ON THE SMALL-TIME LOCAL CONTROLLABILITY OF A KDV SYSTEM FOR CRITICAL LENGTHS

JEAN-MICHEL CORON, ARMAND KOENIG, AND HOAI-MINH NGUYEN

Abstract. This paper is devoted to the local null-controllability of the nonlinear KdV equation equipped the Dirichlet boundary conditions using the Neumann boundary control on the right. Rosier proved that this KdV system is small-time locally controllable for all non-critical lengths and that the uncontrollable space of the linearized system is of finite dimension when the length is critical. Concerning critical lengths, Coron and Crépeau showed that the same result holds when the uncontrollable space of the linearized system is of dimension 1, and later Cerpa, and then Cerpa and Crépeau established that the local controllability holds at a finite time for all other critical lengths. In this paper, we prove that, for a class of critical lengths, the nonlinear KdV system is *not* small-time locally controllable.

Key words. Controllability, nonlinearity, Korteweg–de Vries
AMS subject classification. 93B05, 93C15, 76B15.

Contents

1. Introduction 1
1.1. Bibliography 2
1.2. Statement of the result 2
1.3. Ideas of the analysis 6
1.4. Structure of the paper 7
2. Properties of controls steering 0 at time 0 to 0 at time T 7
3. Attainable directions for small time 12
4. Useful estimates for the linear KdV equations 22
4.1. On the linear KdV-Burgers equations 22
4.2. On the linear KdV equations 26
5. Small time local null-controllability of the KdV system 30
6. Controllability of the KdV system with controls in H^1 37
Appendix A. On symmetric functions of the roots of a polynomial 45
Appendix B. On the real roots of H, the common roots of G and H, and the behavior of $|\det Q|$ 46
References 50

1. Introduction

We are concerned about the local null-controllability of the (nonlinear) KdV equation equipped the Dirichlet boundary conditions using the Neumann boundary control on the right. More precisely, given $L > 0$ and $T > 0$, we consider the following control system

$$\begin{cases} y_t(t,x) + y_x(t,x) + y_{xxx}(t,x) + y(t,x)y_x(t,x) = 0 & \text{for } t \in (0,T),\ x \in (0,L), \\ y(t, x=0) = y(t, x=L) = 0 & \text{for } t \in (0,T), \\ y_x(t, x=L) = u(t) & \text{for } t \in (0,T), \end{cases} \tag{1.1}$$

and

$$y(t = 0, x) = y_0(x) \text{ for } x \in (0, L). \tag{1.2}$$

Here y is the state, y_0 is the initial data, and u is the control. More precisely, we are interested in the *small-time local controllability* property of this system.

The KdV equation has been introduced by Boussinesq [15] and Korteweg and de Vries [30] as a model for propagation of surface water waves along a channel. This equation also furnishes a very useful nonlinear approximation model including a balance between a weak nonlinearity and weak dispersive effects. The KdV equation has been intensively studied from various aspects of mathematics, including the well-posedness, the existence and stability of solitary waves, the integrability, the long-time behavior, etc., see e.g. [46, 33, 29, 44, 31].

1.1. Bibliography.

The controllability properties of system (1.1) and (1.2) (or of its variants) has been studied intensively, see e.g. the surveys [40, 19] and the references therein. Let us briefly review the existing results on (1.1) and (1.2). For initial and final datum in $L^2(0, L)$ and controls in $L^2(0, T)$, Rosier [38] proved that the system is small-time locally controllable around 0 provided that the length L is not critical, i.e., $L \notin \mathcal{N}$, where

$$\mathcal{N} := \left\{ 2\pi \sqrt{\frac{k^2 + kl + l^2}{3}};\ k, l \in \mathbb{N}_* \right\}. \tag{1.3}$$

To this end, he studied the controllability of the linearized system using the Hilbert Uniqueness Method and compactness-uniqueness arguments. Rosier also showed that the linearized system is controllable if $L \notin \mathcal{N}$. He as well established that when $L \in \mathcal{N}$, the linearized system is not controllable. More precisely, he showed that there exists a non-trivial finite-dimensional subspace $\mathcal{M}$ of $L^2(0, L)$ such that its orthogonal space is reachable from 0 whereas $\mathcal{M}$ is not.

To tackle the control problem for the critical length $L \in \mathcal{N}$ with initial and final datum in $L^2(0, L)$ and controls in $L^2(0, T)$, Coron and Crépeau introduced the power series expansion method [24]. The idea is to take into account the effect of the nonlinear term yy_x absent in the linearized system. Using this method, they showed [24] (see also [22, section 8.2]) that system (1.1) and (1.2) is small-time locally controllable if $L = m2\pi$ for $m \in \mathbb{N}_*$ satisfying

$$\nexists (k, l) \in \mathbb{N}_* \times \mathbb{N}_* \text{ with } k^2 + kl + l^2 = 3m^2 \text{ and } k \neq l. \tag{1.4}$$

In this case, $\dim \mathcal{M} = 1$ and $\mathcal{M}$ is spanned by $1 - \cos x$. Cerpa [18] developed the analysis in [24] to prove that system (1.1) and (1.2) is locally controllable at *a finite time* in the case $\dim \mathcal{M} = 2$. This corresponds to the case where

$$L = 2\pi \sqrt{\frac{k^2 + kl + l^2}{3}}$$

for some $k, l \in \mathbb{N}_*$ with $k > l$, and there is no $m, n \in N_*$ with $m > n$ and $m^2 + mn + n^2 = k^2 + kl + l^2$. Later, Crépeau and Cerpa [20] succeeded to extend the ideas in [18] to obtain the local controllability for all other critical lengths at *a finite time*. To summarize, concerning the critical lengths with initial and final datum in $L^2(0, L)$ and controls in $L^2(0, T)$, the small-time local controllability is valid when $\dim \mathcal{M} = 1$ and local controllability in a large enough time holds when $\dim \mathcal{M} \geq 2$.

1.2. Statement of the result.

The control properties of the KdV equations have been intensively studied previously but the following natural question remains open (see [23, Open problem 10], [18, Remark 1.7]):

Open problem 1.1. *Is system* (1.1) *and* (1.2) *small-time locally controllable for all* $L \in \mathcal{N}$*?*

In this paper we give a negative answer to this question. We show that system (1.1) and (1.2) is not small-time locally controllable for a class of critical lengths. More precisely, we have

Theorem 1.2. *Let k, $l \in \mathbb{N}_*$ be such that $2k + l \notin 3\mathbb{N}_*$. Assume that*

$$L = 2\pi\sqrt{\frac{k^2 + kl + l^2}{3}}.$$

Then system (1.1) *and* (1.2) *is not small-time locally null-controllable with controls in H^1 and initial and final datum in $H^3(0,L) \cap H^1_0(0,L)$, i.e., there exist $T_0 > 0$ and $\varepsilon_0 > 0$ such that, for all $\delta > 0$, there is $y_0 \in H^3(0,L) \cap H^1_0(0,L)$ with $\|y_0\|_{H^3(0,L)} < \delta$ such that for all $u \in H^1(0,T_0)$ with $\|u\|_{H^1(0,T_0)} < \varepsilon_0$ and $u(0) = y_0'(L)$, we have*

$$y(T_0, \cdot) \not\equiv 0,$$

where $y \in C\big([0,T_0]; H^3(0,L)\big) \cap L^2\big([0,T_0]; H^4(0,L)\big)$ is the unique solution of (1.1) *and* (1.2)*.*

Open problem 1.3. *We are not able to establish that the control system* (1.1) *and* (1.2) *is not small-time locally controllable with initial and final datum in $L^2(0,L)$ and control in $L^2(0,T)$ for a critical length as in Theorem 1.2. It would be interesting to extend the method in the paper to deal with this problem. It would be also interesting to know what is the smallest s such that system* (1.1) *and* (1.2) *is not small-time locally controllable with controls in $H^s(0,T)$, and initial and final datum in $D(\mathcal{A}^s)$, $\mathcal{A}$ being defined in Lemma 2.1 below.*

Remark 1.4. Concerning Open problem 1.3, may be the smallest s is not an integer, as in the nonlinear parabolic equation studied in [8], a phenomenon which is specific to the infinite dimension as shown in [7]. Note that in [32] a non integer s already appears for an obstruction to small-time local controllability; however it is not known if this s is the optimal one.

Open problem 1.5. *It would be also interesting to know what is the optimal time for the local null controllability. In particular one may ask if $T \le T^>$, with $T^>$ defined in* [20, p. 463]*, then the control system* (1.1) *and* (1.2) *is not locally null controllable in time T (for example with initial and final datum in $H^3(0,L) \cap H^1_0(0,L)$ and control in $H^1(0,T)$) for critical lengths L as in the above theorem.*

Open problem 1.6. *Finally, it would be interesting to know if the assumption $2k + l \notin 3\mathbb{N}_*$ can be replaced by the weaker assumption* $\dim \mathcal{M} > 1$*. In other words, is it true that the control system* (1.1) *and* (1.2) *is not small time locally controllable when* $\dim \mathcal{M} > 1$*?*

In Theorem 1.2, we deal with controls in $H^1(0,T_0)$, and initial and final datum in $H^3(0,L) \cap H^1_0(0,L)$ instead of controls in $L^2(0,T_0)$, and initial and final datum in $L^2(0,L)$ as considered in [38, 24, 18, 20]. For a subclass of the critical lengths considered in Theorem 1.2, we prove later (see Theorem 6.1 in Section 6) that system (1.1) and (1.2) is locally controllable with initial and final datum in $H^3(0,L) \cap H^1_0(0,L)$ and controls in $H^1(0,T)$. It is worth noting that even though the propagation speed of the KdV equation is infinite, some time is needed to reach the zero state.

We emphasize that there are other types of boundary controls for the KdV equations for which there is no critical length, see [38, 39, 28, 19]. There are also results on internal controllability for the KdV equations, see [42], [17] and references therein.

A minimal time of the null-controllability is also required for some linear partial differential equations. This is obviously the case for equations with a finite speed of propagation, such as the transport equation [22, Theorem. 2.6], or the wave equation [3, 16], or hyperbolic systems [25]. But this can also happen for equations with infinite speed of propagation, such as some parabolic systems [2, 11], Grushin-type equations [9, 4, 26], Kolmogorov-type equations [5] or parabolic-transport coupled systems [6], and the references therein. Nevertheless, a minimal time required for the KdV equations using boundary controls is observed and established for the first time in this work to our knowledge. This fact is surprising when compared with known results on internal controls for KdV system (1.1) with $u = 0$. It is known, see [17, 37, 36], that the KdV system (1.1) with $u = 0$ is local controllable using internal controls *whenever* the control region contains an *arbitrary* open subset of $(0, L)$.

However our obstruction to small-time local controllability of our KdV control system is of a different nature than these obstructions to small-time null controllability for linear partial differential equations. It comes from a phenomena which already appears in finite dimension for *nonlinear* control systems. Note that in finite dimension, in contrast to the case of partial differential equations as just pointed above, a linear control system which is controllable in large time is controllable in arbitrary small time. This is no longer the case for nonlinear control systems in finite dimension: There are nonlinear control systems in finite dimension which are locally controllable in large enough time but are not locally controllable in small time. A typical example is the control system

$$\dot{y}_1 = u, \quad \dot{y}_2 = y_3, \quad \dot{y}_3 = -y_2 + 2y_1 u, \tag{1.5}$$

where the state is $(y_1, y_2, y_3)^\mathsf{T} \in \mathbb{R}^3$ and the control is $u \in \mathbb{R}$. There are many powerful necessary conditions for small-time local controllability of nonlinear control systems in finite dimension. Let us mention in particular the Sussmann condition [43, Proposition 6.3]. See also [7] by Beauchard and Marbach for further results, in particular for controls in the Sobolev spaces $H^k(0, T)$, and a different approach. The Sussmann condition [43, Proposition 6.3] tells us that the nonlinear control system (1.5) is not small-time locally controllable (see [22, Example 3.38]): it gives a precise direction, given by an explicit iterated Lie bracket, in which one cannot move in small time. For partial differential equations iterated Lie brackets can sometimes be defined, at least heuristically, for interior controls but are not well understood for boundary controls (see [22, Chapter 5]), which is the type of controls considered here. However, for the simple control system (1.5), an obstruction to small-time local controllability can be obtained by pointing out that if $(y, u) : [0, T] \to \mathbb{R}^3 \times \mathbb{R}$ is a trajectory of the control system (1.5) such that $y(0) = 0$, then

$$y_2(T) = \int_0^T \cos(T - t) y_1^2(t)\, dt, \tag{1.6}$$

$$y_3(T) = y_1(T)^2 - \int_0^T \sin(T - t) y_1^2(t)\, dt. \tag{1.7}$$

Hence,

$$y_2(T) \geq 0 \text{ if } T \in [0, \pi/2] \tag{1.8}$$

$$y_3(T) \leq 0 \text{ if } T \in [0, \pi] \text{ and } y_1(T) = 0, \tag{1.9}$$

which also show that the control system (1.5) is not small-time locally controllable and more precisely, using (1.9), is not locally controllable in time $T \in [0, \pi]$ ((1.8) gives only an obstruction for $T \in [0, \pi/2]$). Note that condition (1.8), at least for $T > 0$ small enough, is the obstruction to small-time local controllability given by [43, Proposition 6.3], while (1.9) is not related to this proposition. For the control system (1.5) one knows that it is locally controllable in a large enough time and the optimal time for local controllability is also known: this control system is locally controllable in time T if and only if $T > \pi$; see [22, Example 6.4]. Moreover, if there are higher order perturbations (with respect to the weight $(r_1, r_2, r_3) = (1, 2, 2)$ for the state and 1 for the control; see [22, Section 12.3]) one can still get an obstruction to small-time local controllability by pointing out that (1.6) and (1.7) respectively imply

$$\text{for every } T \in (0, \pi/2) \text{ there exists } \delta > 0 \text{ such that } y_2(T) \geq \delta |u|^2_{H^{-1}(0,T)}, \tag{1.10}$$

$$\text{for every } T \in (0, \pi] \text{ there exists } \delta > 0 \text{ such that if } y_1(T) = 0, \text{ then } y_3(T) \leq -\delta |u|^2_{H^{-2}(0,T)}. \tag{1.11}$$

Assertion (1.11) follows from the following facts:

$$\int_0^T \Big(\int_0^t y_1(s)\,ds\Big)^2 dt \leq \int_0^T t\int_0^t y_1(s)^2\,ds\,dt \leq T\int_0^T (T-s)y_1(s)^2\,ds,$$

$$\int_0^T \Big(\int_t^T y(s)\,ds\Big)^2 dt \leq \int_0^T (T-s)y(s)^2\,ds,$$

and, since $y_1' = u$ and $y_1(0)=0$,

$$\|u\|^2_{H^{-2}(0,T)} \leq C\int_0^T \Big(\int_0^t y_1(s)\,ds\Big)^2 dt + C\Big(\int_0^T y_1(s)\,ds\Big)^2.$$

Note that inequality (1.10) does not require any condition on the control, while (1.11) requires that the control is such that $y_1(T)=0$. On the other hand it is (1.11) which gives the largest time for the obstruction to local controllability in time T: (1.10) gives an obstruction for $T\in[0,\pi/2)$, while (1.11) gives an obstruction for $T\in[0,\pi]$, which in fact optimal as mentioned above.

There are nonlinear partial differential equations where related inequalities giving an obstruction to small-time local controllability were already proved, namely nonlinear Schrödinger control systems considered by Coron in [21] and by Beauchard and Morancey in [10], a viscous Burgers equation considered by Marbach in [32], and a nonlinear parabolic equation considered by Beauchard and Marbach in [8]. Our obstruction to small-time local controllability is also in the same spirit (see in particular Corollary 3.7). Let us briefly explain some of the main ingredients of these previous works.

- In [21] and [10], the control is interior and one can compute, at least formally, the iterated Lie bracket [43] in which one could not move in small time (see [22, Section 9.3.1]) if the control systems were in finite dimension. Then one checks by suitable computations that it is indeed not possible to move in small time in this direction by proving an inequality analogous to (1.11). The computations are rather explicit due to the fact that the drift[1] of the linearized control system is skew-adjoint with explicit and simple eigenvalues and eigenfunctions.
- In [32] the control is again interior. However the iterated Lie bracket [43] in the direction of which one could not move in small time turns out to be 0. Hence it does not produce any obstruction to small-time local controllability. However an inequality analogous to (1.10) is proved, but with a fractional (non integer) Sobolev norm. An important tool of the proof is a change of time-scale which allows to do an expansion with respect to a new parameter. In the framework of (1.5), this leads to a boundary layer which is analyzed thanks to the maximum principle. Here the drift term of the linearized control system is self-adjoint with explicit and simple eigenvalues and eigenfunctions.
- In [8] the control is again an interior control. Two cases are considered, a case [8, Theorem 3] related to [21] and [10] (already analyzed above) and a case [8, Theorem 4] where classical obstructions relying on iterated Lie brackets fail. Concerning [8, Theorem 4] the proof relies on an inequality of type (1.11). The proof of the inequality of type (1.11) can be performed by explicit computations due to some special structure of the quadratic form one wants to analyze: roughly speaking it corresponds to the case (see [8, (4.17)]) where (3.6) below would be replaced by

$$\int_0^L\int_0^{+\infty} |y(t,x)|^2\varphi_x(x)e^{-ipt}\,dt\,dx = \int_{\mathbb{R}} \hat{u}(z)\overline{\hat{u}(z)}\int_0^L B(z,x)\,dx\,dz, \tag{1.12}$$

  which simplifies the analysis the left hand side of (1.12) (in (3.6) one has $\hat{u}(z)\overline{\hat{u}(z-p)}$ instead of $\hat{u}(z)\overline{\hat{u}(z)}$). The computations are also simplified by the fact that the drift

[1] If the linearized control system is written in the form $\dot{y} = Ay + Bu$, the drift term is the map $y \mapsto Ay$

term of the linearized control system is self-adjoint with, again, explicit eigenvalues and eigenfunctions.

In this article we prove an estimate of type (1.11), instead of (1.10), expecting that with more precise estimates one might get the optimal time for local controllability as for the control system (1.5). The main differences of our study compare with those of these previous articles are the following ones.

- This is the first case dealing with boundary controls. In our case one does not know what are the iterated Lie brackets even heuristically. Let us take this opportunity to point out that, even if they are expected to not leave in the state space (see [22, pages 181–182]), that would be very interesting to understand what are these iterated Lie brackets.
- It sounds difficult to perform the change of time-scale introduced in [32] in our situation. Indeed this change will also lead to a boundary layer. However one can no longer use the maximum principle to study this boundary layer. Moreover if the change of time-scale, if justified, allows simpler computations[2], the advantage for not using it might be to get better or more explicit time for the obstruction to small-time local controllability.
- The linear drift term of the linearized control system (i.e. the operator $\mathcal{A}$ defined in Lemma 2.1) is neither self-adjoint nor skew-adjoint. Moreover its eigenvalues and eigenfunctions are not explicit.
- Finally, (1.12) does not hold.

1.3. **Ideas of the analysis.** Our approach is inspired by the power series expansion method introduced by Coron and Crépeau [24]. The idea of this method is to search/understand a control u of the form

$$u = \varepsilon u_1 + \varepsilon^2 u_2 + \cdots .$$

The corresponding solution then formally has the form

$$y = \varepsilon y_1 + \varepsilon^2 y_2 + \cdots ,$$

and the non-linear term yy_x can be written as

$$yy_x = \varepsilon^2 y_1 y_{1,x} + \cdots .$$

One then obtains the following systems

$$\begin{cases} y_{1,t}(t,x) + y_{1,x}(t,x) + y_{1,xxx}(t,x) = 0 & \text{for } t \in (0,T),\ x \in (0,L), \\ y_1(t, x=0) = y_1(t, x=L) = 0 & \text{for } t \in (0,T), \\ y_{1,x}(t, x=L) = u_1(t) & \text{for } t \in (0,T), \end{cases} \tag{1.13}$$

$$\begin{cases} y_{2,t}(t,x) + y_{2,x}(t,x) + y_{2,xxx}(t,x) + y_1(t,x)y_{1,x}(t,x) = 0 & \text{for } t \in (0,T),\ x \in (0,L), \\ y_2(t, x=0) = y_2(t, x=L) = 0 & \text{for } t \in (0,T), \\ y_{2,x}(t, x=L) = u_2(t) & \text{for } t \in (0,T). \end{cases} \tag{1.14}$$

The idea in [18, 20] with its root in [24] is then to find u_1 and u_2 such that, if $y_1(0,\cdot) = y_2(0,\cdot) = 0$, then $y_1(T,\cdot) = 0$ and the $L^2(0,L)$-orthogonal projection of $y_2(T)$ on $\mathcal{M}$ is a given (non-zero) element in $\mathcal{M}$. In [24], the authors needed to make an expansion up to the order 3 since y_2 belongs to the orthogonal space of $\mathcal{M}$ in this case. To this end, in [24, 18, 20], the authors used delicate contradiction arguments to capture the structure of the KdV systems.

The analysis in this paper has the same root as the ones mentioned above. Nevertheless, instead of using a contradiction argument, our strategy is to characterize all possible u_1 which steers 0 at time 0 to 0 at time T (see Proposition 2.8). This is done by taking the Fourier transform with

[2]This is in particular due to the fact that for the limit problem one has again (1.12)

respect to time of the solution y_1 and applying Paley-Wiener's theorem. Surprisingly, in the case $2k + l \neq 3\mathbb{N}_*$, if the time T is sufficiently small, there are directions in $\mathcal{M}$ which cannot be reached via y_2 (see Corollary 3.7 and Lemma 5.3). This is one of the crucial observations in this paper. Using this observation, we then implement a method to prove the obstruction for the small-time local null-controllability of the KdV system, see Theorem 5.1. The idea is to bring the nonlinear context to the one, based on the power series expansion approach, where the new phenomenon is observed (the context of Corollary 3.7). To be able to reach the result as stated in Theorem 1.2, we establish several new estimates for the linear and nonlinear KdV systems using low regularity data (see Section 4.2 for the linear and Lemma 5.4 for the nonlinear settings). Their proofs partly involve a connection between the linear KdV equation and the linear KdV-Burgers equation as previously used by Bona et al. [13] and inspired by the work of Bourgain [14], and Molinet and Ribaud [34]. To establish the local controllability for a subclass of critical lengths in a finite time (Theorem 6.1), we apply again the power series method and use a fixed point argument. The key point here is first to obtain controls in $H^1(0,T)$ to control directions which can be reached via the linearized system and second to obtain controls in $H^1(0,T)$ for y_1 and y_2 mentioned above. The analysis of the first part is based on a modification of the Hilbert Uniqueness Method and the analysis of the second part is again based on the information obtained in Corollary 3.7 and Lemma 5.3. Our fixed point argument is inspired by [24, 18] but is different, somehow simpler, and, more importantly, relies on the usual Banach fixed point theorem instead of the Brouwer fixed point theorem, which might be interesting to handle nonlinear partial differential equations such that $\mathcal{M}$ is of infinite dimension, as, for example, in [32].

1.4. **Structure of the paper.** The paper is organized as follows. Section 2 is devoted to the study of controls which steers 0 to 0 (motivated by the system of y_1). In Section 3, we study attainable directions for small time via the power series approach (motivated by the system of y_2). The main result in this section is Proposition 3.6 whose consequence (Corollary 3.7) is crucial in the proof of Theorem 1.2. In Section 4, we established several useful estimates for linear KdV systems. In Section 5, we give the proof of Theorem 1.2. In fact, we will establish a result (Theorem 5.1), which implies Theorem 1.2 and reveals a connection with unreachable directions via the power series expansion method. In Section 6, we establish the local controllability for the nonlinear KdV system (1.1) with initial and final datum in $H^3(0,L) \cap H^1_0(0,L)$ and controls in $H^1(0,1)$ for some critical lengths (Theorem 6.1). In the appendix, we establish various results used in Sections 2 to 4.

2. Properties of controls steering 0 at time 0 to 0 at time T

In this section, we characterize the controls that steer 0 to 0 for the linearized KdV system at a given time. This is done by considering the Fourier transform in the t-variable and these conditions are written in terms of Paley-Wiener's conditions. The resolvent of $\partial_x^3 + \partial_x$ hence naturally appears during this analysis. We begin with the discrete property on the spectrum of this operator.

Lemma 2.1. *Set* $D(\mathcal{A}) = \Big\{ v \in H^3(0,L), v(0) = v(L) = v'(L) = 0 \Big\}$ *and let* $\mathcal{A}$ *be the unbounded operator on* $L^2(0,L)$ *with domain* $D(\mathcal{A})$ *and defined by* $\mathcal{A}v = v''' + v'$ *for* $v \in D(\mathcal{A})$. *The spectrum of* $\mathcal{A}$ *is discrete.*

Proof. Since $\mathcal{A}$ is closed, we only have to prove that there exists a discrete set $\mathcal{D} \subset \mathbb{C}$ such that for $z \in \mathbb{C} \setminus \mathcal{D}$ and for $f \in L^2(0,L)$, there exists a unique solution $v \in H^3(0,L)$ of the system

$$\begin{cases} v''' + v' + zv = f \text{ in } (0,L), \\ v(0) = v(L) = v'(L) = 0. \end{cases} \tag{2.1}$$

Step 1: An auxiliary shooting problem. For each $z \in \mathbb{C}$, let $U_{(z)} \in C^3(\mathbb{R};\mathbb{C})$ be the unique solution of the Cauchy problem

$$U_{(z)}''' + U_{(z)}' + zU_{(z)} = 0 \text{ in } (0,L), \quad U_{(z)}'(L) = U_{(z)}(L) = 0, \quad U_{(z)}''(L) = 1. \tag{2.2}$$

Let $\theta\colon \mathbb{C} \to \mathbb{C}$ be defined by $\theta(z) = U_{(z)}(0)$. Then θ is an entire function. We claim that this function does not vanish identically, and $\mathcal{D} := \theta^{-1}(0)$ is therefore a discrete set. Indeed, let us assume that $U_{(1)}(0) = \theta(1) = 0$. Multiplying (2.2) with $z = 1$ (the equation of $U_{(1)}$) by the (real) function $U_{(1)}$ and integrating by parts on $[0,L]$, one gets

$$\frac{1}{2}U_{(1)}'(0)^2 + \int_0^L U_{(1)}^2(x)\,dx = 0, \tag{2.3}$$

which implies $U_{(1)} = 0$ in $[0,L]$. This is in contradiction with $U_{(1)}''(L) = 1$.

Step 2: Uniqueness. Let $z \notin \mathcal{D}$, i.e., $\theta(z) = U_{(z)}(0) \neq 0$. Assume that $v_1, v_2 \in H^3(0,L)$ are two solutions of (2.1). Set $U = v_1 - v_2$. Then $U''' + U' + zU = 0$ and $U(L) = U'(L) = 0$. It follows that $U = U''(L)U_{(z)}$ in $[0,L]$. So, $U(0) = U''(L)U_{(z)}(0) = U''(L)\theta(z)$. Since $\theta(z) \neq 0$ and $U(0) = v_1(0) - v_2(0) = 0$, we conclude that $U''(L) = 0$. Hence $U = 0$ in $[0,L]$, which implies the uniqueness.

Step 3: Existence. Let $z \notin \mathcal{D}$ and $f \in L^2(0,L)$. Let $V \in H^3(0,L)$ be the unique solution of the Cauchy problem

$$\begin{cases} V''' + V' + zV = f \text{ in } (0,L), \\ V(L) = V'(L) = V''(L) = 0. \end{cases} \tag{2.4}$$

Set $v = V - V(0)(\theta(z))^{-1}U_{(z)}$ in $[0,L]$. Then v belongs to $H^3(0,L)$ and satisfies the differential equation $v''' + v' + zv = f$, and the boundary conditions $v(L) = 0$, $v'(L) = 0$, and $v(0) = V(0) - V(0) = 0$. Thus v is a solution of (2.1). □

Before characterizing controls steering 0 at time 0 to 0 at time T, we introduce

Definition 2.2. *For $z \in \mathbb{C}$, let $(\lambda_j)_{1\le j\le 3} = \big(\lambda_j(z)\big)_{1\le j\le 3}$ be the three solutions repeated with the multiplicity of*

$$\lambda^3 + \lambda + iz = 0. \tag{2.5}$$

Set

$$Q = Q(z) := \sum_{j=1}^{3}(\lambda_{j+1} - \lambda_j)e^{\lambda_j L + \lambda_{j+1} L} = \begin{pmatrix} 1 & 1 & 1 \\ e^{\lambda_1 L} & e^{\lambda_2 L} & e^{\lambda_3 L} \\ \lambda_1 e^{\lambda_1 L} & \lambda_2 e^{\lambda_2 L} & \lambda_3 e^{\lambda_3 L} \end{pmatrix}, \tag{2.6}$$

$$P = P(z) := \sum_{j=1}^{3}\lambda_j(e^{\lambda_{j+2}L} - e^{\lambda_{j+1}L}) = \det\begin{pmatrix} 1 & 1 & 1 \\ e^{\lambda_1 L} & e^{\lambda_2 L} & e^{\lambda_3 L} \\ \lambda_1 & \lambda_2 & \lambda_3 \end{pmatrix}, \tag{2.7}$$

and

$$\Xi = \Xi(z) := -(\lambda_2 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_1 - \lambda_3) = \det\begin{pmatrix} 1 & 1 & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 \end{pmatrix}, \tag{2.8}$$

with the convention $\lambda_{j+3} = \lambda_j$ for $j \geq 1$.

Remark 2.3. The matrix Q and the quantities P and Ξ are antisymmetric with respect to λ_j ($j = 1, 2, 3$), and their definitions depend on a choice of the order of $(\lambda_1, \lambda_2, \lambda_3)$. Nevertheless, we later consider a product of either P, Ξ, or $\det Q$ with another antisymmetric function of (λ_j), or deal with $|\det Q|$, and these quantities therefore make sense (see e.g. (2.11), (2.12)). The definitions of P, Ξ, and Q are only understood in these contexts.

In what follows, for an appropriate function v defined on $\mathbb{R}_+ \times (0, L)$, we extend v by 0 on $\mathbb{R}_- \times (0, L)$ and we denote by $\hat{v}$ its Fourier transform with respect to t, i.e., for $z \in \mathbb{C}$,

$$\hat{v}(z, x) = \frac{1}{\sqrt{2\pi}} \int_0^{+\infty} v(t, x) e^{-izt}\, dt.$$

We have

Lemma 2.4. *Let $u \in L^2(0, +\infty)$ and let $y \in C\big([0, +\infty); L^2(0, L)\big) \cap L^2_{\text{loc}}\big([0, +\infty); H^1(0, L)\big)$ be the unique solution of*

$$\left\{ \begin{array}{ll} y_t(t, x) + y_x(t, x) + y_{xxx}(t, x) = 0 & \textit{in } (0, +\infty) \times (0, L), \\ y(t, x = 0) = y(t, x = L) = 0 & \textit{in } (0, +\infty), \\ y_x(t, x = L) = u(t) & \textit{in } (0, +\infty), \end{array} \right. \tag{2.9}$$

with

$$y(t = 0, \cdot) = 0 \textit{ in } (0, L). \tag{2.10}$$

Then, outside of a discrete set $z \in \mathbb{R}$, we have

$$\hat{y}(z, x) = \frac{\hat{u}}{\det Q} \sum_{j=1}^{3} (e^{\lambda_{j+2} L} - e^{\lambda_{j+1} L}) e^{\lambda_j x} \textit{ for a.e. } x \in (0, L), \tag{2.11}$$

and in particular,

$$\partial_x \hat{y}(z, 0) = \frac{\hat{u}(z) P(z)}{\det Q(z)}. \tag{2.12}$$

Remark 2.5. Assume that $\hat{u}(z, \cdot)$ is well-defined for $z \in \mathbb{C}$ (e.g. when u has a compact support). Then the conclusions of Lemma 2.4 hold outside of a discrete set $z \in \mathbb{C}$.

Proof. From the system of y, we have

$$\left\{ \begin{array}{ll} iz\hat{y}(z, x) + \hat{y}_x(z, x) + \hat{y}_{xxx}(z, x) = 0 & \text{in } \mathbb{R} \times (0, L), \\ \hat{y}(z, x = 0) = \hat{y}(z, x = L) = 0 & \text{in } \mathbb{R}, \\ \hat{y}_x(z, x = L) = \hat{u}(z) & \text{in } \mathbb{R}. \end{array} \right. \tag{2.13}$$

Taking into account the equation of $\hat{y}$, we search the solution of the form

$$\hat{y}(z, \cdot) = \sum_{j=1}^{3} a_j e^{\lambda_j x},$$

where $\lambda_j = \lambda_j(z)$ with $j = 1, 2, 3$ are defined in Definition 2.2.

According to the theory of ordinary differential equations with constant coefficients, this is possible if the equation $\lambda^3 + \lambda + iz = 0$ has three distinct solutions, i.e., if the discriminant $-4 + 27z^2$ is not 0. Moreover, if $-iz \notin \mathrm{Sp}(\mathcal{A})$, this solution is unique. Thus, by Lemma 2.1, outside a discrete set in $\mathbb{R}$, $\hat{y}(z,\cdot)$ can be written in this form in a unique way. Using the boundary conditions for $\hat{y}$, we require that

$$\begin{cases} \sum_{j=1}^3 a_j = 0, \\ \sum_{j=1}^3 e^{\lambda_j L} a_j = 0, \\ \sum_{j=1}^3 \lambda_j e^{\lambda_j L} a_j = \hat{u}. \end{cases}$$

This implies, with $Q = Q(z)$ defined in Definition 2.2,

$$Q(a_1, a_2, a_3)^\mathsf{T} = (0, 0, \hat{u})^\mathsf{T}. \tag{2.14}$$

It follows that

$$a_j = \frac{\hat{u}}{\det Q}\big(e^{\lambda_{j+2} L} - e^{\lambda_{j+1} L}\big).$$

This yields

$$\hat{y}(z,x) = \frac{\hat{u}}{\det Q} \sum_{j=1}^3 (e^{\lambda_{j+2} L} - e^{\lambda_{j+1} L}) e^{\lambda_j x}. \tag{2.15}$$

We thus obtain

$$\partial_x \hat{y}(z,0) = \frac{\hat{u}(z) P(z)}{\det Q(z)}. \tag{2.16}$$

□

As mentioned in Remark 2.3, the maps P and $\det Q$ are antisymmetric functions with respect to λ_j. It is hence convenient to consider $\partial_x \hat{y}(z,0)$ under the form

$$\partial_x \hat{y}(z,0) = \frac{\hat{u}(z) G(z)}{H(z)}, \tag{2.17}$$

where, with Ξ defined in (2.8),

$$G(z) = P(z)/\Xi(z) \quad \text{and} \quad H(z) = \det Q(z)/\Xi(z). \tag{2.18}$$

Concerning the functions G and H, we have

Lemma 2.6. *The functions G and H defined in* (2.18) *are entire functions.*

Proof. Note that the maps $z \mapsto \Xi(z)P(z)$, $z \mapsto \Xi(z)\det Q(z)$ and $z \mapsto \Xi(z)^2$ are symmetric functions of the λ_j and are thus well-defined, and even entire functions (see Lemma A.1 in Appendix A). According to the definition of Ξ, $\Xi(z_0) = 0$ if and only if $X^3 + X + iz_0$ has a double root, i.e. $z_0 = \pm 2/(3\sqrt{3})$. Simple computations prove that when ϵ is small,

$$\begin{cases} \lambda_1(z_0 + \varepsilon) = \mp \dfrac{i}{\sqrt{3}} + \dfrac{\sqrt{\mp i}}{3^{1/4}} \sqrt{\epsilon} + O(\varepsilon), \\[2ex] \lambda_2(z_0 + \varepsilon) = \mp \dfrac{i}{\sqrt{3}} - \dfrac{\sqrt{\mp i}}{3^{1/4}} \sqrt{\varepsilon} + O(\varepsilon), \\[2ex] \lambda_3(z_0 + \varepsilon) = \pm \dfrac{2i}{\sqrt{3}} + \dfrac{\varepsilon}{3} + O(\varepsilon^2). \end{cases} \tag{2.19}$$

Indeed, the behavior of λ_3 follows immediately from the expansion of λ_3 near $\pm\frac{2i}{\sqrt{3}}$. The behavior of λ_1 and λ_2 can be then verified using, with $\Delta = -3\lambda_3^2 - 4$,

$$\lambda_1 = \frac{-\lambda_3 + \sqrt{\Delta}}{2} \quad \text{and} \quad \lambda_2 = \frac{-\lambda_3 - \sqrt{\Delta}}{2}.$$

It follows that that $\Xi^2(z_0+\varepsilon) = c_\pm \varepsilon + O(\varepsilon^2)$ for some $c_\pm \neq 0$. This in turn implies that $z_0 = \pm 2/(3\sqrt{3})$ are simple zeros of Ξ^2. When $X^3 + X + iz$ has a double root, the definitions of P and $\det Q$ (Eq. (2.6) and (2.7)) imply

$$|P(z_0)| = |\det Q(z_0)| = 0 \text{ for } z_0 = \pm 2/(3\sqrt{3}).$$

The conclusion follows. □

Remark 2.7. It is interesting to note that

(1) $\big(H(z) = 0$ and $z \neq \pm 2/(3\sqrt{3})\big)$ if and only if $-iz \in \mathrm{Sp}(\mathcal{A})$.

(2) $iz \in \mathrm{Sp}(\mathcal{A})$ and z is real if and only if $L = 2\pi\sqrt{\frac{k^2+kl+l^2}{3}}$, and

$$z = \frac{(2k+l)(k-l)(2l+k)}{3\sqrt{3}(k^2+kl+l^2)^{3/2}}, \tag{2.20}$$

for some $k, l \in \mathbb{N}$ with $1 \le l \le k$.

Indeed, if $L = 2\pi\sqrt{\frac{k^2+kl+l^2}{3}}$ and z is given by the RHS of (2.20), then, from [38], $iz \in \mathrm{Sp}(\mathcal{A})$. On the other hand, if z is real and $iz \in \mathrm{Sp}(\mathcal{A})$, then, by an integration by parts, the corresponding eigenfunction w also satisfies the condition $w_x(0) = 0$. It follows from [38] that $L = 2\pi\sqrt{\frac{k^2+kl+l^2}{3}}$ and z is given by (2.20) for some $k, l \in \mathbb{N}$ with $1 \le l \le k$. We finally note that for $z \neq \pm 2/(3\sqrt{3})$, the solutions of the ordinary differential equation $u''' + u' + izu = 0$ are of the form $u(x) = \sum_{j=1}^3 a_j e^{\lambda_j x}$. This implies that $Q(a_1, a_2, a_3)^\mathsf{T} = (0,0,0)^\mathsf{T}$ if $u(0) = u(L) = u'(L) = 0$. Therefore, for $z \neq \pm 2/(3\sqrt{3})$, $-iz$ is an eigenvalue of $\mathcal{A}$ if and only if $|\det Q(z)| = 0$, i.e., $H(z) = 0$. We finally note that, $\pm 2i/(3\sqrt{3})$ is not a pure imaginary eigenvalue of $\mathcal{A}$ since, for $k \ge l \ge 1$,

$$0 \le \frac{(2k+l)(k-l)(2l+k)}{3\sqrt{3}(k^2+kl+l^2)^{3/2}} = \frac{(2k+l)(k^2+kl-2l^2)}{3\sqrt{3}(k^2+kl+l^2)^{3/2}} < \frac{(2k+l)}{3\sqrt{3}(k^2+kl+l^2)^{1/2}} < \frac{2}{3\sqrt{3}}.$$

We are ready to give the characterization of the controls steering 0 to 0, which is the starting point of our analysis.

Proposition 2.8. *Let $L > 0$, $T > 0$, and $u \in L^2(0,+\infty)$. Assume that u has a compact support in $[0,T]$, and u steers 0 at the time 0 to 0 at the time T, i.e., the unique solution y of* (2.9) *and* (2.10) *satisfies $y(T,\cdot) = 0$ in $(0,L)$. Then $\hat{u}$ and $\hat{u}G/H$ satisfy the assumptions of Paley-Wiener's theorem concerning the support in $[-T,T]$, i.e.,*

$$\hat{u} \text{ and } \hat{u}G/H \text{ are entire functions,}$$

and

$$|\hat{u}(z)| + \left|\frac{\hat{u}G(z)}{H(z)}\right| \le Ce^{T|\Im(z)|},$$

for some positive constant C.

Here and in what follows, for a complex number z, $\Re(z)$, $\Im(z)$, and $\bar{z}$ denote the real part, the imaginary part, and the conjugate of z, respectively.

Proof. Proposition 2.8 is a consequence of Lemma 2.4 and Paley-Wiener's theorem, see e.g. [41, 19.3 Theorem]. The proof is clear from the analysis above in this section and left to the reader. □

3. Attainable directions for small time

In this section, we investigate controls which steer the linear KdV equation from 0 to 0 in some time T, and a quantity related to the quadratic order in the power expansion of the nonlinear KdV equation behaves. Let $u \in L^2(0, +\infty)$ and denote y the corresponding solution of the linear KdV equation (2.9). We assume the initial condition to be 0 and that y satisfies $y(t, \cdot) = 0$ in $(0, L)$ for $t \geq T$. We have, by Lemma 2.4 (and also Remark 2.5), for $z \in \mathbb{C}$ outside a discrete set,

$$\hat{y}(z, x) = \hat{u}(z) \frac{\sum_{j=1}^{3} (e^{\lambda_{j+1} L} - e^{\lambda_j L}) e^{\lambda_{j+2} x}}{\sum_{j=1}^{3} (\lambda_{j+1} - \lambda_j) e^{-\lambda_{j+2} L}}. \tag{3.1}$$

Recall that $\lambda_j = \lambda_j(z)$ for $j = 1, 2, 3$ are the three solutions of the equation

$$x^3 + x = -iz \text{ for } z \in \mathbb{C}. \tag{3.2}$$

Let $\eta_1, \eta_2, \eta_3 \in i\mathbb{R}$, i.e., $\eta_j \in \mathbb{C}$ with $\Re(\eta_j) = 0$ for $j = 1, 2, 3$. Define

$$\varphi(x) = \sum_{j=1}^{3} (\eta_{j+1} - \eta_j) e^{\eta_{j+2} x} \text{ for } x \in [0, L], \tag{3.3}$$

with the convention $\eta_{j+3} = \eta_j$ for $j \geq 1$. The following assumption on η_j is used repeatedly throughout the paper:

$$e^{\eta_1 L} = e^{\eta_2 L} = e^{\eta_3 L}, \tag{3.4}$$

which is equivalent to $\eta_3 - \eta_2, \eta_2 - \eta_1 \in \frac{2\pi i}{L}\mathbb{Z}$. The definition of φ in (3.3) and the assumption on η_j in (3.4) are motivated by the structure of $\mathcal{M}$ [18, 20] and will be clear in Section 5.

We have

Lemma 3.1. *Let $p \in \mathbb{R}$ and let φ be defined by* (3.3)*. Set, for $(z, x) \in \mathbb{C} \times [0, L]$,*

$$B(z, x) = \frac{\sum_{j=1}^{3} (e^{\lambda_{j+1} L} - e^{\lambda_j L}) e^{\lambda_{j+2} x}}{\sum_{j=1}^{3} (\lambda_{j+1} - \lambda_j) e^{-\lambda_{j+2} L}} \cdot \frac{\sum_{j=1}^{3} (e^{\widetilde{\lambda}_{j+1} L} - e^{\widetilde{\lambda}_j L}) e^{\widetilde{\lambda}_{j+2} x}}{\sum_{j=1}^{3} (\widetilde{\lambda}_{j+1} - \widetilde{\lambda}_j) e^{-\widetilde{\lambda}_{j+2} L}} \cdot \varphi_x(x), \tag{3.5}$$

where $\widetilde{\lambda}_j = \widetilde{\lambda}_j(z)$ ($j = 1, 2, 3$) denotes the conjugate of the roots of (3.2) *with z replaced by $z - p$ and with the use of convention $\widetilde{\lambda}_{j+3} = \widetilde{\lambda}_j$ for $j \geq 1$. Let $u \in L^2(0, +\infty)$ and let $y \in C\big([0, +\infty); L^2(0, L)\big) \cap L^2_{\text{loc}}\big([0, +\infty); H^1(0, L)\big)$ be the unique solution of* (2.9) *and* (2.10)*. Then*

$$\int_0^L \int_0^{+\infty} |y(t, x)|^2 \varphi_x(x) e^{-ipt} \, dt \, dx = \int_{\mathbb{R}} \hat{u}(z) \overline{\hat{u}(z - p)} \int_0^L B(z, x) \, dx \, dz. \tag{3.6}$$

Remark 3.2. The LHS of (3.6) is a multiple of the $L^2(0, L)$-projection of the solution $y(T, \cdot)$ into the space spanned by the conjugate of the vector $\varphi(x) e^{-ipT}$ whose real and imaginary parts are in $\mathcal{M}$ for appropriate choices of η_j and p when the initial data is orthogonal to $\mathcal{M}$ (see [24, 18, 20], and also (5.18)).

Proof. We have

$$\begin{aligned}
\int_0^L \int_0^\infty |y(t, x)|^2 \varphi_x(x) e^{-ipt} \, dt \, dx &= \sqrt{2\pi} \int_0^L \varphi_x(x) \widehat{|y|^2}(p, x) \, dx = \int_0^L \varphi_x(x) \hat{y} * \widehat{\bar{y}}(p, x) \, dx \\
&= \int_0^L \varphi_x(x) \int_{\mathbb{R}} \hat{y}(z, x) \widehat{\bar{y}}(p - z, x) \, dz \, dx \\
&= \int_0^L \varphi_x(x) \int_{\mathbb{R}} \hat{y}(z, x) \overline{\hat{y}}(z - p, x) \, dz \, dx.
\end{aligned}$$

Using Fubini's theorem, we derive from (3.1) that

$$\int_0^L \int_0^\infty |y(t,x)|^2 \varphi_x(x) e^{-ipt}\, dt\, dx = \int_{\mathbb{R}} \hat{u}(z)\overline{\hat{u}(z-p)} \int_0^L B(z,x)\, dx\, dz,$$

which is (3.6). □

We next state the behaviors of λ_j and $\widetilde{\lambda}_j$ given in Lemma 3.1 for "large positive" z, which will be used repeatedly in this section and Section 4. These asymptotics are direct consequence of the equation (2.5) satisfied by the λ_j.

Lemma 3.3. *For $p \in \mathbb{R}$ and z in a small enough conic neighborhood of $\mathbb{R}_+$, let λ_j and $\widetilde{\lambda}_j$ with $j = 1, 2, 3$ be given in Lemma 3.1. Consider the convention $\Re(\lambda_1) < \Re(\lambda_2) < \Re(\lambda_3)$ and similarly for $\widetilde{\lambda}_j$. We have, in the limit $|z| \to \infty$,*

$$\lambda_j = \mu_j z^{1/3} - \frac{1}{3\mu_j} z^{-1/3} + O(z^{-2/3}) \quad \textit{with } \mu_j = e^{-i\pi/6 - 2ji\pi/3}, \tag{3.7}$$

$$\widetilde{\lambda}_j = \widetilde{\mu}_j z^{1/3} - \frac{1}{3\widetilde{\mu}_j} z^{-1/3} + O(z^{-2/3}) \quad \textit{with } \widetilde{\mu}_j = e^{i\pi/6 + 2ij\pi/3} \tag{3.8}$$

(see Figure 1 for the geometry of μ_j and $\widetilde{\mu}_j$). Here $z^{1/3}$ denotes the cube root of z with the real part positive.

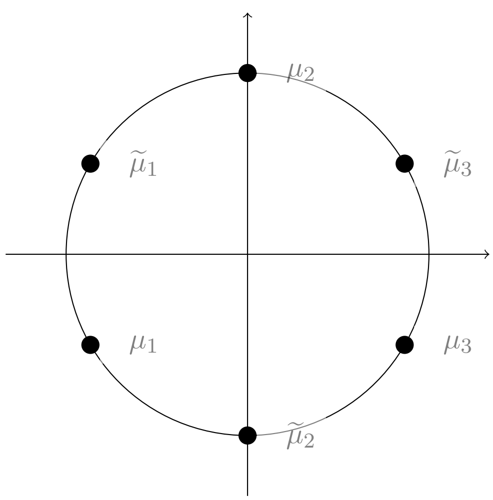

FIGURE 1. The roots λ_j of $\lambda^3 + \lambda + iz = 0$ satisfy, when $z > 0$ is large, $\lambda_j \sim \mu_j z^{1/3}$ where $\mu_j^3 = -i$. When $z < 0$ and $|z|$ is large, then the corresponding roots $\hat{\lambda}_j$ satisfy $\hat{\lambda}_j \sim \widetilde{\mu}_j |z|^{1/3}$ with $\widetilde{\mu}_j = \overline{\mu_j}$. We also have $\widetilde{\lambda}_j \sim \hat{\lambda}_j$.

We are ready to establish the behavior of

$$\int_0^L B(z,x)\, dx$$

for $z \in \mathbb{R}$ with large $|z|$, which is one of the main ingredients for the analysis in this section.

Lemma 3.4. *Let $p \in \mathbb{R}$, and let φ be defined by (3.3). Assume that (3.4) holds and $\eta_j \neq 0$ for $j = 1, 2, 3$. Let B be defined by (3.5). We have*

$$\int_0^L B(z,x)\, dx = \frac{E}{|z|^{4/3}} + O(|z|^{-5/3}) \textit{ for } z \in \mathbb{R} \textit{ with large } |z|, \tag{3.9}$$

where E is defined by

$$E = \frac{1}{3}(e^{\eta_1 L} - 1)\left(-\frac{2}{3}\sum_{j=1}^{3} \eta_{j+2}^2(\eta_{j+1} - \eta_j) - ip \sum_{j=1}^{3} \frac{\eta_{j+1} - \eta_j}{\eta_{j+2}} \right). \tag{3.10}$$

Proof. We first deal with the case where z is positive and large. We use the convention in Lemma 3.3 for λ_j and $\widetilde{\lambda}_j$. Consider the denumerator of $B(z,x)$. We have, by Lemma 3.3,

$$\frac{1}{\sum_{j=1}^3(\lambda_{j+1}-\lambda_j)e^{-\lambda_{j+2}L}}\cdot\frac{1}{\sum_{j=1}^3(\widetilde{\lambda}_{j+1}-\widetilde{\lambda}_j)e^{-\widetilde{\lambda}_{j+2}L}} = \frac{e^{\lambda_1 L}e^{\widetilde{\lambda}_1 L}}{(\lambda_3-\lambda_2)(\widetilde{\lambda}_3-\widetilde{\lambda}_2)}\Big(1+O\big(e^{-C|z|^{1/3}}\big)\Big). \tag{3.11}$$

We next deal with the numerator of $B(z,x)$. Set, for $(z,x)\in\mathbb{R}\times(0,L)$,

$$f(z,x)=\sum_{j=1}^3(e^{\lambda_{j+1}L}-e^{\lambda_j L})e^{\lambda_{j+2}x},\quad g(z,x)=\sum_{j=1}^3(e^{\widetilde{\lambda}_{j+1}L}-e^{\widetilde{\lambda}_j L})e^{\widetilde{\lambda}_{j+2}x}, \tag{3.12}$$

[3]

$$f_m(z,x)=-e^{\lambda_3 L}e^{\lambda_2 x}+e^{\lambda_2 L}e^{\lambda_3 x}+e^{\lambda_3 L}e^{\lambda_1 x},\quad g_m(z,x)=-e^{\widetilde{\lambda}_3 L}e^{\widetilde{\lambda}_2 x}+e^{\widetilde{\lambda}_2 L}e^{\widetilde{\lambda}_3 x}+e^{\widetilde{\lambda}_3 L}e^{\widetilde{\lambda}_1 x}.$$

We have

$$\int_0^L f(z,x)g(z,x)\varphi_x(x)\,dx=\int_0^L f_m(z,x)g_m(z,x)\varphi_x(x)\,dx+\int_0^L(f-f_m)(z,x)g_m(z,x)\varphi_x(x)\,dx$$
$$+\int_0^L f_m(z,x)(g-g_m)(z,x)\varphi_x(x)\,dx+\int_0^L(f-f_m)(z,x)(g-g_m)(z,x)\varphi_x(x)\,dx.$$

It is clear from Lemma 3.3 that

$$\int_0^L|(f-f_m)(z,x)g_m(z,x)\varphi_x(x)|\,dx+\int_0^L|(f-f_m)(z,x)(g-g_m)(z,x)\varphi_x(x)|\,dx + \int_0^L|f_m(z,x)(g-g_m)(z,x)\varphi_x(x)|\,dx\le C|e^{(\lambda_3+\widetilde{\lambda}_3)L}|e^{-C|z|^{1/3}}. \tag{3.13}$$

We next estimate

$$\int_0^L f_m(x,z)g_m(x,z)\varphi_x(x)=\int_0^L f_m(x,z)g_m(x,z)\left(\sum_{j=1}^3\eta_{j+2}(\eta_{j+1}-\eta_j)e^{\eta_{j+2}x}\right)dx. \tag{3.14}$$

We first have, by (3.4) and Lemma 3.3,

$$\int_0^L\Big(-e^{\lambda_3 L}e^{\lambda_2 x}e^{\widetilde{\lambda}_2 L}e^{\widetilde{\lambda}_3 x}-e^{\lambda_2 L}e^{\lambda_3 x}e^{\widetilde{\lambda}_3 L}e^{\widetilde{\lambda}_2 x}+e^{\lambda_2 L}e^{\lambda_3 x}e^{\widetilde{\lambda}_2 L}e^{\widetilde{\lambda}_3 x}\Big) \times\left(\sum_{j=1}^3\eta_{j+2}(\eta_{j+1}-\eta_j)e^{\eta_{j+2}x}\right)dx=e^{(\lambda_3+\widetilde{\lambda}_3+\lambda_2+\widetilde{\lambda}_2)L}\Big(e^{\eta_1 L}T_1(z)+O\big(e^{-C|z|^{1/3}}\big)\Big), \tag{3.15}$$

where

$$T_1(z):=\sum_{j=1}^3\eta_{j+2}(\eta_{j+1}-\eta_j)\left(\frac{1}{\lambda_3+\widetilde{\lambda}_3+\eta_{j+2}}-\frac{1}{\lambda_3+\widetilde{\lambda}_2+\eta_{j+2}}-\frac{1}{\lambda_2+\widetilde{\lambda}_3+\eta_{j+2}}\right). \tag{3.16}$$

[3] The index m stands the main part.

Let us now deal with the terms of (3.14) that contain both $e^{\lambda_3 L+\widetilde{\lambda}_3 L}$ and (either $e^{\lambda_1 x}$ or $e^{\widetilde{\lambda}_1 x}$). We obtain, by (3.4) and Lemma 3.3,

$$\int_0^L \Big(e^{\lambda_3 L}e^{\lambda_1 x}e^{\widetilde{\lambda}_3 L}e^{\widetilde{\lambda}_1 x} - e^{\lambda_3 L}e^{\lambda_1 x}e^{\widetilde{\lambda}_3 L}e^{\widetilde{\lambda}_2 x} - e^{\lambda_3 L}e^{\lambda_2 x}e^{\widetilde{\lambda}_3 L}e^{\widetilde{\lambda}_1 x}\Big) \times \left(\sum_{j=1}^{3}\eta_{j+2}(\eta_{j+1}-\eta_j)e^{\eta_{j+2}x}\right)dx = e^{(\lambda_3+\widetilde{\lambda}_3)L}\Big(T_2(z)+O(e^{-C|z|^{1/3}})\Big), \tag{3.17}$$

where

$$T_2(z) := \sum_{j=1}^{3}\eta_{j+2}(\eta_{j+1}-\eta_j)\left(-\frac{1}{\lambda_1+\widetilde{\lambda}_1+\eta_{j+2}}+\frac{1}{\lambda_1+\widetilde{\lambda}_2+\eta_{j+2}}+\frac{1}{\lambda_2+\widetilde{\lambda}_1+\eta_{j+2}}\right). \tag{3.18}$$

We have, by (3.4),

$$\int_0^L e^{\lambda_3 L}e^{\lambda_2 x}e^{\widetilde{\lambda}_3 L}e^{\widetilde{\lambda}_2 x}\left(\sum_{j=1}^{3}\eta_{j+2}(\eta_{j+1}-\eta_j)e^{\eta_{j+2}x}\right)dx = e^{(\lambda_3+\widetilde{\lambda}_3)L}T_3(z), \tag{3.19}$$

where

$$T_3(z) := \Big(e^{\lambda_2 L+\widetilde{\lambda}_2 L+\eta_1 L}-1\Big)\sum_{j=1}^{3}\frac{\eta_{j+2}(\eta_{j+1}-\eta_j)}{\lambda_2+\widetilde{\lambda}_2+\eta_{j+2}}. \tag{3.20}$$

The other terms of (3.14) are negligible, because we have

$$\left|\int_0^L \Big(e^{\lambda_3 L}e^{\lambda_1 x}e^{\widetilde{\lambda}_2 L}e^{\widetilde{\lambda}_3 x}+e^{\lambda_2 L}e^{\lambda_3 x}e^{\widetilde{\lambda}_3 L}e^{\widetilde{\lambda}_1 x}\Big)\Big(\sum_{j=1}^{3}\eta_{j+2}(\eta_{j+1}-\eta_j)e^{\eta_{j+2}x}\Big)dx\right| = |e^{(\lambda_3+\widetilde{\lambda}_3)L}|O(e^{-Cz^{1/3}}). \tag{3.21}$$

Using Lemma 3.3, we have

$$\begin{cases} \lambda_1+\widetilde{\lambda}_1+\lambda_2+\widetilde{\lambda}_2+\lambda_3+\widetilde{\lambda}_3 = O(z^{-1/3}),\\ \lambda_1+\widetilde{\lambda}_1+\lambda_3+\widetilde{\lambda}_3 = O(z^{-1/3}),\\ (\lambda_3-\lambda_2)(\widetilde{\lambda}_3-\widetilde{\lambda}_2) = 3z^{2/3}(1+O(z^{-1/3})). \end{cases} \tag{3.22}$$

We claim that

$$|T_1(z)|+|T_2(z)|+|T_3(z)| = O(z^{-2/3}) \text{ for large positive } z. \tag{3.23}$$

Assuming (3.23), and combining (3.11), (3.15), (3.17), (3.19), (3.21), and (3.22) yields

$$\int_0^L B(z,x)\,dz = \frac{1}{3|z|^{2/3}}\Big(e^{\eta_1 L}T_1(z)+T_2(z)+T_3(z)+O(z^{-1})\Big). \tag{3.24}$$

We next derive the asymptotic behaviors of $T_1(z)$, $T_2(z)$, and $T_3(z)$, which in particular imply (3.23). We first deal with $T_1(z)$ given in (3.16). Since

$$\sum_{j=1}^{3}\eta_{j+2}(\eta_{j+1}-\eta_j) = 0, \tag{3.25}$$

we obtain

$$T_1(z) = \sum_{j=1}^{3} \eta_{j+2}(\eta_{j+1} - \eta_j) \left(\frac{1}{\lambda_3 + \widetilde{\lambda}_3 + \eta_{j+2}} - \frac{1}{\lambda_3 + \widetilde{\lambda}_3} \right)$$
$$+ \sum_{j=1}^{3} \eta_{j+2}(\eta_{j+1} - \eta_j) \left(-\frac{1}{\lambda_3 + \widetilde{\lambda}_2 + \eta_{j+2}} + \frac{1}{\lambda_3 + \widetilde{\lambda}_2} \right)$$
$$+ \sum_{j=1}^{3} \eta_{j+2}(\eta_{j+1} - \eta_j) \left(-\frac{1}{\lambda_2 + \widetilde{\lambda}_3 + \eta_{j+2}} + \frac{1}{\lambda_2 + \widetilde{\lambda}_3} \right).$$

Using Lemma 3.3, we get

$$T_1(z) = -\sum_{j=1}^{3} \eta_{j+2}^2(\eta_{j+1} - \eta_j) \left(\frac{1}{(\lambda_3 + \widetilde{\lambda}_3)^2} - \frac{1}{(\lambda_3 + \widetilde{\lambda}_2)^2} - \frac{1}{(\lambda_2 + \widetilde{\lambda}_3)^2} \right) + O(z^{-1}).$$

Moreover, we derive from Lemma 3.3 that

$$\frac{1}{(\lambda_3 + \widetilde{\lambda}_3)^2} - \frac{1}{(\lambda_3 + \widetilde{\lambda}_2)^2} - \frac{1}{(\lambda_2 + \widetilde{\lambda}_3)^2} = z^{-2/3}\Big((\mu_3 + \widetilde{\mu}_3)^{-2} - (\mu_3 + \widetilde{\mu}_2)^{-2} - (\mu_2 + \widetilde{\mu}_3)^{-2} \Big) + O(z^{-1})$$
$$= z^{-2/3} \left(\frac{1}{3} - \frac{-1 + i\sqrt{3}}{6} - \frac{-1 - i\sqrt{3}}{6} \right) + O(z^{-1})$$
$$= \frac{2}{3} z^{-2/3} + O(z^{-1}). \tag{3.26}$$

We derive that

$$T_1(z) = -\frac{2}{3} z^{-2/3} \sum_{j=1}^{3} \eta_{j+2}^2(\eta_{j+1} - \eta_j) + O(z^{-1}). \tag{3.27}$$

We next consider $T_2(z)$ given in (3.18). We have, by (3.25),

$$T_2(z) = \sum_{j=1}^{3} \eta_{j+2}(\eta_{j+1} - \eta_j) \left(-\frac{1}{\lambda_1 + \widetilde{\lambda}_1 + \eta_{j+2}} + \frac{1}{\lambda_1 + \widetilde{\lambda}_1} \right)$$
$$+ \sum_{j=1}^{3} \eta_{j+2}(\eta_{j+1} - \eta_j) \left(\frac{1}{\lambda_1 + \widetilde{\lambda}_2 + \eta_{j+2}} - \frac{1}{\lambda_1 + \widetilde{\lambda}_2} \right)$$
$$+ \sum_{j=1}^{3} \eta_{j+2}(\eta_{j+1} - \eta_j) \left(\frac{1}{\lambda_2 + \widetilde{\lambda}_1 + \eta_{j+2}} - \frac{1}{\lambda_2 + \widetilde{\lambda}_1} \right).$$

Using Lemma 3.3, we obtain

$$T_2(z) = \sum_{j=1}^{3} \eta_{j+2}^2(\eta_{j+1} - \eta_j) \left(\frac{1}{(\lambda_1 + \widetilde{\lambda}_1)^2} - \frac{1}{(\lambda_1 + \widetilde{\lambda}_2)^2} - \frac{1}{(\lambda_2 + \widetilde{\lambda}_1)^2} \right) + O(z^{-1}),$$

and

$$\frac{1}{(\lambda_1 + \widetilde{\lambda}_1)^2} - \frac{1}{(\lambda_1 + \widetilde{\lambda}_2)^2} - \frac{1}{(\lambda_2 + \widetilde{\lambda}_1)^2} = z^{-2/3}\Big((\mu_1 + \widetilde{\mu}_1)^{-2} - (\mu_1 + \widetilde{\mu}_2)^{-2} - (\mu_2 + \widetilde{\mu}_1)^{-2} \Big) + O(z^{-1}).$$

By Lemma 3.3, we have

$$(\mu_1 + \widetilde{\mu}_1)^2 = (\mu_3 + \widetilde{\mu}_3)^2 \qquad (\mu_1 + \widetilde{\mu}_2)^2 = (\widetilde{\mu}_3 + \mu_2)^2 \qquad (\widetilde{\mu}_1 + \mu_2)^2 = (\mu_3 + \widetilde{\mu}_2)^2.$$

Combining this with (3.26), we then have

$$T_2(z) = \frac{2}{3} z^{-2/3} \sum_{j=1}^{3} \eta_{j+2}^2 (\eta_{j+1} - \eta_j) + O(z^{-1}). \tag{3.28}$$

We finally consider $T_3(z)$ given in (3.20). We have, by (2.5),

$$\lambda_2^3 + \widetilde{\lambda}_2^3 + \lambda_2 + \widetilde{\lambda}_2 = -iz + i(z - p) = -ip.$$

This yields

$$\lambda_2 + \widetilde{\lambda}_2 = -\frac{ip}{\lambda_2^2 + \widetilde{\lambda}_2^2 + \lambda_2 \widetilde{\lambda}_2}.$$

From Lemma 3.3, we have

$$\lambda_2 + \widetilde{\lambda}_2 = ipz^{-2/3} + O(z^{-1}).$$

It follows that

$$\begin{aligned} \sum_{j=1}^{3} \frac{\eta_{j+2}(\eta_{j+1} - \eta_j)}{\lambda_2 + \widetilde{\lambda}_2 + \eta_{j+2}} &= \sum_{j=1}^{3} \frac{\eta_{j+2}(\eta_{j+1} - \eta_j)}{ipz^{-2/3} + \eta_{j+2}} + O(|z|^{-1}) \\ &= \sum_{j=1}^{3} (\eta_{j+1} - \eta_j) \left(1 - \frac{ipz^{-2/3}}{\eta_{j+2}} \right) + O(|z|^{-1}) \\ &= -ip \sum_{j=1}^{3} \frac{\eta_{j+1} - \eta_j}{\eta_{j+2}} z^{-2/3} + O(z^{-1}). \end{aligned} \tag{3.29}$$

We derive from (3.29) and Lemma 3.3 that

$$T_3 = -ip\Big(e^{\eta_1 L} - 1\Big) \sum_{j=1}^{3} \frac{\eta_{j+1} - \eta_j}{\eta_{j+2}} z^{-2/3} + O(z^{-1}). \tag{3.30}$$

Using (3.27), (3.28), and (3.30), we derive from (3.24) that

$$\int_0^L B(z, x)\, dx = E z^{-4/3} + O(z^{-5/3}),$$

which is the conclusion for large positive z.

The conclusion in the case where z is large and negative can be derived from the case where z is positive and large as follows. Define, for $(z, x) \in \mathbb{R} \times (0, L)$, with large $|z|$,

$$M(z, x) = \frac{\sum_{j=1}^{3} (e^{\lambda_{j+1} L} - e^{\lambda_j L}) e^{\lambda_{j+2} x}}{\sum_{j=1}^{3} (\lambda_{j+1} - \lambda_j) e^{-\lambda_{j+2} L}}.$$

Then

$$B(z, x) = M(z, x) \overline{M(z - p, x)} \varphi_x(x).$$

It is clear from the definition of M that

$$M(-z, x) = \overline{M(z, x)}.$$

We then have

$$B(-z, x) = M(-z, x) \overline{M(-z - p, x)} \varphi_x(x) = \overline{M(z, x) \overline{M(z + p, x)}\, \overline{\varphi_x(x)}}.$$

We thus obtain the result in the case where z is negative and large by taking the conjugate of the corresponding expression for large positive z in which η_j and p are replaced by $-\eta_j$ and $-p$. The conclusion follows. $\square$

As a consequence of Lemmas 3.1 and 3.4, we obtain

Lemma 3.5. *Let $p \in \mathbb{R}$ and let φ be defined by* (3.3)*. Assume that* (3.4) *holds and $\eta_j \neq 0$ for $j = 1, 2, 3$. Let $u \in L^2(0, +\infty)$ and let $y \in C([0, +\infty); L^2(0, L)) \cap L^2_{\mathrm{loc}}\big([0, +\infty); H^1(0, L)\big)$ be the unique solution of* (2.9) *and* (2.10)*. We have*

$$\int_0^{+\infty} \int_0^L |y(t,x)|^2 \varphi_x(x) e^{-ipt}\, dx\, dt = \int_{\mathbb{R}} \hat{u}(z) \overline{\hat{u}(z-p)} \Big(\frac{E}{|z|^{4/3}} + O(|z|^{-5/3}) \Big)\, dz. \tag{3.31}$$

Using Lemma 3.5, we will establish the following result which is the key ingredient for the analysis of the non-null-controllability for small time of the KdV system (1.1).

Proposition 3.6. *Let $p \in \mathbb{R}$ and let φ be defined by* (3.3)*. Assume that* (3.4) *holds and $\eta_j \neq 0$ for $j = 1, 2, 3$. Let $u \in L^2(0, +\infty)$ and let $y \in C([0, +\infty); L^2(0, L)) \cap L^2_{\mathrm{loc}}\big([0, +\infty); H^1(0, L)\big)$ be the unique solution of* (2.9) *and* (2.10)*. Assume that $u \not\equiv 0$, $u(t) = 0$ for $t > T$, and $y(t, \cdot) = 0$ for large t. Then, there exists a real number $N(u) \geq 0$ such that $C^{-1}\|u\|_{H^{-2/3}} \leq N(u) \leq C\|u\|_{H^{-2/3}}$ for some constant $C \geq 1$ depending only on L, and* [4]

$$\int_0^{\infty} \int_0^L |y(t,x)|^2 e^{-ipt} \varphi_x(x)\, dx\, dt = N(u)^2 \big(E + O(1) T^{1/4}\big). \tag{3.32}$$

Here we use the following definition, for $s < 0$ and for $u \in L^2(\mathbb{R}_+)$,

$$\|u\|^2_{H^s(\mathbb{R})} = \int_{\mathbb{R}} |\hat{u}|^2 (1 + |\xi|^2)^s\, d\xi,$$

where $\hat{u}$ is the Fourier transform of the extension of u by 0 for $t < 0$.

Before giving the proof of Proposition 3.6, we present one of its direct consequences. Denote $\xi_1(t, x) = \Re\{\varphi(x) e^{-ipt}\}$ and $\xi_2(t, x) = \Im\{\varphi(x) e^{-ipt}\}$. Then

$$\xi_1(t, x) + i\xi_2(t, x) = \varphi(x) e^{-ipt}. \tag{3.33}$$

Denote $E_1 = \Re(E)$ and $E_2 = \Im(E)$, and set

$$\Psi(t, x) = E_1 \xi_1(t, x) + E_2 \xi_2(t, x). \tag{3.34}$$

Multiplying (3.32) by $\overline{E}$ and normalizing appropriately, we have

Corollary 3.7. *Let $p \in \mathbb{R}$ and let φ be defined by* (3.3)*. Assume that* (3.4) *holds, $\eta_j \neq 0$ for $j = 1, 2, 3$, and $E \neq 0$. There exists $T_* > 0$ such that, for any (real) $u \in L^2(0, +\infty)$ with $u(t) = 0$ for $t > T_*$ and $y(t, \cdot) = 0$ for large t where y is the unique solution of* (2.9) *and* (2.10)*, we have*

$$\int_0^{\infty} \int_0^{+\infty} y^2(t, x) \Psi_x(t, x)\, dx\, dt \geq C\|u\|^2_{H^{-2/3}(\mathbb{R})}. \tag{3.35}$$

We are ready to give the

Proof. [Proof of Proposition 3.6] By Proposition 2.8,

$$\hat{u}G/H \text{ is an entire function.}$$

By Lemma 2.6, G and H are entire functions. The same holds for $\hat{u}$ since $u(t) = 0$ for large t. One can show that the number of common roots of G and H in $\mathbb{C}$ is finite, see Lemma B.2 in Appendix B.

[4]The map $u \mapsto N(u)$ is actually a norm, which is (somewhat) explicitly given in the proof, by $N(u)^2 = \|\hat{w}\|^2_{L^2}$, where w is defined in Eq (3.46).

Let $z_1, \dots, z_k$ be the distinct common roots of G and H in $\mathbb{C}$. There exist $m_1, \dots, m_k \in \mathbb{N}$ such that [5], with

$$\Gamma(z) = \prod_{j=1}^{k} (z - z_j)^{m_j} \quad \text{ in } \mathbb{C},$$

the following two functions are entire

$$\mathcal{G}(z) := \frac{G(z)}{\Gamma(z)} \quad \text{ and } \quad \mathcal{H}(z) := \frac{H(z)}{\Gamma(z)}, \tag{3.36}$$

and $\mathcal{G}$ and $\mathcal{H}$ have no common roots. Since

$$\hat{u}\mathcal{G}/\mathcal{H} = \hat{u}G/H$$

which is an entire function, it follows that the function v defined by

$$v(z) = \hat{u}(z)/\mathcal{H}(z) = \hat{u}(z)\frac{\Gamma(z)\Xi(z)}{\det Q(z)} \text{ in } \mathbb{C} \tag{3.37}$$

is also an entire function.

It is clear that

$$\hat{u}(z) = v(z)\mathcal{H}(z) \text{ in } \mathbb{C}. \tag{3.38}$$

We consider the holomorphic function v restricted to $\mathcal{L}_m := \Big\{ z \in \mathbb{C}; |\Re(z)| \le cm, \ -\big((2m + 1)/(\sqrt{3}L)\big)^3 \le \Im(z) \le \big((2m+1)/(\sqrt{3}L)\big)^3 \Big\}$ with large $m \in \mathbb{N}$. Using Proposition 2.8 to bound $\hat{u}$, and Lemma B.3 in Appendix B to bound $(\det Q(z))^{-1}$, we can bound v on $\partial \mathcal{L}_m$ (and thus also in the interior of $\mathcal{L}_m$) by

$$|v(z)| \le C_\varepsilon e^{(T+\varepsilon/2)\big((2m+1)/(\sqrt{3}L)\big)^3} \text{ in } \mathcal{L}_m, \tag{3.39}$$

for all $\varepsilon > 0$, since, for large $|z|$,

$$|\Xi(z)| \le C|z|.$$

Note that the constant C_ε can be chosen independently of m. Here we used the fact

$$|\hat{u}(z)| \le C e^{T|\Im(z)|} \text{ for } z \in \mathbb{C}.$$

On the other hand, applying Lemma 3.3 and item 2 of Lemma B.3, we have

$$|v(z)| \le C_\varepsilon e^{(T+\varepsilon)|z|} \text{ in } \Big\{ z \in \mathbb{C}; |\Re(z)| \ge cm, \ -\big((2m+1)/(\sqrt{3}L)\big)^3 \le \Im(z) \le \big((2m+1)/(\sqrt{3}L)\big)^3 \Big\}. \tag{3.40}$$

Combining (3.39) and (3.40) yields

$$|v(z)| \le C_\varepsilon e^{(T+\varepsilon)|z|} \text{ in } \mathbb{C}. \tag{3.41}$$

Since $\mathcal{H}$ is a non-constant entire function, there exists $\gamma > 0$ such that

$$\mathcal{H}'(z + i\gamma) \neq 0 \text{ for all } z \in \mathbb{R}. \tag{3.42}$$

Fix such an γ and denote $\mathcal{H}_\gamma(z) = \mathcal{H}(z + i\gamma)$ for $z \in \mathbb{C}$.

Let us prove some asymptotics for $\mathcal{H}_\gamma$. Since $\sum_{j=1}^3 \lambda_j = 0$, it follows from (2.6) that

$$\det Q = (\lambda_2 - \lambda_1)e^{-\lambda_3 L} + (\lambda_3 - \lambda_2)e^{-\lambda_1 L} + (\lambda_1 - \lambda_3)e^{-\lambda_2 L}.$$

We use the convention in Lemma 3.3. Thus, by Lemma 3.3, for fixed $\beta \ge 0$,

$$\mathcal{H}(z + i\beta) = \frac{\det Q(z + i\gamma)}{\Xi(z + i\gamma)\Gamma(z + i\gamma)} = \kappa z^{-2/3 - \sum_{i=1}^k m_j} e^{-\mu_1 L z^{1/3}} \big(1 + O(z^{-1/3})\big), \tag{3.43}$$

[5]One can prove that $m_j = 1$ for $1 \le j \le k$ by Lemma B.1 in Appendix B, but this is not important at this stage.

where

$$\kappa = -\frac{1}{(\mu_2 - \mu_1)(\mu_1 - \mu_3)}.$$

We can also compute the asymptotic expansion of $\mathcal{H}'(z+i\beta)$, either by explicitly computing the asymptotic behavior of $\lambda_j'(z+i\beta)$ for large positive z (formally, one just needs to take the derivative of (3.43) with respect to z), or by using the Cauchy integral formula on the contour $\partial D(z,r)$ for some fixed r to justify differentiating Eq. (3.43). We get:

$$\mathcal{H}'(z+i\beta) = -\frac{\mu_1 L}{3} z^{-2/3} \kappa z^{-2/3 - \sum_{i=1}^k m_j} e^{-\mu_1 L z^{1/3}} \big(1 + O(z^{-1/3})\big).$$

We then get

$$\lim_{z\in\mathbb{R}, z\to+\infty} \mathcal{H}(z)|z|^{-2/3}/\mathcal{H}_\gamma'(z) = \alpha := 3e^{-i\pi/6}/L.$$

Similarly, we obtain

$$\lim_{z\in\mathbb{R}, z\to-\infty} \mathcal{H}(z)|z|^{-2/3}/\mathcal{H}_\gamma'(z) = -\bar{\alpha}.$$

Moreover, we have

$$\big|\mathcal{H}(z)|z|^{-2/3} - \alpha\mathcal{H}_\gamma'(z)\big| \le C|\mathcal{H}(z)||z|^{-1} \le C|\mathcal{H}_\gamma'(z)||z|^{-1/3} \text{ for large positive } z, \tag{3.44}$$

and

$$\big|\mathcal{H}(z)|z|^{-2/3} + \bar{\alpha}\mathcal{H}_\gamma'(z)\big| \le C|\mathcal{H}(z)||z|^{-1} \le C|\mathcal{H}_\gamma'(z)||z|^{-1/3} \text{ for large negative } z. \tag{3.45}$$

Set

$$\hat{w}(z) = v(z)\mathcal{H}_\gamma'(z) = \hat{u}(z)\mathcal{H}_\gamma'(z)\mathcal{H}(z)^{-1}. \tag{3.46}$$

Then $\hat{w}$ is an entire function and satisfies Paley-Wiener's conditions for the interval $(-T-\varepsilon, T+\varepsilon)$ for all $\varepsilon > 0$, see e.g. [41, 19.3 Theorem]. Indeed, this follows from the facts $|\hat{w}(z)| \le C_\varepsilon |v(z)| e^{\varepsilon|z|}$ for $z \in \mathbb{C}$ by Lemma 3.3, $|v(z)| \le C_\varepsilon e^{(T+\varepsilon)|z|}$ for $z \in \mathbb{C}$ by (3.41), $|\mathcal{H}_\gamma'(z) v(z)| = |\mathcal{H}_\gamma'(z)\mathcal{H}(z)^{-1}\hat{u}(z)| \le |\hat{u}(z)|$ for real z with large $|z|$, so that $\int_{\mathbb{R}} |\hat{u}|^2 < +\infty$.

We claim that[6]

$$\left|\int_0^L B(z,x)\,dx\right| \le \frac{C}{(|z|+1)^{4/3}} \text{ for } z \in \mathbb{R}. \tag{3.47}$$

In fact, this inequality follows from Lemma 3.4 for large z, and from Lemma B.1 in Appendix B otherwise since, for if z is a real solution of the equation $H(z) = 0$, which is simple by Lemma B.1, it holds, by Lemma B.1 again,

$$\sum_{j=1}^3 (e^{\lambda_{j+1}L} - e^{\lambda_j L}) e^{\lambda_{j+2}x} \overset{(B.2)}{=} 0.$$

From (3.42), (3.44), (3.45), and (3.47), we derive that

$$\left|\hat{u}(z)\overline{\hat{u}(z-p)} \int_0^L B(z,x)\,dx\right| \le C|\hat{w}(z)||\hat{w}(z-p)| \text{ for } z \in \mathbb{R}. \tag{3.48}$$

[6]Recall that B was defined in Eq.(3.5).

Note that, for $m \geq 1$,

$$\left| \int_{|z|>m} \hat{u}(z)\overline{\hat{u}(z-p)} \int_0^L B(z,x)\,dx\,dz - E|\alpha|^2 \int_{|z|>m} \hat{w}(z)\overline{\hat{w}(z-p)}\,dz \right|$$
$$\leq \int_{|z|>m} \left| \hat{u}(z)\overline{\hat{u}(z-p)} \Big(\int_0^L B(z,x)\,dx - E|z|^{-4/3} \Big) \right| dz$$
$$+ |E| \int_{|z|>m} \left| |\alpha|^2 \hat{w}(z)\overline{\hat{w}(z-p)} - |z|^{-4/3} \hat{u}(z)\overline{\hat{u}(z-p)} \right| dz.$$

Using (3.44) (3.45), and Lemmas 3.1 and 3.4, we derive that

$$\left| \int_{|z|>m} \hat{u}(z)\overline{\hat{u}(z-p)} \int_0^L B(z,x)\,dx\,dz - E|\alpha|^2 \int_{|z|>m} \hat{w}(z)\overline{\hat{w}(z-p)}\,dz \right|$$
$$\leq C \int_{|z|>m} |\hat{w}(z)||\hat{w}(z-p)||z|^{-1/3} dz.$$

We derive from (3.42) and (3.48) that

$$\left| \int_{\mathbb{R}} \hat{u}(z)\overline{\hat{u}(z-p)} \int_0^L B(z,x)\,dx\,dz - E|\alpha|^2 \int_{\mathbb{R}} \hat{w}(z)\overline{\hat{w}(z-p)}\,dz \right|$$
$$\leq C \int_{|z|\leq m} |\hat{w}(z)||\overline{\hat{w}(z-p)}|\,dz + C m^{-1/3} \int_{|z|>m} |\hat{w}(z)||\hat{w}(z-p)|\,dz.$$

Since, for $z \in \mathbb{R}$,

$$|\hat{w}(z)| \leq C\|w\|_{L^1} = C\|w\|_{L^1(-T,T)} \leq C T^{1/2} \|w\|_{L^2(\mathbb{R})},$$

we derive that

$$\left| \int_{\mathbb{R}} \hat{u}(z)\overline{\hat{u}(z-p)} \int_0^L B(z,x)\,dx\,dz - E|\alpha|^2 \int_{\mathbb{R}} \hat{w}(z)\overline{\hat{w}(z-p)}\,dz \right| \leq C \int_{-T}^{T} \Big(Tm + m^{-1/3} \Big) |w|^2.$$

Using the fact

$$\int_{\mathbb{R}} \hat{w}(z)\overline{\hat{w}(z-p)}\,dz = \int_{\mathbb{R}} |w(t)|^2 e^{-itp}\,dt = \int_{-T}^{T} |w(t)|^2 e^{-itp}\,dt,$$

we obtain, by choosing $m = 1/T^{3/4}$,

$$\int_{\mathbb{R}} \hat{u}(z)\overline{\hat{u}(z-p)} \int_0^L B(z,x)\,dx\,dz = E|\alpha|^2 \int_{-T}^{T} |w(t)|^2 (1 + O(1)T^{1/4})\,dt.$$

The conclusion follows by noting that

$$\int_{\mathbb{R}} |w(t)|^2 = \int_{\mathbb{R}} |\hat{w}(z)|^2\,dz \geq C \int_{\mathbb{R}} \frac{|\hat{u}(z)|^2}{1+|z|^{4/3}}\,dz,$$

and by normalizing u such that $|\alpha|\|w\|_{L^2(\mathbb{R})} = 1$. □

4. Useful estimates for the linear KdV equations

In this section, we establish several results for the linear KdV equations which will be used in the proof of Theorem 1.2. Our study of the inhomogeneous KdV equations is based on three elements. The first one is on the information of the KdV equations explored previously. The second one is a connection between the KdV equations and the KdV-Burgers equations, as previously suggested in [29, 13]. The third one is on estimates for the KdV-Burgers equations with periodic boundary condition. This section contains two subsections. The first one is on inhomogeneous KdV-Burgers equations with periodic boundary condition and the second one is on the inhomogeneous KdV equations.

4.1. **On the linear KdV-Burgers equations.**

In this section, we derive several estimates for the solutions of the linear KdV-Burgers equations using low regular data information. The main result of this section is the following result:

Lemma 4.1. *Let $L > 0$ and $f_1 \in L^1\big(\mathbb{R}_+; L^1(0, L)\big)$ and $f_2 \in L^1\big(\mathbb{R}_+; W^{1,1}(0, L)\big)$ be such that*

$$\int_0^L f_1(t, x)\, dx = 0 \text{ for a.e. } t > 0, \tag{4.1}$$

and

$$f_2(t, 0) = f_2(t, L) = 0 \text{ for a.e. } t > 0. \tag{4.2}$$

Set $f = f_1 + f_{2,x}$ and assume that $f \in L^1\big(\mathbb{R}_+; L^2(0, L)\big)$. Let y be the unique solution in $C\big([0, +\infty); L^2(0, L)\big) \cap L^2_{\rm loc}\big([0, +\infty); H^1(0, L)\big)$, which is periodic in space, of the system

$$y_t(t, x) + 4y_x(t, x) + y_{xxx}(t, x) - 3y_{xx}(t, x) = f(t, x) \text{ in } (0, +\infty) \times (0, L), \tag{4.3}$$

and

$$y(t = 0, \cdot) = 0 \text{ in } (0, L). \tag{4.4}$$

We have, for $x \in [0, L]$,

$$\|y(\cdot, x)\|_{L^2(\mathbb{R}_+)} + \|y_x(\cdot, x)\|_{H^{-1/3}(\mathbb{R})} \le C\|f\|_{L^1(\mathbb{R}_+\times(0,L))}, \tag{4.5}$$

and

$$\|y(\cdot, x)\|_{H^{-1/3}(\mathbb{R})} + \|y_x(\cdot, x)\|_{H^{-2/3}(\mathbb{R})} + \|y\|_{L^2(\mathbb{R}_+;H^{-1}(0,L))} \le C\|(f_1, f_2)\|_{L^1(\mathbb{R}_+\times(0,L))}. \tag{4.6}$$

Assume that $f(t, \cdot) = 0$ for $t > T$. We have, for all $\delta > 0$, and for all $t \ge T + \delta$,

$$|y_t(t, x)| + |y_x(t, x)| \le C_\delta\|(f_1, f_2)\|_{L^1(\mathbb{R}_+\times(0,L))} \text{ for } x \in [0, L]. \tag{4.7}$$

Here C (resp. C_δ) denotes a positive constant depending only on L (resp. L and δ).

Remark 4.2. Using the standard energy method, as for the KdV equations, one can prove that if $f \in L^1(\mathbb{R}_+, L^2(0, L))$ with $\int_0^L f(t, x)\, dx = 0$ for a.e. $t > 0$ (this holds by (4.1) and (4.2)), then (4.3)-(4.4) has a unique solution in $C([0, +\infty); L^2(0, L)) \cap L^2([0, +\infty); H^1(0, L))$ which is periodic in space.

In the proof of Lemma 4.1, we use the following elementary estimate, which has its root in the work of Bourgain [14].

Lemma 4.3. *There exists a positive constant C such that, for $j = 0, 1$, and $z \in \mathbb{R}$,*[7]

$$\sum_{n \neq 0} \frac{|n|^j}{|z + 4n - n^3| + n^2} \le \frac{C \ln(|z| + 2)}{(|z| + 2)^{\frac{2-j}{3}}}. \tag{4.8}$$

Proof. For $z \in \mathbb{R}$, let $k \in \mathbb{Z}$ be such that $k^3 \le z < (k+1)^3$. It is clear that

$$\sum_{n \neq 0} \frac{|n|^j}{|z + 4n - n^3| + n^2} = \sum_{m+k \neq 0} \frac{|m + k|^j}{|z + 4(m + k) - (m + k)^3| + (m + k)^2}. \tag{4.9}$$

We split the sum in two parts, one for $|m| \le 2|k| + 2$ and one for $|m| > 2|k| + 2$. Since $k^3 \le z < (k+1)^3$, one can check that, for $m \in \mathbb{Z}$, $m + k \neq 0$, and $|m| \le 2|k| + 2$,

$$|z + 4(m + k) - (m + k)^3| + |m + k|^2 \ge C(|m| + 1)(|k| + 2)^2,$$

and, for $|m| \ge 2|k| + 2$,

$$|z + 4(m + k) - (m + k)^3| + |m + k|^2 \ge C|m|^3$$

(by considering $|k| \ge 10$ and $|k| < 10$). We deduce that

$$\sum_{|m| \le 2|k|+2, m+k \neq 0} \frac{|m + k|^j}{|z + 4(m + k) - (m + k)^3| + (m + k)^2} \le C \sum_{|m| \le 2|k|+2} \frac{1}{(|k| + 2)^{2-j}(|m| + 1)} \le \frac{C \ln(|k| + 2)}{(|k| + 2)^{2-j}}, \tag{4.10}$$

and

$$\sum_{|m| > 2|k|+2} \frac{|m + k|^j}{|z + 4(m + k) - (m + k)^3| + (m + k)^2} \le C \sum_{|m| > 2|k|+2} \frac{1}{|m|^{3-j}} \le \frac{C}{(|k| + 2)^{2-j}}. \tag{4.11}$$

Combining (4.9) - (4.11) yields (4.8). □

In what follows, for an appropriate function ζ defined in $\mathbb{R}_+ \times (0, L)$, we denote

$$\hat{\hat{\zeta}}(z, n) = \frac{1}{L} \int_0^L \hat{\zeta}(z, x) e^{-\frac{i2\pi nx}{L}} \, dx \text{ for } (z, n) \in \mathbb{R} \times \mathbb{Z}.$$

Recall that to define $\hat{\zeta}(z, x)$, we extend ζ by 0 for $t < 0$.

Proof. [Proof of Lemma 4.1] For simplicity of notations, we will assume that $L = 2\pi$. We establish (4.5), (4.6), and (4.7) in Steps 1, 2 and 3 below.

Step 1: Proof of (4.5).

We first estimate $\|y(\cdot, x)\|_{L^2(\mathbb{R}_+)}$ for $x \in [0, L]$. From (4.3) and (4.4), we have

$$\hat{\hat{y}}(z, n) = \frac{\hat{\hat{f}}(z, n)}{i(z + 4n - n^3) + 3n^2} \quad \text{for } (z, n) \in \mathbb{R} \times (\mathbb{Z} \setminus \{0\}), \tag{4.12}$$

and

$$\hat{\hat{y}}(z, 0) = 0 \quad \text{for } z \in \mathbb{R} \tag{4.13}$$

[7]We recall that an absolutely convergent sum is nothing but the integral with the counting measure, which is σ-finite. In the following, we will often exchange sums and integrals without comments, the justification being one of Fubini's theorem.

since $\int_0^L f(t,x)\,dx = 0$ for $t > 0$ by (4.1) and (4.2). By Plancherel's theorem, we obtain

$$\int_{\mathbb{R}_+} |y(t,x)|^2\,dt = \int_{\mathbb{R}} |\hat{y}(z,x)|^2\,dz \le C \int_{\mathbb{R}} \left| \sum_{n\neq 0} \frac{|\hat{\hat{f}}(z,n)|}{|z+4n-n^3|+n^2} \right|^2 dz. \tag{4.14}$$

Since

$$|\hat{\hat{f}}(z,n)| \le C\|f\|_{L^1(\mathbb{R}_+\times(0,L))}, \tag{4.15}$$

it follows from (4.14) that

$$\int_{\mathbb{R}_+} |y(t,x)|^2\,dt \le C\|f\|^2_{L^1(\mathbb{R}_+\times(0,L))} \int_{\mathbb{R}} \left| \sum_{n\neq 0} \frac{1}{|z+4n-n^3|+n^2} \right|^2 dz. \tag{4.16}$$

Applying Lemma 4.3 with $j = 0$, we derive from (4.16) that

$$\int_{\mathbb{R}_+} |y(t,x)|^2\,dt \le C\|f\|^2_{L^1(\mathbb{R}_+\times(0,L))} \int_{\mathbb{R}} \frac{\ln^2(|z|+2)}{(|z|+2)^{4/3}}\,dz,$$

which yields

$$\|y(\cdot,x)\|_{L^2} \le C\|f\|_{L^1(\mathbb{R}_+\times(0,L))}. \tag{4.17}$$

We next estimate $\|y_x(\cdot,x)\|_{H^{-1/3}(\mathbb{R}_+)}$ for $x \in [0,L]$. We have, by (4.12), (4.13), and (4.15),

$$\|y_x(\cdot,x)\|^2_{H^{-1/3}(\mathbb{R}_+)} \le C\|f\|^2_{L^1(\mathbb{R}_+\times(0,L))} \int_{\mathbb{R}} \frac{1}{(1+|z|^2)^{1/3}} \left| \sum_{n\neq 0} \frac{|n|}{|z+4n-n^3|+n^2} \right|^2 dz. \tag{4.18}$$

Applying Lemma 4.3 with $j = 1$, we derive from (4.18) that

$$\|y_x(\cdot,x)\|^2_{H^{-1/3}(\mathbb{R}_+)} \le C\|f\|^2_{L^1(\mathbb{R}_+\times(0,L))} \int_{\mathbb{R}} \frac{\ln^2(|z|+2)}{(|z|+2)^{4/3}}\,dz,$$

which yields

$$\|y_x(\cdot,x)\|_{H^{-1/3}(\mathbb{R})} \le C\|f\|_{L^1(\mathbb{R}_+\times(0,L))}. \tag{4.19}$$

Assertion (4.5) now follows from (4.17) and (4.19).

Step 2: Proof of (4.6). By Step 1, without loss of generality, one might assume that $f_1 = 0$. The proof of the inequality $\|y(\cdot,x)\|_{H^{-1/3}} \le C\|f_2\|_{L^1(\mathbb{R}_+\times(0,L))}$ is similar to the one of (4.19) and is omitted.

To prove

$$\|y_x(\cdot,x)\|_{H^{-2/3}(\mathbb{R})} \le C\|f_2\|_{L^1(\mathbb{R}_+\times(0,L))}, \tag{4.20}$$

we proceed as follows. For $z \in \mathbb{R}$, it holds

$$\hat{y}_x(z,x) = -\frac{1}{L}\int_0^L \hat{f}_2(z,\xi) \sum_{n\neq 0} \frac{n^2 e^{in(x-\xi)}}{i(z+4n-n^3)+3n^2}\,d\xi. \tag{4.21}$$

We have, for some large positive constant c,

$$\left|\sum_{|n|\geq c(|z|+1)} \frac{n^2 e^{in(x-\xi)}}{i(z+4n-n^3)+3n^2} + \sum_{|n|\geq c(|z|+1)} \frac{e^{in(x-\xi)}}{in}\right| \leq C \sum_{|n|\geq c(|z|+1)} \frac{1}{|n|^2} \leq \frac{C}{|z|+1},$$

$$\left|\sum_{0<|n|\leq c(|z|+1)} \frac{e^{in(x-\xi)}}{in}\right| \leq C\ln(|z|+2),$$

and, as in (4.10) in the proof of Lemma 4.3,

$$\left|\sum_{0<|n|\leq c(|z|+1)} \frac{n^2 e^{in(x-\xi)}}{i(z+4n-n^3)+3n^2}\right| \leq C\ln(|z|+2).$$

It follows that

$$\left|\sum_{n\neq 0} \frac{n^2 e^{in(x-\xi)}}{i(z+4n-n^3)+3n^2} + \sum_{n\neq 0} \frac{e^{in(x-\xi)}}{in}\right| \leq \frac{C}{|z|+1} + C\ln(|z|+2). \tag{4.22}$$

Since

$$\sum_{n\neq 0} \frac{e^{in\xi'}}{in} = -\xi' + \pi \text{ for } \xi' \in (0, 2\pi),$$

and

$$\|y_x(\cdot, x)\|^2_{H^{-2/3}(\mathbb{R})} = \int_{\mathbb{R}} \frac{|\hat{y}_x(z,x)|^2}{(1+|z|^2)^{2/3}}\, dz,$$

assertion (4.20) follows from (4.21) and (4.22).

We next deal with

$$\|y\|_{L^2(\mathbb{R}_+; H^{-1}(0,L))} \leq C\|f_2\|_{L^1(\mathbb{R}_+\times(0,L))}.$$

Since

$$\|y\|^2_{L^2(\mathbb{R}_+; H^{-1}(0,L))} \leq C\int_{\mathbb{R}} \sum_{n\neq 0} \left|\frac{\hat{\hat{f}}_2(z,n)}{|i(z+4n-n^3)|+3n^2}\right|^2 dz,$$

the estimate follows from Lemma 4.3. The proof of Step 2 is complete.

Step 3: Proof of (4.7).

For simplicity of the presentation, we will assume that $f_1 = 0$. We have the following representation for the solution:

$$y(t,x) = \sum_{n\neq 0} e^{inx} \int_0^t e^{-\left(i(4n-n^3)+3n^2\right)(t-\tau)} \left(\frac{in}{L}\int_0^L f_2(\tau,\xi)e^{-in\xi}\, d\xi\right) d\tau. \tag{4.23}$$

Let $\mathbb{1}_A$ denote the characteristic function of a set A in $\mathbb{R}$. Assertion (4.7) then follows easily from (4.23) by noting that, for $t \geq T+\delta$

$$\sum_{n\neq 0} \int_0^t |n|^{10} e^{-3n^2(t-\tau)} \mathbb{1}_{\{\tau<T\}}\, d\tau < C_\delta.$$

The proof is complete. □

4.2. **On the linear KdV equations.** In this section, we derive various results on the linear KdV equations using low regularity data information. These will be used in the proof of Theorem 1.2. We begin with

Lemma 4.4. *Let* $h = (h_1, h_2, h_3) \in H^{1/3}(\mathbb{R}_+) \times H^{1/3}(\mathbb{R}_+) \times L^2(\mathbb{R}_+)$, *and let* $y \in C\big([0, +\infty); L^2(0, L)\big) \cap L^2_{\rm loc}\big([0, +\infty); H^1(0, L)\big)$ *be the unique solution of the system*

$$(4.24)\quad \begin{cases} y_t(t, x) + y_x(t, x) + y_{xxx}(t, x) = 0 & \text{in } (0, +\infty) \times (0, L), \\ y(t, x = 0) = h_1(t),\ y(t, x = L) = h_2(t),\ y_x(t, x = L) = h_3(t) & \text{in } (0, +\infty), \end{cases}$$

and

$$(4.25)\quad y(t = 0, \cdot) = 0 \text{ in } (0, L).$$

We have, for $T > 0$,

$$(4.26)\quad \|y\|_{L^2((0,T)\times(0,L))} \le C_{T,L}\Big(\|(h_1, h_2)\|_{L^2(\mathbb{R}_+)} + \|h_3\|_{H^{-1/3}(\mathbb{R})}\Big),$$

and

$$(4.27)\quad \|y\|_{L^2((0,T);H^{-1}(0,L))} \le C_{T,L}\Big(\|(h_1, h_2)\|_{H^{-1/3}(\mathbb{R})} + \|h_3\|_{H^{-2/3}(\mathbb{R})}\Big),$$

for some positive constant $C_{T,L}$ *independent of* h.

Here and in what follows, $H^{-1}(0, L)$ is the dual space of $H^1_0(0, L)$ with the corresponding norm.

Proof. By the linearity and the uniqueness of the system, it suffices to consider the three cases $(h_1, h_2, h_3) = (0, 0, h_3)$, $(h_1, h_2, h_3) = (h_1, 0, 0)$, and $(h_1, h_2, h_3) = (0, h_2, 0)$ separately.

We first consider the case $(h_1, h_2, h_3) = (0, 0, h_3)$. Making a truncation, without loss of generality, one might assume that $h_3 = 0$ for $t > 2T$. This fact is assumed from now on. Let $g_3 \in C^1(\mathbb{R})$ be such that $\operatorname{supp} g_3 \subset [T, 3T]$, and if z is a real solution of the equation $\det Q(z)\Xi(z) = 0$ of order m then z is also a real solution of order m of $\hat{h}_3(z) - \hat{g}_3(z)$, and

$$\|g_3\|_{H^{-1/3}(\mathbb{R})} \le C_{T,L}\|h_3\|_{H^{-2/3}(\mathbb{R})}.$$

The construction of g_3, inspired by the moment method, see e.g. [45], can be done as follows. Set $\eta(t) = e^{-1/(t^2-(T)^2)}\mathbb{1}_{|t|<T}$ for $t \in \mathbb{R}$. Assume that $z_1, \dots, z_k$ are real, distinct solutions of the equation $\det Q(z)\Xi(z) = 0$, and $m_1, \dots, m_k$ are the corresponding orders (the number of real solutions of the equation $\det Q(z)\Xi(z) = 0$ is finite by Lemma B.1 and in fact they are simple; nevertheless, we ignore this point and present a proof without using this information). Set, for $z \in \mathbb{C}$,

$$\zeta(z) = \sum_{i=1}^k \left(\hat{\eta}(z - z_i) \prod_{\substack{j=1 \\ j\neq i}}^k (z - z_j)^{m_j} \Big(\sum_{l=0}^{m_i} c_{i,l}(z - z_i)^l\Big)\right),$$

where $c_{i,l} \in \mathbb{C}$ is chosen such that

$$\frac{d^l}{dz^l}\Big(e^{2iTz}\zeta(z)\Big)_{z=z_i} = \frac{d^l}{dz^l}\hat{h}_3(z_i) \text{ for } 0 \le l \le m_i,\ 1 \le i \le k.$$

This can be done since $\hat{\eta}(0) \neq 0$. Since

$$|\hat{\eta}(z)| \le Ce^{T|\Im(z)|},$$

and, by [45, Lemma 4.3],

$$|\hat{\eta}(z)| \le C_1 e^{-C_2|z|^{1/2}} \text{ for } z \in \mathbb{R},$$

using Paley-Wiener's theorem, one can prove that ζ is the Fourier transform of a function ψ of class C^1; moreover, ψ has the support in $[-T, T]$. Set, for $z \in \mathbb{C}$,

$$g_3(t) = \psi(t + 2T).$$

Using the fact $\hat{g}_3(z) = e^{i2Tz}\zeta(z)$, one can check that $\hat{g}_3 - \hat{h}_3$ has solutions $z_1, \dots, z_k$ with the corresponding orders $m_1, \dots, m_k$. One can check that

$$\|\psi\|_{C^1} \le C_{T,L} \sum_{i=1}^{k} \sum_{l=0}^{m_i} \left| \frac{d^l}{dz^l} \hat{h}_3(z_i) \right|,$$

which yields

$$\|\psi\|_{C^1} \le C_{T,L} \|h_3\|_{H^{-2/3}(\mathbb{R})}.$$

The required properties of g_3 follow.

By considering the solution corresponding to $h_3 - g_3$, without loss of generality, one might assume that if z is a real solution of order m of the equation $\det Q(z)\Xi(z) = 0$ then z is also a real solution of order m of $\hat{h}_3(z)$. This fact is assumed from now on.

We now establish (4.26). We have, by Lemma 2.4,

$$\hat{y}(z, x) = \frac{\hat{h}_3(z)}{\det Q} \sum_{j=1}^{3} \big(e^{\lambda_{j+2}L} - e^{\lambda_{j+1}L}\big) e^{\lambda_j x} \text{ for a.e. } x \in (0, L). \tag{4.28}$$

From the assumption of h_3, we have, for $z \in \mathbb{R}$ and $|z| \le \gamma$,

$$\left| \frac{\hat{h}_3(z)}{\det Q(z)} \sum_{j=1}^{3} \big(e^{\lambda_{j+2}L} - e^{\lambda_{j+1}L}\big) e^{\lambda_j x} \right| \le C_{T,\gamma} \|h_3\|_{H^{-2/3}(\mathbb{R})}, \tag{4.29}$$

and, by Lemma 3.3, for $z \in \mathbb{R}$, $|z| \ge \gamma$ with sufficiently large γ,

$$\left| \frac{1}{\det Q} \sum_{j=1}^{3} \big(e^{\lambda_{j+2}L} - e^{\lambda_{j+1}L}\big) e^{\lambda_j x} \right| \le \frac{C}{(1 + |z|)^{1/3}}. \tag{4.30}$$

Combining (4.29) and (4.30) yields

$$\|\hat{y}\|_{L^2\big(\mathbb{R} \times (0,L)\big)} \le C_T \|h_3\|_{H^{-1/3}(\mathbb{R})},$$

which is (4.26) when $(h_1, h_2, h_3) = (0, 0, h_3)$.

We next deal with (4.27). The proof of (4.27) is similar to the one of (4.26). One just notes that, instead of (4.30), it holds, for $z \in \mathbb{R}$, $|z| \ge \gamma$ with sufficiently large γ,

$$\left\| \frac{1}{\det Q} \sum_{j=1}^{3} \big(e^{\lambda_{j+2}L} - e^{\lambda_{j+1}L}\big) e^{\lambda_j x} \right\|_{H^{-1}(0,L)} \le \frac{C}{(1 + |z|)^{2/3}}. \tag{4.31}$$

The details are omitted.

The proof in the case $(h_1, h_2, h_3) = (h_1, 0, 0)$ or in the case $(h_1, h_2, h_3) = (0, h_2, 0)$ is similar. We only mention here that the solution corresponding to the triple $(h_1, 0, 0)$ is given by

$$\hat{y}(z, x) = \frac{\hat{h}_1(z)}{\det Q} \sum_{j=1}^{3} (\lambda_{j+2} - \lambda_{j+1}) e^{\lambda_j (x - L)} \text{ for a.e. } x \in (0, L),$$

and the solution corresponding to the triple $(0, h_2, 0)$ is given by

$$\hat{y}(z,x) = \frac{\hat{h}_2(z)}{\det Q} \sum_{j=1}^{3} (\lambda_{j+1} e^{\lambda_{j+1} L} - \lambda_{j+2} e^{\lambda_{j+2} L}) e^{\lambda_j x} \text{ for a.e. } x \in (0, L).$$

The details are left to the reader. □

Remark 4.5. The estimates in Lemma 4.4 are in the spirit of the well-posedness results due to Bona et al. in [13] (see also [12]) but quite different. The setting of Lemma 4.4 is below the limiting case in [13], which was not investigated in their work.

We next establish a variant of Lemma 4.4 for inhomogeneous KdV systems.

Lemma 4.6. *Let $L > 0$ and $T > 0$. Let $h = (h_1, h_2, h_3) \in H^{1/3}(\mathbb{R}_+) \times H^{1/3}(\mathbb{R}_+) \times L^2(\mathbb{R}_+)$, $f_1 \in L^1\big((0,T) \times (0,L)\big)$, and $f_2 \in L^1\big((0,T); W^{1,1}(0,L)\big)$ with*

$$f_2(t,0) = f_2(t,L) = 0. \tag{4.32}$$

Set $f = f_1 + f_{2,x}$ and assume that $f \in L^1(\mathbb{R}_+; L^2(0,L))$. Let $y \in C\big([0,+\infty); L^2(0,L)\big) \cap L^2_{\text{loc}}\big([0,+\infty); H^1(0,L)\big)$ be the unique solution of the system

$$\begin{cases} y_t(t,x) + y_x(t,x) + y_{xxx}(t,x) = f(t,x) & \text{in } (0,+\infty) \times (0,L), \\ y(t, x=0) = h_1(t),\ y(t, x=L) = h_2(t),\ y_x(t, x=L) = h_3(t) & \text{in } (0,+\infty), \end{cases} \tag{4.33}$$

and

$$y(t=0,\cdot) = 0 \text{ in } (0,L).$$

We have

$$\|y\|_{L^2\big((0,T)\times(0,L)\big)} \le C_T \Big(\|(h_1,h_2)\|_{L^2(\mathbb{R}_+)} + \|h_3\|_{H^{-1/3}(\mathbb{R})} + \|f\|_{L^1(\mathbb{R}_+\times(0,L))} \Big), \tag{4.34}$$

and

$$\|y\|_{L^2\big((0,T);H^{-1}(0,L)\big)} \le C_T \Big(\|(h_1,h_2)\|_{H^{-1/3}(\mathbb{R})} + \|h_3\|_{H^{-2/3}(\mathbb{R})} + \|(f_1,f_2)\|_{L^1(\mathbb{R}_+\times(0,L))} \Big). \tag{4.35}$$

Assume in addition that $h(t,\cdot) = 0$ and $f(t,\cdot) = 0$ for $t \ge T_1$ for some $0 < T_1 < T$. Then, for any $\delta > 0$ and for $T_1 + \delta \le t \le T$, we have

$$|y_t(t,x)| + |y_x(t,x)| \le C_{T,T_1,\delta} \Big(\|(h_1,h_2)\|_{H^{-1/3}(\mathbb{R})} + \|h_3\|_{H^{-2/3}(\mathbb{R})} + \|(f_1,f_2)\|_{L^1(\mathbb{R}_+\times(0,L))} \Big). \tag{4.36}$$

Here C_T and $C_{T,T_1,\delta}$ denote positive constants independent of h and f.

Proof. The proof is based on a connection between the KdV equations and the KdV-Burgers equations. Set $v(t,x) = e^{-2t+x} y(t,x)$, which is equivalent to $y(t,x) = e^{2t-x} v(t,x)$. Then

$$y_t(t,x) = \big(2v(t,x) + v_t(t,x)\big) e^{2t-x}, \quad y_x(t,x) = \big(- v(t,x) + v_x(t,x)\big) e^{2t-x},$$

$$y_{xxx}(t,x) = \big(v_{xxx}(t,x) - 3v_{xx}(t,x) + 3v_x(t,x) - v(t,x)\big) e^{2t-x}.$$

Hence, if y satisfies the equation

$$y_t(t,x) + y_x(t,x) + y_{xxx}(t,x) = f(t,x) \text{ in } \mathbb{R}_+ \times (0,L),$$

then it holds

$$v_t(t,x) + 4v_x(t,x) + v_{xxx}(t,x) - 3v_{xx}(t,x) = f(t,x) e^{-2t+x} \text{ in } \mathbb{R}_+ \times (0,L).$$

Set, in $\mathbb{R}_+ \times (0,L)$,

$$\psi(t,x) = \psi(t) := \frac{1}{L} \int_0^L f(t,\xi) e^{-2t+\xi} \, d\xi \quad \text{and} \quad g(t,x) := f(t,x) e^{-2t+x} - \psi(t,x). \tag{4.37}$$

Then

$$\int_0^L g(t,x)\,dx = 0.$$

Let $y_1 \in C\big([0,+\infty); L^2(0,L)\big) \cap L^2_{\mathrm{loc}}\big([0,+\infty); H^1(0,L)\big)$ be the unique solution which is periodic in space of the system

$$y_{1,t}(t,x) + 4y_{1,x}(t,x) + y_{1,xxx}(t,x) - 3y_{1,xx}(t,x) = g(t,x) \text{ in } (0,+\infty)\times(0,L), \tag{4.38}$$

and

$$y_1(t=0,\cdot) = 0 \text{ in } (0,L). \tag{4.39}$$

We have, by (4.32),

$$g(t,x) = f_1(t,x)e^{-2t+x} + f_{2,x}(t,x)e^{-2t+x} - \psi(t,x), \tag{4.40}$$

and

$$\psi(t,x) = \frac{1}{L}\int_0^L f_1(t,\xi)e^{-2t+\xi}\,d\xi - \frac{1}{L}\int_0^L f_2(t,\xi)e^{-2t+\xi}\,d\xi. \tag{4.41}$$

Applying Lemma 4.1, we have

$$\|y_1(\cdot,x)\|_{L^2(\mathbb{R}_+)} + \|y_{1,x}(\cdot,x)\|_{H^{-1/3}(\mathbb{R})} \le C\|g\|_{L^1(\mathbb{R}_+\times(0,L))}$$

which yields, by (4.37),

$$\|y_1(\cdot,x)\|_{L^2(\mathbb{R}_+)} + \|y_{1,x}(\cdot,x)\|_{H^{-1/3}(\mathbb{R})} \le C\|f\|_{L^1(\mathbb{R}_+\times(0,L))}. \tag{4.42}$$

Similarly, by noting $f_{2,x}(t,x)e^{-2t+x} = \big(f_2(t,x)e^{-2t+x}\big)_x - f_2(t,x)e^{-2t+x}$, we get

$$\|y_1(\cdot,x)\|_{H^{-1/3}(\mathbb{R})} + \|y_{1,x}(\cdot,x)\|_{H^{-2/3}(\mathbb{R})} \le C\|(f_1,f_2)\|_{L^1(\mathbb{R}_+\times(0,L))}. \tag{4.43}$$

Applying Lemma 4.1 again, we obtain

$$|y_{1,x}(t,x)| + |y_{1,t}(t,x)| \le C_{T,T_1,\delta}\|(f_1,f_2)\|_{L^1(\mathbb{R}_+\times(0,L))} \text{ for } T_1+\delta/2 \le t \le T. \tag{4.44}$$

if $f = 0$ for $t \ge T_1$.

Fix $\varphi \in C(\mathbb{R})$ such that $\varphi = 1$ for $|t| \le T$ and $\varphi = 0$ for $|t| > 2T$. Let $y_2 \in C\big([0,+\infty); L^2(0,L)\big) \cap L^2_{\mathrm{loc}}\big([0,+\infty); H^1(0,L)\big)$ be the unique solution of the system

$$\begin{cases} y_{2,t}(t,x) + y_{2,x}(t,x) + y_{2,xxx}(t,x) = \varphi(t)\psi(t,x) & \text{in } (0,+\infty)\times(0,L),\\ y_2(t,x=0) = h_1(t) - \varphi(t)e^{2t}y_1(t,0) & \text{in } (0,+\infty),\\ y_2(t,x=L) = h_2(t) - \varphi(t)e^{2t-L}y_1(t,L) & \text{in } (0,+\infty),\\ y_{2,x}(t,x=L) = h_3(t) - \varphi(t)\big(e^{2t-\cdot}y_1(t,\cdot)\big)_x(t,L) & \text{in } (0,+\infty), \end{cases}$$

and

$$y_2(t=0,\cdot) = 0 \text{ in } (0,L).$$

Using (4.40) and applying Lemma 4.4 to y_2, from (4.42), we have

$$\|y_2\|_{L^2\big((0,T)\times(0,L)\big)} \le C_T\Big(\|(h_1,h_2)\|_{L^2(\mathbb{R}_+)} + \|h_3\|_{H^{-1/3}(\mathbb{R})} + \|f\|_{L^1(\mathbb{R}_+\times(0,L))}\Big), \tag{4.45}$$

and from (4.43), we obtain

$$\|y_2\|_{L^2\big((0,T);H^{-1}(0,L)\big)} \le C_T\Big(\|(h_1,h_2)\|_{H^{-1/3}(\mathbb{R})} + \|h_3\|_{H^{-2/3}(\mathbb{R})} + \|(f_1,f_2)\|_{L^1(\mathbb{R}_+\times(0,L))}\Big). \tag{4.46}$$

One can verify that $y_1 + y_2$ and y satisfy the same system for $0 \le t \le T$ and they are in the space $C([0,T]; L^2(0,L)) \cap L^2(0,T; H^1(0,L))$. By the well-posedness of the KdV system, one has

$$y = y_1 + y_2 \text{ in } (0,T)\times(0,L).$$

Combining (4.42) and (4.45) yields (4.34), and combining (4.43) and (4.46) yields (4.35). Combining (4.44) and (4.45) yields, for some $T_1 + \delta/2 \le \tau \le T_1 + 3\delta/4$,

$$\|y(\tau,\cdot)\|_{H^{-1}(0,L)} \le C_{T,T_1,\delta}\Big(\|(h_1,h_2)\|_{H^{-1/3}(\mathbb{R})} + \|h_3\|_{H^{-2/3}(\mathbb{R})} + \|(f_1,f_2)\|_{L^1(\mathbb{R}_+\times(0,L))}\Big), \tag{4.47}$$

and assertion (4.36) follows by the standard C^∞ smoothness property of solutions of the linear KdV system (4.33). The proof is complete. □

Remark 4.7. One can check (4.47) by using a variant of (4.7) in Lemma 4.1 in which $f = 0$ however, a non-zero initial condition is considered.

5. Small time local null-controllability of the KdV system

The main result of this section is the following, which implies in particular Theorem 1.2.

Theorem 5.1. *Let $L > 0$, and $k, l \in \mathbb{N}$. Set*

$$p = \frac{(2k+l)(k-l)(2l+k)}{3\sqrt{3}(k^2+kl+l^2)^{3/2}}. \tag{5.1}$$

Assume that

$$L = 2\pi\sqrt{\frac{k^2+kl+l^2}{3}}, \tag{5.2}$$

and

$$2k + l \notin 3\mathbb{N}. \tag{5.3}$$

Let Ψ be defined in (3.34), where

$$\eta_1 = -\frac{2\pi i}{3L}(2k+l), \quad \eta_2 = \eta_1 + \frac{2\pi i}{L}k, \quad \eta_3 = \eta_2 + \frac{2\pi i}{L}l, \tag{5.4}$$

and E is given by (3.10). There exists $\varepsilon_0 > 0$ such that for all $0 < \varepsilon < \varepsilon_0$, for all $0 < T < T_*/2$ [8] *and for all solutions $y \in C\big([0,+\infty); H^2(0,L)\big) \cap L^2_{\rm loc}\big([0,+\infty); H^3(0,L)\big)$ of*

$$\begin{cases} y_t(t,x) + y_x(t,x) + y_{xxx}(t,x) + yy_x(t,x) = 0 & \text{in } (0,+\infty)\times(0,L), \\ y(t,x=0) = y(t,x=L) = 0 & \text{in } (0,+\infty), \\ y_x(t,x=L) = u(t) & \text{in } (0,\infty), \\ y(0,\cdot) = y_0(x) := \varepsilon\Psi(0,\cdot), \end{cases} \tag{5.5}$$

with ($u \in H^{2/3}(\mathbb{R}_+)$, $\|u\|_{H^{2/3}(\mathbb{R})} < \varepsilon_0$, $u(0) = 0$, and $\operatorname{supp} u \subset [0,T]$), we have

$$y(T,\cdot) \neq 0.$$

Remark 5.2. With the choices of p and L in Theorem 5.1, the function $\Psi(t,x)$ given in Corollary 3.7 satisfies the linear KdV system as noted in [18], i.e.,

$$\Psi_t(t,x) + \Psi_{xxx}(t,x) + \Psi_x(t,x) = 0 \text{ in } \mathbb{R}_+\times(0,L), \tag{5.6}$$

and

$$\Psi(t,0) = \Psi(t,L) = \Psi_x(t,0) = \Psi_x(t,L) = 0 \text{ in } \mathbb{R}_+. \tag{5.7}$$

This property can be rechecked using the fact η_1, η_2, η_3 are the roots of $\eta^3 + \eta - ip = 0$.

We first show that E defined by (3.10) with η_j given in (5.4) and with p in (5.1) is not 0 if (5.3) holds. More precisely, we have

[8] T_* is the constant in Corollary 3.7 with p, η_j, and L given previously. Note that $E \neq 0$ by Lemma 5.3 below.

Lemma 5.3. *Let k, $l \in \mathbb{N}$ and let E be given by (3.10) with η_j in (5.4) and with p in (5.1). Assume that (5.2) holds. We have*

$$E = \frac{40\pi^3}{3L^3}(e^{\eta_1 L} - 1)ikl(k+l).$$

Consequently,

$$E \neq 0 \text{ provided that (5.3) holds.}$$

Proof. With $\gamma_j = L\eta_j/(2\pi i)$, we have

$$\gamma_1 = -\frac{2k+l}{3}, \quad \gamma_2 = \frac{k-l}{3}, \quad \gamma_3 = \frac{k+2l}{3}.$$

It follows that

$$\frac{L^3}{(2\pi i)^3}\sum_{j=1}^{3}\eta_{j+2}^2(\eta_{j+1}-\eta_j) = \sum_{j=1}^{3}\gamma_{j+2}^2(\gamma_{j+1}-\gamma_j) = \gamma_3^2 k + \gamma_1^2 l - \gamma_2^2(k+l)$$

$$= (\gamma_3^2 - \gamma_2^2)k - (\gamma_2^2 - \gamma_1^2)l = (k+l)kl,$$

which yields

$$\sum_{j=1}^{3}\eta_{j+2}^2(\eta_{j+1}-\eta_j) = -8\pi^3 ikl(k+l)/L^3.$$

We also have

$$\sum_{j=1}^{3}\frac{\eta_{j+1}-\eta_j}{\eta_{j+2}} = \sum_{j=1}^{3}\frac{\gamma_{j+1}-\gamma_j}{\gamma_{j+2}} = \frac{3k}{k+2l} - \frac{3l}{2k+l} - \frac{3(k+l)}{k-l} = -\frac{27kl(k+l)}{(k+2l)(2k+l)(k-l)}.$$

We then have, by (3.10),

$$E = \frac{1}{3}(e^{\eta_1 L} - 1)\left(\frac{16\pi^3 i}{3L^3}kl(k+l) + \frac{27ipkl(k+l)}{(k-l)(k+2l)(2l+k)}\right). \tag{5.8}$$

From (5.1) and (5.2), we have

$$\frac{p}{(k-l)(k+2l)(2l+k)} = \left(\frac{2\pi}{3L}\right)^3.$$

We derive from (5.8) that

$$E = \frac{40\pi^3}{3L^3}(e^{\eta_1 L} - 1)ikl(k+l).$$

The proof is complete. □

Before giving the proof of Theorem 5.1, we state and establish new estimates for the nonlinear KdV system (1.1) and (1.2) which play a role in the proof of Theorem 5.1.

Lemma 5.4. *Let $L > 0$ and $T > 0$. There exists a constant $\varepsilon_0 > 0$ depending on L and T such that for $y_0 \in L^2(0, L)$ and for $u \in L^2(\mathbb{R}_+)$ with*

$$\|y_0\|_{L^2(0,L)} + \|u\|_{L^2(\mathbb{R}_+)} \leq \varepsilon_0,$$

then the unique solution $y \in C\big([0, +\infty); L^2(0, L)\big) \cap L^2_{\text{loc}}\big([0, +\infty); H^1(0, L)\big)$ of the system

$$\begin{cases} y_t(t,x) + y_x(t,x) + y_{xxx}(t,x) + y(t,x)y_x(t,x) = 0 & \text{in } (0,+\infty)\times(0,L), \\ y(t, x=0) = y(t, x=L) = 0 & \text{in } (0,+\infty), \\ y_x(t, x=L) = u(t) & \text{in } (0,\infty), \end{cases}$$

with $y(0, \cdot) = y_0$, *satisfies*

$$\|y\|_{L^2\big((0,T)\times(0,L)\big)} \le C\Big(\|y_0\|_{L^2(0,L)} + \|u\|_{H^{-1/3}(\mathbb{R})}\Big), \tag{5.9}$$

and

$$\|y\|_{L^2\big((0,T);H^{-1}(0,L)\big)} \le C\Big(\|y_0\|_{L^2(0,L)} + \|u\|_{H^{-2/3}(\mathbb{R})}\Big), \tag{5.10}$$

where C *is a positive constant depending only on* T *and* L*.*

Proof. [Proof of Lemma 5.4] We have, see e.g. [24, Proposition 14] for ε_0 small,

$$\|y_x\|_{L^2\big((0,T)\times(0,L)\big)} \le C_T\Big(\|y_0\|_{L^2(0,L)} + \|u\|_{L^2(\mathbb{R}_+)}\Big),$$

which yields

$$\|y_x\|_{L^2\big((0,T)\times(0,L)\big)} \le C\varepsilon_0. \tag{5.11}$$

Set

$$f(t,x) = -y(t,x)\partial_x y(t,x).$$

The Cauchy–Schwarz inequality and (5.11) yield

$$\|f\|_{L^1(\mathbb{R}_+\times(0,L))} \le C\varepsilon_0\|y\|_{L^2\big(\mathbb{R}_+\times(0,L)\big)}.$$

Applying Lemma 4.6, and more precisely (4.34), we have

$$\|y\|_{L^2(\mathbb{R}_+\times(0,L))} \le C\varepsilon_0\|y\|_{L^2\big(\mathbb{R}_+\times(0,L)\big)} + C\Big(\|y_0\|_{L^2(0,L)} + \|u\|_{H^{-1/3}(\mathbb{R})}\Big).$$

By choosing ε_0 sufficiently small, one can absorb the first term of the RHS by the LHS and assertion (5.9) follows.

To prove (5.10), one notes

$$\|y^2\|_{L^1\big((0,T)\times(0,L)\big)} \le C\|y\|_{L^2\big((0,T);H^{-1}(0,L)\big)}\|y\|_{L^2\big((0,T);H^1(0,L)\big)} \overset{(5.11)}{\le} C\varepsilon_0\|y\|_{L^2\big((0,T);H^{-1}(0,L)\big)}.$$

By Lemma 4.6 (this time Eq. (4.35)), we obtain

$$\|y\|_{L^2((0,T);H^{-1}(0,L))} \le C\varepsilon_0\|y\|_{L^2\big((0,T);H^{-1}(0,L)\big)} + C\Big(\|y_0\|_{L^2(0,L)} + \|u\|_{H^{-2/3}(\mathbb{R})}\Big).$$

By choosing ε_0 sufficiently small, one can absorb the first term of the RHS by the LHS and assertion (5.10) follows. □

We are ready to give the

Proof. [Proof of Theorem 5.1] By Lemma 5.3, the constant E is not 0. Let ε_0 be a small positive constant, which depends only on k and l and is determined later. We prove Theorem 5.1 by contradiction. Assume that there exists a solution $y \in C\big([0,+\infty); H^2(0,L)\big) \cap L^2_{\mathrm{loc}}\big([0,+\infty); H^3(0,L)\big)$ of (5.5) with $y(t,\cdot) = 0$ for $t \ge T$, for some $u \in H^{2/3}(0,+\infty)$, for some $0 < \varepsilon < \varepsilon_0$, and for some $0 < T < T_*/2$ with $\|u\|_{H^{2/3}(\mathbb{R}_+)} < \varepsilon_0$, $u(0) = 0$, and $\operatorname{supp} u \subset [0,T]$.

We have, for ε_0 small, see e.g., [24, Proposition 14],

$$\|y\|_{L^2\big((0,T);H^1(0,L)\big)} \le C\Big(\|y_0\|_{L^2(0,L)} + \|u\|_{L^2(\mathbb{R}_+)}\Big). \tag{5.12}$$

Set

$$y_1(t,x) = y(t,x) - c\int_0^L y(t,\eta)\Psi(t,\eta)\,d\eta\,\Psi(t,x), \tag{5.13}$$

with $c^{-1} := \int_0^L |\Psi(0,\eta)|^2\, d\eta$. Since $y_0(x) = \epsilon\Psi(0,x)$, this choice of c ensures that $y_1(0,\cdot) = 0$ in $(0,L)$. Then $y_1 \in C\big([0,+\infty); L^2(0,L)\big) \cap L^2_{\text{loc}}\big([0,+\infty); H^1(0,L)\big)$ is the solution of

$$\begin{cases} y_{1,t}(t,x) + y_{1,x}(t,x) + y_{1,xxx}(t,x) + f(t,x) = 0 & \text{in } (0,+\infty)\times(0,L),\\ y_1(t,x=0) = y_1(t,x=L) = 0 & \text{in } (0,+\infty),\\ y_{1,x}(t,x=L) = u(t) & \text{in } (0,+\infty),\\ y_1(0,\cdot) = 0, \end{cases}$$

where

$$f(t,x) = f_1(t,x) + f_{2,x}(t,x),$$

with

$$f_1(t,x) = -c\int_0^L yy_x(t,\eta)\Psi(t,\eta)\, d\eta\, \Psi(t,x) = \frac{c}{2}\int_0^L y^2(t,\eta)\Psi_x(t,\eta)\, d\eta\, \Psi(t,x),$$

and

$$f_2(t,x) = \frac{1}{2}y^2(t,x).$$

By Lemma 5.4, we have

$$\|y\|_{L^2\big((0,T)\times(0,L)\big)} \le C\Big(\|y_0\|_{L^2(0,L)} + \|u\|_{H^{-1/3}(\mathbb{R})}\Big), \tag{5.14}$$

and

$$\|y\|_{L^2\big((0,T);H^{-1}(0,L)\big)} \le C\Big(\|y_0\|_{L^2(0,L)} + \|u\|_{H^{-2/3}(\mathbb{R})}\Big). \tag{5.15}$$

From the definition of y_1 in (5.13), and (5.15), after applying Lemma 4.6 to $y - y_1$, we obtain

$$\|y_1\|_{L^2\big((0,T);H^{-1}(0,L)\big)} \le C\Big(\|y_0\|_{L^2(0,L)} + \|u\|_{H^{-2/3}(\mathbb{R})}\Big). \tag{5.16}$$

Let $y_2 \in C\big([0,+\infty); L^2(0,L)\big) \cap L^2_{\text{loc}}\big([0,+\infty); H^1(0,L)\big)$ be the unique solution of

$$\begin{cases} y_{2,t}(t,x) + y_{2,x}(t,x) + y_{2,xxx}(t,x) = -f(t,x) & \text{in } (0,+\infty)\times(0,L),\\ y_2(t,x=0) = y_2(t,x=L) = 0 & \text{in } (0,+\infty),\\ y_{2,x}(t,x=L) = 0 & \text{in } (0,+\infty),\\ y_2(0,\cdot) = 0, \end{cases}$$

and let $y_3 \in C\big([0,+\infty); L^2(0,L)\big) \cap L^2_{\text{loc}}\big([0,+\infty); H^1(0,L)\big)$ be the unique solution of

$$\begin{cases} y_{3,t}(t,x) + y_{3,x}(t,x) + y_{3,xxx}(t,x) = 0 & \text{in } (0,+\infty)\times(0,L),\\ y_3(t,x=0) = y_3(t,x=L) = 0 & \text{in } (0,+\infty),\\ y_{3,x}(t,x=L) = u(t) & \text{in } (0,+\infty),\\ y_3(0,\cdot) = 0. \end{cases}$$

Then

$$y_1 = y_2 + y_3.$$

There exists $u_4 \in L^2(0,+\infty)$ such that $\operatorname{supp} u_4 \subset [2T_*/3, T_*]$,

$$\|u_4\|_{L^2(0,+\infty)} \le C\|y_3(2T_*/3,\cdot)\|_{L^2(2T_*/3,T_*)},$$

and

$$y_4(T_*,\cdot) = 0,$$

where $y_4 \in C\big([0,+\infty); L^2(0,L)\big) \cap L^2_{\rm loc}\big([0,+\infty); H^1(0,L)\big)$ is the unique solution of

$$\begin{cases} y_{4,t}(t,x) + y_{4,x}(t,x) + y_{4,xxx}(t,x) = 0 & \text{in } (2T_*/3, +\infty) \times (0,L), \\ y_4(t, x=0) = y_4(t, x=L) = 0 & \text{in } (2T_*/3, +\infty), \\ y_{4,x}(t, x=L) = u_4(t) & \text{in } (2T_*/3, +\infty), \\ y_4(T_*/2, \cdot) = y_3(2T_*/3, \cdot). \end{cases}$$

Such an u_4 exists since $y_3(2T_*/3, \cdot)$ is generated from zero at time 0, see [38].

Since $y_2(t,\cdot) + y_3(t,\cdot) = 0$ for $t \ge T_*/2$, we have

$$\|u_4\|_{L^2(0,+\infty)} \le C \|y_2(2T_*/3, \cdot)\|_{L^2(0,L)},$$

which yields

$$\text{(5.17)} \quad \|u_4\|_{L^2(0,+\infty)} \overset{Lemma\ 4.6}{\le} C\|(f_1,f_2)\|_{L^1\big(\mathbb{R}_+\times(0,L)\big)}$$
$$\le C \min\Big\{ \|y\|^2_{L^2\big((0,T)\times(0,L)\big)}, \|y\|_{L^2\big((0,T);H^1(0,L)\big)} \|y\|_{L^2\big((0,T);H^{-1}(0,L)\big)} \Big\}$$
$$\overset{(5.12),(5.14),(5.15)}{\le} C \min\Big\{ \Big(\|y_0\|_{L^2(0,L)} + \|u\|_{H^{-1/3}(\mathbb{R})}\Big)^2, \varepsilon_0\Big(\|y_0\|_{L^2(0,L)} + \|u\|_{H^{-2/3}(\mathbb{R})}\Big)\Big\}.$$

Let $\widetilde{y} \in C\big([0,+\infty); L^2(0,L)\big) \cap L^2_{\rm loc}\big([0,+\infty); H^1(0,L)\big)$ be the unique solution of

$$\begin{cases} \widetilde{y}_t(t,x) + \widetilde{y}_x(t,x) + \widetilde{y}_{xxx}(t,x) = 0 & \text{in } (0, +\infty) \times (0,L), \\ \widetilde{y}(t, x=0) = \widetilde{y}(t, x=L) = 0 & \text{in } (0, +\infty), \\ \widetilde{y}_x(t, x=L) = u(t) + u_4(t) & \text{in } (0, +\infty), \\ \widetilde{y}(0, \cdot) = 0, \end{cases}$$

Then, by the choice of u_4,

$$\widetilde{y}(t,\cdot) = 0 \text{ for } t \ge T_*.$$

Multiplying the equation of y with $\Psi(t,x)$, integrating by parts on $[0,L]$, and using (5.6) and (5.7), we have

$$\text{(5.18)} \qquad \frac{d}{dt}\int_0^L y(t,x)\Psi(t,x)\,dx - \frac{1}{2}\int_0^L y^2(t,x)\Psi_x(t,x)\,dx = 0.$$

Integrating (5.18) from 0 to T and using the fact $y(T,\cdot) = 0$ yield

$$\text{(5.19)} \qquad \int_0^L y_0(x)\Psi(0,x)\,dx + \frac{1}{2}\int_0^T\int_0^L y^2(t,x)\Psi_x(t,x)\,dx\,dt = 0.$$

It is clear that

$$\text{(5.20)} \quad \left| \int_0^T\int_0^L y^2(t,x)\Psi_x(t,x)\,dx\,dt - \int_0^{+\infty}\int_0^L \widetilde{y}^2(t,x)\Psi_x(t,x)\,dx\,dt \right|$$
$$\le \left| \int_0^T\int_0^L y^2(t,x)\Psi_x(t,x)\,dx\,dt - \int_0^T\int_0^L y_1^2(t,x)\Psi_x(t,x)\,dx\,dt \right|$$
$$+ \left| \int_0^{+\infty}\int_0^L y_1^2(t,x)\Psi_x(t,x)\,dx\,dt - \int_0^{+\infty}\int_0^L \widetilde{y}^2(t,x)\Psi_x(t,x)\,dx\,dt \right|.$$

We next estimate the two terms of the RHS.

We begin with the first term. We have

$$\left|\int_0^T\int_0^L y^2(t,x)\Psi_x(t,x)\,dx\,dt-\int_0^T\int_0^L y_1^2(t,x)\Psi_x(t,x)\,dx\,dt\right| \\ \le C\|y-y_1\|_{L^2\big((0,T);H^1(0,L)\big)}\|(y,y_1)\|_{L^2\big((0,T);H^{-1}(0,L)\big)}. \tag{5.21}$$

By considering the system of $y-y_1$, we obtain

$$\|y-y_1\|_{L^2\big((0,T);H^1(0,L)\big)}\le C\Big(\|y_0\|_{L^2(0,L)}+\|f_1\|_{L^1\big((0,T);L^2(0,L)\big)}\Big) \\ \le C\|y_0\|_{L^2(0,L)}+C\|y\|^2_{L^2\big((0,T)\times(0,L)\big)}\overset{(5.14)}{\le} C\|y_0\|_{L^2(0,L)}+C\Big(\|y_0\|_{L^2(0,L)}+\|u\|_{H^{-1/3}(\mathbb{R})}\Big)^2. \tag{5.22}$$

Combining (5.15), (5.16), and (5.22), we derive from (5.21) that

$$\left|\int_0^T\int_0^L y^2(t,x)\Psi_x(t,x)\,dx\,dt-\int_0^T\int_0^L y_1^2(t,x)\Psi_x(t,x)\,dx\,dt\right| \\ \le C\varepsilon_0\|y_0\|_{L^2(0,L)}+C\Big(\|y_0\|_{L^2(0,L)}+\|u\|_{H^{-2/3}(\mathbb{R})}\Big)\Big(\|y_0\|_{L^2(0,L)}+\|u\|_{H^{-1/3}(\mathbb{R})}\Big)^2. \tag{5.23}$$

We next estimate the second term of the RHS of (5.20). It is clear that

$$\left|\int_0^{+\infty}\int_0^L y_1^2(t,x)\Psi_x(t,x)\,dx\,dt-\int_0^{+\infty}\int_0^L \widetilde{y}^2(t,x)\Psi_x(t,x)\,dx\,dt\right| \\ \le C\|y_1-\widetilde{y}\|_{L^2\big((0,T_*);H^1(0,L)\big)}\Big(\|y_1\|_{L^2\big((0,T_*);H^{-1}(0,L)\big)}+\|\widetilde{y}\|_{L^2\big((0,T_*);H^{-1}(0,L)\big)}\Big). \tag{5.24}$$

Consider the systems of y_1-y and $\widetilde{y}$. We have

$$\|y_1-\widetilde{y}\|_{L^2\big((0,T_*);H^1(0,L)\big)}\le C\Big(\|f\|_{L^1\big((0,T);L^2(0,L)\big)}+\|u_4\|_{L^2(0,T)}\Big) \\ \overset{(5.17)}{\le} C\|yy_x\|_{L^1\big((0,T);L^2(0,L)\big)}+C\Big(\|y_0\|_{L^2(0,L)}+\|u\|_{H^{-1/3}(\mathbb{R})}\Big)^2 \\ \overset{(5.12)}{\le} C\Big(\|y_0\|_{L^2(0,L)}+\|u\|_{L^2(\mathbb{R}_+)}\Big)^2, \tag{5.25}$$

and, by Lemma 4.6 and (5.17),

$$\|\widetilde{y}\|_{L^2\big((0,T_*);H^{-1}(0,L)\big)}\le C\|(u,u_4)\|_{H^{-2/3}(\mathbb{R})}\le C\Big(\|y_0\|_{L^2(0,L)}+\|u\|_{H^{-2/3}(\mathbb{R})}\Big). \tag{5.26}$$

Using (5.16), (5.25), and (5.26), we derive from (5.24) that

$$\left|\int_0^{+\infty}\int_0^L y_1^2(t,x)\Psi_x(t,x)\,dx\,dt-\int_0^{+\infty}\int_0^L \widetilde{y}^2(t,x)\Psi_x(t,x)\,dx\,dt\right| \\ \le C\Big(\|y_0\|_{L^2(0,L)}+\|u\|_{L^2(\mathbb{R}_+)}\Big)^2\Big(\|y_0\|_{L^2(0,L)}+\|u\|_{H^{-2/3}(\mathbb{R})}\Big). \tag{5.27}$$

Combining (5.20), (5.23), and (5.27) yields

$$(5.28)\quad \left| \int_0^T \int_0^L y^2(t,x)\Psi_x(t,x)\,dx\,dt - \int_0^{+\infty}\int_0^L \widetilde{y}^2(t,x)\Psi_x(t,x)\,dx\,dt \right| \\ \le C\varepsilon_0 \|y_0\|_{L^2(0,L)} + C\Big(\|y_0\|_{L^2(0,L)} + \|u\|_{H^{-2/3}(\mathbb{R})}\Big)\Big(\|y_0\|_{L^2(0,L)} + \|u\|_{L^2(\mathbb{R}_+)}\Big)^2.$$

On the other hand, from Corollary 3.7 and the choice of y_0, we have

$$(5.29)\quad \int_0^L y_0(x)\Psi(0,x)\,dx + \frac{1}{2}\int_0^{+\infty}\int_0^L \widetilde{y}^2(t,x)\Psi_x(t,x)\,dx\,dt \\ \ge C\Big(\|y_0\|_{L^2(0,L)} + \|u+u_4\|^2_{H^{-2/3}(\mathbb{R})}\Big).$$

Using the fact

$$\|u+u_4\|^2_{H^{-2/3}(\mathbb{R})} \ge C\|u\|^2_{H^{-2/3}(\mathbb{R})} - C\|u_4\|^2_{L^2(\mathbb{R})} \overset{(5.17)}{\ge} C\|u\|^2_{H^{-2/3}(\mathbb{R})} - C\Big(\|y_0\|_{L^2(0,L)} + \|u\|_{H^{-1/3}(\mathbb{R})}\Big)^4,$$

we derive from (5.29) that, for small ε_0,

$$(5.30)\quad \int_0^L y_0(x)\Psi(0,x)\,dx + \frac{1}{2}\int_0^{\infty}\int_0^L \widetilde{y}^2(t,x)\Psi_x(t,x)\,dx\,dt \\ \ge C\left(\|y_0\|_{L^2(0,L)} + \|u\|^2_{H^{-2/3}(\mathbb{R})}\right) - C\|u\|^4_{H^{-1/3}(\mathbb{R})}.$$

Combining (5.19), (5.28), and (5.30) yields

$$(5.31)\quad C\varepsilon_0\|y_0\|_{L^2(0,L)} + C\Big(\|y_0\|_{L^2(0,L)} + \|u\|_{H^{-2/3}(\mathbb{R})}\Big)\Big(\|y_0\|_{L^2(0,L)} + \|u\|_{L^2(\mathbb{R}_+)}\Big)^2 \\ \overset{(5.28)}{\ge} \left| \int_0^T \int_0^L y^2(t,x)\Psi_x(t,x)\,dx\,dt - \int_0^{+\infty}\int_0^L \widetilde{y}^2(t,x)\Psi_x(t,x)\,dx\,dt \right| \\ \overset{(5.19)}{\ge} \int_0^L y_0(x)\Psi(0,x)\,dx + \frac{1}{2}\int_0^{\infty}\int_0^L \widetilde{y}^2(t,x)\Psi_x(t,x)\,dx\,dt \\ \overset{(5.30)}{\ge} C\Big(\|y_0\|_{L^2(0,L)} + \|u\|^2_{H^{-2/3}(\mathbb{R})} - C\|u\|^4_{H^{-1/3}(\mathbb{R})}\Big).$$

It follows that, if ε_0 is fixed but sufficiently small,

$$(5.32)\quad \|u\|^4_{H^{-1/3}(\mathbb{R})} + \|u\|_{H^{-2/3}(\mathbb{R})}\|u\|^2_{L^2(\mathbb{R}_+)} \ge C\|u\|^2_{H^{-2/3}(\mathbb{R})}.$$

We have

$$(5.33)\quad \|u\|^2_{H^{-1/3}(\mathbb{R})} \le C\|u\|_{L^2(\mathbb{R})}\|u\|_{H^{-2/3}(\mathbb{R})} \le C\varepsilon_0\|u\|_{H^{-2/3}(\mathbb{R})},$$

and

$$(5.34)\quad \|u\|^2_{L^2(\mathbb{R})} \le C\|u\|_{H^{-2/3}(\mathbb{R})}\|u\|_{H^{2/3}(\mathbb{R})},$$

(recall that we extended u by 0 for $t < 0$). Let U be the even extension of $u\big|_{\mathbb{R}_+}$ in $\mathbb{R}$. Applying the Hardy inequality for fractional Sobolev space $H^{2/3}(\mathbb{R})$ for U after noting that $U(0) = 0$, see e.g. [35, Theorem 1.1] [9], we derive that

$$\| |\cdot|^{-2/3} U(\cdot)\|_{L^2(\mathbb{R})} \le C\|U\|_{H^{2/3}(\mathbb{R})}.$$

We have

$$\|U\|_{H^{2/3}(\mathbb{R})} \le C\|u\|_{H^{2/3}(\mathbb{R}_+)}.$$

since U is an even extension of u, and

$$|U|^2_{H^{2/3}(\mathbb{R})} \sim \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{|U(s)-U(t)|^2}{|s-t|^{1+4/3}}\,ds\,dt, \quad |u|^2_{H^{2/3}(\mathbb{R})} \sim \int_{\mathbb{R}_+}\int_{\mathbb{R}_+} \frac{|u(s)-u(t)|^2}{|s-t|^{1+4/3}}\,ds\,dt.$$

We derive that

$$\| |\cdot|^{-2/3} u(\cdot)\|_{L^2(\mathbb{R})} \le C\|u\|_{H^{2/3}(\mathbb{R}_+)}.$$

Since

$$\begin{aligned} |u|^2_{H^{2/3}(\mathbb{R})} &\sim \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{|u(s)-u(t)|^2}{|s-t|^{1+4/3}}\,ds\,dt \\ &\overset{u(s)=0,\ s<0}{\le} \int_{\mathbb{R}_+}\int_{\mathbb{R}_+} \frac{|u(s)-u(t)|^2}{|s-t|^{1+4/3}}\,dx\,dy + C\int_{\mathbb{R}_+} \frac{|u(t)|^2}{t^{4/3}}\,dt \\ &\le C\|u\|^2_{H^{2/3}(\mathbb{R}_+)} + C\int_{\mathbb{R}_+} \frac{|u(t)|^2}{t^{4/3}}\,dt, \end{aligned}$$

it follows that

$$\|u\|_{H^{2/3}(\mathbb{R})} \le C\|u\|_{H^{2/3}(\mathbb{R}_+)}. \tag{5.35}$$

Here we also used the fact $u=0$ in $\mathbb{R}_-$. Combining (5.34) and (5.35) yields

$$\|u\|^2_{L^2(\mathbb{R})} \le C\varepsilon_0\|u\|_{H^{-2/3}(\mathbb{R})}. \tag{5.36}$$

Using (5.33) and (5.36), we derive from (5.32) that, $\|u\|^2_{H^{-2/3}} \le C\epsilon_0^2\|u\|^2_{H^{-2/3}} + C\epsilon_0\|u\|^2_{H^{-2/3}}$. So, for fixed sufficiently small ε_0,

$$u = 0.$$

As a consequence, we obtain

$$\|y(t,\cdot) - \varepsilon\Psi(T_*/2,\cdot)\|_{L^2(0,L)} \le C\varepsilon^2.$$

One has a contradiction if ε_0 is sufficiently small. The proof is complete. □

Remark 5.5. Viewing the proof of Theorem 5.1, it is natural to ask whether or not one needs to derive estimates for the (linear and nonlinear) KdV systems using low regular data. In fact, without using these estimates, one might require that $\|u\|_{H^2(0,T)}$ or even $\|u\|_{H^3(0,T)}$ is small.

6. Controllability of the KdV system with controls in H^1

For $T>0$, set

$$X = C\big([0,T];Y\big) \cap L^2\big((0,T);H^4([0,L])\big)$$

with the corresponding norm. Here we denote

$$Y = H^3(0,L) \cap H^1_0(0,L),$$

which is a Hilbert space with the corresponding scalar product.

In this section, we prove the following local controllability of the KdV system (1.1) and (1.2):

[9]We here apply [35, ii) of Theorem 1.1] with $\gamma=-2/3$, $\tau=p=2$, $s=2/3$, $a=1$, $\alpha=0$.

Theorem 6.1. *Let $L > 0$, and $k, l \in \mathbb{N}$. Let p be defined by (5.1). Assume that (5.2) holds, $2k + l \notin 3\mathbb{N}$, and the dimension of $\mathcal{M}$ is 2. Given $T > \pi/p$, there exists $\varepsilon_0 > 0$ such that for $y_0, y_T \in Y$ with*

$$\|(y_0, y_T)\|_Y \le \varepsilon_0,$$

there exists $u \in H^1(0,T)$ such that $u(0) = y_0'(L)$,

$$\|u\|_{H^1(0,T)} \le C\|(y_0, y_T)\|_Y^{1/2},$$

and the corresponding solution $y \in X$ of the nonlinear system (1.1) with $y(t = 0, \cdot) = y_0$ satisfies $y(t = T, \cdot) = y_T$.

We recall a result in [12] ([12, Lemma 3.3] applied to $s = 3$) on the well-posedness and the stability of the linearized system of (1.1).

Lemma 6.2. *Let $L > 0$ and $T > 0$. For $y_0 \in H^3(0,L) \cap H^1_0(0,L)$, $f \in W^{1,1}\big([0,T]; L^2(0,L)\big)$, and $u \in H^1(0,T)$ with $u(0) = y_0'(L)$. There exists a unique solution $y \in X$ of the system*

$$\begin{cases} y_t(t,x) + y_x(t,x) + y_{xxx}(t,x) = f(t,x) & \text{for } t \in (0,T),\ x \in (0,L), \\ y(t, x = 0) = y(t, x = L) = 0 & \text{for } t \in (0,T), \\ y_x(t, x = L) = u(t) & \text{for } t \in (0,T), \\ y(t = 0, \cdot) = y_0 & \text{for } x \in (0,L). \end{cases} \tag{6.1}$$

Moreover,

$$\|y\|_X \le C\Big(\|f\|_{W^{1,1}\big([0,T];L^2(0,L)\big)} + \|u\|_{H^1(0,1)}\Big),$$

for some positive constant C depending only on L and T.

Remark 6.3. By the same method, the conclusion also holds for the non-linear KdV equations if $\|f\|_{W^{1,1}\big((0,T);L^2(0,L)\big)} + \|u_0\|_{H^1(0,L)}$ is small.

In what follows in this section, $\mathcal{M}^\perp$ denotes all elements of Y orthogonal to $\mathcal{M}$ with respect to $L^2(0,L)$-scalar product. We also denote $P_\mathcal{M}$ and $P_{\mathcal{M}^\perp}$ the projections into $\mathcal{M}$ and $\mathcal{M}^\perp$ with respect to $L^2(0,L)$-scalar product. Before giving the proof of Theorem 6.1, let us establish two lemmas used in its proof. The first one is a consequence of the Hilbert Uniqueness Method for controls in H^1 and solutions in X.

Lemma 6.4. *Let $L \in \mathcal{N}$ and $T > 0$. There is a continuous linear map $\mathcal{L} : \mathcal{M}^\perp \to H^1(0,T)$ such that for $\varphi \in \mathcal{M}^\perp$ and $u = \mathcal{L}(\varphi)$, then $u(0) = 0$, and the unique solution $y \in X$ of*

$$\begin{cases} y_t(t,x) + y_x(t,x) + y_{xxx}(t,x) = 0 & \text{for } t \in (0,T),\ x \in (0,L), \\ y(t, x = 0) = y(t, x = L) = 0 & \text{for } t \in (0,T), \\ y_x(t, x = L) = u(t) & \text{for } t \in (0,T), \\ y(t = 0, \cdot) = 0, \end{cases} \tag{6.2}$$

satisfies $y(T, \cdot) = \varphi$.

Proof. Set

$$\mathcal{M}_1^\perp = \Big\{w \in \mathcal{M}^\perp; w_x(0) = 0\Big\}.$$

For $\psi \in \mathcal{M}_1^\perp$, by Lemma 6.2, there exists a unique solution $y^* \in X$ of the backward KdV system

$$
\begin{cases}
y_t^*(t,x) + y_x^*(t,x) + y_{xxx}^*(t,x) = 0 & \text{for } t \in (0,T),\ x \in (0,L),\\
y^*(t, x=0) = y^*(t, x=L) = 0 & \text{for } t \in (0,T),\\
y_x^*(t, x=0) = 0 & \text{for } t \in (0,T),\\
y^*(T,\cdot) = \psi.
\end{cases} \tag{6.3}
$$

Applying the observability inequality to y^* and y_t^* (see e.g. [18, Theorem 2.4] and also [38, the proof of Proposition 3.9]), we have, for $\gamma \ge 1$,

$$
\int_{T/2}^T \gamma |y_x^*(t,L)\eta|^2 + |y_{tx}^*(t,L)|^2\, dt \ge C \int_0^L \gamma |y^*(T,x)|^2 + |y_t^*(T,x)|^2\, dx,
$$

where in the last inequality, we used the fact that if $\psi \in \mathcal{M}^\perp$ then $\psi''' + \psi'$ is also in $\mathcal{M}^\perp$ (this can be proved through integration by part arguments; recall that $\mathcal{M}^\perp$ is defined via $L^2(0,L)$-scalar product). In other words,

$$
\int_{T/2}^T \gamma |y_x^*(t,L)|^2 + |y_{tx}^*(t,L)|^2\, dt \ge C \int_0^L \gamma |\psi|^2 + |\psi''' + \psi'|^2\, dx. \tag{6.4}
$$

Fix a non-negative function $\eta \in C^1([0,T])$ such that $\eta = 1$ in $[T/2, T]$ and $\eta = 0$ in $[0, T/3]$. Since

$$
\int_0^L \gamma |\psi|^2 + |\psi''' + \psi'|^2\, dx = \int_0^L \gamma |\psi|^2 + |\psi'''|^2 + |\psi'|^2 + 2\psi'''\psi'\, dx,
$$

and, for all $\varepsilon > 0$,

$$
\int_0^L |\psi'|^2\, dx \le \int_0^L \varepsilon |\psi'''|^2 + C_\varepsilon |\psi|^2\, dx,
$$

it follows that, for large γ,

$$
\int_0^L \gamma |\psi|^2 + |\psi''' + \psi'|^2\, dx \ge C \|\psi\|_{H^3(0,L)}^2. \tag{6.5}
$$

We have

$$
\int_0^T |y_x^*(t,L) y_{tx}^*(t,L)|\, dt \le \int_0^T \varepsilon^{-1} |y_x^*|^2 + \varepsilon |y_{tx}^*|^2\, dt \le C \int_0^L \varepsilon^{-1} |\psi|^2 + \varepsilon |\psi''' + \psi'|^2\, dx.
$$

Here in the last inequalitiy, we applied [38, (58) in the proof of Proposition 3.7] (see also [18, Proposition 2]) to y^* and y_t^*. It follows from (6.4) and (6.5), for γ large enough, that

$$
\int_0^T \gamma \eta(t) |y_x^*(t,L)|^2 + y_{tx}^*(t,L) \big(\eta y_x^*(t,L)\big)_t\, dt \ge C_\gamma \|\psi\|_{H^3(0,L)}^2. \tag{6.6}
$$

For a given $\varphi \in \mathcal{M}_1^\perp$, by the Lax-Milgram's theorem and (6.6), there exists a unique $\Phi \in \mathcal{M}_1^\perp$ such that

$$
\int_0^L \gamma \varphi \psi + (\varphi''' + \varphi')(\psi''' + \psi')\, dx = \int_0^T \gamma y_x^* \eta Y_x^* + y_{tx}^* (\eta Y_x^*)_t\, dt \quad \forall\, \psi \in \mathcal{M}_1^\perp, \tag{6.7}
$$

where Y^* is the solution of (6.3) with $\psi = \Phi$.

Let $y \in X$ be the solution of (6.2) with $u(\cdot) = \mathcal{L}_1(\varphi) = \eta(\cdot) Y_x^*(\cdot, L)$. Then, by integration by parts,

$$
\int_0^L \gamma \psi y(T,\cdot) + (\psi''' + \psi')\big(y_{xxx}(T,\cdot) + y_x(T,\cdot)\big)\, dx = \int_0^T \gamma y_x^* \eta Y_x^* + y_{tx}^* (\eta Y_x^*)_t\, dt \quad \forall\, \psi \in \mathcal{M}_1^\perp. \tag{6.8}
$$

From (6.7) and (6.8), we obtain

$$\int_0^L \gamma\varphi\psi + (\varphi''' + \varphi')(\psi''' + \psi') = \int_0^L \gamma\psi y(T,\cdot) + (\psi''' + \psi')(y_{xxx}(T,\cdot) + y_x(T,\cdot)) \quad \forall\, \psi \in \mathcal{M}_1^{\perp}.$$

Since y and Y^* satisfies system (6.2) with the same u for $t \in [T/2, T]$, it follows that $y(t,\cdot) - Y^*(t,\cdot) \in \mathcal{M}$ for $t \in [T/2, T]$. In particular, $y(T,\cdot) \in \mathcal{M}_1^{\perp}$ since $Y^*(T,\cdot) \in \mathcal{M}_1^{\perp}$. Combining this with the fact that $\varphi \in \mathcal{M}_1^{\perp}$, we then derive from (6.5) that

$$y(T,\cdot) = \varphi.$$

The conclusion for $2T$ (instead of T) is now as follows. Fix $\zeta \in C^1([0, 2T])$ with $\zeta(2T) = 1$ and $\zeta(t) = 0$ for $t \le 5T/4$. For $\varphi \in \mathcal{M}^{\perp}$, let $\widetilde{y}^*$ be the unique solution of

$$\begin{cases} \widetilde{y}^*_t(t,x) + \widetilde{y}^*_x(t,x) + \widetilde{y}^*_{xxx}(t,x) = 0 & \text{for } t \in (T, 2T),\ x \in (0, L), \\ \widetilde{y}^*(t, x = 0) = \widetilde{y}^*(t, x = L) = 0 & \text{for } t \in (T, 2T), \\ \widetilde{y}^*_x(t, x = 0) = \varphi_x(2T, 0)\zeta(t) & \text{for } t \in (T, 2T), \\ \widetilde{y}^*(2T, \cdot) = \varphi. \end{cases}$$

One can check that $\widetilde{y}^*(T,\cdot) \in \mathcal{M}_1^{\perp}$. Set

$$\mathcal{L}(\varphi)(t) = \begin{cases} \widetilde{y}^*_x(t, L) & \text{for } t \in (T, 2T), \\ \mathcal{L}_1(\widetilde{y}^*(T,\cdot))(t) & \text{for } t \in (0, T). \end{cases} \tag{6.9}$$

It is clear that $\mathcal{L}(\varphi) \in H^1(0, 2T)$ since $\widetilde{y}_x(\cdot, L) \in H^1(T, 2T)$, $\mathcal{L}_1(\widetilde{y}^*(T,\cdot)) \in H^1(0,T)$, and $\mathcal{L}_1(\widetilde{y}^*(T,\cdot))(T) = \widetilde{y}^*_x(T, L)$, and that the corresponding solution at the time $2T$ is φ. The proof is complete. □

For $r > 0$ and an element $e \in Y$, we denote $B_r(e)$ the ball in Y centered at e with radius r, and $\overline{B_r(e)}$ its closure in Y. The second lemma is a consequence of the power series method and the information derived in Sections 3 and 5.

Lemma 6.5. *Let $L > 0$, and $k, l \in \mathbb{N}$. Let p be defined by (5.1). Assume that (5.2) holds, $2k + l \notin 3\mathbb{N}$, and the dimension of $\mathcal{M}$ is 2. Let $T > \pi/p$ and $0 < c_1 < c_2$. Fix $\varphi \in \mathcal{M}$ with $c_1 \le \|\varphi\|_Y \le c_2$. There exist a constant $0 < c_3 < c_1/2$, and two maps $U_1 : B_{c_3}(\varphi) \to H^1(0,T)$ and $U_2 : B_{c_3}(\varphi) \to H^1(0,T)$ such that for $\psi \in B_{c_3}(\varphi)$, $U_1(\varphi)(0) = U_2(\varphi)(0) = 0$, and the unique solutions y_1 and y_2 in X of the following two systems, with $u_1 = U_1(\varphi)$ and $u_2 = U_2(\varphi)$,*

$$\begin{cases} y_{1,t}(t,x) + y_{1,x}(t,x) + y_{1,xxx}(t,x) = 0 & \textit{for } t \in (0,T),\ x \in (0,L), \\ y_1(t, x = 0) = y_1(t, x = L) = 0 & \textit{for } t \in (0,T), \\ y_{1,x}(t, x = L) = u_1(t) & \textit{for } t \in (0,T), \\ y_1(t = 0, \cdot) = 0 & \textit{for } t \in (0,T), \end{cases} \tag{6.10}$$

$$\begin{cases} y_{2,t}(t,x) + y_{2,x}(t,x) + y_{2,xxx}(t,x) + y_1(t,x)y_{1,x}(t,x) = 0 & \textit{for } t \in (0,T),\ x \in (0,L), \\ y_2(t, x = 0) = y_2(t, x = L) = 0 & \textit{for } t \in (0,T), \\ y_{2,x}(t, x = L) = u_2(t) & \textit{for } t \in (0,T), \\ y_1(t = 0, \cdot) = 0 & \textit{for } t \in (0,T), \end{cases} \tag{6.11}$$

satisfy

$$y_1(T,\cdot) = 0 \quad \textit{and} \quad y_2(T,\cdot) = \psi.$$

Moreover, for ψ, $\widetilde{\psi} \in B_{c_3}(\varphi)$,

$$\|U_1(\psi) - U_1(\widetilde{\psi})\|_{H^1(0,T)} \le C\|\psi - \widetilde{\psi}\|_Y \tag{6.12}$$

and

$$\|U_2(\psi) - U_2(\widetilde{\psi})\|_{H^1(0,T)} \le C\|\psi - \widetilde{\psi}\|_Y, \tag{6.13}$$

for some positive constant C *depending only on* L, T, c_1, *and* c_2.

Proof. By Lemma 5.3 and Corollary 3.7, for all $\tau > 0$, there exists $v_1 \in H^2_0(0,\tau)$ such that if $y_1 \in X$ is the solution of (6.10) with $u_1 = v_1$ and $y_2 \in X$ is the solution of (6.11) with $u_2 = 0$ then

$$y_2(\tau, \cdot) \in \mathcal{M} \setminus \{0\}.$$

Since c_3 is small, $\dim \mathcal{M} = 2$, and $v_1 \in H^2_0(0,L)$, by using rotations (see also [18, the proof of Proposition 13]) there exists $U_1(\psi)$ with $U_1(\psi)(0) = 0$ satisfying (6.12) such that if $y_1 \in X$ is the solution of (6.10) with $u_1 = U_1(\psi)$ and $\hat{y}_2 \in X$ is the solution of (6.11) with $u_2 = 0$ then

$$\hat{y}_2 = P_{\mathcal{M}}\psi.$$

We then choose

$$u_2 = \mathcal{L}(\hat{y}_2 - P_{\mathcal{M}}\psi),$$

where $\mathcal{L}$ is a map given by Lemma 6.4. □

We are ready to give the

Proof. [Proof of Theorem 6.1]

Fix $y_0, y_T \in Y$ with small norms. For simplicity of the presentation, we will assume that $\|y_0\|_Y \le \|y_T\|_Y$ (the other case also follows from this case by e.g. reversing the time: $t \to T - t$ and noting that $y_x(\cdot, 0)$ is in $H^1(0,T)$; this can be derived by considering the equation for y_t [10]). Set $\rho = \|y_T\|_Y$ and assume that $\rho > 0$ otherwise, one just takes the zero control and the conclusion follows.

Let w_0 be the state at the time T of the solution of the linear system (6.2) with the zero control starting from $P_{\mathcal{M}}y_0$ at the time 0. We first consider the case where

$$\|P_{\mathcal{M}}y_T - w_0\|_{H^2(0,L)} \ge 2c\rho, \tag{6.14}$$

for some small constant c independent of ρ and defined later.

Set

$$\begin{array}{rccc} \mathbb{G}: & Y \cap B_{c\rho}(y_T) & \to & H^1(0,T) \\ & \varphi & \mapsto & \rho\mathbf{u}_0 + \rho^{1/2}u_1 + \rho u_2. \end{array}$$

Here we decompose φ as

$$\varphi = P_{\mathcal{M}^\perp}\varphi + P_{\mathcal{M}}\varphi,$$

$\mathbf{u}_0 \in H^1(0,T)$ is a control for which the corresponding solution $\mathbf{y}_0$ in X of the linear system (6.2) starting from $P_{\mathcal{M}^\perp}y_0/\rho$ at 0 and arriving $P_{\mathcal{M}^\perp}\varphi/\rho$ at the time T, and u_1 and u_2 are controls for which the solutions $y_1 \in X$ and $y_2 \in X$ of the system (6.10) and $\big($(6.11) with the initial data $P_{\mathcal{M}}y_0/\rho$ instead of 0$\big)$ satisfies $y_1(T,\cdot) = 0$ and $y_2(T,\cdot) = P_{\mathcal{M}}\varphi/\rho$. Moreover, by Lemma 6.4, one can choose $\mathbf{u}_0$ in such a way that $\mathbf{u}_0 = \mathbf{u}_0(\varphi)$ is a Lipschitz function of φ with the Lipschitz constant bounded by a positive constant independent of ρ, and by Lemma 6.5 one can choose $u_1 = u_1(\varphi)$ and $u_2 = u_2(\varphi)$ as Lipschitz functions of $P_{\mathcal{M}}\varphi/\rho$ with the Lipschitz constants bounded by positive constants independent of ρ.

[10] The compatibility condition is automatic.

Set

$$\begin{array}{rccc} \mathbb{P}\colon & \Big\{ w \in H^1(0,T); w(0) = y_0'(L) \Big\} & \to & H^3(0,L) \\ & w & \mapsto & y(T,\cdot), \end{array}$$

where $y \in X$ is the unique solution of the nonlinear system (1.1) with $u = w$ starting from y_0 at time 0. Consider the map

$$\begin{array}{rccc} \Lambda\colon & Y \cap \overline{B_{c\rho}(y_T)} & \to & Y \\ & \varphi & \mapsto & \varphi - \mathbb{P} \circ \mathbb{G}(\varphi) + y_T. \end{array}$$

We will prove that

$$\Lambda(\varphi) \in \overline{B_{c\rho}(y_T)}, \tag{6.15}$$

and

$$\|\Lambda(\varphi) - \Lambda(\phi)\|_Y \le \lambda \|\varphi - \phi\|_Y, \tag{6.16}$$

for some $\lambda \in (0,1)$. Assuming this, one derives from the contraction mapping theorem that there exists a unique $\varphi_0 \in Y \cap \overline{B_{c\rho}(y_T)}$ such that $\Lambda(\varphi_0) = \varphi_0$. As a consequence,

$$y_T = \mathbb{P} \circ \mathbb{G}(\varphi_0),$$

and $\mathbb{G}(\varphi_0)$ is hence a required control.

We next establish (6.15) and (6.16). Indeed, assertion (6.15) follows from the fact

$$\|\varphi - \mathbb{P} \circ \mathbb{G}(\varphi)\|_Y \le C \|\varphi\|_Y^{3/2} \text{ for } Y \cap \overline{B_{\rho/2}(y_T)}.$$

This can be proved using the approximation via the power series method as follows. Set [11]

$$u = \rho \mathbf{u}_0 + \rho^{1/2} u_1 + \rho u_2 \quad \text{ and } \quad y_a = \rho \mathbf{y}_0 + \rho^{1/2} y_1 + \rho y_2.$$

Let $y \in X$ be the solution of the nonlinear KdV system (1.1) with $y(t = 0, \cdot) = y_0$ and with u defined above. Then

$$(y - y_a)_t + (y - y_a)_x + (y - y_a)_{xxx} + y y_x - y_a y_{a,x} = f(t,x),$$

where

$$-f(t,x) = \rho^{3/2} (y_1 y_2)_x + \rho^2 y_2 y_{2,x} + \rho^2 \mathbf{y}_0 \mathbf{y}_{0,x} + \rho^{3/2} \Big(\mathbf{y}_0 (y_1 + \rho^{1/2} y_2) \Big)_x.$$

Since

$$y y_x - y_a y_{a,x} = (y - y_a) y_x + y_a (y_x - y_{a,x}),$$

applying Lemma 6.2, we obtain, for small ρ,

$$\|y - y_a\|_X \le C \|f\|_{W^{1,1}\big((0,T); L^2(0,L)\big)} \le C \rho^{3/2}. \tag{6.17}$$

Assertion (6.15) follows since $y(T,\cdot) = \mathbb{P} \circ \mathbb{G}(\varphi)$ and $y_a(T,\cdot) = \varphi$.

We next establish (6.16). To this end, we estimate

$$\Big(\varphi - \mathbb{P} \circ \mathbb{G}(\varphi)\Big) - \Big(\widetilde{\varphi} - \mathbb{P} \circ \mathbb{G}(\widetilde{\varphi})\Big).$$

Denote $\widetilde{\mathbf{u}}_0, \widetilde{u}_1, \widetilde{u}_2, \widetilde{u}$ and $\widetilde{\mathbf{y}}_0, \widetilde{y}_1, \widetilde{y}_2, \widetilde{y}_a, \widetilde{y}$ the functions corresponding to $\widetilde{\varphi}$ which are defined in the same way as the functions $\mathbf{u}_0, u_1, u_2, u$ and $\mathbf{y}_0, y_1, y_2, y_a, y$ defined for φ.

We have

$$(y - \widetilde{y})_t + (y - \widetilde{y})_x + (y - \widetilde{y})_{xxx} + y y_x - \widetilde{y} \widetilde{y}_x = 0,$$
$$(y_a - \widetilde{y}_a)_t + (y_a - \widetilde{y}_a)_x + (y_a - \widetilde{y}_a)_{xxx} + y_a y_{a,x} - \widetilde{y}_a \widetilde{y}_{a,x} = g(t,x),$$

[11]The index a stands the approximation.

where

$$g(t,x) = \rho^{3/2}\Big((y_1y_2)_x - (\widetilde{y}_1\widetilde{y}_2)_x\Big) + \rho^2\Big(y_2y_{2,x} - \widetilde{y}_2\widetilde{y}_{2,x}\Big) + \rho^2\Big(\mathbf{y}_0\mathbf{y}_{0,x} - \widetilde{\mathbf{y}}_0\widetilde{\mathbf{y}}_{0,x}\Big) + \rho^{3/2}\Big(\mathbf{y}_0(y_1+\rho^{1/2}y_2) - \widetilde{\mathbf{y}}_0(\widetilde{y}_1+\rho^{1/2}\widetilde{y}_2)\Big)_x. \tag{6.18}$$

This implies

$$\begin{aligned}
(y-&y_a - \widetilde{y} + \widetilde{y}_a)_t + (y - y_a - \widetilde{y} + \widetilde{y}_a)_x + (y - y_a - \widetilde{y} + \widetilde{y}_a)_{xxx}\\
&= -\Big((y-y_a)y_x + y_a(y-y_a)_x - (\widetilde{y}-\widetilde{y}_a)\widetilde{y}_x - \widetilde{y}_a(\widetilde{y}-\widetilde{y}_a)_x + g(t,x)\Big)\\
&= -\Big((y - y_a - \widetilde{y} + \widetilde{y}_a)y_x + (y_x - \widetilde{y}_x)(\widetilde{y}-\widetilde{y}_a) + y_a(y - y_a - \widetilde{y} + \widetilde{y}_a)_x\\
&\qquad + (y_a - \widetilde{y}_a)(\widetilde{y}-\widetilde{y}_a)_x + g(t,x)\Big)\\
&= -\Big((y - y_a - \widetilde{y} + \widetilde{y}_a)y_x + y_a(y - y_a - \widetilde{y} + \widetilde{y}_a)_x + (y_x - y_{a,x} - \widetilde{y}_x + \widetilde{y}_{a,x})(\widetilde{y}-\widetilde{y}_a) + h(t,x)\Big),
\end{aligned}$$

where

$$h(t,x) = g(t,x) + (y_{a,x} - \widetilde{y}_{a,x})(\widetilde{y}-\widetilde{y}_a) + (y_a - \widetilde{y}_a)(\widetilde{y}-\widetilde{y}_a)_x.$$

Using Lemma 6.2, we derive that, for ρ small,

$$\|y - y_a - \widetilde{y} + \widetilde{y}_a\|_X \le C\|h(t,x)\|_{W^{1,1}\big((0,T);L^2(0,L)\big)}. \tag{6.19}$$

We have

$$\|(y-y_a, \widetilde{y}-\widetilde{y}_a)\|_X \overset{(6.17)}{\le} C\rho^{3/2}, \quad \|y_a - \widetilde{y}_a\|_X \le C\rho^{-1/2}\|\varphi - \widetilde{\varphi}\|_Y,$$

and

$$\|g(t,x)\|_{W^{1,1}\big((0,T);L^2(0,L)\big)} \le C\rho^{1/2}\|\varphi - \widetilde{\varphi}\|_Y.$$

It follows that

$$\|h(t,x)\|_{W^{1,1}\big((0,T);L^2(0,L)\big)} \le C\rho^{1/2}\|\varphi - \phi\|_Y, \tag{6.20}$$

which yields, by (6.19),

$$\|(y - y_a - \widetilde{y} + \widetilde{y}_a)(T,\cdot)\|_Y \le C\rho^{1/2}\|\varphi - \phi\|_Y.$$

Assertion (6.16) follows.

We next consider the case $\|P_{\mathcal{M}}y_T - w_0\|_{H^3(0,L)} \le 2c\|y_T\|_{H^3(0,L)}$. In fact, one can bring this case to the previous case as follows. Fix $\varepsilon > 0$ small. By Lemma 5.3 and Corollary 3.7, there exists $v_1 \in H^2_0(0,\varepsilon)$ such that if $y_1 \in X$ (with $T = \varepsilon$) is the solution of (6.10) with $u_1 = v_1$ and $y_2 \in X$ is the solution of (6.11) with $u_2 = 0$ then

$$y_2(\varepsilon,\cdot) \in \mathcal{M} \setminus \{0\}.$$

Let $u_{0,T}$, $u_{1,T}$, $u_{2,T}$ be such that $u_{0,T}$ is a control for which the corresponding solution in X of the linear system (6.2) starting from $y_T(L-\cdot)/\rho$ at 0 and arriving 0 at the time ε, $u_{1,T} = \gamma v_1$, $u_{2,T} = \gamma^2 v_2$ for some $\gamma > 0$ defined later. Let $\mathbf{y}$ be the unique solution of the nonlinear KdV system in the time interval $[T, T+\varepsilon]$ using the control

$$\rho u_0(\cdot - T) + \rho^{1/2}u_1(\cdot - T) + \rho u_2(\cdot - T),$$

with $\mathbf{y}(T,\cdot) = y_T(L-\cdot)$. By choosing γ large enough, y_0 and $\mathbf{y}(T+\varepsilon, L-\cdot)$ satisfy the setting of the previous case for the time interval $[0, T+\varepsilon]$ (instead of [0, T]). One now considers the control (for the nonlinear KdV system) in the time interval $[0, T+2\varepsilon]$ which is equal to the one which brings y_0 at the time 0 to $\mathbf{y}(T+\varepsilon, L-\cdot)$ at the time $T+\varepsilon$ obtained in the previous case in the time interval $[0, T+\varepsilon]$, and is equal to $-\mathbf{y}_x\big(2(T+\varepsilon)-t, 0\big)$ for $t \in [T+\varepsilon, T+2\varepsilon]$. It is clear that the solution of the nonlinear KdV system at the time $T+2\varepsilon$ is y_T. The proof is complete by changing $T+2\varepsilon$ to T. □

Remark 6.6. Similar result as the one in Theorem 6.1 also holds for $y_0, y_T \in H^2(0,L) \cap H^1_0(0,L)$ and $u \in H^{2/3}(0,T)$. More precisely, one has the following result. Let $L > 0$, and $k, l \in \mathbb{N}$. Let p be defined by (5.1). Assume that (5.2) holds, $2k + l \not\in 3\mathbb{N}$, and the dimension of $\mathcal{M}$ is 2. Given $T > \pi/p$, there exists $\varepsilon_0 > 0$ such that for $y_0, y_T \in H^2(0,L) \cap H^1_0(0,L)$ with

$$\|(y_0, y_T)\|_{H^2(0,L)} \le \varepsilon_0,$$

there exists $u \in H^{2/3}(0,T)$ such that $u(0) = y_0'(L)$,

$$\|u\|_{H^{2/3}(0,T)} \le C\|(y_0, y_T)\|_{H^2}^{1/2},$$

and the corresponding solution $y \in C\big([0,T]; H^2(0,L)\big) \cap L^2\big((0,T); H^3[0,L]\big)$ of the nonlinear system (1.1) with $y(t=0,\cdot) = y_0$ satisfies $y(t=T,\cdot) = y_T$. This is complementary to Theorem 5.1. The only important modification in comparison with the proof of Theorem 6.1 is Lemma 6.4. Nevertheless, the method presented in its proof can be extended to cover the setting mentioned here (initial and final datum in $H^2(0,L) \cap H^1_0(0,1)$ and controls in $H^{2/3}(0,T)$). We also have

$$\|y_x(\cdot, 0)\|_{H^{2/3}(0,T)} \le C\Big(\|y(0,\cdot)\|_{H^2(0,L)} + \|y_x(\cdot, L)\|_{H^{2/3}(0,T)}\Big), \tag{6.21}$$

for solutions $y \in C\big([0,T]; H^2(0,L)\big) \cap L^2\big((0,T); H^3[0,L]\big)$ of (1.1) with small norm. Assertion (6.21) would follow from [12] applied to $s = 2$. Here is another way to see it. Split y into two parts y_1 and y_2 where y_1 is the solution of the linearized system with zero initial data and $y_{1,x}(\cdot, L) = y_x(\cdot, L)$. As in the proof of Lemma 4.4, one can prove

$$\|y_{1,x}(\cdot, 0)\|_{H^{2/3}(0,T)} \le C\|y_x(\cdot, L)\|_{H^{2/3}(0,T)}. \tag{6.22}$$

Concerning y_2, by considering yy_x as a source term, similar to the proof of Lemma 4.6, one can prove

$$\|y_{2,x}(\cdot, 0)\|_{H^{2/3}(0,T)} \le C\Big(\|y(0,\cdot)\|_{H^2(0,L)} + \|yy_x\|_{L^2\big((0,T); H^2(0,L)\big)}\Big). \tag{6.23}$$

Since

$$\|yy_x\|_{L^2\big((0,T);H^2(0,L)\big)} \le C\|y\|^2_{C\big([0,T];H^2(0,L)\big)\cap L^2\big((0,T);H^3[0,L]\big)} \quad \text{(by the embedding theorem)}$$

$$\le C\Big(\|y(0,\cdot)\|_{H^2(0,L)} + \|y_x(\cdot, L)\|_{H^{2/3}(0,T)}\Big)^2 \quad \text{(by [12, Theorem 3.4] applied to } s = 2\text{)},$$

assertion (6.21) follows from (6.22) and (6.23). Therefore, the arguments using the backward systems also work in this case.

Remark 6.7. The proof given in Theorem 6.1 can be extended easily to the case $L \not\in \mathcal{N}$ to yield the small-time local controllability of (1.1) with initial final and initial datum in $H^3(0,L) \cap H^1_0(0,L)$ (resp. $H^2(0,L) \cap H^1_0(0,L)$) and controls in $H^1(0,T)$ (resp. $H^{2/3}(0,T)$).

Remark 6.8. Let $L \in \mathcal{N}$. Assume that $\dim \mathcal{M}$ is pair and for all $(k,l) \in \mathbb{N}^2$ with $k > l \ge 1$ and $L = \frac{1}{2\pi}\sqrt{\frac{k^2+l^2+kl}{3}}$, it holds $2k + l \not\in 3\mathbb{N}$. Then, using the same method in the proof of Theorem 6.1, and involving the ideas in [20], one can prove that the system (1.1) and (1.2) is controllable at the time given in [20].

Remark 6.9. The mappings $\mathbb{G}$ and Λ have their roots in [24] (see also [18]).

Remark 6.10. Lemma 6.4 is motivated by the Hilbert Uniqueness Method and inspired by the construction of smooth controls (for different contexts, e.g. the context of the wave equation) in [27]. The function η used there is inspired from [27]. Nevertheless, we cannot take $\eta = 0$ near T as in [27]. We also add a large parameter λ in the proof.

Remark 6.11. In the proof of Lemma 6.5, we use essentially the fact that for all $\tau > 0$, there exists $v_1 \in H^2_0(0,\tau)$ such that if $y_1 \in X$ is the solution of (6.10) with $u_1 = v_1$ and $y_2 \in X$ is the solution of (6.11) with $u_2 = 0$ then

$$y_2(\tau, \cdot) \in \mathcal{M} \setminus \{0\}.$$

This is a consequence of Lemma 5.3 and Corollary 3.7. It is not clear for us how to use a contradiction argument as in [24, 18, 20] to obtain such a function v_1. This is why we cannot implement the strategy in [24, 18, 20] to derive the local controllability for initial and final datum in $H^3(0,L) \cap H^1_0(0,L)$ with controls in $H^1(0,T)$ for all critical lengths and for small time when $\dim \mathcal{M} = 1$ and for finite time otherwise.

Remark 6.12. We emphasize that the way to implement the fixed point argument for Λ presented in this paper is somehow different from the one in [18]. We only apply the fixed point arguments once instead of twice, first for $P_{\mathcal{M}^\perp}\Lambda$ and then for $P_{\mathcal{M}}\Lambda$ as in [18]. The Brouwer fixed point theorem is not required in our analysis.

Appendix A. On symmetric functions of the roots of a polynomial

This is standard for people knowing algebraic functions [1, Ch. 8 §2], but for the sake of completeness, we justify that an analytic symmetric function of the roots $\lambda_j(z)$ of $\lambda^3 + \lambda + iz = 0$ is an entire function.

Lemma A.1. *Let $(\lambda_1(z), \lambda_2(z), \lambda_3(z))$ be the three roots of $\lambda^3 + \lambda + iz = 0$. Let $F\colon \mathbb{C}^3 \to \mathbb{C}$ be holomorphic in $\mathbb{C}^3$ and symmetric, i.e., for every permutation $\sigma \in \mathfrak{S}_3$, $F(z_{\sigma(1)}, z_{\sigma(2)}, z_{\sigma(3)}) = F(z_1, z_2, z_3)$. Then, the function $G\colon z \in \mathbb{C} \mapsto F(\lambda_1(z), \lambda_2(z), \lambda_3(z))$ is entire.*

Note that the ordering $\lambda_1(z), \lambda_2(z), \lambda_3(z)$ is not unique (and we could prove that we cannot chose an ordering that makes any of the λ_j entire), but since F is symmetric, the value $F(\lambda_1(z), \lambda_2(z), \lambda_3(z))$ does not depend on the ordering.

Proof. Note that, for $z_0 \neq \pm 2/(3\sqrt{3})$, the discriminant of $X^3 + X + iz$ is nonzero, and thus the roots of $X^3 + X + iz_0$ are simple. By the implicit function theorem, there exists some complex neighborhood U of z_0, some neighborhood V_j of $\lambda_j(z_0)$ $(1 \le j \le 3)$, and three holomorphic functions $\mu_j\colon U \to V_j$ such that $\mu_1(z)$, $\mu_2(z), \mu_3(z)$ are the three distinct roots. Since F is symmetric, it follows that $G(z) = F(\mu_1(z), \mu_2(z), \mu_3(z))$ and is therefore analytic in U. Consequently, G is analytic in $\mathbb{C} \setminus \{\pm 2/(3\sqrt{3})\}$.

It suffices then to prove that G is continuous at $\pm 2/3\sqrt{3}$. The roots $\lambda_j(z)$ are continuous, even around at $\pm\sqrt{4/27}$, in the sense that for every $\epsilon > 0$, there exists $\delta > 0$ such that for every $|z - z_0| < \delta$, there exists some ordering of the $\lambda_{k_j}(z)$ such that $|\lambda_{k_1}(z) - \lambda_1(z_0)| + \cdots + |\lambda_{k_3}(z) - \lambda_3(z_0)| < \epsilon$ (this can be seen e.g. thanks to Cardano's formula). Thus $G(z)$ is continuous at $z_0 = \pm\sqrt{4/27}$ and $\pm\sqrt{4/27}$. □

Remark A.2. A variant of Lemma A.1 still holds for more general polynomial equations $P(z, \lambda) = 0$, but we wanted to avoid some technicalities of such a general equation. The general case would be a consequence of the fact that the solutions of $P(z, \lambda) = 0$ define a finite number of algebraic functions, see [1, Ch. 8 §2].

Appendix B. On the real roots of H, the common roots of G and H, and the behavior of $|\det Q|$

We begin with

Lemma B.1. *Let $z \in \mathbb{R}$. We have*

1) *if $z \neq \pm 2/(3\sqrt{3})$ and $H(z) = 0$, then, for some $k, l \in \mathbb{N}$ with $1 \le l \le k$, $L = 2\pi\sqrt{\frac{k^2+kl+l^2}{3}}$, and*

$$z = -\frac{(2k+l)(k-l)(2l+k)}{3\sqrt{3}(k^2+kl+l^2)^{3/2}}. \tag{B.1}$$

Moreover,

$$\lambda_1(z) = -\frac{2\pi i}{3L}(2k+l), \quad \lambda_2(z) = \lambda_1(z) + \frac{2\pi i}{L}k, \quad \lambda_3(z) = \lambda_2(z) + \frac{2\pi i}{L}l, \tag{B.2}$$

and z is a simple zero of the equation H.

2) *if $z = \pm 2/(3\sqrt{3})$ then*

$$\lambda_1(z) = \mp\frac{i}{\sqrt{3}}, \quad \lambda_2(z) = \mp\frac{i}{\sqrt{3}}, \quad \lambda_3(z) = \pm\frac{2i}{\sqrt{3}}, \tag{B.3}$$

z is not a zero of H, and z is a simple solution of the equation $\det Q(z)\Xi(z) = 0$.

Proof. We begin with 1). By Remark 2.7, assertion (B.1) holds. Assertion (B.2) then follows from [38]. To prove that z is then a simple root of the equation $H(z) = 0$ in the case $z \neq \pm 2/(3\sqrt{3})$, we proceed as follows. We have

$$\lambda_j(z+\varepsilon) = \lambda_j(z) - \frac{i\varepsilon}{3\lambda_j^2+1} + O(\varepsilon^2).$$

It follows that

$$\det Q(z+\varepsilon) = \sum_{j=1}^{3} \big(\lambda_{j+1}(z+\varepsilon) - \lambda_j(z+\varepsilon)\big)e^{-\lambda_{j+2}(z+\varepsilon)L}$$

$$= \sum_{j=1}^{3} \Big(\lambda_{j+1}(z) - \lambda_j(z) - \frac{i\varepsilon}{3\lambda_{j+1}^2+1} + \frac{i\varepsilon}{3\lambda_j^2+1} + O(\varepsilon^2)\Big)e^{-\lambda_{j+2}(z)L}\Big(1 + \frac{i\varepsilon L}{3\lambda_{j+2}^2+1} + O(\varepsilon^2)\Big).$$

Since

$$e^{-\lambda_1(z)L} = e^{-\lambda_2(z)L} = e^{-\lambda_3(z)L},$$

we derive that

$$\det Q(z+\varepsilon) = i\varepsilon L e^{-\lambda_1(z)L} \sum_{j=1}^{3} \frac{\lambda_{j+1}(z) - \lambda_j(z)}{3\lambda_{j+2}^2(z)+1} + O(\varepsilon^2). \tag{B.4}$$

In what follows, for notational ease, we denote $\lambda_j(z)$ by λ_j. We have

$$\sum_{j=1}^{3} \frac{\lambda_{j+1} - \lambda_j}{3\lambda_{j+2}^2 + 1} = \frac{2\pi i}{L} \left(\frac{k}{3\lambda_3^2 + 1} + \frac{l}{3\lambda_1^2 + 1} - \frac{k+l}{3\lambda_2^2 + 1} \right) \tag{B.5}$$

$$= \frac{2\pi i}{L} \left(\frac{3k(\lambda_2^2 - \lambda_3^2)}{(3\lambda_3^2 + 1)(3\lambda_2^2 + 1)} + \frac{3l(\lambda_2^2 - \lambda_1^2)}{(3\lambda_1^2 + 1)(3\lambda_2^2 + 1)} \right)$$

$$= \left(\frac{2\pi i}{L}\right)^2 \left(- \frac{3kl(\lambda_2 + \lambda_3)}{(3\lambda_3^2 + 1)(3\lambda_2^2 + 1)} + \frac{3kl(\lambda_2 + \lambda_1)}{(3\lambda_1^2 + 1)(3\lambda_2^2 + 1)} \right).$$

Note that

$$(\lambda_2 + \lambda_1)(3\lambda_3^2 + 1) - (\lambda_2 + \lambda_3)(3\lambda_1^2 + 1) \tag{B.6}$$

$$= (\lambda_1 - \lambda_3) + 3(\lambda_3 - \lambda_1)(\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3) = 2(\lambda_3 - \lambda_1),$$

since $\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3 = 1$. From (B.4), (B.5), and (B.6), we derive that z is a simple root of $H(z)$.

We next consider 2). We only consider the case $z = 2/(3\sqrt{3})$, the other case follows similarly. By (2.19) in the proof of Lemma 2.6, we have

$$\lambda_1(z+\varepsilon) = -\frac{i}{\sqrt{3}} + \frac{\sqrt{-i}}{3^{1/4}}\sqrt{\epsilon} + O(\epsilon), \; \lambda_2(z+\varepsilon) = -\frac{i}{\sqrt{3}} - \frac{\sqrt{-i}}{3^{1/4}}\sqrt{\epsilon} + O(\epsilon), \; \lambda_3(z+\epsilon) = \frac{2i}{\sqrt{3}} + O(\epsilon). \tag{B.7}$$

It follows that

$$\det Q(z+\varepsilon) = -\frac{2Li}{\sqrt{3}} \frac{\sqrt{-i}}{3^{1/4}} \sqrt{\varepsilon} + O(\varepsilon).$$

Since $\Xi(z+\varepsilon) = c_+\sqrt{\varepsilon}$ for some $c_+ \neq 0$ by (B.7), $z = 2/(3\sqrt{3})$ is not a root of the equation $H(z) = 0$ and z is a simple root of the equation $\det Q(z)\Xi(z) = 0$. The proof is complete. $\square$

Lemma B.2. *Let $z \in \mathbb{C}$ be such that $z \neq \pm 2/(3\sqrt{3})$. Assume that $H(z) = G(z) = 0$. Then, for some $k, l \in \mathbb{N}$ with $k \geq l \geq 1$, we have*

$$L = 2\pi\sqrt{\frac{k^2 + kl + l^2}{3}}, \tag{B.8}$$

and

$$z = -\frac{(2k + l)(k - l)(2l + k)}{3\sqrt{3}(k^2 + kl + l^2)^{3/2}}. \tag{B.9}$$

Proof. By Remark 2.7 (see also Lemma B.1), it suffices to prove that if $z \in \mathbb{C}$ is such that $z \neq \pm 2/(3\sqrt{3})$, and $H(z) = G(z) = 0$, then z is real. Indeed, note that

$$\det Q(z) = (\lambda_1 - \lambda_3)(e^{-\lambda_2 L} - e^{-\lambda_3 L}) + (\lambda_3 - \lambda_2)(e^{-\lambda_1 L} - e^{-\lambda_3 L}),$$

and

$$-P(z) = (\lambda_1 - \lambda_3)(e^{\lambda_2 L} - e^{\lambda_3 L}) + (\lambda_3 - \lambda_2)(e^{\lambda_1 L} - e^{\lambda_3 L}).$$

It follows that

$$|\det Q(z)| = 0 \text{ if and only if } (\lambda_3 - \lambda_1)(e^{(\lambda_3 - \lambda_2)L} - 1) = (\lambda_3 - \lambda_2)(e^{(\lambda_3 - \lambda_1)L} - 1), \tag{B.10}$$

and

$$|P(z)| = 0 \text{ if and only if } (\lambda_3 - \lambda_1)(e^{-(\lambda_3 - \lambda_2)L} - 1) = (\lambda_3 - \lambda_2)(e^{-(\lambda_3 - \lambda_1)L} - 1). \tag{B.11}$$

Solving the system

$$\left\{\begin{array}{c}\sum_{j=1}^{3} \lambda_j = 0,\\ \sum_{j=1}^{3} \lambda_j \lambda_{j+1} = 1,\end{array}\right. \tag{B.12}$$

in which λ_3 is a parameter, one has, with $\Delta = -3\lambda_3^2 - 4$,

$$\lambda_1 = \frac{-\lambda_3 + \sqrt{\Delta}}{2} \quad \text{ and } \quad \lambda_2 = \frac{-\lambda_3 - \sqrt{\Delta}}{2}.$$

This implies

$$\alpha = \alpha(\lambda_3) = \lambda_3 - \lambda_1 = \frac{3\lambda_3 - \sqrt{\Delta}}{2} \quad \text{ and } \quad \beta = \beta(\lambda_3) = \lambda_3 - \lambda_2 = \frac{3\lambda_3 + \sqrt{\Delta}}{2}. \tag{B.13}$$

Thus, if z is a common root of $|\det Q|$ and $|P|$ and $\lambda_i(z) \neq \lambda_j(z)$ for $i \neq j$ $(1 \leq i, j \leq 3)$, then, by (B.10) and (B.11),

$$(e^{\alpha L} - 1)(e^{-\beta L} - 1) = (e^{-\alpha L} - 1)(e^{\beta L} - 1),$$

which is equivalent to

$$(e^{\alpha L} - e^{\beta L})(e^{\alpha L} - 1)(e^{\beta L} - 1) = 0.$$

This implies that either $e^{\alpha L} = e^{\beta L}$, or $e^{\alpha L} = 1$, or $e^{\beta L} = 1$. Since $\lambda_1, \lambda_2, \lambda_3$ are distinct, it follows from (B.10) and (B.11) that

$$e^{\alpha L} = e^{\beta L} = 1. \tag{B.14}$$

We derive from (B.13) that

$$3\lambda_3 \in 2\pi i \mathbb{Z}/L.$$

Since

$$\lambda_3^3 + \lambda_3 = -iz,$$

it follows that z is real. The proof is complete. □

We finally establish

Lemma B.3. *There exist $c, C > 0$ and $m_0 \in \mathbb{N}$ such that*

1) for $m \in \mathbb{Z}$ with $|m| \geq m_0$, we have

$$|\det Q(z)| \geq C e^{-c|z|^{1/3}} \text{ if } \Im(z) = \Big((2m+1)\pi/(\sqrt{3}L)\Big)^3.$$

2) for $z \in \mathbb{C}$ with $|z| \geq m_0$ and $|\Re(z)| \geq c|z|^{1/3}$, we have

$$|\det Q(z)| \geq C e^{-c|z|^{1/3}}.$$

Proof. For $z \in \mathbb{C}$ with large $|z|$, denote $\lambda_1, \lambda_2, \lambda_3$ be the three roots of the equation

$$\lambda^3 + \lambda = -iz,$$

with the convention $\Re(\lambda_3) \geq \max\big\{\Re(\lambda_1), \Re(\lambda_2)\big\}$, and, with $\Delta = -3\lambda_3^2 - 4$,

$$\lambda_1 = \frac{-\lambda_3 + \sqrt{\Delta}}{2}, \quad \text{ and } \quad \lambda_2 = \frac{-\lambda_3 - \sqrt{\Delta}}{2}.$$

This is possible since

$$\left\{\begin{array}{l}\lambda_1 + \lambda_2 = -\lambda_3,\\ \lambda_1 \lambda_2 = 1 + \lambda_3^2.\end{array}\right.$$

We have

$$|\lambda_3^{-1} \det Q(z) e^{\lambda_3 L}| = |f(\lambda_3)|,$$

where

$$f(\lambda_3) := \frac{3\lambda_3 - \sqrt{\Delta}}{2\lambda_3}(e^{\frac{3\lambda_3+\sqrt{\Delta}}{2}L} - 1) - \frac{3\lambda_3 + \sqrt{\Delta}}{2\lambda_3}(e^{\frac{3\lambda_3-\sqrt{\Delta}}{2}L} - 1). \tag{B.15}$$

Since λ_3 is large, we have

$$\left(\frac{3 - i\sqrt{3}}{2}\right)^{-1} f(\lambda_3) = [1 + O(\lambda_3^{-2})](e^{\frac{3+i\sqrt{3}}{2}\lambda_3 L + O(\lambda_3^{-1})} - 1) \\ - [1 + O(\lambda_3^{-2})]e^{i\varphi_0}(e^{\frac{3-i\sqrt{3}}{2}\lambda_3 L + O(\lambda_3^{-1})} - 1), \tag{B.16}$$

where $\varphi_0 = \pi/3$ since $\frac{3+i\sqrt{3}}{2}/\frac{3-i\sqrt{3}}{2} = e^{i\varphi_0}$.

We begin with 1). It suffices to prove, for $z \in \mathbb{C}$ with $\Im(z) = \Big((2m+1)\pi/(\sqrt{3}L)\Big)^3$ with large $|m|$ ($m \in \mathbb{Z}$), that

$$|\lambda_3^{-1} \det Q(z) e^{\lambda_3 L}| \geq 1. \tag{B.17}$$

Assume that (B.17) does not hold. Then for some $m \in \mathbb{Z}$ with large modulus and for some $z \in \mathbb{C}$ with $\Im(z) = \Big((2m+1)\pi/(\sqrt{3}L)\Big)^3$, we have

$$|f(\lambda_3)| \leq 1.$$

Since $\Re(\lambda_3) > 0$ and is large, it follows that

$$|e^{\frac{3+i\sqrt{3}}{2}\lambda_3 L}| = (1 + O(\lambda_3^{-1}))|e^{\frac{3-i\sqrt{3}}{2}\lambda_3 L}|.$$

One derives that, if $\lambda_3 = a + ib$ with $a, b \in \mathbb{R}$,

$$a \text{ is large and } |b| = O(\lambda_3^{-1}). \tag{B.18}$$

It follows that

$$e^{\frac{3+i\sqrt{3}}{2}\lambda_3 L} = e^{\frac{3aL}{2}} e^{i\frac{\sqrt{3}aL}{2}} e^{O(\lambda_3^{-1})}$$

and

$$e^{\frac{3-i\sqrt{3}}{2}\lambda_3 L} = e^{\frac{3aL}{2}} e^{-i\frac{\sqrt{3}aL}{2}} e^{O(\lambda_3^{-1})}.$$

Using (B.16), and the fact $|f(\lambda_3)| \leq 1$ and $\Im(z) = \Big((2m+1)\pi/(\sqrt{3}L)\Big)^3$, we obtain a contradiction. Hence (B.17) holds. The proof of 1) is complete.

To establish 2), it suffices to prove (B.17) for $z \in \mathbb{C}$ with $|z| \geq m_0$ and $|\Re(z)| \geq c|z|^{1/3}$ for some $c > 0$. This indeed follows from the fact if $|z|$ is large and $|f(\lambda_3)| \leq 1$, then (B.18) holds. The proof is complete. □

Acknowledgments. The authors were partially supported by ANR Finite4SoS ANR-15-CE23-0007. A. Koenig is supported by a public grant overseen by the French National Research Agency (ANR) as part of the "Investissements d'Avenir"'s program of the Idex PSL reference ANR-10-IDEX-0001-02 PSL. H.-M. Nguyen thanks Fondation des Sciences Mathématiques de Paris (FSMP) for the Chaire d'excellence which allows him to visit Laboratoire Jacques Louis Lions and Mines ParisTech. This work has been done during this visit.

References

[1] Lars V. Ahlfors, *Complex analysis*, third ed., McGraw-Hill Book Co., New York, 1978, An introduction to the theory of analytic functions of one complex variable, International Series in Pure and Applied Mathematics. MR 510197

[2] Farid Ammar-Khodja, Assia Benabdallah, Manuel González-Burgos, and Luz de Teresa, *Recent results on the controllability of linear coupled parabolic problems: a survey*, Math. Control Relat. Fields **1** (2011), no. 3, 267–306. MR 2846087

[3] Claude Bardos, Gilles Lebeau, and Jeffrey Rauch, *Sharp sufficient conditions for the observation, control, and stabilization of waves from the boundary*, SIAM J. Control Optim. **30** (1992), no. 5, 1024–1065. MR 1178650

[4] Karine Beauchard, Jérémi Dardé, and Sylvain Ervedoza, *Minimal time issues for the observability of Grushin-type equations*, Ann. Inst. Fourier (Grenoble) **70** (2020), no. 1, 247–313. MR 4105940

[5] Karine Beauchard, Bernard Helffer, Raphael Henry, and Luc Robbiano, *Degenerate parabolic operators of Kolmogorov type with a geometric control condition*, ESAIM Control Optim. Calc. Var. **21** (2015), no. 2, 487–512. MR 3348409

[6] Karine Beauchard, Armand Koenig, and Kevin Le Balc'h, *Null-controllability of linear parabolic-transport systems*, J. Éc. polytech. Math, to appear.

[7] Karine Beauchard and Frédéric Marbach, *Quadratic obstructions to small-time local controllability for scalar-input systems*, J. Differential Equations **264** (2018), no. 5, 3704–3774. MR 3741402

[8] ______, *Unexpected quadratic behaviors for the small-time local null controllability of scalar-input parabolic equations*, J. Math. Pures Appl. (9) **136** (2020), 22–91. MR 4076969

[9] Karine Beauchard, Luc Miller, and Morgan Morancey, *2D Grushin-type equations: minimal time and null controllable data*, J. Differential Equations **259** (2015), no. 11, 5813–5845. MR 3397310

[10] Karine Beauchard and Morgan Morancey, *Local controllability of 1D Schrödinger equations with bilinear control and minimal time*, Math. Control Relat. Fields **4** (2014), no. 2, 125–160. MR 3167929

[11] Assia Benabdallah, Franck Boyer, and Morgan Morancey, *A block moment method to handle spectral condensation phenomenon in parabolic control problems*, Annales Henri Lebesgue **3** (2020), 717–793.

[12] Jerry L. Bona, Shu Ming Sun, and Bing-Yu Zhang, *A nonhomogeneous boundary-value problem for the Korteweg-de Vries equation posed on a finite domain*, Comm. Partial Differential Equations **28** (2003), no. 7-8, 1391–1436. MR 1998942

[13] ______, *A non-homogeneous boundary-value problem for the Korteweg-de Vries equation posed on a finite domain. II*, J. Differential Equations **247** (2009), no. 9, 2558–2596. MR 2568064

[14] Jean Bourgain, *Fourier transform restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations. I. Schrödinger equations*, Geom. Funct. Anal. **3** (1993), no. 2, 107–156. MR 1209299

[15] Joseph Boussinesq, *Essai sur la théorie des eaux courantes*, Mémoires présentés par divers savants à l'Acad. des Sci. Inst. Nat. France, XXIII, pp. 1–680 (1877), 1–680.

[16] Nicolas Burq and Patrick Gérard, *Condition nécessaire et suffisante pour la contrôlabilité exacte des ondes*, C. R. Acad. Sci. Paris Sér. I Math. **325** (1997), no. 7, 749–752. MR 1483711

[17] Roberto A. Capistrano-Filho, Ademir F. Pazoto, and Lionel Rosier, *Internal controllability of the Korteweg–de Vries equation on a bounded domain*, ESAIM Control Optim. Calc. Var. **21** (2015), no. 4, 1076–1107. MR 3395756

[18] Eduardo Cerpa, *Exact controllability of a nonlinear Korteweg-de Vries equation on a critical spatial domain*, SIAM J. Control Optim. **46** (2007), no. 3, 877–899. MR 2338431

[19] ______, *Control of a Korteweg-de Vries equation: a tutorial*, Math. Control Relat. Fields **4** (2014), no. 1, 45–99. MR 3191303

[20] Eduardo Cerpa and Emmanuelle Crépeau, *Boundary controllability for the nonlinear Korteweg-de Vries equation on any critical domain*, Ann. Inst. H. Poincaré Anal. Non Linéaire **26** (2009), no. 2, 457–475. MR 2504039

[21] Jean-Michel Coron, *On the small-time local controllability of a quantum particle in a moving one-dimensional infinite square potential well*, C. R. Math. Acad. Sci. Paris **342** (2006), no. 2, 103–108. MR 2193655

[22] ______, *Control and nonlinearity*, Mathematical Surveys and Monographs, vol. 136, American Mathematical Society, Providence, RI, 2007. MR 2302744

[23] ______, *Some open problems on the control of nonlinear partial differential equations*, Perspectives in nonlinear partial differential equations, Contemp. Math., vol. 446, Amer. Math. Soc., Providence, RI, 2007, pp. 215–243. MR 2376661

[24] Jean-Michel Coron and Emmanuelle Crépeau, *Exact boundary controllability of a nonlinear KdV equation with critical lengths*, J. Eur. Math. Soc. (JEMS) **6** (2004), no. 3, 367–398. MR 2060480

[25] Jean-Michel Coron and Hoai-Minh Nguyen, *Optimal time for the controllability of linear hyperbolic systems in one-dimensional space*, SIAM J. Control Optim. **57** (2019), no. 2, 1127–1156. MR 3932617

[26] Michel Duprez and Armand Koenig, *Control of the Grushin equation: non-rectangular control region and minimal time*, ESAIM Control Optim. Calc. Var. **26** (2020), Paper No. 3, 18. MR 4050579

[27] Sylvain Ervedoza and Enrique Zuazua, *A systematic method for building smooth controls for smooth data*, Discrete Contin. Dyn. Syst. Ser. B **14** (2010), no. 4, 1375–1401. MR 2679646

[28] Olivier Glass and Sergio Guerrero, *Some exact controllability results for the linear KdV equation and uniform controllability in the zero-dispersion limit*, Asymptot. Anal. **60** (2008), no. 1-2, 61–100. MR 2463799

[29] Tosio Kato, *On the Cauchy problem for the (generalized) Korteweg-de Vries equation*, Studies in applied mathematics, Adv. Math. Suppl. Stud., vol. 8, Academic Press, New York, 1983, pp. 93–128. MR 759907

[30] Diederik J. Korteweg and Gustave de Vries, *On the change of form of long waves advancing in a rectangular canal, and on a new type of long stationary waves*, Philos. Mag. (5) **39** (1895), no. 240, 422–443. MR 3363408

[31] Felipe Linares and Gustavo Ponce, *Introduction to nonlinear dispersive equations*, second ed., Universitext, Springer, New York, 2015. MR 3308874

[32] Frédéric Marbach, *An obstruction to small-time local null controllability for a viscous Burgers' equation*, Ann. Sci. Éc. Norm. Supér. (4) **51** (2018), no. 5, 1129–1177. MR 3942039

[33] Robert M. Miura, *The Korteweg-de Vries equation: a survey of results*, SIAM Rev. **18** (1976), no. 3, 412–459. MR 404890

[34] Luc Molinet and Francis Ribaud, *On the low regularity of the Korteweg-de Vries-Burgers equation*, Int. Math. Res. Not. (2002), no. 37, 1979–2005. MR 1918236

[35] Hoai-Minh Nguyen and Marco Squassina, *Fractional Caffarelli-Kohn-Nirenberg inequalities*, J. Funct. Anal. **274** (2018), no. 9, 2661–2672. MR 3771839

[36] Ademir Fernando Pazoto, *Unique continuation and decay for the Korteweg-de Vries equation with localized damping*, ESAIM Control Optim. Calc. Var. **11** (2005), no. 3, 473–486. MR 2148854

[37] Gustavo Perla Menzala, Carlos F. Vasconcellos, and Enrique Zuazua, *Stabilization of the Korteweg-de Vries equation with localized damping*, Quart. Appl. Math. **60** (2002), no. 1, 111–129. MR 1878262

[38] Lionel Rosier, *Exact boundary controllability for the Korteweg-de Vries equation on a bounded domain*, ESAIM Control Optim. Calc. Var. **2** (1997), 33–55. MR 1440078

[39] ______, *Control of the surface of a fluid by a wavemaker*, ESAIM Control Optim. Calc. Var. **10** (2004), no. 3, 346–380. MR 2084328

[40] Lionel Rosier and Bing-Yu Zhang, *Control and stabilization of the Korteweg-de Vries equation: recent progresses*, J. Syst. Sci. Complex. **22** (2009), no. 4, 647–682. MR 2565262

[41] W. Rudin, *Real and Complex Analysis*, third ed., McGraw-Hill Series in Higher Mathematics, McGraw-Hill, 1987.

[42] David L. Russell and Bing Yu Zhang, *Exact controllability and stabilizability of the Korteweg-de Vries equation*, Trans. Amer. Math. Soc. **348** (1996), no. 9, 3643–3672. MR 1360229

[43] Héctor J. Sussmann, *A general theorem on local controllability*, SIAM J. Control Optim. **25** (1987), no. 1, 158–194. MR 872457

[44] Terence Tao, *Nonlinear dispersive equations*, CBMS Regional Conference Series in Mathematics, vol. 106, Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 2006, Local and global analysis. MR 2233925

[45] Georges Tenenbaum and Marius Tucsnak, *New blow-up rates for fast controls of Schrödinger and heat equations*, J. Differential Equations **243** (2007), no. 1, 70–100. MR 2363470

[46] Gerald Beresford Whitham, *Linear and nonlinear waves*, Wiley-Interscience [John Wiley & Sons], New York-London-Sydney, 1974, Pure and Applied Mathematics. MR 0483954

(Jean-Michel Coron) Sorbonne Université, Université de Paris, CNRS, INRIA, Laboratoire Jacques-Louis Lions, équipe Cage, Paris, France.
Email address: `coron@ann.jussieu.fr.`

(Armand Koenig) Université Côte d'Azur, CNRS, LJAD, France.
Email address: `Armand.KOENIG@univ-cotedazur.fr`

(Hoai-Minh Nguyen) Ecole Polytechnique Fédérale de Lausanne, EPFL, CAMA, Station 8, CH-1015 Lausanne, SwitzerSwitzerland.
Email address: `hoai-minh.nguyen@epfl.ch`