{34} \to 163, & \mathbb{T}^0_{35} \to 147, \\
\mathbb{T}^0_{36} \to 130, & \mathbb{T}^0_{37} \to 129, & \mathbb{T}^0_{38} \to 101, & \mathbb{T}^0_{39} \to 99, & \mathbb{T}^0_{40} \to 83, \\
\mathbb{T}^0_{41} \to 66, & \mathbb{T}^0_{42} \to 65, & \mathbb{T}^0_{43} \to 33, & \mathbb{T}^0_{44} \to 222, & \mathbb{T}^0_{45} \to 190, \\
\mathbb{T}^0_{46} \to 189, & \mathbb{T}^0_{47} \to 172, & \mathbb{T}^0_{48} \to 156, & \mathbb{T}^0_{49} \to 154, & \mathbb{T}^0_{50} \to 126, \\
\mathbb{T}^0_{51} \to 125, & \mathbb{T}^0_{52} \to 108, & \mathbb{T}^0_{53} \to 92, & \mathbb{T}^0_{54} \to 60, & \mathbb{T}^0_{55} \to 123, \\
\mathbb{T}^0_{56} \to 106, & \mathbb{T}^0_{57} \to 90, & \mathbb{T}^0_{58} \to 105, & \mathbb{T}^0_{59} \to 89, & \mathbb{T}^0_{60} \to 72, \\
\mathbb{T}^0_{61} \to 58, & \mathbb{T}^0_{62} \to 57, & \mathbb{T}^0_{63} \to 40, & \mathbb{T}^0_{64} \to 24, & \mathbb{T}^0_{65} \to 86,
\end{array}$$

$$\mathbb{T}^0_{66} \to 54\,, \qquad \mathbb{T}^0_{67} \to 53\,, \qquad \mathbb{T}^0_{68} \to 36\,, \qquad \mathbb{T}^0_{69} \to 20\,, \qquad \mathbb{T}^0_{70} \to 18\,. \tag{D.10}$$

The ordering that we chose is the default one given by `n4ListStructures` in `CFTs4D`. These 70 structures do not satisfy the constraints of the kinematic permutation symmetry $\pi \in$ (6.6). Imposing these reduces the independent structures to 22. Since spacetime parity and permutations do simply permute the charges, $q_i, \bar{q}_i$, we can easily classify the orbits under these permutations and define the structures by taking a representative of each orbit and symmetrizing it. Define the total charge to be $|\sum_i q_i| = |\sum_i \bar{q}_i|$. We see that there is one orbit at charge two, four orbits at charge one, nine orbits at charge zero and obviously the same number orbits for negative charges. In total they are 19. We can symmetrize them under permutations and parity, to obtain the 19^+ functions. We can also antisymmetrize for parity and symmetrize for permutation symmetry. There are only three structures in the zero charge orbit that survive. All the others are mapped to zero. We therefore obtain the 3^- odd structures. Next we analyze one by one the 22 structures obtained and we take appropriate linear combinations of them so that the result is either even or odd under $z \leftrightarrow \bar{z}$. The final result is

$$\begin{aligned}
\mathbb{T}_1 &= 8\big(\mathbb{T}^0_1 + \mathbb{T}^0_{16}\big)\,,\\
\mathbb{T}_2 &= \frac{2(\bar{z}-1)\bar{z}}{(z-1)z}\mathbb{T}^0_{12} + 2\big(\mathbb{T}^0_2 + \mathbb{T}^0_{15}\big) + \frac{2(z-1)}{\bar{z}-1}\mathbb{T}^0_3 + \frac{2(\bar{z}-1)}{z-1}\mathbb{T}^0_{14} + \frac{2z}{\bar{z}}\mathbb{T}^0_9 + \frac{2(z-1)z}{(\bar{z}-1)\bar{z}}\mathbb{T}^0_5\\
&\quad + \frac{2\bar{z}}{z}\mathbb{T}^0_8\,,\\
\mathbb{T}_3 &= \frac{2\bar{z}}{z}\big(\mathbb{T}^0_{33} + \mathbb{T}^0_{60}\big) + 2\big(\mathbb{T}^0_{17} + \mathbb{T}^0_{43} + \mathbb{T}^0_{44} + \mathbb{T}^0_{70}\big) + \frac{2z}{\bar{z}}\big(\mathbb{T}^0_{28} + \mathbb{T}^0_{55}\big)\,,\\
\mathbb{T}_4 &= 2\big(\mathbb{T}^0_{18} + \mathbb{T}^0_{24} + \mathbb{T}^0_{36} + \mathbb{T}^0_{42} + \mathbb{T}^0_{45} + \mathbb{T}^0_{51} + \mathbb{T}^0_{63} + \mathbb{T}^0_{69}\big)\,,\\
\mathbb{T}_5 &= \frac{2(\bar{z}-1)}{z-1}\big(\mathbb{T}^0_{41} + \mathbb{T}^0_{68}\big) + 2\big(\mathbb{T}^0_{23} + \mathbb{T}^0_{37} + \mathbb{T}^0_{50} + \mathbb{T}^0_{64}\big) + \frac{2(z-1)}{\bar{z}-1}\big(\mathbb{T}^0_{19} + \mathbb{T}^0_{46}\big)\,,\\
\mathbb{T}_6 &= 4z^2\mathbb{T}^0_{13} + 4\bar{z}^2\mathbb{T}^0_4\,,\\
\mathbb{T}_7 &= 4\big(\mathbb{T}^0_6 + \mathbb{T}^0_{11}\big)\,,\\
\mathbb{T}_8 &= 4(z-1)^2\mathbb{T}^0_7 + 4(\bar{z}-1)^2\mathbb{T}^0_{10}\,,\\
\mathbb{T}_9 &= 4\big(\mathbb{T}^0_{27} + \mathbb{T}^0_{54}\big)\,,\\
\mathbb{T}_{10} &= 4\big(\mathbb{T}^0_{30} + \mathbb{T}^0_{57}\big)\,,\\
\mathbb{T}_{11} &= 4\big(\mathbb{T}^0_{31} + \mathbb{T}^0_{58}\big)\,,\\
\mathbb{T}_{12} &= z\big(\mathbb{T}^0_{34} + \mathbb{T}^0_{40} + \mathbb{T}^0_{61} + \mathbb{T}^0_{67}\big) + \bar{z}\big(\mathbb{T}^0_{20} + \mathbb{T}^0_{26} + \mathbb{T}^0_{47} + \mathbb{T}^0_{53}\big)\,,\\
\mathbb{T}_{13} &= -(z-1)z\big(\mathbb{T}^0_{35} + \mathbb{T}^0_{62}\big) - z(\bar{z}-1)\big(\mathbb{T}^0_{39} + \mathbb{T}^0_{66}\big) - (z-1)\bar{z}\big(\mathbb{T}^0_{21} + \mathbb{T}^0_{48}\big)\\
&\quad - (\bar{z}-1)\bar{z}\big(\mathbb{T}^0_{25} + \mathbb{T}^0_{52}\big)\,,\\
\mathbb{T}_{14} &= (1-z)\big(\mathbb{T}^0_{22} + \mathbb{T}^0_{32} + \mathbb{T}^0_{49} + \mathbb{T}^0_{59}\big) + (1-\bar{z})\big(\mathbb{T}^0_{29} + \mathbb{T}^0_{38} + \mathbb{T}^0_{56} + \mathbb{T}^0_{65}\big)\,,\\
\mathbb{T}_{15} &= i(z-1)z\big(\mathbb{T}^0_{35} - \mathbb{T}^0_{62}\big) + iz(\bar{z}-1)\big(\mathbb{T}^0_{39} - \mathbb{T}^0_{66}\big) - i(z-1)\bar{z}\big(\mathbb{T}^0_{21} - \mathbb{T}^0_{48}\big)
\end{aligned}$$

$$
\begin{aligned}
&\quad - i(\bar{z}-1)\bar{z}\big(\mathbb{T}^0_{25}-\mathbb{T}^0_{52}\big)\,,\\
\mathbb{T}_{16} &= i(z-1)\big(\mathbb{T}^0_{22}-\mathbb{T}^0_{32}-\mathbb{T}^0_{49}+\mathbb{T}^0_{59}\big)-iz\big(\mathbb{T}^0_{34}+\mathbb{T}^0_{40}-\mathbb{T}^0_{61}-\mathbb{T}^0_{67}\big)\\
&\quad + i(\bar{z}-1)\big(\mathbb{T}^0_{29}-\mathbb{T}^0_{38}-\mathbb{T}^0_{56}+\mathbb{T}^0_{65}\big)+i\bar{z}\big(\mathbb{T}^0_{20}+\mathbb{T}^0_{26}-\mathbb{T}^0_{47}-\mathbb{T}^0_{53}\big)\,,\\
\mathbb{T}_{17} &= -i(z-1)\big(\mathbb{T}^0_{22}-\mathbb{T}^0_{32}-\mathbb{T}^0_{49}+\mathbb{T}^0_{59}\big)-iz\big(\mathbb{T}^0_{34}+\mathbb{T}^0_{40}-\mathbb{T}^0_{61}-\mathbb{T}^0_{67}\big)\\
&\quad - i(\bar{z}-1)\big(\mathbb{T}^0_{29}-\mathbb{T}^0_{38}-\mathbb{T}^0_{56}+\mathbb{T}^0_{65}\big)+i\bar{z}\big(\mathbb{T}^0_{20}+\mathbb{T}^0_{26}-\mathbb{T}^0_{47}-\mathbb{T}^0_{53}\big)\,,\\
\mathbb{T}_{18} &= 8\big(\mathbb{T}^0_{1}-\mathbb{T}^0_{16}\big)\,,\\
\mathbb{T}_{19} &= -\frac{2(\bar{z}-1)\bar{z}}{(z-1)z}\mathbb{T}^0_{12}+2\big(\mathbb{T}^0_{2}-\mathbb{T}^0_{15}\big)+\frac{2(z-1)}{\bar{z}-1}\mathbb{T}^0_{3}-\frac{2(\bar{z}-1)}{z-1}\mathbb{T}^0_{14}+\frac{2z}{\bar{z}}\mathbb{T}^0_{9}\\
&\quad + \frac{2(z-1)z}{(\bar{z}-1)\bar{z}}\mathbb{T}^0_{5}-\frac{2\bar{z}}{z}\mathbb{T}^0_{8}\,,\\
\mathbb{T}_{20} &= -\frac{2\bar{z}}{z}\big(\mathbb{T}^0_{33}+\mathbb{T}^0_{60}\big)+2\big(\mathbb{T}^0_{17}-\mathbb{T}^0_{43}+\mathbb{T}^0_{44}-\mathbb{T}^0_{70}\big)+\frac{2z}{\bar{z}}\big(\mathbb{T}^0_{28}+\mathbb{T}^0_{55}\big)\,,\\
\mathbb{T}_{21} &= 2\big(\mathbb{T}^0_{18}+\mathbb{T}^0_{24}-\mathbb{T}^0_{36}-\mathbb{T}^0_{42}+\mathbb{T}^0_{45}+\mathbb{T}^0_{51}-\mathbb{T}^0_{63}-\mathbb{T}^0_{69}\big)\,,\\
\mathbb{T}_{22} &= -\frac{2(\bar{z}-1)}{z-1}\big(\mathbb{T}^0_{41}+\mathbb{T}^0_{68}\big)+2\big(\mathbb{T}^0_{23}-\mathbb{T}^0_{37}+\mathbb{T}^0_{50}-\mathbb{T}^0_{64}\big)+\frac{2(z-1)}{\bar{z}-1}\big(\mathbb{T}^0_{19}+\mathbb{T}^0_{46}\big)\,.
\end{aligned}
\tag{D.11}
$$

In this basis, the structures $\mathbb{T}_{1,2,\ldots 14}$ are P even and $z \leftrightarrow \bar{z}$ even; $\mathbb{T}_{15,16,17}$ are P odd and $z \leftrightarrow \bar{z}$ even and finally $\mathbb{T}_{18,19,\ldots 22}$ are P even and $z \leftrightarrow \bar{z}$ odd. All structures are CPT invariant, which means that the associated functions are real. In particular, this explains the presence of the i factor in $\mathbb{T}_{15,16,17}$.

D.3 Superconformal blocks

In this appendix we will report all the values for the coefficients $\mathcal{S}^{\hat{a},\hat{b}}_{k,s,t|\hat{c},\hat{d}}$ that were defined in (7.114). We will proceed in order with $p = 0, 2$ and 4.

D.3.1 $\mathcal{S}$ coefficients for $p = 0$

For $\ell \geqslant 2$ even the results can be found in Table D.1. We named Φ_i some recurring factors in the numerators for brevity. Here is their definition:

$$\begin{aligned}
\Phi_1 &= \Delta^2 - \ell^3 + 2\Delta\ell^2 - 11\ell^2 - \Delta^2\ell + 14\Delta\ell - 40\ell - 12\,,\\
\Phi_2 &= \Delta^2 - 6\Delta + \ell^2 + 4\ell + 12\,,\\
\Phi_3 &= \Delta^2 - 6\Delta + \ell^2 + 8\,,\\
\Phi_4 &= \Delta^3 - 6\Delta^2 + 11\Delta - 2\Delta\ell + 4\ell - 8\,,\\
\Phi_5 &= \Delta^2 - 5\Delta - \Delta\ell + 2\ell + 5\,,\\
\Phi_6 &= \Delta^5 - 12\Delta^4 + 54\Delta^3 - 120\Delta^2 + 145\Delta - 8\Delta\ell^2 + 16\ell^2\\
&\quad + 8\Delta^2\ell - 32\Delta\ell + 24\ell - 76\,,\\
\Phi_7 &= \Delta^2 - 6\Delta + 2\ell + 7\,.
\end{aligned} \tag{D.12}$$

Coefficient	Value
$\mathcal{S}^{11}_{0\|11}$	1
$\mathcal{S}^{12}_{0\|12}$	$\frac{1}{2}$
$\mathcal{S}^{22}_{0\|22}$	1
$\mathcal{S}^{11}_{1,-,-\|11}$	$\frac{\Phi_1^2(\Delta-\ell-2)}{2(\ell+1)^2(\Delta-\ell-1)}$
$\mathcal{S}^{11}_{1,+,-\|11}$	$\frac{(\Delta-3)^2\ell(\Delta-\ell-2)(\Delta+\ell)}{2(\Delta-1)\Delta(\ell+1)}$
$\mathcal{S}^{11}_{1,+,+\|11}$	$\frac{(\Delta-\ell-4)^4(\Delta+\ell)}{4(\Delta+\ell+1)}$
$\mathcal{S}^{12}_{1,-,-\|11}$	$\frac{\Phi_1\Phi_2(\ell-1)(\Delta-\ell-2)}{(\ell+1)^2(\Delta-\ell-1)}$
$\mathcal{S}^{12}_{1,+,-\|11}$	$-\frac{(\Delta-3)^2\ell(\Delta-\ell-2)(\Delta+\ell)}{(\Delta-1)\Delta(\ell+1)}$
$\mathcal{S}^{12}_{1,+,+\|11}$	$-\frac{\Phi_3(\Delta-\ell-4)^2(\Delta+\ell)}{2(\Delta+\ell+1)}$
$\mathcal{S}^{22}_{1,-,-\|11}$	$\frac{2\Phi_2^2(\ell-1)^2(\Delta-\ell-2)}{(\ell+1)^2(\Delta-\ell-1)}$

Table D.1. Superconformal block coefficients for $\mathcal{G}_{i,\Delta,(\ell,\ell)}$ with ℓ even (continues).

Coefficient	Value (continued)
$\mathcal{S}^{22}_{1,+,-\|11}$	$\frac{2(\Delta-3)^2\ell(\Delta-\ell-2)(\Delta+\ell)}{(\Delta-1)\Delta(\ell+1)}$
$\mathcal{S}^{22}_{1,+,+\|11}$	$\frac{\Phi_3^2(\Delta+\ell)}{\Delta+\ell+1}$
$\mathcal{S}^{11}_{2\|11}$	$\frac{\Phi_4^2(\Delta-\ell-2)(\Delta+\ell)}{16(\Delta-1)^4\Delta^2(\Delta-\ell-1)(\Delta+\ell+1)}$
$\mathcal{S}^{11}_{2\|12}$	$\frac{\Phi_4\Phi_5(\Delta-\ell-2)(\Delta+\ell)}{16(\Delta-1)^4\Delta^2(\Delta-\ell-1)(\Delta+\ell+1)}$
$\mathcal{S}^{11}_{2\|22}$	$\frac{\Phi_5^2(\Delta-\ell-2)(\Delta+\ell)}{16(\Delta-1)^4\Delta^2(\Delta-\ell-1)(\Delta+\ell+1)}$
$\mathcal{S}^{12}_{2\|11}$	$\frac{(\Delta-2)\Phi_4(\ell-1)(\Delta-\ell-2)(\Delta+\ell)}{4(\Delta-1)^4\Delta^2(\Delta-\ell-1)(\Delta+\ell+1)}$
$\mathcal{S}^{12}_{2\|12}$	$\frac{(\Delta-2)\Phi_6(\Delta-\ell-2)(\Delta+\ell)}{32(\Delta-1)^4\Delta^2(\Delta-\ell-1)(\Delta+\ell+1)}$
$\mathcal{S}^{12}_{2\|22}$	$\frac{(\Delta-2)\Phi_5\Phi_7(\Delta-\ell-2)(\Delta+\ell)}{16(\Delta-1)^4\Delta^2(\Delta-\ell-1)(\Delta+\ell+1)}$
$\mathcal{S}^{22}_{2\|11}$	$\frac{(\Delta-2)^2(\ell-1)^2(\Delta-\ell-2)(\Delta+\ell)}{(\Delta-1)^4\Delta^2(\Delta-\ell-1)(\Delta+\ell+1)}$
$\mathcal{S}^{22}_{2\|12}$	$\frac{(\Delta-2)^2\Phi_7(\ell-1)(\Delta-\ell-2)(\Delta+\ell)}{4(\Delta-1)^4\Delta^2(\Delta-\ell-1)(\Delta+\ell+1)}$
$\mathcal{S}^{22}_{2\|22}$	$\frac{(\Delta-2)^2\Phi_7^2(\Delta-\ell-2)(\Delta+\ell)}{16(\Delta-1)^4\Delta^2(\Delta-\ell-1)(\Delta+\ell+1)}$

Table D.1. Superconformal block coefficients for $\mathcal{G}_{i,\Delta,(\ell,\ell)}$ with ℓ even.

The results for $\ell \geqslant 3$ odd are in Table D.2. Also in this case we defined some factors Ψ_i as follows

$$\Psi_1 = -5\Delta^2 + 20\Delta + 3\ell^2 + 6\ell - 24\,,$$
$$\Psi_2 = 2\Delta^2 - 16\Delta + \ell^3 - \Delta\ell^2 + 7\ell^2 - \Delta^2\ell + 10\ell + 24\,,$$
$$\Psi_3 = \Delta^2 - 6\Delta - \ell^2 + \Delta\ell - 3\ell + 10\,,$$
$$\Psi_4 = -3\Delta^2 + 6\Delta + \ell^2 - 4\,,$$
$$\Psi_5 = -3\Delta^2 + 6\Delta + \ell^2 + 4\ell\,,$$

$$
\begin{aligned}
\Psi_6 &= -12\Delta^4 + 110\Delta^3 - 292\Delta^2 + 288\Delta + 2\ell^5 + \Delta\ell^4 + 8\ell^4 - 8\Delta^2\ell^3 + 27\Delta\ell^3 \\
&\quad - 18\ell^3 + \Delta^3\ell^2 - 20\Delta^2\ell^2 + 20\Delta\ell^2 - 8\ell^2 + 6\Delta^4\ell - 27\Delta^3\ell + 38\Delta^2\ell \\
&\quad - 60\Delta\ell + 40\ell - 96\,, \\
\Psi_7 &= 6\Delta^4 - 53\Delta^3 + 162\Delta^2 - 184\Delta + 2\ell^4 + \Delta\ell^3 + 8\ell^3 - 8\Delta^2\ell^2 + 25\Delta\ell^2 - 14\ell^2 \\
&\quad + \Delta^3\ell - 8\Delta^2\ell + 26\Delta\ell - 44\ell + 48\,, \\
\Psi_8 &= -5\Delta^4 + 35\Delta^3 - 84\Delta^2 + 84\Delta + 3\Delta^2\ell^2 - 13\Delta\ell^2 + 16\ell^2 + 6\Delta^2\ell - 26\Delta\ell \\
&\quad + 32\ell - 48\,.
\end{aligned} \tag{D.13}
$$

Coefficient	Value
$\mathcal{S}^{11}_{0\|11}$	1
$\mathcal{S}^{11}_{1,-,-\|11}$	$\frac{(\ell-2)^2(\Delta-\ell-5)^2(\Delta-\ell-2)(\Delta+\ell)^2}{2(\Delta-2)^2\ell^2(\ell+1)^2\Psi_1^2(\Delta-\ell-1)}$
$\mathcal{S}^{11}_{1,+,-\|11}$	$\frac{(\Delta-4)^2(\ell-1)^2(\Delta-\ell-2)(\Delta+\ell)}{2(\Delta-1)\Delta\ell(\ell+1)\Psi_1^2}$
$\mathcal{S}^{11}_{1,+,+\|11}$	$\frac{(\Delta-\ell-2)^2(\Delta+\ell-3)^2(\Delta+\ell)}{4(\Delta-2)^2(\ell+1)^2\Psi_1^2(\Delta+\ell+1)}$
$\mathcal{S}^{11}_{1,-,-\|12}$	$-\frac{(\ell-2)\Psi_2(\Delta-\ell-5)(\Delta-\ell-2)(\Delta+\ell)}{4(\Delta-2)^2\ell^2(\ell+1)^2\Psi_1^2(\Delta-\ell-1)}$
$\mathcal{S}^{11}_{1,+,+\|12}$	$\frac{\Psi_3(\Delta-\ell-2)(\Delta+\ell-3)(\Delta+\ell)}{8(\Delta-2)^2(\ell+1)^2\Psi_1^2(\Delta+\ell+1)}$
$\mathcal{S}^{11}_{1,-,-\|22}$	$\frac{\Psi_2^2(\Delta-\ell-2)}{8(\Delta-2)^2\ell^2(\ell+1)^2\Psi_1^2(\Delta-\ell-1)}$
$\mathcal{S}^{11}_{1,+,+\|22}$	$\frac{\Psi_3^2(\Delta+\ell)}{16(\Delta-2)^2(\ell+1)^2\Psi_1^2(\Delta+\ell+1)}$
$\mathcal{S}^{12}_{1,-,-\|11}$	$\frac{(\ell-2)(\ell+3)(\ell+4)\Psi_4(\Delta-\ell-5)(\Delta-\ell-2)(\Delta+\ell)}{16(\Delta-2)^3(\ell-1)\ell^2(\ell+1)^2\Psi_1^2(\Delta-\ell-1)}$
$\mathcal{S}^{12}_{1,+,-\|11}$	$\frac{(\Delta-4)(\ell-1)(\ell+4)(\Delta-\ell-2)(\Delta+\ell)}{8(\Delta-2)\Delta\ell(\ell+1)\Psi_1^2}$
$\mathcal{S}^{12}_{1,+,+\|11}$	$\frac{\Psi_5(\Delta-\ell-2)(\Delta+\ell-3)(\Delta+\ell)}{32(\Delta-2)^3(\ell+1)^2\Psi_1^2(\Delta+\ell+1)}$

Table D.2. Superconformal block coefficients for $\mathcal{G}_{i,\Delta,(\ell,\ell)}$ with ℓ odd (continues).

Coefficient	Value (continued)
$\mathcal{S}^{12}_{1,-,-\|12}$	$-\frac{(\ell+3)(\ell+4)\Psi_6(\Delta-\ell-2)}{64(\Delta-2)^3(\ell-1)\ell^2(\ell+1)^2\Psi_1^2(\Delta-\ell-1)}$
$\mathcal{S}^{12}_{1,+,+\|12}$	$-\frac{\Psi_7(\Delta+\ell)}{128(\Delta-2)^3(\ell+1)^2\Psi_1^2(\Delta+\ell+1)}$
$\mathcal{S}^{12}_{1,-,-\|22}$	$\frac{(\ell+3)(\ell+4)\Psi_2(\Delta-\ell-2)^2(3\Delta+\ell-4)}{64(\Delta-2)^3(\ell-1)\ell^2(\ell+1)^2\Psi_1^2(\Delta-\ell-1)}$
$\mathcal{S}^{12}_{1,+,+\|22}$	$-\frac{\Psi_3(\Delta-\ell-4)(\Delta+\ell)(3\Delta+\ell-2)}{128(\Delta-2)^3(\ell+1)^2\Psi_1^2(\Delta+\ell+1)}$
$\mathcal{S}^{22}_{1,-,-\|11}$	$\frac{(\ell+3)^2(\ell+4)^2\Psi_4^2(\Delta-\ell-2)}{128(\Delta-2)^4(\ell-1)^2\ell^2(\ell+1)^2\Psi_1^2(\Delta-\ell-1)}$
$\mathcal{S}^{22}_{1,+,-\|11}$	$\frac{(\Delta-1)(\ell+4)^2(\Delta-\ell-2)(\Delta+\ell)}{32(\Delta-2)^2\Delta\ell(\ell+1)\Psi_1^2}$
$\mathcal{S}^{22}_{1,+,+\|11}$	$\frac{\Psi_5^2(\Delta+\ell)}{256(\Delta-2)^4(\ell+1)^2\Psi_1^2(\Delta+\ell+1)}$
$\mathcal{S}^{22}_{1,-,-\|12}$	$-\frac{(\ell+3)^2(\ell+4)^2\Psi_4(\Delta-\ell-2)^2(3\Delta+\ell-4)}{256(\Delta-2)^4(\ell-1)^2\ell^2(\ell+1)^2\Psi_1^2(\Delta-\ell-1)}$
$\mathcal{S}^{22}_{1,+,+\|12}$	$-\frac{\Psi_5(\Delta-\ell-4)(\Delta+\ell)(3\Delta+\ell-2)}{512(\Delta-2)^4(\ell+1)^2\Psi_1^2(\Delta+\ell+1)}$
$\mathcal{S}^{22}_{1,-,-\|22}$	$\frac{(\ell+3)^2(\ell+4)^2(\Delta-\ell-2)^3(3\Delta+\ell-4)^2}{512(\Delta-2)^4(\ell-1)^2\ell^2(\ell+1)^2\Psi_1^2(\Delta-\ell-1)}$
$\mathcal{S}^{22}_{1,+,+\|22}$	$\frac{(\Delta-\ell-4)^2(\Delta+\ell)(3\Delta+\ell-2)^2}{1024(\Delta-2)^4(\ell+1)^2\Psi_1^2(\Delta+\ell+1)}$
$\mathcal{S}^{11}_{2\|11}$	$\frac{\Psi_8^2(\Delta-\ell-2)(\Delta+\ell)}{16(\Delta-1)^2\Delta^2\Psi_1^2(\Delta-\ell-1)(\Delta+\ell+1)}$
$\mathcal{S}^{12}_{2\|11}$	$-\frac{(\ell+3)(\ell+4)\Psi_8(\Delta-\ell-2)(\Delta+\ell)}{16(\Delta-2)(\Delta-1)\Delta^2\Psi_1^2(\Delta-\ell-1)(\Delta+\ell+1)}$
$\mathcal{S}^{22}_{2\|11}$	$\frac{(\ell+3)^2(\ell+4)^2(\Delta-\ell-2)(\Delta+\ell)}{16(\Delta-2)^2\Delta^2\Psi_1^2(\Delta-\ell-1)(\Delta+\ell+1)}$

Table D.2. Superconformal block coefficients for $\mathcal{G}_{i,\Delta,(\ell,\ell)}$ with ℓ odd.

Next we have the special case for $\ell = 0$

Coefficient	Value
$\mathcal{S}^{11}_{0\|11}$	1
$\mathcal{S}^{11}_{1,+,+\|11}$	$\frac{(\Delta-4)^2\Delta}{4(\Delta+1)}$
$\mathcal{S}^{11}_{2\|11}$	$\frac{(\Delta-4)^2(\Delta-2)}{16(\Delta-1)\Delta(\Delta+1)}$

Table D.3. Superconformal block coefficients for $\mathcal{G}_{i,\Delta,(0,0)}$.

And finally for $\ell = 1$

Coefficient	Value
$\mathcal{S}^{11}_{0\|11}$	1
$\mathcal{S}^{11}_{1,-,-\|11}$	$\frac{\Delta-3}{2(\Delta-2)(\Delta-1)^2}$
$\mathcal{S}^{11}_{1,+,+\|11}$	$\frac{\Delta+1}{400(\Delta-1)^2(\Delta+2)}$
$\mathcal{S}^{11}_{1,+,+\|12}$	$\frac{\Delta+1}{800(\Delta-1)^2(\Delta+2)}$
$\mathcal{S}^{11}_{1,+,+\|22}$	$\frac{\Delta+1}{1600(\Delta-1)^2(\Delta+2)}$
$\mathcal{S}^{11}_{2\|11}$	$\frac{(\Delta-3)^3(\Delta+1)}{16(\Delta-2)(\Delta-1)^2(\Delta+2)}$

Table D.4. Superconformal block coefficients for $\mathcal{G}_{i,\Delta,(1,1)}$.

D.3.2 $\mathcal{S}$ coefficients for $p = 2$

In Table D.5 we show the coefficients for spin $(\ell+2,\ell)$ where ℓ is even

Coefficient	Value
$\mathcal{S}^{11}_{0\|11}$	1

Table D.5. Superconformal block coefficients for $\mathcal{G}_{i,\Delta,(\ell+2,\ell)}$ for ℓ even (continues).

Coefficient	Value (continued)
$\mathcal{S}^{11}_{1,-,-\|11}$	$\frac{(\ell+3)(\ell+4)^2(\Delta-\ell-4)}{2\ell(\ell+1)(\ell+2)(\Delta-\ell-3)}$
$\mathcal{S}^{11}_{1,-,+\|11}$	$\frac{(\Delta-3)^2(\ell+4)^2(\Delta-\ell-4)(\Delta+\ell)}{(\Delta-2)(\Delta-1)(\ell+2)(\ell+3)}$
$\mathcal{S}^{11}_{1,+,+\|11}$	$\frac{\Delta+\ell}{4(\Delta+\ell+1)}$
$\mathcal{S}^{11}_{2\|11}$	$\frac{(\Delta-3)^2(\Delta-2)(\Delta-\ell-4)(\Delta+\ell)}{16(\Delta-1)\Delta(\Delta+1)(\Delta-\ell-3)(\Delta+\ell+1)}$

Table D.5. Superconformal block coefficients for $\mathcal{G}_{i,\Delta,(\ell+2,\ell)}$ for ℓ even.

The results for odd ℓ are significantly more involved instead. We defined some recurring factors as Ω_i, written below. The coefficients can be found in Table D.6 for $\ell \geqslant 3$ and in Table D.7 for $\ell = 1$.

$$
\begin{aligned}
\Omega_1 &= -5\Delta^2 + 20\Delta + \ell^2 + 4\ell - 20\,,\\
\Omega_2 &= 3\Delta^2 - 16\Delta + 5\ell^2 + 4\Delta\ell + 4\ell + 36\,,\\
\Omega_3 &= -12\Delta^4 + 96\Delta^3 - 360\Delta^2 + 672\Delta + \Delta\ell^4 + \ell^4 + 3\Delta^2\ell^3 - 13\Delta\ell^3 + 44\ell^3 + 3\Delta^3\ell^2\\
&\quad - 45\Delta^2\ell^2 + 128\Delta\ell^2 - 4\ell^2 - 3\Delta^4\ell + 21\Delta^3\ell - 168\Delta^2\ell + 532\Delta\ell - 416\ell - 480\,,\\
\Omega_4 &= 12\Delta^2 - 58\Delta + 2\Delta\ell^2 - 3\ell^2 + 3\Delta^2\ell - 4\Delta\ell - 12\ell + 60\,,\\
\Omega_5 &= 60\Delta^2 - 360\Delta + 5\ell^4 + 4\Delta\ell^3 + 19\ell^3 + 3\Delta^2\ell^2 - 14\Delta\ell^2 + 58\ell^2 + 27\Delta^2\ell - 170\Delta\ell\\
&\quad + 248\ell + 480\,,\\
\Omega_6 &= 120\Delta^3 - 960\Delta^2 + 2400\Delta + \Delta\ell^5 + \ell^5 + 11\Delta^2\ell^4 - 49\Delta\ell^4 + 90\ell^4 + 15\Delta^3\ell^3 - 65\Delta^2\ell^3\\
&\quad + 2\Delta\ell^3 + 292\ell^3 - 3\Delta^4\ell^2 + 117\Delta^3\ell^2 - 722\Delta^2\ell^2 + 1372\Delta\ell^2 - 456\ell^2 - 12\Delta^4\ell\\
&\quad + 318\Delta^3\ell - 1984\Delta^2\ell + 4344\Delta\ell - 2912\ell - 1920\,,\\
\Omega_7 &= -3\Delta^2 + 6\Delta + \ell^2 + 4\ell\,,\\
\Omega_8 &= -180\Delta^5 + 1800\Delta^4 - 6480\Delta^3 + 10080\Delta^2 - 5760\Delta + \Delta\ell^7 + \ell^7 + 8\Delta^2\ell^6 - 27\Delta\ell^6\\
&\quad + 70\ell^6 + 9\Delta^3\ell^5 - 36\Delta^2\ell^5 + 4\Delta\ell^5 + 364\ell^5 - 21\Delta^4\ell^4 + 204\Delta^3\ell^4 - 943\Delta^2\ell^4\\
&\quad + 1710\Delta\ell^4 - 32\ell^4 - 30\Delta^5\ell^3 + 231\Delta^4\ell^3 - 423\Delta^3\ell^3 - 1602\Delta^2\ell^3 + 5704\Delta\ell^3\\
&\quad - 3008\ell^3 + 9\Delta^6\ell^2 - 249\Delta^5\ell^2 + 1905\Delta^4\ell^2 - 5850\Delta^3\ell^2 + 5684\Delta^2\ell^2 + 3672\Delta\ell^2\\
&\quad - 5600\ell^2 + 36\Delta^6\ell - 741\Delta^5\ell + 5190\Delta^4\ell - 16140\Delta^3\ell + 22344\Delta^2\ell - 9984\Delta\ell - 1920\ell\,,
\end{aligned}
$$

$$\Omega_9 = -15\Delta^5 + 135\Delta^4 - 390\Delta^3 + 300\Delta^2 + 360\Delta + 3\Delta^2\ell^3 - 31\Delta\ell^3 + 46\ell^3 + 3\Delta^3\ell^2 - 27\Delta^2\ell^2 + 2\Delta\ell^2 + 112\ell^2 - 15\Delta^4\ell + 147\Delta^3\ell - 546\Delta^2\ell + 884\Delta\ell - 488\ell - 480\,. \tag{D.14}$$

Coefficient	Value
$\mathcal{S}^{11}_{0\|11}$	1
$\mathcal{S}^{11}_{1,-,-\|11}$	$\frac{(\ell-1)^2\ell\Omega_2^2(\Delta-\ell-4)}{18(\ell+1)(\ell+2)(\ell+3)\Omega_1^2(\Delta-\ell-3)}$
$\mathcal{S}^{11}_{1,-,+\|11}$	$\frac{\ell^2\Omega_3^2(\Delta-\ell-4)}{36(\Delta-2)^3(\Delta-1)(\ell+1)^2(\ell+2)(\ell+3)(\ell+4)^2\Omega_1^2(\Delta+\ell)}$
$\mathcal{S}^{11}_{1,+,-\|11}$	$\frac{4\ell\Omega_4^2(\Delta-\ell-4)^2(\Delta-\ell-2)(\Delta+\ell+2)}{9\Delta(\Delta+1)(\ell+1)(\ell+4)^2\Omega_1^2}$
$\mathcal{S}^{11}_{1,+,+\|11}$	$\frac{\Omega_5^2(\Delta+\ell-2)^2}{36(\ell+4)^2(\ell+5)^2\Omega_1^2(\Delta+\ell)(\Delta+\ell+1)}$
$\mathcal{S}^{11}_{1,-,+\|12}$	$\frac{\ell\Omega_3\Omega_6(\Delta-\ell-4)}{72(\Delta-2)^3(\Delta-1)(\ell+1)^2(\ell+2)(\ell+3)(\ell+4)^2\Omega_1^2(\Delta+\ell)}$
$\mathcal{S}^{11}_{1,-,+\|22}$	$\frac{\Omega_6^2(\Delta-\ell-4)}{144(\Delta-2)^3(\Delta-1)(\ell+1)^2(\ell+2)(\ell+3)(\ell+4)^2\Omega_1^2(\Delta+\ell)}$
$\mathcal{S}^{12}_{1,-,-\|11}$	$\frac{(\ell-1)(\ell+4)(\ell+5)\Omega_2(\Delta-\ell-4)}{9(\ell+1)(\ell+2)(\ell+3)\Omega_1^2}$
$\mathcal{S}^{12}_{1,-,+\|11}$	$-\frac{\ell(\ell+5)\Omega_3\Omega_7(\Delta-\ell-4)}{36(\Delta-2)^2(\Delta-1)(\ell+1)^2(\ell+2)(\ell+3)(\ell+4)\Omega_1^2(\Delta+\ell)}$
$\mathcal{S}^{12}_{1,+,-\|11}$	$-\frac{2(\ell+5)\Omega_4(\Delta-\ell-4)^2(\Delta-\ell-2)(\Delta+\ell+2)}{9(\Delta+1)(\ell+1)(\ell+4)\Omega_1^2}$
$\mathcal{S}^{12}_{1,+,+\|11}$	$\frac{\Omega_5(\Delta-\ell-4)(\Delta+\ell-2)}{18(\ell+4)(\ell+5)\Omega_1^2(\Delta+\ell)}$
$\mathcal{S}^{12}_{1,-,+\|12}$	$-\frac{(\ell+5)\Omega_8(\Delta-\ell-4)}{72(\Delta-2)^2(\Delta-1)(\ell+1)^2(\ell+2)(\ell+3)(\ell+4)\Omega_1^2(\Delta+\ell)}$
$\mathcal{S}^{12}_{1,-,+\|22}$	$\frac{(\ell+5)\Omega_6(\Delta-\ell-4)^2(3\Delta+\ell-2)}{144(\Delta-2)^2(\Delta-1)(\ell+1)^2(\ell+2)(\ell+3)(\ell+4)\Omega_1^2(\Delta+\ell)}$

Table D.6. Superconformal block coefficients for $\mathcal{G}_{i,\Delta,(\ell+2,\ell)}$ for $\ell \geqslant 3$ odd (continues).

Coefficient	Value (continued)
$\mathcal{S}^{22}_{1,-,-\|11}$	$\frac{2(\ell+4)^2(\ell+5)^2(\Delta-\ell-4)(\Delta-\ell-3)}{9\ell(\ell+1)(\ell+2)(\ell+3)\Omega_1^2}$
$\mathcal{S}^{22}_{1,-,+\|11}$	$\frac{(\ell+5)^2\Omega_7^2(\Delta-\ell-4)}{36(\Delta-2)(\Delta-1)(\ell+1)^2(\ell+2)(\ell+3)\Omega_1^2(\Delta+\ell)}$
$\mathcal{S}^{22}_{1,+,-\|11}$	$\frac{\Delta(\ell+5)^2(\Delta-\ell-4)^2(\Delta-\ell-2)(\Delta+\ell+2)}{9(\Delta+1)\ell(\ell+1)\Omega_1^2}$
$\mathcal{S}^{22}_{1,+,+\|11}$	$\frac{(\Delta-\ell-4)^2(\Delta+\ell+1)}{9\Omega_1^2(\Delta+\ell)}$
$\mathcal{S}^{22}_{1,-,+\|12}$	$-\frac{(\ell+5)^2\Omega_7(\Delta-\ell-4)^2(3\Delta+\ell-2)}{72(\Delta-2)(\Delta-1)(\ell+1)^2(\ell+2)(\ell+3)\Omega_1^2(\Delta+\ell)}$
$\mathcal{S}^{22}_{1,-,+\|22}$	$\frac{(\ell+5)^2(\Delta-\ell-4)^3(3\Delta+\ell-2)^2}{144(\Delta-2)(\Delta-1)(\ell+1)^2(\ell+2)(\ell+3)\Omega_1^2(\Delta+\ell)}$
$\mathcal{S}^{11}_{2\|11}$	$\frac{\Omega_9^2(\Delta-\ell-4)}{144(\Delta-2)(\Delta-1)\Delta(\Delta+1)\Omega_1^2(\Delta-\ell-3)(\Delta+\ell)(\Delta+\ell+1)}$
$\mathcal{S}^{12}_{2\|11}$	$\frac{(\ell+4)(\ell+5)\Omega_9(\Delta-\ell-4)}{36(\Delta-2)\Delta(\Delta+1)\Omega_1^2(\Delta-\ell-3)(\Delta+\ell)(\Delta+\ell+1)}$
$\mathcal{S}^{22}_{2\|11}$	$\frac{(\Delta-1)(\ell+4)^2(\ell+5)^2(\Delta-\ell-4)}{9(\Delta-2)\Delta(\Delta+1)\Omega_1^2(\Delta-\ell-3)(\Delta+\ell)(\Delta+\ell+1)}$

Table D.6. Superconformal block coefficients for $\mathcal{G}_{i,\Delta,(\ell+2,\ell)}$ for $\ell \geqslant 3$ odd.

Coefficient	Value
$\mathcal{S}^{11}_{0\|11}$	1
$\mathcal{S}^{11}_{1,-,+\|11}$	$-\frac{(\Delta-5)\left(\Delta^2+4\Delta-19\right)^2}{4800(\Delta-2)^3(\Delta-1)(\Delta+1)}$
$\mathcal{S}^{11}_{1,+,-\|11}$	$-\frac{2(\Delta-5)^2(\Delta-3)(\Delta+3)}{25\Delta(\Delta+1)}$
$\mathcal{S}^{11}_{1,+,+\|11}$	$-\frac{(\Delta-3)^2}{100(\Delta+1)(\Delta+2)}$
$\mathcal{S}^{11}_{1,-,+\|12}$	$-\frac{(\Delta-5)\left(\Delta^2+4\Delta-19\right)\left(\Delta^2+34\Delta-109\right)}{9600(\Delta-2)^3(\Delta-1)(\Delta+1)}$

Table D.7. Superconformal block coefficients for $\mathcal{G}_{i,\Delta,(3,1)}$ (continues).

Coefficient	Value (continued)
$\mathcal{S}^{11}_{1,-,+\|22}$	$-\frac{(\Delta-5)\left(\Delta^2+34\Delta-109\right)^2}{19200(\Delta-2)^3(\Delta-1)(\Delta+1)}$
$\mathcal{S}^{11}_{2\|11}$	$\frac{(\Delta-5)(\Delta-3)^4(\Delta+2)}{16(\Delta-4)(\Delta-2)(\Delta-1)\Delta(\Delta+1)^2}$

Table D.7. Superconformal block coefficients for $\mathcal{G}_{i,\Delta,(3,1)}$.

D.3.3 $\mathcal{S}$ coefficients for $p = 4$

To conclude this appendix we present the results for $(\ell+4,\ell)$. In Table D.8 are shown the coefficients for ℓ even

Coefficient	Value
$\mathcal{S}^{11}_{0\|11}$	$(\Delta-\ell-6)^2$
$\mathcal{S}^{11}_{1,-,+\|11}$	$\frac{(\Delta-4)^2(\ell+5)(\Delta-\ell-6)}{4(\Delta-3)(\Delta-2)(\ell+4)(\Delta+\ell)}$
$\mathcal{S}^{11}_{2\|11}$	$\frac{(\Delta-4)^2(\Delta-3)(\Delta-\ell-6)(\Delta-\ell-4)(\Delta-\ell-2)}{16(\Delta-2)(\Delta+1)(\Delta+2)(\Delta-\ell-3)(\Delta+\ell)(\Delta+\ell+3)}$ $\times(\Delta+\ell+2)(\Delta+\ell+4)$

Table D.8. Superconformal block coefficients for $\mathcal{G}_{i,\Delta,(\ell+4,\ell)}$ for ℓ even.

Next we show in Table D.9 the results for ℓ odd.

Coefficient	Value
$\mathcal{S}^{11}_{1,-,-\|11}$	$\frac{(\ell+5)(\ell+6)^2(\Delta-\ell-6)(\Delta-\ell-4)(\Delta-\ell-2)}{8(\Delta-2)^2\ell(\ell+1)(\ell+4)(\Delta-\ell-3)}$
$\mathcal{S}^{11}_{1,-,+\|11}$	$\frac{(\Delta-3)(\ell+6)^2(\Delta-\ell-6)}{16(\Delta-2)^3(\ell+4)(\ell+5)(\Delta+\ell)}$
$\mathcal{S}^{11}_{1,+,+\|11}$	$\frac{(\Delta-\ell-6)^2(\Delta+\ell+2)(\Delta+\ell+4)}{16(\Delta-2)^2(\Delta+\ell)(\Delta+\ell+3)}$

Table D.9. Superconformal block coefficients for $\mathcal{G}_{i,\Delta,(\ell+4,\ell)}$ for ℓ odd.

And finally here follow the results for $\ell = 0$

Coefficient	Value
$\mathcal{S}^{11}_{0\|11}$	$(\Delta-6)^2$
$\mathcal{S}^{11}_{1,-,+\|11}$	$\frac{5(\Delta-6)(\Delta-4)^2}{16(\Delta-3)(\Delta-2)\Delta}$
$\mathcal{S}^{11}_{2\|11}$	$\frac{(\Delta-6)(\Delta-4)^3(\Delta+4)}{16\Delta(\Delta+1)(\Delta+3)}$

Table D.10. Superconformal block coefficients for $\mathcal{G}_{i,\Delta,(4,0)}$.

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Curriculum Vitae

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Academic experience

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PhD in Physics 1/10/2016 — 9/2020
- Thesis: Bootstrapping superconformal field theories in four dimensions
- Supervisor: Prof. Alessandro Vichi

Teaching assistant
- General Physics II – Prof. Tatsuya Nakada 2/2018
- Classical Electrodynamics – Prof. João Penedones 10/2017
- Numerical Analysis – Prof. Marco Picasso 2/2017

Simons Center for Geometry and Physics, Stony Brook, New York, USA

Visiting scholar 1/10/2018 — 26/9/2019

Publications

[1] L. Bartolini, F. Bigazzi, S. Bolognesi, A. L. Cotrone and A. Manenti, *Neutron electric dipole moment from gauge/string duality*, *Phys. Rev. Lett.* **118** (2017) 091601 [1609.09513].

[2] L. Bartolini, F. Bigazzi, S. Bolognesi, A. L. Cotrone and A. Manenti, *Theta dependence in Holographic QCD*, *JHEP* **02** (2017) 029 [1611.00048].

[3] A. Manenti, A. Stergiou and A. Vichi, *R-current three-point functions in 4d $\mathcal{N} = 1$ superconformal theories*, *JHEP* **12** (2018) 108 [1804.09717].

[4] A. Manenti, *Thermal CFTs in momentum space*, *JHEP* **01** (2020) 009 [1905.01355].

[5] A. Manenti, A. Stergiou and A. Vichi, *Implications of ANEC for SCFTs in four dimensions*, *JHEP* **01** (2020) 093 [1905.09293].

[6] A. Manenti, *Differential operators for superconformal correlation functions*, *JHEP* **04** (2020) 145 [1910.12869].

Education

University of Pisa, Pisa, Italy

Master's degree (*Laurea Magistrale*) in Physics 30/7/2014 — 21/7/2016
- Thesis: Holographic computation of the neutron electric dipole moment
- Supervisor: Prof. Francesco Bigazzi

Bachelor's degree (*Laurea Triennale*) in Physics 1/10/2011 — 29/7/2014
- Thesis: Hamilton-Jacobi equations
- Supervisor: Prof. Damiano Anselmi

Talks

Seminar, Caltech, Pasadena, California, USA 11/2019
Seminar, Perimeter Institute, Waterloo, Ontario, Canada 9/2019
Seminar, New York University, New York, USA 9/2019
Seminar, Yale University, New Haven, Connecticut, USA 9/2019
Seminar, Cornell University, Ithaca, New York, USA 9/2019
Journal Club, Uppsala University, Uppsala, Sweden 5/2019
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SCHOOLS	Caltech, Doctoral School	7/2018
	ICTP-SAIFR, Doctoral School	5/2017
	Solvay Institutes, Doctoral School	10/2016 - 12/2016
	LACES, Doctoral School	11/2015
AWARDS	Swiss National Science Foundation, Mobility grant	8/2018
	Fondazione Occhialini, Fourth classified – Scholarship	7/2011
LANGUAGES	Italian: Native language.	
	English: Fluent (speaking, reading, writing).	
	French: Intermediate (reading); basic (speaking, writing).	
COMPUTER SKILLS	Programming languages: Mathematica, Python.	
	High-Performance-Computing: Slurm Workload Manager.	