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The Weyl law of transmission eigenvalues and the completeness of generalized transmission eigenfunctions

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ABSTRACT

The transmission problem is a system of two second-order elliptic equations of two unknowns equipped with the Cauchy data on the boundary. After four decades of research motivated by scattering theory, the spectral properties of this problem are now known to depend on a type of contrast between coefficients near the boundary. Previously, we established the discreteness of eigenvalues for a large class of anisotropic coefficients which is related to the celebrated complementing conditions due to Agmon, Douglis, and Nirenberg. In this work, we establish the Weyl law for the eigenvalues and the completeness of the generalized eigenfunctions for this class of coefficients under an additional mild assumption on the continuity of the coefficients. The analysis is new and based on the L^p regularity theory for the transmission problem established here. It also involves a subtle application of the spectral theory for the Hilbert Schmidt operators. Our

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work extends largely known results in the literature which are mainly devoted to the isotropic case with C^∞-coefficients.

1. Introduction

The transmission eigenvalue problem plays a role in the inverse scattering theory for inhomogeneous media. This eigenvalue problem is connected to the injectivity of the relative scattering operator [12], [18]. Transmission eigenvalues are related to interrogating frequencies for which there is an incident field that is not scattered by the medium. In the acoustic setting, the transmission problem is a system of two second-order elliptic equations of two unknowns equipped with the Cauchy data on the boundary. After four decades of extensive study, the spectral properties are known to depend on a type of contrasts of the media near the boundary (i.e., a difference of some relation of the respective coefficients in each of the equations). Natural and interesting questions on the inverse scattering theory include: *discreteness* of the spectrum (see e.g. [7,6,39,19,32]) *location* of transmission eigenvalues (see [9,22,40,41], and also [10] for the application in time domain), and the *Weyl law* of transmission eigenvalues and the *completeness* of the generalized eigenfunctions (see e.g. [19,20,5,21,38]). We refer the reader to [8] for a recent, and self-contained introduction to the transmission problem and its applications.

This paper concerns the Weyl law of eigenvalues and the completeness of the generalized eigenfunctions of the transmission problem in the time-harmonic acoustic setting. Let us introduce its mathematical formulation. Let Ω be a bounded, simply connected, open subset of $\mathbb{R}^d$ of class C^2 with $d \geqslant 2$. Let A_1, A_2 be two real, symmetric matrix-valued functions, and let Σ_1, Σ_2 be two bounded positive functions that are all defined in Ω. Assume that A_1 and A_2 are uniformly elliptic, and Σ_1 and Σ_2 are bounded below by a positive constant in Ω, i.e., for some constant $\Lambda \geqslant 1$, one has, for $j = 1, 2$,

$$\Lambda^{-1}|\xi|^2 \leqslant \langle A_j(x)\xi, \xi\rangle \leqslant \Lambda|\xi|^2 \quad \text{for all } \xi \in \mathbb{R}^d, \text{ for a.e. } x \in \Omega, \tag{1.1}$$

and

$$\Lambda^{-1} \leqslant \Sigma_j(x) \leqslant \Lambda \text{ for a.e. } x \in \Omega. \tag{1.2}$$

Here and in what follows, $\langle \cdot, \cdot \rangle$ denotes the Euclidean scalar product in $\mathbb{R}^d$ and $|\cdot|$ is the corresponding norm. A complex number λ is called an eigenvalue of the transmission eigenvalue problem associated with the pairs (A_1, Σ_1) and (A_2, Σ_2) in Ω if there is a non-zero pair of functions $(u_1, u_2) \in [H^1(\Omega)]^2$ that satisfy the system

$$\begin{cases} \operatorname{div}(A_1 \nabla u_1) - \lambda \Sigma_1 u_1 = 0 & \text{in } \Omega, \\ \operatorname{div}(A_2 \nabla u_2) - \lambda \Sigma_2 u_2 = 0 & \text{in } \Omega, \\ u_1 = u_2, \quad A_1 \nabla u_1 \cdot \nu = A_2 \nabla u_2 \cdot \nu & \text{on } \Gamma. \end{cases} \tag{1.3}$$

Here and in what follows, Γ denotes $\partial\Omega$, and ν denotes the outward, normal, unit vector on Γ. Such a pair (u_1, u_2) is then called an eigenfunction pair of (1.3).

The Weyl law of transmission eigenvalues has been investigated under various assumptions on (A_1, Σ_1) and (A_2, Σ_2). Robbiano [38] (see also [37]) gives the sharp order of the counting number when $A_1 = A_2 = I$, and $\Sigma_2 \neq \Sigma_1 = 1$ near the boundary and Σ_2 is smooth. The analysis is based on both the microanalysis (see e.g. [15,45]) and the regularity theory for the transmission problem. In [20], Lakshtanov and Vainberg obtained similar results when $A_1 = I$, $\Sigma_1 = 1$, under certain assumptions on A_2 and Σ_2. In particular, they required that $\Sigma_2^{-1} A_2 - I$ is positive definite or negative definite in the whole domain Ω. They also investigated the order of the counting functions for positive and negative eigenvalues under different assumptions on A_2 and Σ_2 (see also [36,21]) via concepts on billiard trajectories. In the isotropic case, the Weyl law for the remainder was established by Petkov and Vodev [35] and Vodev [41–43] for C^∞ coefficients that satisfy the conditions (1.4) and (1.5) below. The case where $A_1 = A_2$ and represent scalar functions was also investigated in their work. Their analysis is heavily based on microanalysis and required a strong smoothness condition. In addition, their work involved a delicate analysis on the Dirichlet to Neumann maps using non-standard parametrix construction initiated by Vodev [40]. It is not clear how one can improve the C^∞ condition and extend their results to the anisotropic setting using their analysis. Concerning the completeness of the generalized eigenfunctions, we want to mention the work of Robbiano [37] where $A_1 = A_2 = I$ and $\Sigma_2 \neq \Sigma_1 = 1$, and the work of Blästen and Päivärinta [5] where $A_1 = A_2 = I$, and $\Sigma_2 - \Sigma_1 = \Sigma_2 - 1 > 0$ and smooth in $\bar{\Omega}$.

In this paper, we investigate the Weyl law of eigenvalues and the completeness of the generalized eigenfunctions for transmission problem under quite general assumptions on A_1, A_2, Σ_1, Σ_2. These are only imposed on the boundary of $\partial\Omega$ except for the continuity requirement. The starting point and one of the main motivations of our work are our discreteness result established in [32]. We demonstrated the discreteness holds if A_1, A_2, Σ_1, Σ_2 are continuous in a neighborhood of the boundary Γ, and satisfy the following two conditions, with $\nu = \nu(x)$:

$$\langle A_2(x)\nu, \nu\rangle \langle A_2(x)\xi, \xi\rangle - \langle A_2(x)\nu, \xi\rangle^2 \neq \langle A_1(x)\nu, \nu\rangle \langle A_1(x)\xi, \xi\rangle - \langle A_1(x)\nu, \xi\rangle^2, \tag{1.4}$$

for all $x \in \Gamma$ and for all $\xi \in \mathbb{R}^d \setminus \{0\}$ with $\langle \xi, \nu\rangle = 0$, and

$$\big\langle A_2(x)\nu, \nu\big\rangle \Sigma_2(x) \neq \big\langle A_1(x)\nu, \nu\big\rangle \Sigma_1(x), \ \forall x \in \Gamma. \tag{1.5}$$

Condition (1.4) is equivalent to the celebrated complementing condition due to Agmon, Douglis, and Nirenberg [4] (see also [3]). The explicit formula given here was derived in [27].

In this paper, we establish that if conditions (1.4) and (1.5) hold then the Weyl law for transmission eigenvalues and the completeness of the generalized eigenfunctions hold as well, under the mild assumption that the coefficients are continuous in $\bar{\Omega}$. More precisely, we have

Theorem 1. *Assume that A_1, A_2, Σ_1, $\Sigma_2 \in C^0(\bar{\Omega})$, and (1.4) and (1.5) hold. Then*

$$N(t) := \#\Big\{k \in \mathbb{N} : |\lambda_k| \leqslant t\Big\} = \mathbf{c}t^{\frac{d}{2}} + o(t^{\frac{d}{2}}) \ \text{as} \ \ t \to +\infty, \tag{1.6}$$

where

$$\mathbf{c} = \frac{1}{(2\pi)^d} \sum_{j=1}^{2} \int_{\Omega} \Big|\Big\{\xi \in \mathbb{R}^d : \langle A_j(x)\xi, \xi\rangle < \Sigma_j(x)\Big\}\Big| \, dx. \tag{1.7}$$

For a measurable subset D of $\mathbb{R}^d$, we denote $|D|$ its (Lebesgue) measure.

We also have

Theorem 2. *Assume that A_1, A_2, Σ_1, $\Sigma_2 \in C^0(\bar{\Omega})$, and (1.4) and (1.5) hold. Then the generalized eigenfunctions are complete in $[L^2(\Omega)]^2$.*

Remark 1. As a direct consequence of either Theorem 1 or Theorem 2, the number of eigenvalues of the transmission problem is infinite. As far as we know, this fact is new under the general assumptions stated here.

Some comments on Theorem 1 and Theorem 2 are in order. In the conclusion of Theorem 1, the multiplicity of eigenvalues is taken into account. The meaning of the multiplicity is understood as follows. One can show (see [32], and also Theorem 3) that the well-posedness of the following system in $[H^1(\Omega)]^2$:

$$\begin{cases} \operatorname{div}(A_1 \nabla u_1) - \lambda \Sigma_1 u_1 = \Sigma_1 f_1 & \text{in } \Omega, \\ \operatorname{div}(A_2 \nabla u_2) - \lambda \Sigma_2 u_2 = \Sigma_2 f_2 & \text{in } \Omega, \\ u_1 = u_2, \quad A_1 \nabla u_1 \cdot \nu = A_2 \nabla u_2 \cdot \nu & \text{on } \Gamma, \end{cases} \tag{1.8}$$

holds for all $(f_1, f_2) \in [L^2(\Omega)]^2$ and for some $\lambda \in \mathbb{C}$ under the assumptions of Theorem 1. We then define the operator $T_\lambda : [L^2(\Omega)]^2 \to [L^2(\Omega)]^2$ by

$$T_\lambda(f_1, f_2) := (u_1, u_2) \text{ where } (u_1, u_2) \text{ is the unique solution of (1.8).} \tag{1.9}$$

We can also prove that such a T_λ is compact using a priori estimates. If λ_j is an eigenvalue of the transmission problem, then $\lambda_j \neq \lambda$, and $\lambda_j - \lambda$ is a characteristic value of T_λ (i.e.,

$(\lambda_j - \lambda)^{-1}$ is its eigenvalue) and conversely. One can show that the multiplicity of the characteristic values $\lambda_j - \lambda$ and $\lambda_j - \hat{\lambda}$ (which are the multiplicity of $(\lambda_j - \lambda)^{-1}$ and $(\lambda_j - \hat{\lambda})^{-1}$, see Definition 1 below) associated with T_λ and $T_{\hat{\lambda}}$ are the same as long as T_λ and $T_{\hat{\lambda}}$ are well-defined (see Remark 11). Hence, the multiplicity of eigenvalues that are associated with T_λ is independent of λ and it is used in Assertion (1.6). One can also prove that T_λ and $T_{\hat{\lambda}}$ have the same set of the generalized eigenfunctions. In Theorem 2, the generalized eigenfunctions are associated to such a T_λ. We recall that the generalized eigenfunctions are complete in $[L^2(\Omega)]^2$ if the subspace spanned by them is dense in $[L^2(\Omega)]^2$.

Recall that, see e.g. [2, Definition 12.5]:

Definition 1. Let γ be an eigenvalue of a linear continuous operator $A : H \to H$ where H is a Hilbert space. A non-zero vector v is a generalized eigenvector of A corresponding to γ if $(\gamma I - A)^k v = 0$ holds for some positive integer k. The set of all generalized eigenvectors of A corresponding to the eigenvalue γ together with the origin in H, forms a subspace of H, whose dimension is the multiplicity of γ.

Theorem 1 gives the order of the counting function $N(t)$ and its first-order approximation. Theorem 1 and Theorem 2 provide new general conditions on the coefficients for which the Weyl law and the completeness of the generalized eigenfunctions hold. These conditions are imposed only on the boundary and the regularity assumption is very mild.

Remark 2. It is worth noting that the convention of eigenvalues of the transmission problem in the work of Lakshtanov and Vainberg is similar to ours and different from that of Robbiano (also the work Petkov and Vodev, and Vodev mentioned above) where λ^2 is used in (1.3) instead of λ (where λ is used but t^2 is considered instead of t in the formula of the counting function).

Remark 3. In [35], Petkov and Vodev considered the isotropic setting and obtained a shaper estimate for the remainder of (1.6) as in the spirit of Hörmander [17]. Other refined estimates were given in [41–43] and are obtained under the C^∞ smoothness assumption. Under the continuity assumption on the smoothness of coefficients, a better estimate for the remainder of (1.6) as in [35] is implausible. Nevertheless, it is interesting to obtain better estimates for the remainder as in [35,41–43] for sufficiently regular coefficients and/or for the anisotropic setting.

Our strategy of the analysis is to develop the approach in [32] at the level where one can apply the general spectral theory for Hilbert-Schmidt operators in Hilbert space as given in Agmon [2] (see also [1]). Two important steps are follows. One is on sharp estimates for $\|T_\lambda\|_{L^p \to W^{1,p}}$ for $p > 1$ and its consequences (see Theorem 3) for large $|\lambda|$ with an appropriate direction. This, in particular, shows that $\mathbf{T} := T_{\hat{\mu}_1} \circ \cdots \circ T_{\hat{\mu}_{k+1}}$ with $k = [d/2]$ is a Hilbert - Schmidt operator (see Proposition 1) for an appropriate choice of

$\hat{\mu}_j \in \mathbb{C}$. The analysis of this part is on the regularity theory of the transmission problems in L^p-scale. This is one of the cores of this paper and has its own interest. To this end, we first investigate the corresponding problems in the whole space and in a half space with constant coefficients, and then use the freezing-coefficient technique. The analysis also involves the Mikhlin-Hörmander multiplier theorem (in particular the theory of singular integrals) and Gagliardo-Nirenberg interpolation inequalities. The second step is to apply the spectral theory for Hilbert-Schmidt operators. To this end, we use the estimates for T_λ to obtain an approximation of the trace of the kernel of the product of $\mathbf{T}$ and its appropriate modified operator (see Proposition 3). The approximation of the trace of the kernel is then used to derive information for the Weyl law via a formula for eigenvalues established in Proposition 2. This formula is derived from the spectral theory of Hilbert-Schmidt operator and is interesting itself. The completeness of the generalized eigenfunctions follows directly from the estimates for T_λ in Theorem 3 where we pay special attention to the possible directions of λ where the information can be derived, after applying the spectral theory in [2].

Remark 4. We use the regularity theory and spectral theory for Hilbert-Schmidt operators to investigate the Weyl law, which was also presented by Robbiano [37]. Nevertheless, the way we derive the regularity theory in this paper is distinct from [37], which involved Carleman's inequalities and the theory of microanalysis. The way we explore the information of Hilbert-Schmidt operators allows us to exactly obtain the first term of the Weyl Law in (1.6) instead of its magnitude order as in [37].

We propose a new approach to establish the Weyl law of eigenvalues and the completeness of the generalized eigenfunctions. This allows us to obtain new significant results and strongly weaken the smoothness assumption in various known results, that is out of reach previously. The transmission problem also appears naturally for electromagnetic waves. In this case, it is a system of two Maxwell systems equipped the Cauchy data on the boundary. The spectral theory of the transmission problem for electromagnetic waves is much less known. On this aspect, we point the reader to [11] on the discreteness, and to [16] on the completeness. More information can be found in the references therein. The analysis in this paper will be developed for the Maxwell setting in our forthcoming work.

The transmission problem has an interesting connection with the study of negative-index materials which are modeled by the Helmholtz or Maxwell equations with sign changing coefficients. In fact, our work has its roots in [27] where the stability of solutions of the Helmholtz equations with sign changing coefficients was studied. Concerning the Maxwell equations, the stability was studied in [33]. It is not coincident that the transmission problem and the Helmholtz equations with sign-changing coefficients share some common analysis. In fact, using reflections (a class of changes of variables), the Cauchy problems appear naturally in the context of the Helmholtz with sign-changing coefficients as first observed in [23] (see also [29] for the Maxwell setting). Other properties of the Cauchy problems related to resonant (unstable) aspects and applications of negative-index materials such as cloaking and superlensing can be found in [24–26,28,30,31] and the references therein.

The paper is organized as follows. In Section 2, we introduce several notations used throughout the paper. In Section 3, we establish Theorem 3, which describes the regularity theory for the transmission problem in L^p-scale. In Section 4, we recall some definitions, properties of Hilbert-Schmidt operators, and their finite double-norms. We then derive their applications in the context of the transmission problem. The main result of this section is Proposition 2, which is derived from Theorem 3. The Weyl law and the completeness are then established in Section 5 and in Section 6, respectively.

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2. Notations

We denote, for $\tau > 0$,

$$\Omega_\tau = \Big\{ x \in \Omega : \operatorname{dist}(x, \Gamma) < \tau \Big\}.$$

For $d \geqslant 2$, set

$$\mathbb{R}^d_+ = \Big\{ x \in \mathbb{R}^d ; x_d > 0 \Big\} \quad \text{and} \quad \mathbb{R}^d_0 = \Big\{ x \in \mathbb{R}^d ; x_d = 0 \Big\}.$$

We will identify $\mathbb{R}^d_0$ with $\mathbb{R}^{d-1}$ in several places.

For $\theta \in \mathbb{R}$ and $a > 0$, denote

$$\mathcal{L}(\theta, a) = \Big\{ re^{i\theta} \in \mathbb{C} : r \geqslant a \Big\}. \tag{2.1}$$

3. Regularity theory for transmission problems

In this section, we establish several estimates for T_λ for appropriate values of λ. The main results are as follows.

Theorem 3. *Let $\varepsilon_0 > 0$ and $\Lambda \geqslant 1$. Assume that (1.1) and (1.2) hold, and A_1, A_2, Σ_1, Σ_2 are continuous in $\bar{\Omega}$. Assume that (1.4) and (1.5) hold in the following sense, with $\nu = \nu(x)$,*

$$\Big| \langle A_2(x)\nu, \nu\rangle \langle A_2(x)\xi, \xi\rangle - \langle A_2(x)\nu, \xi\rangle^2 - \langle A_1(x)\nu, \nu\rangle \langle A_1(x)\xi, \xi\rangle + \langle A_1(x)\nu, \xi\rangle^2 \Big| \geqslant \Lambda^{-1}|\xi|^2, \tag{3.1}$$

for all $x \in \Gamma$ and for all $\xi \in \mathbb{R}^d \setminus \{0\}$ with $\langle \xi, \nu\rangle = 0$, and

$$\Big|\langle A_2(x)\nu,\nu\rangle\Sigma_2(x) - \langle A_1(x)\nu,\nu\rangle\Sigma_1(x)\Big| \geqslant \Lambda^{-1}, \ \forall x \in \Gamma. \tag{3.2}$$

There exist two positive constants Λ_0 and C depending only on Λ, ε_0, Ω, and the continuity modulus of A_1, A_2, Σ_1, and Σ_2 in $\bar{\Omega}$ such that for $\theta \in \mathbb{R}$ with $\inf_{n\in\mathbb{Z}} |\theta - n\pi| \geqslant \varepsilon_0$, and for $\lambda \in \mathcal{L}(\theta,\Lambda_0)$, the following fact holds: for $g = (g_1, g_2) \in [L^2(\Omega)]^2$, there exists a unique solution $u = (u_1, u_2) \in [H^1(\Omega)]^2$ of the system

$$\begin{cases} \operatorname{div}(A_1\nabla u_1) - \lambda\Sigma_1 u_1 = g_1 & \text{in } \Omega, \\ \operatorname{div}(A_2\nabla u_2) - \lambda\Sigma_2 u_2 = g_2 & \text{in } \Omega, \\ u_1 - u_2 = 0, \quad (A_1\nabla u_1 - A_2\nabla u_2)\cdot\nu = 0 & \text{on } \Gamma. \end{cases} \tag{3.3}$$

Moreover, for $1 < p < \infty$,

$$\|\nabla u\|_{L^p(\Omega)} + |\lambda|^{1/2}\|u\|_{L^p(\Omega)} \leqslant C|\lambda|^{-\frac{1}{2}}\|g\|_{L^p(\Omega)}. \tag{3.4}$$

As a consequence, we have

- *for $1 < p < d$ and $p \leqslant q \leqslant \frac{dp}{d-p}$,*

$$||u||_{L^q(\Omega)} \leqslant C|\lambda|^{-1+\frac{d}{2}\left(\frac{1}{p}-\frac{1}{q}\right)}\|g\|_{L^p(\Omega)}, \tag{3.5}$$

- *for $p > d$,*

$$\|u\|_{L^\infty(\Omega)} \leqslant C|\lambda|^{-1+\frac{d}{2p}}\|g\|_{L^p(\Omega)}, \tag{3.6}$$

- *for $p > d$ and $q = \frac{p}{p-1}$,*

$$\|u\|_{L^q(\Omega)} \leqslant C|\lambda|^{-1+\frac{d}{2}-\frac{d}{2q}}\|g\|_{L^1(\Omega)}. \tag{3.7}$$

The remainder of this section contains two subsections, which are organized as follows. In the first subsection, we establish several lemmas used in the proof of Theorem 3. The proof of Theorem 3 is given in the second subsection.

3.1. Preliminaries

In this section, we establish several results used in the proof of Theorem 3, which is based on freezing coefficient technique. We begin with the corresponding settings/variants with constant coefficients in $\mathbb{R}^d$ and in $\mathbb{R}^d_+$. The first one is

Lemma 1. *Let $d \geqslant 2$, $\Lambda \geqslant 1$, $\varepsilon_0 > 0$, $1 < p < \infty$, and let A and Σ be a symmetric matrix and a non-zero real constant, respectively. Assume that $\inf_{n\in\mathbb{Z}} |\theta - n\pi| \geqslant \varepsilon_0$ and $\lambda \in \mathcal{L}(\theta, 1)$,*

$$\Lambda^{-1} \leqslant A \leqslant \Lambda \quad \text{and} \quad \Lambda^{-1} \leqslant |\Sigma| \leqslant \Lambda. \tag{3.8}$$

For $g \in L^p(\mathbb{R}^d)$ and $G \in [L^p(\mathbb{R}^d)]^d$, let $u \in W^{1,p}(\mathbb{R}^d)$ be the unique solution of

$$\operatorname{div}(A\nabla u) - \lambda\Sigma u = g + \operatorname{div}(G) \text{ in } \mathbb{R}^d.$$

We have

$$|\lambda|^{1/2}\|\nabla u\|_{L^p(\mathbb{R}^d)} + |\lambda|\|u\|_{L^p(\mathbb{R}^d)} \leqslant C\Big(\|g\|_{L^p(\mathbb{R}^d)} + |\lambda|^{1/2}\|G\|_{L^p(\mathbb{R}^d)}\Big), \tag{3.9}$$

and, if $G = 0$,

$$\|\nabla^2 u\|_{L^p(\mathbb{R}^d)} \leqslant C\|g\|_{L^p(\mathbb{R}^d)}. \tag{3.10}$$

Here C denotes a positive constant depending only on p, d, Λ, and ε_0.

Here and in what follows, for two $d \times d$ symmetric matrices M_1 and M_2, we denote $M_1 \geqslant M_2$ (resp. $M_1 \leqslant M_2$) if $\langle M_1\xi, \xi\rangle \geqslant \langle M_2\xi, \xi\rangle$ (resp. $\langle M_1\xi, \xi\rangle \leqslant \langle M_2\xi, \xi\rangle$) for all $\xi \in \mathbb{R}^d$.

Proof. For an appropriate function/vector field f defined in $\mathbb{R}^d$, let $\mathcal{F}f$ denote its Fourier transform. We have

$$\mathcal{F}u(\xi) = -\frac{\mathcal{F}g(\xi) + i\xi \cdot \mathcal{F}G(\xi)}{\langle A\xi, \xi\rangle + \lambda\Sigma}.$$

Set

$$m(\xi) = \frac{1}{\langle A\xi, \xi\rangle + \lambda\Sigma}.$$

One can check that

$$|\xi|^\ell |\nabla^\ell m(\xi)| \leqslant C_\ell |\lambda|^{-1} \text{ for } \ell \in \mathbb{N}.$$

It follows from Mikhlin-Hörmander's multiplier theorem, see e.g. [14, Theorem 5.2.7], that

$$\|u\|_{L^p(\mathbb{R}^d)} \leqslant C|\lambda|^{-1}\|g\|_{L^p(\mathbb{R}^d)}.$$

The other estimates in Assertion (3.9) and (3.10) can be derived in the same manner. The proof is complete. □

Here is a result on a half space.

Lemma 2. *Let A_1, A_2 be two constant, symmetric matrices, and let Σ_1, Σ_2 be two non-zero, real constants. Assume that, for some $\Lambda \geqslant 1$,*

$$\Lambda^{-1} \leqslant A_1, A_2 \leqslant \Lambda, \quad \Lambda^{-1} \leqslant |\Sigma_1|, |\Sigma_2| \leqslant \Lambda, \tag{3.11}$$

$$\Big|\langle A_2 e_d, e_d\rangle\langle A_2\xi,\xi\rangle - \langle A_2 e_d, \xi\rangle^2 - \langle A_1 e_d, e_d\rangle\langle A_1\xi,\xi\rangle + \langle A_1 e_d,\xi\rangle^2\Big| \geqslant \Lambda^{-1}|\xi|^2 \quad \forall \xi \in \mathcal{P}, \tag{3.12}$$

where $\mathcal{P} = \big\{\xi \in \mathbb{R}^d; \langle \xi, e_d\rangle = 0\big\}$, and

$$\Big|\big\langle A_2 e_d, e_d\big\rangle\Sigma_2 - \big\langle A_1 e_d, e_d\big\rangle\Sigma_1\Big| \geqslant \Lambda^{-1}. \tag{3.13}$$

Let $p > 1$, $\varepsilon_0 > 0$, $g_1, g_2 \in L^p(\mathbb{R}^d_+)$, $G_1, G_2 \in [L^p(\mathbb{R}^d_+)]^d$, and $\varphi \in W^{1-1/p,p}(\mathbb{R}^d_0)$. There exist two positive constants C and Λ_0 depending only on Λ and ε_0 such that for $\theta \in \mathbb{R}$ with $\min_{n\in\mathbb{Z}} |\theta - n\pi| \geqslant \varepsilon_0$ and for $\lambda \in \mathcal{L}(\theta, \Lambda_0)$, there exists a unique solution $u = (u_1, u_2) \in [W^{1,p}(\mathbb{R}^d_+)]^2$ of the system

$$\begin{cases} \operatorname{div}(A_1\nabla v_1) - \lambda\Sigma_1 v_1 = g_1 + \operatorname{div}(G_1) & \text{in } \mathbb{R}^d_+, \\ \operatorname{div}(A_2\nabla v_2) - \lambda\Sigma_2 v_2 = g_2 + \operatorname{div}(G_2) & \text{in } \mathbb{R}^d_+, \\ v_1 - v_2 = \varphi, \quad (A_1\nabla v_1 - G_1)\cdot e_d - (A_2\nabla v_2 - G_2)\cdot e_d = 0 & \text{on } \mathbb{R}^d_0. \end{cases} \tag{3.14}$$

Moreover,

$$\|\nabla v\|_{L^p(\mathbb{R}^d_+)} + |\lambda|^{1/2}\|v\|_{L^p(\mathbb{R}^d_+)} \leqslant C\left(|\lambda|^{-1/2}\|g\|_{L^p(\mathbb{R}^d_+)} + \|G\|_{L^p(\mathbb{R}^d_+)}\right.$$
$$\left. + \lambda^{1/2-1/(2p)}\|\varphi\|_{L^p(\mathbb{R}^d_0)} + \|\varphi\|_{\dot W^{1-1/p,p}(\mathbb{R}^d_0)}\right). \tag{3.15}$$

Proof. We only establish (3.15). The uniqueness for (3.14) is a consequence of (3.15). The existence of (v_1, v_2) follows from the proof of (3.15) and is omitted. Let $u_j \in W^{1,p}(\mathbb{R}^d)$ be the unique solution of the equation

$$\operatorname{div}(A_j\nabla u_j) - \lambda\Sigma_j u_j = g_j \mathbb{1}_{\mathbb{R}^d_+} + \operatorname{div}(G_j\mathbb{1}_{\mathbb{R}^d_+}) \text{ in } \mathbb{R}^d.$$

It follows from Lemma 1 that

$$\|\nabla u_j\|_{L^p(\mathbb{R}^d)} + |\lambda|^{1/2}\|u_j\|_{L^p(\mathbb{R}^d)} \leqslant C\Big(|\lambda|^{-1/2}\|g_j\|_{L^p(\mathbb{R}^d_+)} + \|G_j\|_{L^p(\mathbb{R}^d_+)}\Big).$$

We have

$$|\lambda|^{1/2-1/(2p)}\|u_j\|_{L^p(\mathbb{R}^d_0)} \leqslant C\Big(\|\nabla u_j\|_{L^p(\mathbb{R}^d_+)} + |\lambda|^{1/2}\|u_j\|_{L^p(\mathbb{R}^d_+)}\Big),$$
$$(A_j\nabla u_j - G_j)\cdot e_d = 0 \text{ on } \mathbb{R}^d_0,$$

and, by the trace theory,

$$\|u_j\|_{W^{1-1/p,p}(\mathbb{R}^d_0)} \leqslant C\|u_j\|_{W^{1,p}(\mathbb{R}^d_+)}.$$

Therefore, without loss of generality, one might assume that $g_1 = g_2 = 0$ and $G_1 = G_2 = 0$. This will be assumed from now on.

Let $\hat{v}_j(\xi', t)$ for $j = 1, 2$ and $\hat{\varphi}(\xi', t)$ be the Fourier transform of v_j and φ with respect to $x' \in \mathbb{R}^{d-1}$, i.e., for $(\xi', t) \in \mathbb{R}^{d-1} \times (0, +\infty)$,

$$\hat{v}_j(\xi', t) = \int_{\mathbb{R}^{d-1}} v_j(x', t)e^{-ix'\cdot\xi'}\, dx' \quad \text{and} \quad \hat{\varphi}(\xi', t) = \int_{\mathbb{R}^{d-1}} \varphi(x')e^{-ix'\cdot\xi'}\, dx'.$$

Since

$$\operatorname{div}(A_j\nabla v_j) - \lambda\Sigma_j v_j = 0 \text{ in } \mathbb{R}^d_+,$$

it follows that

$$a_j v_j''(t) + 2ib_j v_j'(t) - (c_j + \lambda\Sigma_j)v_j(t) = 0 \text{ for } t > 0,$$

where

$$a_j = (A_j)_{d,d}, \quad b_j = \sum_{k=1}^{d-1}(A_j)_{d,k}\xi_k, \quad \text{and} \quad c_j = \sum_{k=1}^{d-1}\sum_{l=1}^{d-1}(A_j)_{k,l}\xi_k\xi_l.$$

Here $(A_j)_{k,l}$ denotes the (k, l) component of A_j for $j = 1, 2$ and the symmetry of A_j is used. Define, for $j = 1, 2$,

$$\Delta_j = -b_j^2 + a_j(c_j + \lambda\Sigma_j). \tag{3.16}$$

Denote $\xi = (\xi', 0)$. Since A_j is symmetric and positive, it is clear that, for $j = 1, 2$,

$$a_j = \langle A_j e_d, e_d\rangle, \;\; b_j = \langle A_j\xi, e_d\rangle, \;\; c_j = \langle A_j\xi, \xi\rangle, \quad \text{and} \quad a_jc_j - b_j^2 > 0. \tag{3.17}$$

Since $\hat{v}_j(\xi', t) \in L^2(\mathbb{R}^d_+)$, we have

$$\hat{v}_j(\xi', t) = \alpha_j(\xi')e^{\eta_j(\xi')t},$$

for some $\alpha_j(\xi') \in \mathbb{C}$, where

$$\eta_j = (-ib_j - \sqrt{\Delta_j})/a_j.$$

Here $\sqrt{\Delta_j}$ denotes the square root of Δ_j with positive real part. Using the fact that $v_1 - v_2 = \varphi$ and $A_1\nabla v_1 \cdot e_d - A_2\nabla v_2 \cdot e_d = 0$ on $\mathbb{R}^d_0$, we derive that

$$\alpha_1(\xi') - \alpha_2(\xi') = \hat{\varphi}(\xi') \quad \text{and} \quad \alpha_1(\xi')\langle iA_1\xi + \eta_1 A_1 e_d, e_d\rangle - \alpha_2(\xi')\langle iA_2\xi + \eta_2 A_2 e_d, e_d\rangle = 0. \tag{3.18}$$

Note that, by (3.17),

$$\langle A_j\xi, e_d\rangle - \langle A_j e_d, e_d\rangle b_j/a_j = 0.$$

The last identity of (3.18) is equivalent to

$$\alpha_1(\xi')\sqrt{\Delta_1} = \alpha_2(\xi')\sqrt{\Delta_2}.$$

Combining this identity and the first one of (3.18) yields

$$\alpha_1(\xi') = \frac{\hat{\varphi}(\xi')\sqrt{\Delta_2}}{\sqrt{\Delta_2} - \sqrt{\Delta_1}}. \tag{3.19}$$

Extend $v_1(x', t)$ by 0 for $t < 0$. We then obtain

$$\mathcal{F}v_1(\xi) = -\hat{\varphi}(\xi')\frac{\sqrt{\Delta_2}}{\sqrt{\Delta_2} - \sqrt{\Delta_1}}\frac{1}{\eta_j - i\xi_d}.$$

Here, $\mathcal{F}$ is the Fourier transform in $\mathbb{R}^d$. Set

$$g(t) = e^{-|\lambda|^{1/2}t}\mathbb{1}_{t\geqslant 0} \text{ for } t \in \mathbb{R} \quad \text{and} \quad \Phi(x) = \varphi(x')g(x_d) \text{ for } x \in \mathbb{R}^d.$$

It follows that

$$\mathcal{F}v_1(\xi) = \mathcal{F}\Phi(\xi)\frac{\sqrt{\Delta_2}}{\sqrt{\Delta_2} - \sqrt{\Delta_1}}\frac{|\lambda|^{1/2} + i\xi_d}{-\eta_j + i\xi_d}.$$

We have

$$|\Delta_2 - \Delta_1|^2 \geqslant C(|\xi'|^4 + |\lambda|^2), \quad |\Delta_j| \leqslant C(|\xi'|^2 + |\lambda|),$$

and

$$|\Re(\eta_j)| \geqslant C(|\xi'| + |\lambda|^{1/2}).$$

As in the proof of Lemma 1, by Mikhlin-Hörmander's multiplier theorem, see e.g. [14, Theorem 5.2.7], one has

$$\|v_1\|_{L^p(\mathbb{R}^d)} \leqslant C\|\Phi\|_{L^p(\mathbb{R}^d)} \leqslant C|\lambda|^{-1/(2p)}\|\varphi\|_{L^p(\mathbb{R}^{d-1})}. \tag{3.20}$$

We next deal with ∇v_1. We have

$$\partial_t \hat{v}_1(\xi', t) = \eta_1 \hat{v}_1(\xi', t) \text{ in } \mathbb{R}^d_+. \tag{3.21}$$

It is clear that

$$\eta_1 = \eta_{1,1} + \eta_{1,2},$$

where

$$\eta_{1,1} = -\frac{\sqrt{a_1\lambda\Sigma_1}}{a_1}, \quad \text{and} \quad \eta_{1,2} = -\frac{ib_1}{a_1} - \frac{\sqrt{\Delta_1} - \sqrt{a_1\lambda\Sigma_1}}{a_1}.$$

As above, one can prove that

$$\|\hat{\mathcal{F}}^{-1}(\eta_{1,1}\hat{v}_1)\|_{L^p(\mathbb{R}^d_+)} \leqslant C|\lambda|^{1/2}\|\Phi\|_{L^p(\mathbb{R}^d)} \leqslant C|\lambda|^{1/2-1/(2p)}\|\varphi\|_{L^p(\mathbb{R}^{d-1})}, \tag{3.22}$$

and, for some $\gamma > 0$,

$$\|\hat{\mathcal{F}}^{-1}(\eta_{1,2}\hat{v}_1)\|_{L^p(\mathbb{R}^d_+)} \leqslant C\|g\|_{L^p(\mathbb{R}^d_+)},$$

where $\hat{\mathcal{F}}^{-1}$ denotes the Fourier inverse with respect to ξ' in $\mathbb{R}^{d-1}$ and

$$\hat{g}(\xi', t) = i\xi'\hat{\varphi}(\xi')e^{-\gamma|\xi'|t}.$$

It is clear that $g(x) = \nabla_{x'}v(x)$, where v is the unique solution of the system

$$\Delta_{x'}v + \gamma\partial^2_{x_d}v = 0 \text{ in } \mathbb{R}^d_+ \quad \text{and} \quad v = \varphi \text{ on } \mathbb{R}^d_0$$

for $\gamma > 0$. It follows that, see e.g. [3, Theorem 3.3], we have

$$\|g\|_{L^p(\mathbb{R}^d_+)} \leqslant C\|\varphi\|_{\dot{W}^{1/p-1,p}(\mathbb{R}^{d-1})}. \tag{3.23}$$

Combining (3.22) and (3.23) yields

$$\|\partial_{x_d}v_1\|_{L^p(\mathbb{R}^d_+)} \leqslant C\|\varphi\|_{\dot{W}^{1/p-1,p}(\mathbb{R}^{d-1})} + C|\lambda|^{1/2-1/(2p)}\|\varphi\|_{L^p(\mathbb{R}^{d-1})}. \tag{3.24}$$

By the same manner, we also obtain

$$\|\nabla_{x'}v_1\|_{L^p(\mathbb{R}^d_+)} \leqslant C\|\varphi\|_{\dot{W}^{1/p-1,p}(\mathbb{R}^{d-1})}. \tag{3.25}$$

From (3.20), (3.24), and (3.25), we obtain

$$|\lambda|^{1/2}\|v_1\|_{L^p(\mathbb{R}^d_+)} + \|\nabla_x v_1\|_{L^p(\mathbb{R}^d_+)} \leqslant C\|\varphi\|_{\dot{W}^{1/p-1,p}(\mathbb{R}^{d-1})} + C|\lambda|^{1/2-1/(2p)}\|\varphi\|_{L^p(\mathbb{R}^{d-1})}. \tag{3.26}$$

Similar to (3.26), we also get

$$|\lambda|^{1/2}\|v_2\|_{L^p(\mathbb{R}^d_+)} + \|\nabla_x v_2\|_{L^p(\mathbb{R}^d_+)} \leqslant C\|\varphi\|_{\dot{W}^{1/p-1,p}(\mathbb{R}^{d-1})} + C|\lambda|^{1/2-1/(2p)}\|\varphi\|_{L^p(\mathbb{R}^{d-1})}. \tag{3.27}$$

The conclusion thus follows from (3.26) and (3.27). The proof is complete. □

Remark 5. Assertion (3.14) was previously established in [32] for $p = 2$ (see [32, the proof of Theorem 4]). The analysis given here has its root in [27,32]. Nevertheless, instead of using Parseval's theorem to derive L^2-estimates, the new ingredient involves Mikhlin-Hörmander's multiplier theory.

We now derive consequences of Lemmas 1 and 2 via the freezing-coefficient technique. As a consequence of Lemma 1, we have

Corollary 1. *Let $d \geqslant 2$, $p > 1$, $\Lambda \geqslant 1$, $\varepsilon_0 > 0$, and let A be a symmetric, matrix-valued function, and let Σ be a real function defined in Ω. Assume that A and Σ are continuous in $\bar{\Omega}$,*

$$\Lambda^{-1} \leqslant A \leqslant \Lambda \quad \text{and} \quad \Lambda^{-1} \leqslant |\Sigma| \leqslant \Lambda \text{ in } \Omega, \tag{3.28}$$

for some $\Lambda \geqslant 1$, $\inf_{n\in\mathbb{Z}} |\theta - n\pi| \geqslant \varepsilon_0$, and $\lambda \in \mathcal{L}(\theta, 1)$. For $g \in L^p(\Omega)$ and $G \in [L^p(\Omega)]^d$, let $u \in W^{1,p}(\Omega)$ be a solution of

$$\operatorname{div}(A\nabla u) - \lambda\Sigma u = g + \operatorname{div}(G) \text{ in } \Omega.$$

We have, for $\tau > 0$,

$$\begin{aligned} &\|\nabla u\|_{L^p(\Omega\setminus\Omega_\tau)} + |\lambda|^{1/2}\|u\|_{L^p(\Omega\setminus\Omega_\tau)} \\ &\quad \leqslant C\Big(|\lambda|^{-1/2}\|g\|_{L^p(\Omega)} + \|G\|_{L^p(\Omega)}\Big) + C|\lambda|^{-\frac{1}{2}}\Big(\|\nabla u\|_{L^p(\Omega)} + |\lambda|^{1/2}\|u\|_{L^p(\Omega)}\Big). \end{aligned} \tag{3.29}$$

Here C denotes a positive constant depending only on Λ, p, ε_0, τ, Ω, and the continuity modulus of A_1, A_2, Σ_1, and Σ_2 in $\overline{\Omega}$.

Proof. Let χ be an arbitrary smooth function with support in Ω. Set $v = \chi u$ in Ω. We have

$$\operatorname{div}(A\nabla v) - \lambda\Sigma v = f + \operatorname{div} F \text{ in } \Omega,$$

where

$$f = \chi g + A\nabla u \nabla\chi - F \cdot \nabla\chi \quad \text{and} \quad F = \chi F + uA\nabla\varphi.$$

The conclusion follows from Lemma 1 by the freezing-coefficient technique and the computations above. □

Similarly, as a consequence of Lemma 2, we obtain

Corollary 2. *Let $d \geqslant 2$, $p > 1$, $\varepsilon_0 > 0$, $\tau > 0$, and $\Lambda \geqslant 1$, and let A_1, A_2 be two symmetric, matrix-valued functions, and let Σ_1, Σ_2 be two real functions defined in Ω. Assume that A_1, A_2, Σ_1, Σ_2 are continuous in $\overline{\Omega}_{2\tau}$, (3.28) holds, and (1.4) and (1.5) are satisfied. There exist two positive constants Λ_0 and C, depending only on Λ, ε_0, and the continuity of A_1, A_2, Σ_1, and Σ_2 in $\overline{\Omega}_{2\tau}$ such that for $\theta \in \mathbb{R}$, for $\lambda \in \mathcal{L}(\theta, \Lambda_0)$ with $\inf_{n \in \mathbb{Z}} |\theta - n\pi| \geqslant \varepsilon_0$, and for $g = (g_1, g_2) \in [L^p(\Omega)]^2$, and $G = (G_1, G_2) \in [L^p(\Omega)]^d \times [L^p(\Omega)]^d$, let $u = (u_1, u_2) \in [W^{1,p}(\Omega)]^2$ be a solution of the system*

$$\begin{cases} \operatorname{div}(A_1 \nabla u_1) - \lambda \Sigma_1 u_1 = g_1 + \operatorname{div}(G_1) & \text{in } \Omega, \\ \operatorname{div}(A_2 \nabla u_2) - \lambda \Sigma_2 u_2 = g_2 + \operatorname{div}(G_2) & \text{in } \Omega, \\ u_2 - u_1 = 0, \quad (A_2 \nabla u_2 - A_1 \nabla u_1 - G_2 + G_1) \cdot \nu = 0 & \text{on } \Gamma. \end{cases} \tag{3.30}$$

Moreover, we have

$$\|\nabla v\|_{L^p(\Omega_\tau)} + |\lambda|^{1/2} \|v\|_{L^p(\Omega_\tau)} \leqslant C\Big(|\lambda|^{-1/2} \|g\|_{L^p(\Omega)} + \|G\|_{L^p(\Omega)}\Big) + C|\lambda|^{-\frac{1}{2}} \Big(\|\nabla v\|_{L^p(\Omega)} + |\lambda|^{1/2} \|v\|_{L^p(\Omega)}\Big). \tag{3.31}$$

Here C denotes a positive constant depending only on Λ, p, ε_0, τ, Ω, and the continuity modulus of A_1, A_2, Σ_1, and Σ_2 in $\overline{\Omega}_{2\tau}$.

3.2. Proof of Theorem 3

We first assume the well-posedness of (3.3) and establish (3.4) - (3.7).

It is clear that (3.4) is a consequence of Corollary 1 and Corollary 2.

We next deal with (3.5) and (3.6). By Gagliardo-Nirenberg's interpolation inequalities [13,34], if $p > d$ and $u \in W^{1,p}(\Omega)$, then $u \in C(\overline{\Omega})$ and

$$\|u\|_{L^\infty(\Omega)} \leqslant C \|u\|_{W^{1,p}(\Omega)}^{\frac{d}{p}} \|u\|_{L^p(\Omega)}^{1 - \frac{d}{p}}, \tag{3.32}$$

and if $1 < p < d$, and $u \in W^{1,p}(\Omega)$, then, for $p \leqslant q < \frac{dp}{d-p}$,

$$\|u\|_{L^q(\Omega)} \leqslant C \|u\|_{W^{1,p}(\Omega)}^{d(\frac{1}{p} - \frac{1}{q})} \|u\|_{L^p(\Omega)}^{1 - d(\frac{1}{p} - \frac{1}{q})}. \tag{3.33}$$

Assertions (3.5) and (3.6) now follow from (3.4), (3.32), (3.33), and Hölder's inequality.

We finally establish (3.7). Let

$$\begin{array}{rccc} \mathcal{T}_\delta: & [L^2(\Omega)]^2 & \to & [L^2(\Omega)]^2 \\ & g & \mapsto & u, \end{array}$$

where $u = (u_1, u_2) \in [H^1(\Omega)]^2$ is the unique solution of (3.3) with $(g_1, g_2) = g$. We have, for $q = \frac{p}{p-1}$ and $p > d$,

$$\|u\|_{L^q(\Omega)} = \sup_{f \in [L^2(\Omega)]^2; \|f\|_{L^p(\Omega)} \leqslant 1} |\langle u, f \rangle|,$$

and

$$\langle u, f \rangle = \langle \mathcal{T}_\lambda(g), f \rangle = \langle g, \mathcal{T}_\lambda^*(f) \rangle.$$

One can check that

$$\mathcal{T}_\lambda^* = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mathcal{T}_{\bar{\lambda}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{3.34}$$

It follows that

$$|\langle u, f \rangle| \leqslant \|g\|_{L^1(\Omega)} \|\mathcal{T}_{\bar{\lambda}}(f)\|_{L^\infty(\Omega)} \overset{(3.6)}{\leqslant} C|\lambda|^{-1+\frac{d}{2p}} \|g\|_{L^1(\Omega)} \|f\|_{L^p(\Omega)}.$$

Assertion (3.7) follows.

It remains to prove the well-posedness of (3.3). It is clear that the uniqueness of (3.3) follows from (3.4). To establish the existence for (3.3), we use the principle of limiting absorption and the Fredholm theory. We only consider the case where $\Im(\lambda) < 0$; the other case can be proved similarly. For $\delta > 0$, by the Lax-Milgram theory, there exists a unique solution $v_\delta = (v_{1,\delta}, v_{2,\delta}) \in [H^1(\Omega)]^2$ of the system

$$\begin{cases} \operatorname{div}\big((1 - i\delta) A_1 \nabla v_{1,\delta}\big) - \lambda \Sigma_1 v_{1,\delta} = g_1 & \text{in } \Omega, \\ \operatorname{div}\big((1 + i\delta) A_2 \nabla v_{2,\delta}\big) + \lambda \Sigma_2 v_{2,\delta} = g_2 & \text{in } \Omega, \\ v_{2,\delta} - v_{1,\delta} = 0, \quad \big((1 + i\delta) A_2 \nabla v_{2,\delta} - (1 - i\delta) A_1 \nabla v_{1,\delta}\big) \cdot \nu = 0 & \text{on } \Gamma. \end{cases}$$

Moreover, by Corollaries 1 and 2, applied with $G_1 = i\delta A_1 \nabla v_{1,\delta}$ and $G_2 = -i\delta A_2 \nabla v_{2,\delta}$, we have, for sufficiently small δ,

$$\|\nabla u_\delta\|_{L^2(\Omega)} + |\lambda|^{1/2} \|u_\delta\|_{L^2(\Omega)} \leqslant C|\lambda|^{-1/2} \|g\|_{L^2(\Omega)}.$$

By taking $\delta \to 0_+$, one derives the existence of a solution $v = (v_1, v_2) \in [H^1(\Omega)]^2$ of the system, with $\hat{\Sigma}_2 = -\Sigma_2$

$$\begin{cases} \operatorname{div}\big(A_1 \nabla v_1\big) - \lambda \Sigma_1 v_1 = g_1 & \text{in } \Omega, \\ \operatorname{div}\big(A_2 \nabla v_2\big) - \lambda \hat{\Sigma}_2 v_2 = g_2 & \text{in } \Omega, \\ v_2 - v_1 = 0, \quad \big(A_2 \nabla v_2 - A_1 \nabla v_1\big) \cdot \nu = 0 & \text{on } \Gamma, \end{cases} \tag{3.35}$$

which satisfies

$$\|\nabla v\|_{L^2(\Omega)} + |\lambda|^{1/2}\|v\|_{L^2(\Omega)} \leqslant C|\lambda|^{-1/2}\|g\|_{L^2(\Omega)}. \tag{3.36}$$

The uniqueness of (3.35) is again a consequence of Corollary 1 and Corollary 2.

Define

$$\begin{array}{cccc} \hat{\mathcal{T}}: & [L^2(\Omega)]^2 & \to & [L^2(\Omega)]^2 \\ & g & \mapsto & v, \end{array}$$

where $v = (v_1, v_2) \in [H^1(\Omega)]^2$ is the unique solution of (3.35). It follows from (3.36) that $\hat{\mathcal{T}}$ is compact.

It is clear that $u = (u_1, u_2) \in [H^1(\Omega)]^2$ is a solution of (3.3) if and only if

$$\begin{cases} \operatorname{div}(A_1 \nabla u_1) - \lambda \Sigma_1 u_1 = g_1 & \text{in } \Omega, \\ \operatorname{div}(A_2 \nabla u_2) - \lambda \hat{\Sigma}_2 u_2 = g_2 + 2\lambda \Sigma_2 u_2 & \text{in } \Omega, \\ u_1 - u_2 = 0, \quad (A_1 \nabla u_1 - A_2 \nabla u_2) \cdot \nu = 0 & \text{on } \Gamma. \end{cases}$$

In other words,

$$(u_1, u_2) = \hat{\mathcal{T}}(g_1, g_2) + \hat{\mathcal{T}}(0, 2\lambda \Sigma_2 u_2).$$

Since this equation has at most one solution and $\hat{\mathcal{T}}$ is compact, this equation has a unique solution by the Fredholm theory. The proof is complete. □

Remark 6. Note from (3.34) that $\mathcal{T}_\lambda$ is not self-adjoint.

4. Hilbert-Schmidt operators

We now devote two subsections to the applications of Hilbert-Schmidt operators for the transmission problem. In the first subsection, we recall some basis facts on Hilbert-Schmidt operators and the finite double norms. In the second subsection, we derive their applications for the transmission problem. The main result here is Proposition 2.

4.1. Some basic facts on Hilbert-Schmidt operators

In this section, we recall the definition and several properties of Hilbert-Schmidt operators. We begin with

Definition 2. Let H be a separable Hilbert space and let $(\phi_k)_{k=1}^\infty$ be an orthogonal basis. A bounded linear operator $\mathbf{T} : H \to H$ is Hilbert Schmidt if its finite double norm

$$|||\mathbf{T}||| := \left(\sum_{k=1}^\infty \|\mathbf{T}(\phi_k)\|_H^2\right)^{1/2} < +\infty. \tag{4.1}$$

The trace of $\mathbf{T}$ is then defined by

$$\operatorname{trace}(\mathbf{T}) = \sum_{k=1}^{\infty} \langle \mathbf{T}(\phi_k), \phi_k \rangle. \tag{4.2}$$

Remark 7. The definition of $|||\mathbf{T}|||$ and of trace($\mathbf{T}$) do not depend on the choice of (ϕ_k), see e.g. [2, Chapter 12].

One can check, see [2, Theorem 12.12], that if $\mathbf{T}_1$ and $\mathbf{T}_2$ are Hilbert Schmidt then $\mathbf{T}_1\mathbf{T}_2$ is also Hilbert Schmidt, and

$$|\operatorname{trace}(\mathbf{T}_1\mathbf{T}_2)| \leqslant |||\mathbf{T}_1||| \, |||\mathbf{T}_2|||. \tag{4.3}$$

Let $m \in \mathbb{N}$ and $\mathbf{T} : [L^2(\Omega)]^m \to [L^2(\Omega)]^m$ be a Hilbert Schmidt operator. There exists a unique kernel $\mathbf{K} \in [L^2(\Omega \times \Omega)]^{m\times m}$, see e.g. [2, Theorems 12.18 and 12.19], such that

$$(\mathbf{T}u)(x) = \langle \mathbf{K}(x, .), u\rangle \quad \text{for a.e. } x \in \Omega, \text{ for all } u \in [L^2(\Omega)]^m. \tag{4.4}$$

Moreover,

$$|||\mathbf{T}|||^2 = \iint_{\Omega\times\Omega} |\mathbf{K}(x, y)|^2 \, dx \, dy. \tag{4.5}$$

Note that [2, Theorems 2.18 and 12.19] state for $m = 1$, nevertheless, the same arguments hold for $m \in \mathbb{N}$.

We have

Lemma 3. *Let $d \geqslant 2$, $m \in \mathbb{N}$, and $\mathbf{T} : [L^2(\Omega)]^m \to [L^2(\Omega)]^m$ be such that $\mathbf{T}(\phi) \in \mathbf{C}(\bar{\Omega})$ for $\varphi \in [L^2(\Omega)]^m$, and*

$$\|\mathbf{T}(\phi)\|_{L^\infty(\Omega)} \leqslant M\|\phi\|_{L^2(\Omega)}, \tag{4.6}$$

for some $M \geqslant 0$. Then $\mathbf{T}$ is a Hilbert-Schmidt operator,

$$|||\mathbf{T}||| \leqslant C_m|\Omega|^{1/2}M, \tag{4.7}$$

and the kernel $\mathbf{K}$ of $\mathbf{T}$ satisfies

$$\sup_{x\in\Omega} \left(\int_\Omega |\mathbf{K}(x, y)|^2 dy \right)^{1/2} \leqslant C_m|\Omega|^{1/2}M. \tag{4.8}$$

Assume in addition that

$$\|\mathbf{T}(\phi)\|_{L^\infty(\Omega)} \leqslant \widetilde{M}||\phi||_{L^1(\Omega)} \textit{ for } \phi \in [L^2(\Omega)]^m, \tag{4.9}$$

for some $\widetilde{M} \geqslant 0$, *then the kernel* $\mathbf{K}$ *of* $\mathbf{T}$ *satisfies*

$$|\mathbf{K}(x,y)| \leqslant \widetilde{M} \quad \forall x, y \in \Omega. \tag{4.10}$$

Here C_m *denotes a positive constant depending only on* m.

Proof. The proof is quite standard as in [2]. We present the details of this proof for the convenience of the reader. Let $(\phi_k)_{k=1}^\infty$ be an orthonormal basis of $[L^2(\Omega)]^m$ and set $\varphi_j = \mathbf{T}(\phi_j)$. Let $a_1, ..., a_N \in \mathbb{C}^m$ be arbitrary. By (4.6), we have

$$\Big|\sum_{j=1}^N a_j \cdot \varphi_j(x)\Big| \leqslant M\|\sum_{j=1}^N a_j \cdot \phi_j\|_{L^2(\Omega)} \leqslant C_m M \left(\sum_{j=1}^N |a_j|^2\right)^{1/2} \quad \forall x \in \Omega.$$

Choosing $a_j = \overline{\varphi_j(x)}$ yields

$$\sum_{j=1}^N |\varphi_j(x)|^2 \leqslant C_m M^2 \quad \forall x \in \Omega.$$

Integrating over Ω, we obtain

$$\sum_{j=1}^N \|\varphi^j\|^2_{L^2(\Omega)} \leqslant C_m |\Omega| M^2,$$

which implies (4.7).

Assertion (4.8) follows from (4.6) by (4.4).

It is clear that (4.10) is a consequence of (4.9) by the definition of the kernel. □

We next recall a basic, useful property of a Hilbert-Schmidt operator, see e.g., [2, Theorem 12.21][1]:

Lemma 4. *Let* $m \in \mathbb{N}$ *and let* $\mathbf{T}_1, \mathbf{T}_2$ *be two Hilbert-Schmidt operators in* $[L^2(\Omega)]^m$ *with the corresponding kernels* $\mathbf{K}_1$ *and* $\mathbf{K}_2$. *Then* $\mathbf{T} := \mathbf{T}_1\mathbf{T}_2$ *is a Hilbert-Schmidt operator with the kernel* $\mathbf{K}$ *given by*

$$\mathbf{K}(x,y) = \int_\Omega \mathbf{K}_1(x,z)\mathbf{K}_2(z,y)\, dz. \tag{4.11}$$

[1] Note that [2, Theorems 2.21] states for $m = 1$, nevertheless, the same arguments hold for $m \in \mathbb{N}$.

Moreover,

$$trace(\mathbf{T}_1\mathbf{T}_2) = \int_\Omega trace\ \mathbf{K}(x,x)dx. \tag{4.12}$$

Remark 8. Using (4.11), one can check that

$$\int_\Omega |\mathbf{K}(x,x)|\,dx \leqslant \int_\Omega |\mathbf{K}_1(x,z)||\mathbf{K}_2(z,x)|\,dz\,dx \leqslant \|\mathbf{K}_1\|_{L^2(\Omega\times\Omega)}\|\mathbf{K}_2\|_{L^2(\Omega\times\Omega)}.$$

Hence $\mathbf{K}(x,x) \in [L^1(\Omega)]^{m\times m}$.

4.2. Applications of the theory of Hilbert-Schmidt operators

In this section, we apply the theory of Hilbert-Schmidt operators to the operator T_λ mentioned in the introduction. The main ingredient of the analysis is Theorem 3. We begin with

Definition 3. Let $\varepsilon_0 > 0$ and $\Lambda \geqslant 1$. Assume the assumptions of Theorem 3 hold. Let Λ_0 and C be the constants in Theorem 3. For $\lambda \in \mathcal{L}(\theta, \Lambda_0)$ with $\inf_{n\in\mathbb{Z}} |\theta - n\pi| > \varepsilon_0$, define

$$\begin{array}{rccc} T_\lambda: & [L^2(\Omega)]^2 & \to & [L^2(\Omega)]^2 \\ & f & \mapsto & u, \end{array}$$

where $u = (u_1, u_2) \in [H^1(\Omega)]^2$ is the unique solution of, with $(f_1, f_2) = f$,

$$\begin{cases} \operatorname{div}(A_1\nabla u_1) - \lambda\Sigma_1 u_1 = \Sigma_1 f_1 & \text{in } \Omega, \\ \operatorname{div}(A_2\nabla u_2) - \lambda\Sigma_2 u_2 = \Sigma_2 f_2 & \text{in } \Omega, \\ u_1 = u_2, \quad A_1\nabla u_1\cdot\nu = A_2\nabla u_2\cdot\nu & \text{on } \Gamma. \end{cases} \tag{4.13}$$

From now on, we fix the constant Λ_0 as required in Definition 3 for a given ε_0 and set

$$\lambda_0 = \Lambda_0 e^{i\pi/2}. \tag{4.14}$$

Remark 9. Let $T_\lambda^\star$ be the adjoint operator of T_λ, i.e., $\langle T_\lambda(f), g\rangle = \langle f, T_\lambda^*(g)\rangle$ for any f, $g \in [L^2(\Omega)]^2$. Integrating by parts, one has

$$\left(\int_\Omega \operatorname{div}(A_1\nabla u_1)v_1 - \operatorname{div}(A_2\nabla u_2)v_2\right) - \left(\int_\Omega u_1\operatorname{div}(A_1\nabla v_1) - u_2\operatorname{div}(A_2\nabla v_2)\right)$$
$$= \left(\int_\Gamma v_1.A_1\nabla u_1\cdot\nu - v_2.A_2\nabla u_2\cdot\nu\right) - \left(\int_\Gamma u_1A_1\nabla v_1\cdot\nu - u_2A_2\nabla v_2\cdot\nu\right).$$

This implies

$$T_\lambda^* = \begin{pmatrix} \Sigma_1 & 0 \\ 0 & -\Sigma_2 \end{pmatrix} T_{\overline{\lambda}} \begin{pmatrix} 1/\Sigma_1 & 0 \\ 0 & -1/\Sigma_2 \end{pmatrix}. \tag{4.15}$$

Thus T_λ is not self-adjoint.

For the operator T_λ defined above, the following estimates hold:

Proposition 1. *We have*

$$\|T_\lambda\|_{L^p \to L^\infty} \leqslant C|\lambda|^{-1+\frac{d}{2p}} \quad \text{if} \;\; p > d, \tag{4.16}$$

$$\|T_\lambda\|_{L^p \to L^q} \leqslant C|\lambda|^{-1+\frac{d}{2}\left(\frac{1}{p}-\frac{1}{q}\right)} \quad \text{if} \;\; 1 < p < d, \;\; p \leqslant q < \frac{dp}{d-p}, \tag{4.17}$$

$$\|T_\lambda\|_{L^1 \to L^q} \leqslant C|\lambda|^{-1+\frac{d}{2}-\frac{d}{2q}} \quad \text{if} \;\; 1 < q < \frac{d}{d-1}. \tag{4.18}$$

Assume that $\lambda_1, ..., \lambda_{k+1}$ satisfy the assumption of Theorem 3 with $k = k_d = \left[\frac{d}{2}\right]$, and $|\lambda_1| \sim |\lambda_2| \sim ... \sim |\lambda_{k+1}| = t$. Then operator $\prod_{j=1}^{k+1} T_{\lambda_j} = T_{\lambda_{k+1}} \circ T_{\lambda_k} \circ ... \circ T_{\lambda_1}$ is Hilbert-Schmidt, and

$$\left|\!\left|\!\left| \prod_{j=1}^{k+1} T_{\lambda_j} \right|\!\right|\!\right| \leqslant C t^{\frac{d}{4}-1-k}. \tag{4.19}$$

Proof. Clearly, (4.16) - (4.18) follow from (3.5) - (3.7). Fix $p_1 = 2 < p_2 < ... < p_k < p_{k+1} < +\infty$ with $p_{k+1} > d$ and $p_j < \frac{dp_{j-1}}{d-p_{j-1}}$. By (4.16) and (4.17), we obtain

$$\left\| \prod_{j=1}^{k+1} T_{\lambda_j} \right\|_{L^2 \to L^\infty} \leqslant \|T_{\lambda_1}\|_{L^{p_1} \to L^{p_2}} .... \|T_{\lambda_k}\|_{L^{p_k} \to L^{p_{k+1}}} \|T_{\lambda_{k+1}}\|_{L^{p_{k+1}} \to L^\infty}$$

$$\leqslant C \left(\prod_{j=1}^{k} |\lambda|^{-1+\frac{d}{2}\left(\frac{1}{p_j}-\frac{1}{p_{j+1}}\right)} \right) |\lambda|^{-1+\frac{d}{2p_{k+1}}} = C|\lambda|^{\frac{d}{4}-(k+1)}.$$

The conclusions now follow from Lemma 3. □

The following is the main result of this section and plays a crucial role in our analysis.

Proposition 2. *Let $k = k_d = \left[\frac{d}{2}\right]$ and denote*

$$\theta_j = \left(\frac{1}{4} + 2(j-1)\right)\frac{\pi}{k+1} \;\; \text{and} \quad \theta_{k+1+j} = \left(\frac{5}{4} + 2(j-1)\right)\frac{\pi}{k+1} \qquad \text{for } 1 \leqslant j \leqslant k+1.$$

Let $t > 10\Lambda_0$ and set $\mu_j = \lambda_0 + t z_j$ with $z_j = e^{i\theta_j}$ for $j = 1, \ldots, 2(k+1)$. We have

$$trace\ (T_{\mu_{2(k+1)}} \circ T_{\mu_{2k+1}} \circ ... \circ T_{\mu_1}) = \sum_j \frac{1}{\widetilde{\lambda}_j^{2(k+1)} - it^{2(k+1)}}, \tag{4.20}$$

where each characteristic value $\widetilde{\lambda}_j$ *of* T_{λ_0} *is repeated a number of times equal to its multiplicity.*

Remark 10. In Proposition 2, Λ_0 is chosen large and corresponds with $\varepsilon_0 = \frac{\pi}{8(k+1)}$.

Proof. It is clear that $z_1, \dots, z_{k+1}$ are the solutions of $z^{k+1} - e^{i\frac{\pi}{4}} = 0$ in $\mathbb{C}$ and $z_{k+2}, \dots, z_{2(k+1)}$ are the solutions of $z^{k+1} - e^{i\frac{5\pi}{4}} = 0$ in $\mathbb{C}$. One then has, for $z \in \mathbb{C}$,

$$\prod_{j=1}^{k+1}(z - z_j) = z^{k+1} - e^{i\frac{\pi}{4}}, \quad \prod_{j=1}^{k+1}(1 - z_j z) = 1 - e^{i\frac{\pi}{4}} z^{k+1} \tag{4.21}$$

and

$$\prod_{j=k+2}^{2(k+1)}(z - z_j) = z^{k+1} - e^{i\frac{5\pi}{4}}, \quad \prod_{j=k+2}^{2(k+1)}(1 - z_j z) = 1 - e^{i\frac{5\pi}{4}} z^{k+1}. \tag{4.22}$$

Note that, if T_λ and $T_{\lambda+s}$ exist, and T_λ is compact, then s is not a characteristic value of T_λ, and

$$T_{\lambda+s} = T_\lambda(I - sT_\lambda)^{-1} = (I - sT_\lambda)^{-1}T_\lambda. \tag{4.23}$$

Indeed, if T_λ and $T_{\lambda+s}$ exist, one can check that $I - sT_\lambda$ is injective, and therefore subjective since T_λ is compact. One can then show that (4.23) holds.

As a consequence of (4.23), $T_{\lambda+s}$ is the modified operator of T_λ with respect to s.

Set

$$\mathbf{T} = T_{\mu_{2(k+1)}} \circ ... \circ T_{\mu_{k+2}}.$$

It follows from Proposition 1 that $\mathbf{T}$ is Hilbert-Schmidt, and

$$|||\mathbf{T}||| \leqslant Ct^{\frac{d}{4}-1-k}, \quad \text{and} \quad \|\mathbf{T}\|_{L^2 \to L^2} \leqslant Ct^{-k-1}.$$

Let $s_1, s_2, \dots$ be the characteristic values of $\mathbf{T}$ repeated a number of times equal to their multiplicities. Thanks to [2, Theorem 12.17], one has, for a non-characteristic value λ of $\mathbf{T}$,

$$\text{trace }(\mathbf{T} \circ (\mathbf{T})_\lambda) = \sum_j \frac{1}{s_j(s_j - \lambda)} + c_t, \tag{4.24}$$

where $(\mathbf{T})_\lambda$ is the modified operator associated with $\mathbf{T}$ and λ, i.e., $(\mathbf{T})_\lambda := \mathbf{T}(I - \lambda\mathbf{T})^{-1}$, for some $c_t \in \mathbb{C}$.

By applying (4.24) with $\lambda = 2e^{\frac{i\pi}{4}} t^{k+1}$, it suffices to establish

$$(\mathbf{T})_\lambda = T_{\mu_{k+1}} \circ \ldots \circ T_{\mu_1} \text{ for } \lambda = 2e^{\frac{i\pi}{4}} t^{k+1}, \tag{4.25}$$

$$s_j = \widetilde{\lambda}_\ell^{k+1} - e^{i\frac{5\pi}{4}} t^{k+1}, \tag{4.26}$$

for some ℓ, and

the multiplicity of s_j is equal to the sum of the multiplicity of $\widetilde{\lambda}_\ell$ such that (4.26) holds, (4.27)

and

$$c_t = 0. \tag{4.28}$$

This will be done in the next three steps.

Step 1: Proof of (4.25). Since $\mu_j - \lambda_0 = z_j t$, it follows from the second identity in (4.21) that

$$\prod_{l=1}^{k+1} \big(1 - (\mu_l - \lambda_0) z\big) = 1 - e^{i\frac{\pi}{4}} t^{k+1} z^{k+1}. \tag{4.29}$$

One has

$$T_{\mu_{k+1}} \circ \ldots \circ T_{\mu_1} \overset{(4.23)}{=} T_{\lambda_0} \left(I - (\mu_{k+1} - \lambda_0) T_{\lambda_0}\right)^{-1} \circ \ldots \circ T_{\lambda_0} \left(I - (\mu_1 - \lambda_0) T_{\lambda_0}\right)^{-1}$$

$$\overset{(4.23)}{=} T_{\lambda_0}^{k+1} \prod_{l=1}^{k+1} (I - (\mu_l - \lambda_0) T_{\lambda_0})^{-1} \overset{(4.29)}{=} T_{\lambda_0}^{k+1} \left(I - e^{i\frac{\pi}{4}} t^{k+1} T_{\lambda_0}^{k+1}\right)^{-1}. \tag{4.30}$$

In other words, we have

$$T_{\mu_{k+1}} \circ \ldots \circ T_{\mu_1} = \left(T_{\lambda_0}^{k+1}\right)_{e^{i\frac{\pi}{4}} t^{k+1}}. \tag{4.31}$$

Similarly, we obtain

$$\mathbf{T} = T_{\mu_{2(k+1)}} \circ \ldots \circ T_{\mu_{k+2}} = \left(T_{\lambda_0}^{k+1}\right)_{e^{i\frac{5\pi}{4}} t^{k+1}}. \tag{4.32}$$

Using the property

$$\Big(\left(T_{\lambda_0}^{k+1}\right)_{\gamma_1} \Big)_{\gamma_2} = \left(T_{\lambda_0}^{k+1}\right)_{\gamma_1 + \gamma_2},$$

for γ_1 and $\gamma_1 + \gamma_2$ non-characteristic values of $T_{\lambda_0}^{k+1}$, we derive from (4.31) and (4.32) that

$$(\mathbf{T})_\lambda = \left(T_{\lambda_0}^{k+1}\right)_{e^{i\frac{5\pi}{4}} t^{k+1} + \lambda} = \left(T_{\lambda_0}^{k+1}\right)_{e^{i\frac{\pi}{4}} t^{k+1}} = T_{\mu_{k+1}} \circ \ldots \circ T_{\mu_1},$$

and (4.25) follows.
Step 2: Proof of (4.26) and (4.27). Since $\mathbf{T} = (T_{\lambda_0}^{k+1})_{e^{i5\pi/4}t^{k+1}}$, it follows, see e.g. [2, Theorem 12.4], that s_j^{-1} is an eigenvalue of $\mathbf{T}$ that is not equal to $-e^{-i5\pi/4}t^{-(k+1)}$ if and only if $\frac{s_j^{-1}}{1+s_j^{-1}e^{i5\pi/4}t^{k+1}} = \frac{1}{s_j+e^{i5\pi/4}t^{k+1}}$ is an eigenvalue of $T_{\lambda_0}^{k+1}$ (or equivalently $s_j + e^{i5\pi/4}t^{k+1}$ is a characteristic value of $T_{\lambda_0}^{k+1}$), and they have the same multiplicity. One can check that $-e^{-i5\pi/4}t^{-(k+1)}$ is not an eigenvalue of $\mathbf{T}$. Assertions (4.26) and (4.27) follow.
Step 3: Proof of (4.28). For $z \in \mathcal{L}(\theta, 1)$ with $\inf_{n\in\mathbb{Z}} |\theta - n\pi| > \varepsilon_0$ and $|z|$ large enough, let $\tau_1, \cdots, \tau_{k+1}$ be the $k+1$ distinct roots in $\mathbb{C}$ of the equation $x^{k+1} = z$. Set

$$\eta_l = \lambda_0 + \tau_l \text{ for } 1 \leqslant l \leqslant k+1.$$

As in the proof of (4.30), one has

$$T_{\eta_{k+1}} \circ \cdots \circ T_{\eta_1} = T_{\lambda_0}^{k+1}\Big(I - zT_{\lambda_0}^{k+1}\Big)^{-1}.$$

It follows that

$$T_{\eta_{k+1}} \circ \cdots \circ T_{\eta_1} = \Big(T_{\lambda_0}^{k+1}\Big)_z.$$

Consider λ defined by $e^{i\frac{5\pi}{4}}t^{k+1} + \lambda = z$. We have, for large $|z|$,

$$|\text{trace } (\mathbf{T} \circ (\mathbf{T})_\lambda)| \overset{(4.3)}{\leqslant} \|\|\mathbf{T}\|\| \|\|\mathbf{T}_\lambda\|\| \overset{(4.19)}{\leqslant} C_t|z|^{\frac{d}{4}-1-k} \to 0 \quad \text{as} \quad |z| \to +\infty, \tag{4.33}$$

and

$$\left|\sum_j \frac{1}{s_j(s_j - \lambda)}\right| \leqslant \left(\sum_j |s_j|^{-2}\right)^{1/2} \left(\sum_j |s_j - \lambda|^{-2}\right)^{1/2}. \tag{4.34}$$

Applying [2, Theorem 12.14], we have

$$\sum_j |s_j|^{-2} \leqslant \|\|\mathbf{T}\|\| \leqslant C_t, \tag{4.35}$$

and applying [2, Theorems 12.4 and 12.14], we obtain

$$\sum_j |s_j - \lambda|^{-2} \leqslant \|\|\mathbf{T}_\lambda\|\| \overset{(4.19)}{\leqslant} C_t|z|^{\frac{d}{4}-1-k} \to 0 \quad \text{as} \quad |z| \to +\infty. \tag{4.36}$$

We derive from (4.34), (4.35), and (4.36) that

$$\sum_j \frac{1}{s_j(s_j-\lambda)} \to 0 \text{ as } |z| \to +\infty. \tag{4.37}$$

Combining (4.33) and (4.37) yields $c_t = 0$.

The proof is complete. □

Remark 11. Let λ_j be an eigenvalue of the transmission problem. Then $\lambda_j - \lambda$ and $\lambda_j - \hat{\lambda}$ are the characteristic values of T_λ and $T_{\hat{\lambda}}$ respectively, provided that T_λ and $T_{\hat{\lambda}}$ exist. Using (4.23) and applying [2, Theorem 12.4], one can show that the multiplicity of $\lambda_j - \lambda$ and the multiplicity of $\lambda_j - \hat{\lambda}$ are the same.

5. The Weyl law for eigenvalues of the transmission problem - Proof of Theorem 1

5.1. Approximation of the trace of the kernel and their applications

For $\lambda \in \mathcal{L}(\theta, 1)$ with $\theta \neq n\pi$ for all $n \in \mathbb{Z}$, and $x_0 \in \Omega$, set

$$\begin{array}{rcl} S_{j,\lambda,x_0} : L^2(\mathbb{R}^d) & \to & L^2(\mathbb{R}^d) \\ f_j & \mapsto & v_j, \end{array} \tag{5.1}$$

where $v_j \in H^1(\mathbb{R}^d)$ is the unique solution of

$$\operatorname{div}(A_j(x_0)\nabla v_j) - \lambda \Sigma_j(x_0) v_j = \Sigma_j(x_0) f_j \quad \text{ in } \mathbb{R}^d.$$

We also define

$$\begin{array}{rcl} S_{\lambda,x_0} : [L^2(\mathbb{R}^d)]^2 & \to & [L^2(\mathbb{R}^d)]^2 \\ (f_1, f_2) & \mapsto & (S_{1,\lambda,x_0} f_1, S_{2,\lambda,x_0} f_2). \end{array} \tag{5.2}$$

One then has

$$S_{j,\lambda,x_0} f(x) = \int_{\mathbb{R}^d} F_{j,\lambda}(x_0, x-y) f_j(y) dy,$$

where

$$F_{j,\lambda}(x_0, z) = -\frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \frac{e^{iz\xi}}{\Sigma_j(x_0)^{-1}\langle A_j(x_0)\xi, \xi\rangle + \lambda} d\xi \quad \text{for} \quad z \in \mathbb{R}^d.$$

By Lemma 1, we get, for $1 < p < +\infty$,

$$\|\nabla^2 S_{j,\lambda,x_0} f_j\|_{L^p(\mathbb{R}^d)} + |\lambda|^{1/2} \|\nabla S_{j,\lambda,x_0} f_j\|_{L^p(\mathbb{R}^d)} + |\lambda| \|S_{j,\lambda,x_0} f_j\|_{L^p(\mathbb{R}^d)} \leqslant C \|f_j\|_{L^p(\mathbb{R}^d)}. \tag{5.3}$$

As in the proof of Theorem 3, we obtain from the interpolation inequalities (3.32) and (3.33) that

$$\|S_{\lambda,x_0}\|_{L^p\to L^\infty} \leqslant C|\lambda|^{-1+\frac{d}{2p}} \quad \text{if} \quad p > d, \tag{5.4}$$

$$\|S_{\lambda,x_0}\|_{L^p\to L^q} \leqslant C|\lambda|^{-1+\frac{d}{2}\left(\frac{1}{p}-\frac{1}{q}\right)} \quad \text{if} \quad 1 < p < d, \quad p \leqslant q < \frac{dp}{d-p}, \tag{5.5}$$

$$\|S_{\lambda,x_0}\|_{L^1\to L^q} \leqslant C|\lambda|^{-1+\frac{d}{2}-\frac{d}{2q}} \quad \text{if} \quad 1 < q < \frac{d}{d-1}. \tag{5.6}$$

Let $t > 10\Lambda_0$ and let $\mu_1, \dots, \mu_{2(k+1)}$ be defined in Proposition 2. Set, for $z \in \mathbb{R}^d$,

$$\mathcal{F}_{j,t}(x_0, z) = \frac{1}{(2\pi)^d} \int\limits_{\mathbb{R}^d} \frac{e^{iz\xi}}{\prod_{l=1}^{2(k+1)} \left(\Sigma_j(x_0)^{-1}\langle A_j(x_0)\xi, \xi\rangle + \mu_l\right)} \, d\xi, \tag{5.7}$$

and define

$$\mathcal{S}_{j,t,x_0} = \prod_{l=1}^{2(k+1)} S_{j,\mu_l,x_0}.$$

Then

$$\mathcal{S}_{j,t,x_0} f_j(x) = \int\limits_{\mathbb{R}^d} \mathcal{F}_{j,t}(x_0, x-y) f_j(y) dy.$$

Since, by the definition of μ_l and z_l,

$$\prod_{l=1}^{2(k+1)} \left(\Sigma_j(x_0)^{-1}\langle A_j(x_0)\xi, \xi\rangle + \mu_l\right) = \prod_{l=1}^{2(k+1)} \left(\Sigma_j(x_0)^{-1}\langle A_j(x_0)\xi, \xi\rangle + \lambda_0 + tz_l\right)$$
$$= \left(\Sigma_j(x_0)^{-1}\langle A_j(x_0)\xi, \xi\rangle + \lambda_0\right)^{2(k+1)} - it^{2(k+1)},$$

it follows from (5.7) that

$$\mathcal{F}_{j,t}(x_0, z) = \frac{1}{(2\pi)^d} \int\limits_{\mathbb{R}^d} \frac{e^{iz\xi}}{(\Sigma_j^{-1}(x_0)\langle A_j(x_0)\xi, \xi\rangle + \lambda_0)^{2(k+1)} - it^{2(k+1)}} \, d\xi. \tag{5.8}$$

As a consequence of (5.8), we obtain, by a change of variables,

$$\mathcal{F}_{j,t}(x_0, 0) = \frac{1}{(2\pi)^d} \int\limits_{\mathbb{R}^d} \frac{1}{\left(\Sigma_j^{-1}(x_0)\langle A_j(x_0)\xi, \xi\rangle + \lambda_0\right)^{2(k+1)} - it^{2(k+1)}} \, d\xi$$

$$= \frac{t^{\frac{d}{2}-2(k+1)}}{(2\pi)^d} \int\limits_{\mathbb{R}^d} \frac{1}{\left(\Sigma_j^{-1}(x_0)\langle A_j(x_0)\xi, \xi\rangle + t^{-1}\lambda_0\right)^{2(k+1)} - i}\, d\xi. \quad (5.9)$$

This implies, by the dominated convergence theorem,

$$\mathcal{F}_{j,t}(x_0, 0) = \frac{t^{\frac{d}{2}-2(k+1)}}{(2\pi)^d} \int\limits_{\mathbb{R}^d} \frac{1}{\left(\Sigma_j^{-1}(x_0)\langle A_j(x_0)\xi, \xi\rangle\right)^{2(k+1)} - i}\, d\xi + \mathcal{O}(t^{\frac{d}{2}-2(k+1)-1}). \quad (5.10)$$

We next introduce $\mathcal{S}_{t,x_0} : [L^2(\mathbb{R}^d)]^2 \to [L^2(\mathbb{R}^d)]^2$ by

$$\mathcal{S}_{t,x_0} = S_{\mu_{2(k+1)},x_0} \circ \cdots \circ S_{\mu_1,x_0}, \quad \text{where} \quad S_{\mu_l,x_0} = \begin{pmatrix} S_{1,\mu_l,x_0} & 0 \\ 0 & S_{2,\mu_l,x_0} \end{pmatrix}.$$

Set

$$\mathcal{F}_t(x_0, \cdot) = \begin{pmatrix} \mathcal{F}_{1,t}(x_0, \cdot) & 0 \\ 0 & \mathcal{F}_{2,t}(x_0, \cdot) \end{pmatrix}.$$

We then have

$$\mathcal{S}_{t,x_0} f(x) = \int\limits_{\mathbb{R}^d} \mathcal{F}_t(x_0, x - y) f(y) dy.$$

Let $\mathcal{K}_t$ denote the kernel corresponding to $\prod_{l=1}^{2(k+1)} T_{\mu_l} = T_{\mu_{2(k+1)}} \circ \cdots \circ T_{\mu_1}$. Here is the main result of this section.

Proposition 3. *We have*

$$\int\limits_{\Omega} \mathit{trace}\ \mathcal{K}_t(x, x) dx = \hat{\mathbf{c}} t^{\frac{d}{2}-2(k+1)} + o(t^{\frac{d}{2}-2(k+1)}) \quad \mathit{as} \quad t \to \infty, \quad (5.11)$$

where

$$\hat{\mathbf{c}} = \frac{1}{(2\pi)^d} \sum_{j=1}^{2} \int\limits_{\Omega} \int\limits_{\mathbb{R}^d} \frac{1}{(\Sigma_j^{-1}(x)\langle A_j(x)\xi, \xi\rangle)^{2(k+1)} - i}\, d\xi\, dx. \quad (5.12)$$

Proof. We claim that

$$\int\limits_{\Omega} \text{trace } \mathcal{K}_t(x, x) dx = \int\limits_{\Omega} \text{trace } \mathcal{F}_t(x, 0) dx + o(t^{\frac{d}{2}-2(k+1)}) \quad \text{as} \quad t \to \infty. \quad (5.13)$$

The conclusion then follows from (5.10).

The main point of the proof is to establish (5.13). Let φ be a function in $C^\infty(\mathbb{R}^d)$ such that $0 \leqslant \varphi \leqslant 1$, $\varphi = 1$ in $B_{\frac{1}{2}}$ and $\operatorname{supp} \varphi \subset B_{\frac{1}{2}+\frac{1}{100d}}$. Let $\delta_0 > 0$ and $x_0 \in \Omega$ be such that $\operatorname{dist}(x_0, \partial\Omega) > \delta_0$. For $\delta \in (0, 10^{-2}\delta_0)$, set $\varphi_\delta(x) = \varphi(\delta^{-1}(x - x_0))$ and

$$\Phi(\delta, x_0) = \sup_{B_{10\delta}(x_0)} \sum_{j=1}^{2} \left(|A_j(x) - A_j(x_0)| + |\Sigma_j(x) - \Sigma_j(x_0)|\right).$$

The essential ingredient of the analysis is the following estimate, for $t > \delta^{-4}$:

$$\|\varphi_{2\delta} \left(T_{\mu_{2(k+1)}} \circ ... \circ T_{\mu_1} - \mathcal{S}_{t,x_0}\right) \varphi_\delta\|_{L^1 \to L^\infty} \leqslant C_{\delta_0} \left(\Phi(\delta, x_0) + \delta^{-1} t^{-1/2}\right) t^{\frac{d}{2} - 2(k+1)}. \tag{5.14}$$

We first assume (5.14) and continue the proof. We have

$$\begin{aligned}&\Big(\varphi_{2\delta} \left(T_{\mu_{2(k+1)}} \circ ... \circ T_{\mu_1} - \mathcal{S}_{t,x_0}\right) \varphi_\delta\Big)(f)(x) \\ &\quad = \varphi_{2\delta}(x) \int_\Omega \Big(\mathcal{K}_t(x, y) - \mathcal{F}_t(x_0, x - y)\Big) \varphi_\delta(y) f(y) dy.\end{aligned}$$

It follows from (5.14) that, for $x, y \in \Omega$ and for $t > \delta^{-4}$,

$$\Big|\varphi_{2\delta}(x)\varphi_\delta(y)\big(\mathcal{K}_t(x, y) - \mathcal{F}_t(x_0, x - y)\big)\Big| \leqslant C_{\delta_0} \left(\Phi(\delta, x_0) + \delta^{-1} t^{-1/2}\right) t^{\frac{d}{2} - 2(k+1)}.$$

This implies that, for $t > \delta^{-4}$,

$$|\operatorname{trace} \mathcal{K}_t(x_0, x_0) - \operatorname{trace} \mathcal{F}_t(x_0, 0)| \leqslant C_{\delta_0} \left(\Phi(\delta, x_0) + \delta^{-1} t^{-1/2}\right) t^{\frac{d}{2} - 2(k+1)}. \tag{5.15}$$

Here we used the fact that $\varphi_{2\delta}(x_0) = \varphi_\delta(x_0) = 1$. Using Proposition 1 and (4.9)-(4.10), we have

$$|\mathcal{K}_t(x, x)| \leqslant C t^{\frac{d}{2} - 2(k+1)} \text{ for } x \in \Omega. \tag{5.16}$$

By (5.10), we obtain

$$|\mathcal{F}_t(x, 0)| \leqslant C t^{\frac{d}{2} - 2(k+1)} \text{ for } x \in \Omega. \tag{5.17}$$

Assertion (5.13) now follows from (5.15), (5.16), and (5.17) by noting that $\sup_{x \in \Omega} \Phi(\delta, x) \to 0$ as $\delta \to 0$.

It remains to prove (5.14). We have

$$\begin{aligned}&\varphi_{2\delta} \left(T_{\mu_{2(k+1)}} \circ \cdots \circ T_{\mu_1} - \mathcal{S}_{t,x_0}\right) \varphi_\delta \\ &\quad = \sum_{l=1}^{2(k+1)} \varphi_{2\delta} \left(T_{\mu_{2(k+1)}} \circ \dots T_{\mu_{l+1}} \circ (T_{\mu_l} - S_{\mu_l, x_0}) \circ S_{\mu_{l-1}, x_0} \cdots \circ S_{\mu_1, x_0}\right) \varphi_\delta.\end{aligned} \tag{5.18}$$

Fix $\beta_0 = 1 < \beta_1 < ... < \beta_{2k-1} < \beta_{2(k+1)} = 2$ with $\beta_{l+1} - \beta_l > 1/(10d)$. Set

$$S_{\mu_l, x_0, 1} = \varphi_{\beta_l \delta} S_{\mu_l, x_0}, \text{ and } \quad S_{\mu_l, x_0, 2} = (1 - \varphi_{\beta_l \delta}) S_{\mu_l, x_0}.$$

Then

$$\left(S_{\mu_{l-1}, x_0} \circ \cdots \circ S_{\mu_1, x_0}\right) \varphi_\delta = \left((S_{\mu_{l-1}, x_0, 1} + S_{\mu_{l-1}, x_0, 2}) \circ \cdots \circ (S_{\mu_1, x_0, 1} + S_{\mu_1, x_0, 2})\right) \varphi_\delta.$$

Since $\varphi_{\beta_{l-1}\delta} = \varphi_{\beta_l \delta} \varphi_{\beta_{l-1}\delta}$, it follows from (5.18) that

$$\begin{aligned} &\varphi_{2\delta} \left(T_{\mu_{2(k+1)}} \circ \cdots \circ T_{\mu_1} - \mathcal{S}_{t, x_0}\right) \varphi_\delta \\ &= \sum_{l=1}^{2(k+1)} \varphi_{2\delta} \left(T_{\mu_{2(k+1)}} \circ \cdots \circ T_{\mu_{l+1}}\right) \circ \left((T_{\mu_l} - S_{\mu_l, x_0}) \varphi_{\beta_l \delta}\right) \\ &\quad \circ \left(S_{\mu_{l-1}, x_0, 1} \circ \cdots \circ S_{\mu_1, x_0, 1} \varphi_\delta\right) \\ &+ \sum_{l=1}^{2(k+1)} \varphi_{2\delta} \left(T_{\mu_{2(k+1)}} \circ \cdots \circ T_{\mu_{l+1}}\right) \circ \left(T_{\mu_l} - S_{\mu_l, x_0}\right) \\ &\quad \circ \left(S_{\mu_{l-1}, x_0} \circ \cdots \circ S_{\mu_1, x_0} - S_{\mu_{l-1}, x_0, 1} \circ \cdots \circ S_{\mu_1, x_0, 1}\right) \varphi_\delta. \end{aligned} \tag{5.19}$$

Let $p_1 = 1 < p_2 < ... < p_{2k} < d < p_{2(k+1)} < +\infty$ be such that $p_{l+1} < p_l d/(d - p_l)$ for $1 \leqslant l \leqslant 2k+1$. Using the exponential decay property: for $\gamma > 1$, $r > 0$, $y \in \mathbb{R}^d$, and for f with $\operatorname{supp} f \subset B_r$, it holds, for $t > r^{-3/2}$

$$\|S_{\mu_l} f\|_{L^\infty(\Omega \setminus B_{\gamma r}(y))} \leqslant C_\gamma e^{-c_\gamma r t} \|f\|_{L^q(B_r(y))}, \tag{5.20}$$

one has for $l = 2, ..., 2(k+2)+1$,

$$\| \left(S_{\mu_{l-1}, x_0} \cdots \circ S_{\mu_1, x_0}\right) \varphi_\delta - \left(S_{\mu_{l-1}, x_0, 1} \cdots \circ S_{\mu_1, x_0, 1}\right) \varphi_\delta \|_{L^1 \to L^{p_{l+1}}} \leqslant C e^{-c\delta t}. \tag{5.21}$$

Combining (5.19)-(5.21), and using (4.16)-(4.18) for T_{μ_l}, and (5.4)-(5.6) for S_{μ_l}, it suffices to prove that

$$\|\varphi_{2\delta} \left(T_{\mu_{2(k+1)}} - S_{\mu_{2(k+1)}, x_0}\right) \|_{L^{p_{2(k+1)}} \to L^\infty} \leqslant C_{\delta_0} \Big(\Phi(\delta, x_0) + t^{-1/2} \delta^{-1}\Big) t^{-1 + \frac{d}{2 p_{2(k+1)}}}, \tag{5.22}$$

and for $l = 1, 2, ..., 2k+1$,

$$\| \left(T_{\mu_l} - S_{\mu_l, x_0}\right) \varphi_{\beta_l \delta} \|_{L^{p_l} \to L^{p_{l+1}}} \leqslant C_{\delta_0} \left(\Phi(\delta, x_0) + \delta^{-1} t^{-1/2}\right) t^{-1 + \frac{d}{2}\left(\frac{1}{p_l} - \frac{1}{p_{l+1}}\right)}. \tag{5.23}$$

Step 1: Proof of (5.22). We will prove the following stronger result, which will be used in the proof of (5.23): for $\lambda \in \mathcal{L}(\theta, \Lambda_0)$ and $\sup_{n \in \mathbb{Z}} |\theta - n\pi| > \varepsilon_0$, $\beta \in [1, 2]$; and for $1 < p < d$, $p \leqslant q < \frac{pd}{d-p}$ or for $d > p$ and $q = +\infty$:

$$\|\varphi_{\beta\delta}\left(T_\lambda - S_{\lambda,x_0}\right)\|_{L^p\to L^q} \leqslant C_{\delta_0,\varepsilon_0}\Big(\Phi(\delta,x_0) + |\lambda|^{-1/2}\delta^{-1}\Big)|\lambda|^{-1+\frac{d}{2}\left(\frac{1}{p}-\frac{1}{q}\right)}. \tag{5.24}$$

Denote

$$u = T_\lambda(f) \qquad \text{and} \qquad v = S_{\lambda,x_0}f.$$

Set

$$u_{j,\delta} = \varphi_{\beta\delta}u_j \quad \text{and} \quad v_{j,\delta} = \varphi_{\beta\delta}v_j.$$

Since, in Ω,

$$\operatorname{div}(A_j\nabla u_j) - \lambda\Sigma_j u_j = \Sigma_j f_j,$$

and

$$\operatorname{div}(A_j(x_0)\nabla v_j) - \lambda\Sigma_j(x_0)v_j = \Sigma_j(x_0)f_j,$$

we have, in Ω,

$$\operatorname{div}(A_j(x_0)\nabla u_{j,\delta}) - \lambda\Sigma_j(x_0)u_{j,\delta} = f_{j,\delta} \quad \text{and} \quad \operatorname{div}(A_j(x_0)\nabla u_{j,\delta}) - \lambda\Sigma_j(x_0)u_{j,\delta} = g_{j,\delta},$$

where

$$f_{j,\delta} = \widetilde{f}_{j,\delta} + \operatorname{div}\widetilde{F}_{j,\delta} \quad \text{and} \quad g_{j,\delta} = \widetilde{g}_{j,\delta} + \operatorname{div}\widetilde{G}_{j,\delta},$$

with

$$\begin{gathered}\widetilde{f}_{j,\delta} = \varphi_{\beta\delta}\Sigma_j f_j + A_j\nabla u_j\nabla\varphi_{\beta\delta} - \lambda\big(\Sigma_j(x_0) - \Sigma_j(x)\big)u_{j,\delta},\\ \widetilde{F}_{j,\delta} = u_jA_j\nabla\varphi_{\beta\delta} + \big(A_j(x_0) - A_j(x)\big)\nabla u_{j,\delta},\\ \widetilde{g}_{j,\delta} = \varphi_{\beta\delta}\Sigma_j(x_0)f_j + A_j(x_0)\nabla v_j\nabla\varphi_{\beta\delta}, \quad \text{and} \quad \widetilde{G}_{j,\delta} = v_jA_j(x_0)\nabla\varphi_{\beta\delta}.\end{gathered}$$

By Theorem 3 and (5.3), we have for $1 < p < +\infty$,

$$\|\nabla u\|_{L^p(\Omega)} + \|\nabla v\|_{L^p(\mathbb{R}^d)} + |\lambda|^{1/2}\left(\|u\|_{L^p(\Omega)} + \|v\|_{L^p(\mathbb{R}^d)}\right) \leqslant C|\lambda|^{-\frac{1}{2}}\|f\|_{L^p(\Omega)}. \tag{5.25}$$

By Lemma 1, we obtain for $1 < p < +\infty$,

$$\begin{aligned}&|\lambda|^{1/2}\|\nabla(u_{j,\delta} - v_{j,\delta})\|_{L^p(\mathbb{R}^d)} + |\lambda|\|u_{j,\delta} - v_{j,\delta}\|_{L^p(\mathbb{R}^d)}\\ &\qquad\leqslant C\Big(\|\widetilde{f}_{j,\delta} - \widetilde{g}_{j,\delta}\|_{L^p(\mathbb{R}^d)} + |\lambda|^{1/2}\|\widetilde{F}_{j,\delta} - \widetilde{G}_{j,\delta}\|_{L^p(\mathbb{R}^d)}\Big).\end{aligned} \tag{5.26}$$

Using (5.25), we derive, for $1 < p < +\infty$, that, with $u_\delta = (u_{1,\delta}, u_{2,\delta})$ and $v_\delta = (v_{1,\delta}, v_{2,\delta})$,

$$\|\nabla(u_\delta - v_\delta)\|_{L^p(\mathbb{R}^d)} + |\lambda|^{1/2}\|u_\delta - v_\delta\|_{L^p(\mathbb{R}^d)} \leqslant C_{\delta_0}\Big(\Phi(\delta, x_0) + |\lambda|^{-1/2}\delta^{-1}\Big)|\lambda|^{-1/2}\|f\|_{L^p(\Omega)}.$$

By Gagliardo-Nirenberg's interpolation inequalities, one gets that for $1 < p < d$ and $p \leqslant q \leqslant \frac{dp}{d-p}$

$$\|u_\delta - v_\delta\|_{L^q(\mathbb{R}^d)} \leqslant C_{\delta_0}\Big(\Phi(\delta, x_0) + |\lambda|^{-1/2}\delta^{-1}\Big)|\lambda|^{-1+\frac{d}{2}\left(\frac{1}{p}-\frac{1}{q}\right)}\|f\|_{L^p(\Omega)},$$

and for $p > d$,

$$\|u_\delta - v_\delta\|_{L^\infty(\Omega)} \leqslant C_{\delta_0}\Big(\Phi(\delta, x_0) + |\lambda|^{-1/2}\delta^{-1}\Big)|\lambda|^{-1+\frac{d}{2p}}\|f\|_{L^p(\Omega)},$$

and assertion (5.24) follows.
Step 2: Proof of (5.23). By (4.15),

$$T^\star_\lambda - S^\star_{\lambda,x_0} = \begin{pmatrix} \Sigma_1 & 0 \\ 0 & -\Sigma_2 \end{pmatrix} (T_{\overline{\lambda}} - S_{\overline{\lambda},x_0}) \begin{pmatrix} 1/\Sigma_1 & 0 \\ 0 & -1/\Sigma_2 \end{pmatrix},$$

it follows that by (5.24), for $\overline{\lambda} \in \mathcal{L}(\theta, \Lambda_0)$ with $\sup_{n\in\mathbb{Z}} |\theta - n\pi| > \varepsilon_0$, $\beta \in [1, 2]$; and for $1 < p < d$, $p \leqslant q < \frac{pd}{d-p}$ or for $d > p$ and $q = +\infty$:

$$\|\varphi_{\beta\delta}\left(T^*_\lambda - S^*_{\lambda,x_0}\right)\|_{L^p\to L^q} \leqslant C_{\delta_0,\varepsilon_0}\Big(\Phi(\delta, x_0) + |\lambda|^{-1/2}\delta^{-1}\Big)|\lambda|^{-1+\frac{d}{2}\left(\frac{1}{p}-\frac{1}{q}\right)},$$

which implies

$$\|\left(T_\lambda - S_{\lambda,x_0}\right)\varphi_{\beta\delta}\|_{L^{\frac{q}{q-1}}\to L^{\frac{p}{p-1}}} \leqslant C_{\delta_0,\varepsilon_0}\Big(\Phi(\delta, x_0) + |\lambda|^{-1/2}\delta^{-1}\Big)|\lambda|^{-1+\frac{d}{2}\left(\frac{q-1}{q}-\frac{p-1}{p}\right)}. \tag{5.27}$$

This gives (5.23). The proof is complete. □

As a consequence of Proposition 2 and Proposition 3, we obtain

Corollary 3. *We have*

$$\sum_j \frac{1}{|\widetilde{\lambda}_j|^{2(k+1)} - it^{2(k+1)}} = \hat{\mathbf{c}}t^{\frac{d}{2}-2(k+1)} + o(t^{\frac{d}{2}-2(k+1)}) \quad as \quad t \to \infty,$$

where each characteristic value $\widetilde{\lambda}_j$ of T_{λ_0} is repeated a number of times equal to its multiplicity, and $\hat{\mathbf{c}}$ is defined by (5.12).

Proof. By Propositions 2 and 3, and (4.12) in Lemma 4, we have

$$\sum_j \frac{1}{\widetilde{\lambda}_j^{2(k+1)} - it^{2(k+1)}} = \hat{\mathbf{c}} t^{\frac{d}{2}-2(k+1)} + o(t^{\frac{d}{2}-2(k+1)}) \quad \text{as} \quad t \to \infty.$$

For $j_0 \in \mathbb{N}$ large and for $t \geqslant 2|\widetilde{\lambda}_{j_0}|$, we have

$$\left| \sum_{j=1}^{\infty} \frac{1}{\widetilde{\lambda}_j^{2(k+1)} - t^{2(k+1)} i} - \sum_{j=1}^{\infty} \frac{1}{|\widetilde{\lambda}_j|^{2(k+1)} - t^{2(k+1)} i} \right|$$

$$\leqslant \sum_{j=1}^{j_0} \frac{1}{|\widetilde{\lambda}_j^{2(k+1)} - t^{2(k+1)} i|} + \frac{1}{||\widetilde{\lambda}_j|^{2(k+1)} - t^{2(k+1)} i|}$$

$$+ \sum_{j=j_0+1}^{\infty} \frac{C|\widetilde{\lambda}_j|^{2k+1} |\Im \widetilde{\lambda}_j|}{|\widetilde{\lambda}_j^{2(k+1)} - t^{2(k+1)} i| \, ||\widetilde{\lambda}_j|^{2(k+1)} - t^{2(k+1)} i|}$$

$$\leqslant 2 j_0 t^{-2(k+1)} + \left(\sup_{j \geqslant j_0} \frac{|\Im \widetilde{\lambda}_j|}{|\widetilde{\lambda}_j|} \right) \sum_{j=j_0+1}^{\infty} \frac{1}{(|\widetilde{\lambda}_j| + t)^{2(k+1)}}. \tag{5.28}$$

By [2, Theorem 12.14] and Theorem 3, we have

$$\sum_{j \geqslant j_0}^{\infty} (|\widetilde{\lambda}_j| + |t|)^{-2(k+1)} \leqslant C |||T_{-it}^{k+1}|||^2, \tag{5.29}$$

and by Proposition 1, we obtain

$$|||T_{-it}|||^2 \leqslant C t^{\frac{d}{2}-2(k+1)}. \tag{5.30}$$

Since, by Theorem 3,

$$\left(\sup_{j \geqslant j_0} \frac{|\Im \widetilde{\lambda}_j|}{|\widetilde{\lambda}_j|} \right) \to 0 \text{ as } j_0 \to +\infty,$$

it follows from (5.28), (5.29), and (5.30) that

$$\left| \sum_{j=1}^{\infty} \frac{1}{\widetilde{\lambda}_j^{2(k+1)} - t^{2(k+1)} i} - \sum_{j=1}^{\infty} \frac{1}{|\widetilde{\lambda}_j|^{2(k+1)} - t^{2(k+1)} i} \right| = \circ(1) t^{\frac{d}{2}-2(k+1)} \quad \text{as } t \to \infty.$$

The proof is complete. □

5.2. Proof of Theorem 1

Before giving the proof of Theorem 1, we recall a Tauberian theorem of Hardy and Littlewood, see e.g. [44, Theorem 2a] or [2, Theorem 14.5].

Lemma 5. *Let $\sigma(s)$ be a non-decreasing function for $s > 0$, let $a \in (0,1)$ and $P \geqslant 0$. Then, as $t \to \infty$,*

$$\int_0^\infty \frac{d\sigma(s)}{s+t} = Pt^{a-1} + \circ(t^{a-1}),$$

if and only if, as $s \to \infty$,

$$\sigma(s) = \frac{P}{a\int_0^\infty t^{a-1}(1+t)^{-1}dt} s^a + \circ(s^a).$$

We are ready to give

Proof of Theorem 1. We have, by Corollary 3,

$$\sum_{j=1}^\infty \frac{1}{|\widetilde{\lambda}_j|^{2(k+1)} - t^{2(k+1)}i} = \hat{\mathbf{c}} t^{\frac{d}{2}-2(k+1)} + o(t^{\frac{d}{2}-2(k+1)}) \quad \text{as} \quad t \to \infty,$$

where $\hat{\mathbf{c}}$ is given by (5.12):

$$\hat{\mathbf{c}} = \frac{1}{(2\pi)^d} \int_\Omega \sum_{j=1,2} \int_{\mathbb{R}^d} \frac{1}{(\Sigma_j(x)^{-1}\langle A_j(x)\xi, \xi\rangle)^{2(k+1)} - i} d\xi dx. \tag{5.31}$$

Considering the imaginary part yields,

$$\sum_{j=1}^\infty \frac{t^{2(k+1)}}{|\widetilde{\lambda}_j|^{4(k+1)} + t^{4(k+1)}} = \hat{\mathbf{c}}_1 t^{\frac{d}{2}-2(k+1)} + o(t^{\frac{d}{2}-2(k+1)}) \quad \text{as} \quad t \to +\infty,$$

where

$$\hat{\mathbf{c}}_1 = \Im(\hat{\mathbf{c}}) = \frac{1}{(2\pi)^d} \sum_{j=1,2} \int_\Omega \int_{\mathbb{R}^d} \frac{1}{(\Sigma_j(x)^{-1}\langle A_j(x)\xi, \xi\rangle)^{4(k+1)} + 1} d\xi dx.$$

This implies, by replacing $t^{4(k+1)}$ by t,

$$\sum_{j=1}^\infty \frac{1}{|\widetilde{\lambda}_j|^{4(k+1)} + t} = \hat{\mathbf{c}}_1 t^{\frac{d}{8(k+1)}-1} + o(t^{\frac{d}{8(k+1)}-1}) \quad \text{as} \quad t \to +\infty.$$

Since $\widetilde{\lambda}_j = \lambda_j - \lambda_0$, one obtains

$$\sum_{j=1}^{\infty} \frac{1}{|\lambda_j|^{4(k+1)} + t} = \hat{\mathbf{c}}_1 t^{\frac{d}{8(k+1)}-1} + o(t^{\frac{d}{8(k+1)}-1}) \quad \text{as} \quad t \to +\infty.$$

We can write this identity under the form

$$\int_0^{\infty} \frac{dN(s^{\frac{1}{4(k+1)}})}{s+t} = \hat{\mathbf{c}}_1 t^{\frac{d}{8(k+1)}-1} + o(t^{\frac{d}{8(k+1)}-1}) \quad \text{as} \quad t \to \infty.$$

By Lemma 5, one has

$$N(t) = \mathbf{c} t^{\frac{d}{2}} + \circ(t^{\frac{d}{2}}), \tag{5.32}$$

where

$$\mathbf{c} = \frac{\hat{\mathbf{c}}_1}{\frac{d}{8(k+1)} \int_0^{\infty} t^{\frac{d}{8(k+1)}-1}(1+t)^{-1} dt}.$$

We have, by Fubini's theorem,

$$\begin{aligned} &\int_{\mathbb{R}^d} \frac{1}{(\Sigma_j(x)^{-1}\langle A_j(x)\xi, \xi\rangle)^{4(k+1)} + 1} d\xi \\ &\qquad = \int_0^{\infty} \left|\left\{\xi : (\Sigma_j(x)^{-1}\langle A_j(x)\xi, \xi\rangle)^{4(k+1)} < t\right\}\right| \frac{dt}{(t+1)^2}. \end{aligned}$$

Since

$$\left|\left\{\xi : (\Sigma_j(x)^{-1}\langle A_j(x)\xi, \xi\rangle)^{4(k+1)} < t\right\}\right| = t^{\frac{d}{8(k+1)}} \left|\left\{\xi : \langle A_j(x)\xi, \xi\rangle < \Sigma_j(x)\right\}\right|,$$

it follows that

$$\begin{aligned} &\int_{\mathbb{R}^d} \frac{1}{(\Sigma_j(x)^{-1}\langle A_j(x)\xi, \xi\rangle)^{4(k+1)} + 1} d\xi \\ &\quad = \left|\left\{\xi : \langle A_j(x)\xi, \xi\rangle < \Sigma_j(x)\right\}\right| \int_0^{\infty} \frac{t^{\frac{d}{8(k+1)}} dt}{(t+1)^2} \\ &\quad = \left|\left\{\xi : \langle A_j(x)\xi, \xi\rangle < \Sigma_j(x)\right\}\right| \frac{d}{8(k+1)} \int_0^{\infty} t^{\frac{d}{8(k+1)}-1}(1+t)^{-1} dt. \end{aligned}$$

Here in the last identity, an integration by parts is used. We therefore have

$$\mathbf{c} = \frac{1}{(2\pi)^d} \sum_{j=1,2} \int_\Omega \Big| \Big\{ \xi : \langle A_j(x)\xi, \xi \rangle < \Sigma_j(x) \Big\} \Big| dx.$$

The proof is complete. □

6. Completeness of generalized eigenfunctions of the transmission problem - Proof of Theorem 2

Fix $\varepsilon_0 > 0$. For $z \in \mathcal{L}(\theta, 1)$ with $\inf_{n\in\mathbb{Z}} |\theta - n\pi| > \varepsilon_0$ and $|z|$ large enough, let $\tau_1, \cdots, \tau_{k+1}$ with $k = k_d = [d/2]$ be the $k+1$ distinct roots in $\mathbb{C}$ of the equation $x^{k+1} = z$. Set

$$\eta_l = \lambda_0 + \tau_l \text{ for } 1 \leqslant l \leqslant k+1.$$

As in the proof of (4.30), one has

$$T_{\eta_{k+1}} \circ \cdots \circ T_{\eta_1} = T_{\lambda_0}^{k+1} \Big(I - z T_{\lambda_0}^{k+1} \Big)^{-1}.$$

It follows that

$$T_{\eta_{k+1}} \circ \cdots \circ T_{\eta_1} = \Big(T_{\lambda_0}^{k+1} \Big)_z.$$

Since $T_{\lambda_0}^{k+1}$ is a Hilbert-Schmidt operator, it follows from [2, Theorem 16.4] that:

1) the space spanned by the general eigenfunctions of $T_{\lambda_0}^{k+1}$ is equal to $\overline{\mathbf{R}(T_{\lambda_0}^{k+1})}$, the closure of the range of $T_{\lambda_0}^{k+1}$ with respect to the L^2-topology.

On the other hand, we have

2) the range $\mathbf{R}(T_{\lambda_0}^{k+1})$ of $T_{\lambda_0}^{k+1}$ is dense in $[L^2(\Omega)]^2$, since $\mathbf{R}(T_{\lambda_0})$ is dense in $[L^2(\Omega)]^2$ and T_{λ_0} is continuous,

3) the space spanned by the general eigenfunctions of $T_{\lambda_0}^{k+1}$ associated to the non-zero eigenvalues of $T_{\lambda_0}^{k+1}$ is equal to the space spanned by the general eigenfunctions of T_{λ_0} associated to the non-zero eigenvalues of T_{λ_0}. This can be done as in the last part of the proof of [2, Theorem 16.5]. Consequently, the space spanned by all generalized eigenfunctions of $T_{\lambda_0}^{k+1}$ is equal to the space spanned by all generalized eigenfunctions of T_{λ_0}.

The conclusion now follows from 1), 2), and 3). □

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