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Decay for the nonlinear KdV equations at critical lengths

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Abstract

The nonlinear KdV equation in a bounded interval equipped with the Dirichlet boundary condition and the Neumann boundary condition on the right is considered. It is known that there is a set of critical lengths for which the solutions of the linearized system conserve the L^2-norm if their initial data belong to a finite dimensional space $\mathcal{M}$. We show that all solutions of the nonlinear system decay to 0 at least with the rate $1/t^{1/2}$ when $\dim \mathcal{M} = 1$ or when $\dim \mathcal{M}$ is even and a specific condition is satisfied, for sufficiently small initial data. Our analysis is inspired by the power series expansion approach and involves the theory of quasi-periodic functions. Consequently, we rediscover all known results by a different approach and obtain new results. We also show that the decay rate is not slower than $\ln(t+2)/t$ for all critical lengths.

MSC: 35B40; 35C20; 35Q53; 93B05

Keywords: KdV equations; Critical lengths; Decay of solutions; Asymptotically stable; Quasi-periodic functions

1. Introduction

1.1. Introduction and statement of the main results

We consider the nonlinear Korteweg-de Vries (KdV) equation in a bounded interval $(0, L)$ equipped with the Dirichlet boundary condition and the Neumann boundary condition on the right:

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$$\begin{cases} u_t(t,x)+u_x(t,x)+u_{xxx}(t,x)+u(t,x)u_x(t,x)=0 & \text{for } t\in(0,+\infty),\ x\in(0,L),\\ u(t,x=0)=u(t,x=L)=u_x(t,x=L)=0 & \text{for } t\in(0,+\infty),\\ u(t=0,\cdot)=u_0 & \text{in } (0,L), \end{cases} \tag{1.1}$$

where $u_0 \in L^2(0, L)$ is the initial data. The KdV equation has been introduced by Boussinesq [2] and Korteweg and de Vries [13] as a model for propagation of surface water waves along a channel. This equation also furnishes a very useful nonlinear approximation model including a balance between a weak nonlinearity and weak dispersive effects and has been studied extensively, see e.g. [22,16].

Regarding (1.1), Rosier [18] introduced a set of critical lengths $\mathcal{N}$ defined by

$$\mathcal{N} := \left\{ 2\pi\sqrt{\frac{k^2+kl+l^2}{3}};\ k,l\in\mathbb{N}_* \right\}. \tag{1.2}$$

This set plays an important role in both the decay property of the solution u of (1.1) and the controllability property of the system associated with (1.1) where $u_x(t, L)$ is a control instead of 0.

Let us briefly review the known results on the controllability of (1.1) where $u_x(t, L)$ is a control:

$$\begin{cases} u_t(t,x)+u_x(t,x)+u_{xxx}(t,x)+u(t,x)u_x(t,x)=0 & \text{for } t\in(0,+\infty),\ x\in(0,L),\\ u(t,x=0)=u(t,x=L)=0 & \text{for } t\in(0,+\infty),\\ u_x(\cdot,x=L):\ \text{is a control}, & \\ u(t=0,\cdot)=u_0 & \text{in } (0,L). \end{cases} \tag{1.3}$$

For initial and final datum in $L^2(0, L)$ and controls in $L^2(0, T)$, Rosier [18] proved that system (1.3) is small-time locally controllable around 0 provided that the length L is not critical, i.e., $L \notin \mathcal{N}$. To this end, he studied the controllability of the linearized system using the Hilbert Uniqueness Method and compactness-uniqueness arguments. He also established that when the length L is critical, i.e., $L \in \mathcal{N}$, the linearized system is not controllable. More precisely, he showed that there exists a non-trivial finite-dimensional subspace $\mathcal{M}$ $(= \mathcal{M}_L)$ of $L^2(0, L)$ such that its orthogonal space in $L^2(0, L)$ is reachable from 0 whereas $\mathcal{M}$ is not. To tackle the control problem for the critical length $L \in \mathcal{N}$, Coron and Crépeau introduced the power series expansion method [9]. The idea is to take into account the effect of the nonlinear term uu_x absent in the linearized system. Using this method, they showed [9] (see also [8, section 8.2]) that system (1.3) is small-time locally controllable if $L = m2\pi$ for $m \in \mathbb{N}_*$ satisfying

$$\nexists (k,l)\in\mathbb{N}_*\times\mathbb{N}_* \text{ with } k^2+kl+l^2=3m^2 \text{ and } k\neq l, \tag{1.4}$$

with initial and final datum in $L^2(0, L)$ and controls in $L^2(0, T)$. In this case, $\dim \mathcal{M} = 1$ and $\mathcal{M}$ is spanned by $1 - \cos x$. Cerpa [4] developed the analysis in [9] to prove that (1.3) is locally controllable at *a finite time* in the case $\dim \mathcal{M} = 2$. This corresponds to the case where

$$L = 2\pi\sqrt{\frac{k^2+kl+l^2}{3}}$$

for some k, $l \in \mathbb{N}_*$ with $k > l$, and there is no $(m, n) \in \mathbb{N}_* \times \mathbb{N}_*$ with $m > n$ and $m^2+mn+n^2 = k^2+kl+l^2$. Later, Crépeau and Cerpa [6] succeeded to extend the ideas in [4] to obtain the local controllability for all other critical lengths at *a finite time*. Recently, Coron, Koenig, and Nguyen [10] prove that when $(2k+l)/3 \notin \mathbb{N}_*$, one cannot achieve the small time local controllability for initial datum in $H^3(0, L)$ and controls in H^1 (in time). We also establish the local controllability for finite time of (1.3) for some subclass of these pairs (k,l) with initial datum in $H^3(0, L)$ and the controls in $H^1(0, T)$. This is surprising when compared with known results on internal controls for system (1.1). It is known, see [3,15,17], that system (1.1) is locally controllable using internal controls *whenever* the control region contains an *arbitrary* open subset of $(0, L)$.

We next discuss the decay property of (1.1). Multiplying the equation of u (real) with u and integrating by parts, one obtains

$$\int_0^L |u(t,x)|^2\,dx + \int_0^t |u_x(s,0)|^2\,ds = \int_0^L |u(0,x)|^2\,dx \text{ for all } t > 0. \tag{1.5}$$

As a consequence of (1.5), one has

$$\int_0^L |u(t,x)|^2\,dx \le \int_0^L |u(0,x)|^2\,dx \text{ for all } t > 0. \tag{1.6}$$

In the case $L \notin \mathcal{N}$, Menzala, Vasconcellos, and Zuazua [15] proved that the solutions of (1.1) with small initial datum in $L^2(0, L)$ decay exponentially to 0. Their analysis is based on the exponential decay of the linearized system for which it holds, see [18, Proposition 3.3],

$$\int_0^t |u_x(s,0)|^2\,ds \ge c_t \int_0^L |u(0,x)|^2 \text{ for all } t > 0. \tag{1.7}$$

When a local damping was added, they also obtained the global exponential stability using the multiplier technique, compactness arguments, and the unique continuation for the KdV equations. Related results on modified nonlinear KdV equations can be found in [19,14]. It is known from the work of Rosier [18] that for $u_0 \in \mathcal{M}$, the solution u of the linearized system satisfies

$$\int_0^t |u_x(s,0)|^2\,ds = 0 \text{ for all } t > 0, \tag{1.8}$$

which implies in particular that (1.7) does not hold for any $t > 0$. The work of Menzala, Vasconcellos, and Zuazua naturally raises the question whether or not the solutions of (1.1) go to 0 as the time goes to infinity (see [15, Section 4] and also [17, Section 5]). Quite recently, progress has been made for this problem. Concerning the decay property of (1.1) for critical lengths, when

$\dim \mathcal{M} = 1$, Chu, Coron, and Shang [7] showed that the solution $u(t, \cdot)$ goes to 0 as $t \to +\infty$ for all small initial data in $L^2(0, L)$. Moreover, they showed that there exists a constant C depending only on L such that

$$\|u(t, \cdot)\|_{L^2(0,L)} \le \frac{C}{\sqrt{t}} \text{ for } t > 0. \tag{1.9}$$

It is worth mentioning that the set of $L \in \mathcal{N}$ such that $\dim \mathcal{M} = 1$ is infinite [9]. When $k = 2$ and $l = 2$ (the smallest length for which $\dim \mathcal{M} = 2$), Tang, Chu, Sang, and Coron [20] also established the decay to 0 of the solutions by establishing an estimate equivalent to (1.9) (see [20, (1.20) in Theorem 1.1]). The analysis in [7,20] is based on the center manifold theory in infinite dimensions, see e.g. [12], in particular the work [21]. To this end, the authors showed the existence and smoothness of a center manifold associated with (1.1), which have their own interests.

In this paper, we show that all solutions of (1.1) decay to 0 at least with a rate $1/t^{1/2}$ provided their initial data in $L^2(0, L)$ is small enough when $\dim \mathcal{M} = 1$ or when condition (1.14) below holds (this requires in particular that $\dim \mathcal{M}$ is even). Given a critical length L, condition (1.14) can be checked numerically, a scilab program is given in the appendix (see Corollary 1.1 for a range of validation). Our approach is inspired by the spirit of the power series expansion due to Coron and Crépeau [9] and involves the theory of quasi-periodic functions.

Before stating our results, let us introduce some notations associated with the structure of $\mathcal{M}$, see e.g. [18,9,5]. Recall that, for each $L \in \mathcal{N}$, there exists exactly $n_L \in \mathbb{N}_*$ pairs $(k_m, l_m) \in \mathbb{N}_* \times \mathbb{N}_*$ $(1 \le m \le n_L)$ such that $k_m \ge l_m$, and

$$L = 2\pi \sqrt{\frac{k_m^2 + k_m l_m + l_m^2}{3}}. \tag{1.10}$$

For $1 \le m \le n_L$, set

$$p_m = p(k_m, l_m) = \frac{(2k_m + l_m)(k_m - l_m)(2l_m + k_m)}{3\sqrt{3}(k_m^2 + k_m l_m + l_m^2)^{3/2}}, \tag{1.11}$$

and denote

$$\mathcal{P}_L = \Big\{ p_m \text{ given by (1.11)}; 1 \le m \le n_L \Big\}. \tag{1.12}$$

For $L \in \mathcal{N}$ and $1 \le m \le n_L$ with $p_m \neq 0$, let $\sigma_{j,m}$ $(1 \le j \le 3)$ be the solutions of

$$\sigma^3 - 3(k_m^2 + k_m l_m + l_m^2)\sigma + 2(2k_m + l_m)(2l_m + k_m)(k_m - l_m) = 0,$$

and set, with the convention $\sigma_{j+3,m} = \sigma_{j,m}$ for $j \ge 1$,

$$s_m = s(k_m, l_m) := \sum_{j=1}^{3} \sigma_{j,m}(\sigma_{j+2,m} - \sigma_{j+1,m}) \Big(e^{\frac{4\pi i (k_m - l_m)}{3}} e^{2\pi i \sigma_{j,m}} + e^{-2\pi i \sigma_{j,m}} \Big). \tag{1.13}$$

We are ready to state the main result of the paper:

Theorem 1.1. *Let $L \in \mathcal{N}$. Assume that either* $\dim \mathcal{M} = 1$ *or*

$$p_m \neq 0 \quad \text{and} \quad s_m \neq 0 \quad \text{for all } 1 \leq m \leq n_L. \tag{1.14}$$

There exists $\varepsilon_0 > 0$ depending only on L such that for all (real) $u_0 \in L^2(0, L)$ with $\|u_0\|_{L^2(0,L)} \leq \varepsilon_0$, the unique solution $u \in C\big([0, +\infty); L^2(0, L)\big) \cap L^2_{\text{loc}}\big([0, +\infty); H^1(0, L)\big)$ of (1.1) satisfies

$$\lim_{t \to 0} \|u(t, \cdot)\|_{L^2(0,L)} = 0. \tag{1.15}$$

More precisely, there exists a constant C depending only on L such that, for $t \geq C/\|u_0\|^2_{L^2(0,L)}$ and $\|u_0\|_{L^2(0,L)} \leq \varepsilon_0$, it holds

$$\|u(t, \cdot)\|_{L^2(0,L)} \leq \frac{1}{2}\|u(0, \cdot)\|_{L^2(0,L)}. \tag{1.16}$$

As a consequence, we have

$$\|u(t, \cdot)\|_{L^2(0,L)} \leq c/t^{1/2} \text{ for } t \geq 0, \tag{1.17}$$

for some positive constant c depending only on L.

Remark 1.1. Let $L \in \mathbb{N}$. Condition $p_m \neq 0$ for all $1 \leq m \leq n_L$ is equivalent to the fact that $\dim \mathcal{M}$ is even, see e.g. [5].

Remark 1.2. Note that s_m is a antisymmetric function of $(\sigma_{1,m}, \sigma_{2,m}, \sigma_{3,m})$ and hence the condition (1.14) does not depend on the order of $(\sigma_{1,m}, \sigma_{2,m}, \sigma_{3,m})$.

Remark 1.3. When $p_m \neq 0$ for all $1 \leq m \leq n_L$, the condition $s_m \neq 0$ for all $1 \leq m \leq n_L$ is almost equivalent to the fact that the second order approximation of solutions with initial conditions in $\mathcal{M}_L$ decays (the first oder approximation conserves the L^2-norm as shown by Rosier).

Remark 1.4. Assume (1.14). Applying Theorem 1.1, one derives from (1.6) that 0 is (locally) asymptotically stable with respect to $L^2(0, L)$-norm for system (1.1).

Remark 1.5. Assume that (1.16) holds. By the regularity properties of the KdV equations, one derives that the same rate of decay holds for $t > 1$ when $\| \cdot \|_{L^2(0,L)}$ is replaced by $\| \cdot \|_{H^m(0,L)}$ for $m \geq 1$.

Condition (1.14) can be checked numerically. For example, using scilab (the program is given in the appendix), we can check $s_m \neq 0$ for all $(k_m, l_m) \in \mathbb{N}_*$ with $1 \leq l_m < k_m < 2000$. As a consequence, we have

Corollary 1.1. *Let $L \in \mathcal{N}$. Assume that either* $\dim \mathcal{M} = 1$ *or $1 \leq k_m, l_m \leq 1000$ for some $1 \leq m \leq n_L$. Then (1.17) holds if $p_m \neq 0$ for all $1 \leq m \leq n_L$.*

We thus rediscover the decay results in [7,20] by a different approach and obtain new results.

Remark 1.6. Concerning (1.14), we expect that $s_m \neq 0$ holds for all $L \in \mathcal{N}$ but we are not able to show it.

Remark 1.7. In Appendix C, we show that $s_m \neq 0$ for a class of (k_m, l_m).

The optimality of the decay rate $1/t^{1/2}$ given in (1.17) is open. However, we can establish the following result for all critical lengths.

Proposition 1.1. *Let $L \in \mathcal{N}$. There exists $c > 0$ such that for all $\varepsilon > 0$, there exists $u_0 \in L^2(0, L)$ such that*

$$\|u_0\|_{L^2(0,L)} \leq \varepsilon \quad \textit{and} \quad \|u(t, \cdot)\|_{L^2(0,L)} \geq c \ln(t+2)/t \textit{ for some } t > 0.$$

It is natural to ask if the decay holds globally, i.e., without the assumption on the smallness of the initial data. In fact, this cannot hold even for non-critical lengths. More precisely, Doronin and Natali [11] showed that there exist (infinite) stationary states of (1.1) for any length L, which is critical or not.

1.2. Ideas of the analysis and structure of the paper

The key of the analysis of Theorem 1.1 is to (observe and) establish the following fact (see Lemma 5.1): Let $L \in \mathcal{N}$. Under condition (1.14) or $\dim \mathcal{M} = 1$, there exist two constants $T_0 > 0$ and $C > 0$ depending only on L such that for $T \geq T_0$, one has, for all $u_0 \in L^2(0, L)$ with $\|u_0\|_{L^2(0,L)}$ sufficiently small,

$$\|u(T, \cdot)\|_{L^2(0,L)} \leq \|u_0\|_{L^2(0,L)}\Big(1 - C\|u_0\|^2_{L^2(0,L)}\Big) \text{ for } T \geq T_0, \tag{1.18}$$

where u is the unique solution of (1.1).

To get an idea of how to prove (1.18), let us consider the case $u_0 \in \mathcal{M} \setminus \{0\}$, which is somehow the worst case. The analysis is inspired by the spirit of the power expansion method [9]. Let $\widetilde{u}_1$ be the unique solution of

$$\begin{cases} \widetilde{u}_{1,t}(t,x) + \widetilde{u}_{1,x}(t,x) + \widetilde{u}_{1,xxx}(t,x) = 0 & \text{for } t \in (0, +\infty),\ x \in (0, L), \\ \widetilde{u}_1(t, x=0) = \widetilde{u}_1(t, x=L) = \widetilde{u}_{1,x}(t, x=L) = 0 & \text{for } t \in (0, +\infty), \\ \widetilde{u}_1(t=0, \cdot) = u_0/\varepsilon & \text{in } (0, L), \end{cases} \tag{1.19}$$

with $\varepsilon = \|u_0\|_{L^2(0,L)} > 0$, and let $\widetilde{u}_2$ be the unique solution of

$$\begin{cases} \widetilde{u}_{2,t}(t,x) + \widetilde{u}_{2,x}(t,x) + \widetilde{u}_{2,xxx}(t,x) + \widetilde{u}_{1,x}(t,x)\widetilde{u}_1(t,x) = 0 & \text{for } t \in (0, +\infty),\ x \in (0, L), \\ \widetilde{u}_2(t, x=0) = \widetilde{u}_2(t, x=L) = \widetilde{u}_{2,x}(t, x=L) = 0 & \text{for } t \in (0, +\infty), \\ \widetilde{u}_2(t=0, \cdot) = 0 & \text{in } (0, L). \end{cases} \tag{1.20}$$

By considering the system of $\varepsilon\widetilde{u}_1 + \varepsilon^2\widetilde{u}_2 - u$, we can prove that, for arbitrary $\tau > 0$,

$$\|(\varepsilon\widetilde{u}_1 + \varepsilon^2\widetilde{u}_2 - u)_x(\cdot, 0)\|_{L^2(0,\tau)} \le c_\tau \varepsilon^3, \tag{1.21}$$

for some $c_\tau > 0$ depending only on τ and L, provided that ε is sufficiently small. Since $\widetilde{u}_1(t, \cdot) \in \mathcal{M}$ for all $t > 0$, one can then derive that

$$\widetilde{u}_{1,x}(t, 0) = 0 \text{ for } t \ge 0.$$

Thus, if one can show that, for some $\tau_0 > 0$ and for some $c_0 > 0$

$$\|\widetilde{u}_{2,x}(\cdot, 0)\|_{L^2(0,\tau_0)} \ge c_0, \tag{1.22}$$

then from (1.21) one has, for ε small enough,

$$\|u_x(\cdot, 0)\|_{L^2(0,\tau_0)} \ge c_0\varepsilon^2.$$

This implies (1.18) with $T_0 = \tau_0$ by (1.5).

To establish (1.22), we first construct a special solution W of the system

$$\begin{cases} W_t(t,x) + W_x(t,x) + W_{xxx}(t,x) + \widetilde{u}_{1,x}(t,x)\widetilde{u}_1(t,x) = 0 & \text{for } t \in (0,+\infty),\ x \in (0,L), \\ W(t,x=0) = W(t,x=L) = W_x(t,x=L) = 0 & \text{for } t \in (0,+\infty), \end{cases} \tag{1.23}$$

via a separation-of-variable process. Moreover, we can prove for such a solution W that

$$W \text{ is bounded by } \|\widetilde{u}_1(0,\cdot)\|_{L^2(0,L)} \text{ up to a positive constant,}$$
$$\text{and } W_x(\cdot, 0) \text{ is a non-trivial quasi-periodic function.} \tag{1.24}$$

The proof of this property is based on some useful observations on p_m and the boundary conditions considered in (1.1), and involves some arithmetic arguments. It is in the proof of the existence of W and the second fact of (1.24) that assumption (1.14) or $\dim \mathcal{M} = 1$ is required. Note that, for all $\delta > 0$, there exists $T_\delta > 0$ such that it holds, for $\tau \ge T_\delta$,

$$\|y_x(\cdot, 0)\|_{L^2(\tau,2\tau)} \le \delta\|y_0\|_{L^2(0,L)}, \tag{1.25}$$

for all solution $y \in C\big([0,+\infty); L^2(0,L)\big) \cap L^2_{\text{loc}}\big([0,+\infty); H^1(0,L)\big)$ of the system

$$\begin{cases} y_t(t,x) + y_x(t,x) + y_{xxx}(t,x) = 0 & \text{for } t \in (0,+\infty),\ x \in (0,L), \\ y(t,x=0) = y(t,x=L) = y_x(t,x=L) = 0 & \text{for } t \in (0,+\infty). \end{cases}$$

Combining (1.24) and (1.25), we can derive (1.22) after applying the theory of quasi-periodic functions, see e.g. [1].

The proof of Proposition 1.1 is inspired by the approach which is used to prove Theorem 1.1 and is mentioned above.

The paper is organized as follows. The elements for the construction of W are given in Section 2 and the elements for the proof of (1.24) are given in Section 3. The proof of Theorem 1.1 is given in Section 5 where (1.18) is formulated in Lemma 5.1. The proof of Proposition 1.1 is given in Section 6. In the appendix, we reproduce a proof of a technical result, which is obtained in [10], and provide the scilab code.

2. Construction of auxiliary functions

Let us begin with recalling and introducing some useful notations motivated by the structure of $\mathcal{M}$, see e.g. [18,9,5]. For $L \in \mathcal{N}$ and for $1 \le m \le n_L$, denote

$$\begin{cases} \eta_{1,m} = -\dfrac{2\pi i(2k_m + l_m)}{3L}, \\ \eta_{2,m} = \eta_{1,m} + \dfrac{2\pi i}{L}k_m = \dfrac{2\pi i(k_m - l_m)}{3L}, \\ \eta_{3,m} = \eta_{2,m} + \dfrac{2\pi i}{L}l_m = \dfrac{2\pi i(k_m + 2l_m)}{3L}. \end{cases} \tag{2.1}$$

Set

$$\begin{cases} \psi_m(x) = \sum_{j=1}^{3}(\eta_{j+1,m} - \eta_{j,m})e^{\eta_{j+2,m}x} & \text{for } x \in [0, L], \\ \Psi_m(t, x) = e^{-itp_m}\psi_m(x) & \text{for } (t, x) \in \mathbb{R} \times [0, L], \end{cases} \tag{2.2}$$

(recall that p_m is defined in (1.11)). It is clear from the definition of $\eta_{j,m}$ in (2.1) that

$$e^{\eta_{1,m}L} = e^{\eta_{2,m}L} = e^{\eta_{3,m}L}. \tag{2.3}$$

This property of $\eta_{j,m}$ associated with L is used several times in our analysis.

Remark 2.1. One can check that $\eta_{j,m}$ are the solutions of the equation

$$\lambda^3 + \lambda - ip_m\lambda = 0.$$

This implies in particular that $p_{m_1} \neq p_{m_2}$ if $(k_{m_1}, l_{m_1}) \neq (k_{m_2}, l_{m_2})$ as observed in [4].

It is known that Ψ_m is a solution of the linearized KdV system; moreover,

$$\Psi_{m,x}(\cdot, 0) = 0,$$

i.e.,

$$\begin{cases} \Psi_{m,t}(t, x) + \Psi_{m,x}(t, x) + \Psi_{m,xxx}(t, x) = 0 & \text{for } t \in (0, +\infty),\ x \in (0, L), \\ \Psi_m(t, 0) = \Psi_m(t, L) = \Psi_{m,x}(t, 0) = \Psi_{m,x}(t, L) = 0 & \text{for } t \in (0, +\infty). \end{cases} \tag{2.4}$$

These properties of Ψ_m can be easily checked. It is known that, see e.g. [5],

$$\mathcal{M} = \operatorname{span}\Big\{\big\{\Re(\psi_m(x));\ 1 \le m \le n_L\big\} \cup \big\{\Im(\psi_m(x));\ 1 \le m \le n_L\big\}\Big\}. \tag{2.5}$$

Here and in what follows, for a complex number z, we denote $\Re z$, $\Im z$, and $\bar{z}$ its real part, its imaginary part, and its conjugate, respectively.

In this section, we prepare elements to construct the function W mentioned in the introduction. Assume that $u_0 \in \mathcal{M} \setminus \{0\}$ and let $\varepsilon = \|u_0\|_{L^2(0,L)}$. By (2.5), there exists $(\alpha_m)_{m=1}^{n_L} \subset \mathbb{C}$ such that

$$\frac{1}{\varepsilon} u_0 = \Re \left\{ \sum_{m=1}^{n_L} \alpha_m \Psi_m(0, x) \right\}. \tag{2.6}$$

The function $\widetilde{u}_1$ defined by (1.19) is then given by

$$\widetilde{u}_1(t, x) = \Re \left\{ \sum_{m=1}^{n_L} \alpha_m \Psi_m(t, x) \right\} = \Re \left\{ \sum_{m=1}^{n_L} \alpha_m e^{-ip_m t} \psi_m(x) \right\}.$$

Using the fact, for an appropriate complex function f,

$$\Re f(t, x) \Re f_x(t, x) = \frac{1}{2}\Big((\Re f(t, x))^2\Big)_x = \frac{1}{8}\Big(\big(f(t, x)^2\big)_x + \big(\bar{f}(t, x)^2\big)_x + 2(|f(t, x)|^2)_x\Big),$$

we derive from (2.2) and (2.6) that

$$\begin{aligned} \widetilde{u}_{1,x}(t, x)\widetilde{u}_1(t, x) =& \frac{1}{8} \sum_{m_1=1}^{n_L} \sum_{m_2=1}^{n_L} \Big(\alpha_{m_1}\alpha_{m_2} e^{-i(p_{m_1}+p_{m_2})t} \psi_{m_1}(x)\psi_{m_2}(x)\Big)_x \\ &+ \frac{1}{8} \sum_{m_1=1}^{n_L} \sum_{m_2=1}^{n_L} \Big(\overline{\alpha_{m_1}\alpha_{m_2} e^{-i(p_{m_1}+p_{m_2})t} \psi_{m_1}(x)\psi_{m_2}(x)}\Big)_x \\ &+ \frac{1}{4} \sum_{m_1=1}^{n_L} \sum_{m_2=1}^{n_L} \Big(\alpha_{m_1}\bar{\alpha}_{m_2} e^{-i(p_{m_1}-p_{m_2})t} \psi_{m_1}(x)\bar{\psi}_{m_2}(x)\Big)_x. \end{aligned} \tag{2.7}$$

Motivated by (2.7), in this section, we construct solutions of system (2.12)-(2.13) and system (2.33)-(2.34) below.

We begin with the following simple result whose proof is omitted.

Lemma 2.1. *Let $L \in \mathcal{N}$ and $1 \le m_1, m_2 \le n_L$. We have, in $[0, L]$,*

$$\begin{aligned} &\Big(\psi_{m_1}\psi_{m_2}\Big)'(x) \\ &= \sum_{j=1}^{3}\sum_{k=1}^{3} (\eta_{j+1,m_1} - \eta_{j,m_1})(\eta_{k+1,m_2} - \eta_{k,m_2})(\eta_{j+2,m_1} + \eta_{k+2,m_2}) e^{(\eta_{j+2,m_1} + \eta_{k+2,m_2})x}, \end{aligned} \tag{2.8}$$

and

$$\left(\psi_{m_1}\bar{\psi}_{m_2}\right)'(x)$$

$$= \sum_{j=1}^{3}\sum_{k=1}^{3}(\eta_{j+1,m_1} - \eta_{j,m_1})(\bar{\eta}_{k+1,m_2} - \bar{\eta}_{k,m_2})(\eta_{j+2,m_1} + \bar{\eta}_{k+2,m_2})e^{(\eta_{j+2,m_1}+\bar{\eta}_{k+2,m_2})x}. \tag{2.9}$$

We next introduce

Definition 2.1. For $z \in \mathbb{C}$, let $\lambda_j = \lambda_j(z)$ $(1 \le j \le 3)$ be the roots of the equation

$$\lambda^3 + \lambda - iz = 0, \tag{2.10}$$

and set

$$Q(z) = \begin{pmatrix} 1 & 1 & 1 \\ e^{\lambda_1 L} & e^{\lambda_2 L} & e^{\lambda_3 L} \\ \lambda_1 e^{\lambda_1 L} & \lambda_2 e^{\lambda_2 L} & \lambda_3 e^{\lambda_3 L} \end{pmatrix}. \tag{2.11}$$

Remark 2.2. Some comments on the definition of Q are in order. The matrix Q is antisymmetric with respect to λ_j $(j = 1, 2, 3)$, and its definitions depend on a choice of the order of $(\lambda_1, \lambda_2, \lambda_3)$. Nevertheless, we later consider either the equation $\det Q = 0$ or a quantity depending on Q in such a way that the order of $(\lambda_1, \lambda_2, \lambda_3)$ does not matter. The definition of Q is only considered in these contexts.

Remark 2.3. The definition of $\lambda_j(z)$ in Definition 2.1 is slightly different from the one given in [10] where iz is used instead of $-iz$ in (2.10).

Remark 2.4. It is known that if $z \in \mathcal{P}_L$ for some $L \in \mathcal{N}$, then

$$\lambda_j = \eta_{j,m} \text{ for some } 1 \le m \le n_L.$$

Hence, by (2.3),

$$e^{\lambda_1 L} = e^{\lambda_2 L} = e^{\lambda_3 L}.$$

Remark 2.5. Note that (2.10) has simple roots for $z \neq \pm 2/(3\sqrt{3})$. Thus, a general solution of the equation

$$y'''(x) + y'(x) - izy(x) = 0 \text{ in } [0, L],$$

is of the form $\sum_{j=1}^{3} a_j e^{\lambda_j(z)x}$ when $z \neq \pm 2/(3\sqrt{3})$. For $z = \pm 2/(3\sqrt{3})$, equation (2.10) has three roots

$$\lambda_1 = \mp 2i/\sqrt{3} \quad \text{and} \quad \lambda_2 = \lambda_3 = \pm i/\sqrt{3}.$$

We now recall a useful property of solutions of the equation $\det Q = 0$ which is established in [10] (a consequence of [10, Remark 2.7]).

Lemma 2.2. *Let $z \in \mathbb{R}$. Then $\det Q(z) = 0$ if and only if either $z = \pm 2/\sqrt{3}$ or ($L \in \mathcal{N}$ and $z \in \mathcal{P}_L$). Moreover,*

$$\Big\{\pm 2/\sqrt{3}\Big\} \cap \mathcal{P}_L = \emptyset \text{ for all } L \in \mathcal{N}.$$

The proof of Lemma 2.2 is reproduced in the appendix for the convenience of the reader.

Let $L \in \mathcal{N}$ and $1 \le m_1, m_2 \le n_L$. As mentioned above, we are interested in constructing a solution of the system

$$-i(p_{m_1} + p_{m_2})\varphi_{m_1,m_2}(x) + \varphi'_{m_1,m_2}(x) + \varphi'''_{m_1,m_2}(x) + \Big(\psi_{m_1}\psi_{m_2}\Big)'(x) = 0 \text{ in } (0, L), \tag{2.12}$$

and

$$\varphi_{m_1,m_2}(0) = \varphi_{m_1,m_2}(L) = \varphi'_{m_1,m_2}(L) = 0. \tag{2.13}$$

We have

Proposition 2.1. *Let $L \in \mathcal{N}$ and $1 \le m_1, m_2 \le n_L$. Let $\lambda_j = \lambda_j(p_{m_1} + p_{m_2})$ and $\mathcal{Q} = Q(ip_{m_1} + ip_{m_2})$ where λ_j and Q are defined by (2.10) and (2.11). When $p_{m_1} \ne 0$ and $p_{m_2} \ne 0$, set*

$$D = D_{m_1,m_2} = \sum_{j=1}^{3}\sum_{k=1}^{3} \frac{(\eta_{j+1,m_1} - \eta_{j,m_1})(\eta_{k+1,m_2} - \eta_{k,m_2})}{3\eta_{j+2,m_1}\eta_{k+2,m_2}}, \tag{2.14}$$

and

$$\chi_{m_1,m_2}(x) = -\sum_{j=1}^{3}\sum_{k=1}^{3} \frac{(\eta_{j+1,m_1} - \eta_{j,m_1})(\eta_{k+1,m_2} - \eta_{k,m_2})}{3\eta_{j+2,m_1}\eta_{k+2,m_2}} e^{(\eta_{j+2,m_1}+\eta_{k+2,m_2})x} \text{ in } [0, L]. \tag{2.15}$$

We have

1) *Assume that $p_{m_1} \ne 0$, $p_{m_2} \ne 0$, and $p_{m_1} + p_{m_2} \notin \mathcal{P}_L \cup \Big\{2/(3\sqrt{3})\Big\}$. The unique solution of system (2.12)-(2.13) is given by*

$$\varphi_{m_1,m_2}(x) = \chi_{m_1,m_2}(x) + \sum_{j=1}^{3} a_j e^{\lambda_j x}, \tag{2.16}$$

where (a_1, a_2, a_3) is uniquely determined via (2.13), i.e.,

$$\mathcal{Q}(a_1, a_2, a_3)^\mathsf{T} = D(1, e^{(\eta_{1,m_1}+\eta_{1,m_2})L}, 0)^\mathsf{T}. \tag{2.17}$$

2) *Assume that* $p_{m_1} \neq 0$, $p_{m_2} \neq 0$, *and* $p_{m_1} + p_{m_2} \in \mathcal{P}_L$. *A solution of system* (2.12)-(2.13) *is given by* (2.16) *where* (a_1, a_2, a_3) *satisfies*

$$a_1 + a_2 + a_3 = D \quad \text{and} \quad \lambda_1 a_1 + \lambda_2 a_2 + \lambda_3 a_3 = 0. \tag{2.18}$$

3) *Assume that* $p_{m_1} \neq 0$, $p_{m_2} \neq 0$, *and* $p_{m_1} + p_{m_2} = 2/(3\sqrt{3})$. *Consider the convention*

$$\lambda_1 = -2i/\sqrt{3} \quad \text{and} \quad \lambda_2 = \lambda_3 = i/\sqrt{3}. \tag{2.19}$$

System (2.12)-(2.13) *has a unique solution given by*

$$\varphi_{m_1,m_2}(x) = \chi_{m_1,m_2}(x) + a_1 e^{\lambda_1 x} + (a_2 + a_3 x) e^{\lambda_2 x}, \tag{2.20}$$

where (a_1, a_2, a_3) *is uniquely determined via* (2.13), *i.e.,*

$$\mathcal{Q}_1 (a_1, a_2, a_3)^{\mathsf{T}} = D(1, e^{(\eta_{1,m_1} + \eta_{1,m_2})L}, 0)^{\mathsf{T}}, \tag{2.21}$$

where

$$\mathcal{Q}_1 = \begin{pmatrix} 1 & 1 & 0 \\ e^{\lambda_1 L} & e^{\lambda_2 L} & L e^{\lambda_2 L} \\ \lambda_1 e^{\lambda_1 L} & \lambda_2 e^{\lambda_2 L} & (\lambda_2 L + 1) e^{\lambda_2 L} \end{pmatrix}. \tag{2.22}$$

4) *Assume that* $p_{m_1} = p_{m_2} = 0$ *and thus* $m_1 = m_2 = m$. *A solution of system* (2.12)-(2.13) *is*

$$\varphi_{m,m}(x) = 4\left(L \sin x + \frac{1}{6} - x \sin x - \frac{1}{6}\cos(2x)\right). \tag{2.23}$$

Proof. We proceed with the proof of 1), 2), 3), and 4) in Steps 1, 2, 3, and 4 below, respectively.

Step 1: Proof of 1). Since $\eta = \eta_{j,m}$ $(1 \le j \le 3)$ is a root of the equation

$$\eta^3 + \eta - i p_m = 0,$$

it follows that

$$\eta_{j,m_1} \neq -\eta_{k,m_2}$$

(since otherwise $p_{m_1} = -p_{m_2}$ which is impossible), and

$$(\eta_{j,m_1} + \eta_{k,m_2})^3 + (\eta_{j,m_1} + \eta_{k,m_2}) - i(p_{m_1} + p_{m_2}) = 3\eta_{j,m_1}\eta_{k,m_2}(\eta_{j,m_1} + \eta_{k,m_2}).$$

Since $p_{m_1} \neq 0$ and $p_{m_2} \neq 0$, we derive from Lemma 2.1 that χ_{m_1,m_2} is a solution of (2.12). Since a general solution of the equation $\xi''' + \xi' = i(p_{m_1} + p_{m_2})\xi$ is of the form $\sum_{j=1}^{3} a_j e^{\lambda_j x}$ by Remark 2.5, it follows that

$$\text{a general solution of (2.12) is of the form } \chi_{m_1,m_2}(x) + \sum_{j=1}^{3} a_j e^{\lambda_j x}. \tag{2.24}$$

We have

$$-\chi_{m_1,m_2}(0) = D, \quad -\chi_{m_1,m_2}(L) \overset{(2.3)}{=} De^{(\eta_{1,m_1}+\eta_{1,m_2})L}, \quad \text{and} \quad -\chi_{m_1,m_2,x}(L) \overset{(2.3)}{=} 0. \tag{2.25}$$

It follows that a function of the form $\chi_{m_1,m_2}(x) + \sum_{j=1}^{3} a_j e^{\lambda_j x}$ satisfies (2.13) if and only if

$$\sum_{j=1}^{3} a_j = D, \quad \sum_{j=1}^{3} a_j e^{\lambda_j L} = De^{(\eta_{1,m_1}+\eta_{1,m_2})L}, \quad \sum_{j=1}^{3} a_j \lambda_j e^{\lambda_j L} = 0,$$

which is equivalent to (2.17). Since $p_{m_1} + p_{m_2} \notin \mathcal{P}_L \cup \{2/(3\sqrt{3})\}$ and $p_{m_1} + p_{m_2} > 0$, it follows from Lemma 2.2 that $\det \mathcal{Q} \neq 0$. Therefore, one obtains 1).

Step 2: Proof of 2). A solution of (2.12) is of the form $\chi_{m_1,m_2}(x) + \sum_{j=1}^{3} a_j e^{\lambda_j x}$. This function satisfies (2.13) if and only if, by Remark 2.4 (recall that $p_{m_1} + p_{m_2} \in \mathcal{P}_L$),

$$\sum_{j=1}^{3} a_j = D, \quad e^{\lambda_1 L} \sum_{j=1}^{3} a_j \overset{(2.3)}{=} De^{(\eta_{1,m_1}+\eta_{1,m_2})L}, \quad \sum_{j=1}^{3} a_j \lambda_j \overset{(2.3)}{=} 0.$$

This system has a solution if

$$e^{\lambda_1 L} = e^{(\eta_{1,m_1}+\eta_{1,m_2})L}, \tag{2.26}$$

and a solution is given by (2.16) where (a_1, a_2, a_3) satisfies (2.18).

It remains to prove (2.26). Assume, for some $p_{m_3} \in \mathcal{P}_L$, that

$$p_{m_1} + p_{m_2} = p_{m_3}. \tag{2.27}$$

To establish (2.26), it suffices to prove that, by (2.3) and Remark 2.4,

$$e^{(\eta_{2,m_1}+\eta_{2,m_2})L} = e^{\eta_{2,m_3} L}$$

which is equivalent to the fact, by (2.1),

$$\frac{k_{m_3} - l_{m_3}}{3} - \frac{k_{m_1} - l_{m_1}}{3} - \frac{k_{m_2} - l_{m_2}}{3} \in \mathbb{Z}. \tag{2.28}$$

From (2.27) and the definition of p_m in (1.11), we have

$$\begin{aligned}&(k_{m_3} - l_{m_3})(2k_{m_3} + l_{m_3})(2l_{m_3} + k_{m_3}) \\ &\quad = (k_{m_1} - l_{m_1})(2k_{m_1} + l_{m_1})(2l_{m_1} + k_{m_1}) + (k_{m_2} - l_{m_2})(2k_{m_2} + l_{m_2})(2l_{m_2} + k_{m_2}).\end{aligned} \tag{2.29}$$

Since

$$(k_{m_j} - l_{m_j})(2k_{m_j} + l_{m_j})(2l_{m_j} + k_{m_j}) = l_{m_j} - k_{m_j} \mod 3,$$

it follows from (2.29) that

$$k_{m_3} - l_{m_3} = k_{m_1} - l_{m_1} + \big(k_{m_2} - l_{m_2}\big) \mod 3,$$

which yields (2.28). The proof of Step 2 is complete.

Step 3: Proof of 3). A solution of (2.12) is of the form $\chi_{m_1,m_2}(x) + a_1 e^{\lambda_1 x} + (a_2 + a_3 x)e^{\lambda_2 x}$. This function satisfies (2.13) if and only if, by (2.25),

$$a_1 + a_2 = D, \quad a_1 e^{\lambda_1 L} + a_2 e^{\lambda_2 L} + a_3 L e^{\lambda_2 L} = D e^{(\eta_{1,m_1} + \eta_{1,m_2})L},$$

and

$$a_1 \lambda_1 e^{\lambda_1 L} + a_2 \lambda_2 e^{\lambda_2 L} + a_3 (\lambda_2 L + 1) e^{\lambda_2 L} = 0,$$

which is equivalent to (2.21).

Hence, it suffices to prove that $\mathcal{Q}_1$ is invertible. Replacing the third row of $\mathcal{Q}_1$ by itself minus λ_2 times the second row, we obtain

$$\mathcal{Q}_2 = \begin{pmatrix} 1 & 1 & 0 \\ e^{\lambda_1 L} & e^{\lambda_2 L} & L e^{\lambda_2 L} \\ (\lambda_1 - \lambda_2) e^{\lambda_1 L} & 0 & e^{\lambda_2 L} \end{pmatrix}. \tag{2.30}$$

We have

$$\det \mathcal{Q}_2 = e^{2\lambda_2 L} - \big(1 - L(\lambda_1 - \lambda_2)\big) e^{(\lambda_1 + \lambda_2)L}.$$

Using (2.19), we derive that $\det \mathcal{Q}_2 = 0$ if and only if

$$e^{3\lambda_2 L} = 1 + 3\lambda_2 L.$$

Since the equation $e^{ix} = 1 + ix$ has only one solution $x = 0$ in the real line, one derives that $\det \mathcal{Q}_2 \neq 0$. Therefore, $\mathcal{Q}_1$ is invertible. The proof of Step 3) is complete.

Step 4: Proof of 4). Since $p_m = 0$, it follows that $k_m = l_m$, and $L = 2\pi k_m$. One then has

$$\eta_{1,m} = -i, \quad \eta_{2,m} = 0, \quad \eta_{3,m} = i.$$

It follows from the definition of ψ_m in (2.2) that

$$\psi_m(x) = 2i(\cos x - 1). \tag{2.31}$$

This implies

$$\left(\psi_m^2(x)\right)_x = 8(\cos x - 1)\sin x.$$

A straightforward computation gives the conclusion.

The proof of Proposition 2.1 is complete. □

Remark 2.6. In the case, $p_{m_1} = 0$ and $p_{m_2} \neq 0$, one cannot construct a solution of (2.12)-(2.13) in general. In fact, one can check that

$$\chi_{m_1,m_2}(x) = -\sum_{j=1,2}\sum_{k=1}^{3} \frac{(\eta_{j+1,m_1} - \eta_{j,m_1})(\eta_{k+1,m_2} - \eta_{k,m_2})}{3\eta_{j+2,m_1}\eta_{k+2,m_2}} e^{(\eta_{j+2,m_1}+\eta_{k+2,m_2})x}$$

$$- \sum_{k=1}^{3} \frac{(\eta_{1,m_1} - \eta_{3,m_1})(\eta_{k+1,m_2} - \eta_{k,m_2})\eta_{k+2,m_2}}{3\eta_{k+2,m_2}{}^2 + 1} x e^{\eta_{k+2,m_2}x} \qquad (2.32)$$

is a solution of (2.12). However,

$$\chi_{m_1,m_2}(0) \neq e^{-\eta_{1,m_2}L}\chi_{m_1,m_2}(L)$$

since, in general,

$$\sum_{k=1}^{3} \frac{(\eta_{k+1,m_2} - \eta_{k,m_2})\eta_{k+2,m_2}}{3\eta_{k+2,m_2}{}^2 + 1} \neq 0.$$

Hence one cannot find $(a_1, a_2, a_3) \in \mathbb{C}^3$ such that the function $\chi_{m_1,m_2}(x) + \sum_{j=1}^{3} a_j e^{\lambda_j x}$, with $\lambda_j = \lambda_j(p_{m_2})$, verifies (2.13).

Let $L \in \mathcal{N}$ and $1 \leq m_1, m_2 \leq n_L$. We are next interested in constructing a solution of the system

$$-i(p_{m_1} - p_{m_2})\phi_{m_1,m_2}(x) + \phi'_{m_1,m_2}(x) + \phi'''_{m_1,m_2}(x) + \left(\psi_{m_1}\bar{\psi}_{m_2}\right)'(x) = 0 \text{ in } (0, L), \qquad (2.33)$$

and

$$\phi_{m_1,m_2}(0) = \phi_{m_1,m_2}(L) = \phi'_{m_1,m_2}(L) = 0. \qquad (2.34)$$

We have

Proposition 2.2. *Let $L \in \mathcal{N}$ and $1 \leq m_1, m_2 \leq n_L$. Let $\widetilde{\lambda}_j = \lambda_j(p_{m_1} - p_{m_2})$ and $\widetilde{Q} = Q(ip_{m_1} - ip_{m_2})$ where λ_j and Q are defined by (2.10) and (2.11). When $p_{m_1} \neq 0$ and $p_{m_2} \neq 0$, set*

$$\widetilde{D} = \widetilde{D}_{m_1,m_2} = \sum_{j=1}^{3}\sum_{k=1}^{3} \frac{(\eta_{j+1,m_1} - \eta_{j,m_1})(\bar{\eta}_{k+1,m_2} - \bar{\eta}_{k,m_2})}{3\eta_{j+2,m_1}\bar{\eta}_{k+2,m_2}} \qquad (2.35)$$

and

$$\widetilde{\chi}_{m_1,m_2}(x) = -\sum_{j=1}^{3}\sum_{k=1}^{3} \frac{(\eta_{j+1,m_1} - \eta_{j,m_1})(\bar{\eta}_{k+1,m_2} - \bar{\eta}_{k,m_2})}{3\eta_{j+2,m_1}\bar{\eta}_{k+2,m_2}} e^{(\eta_{j+2,m_1}+\bar{\eta}_{k+2,m_2})x} \text{ in } [0, L]. \tag{2.36}$$

We have

1) *Assume that* $p_{m_1} \neq 0$, $p_{m_2} \neq 0$, $p_{m_1} \neq p_{m_2}$, *and* $p_{m_1} - p_{m_2} \notin \mathcal{P}_L$. *The unique solution of system* (2.33)-(2.34) *is given by*

$$\phi_{m_1,m_2}(x) = \widetilde{\chi}_{m_1,m_2}(x) + \sum_{j=1}^{3} a_j e^{\widetilde{\lambda}_j x}, \tag{2.37}$$

where (a_1, a_2, a_3) *is uniquely determined via* (2.34), *i.e.,*

$$\widetilde{\mathcal{Q}}(a_1, a_2, a_3)^{\mathsf{T}} = \widetilde{D}(1, e^{(\eta_{1,m_1}+\bar{\eta}_{1,m_2})L}, 0)^{\mathsf{T}}. \tag{2.38}$$

2) *Assume that* $p_{m_1} \neq 0$, $p_{m_2} \neq 0$, $p_{m_1} \neq p_{m_2}$, *and* $p_{m_1} - p_{m_2} \in \mathcal{P}_L$. *A solution of system* (2.33)-(2.34) *is given by* (2.37) *where* (a_1, a_2, a_3) *satisfies*

$$a_1 + a_2 + a_3 = \widetilde{D} \quad \text{and } \widetilde{\lambda}_1 a_1 + \widetilde{\lambda}_2 a_2 + \widetilde{\lambda}_3 a_3 = 0. \tag{2.39}$$

3) *Assume that* $p_{m_1} = p_{m_2} \neq 0$ *and thus* $m_1 = m_2 = m$. *System* (2.33)-(2.34) *has a unique solution*

$$\phi_{m,m}(x) = -\sum_{j=1}^{3}\sum_{k=1}^{3} \frac{(\eta_{j+1,m} - \eta_{j,m})(\bar{\eta}_{k+1,m} - \bar{\eta}_{k,m})}{3\eta_{j+2,m}\bar{\eta}_{k+2,m}} e^{(\eta_{j+2,m}+\bar{\eta}_{k+2,m})x} + \sum_{j=1}^{3}\sum_{k=1}^{3} \frac{(\eta_{j+1,m} - \eta_{j,m})(\bar{\eta}_{k+1,m} - \bar{\eta}_{k,m})}{3\eta_{j+2,m}\bar{\eta}_{k+2,m}}.$$

4) *Assume that* $p_{m_1} = p_{m_2} = 0$ *and thus* $m_1 = m_2 = m$. *A solution of system* (2.33)-(2.34) *is*

$$\phi_{m,m}(x) = -4\left(L\sin x + \frac{1}{6} - x\sin x - \frac{1}{6}\cos(2x)\right). \tag{2.40}$$

Proof. We proceed with the proof of 1), 2), 3), and 4) in Steps 1, 2, 3, and 4 below, respectively.

Step 1: Proof of 1). The proof is similar to Step 1 in the proof of Proposition 2.1. One just notes that

$$(\eta_{j,m_1} + \bar{\eta}_{k,m_2})^3 + (\eta_{j,m_1} + \bar{\eta}_{k,m_2}) - i(p_{m_1} - p_{m_2}) = 3\eta_{j,m_1}\bar{\eta}_{k,m_2}(\eta_{j,m_1} + \bar{\eta}_{k,m_2}),$$

and

$$\eta_{j,m_1} + \bar{\eta}_{k,m_2} \neq 0$$

since $p_{m_1} \neq p_{m_2}$.

Step 2: Proof of 2). The proof is almost the same as Step 2 in the proof of Proposition 2.1. The details are omitted.

Step 3: Proof of 3). One can check that $\phi_{m,m}$ is a solution of (2.33)-(2.34). The uniqueness follows from the fact that equation (2.10) has simple roots for $z = 0$.

Step 4: Proof of 4). The conclusion is from 4) of Proposition 2.1 by noting that

$$|\psi_m(x)|^2 \overset{(2.31)}{=} -\psi_m(x)^2 \text{ if } p_m = 0.$$

The proof is complete. □

3. Properties of auxiliary functions

The main goal of this section is to establish, for $L \in \mathcal{N}$ and $1 \leq m \leq n_L$ with $p_m \neq 0$, that

$$\varphi'_{m,m}(0) \neq 0 \tag{3.1}$$

provided (1.14) holds (see Proposition 3.1) where $\varphi_{m,m}$ is determined in Proposition 2.1. We begin with

Lemma 3.1. *Let $L \in \mathcal{N}$ and $1 \leq m \leq n_L$ with $p_m \neq 0$. Set*

$$E_m := \sum_{j=1}^{3} \frac{\eta_{j+1,m} - \eta_{j,m}}{\eta_{j+2,m}}. \tag{3.2}$$

We have

$$D_{m,m} = -\chi_{m,m}(0) = \frac{1}{3} E_m^2, \tag{3.3}$$

and

$$E_m = -\frac{27 k_m l_m (k_m + l_m)}{(k_m + 2l_m)(2k_m + l_m)(k_m - l_m)} \neq 0. \tag{3.4}$$

Proof. It is clear to see from (2.15) that

$$D_{m,m} = -\chi_{m,m}(0) = \frac{1}{3} E_m^2.$$

With the notation $\gamma_{j,m} = L\eta_{j,m}/(2\pi i)$, we have

$$\gamma_{1,m} = -\frac{2k_m + l_m}{3}, \quad \gamma_{2,m} = \frac{k_m - l_m}{3}, \quad \gamma_{3,m} = \frac{k_m + 2l_m}{3}. \tag{3.5}$$

It follows that

$$E_m = \sum_{j=1}^{3} \frac{\gamma_{j+1,m} - \gamma_{j,m}}{\gamma_{j+2,m}} = \frac{3k_m}{k_m + 2l_m} - \frac{3l_m}{2k_m + l_m} - \frac{3(k_m + l_m)}{k_m - l_m}.$$

Since

$$\begin{aligned} k_m(2k_m + l_m)(k_m - l_m) - l_m(k_m + 2l_m)(k_m - l_m) - (k_m + l_m)(k_m + 2l_m)(2k_m + l_m) \\ = 2(k_m^2 - l_m^2)(k_m - l_m) - (k_m + l_m)(k_m + 2l_m)(2k_m + l_m) \\ = (k_m + l_m)\Big(2k_m^2 - 4k_m l_m + 2k_m^2 - 2k_m^2 - 2l_m^2 - 5k_m l_m\Big) = -9k_m l_m (k_m + l_m), \end{aligned}$$

we derive that

$$E_m = -\frac{27 k_m l_m (k_m + l_m)}{(k_m + 2l_m)(2k_m + l_m)(k_m - l_m)} \neq 0.$$

The proof is complete. □

We next show in Lemmas 3.2 and 3.3 below that for $L \in \mathcal{N}$ and for $1 \le m \le n_L$ with $p_m \neq 0$, it holds

$$2p_m \neq 2/(3\sqrt{3}) \quad \text{and} \quad 2p_m \notin \mathcal{P}_L.$$

As a consequence $\varphi_{m,m}$ is constructed via 1) and 4) in Proposition 2.1. We begin with

Lemma 3.2. *Let $L \in \mathcal{N}$ and $1 \le m \le n_L$. Then*

$$2p_m \neq 2/(3\sqrt{3}).$$

Proof. We first claim that there is no $k, l \in \mathbb{N}_*$ with $k \ge l$ such that

$$(2k + l)(2l + k)(k - l) = (k^2 + l^2 + kl)^{3/2}. \tag{3.6}$$

We prove this by contradiction. Assume that there exists such a pair (k, l). Set

$$H = \Big\{(k, l) \in \mathbb{N}_* \times \mathbb{N}_*,\ k \ge l,\ \text{and (3.6) holds}\Big\}.$$

Set

$$h = \min \Big\{k + l; (k, l) \in H\Big\} > 0.$$

Fix $(k, l) \in H$ such that $k + l = h$. Since

$$(2k + l)(2l + k)(k - l) \text{ is even},$$

it follows from (3.6) that $k^2 + l^2 + kl$ is even. Hence both k and l are even. We write $k = 2k_1$ and $l = 2l_1$ for some $k_1, l_1 \in \mathbb{N}_*$. It is clear that

$$k_1 \geq l_1,$$

and

$$(2k_1 + l_1)(2l_1 + k_1)(k_1 - l_1) = (k_1^2 + l_1^2 + k_1 l_1)^{3/2}.$$

This implies

$$(k_1, l_1) \in H.$$

We have

$$k_1 + l_1 = (k + l)/2 = h/2 \quad \text{and} \quad h > 0.$$

This contradicts the definition of h. The claim is proved.

We are ready to derive the conclusion of Lemma 3.2. Since $2p_m = 2/(3\sqrt{3})$ for some $1 \leq m \leq n_L$ and for some $L \in \mathcal{N}$ if and only if, by the definition of p_m in (1.11),

$$(2k_m + l_m)(k_m - l_m)(2l_m + k_m) = (k_m^2 + l_m^2 + k_m l_m)^{3/2},$$

the conclusion follows from the claim. □

We next prove

Lemma 3.3. *There is no quadruple $(k_1, l_1, k_2, l_2) \in \mathbb{N}_*^4$ satisfying the system*

$$\begin{cases} k_1 > l_1, \quad k_2 > l_2, \\ k_1^2 + k_1 l_1 + l_1^2 = k_2^2 + k_2 l_2 + l_2^2, \\ (2k_2 + l_2)(2l_2 + k_2)(k_2 - l_2) = 2(2k_1 + l_1)(2l_1 + k_1)(k_1 - l_1). \end{cases} \tag{3.7}$$

Consequently, for $L \in \mathcal{N}$ and $1 \leq m \leq n_L$, we have

$$2p_m \notin \mathcal{P}_L \text{ if } p_m \neq 0. \tag{3.8}$$

Proof. We prove the non-existence by contradiction. Assume that there exists a quadruple $(k_1, l_1, k_2, l_2) \in \mathbb{N}_*^4$ satisfying (3.7). Set

$$G = \Big\{(k_1, l_1, k_2, l_2) \in \mathbb{N}_*^4; (3.7) \text{ holds}\Big\}, \tag{3.9}$$

and let

$$g = \min\Big\{k_1 + l_1 + k_2 + l_2; (k_1, l_1, k_2, l_2) \in G\Big\} > 0. \tag{3.10}$$

Fix $(k_1, l_1, k_2, l_2) \in G$ such that $k_1 + l_1 + k_2 + l_2 = g$. Set

$$A := k_1^2 + k_1 l_1 + l_1^2 = k_2^2 + k_2 l_2 + l_2^2 \quad \text{(by the second line of (3.7))}. \tag{3.11}$$

Since, for $(k, l) \in \mathbb{R}$,

$$(2k + l)(2l + k) = 2(k^2 + kl + l^2) + 3kl \quad \text{and} \quad (k - l)^2 = (k^2 + kl + l^2) - 3kl,$$

it follows from the square of the last line of (3.7), with

$$x_1 = 3k_1 l_1 \quad \text{and} \quad x_2 = 3k_2 l_2, \tag{3.12}$$

that

$$(2A + x_2)^2 (A - x_2) = 4(2A + x_1)^2 (A - x_1).$$

This implies

$$(4A^3 - 3Ax_2^2 - x_2^3) = 4(4A^3 - 3Ax_1^2 - x_1^3), \tag{3.13}$$

or equivalently

$$12A^3 = 3A(4x_1^2 - x_2^2) + 4x_1^3 - x_2^3. \tag{3.14}$$

Using (3.12), we derive that $A^3 = 0 \mod 3$, which yields

$$A = 0 \mod 3.$$

Putting this information into (3.14) and using again (3.12), we obtain

$$x_1^3 - x_2^3 = 0 \mod 3^4.$$

We deduce from (3.12) that

$$(k_1 l_1)^3 - (k_2 l_2)^3 = 0 \mod 3. \tag{3.15}$$

By writing $k_1 l_1$ under the form $k_2 l_2 + 3q + r$ with $q \in \mathbb{Z}$ and $r \in \mathbb{N}$ with $0 \le r \le 2$, we have

$$(k_1 l_1)^3 - (k_2 l_2)^3 = 3k_2^2 l_2^2 (3q + r) + 3k_2 l_2 (3q + r)^2 + (3q + r)^3. \tag{3.16}$$

Combining (3.15) and (3.16) yields that $r = 0$. Putting this information into (3.14), we obtain

$$A^3 = 0 \mod 3^4.$$

This implies

$$A = 0 \mod 9.$$

We deduce from (3.11) that

$$k_1 = 0 \mod 3, \quad l_1 = 0 \mod 3, \quad k_2 = 0 \mod 3, \quad l_2 = 0 \mod 3.$$

Let $\hat{k}_1, \hat{l}_1, \hat{k}_2, \hat{l}_2 \in \mathbb{N}_*$ be such that

$$k_1 = 3\hat{k}_1, \quad l_1 = 3\hat{l}_1, \quad k_2 = 3\hat{k}_2, \quad l_2 = 3\hat{l}_2.$$

One can easily check that $(\hat{k}_1, \hat{l}_1, \hat{k}_2, \hat{l}_2) \in G$ and

$$\hat{k}_1 + \hat{l}_1 + \hat{k}_2 + \hat{l}_2 = g/3 < g.$$

We obtain a contradiction. The non-existence associated with (3.7) is proved.

It is clear that (3.8) is just a consequence of the non-existence by the definition of L and p_m as a function of k_m and l_m in (1.10) and (1.11). The proof is complete. □

We are ready to state and prove the main result of this section:

Proposition 3.1. *Let $L \in \mathcal{N}$ and $1 \le m \le n_L$. Then*

$$\varphi'_{m,m}(0) = 4\pi L = -\phi'_{m,m}(0) \textit{ if } p_m = 0, \tag{3.17}$$

and, if $p_m \neq 0$ and $s_m \neq 0$ then

$$\varphi'_{m,m}(0) \neq 0. \tag{3.18}$$

Proof. Assertion (3.17) follows immediately from 4) of Propositions 2.1 and 2.2. We next consider the case $p_m \neq 0$. By Lemmas 3.2 and 3.3, we have

$$\varphi'_{m,m}(0) = 0$$

only if, with $\alpha = e^{2\eta_{2,m}L}$ and $\lambda_j = \lambda_j(2p_m)$,

$$\begin{cases} \sum_{j=1}^3 \lambda_j a_j = 0 \quad (= \varphi'_{m,m}(0) \text{ since } \chi'_{m,m}(0) = 0), \\ \sum_{j=1}^3 \lambda_j e^{\lambda_j L} a_j = 0 \quad (= \varphi'_{m,m}(L) \text{ since } \chi'_{m,m}(L) = 0), \\ \sum_{j=1}^3 (e^{\lambda_j L} - \alpha) a_j = 0 \quad (= -\chi_{m,m}(L) + \alpha\chi_{m,m}(0) \text{ since } \chi_{m,m}(L) = \alpha\chi_{m,m}(0)). \end{cases} \tag{3.19}$$

Since $E_m \neq 0$ by Lemma 3.1, one has a non-trivial solution (a_1, a_2, a_3) of this system. This implies

$$\det K_1 = 0 \quad \text{where } K_1 := \begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_1 e^{\lambda_1 L} & \lambda_2 e^{\lambda_2 L} & \lambda_3 e^{\lambda_3 L} \\ e^{\lambda_1 L} - \alpha & e^{\lambda_2 L} - \alpha & e^{\lambda_3 L} - \alpha \end{pmatrix}. \tag{3.20}$$

Set

$$\hat{\lambda}_j = \lambda_j L.$$

Condition (3.20) is equivalent to

$$\det K_2 = 0 \quad \text{where} \quad K_2 := \begin{pmatrix} \hat{\lambda}_1 & \hat{\lambda}_2 & \hat{\lambda}_3 \\ \hat{\lambda}_1 e^{\hat{\lambda}_1} & \hat{\lambda}_2 e^{\hat{\lambda}_2} & \hat{\lambda}_3 e^{\hat{\lambda}_3} \\ e^{\hat{\lambda}_1} - \alpha & e^{\hat{\lambda}_2} - \alpha & e^{\hat{\lambda}_3} - \alpha \end{pmatrix}. \tag{3.21}$$

A computation yields

$$\det K_2 = \sum_{j=1}^{3} \hat{\lambda}_j \Big((\hat{\lambda}_{j+1} - \hat{\lambda}_{j+2}) e^{\hat{\lambda}_{j+1} + \hat{\lambda}_{j+2}} - \alpha (\hat{\lambda}_{j+1} e^{\hat{\lambda}_{j+1}} - \hat{\lambda}_{j+2} e^{\hat{\lambda}_{j+2}}) \Big),$$

which implies

$$\det K_2 = \sum_{j=1}^{3} \hat{\lambda}_j (\hat{\lambda}_{j+1} - \hat{\lambda}_{j+2}) \Big(e^{-\hat{\lambda}_j} + \alpha e^{\hat{\lambda}_j} \Big). \tag{3.22}$$

Here we used the fact $\sum_{j=1}^{3} \hat{\lambda}_j = L \sum_{j=1}^{3} \lambda_j = 0$. From the definition of $\lambda_j = \lambda_j(2p_m)$ given in Definition 2.1, we have

$$\begin{cases} \hat{\lambda}_1 + \hat{\lambda}_2 + \hat{\lambda}_3 = 0, \\ \hat{\lambda}_1 \hat{\lambda}_2 + \hat{\lambda}_1 \hat{\lambda}_3 + \hat{\lambda}_2 \hat{\lambda}_3 = L^2, \\ \hat{\lambda}_1 \hat{\lambda}_2 \hat{\lambda}_3 = 2 i p_m L^3. \end{cases}$$

Define $\sigma_{j,m}$ by

$$\hat{\lambda}_j = \frac{2\pi i \sigma_{j,m}}{3}.$$

We then have

$$\begin{cases} \sigma_{1,m} + \sigma_{2,m} + \sigma_{3,m} = 0, \\ \sigma_{1,m}\sigma_{2,m} + \sigma_{1,m}\sigma_{3,m} + \sigma_{2,m}\sigma_{3,m} = -3(k_m^2 + l_m^2 + k_m l_m), \\ \sigma_{1,m}\sigma_{2,m}\sigma_{3,m} = -2(2k_m + l_m)(2l_m + k_m)(k_m - l_m), \end{cases}$$

where in the last identity, we used the fact

$$p_m L^3 = \frac{1}{27} (2\pi)^3 (2k_m + l_m)(2l_m + k_m)(k_m - l_m).$$

It is clear that $\det K_2 = 0$ if and only if (1.14) holds. The proof is complete. □

4. Useful properties related to quasi-periodic functions

In this section, we derive some properties for $W_x(\cdot, 0)$ given in the introduction using the quasi-periodic-function theory. The main result of this section is Proposition 4.1. We begin with its weaker version.

Lemma 4.1. *Let $\ell \in \mathbb{N}_*$, $a_j \in \mathbb{C}$, $q_j \geq 0$ for $1 \leq j \leq \ell$, and $M_{j_1,j_2}, N_{j_1,j_2} \in \mathbb{C}$ with $1 \leq j_1, j_2 \leq \ell$. Assume that*

$$\begin{cases} q_{j_1} \neq q_{j_2} \text{ for } 1 \leq j_1 \neq j_2 \leq \ell, \\ M_{j,j} \neq 0 \text{ for } 1 \leq j \leq \ell, \\ (M_{j,j} \text{ is real and } N_{j,j} \neq 0) \text{ if } q_j = 0, \\ a_j \in i\mathbb{R} \text{ if } q_j = 0, \end{cases} \tag{4.1}$$

and

$$\sum_{j=1}^{\ell} |a_j|^2 > 0. \tag{4.2}$$

Set, for $t \in \mathbb{R}$,

$$g(t) := \sum_{j_1=1}^{\ell} \sum_{j_2=1}^{\ell} \Big(a_{j_1} a_{j_2} M_{j_1,j_2} e^{-i(q_{j_1}+q_{j_2})t} + \bar{a}_{j_1} \bar{a}_{j_2} \bar{M}_{j_1,j_2} e^{i(q_{j_1}+q_{j_2})t} + 2 a_{j_1} \bar{a}_{j_2} N_{j_1,j_2} e^{-i(q_{j_1}-q_{j_2})} \Big). \tag{4.3}$$

There exists $t \in \mathbb{R}_+$ such that

$$g(t) \neq 0. \tag{4.4}$$

Proof. We prove (4.4) by recurrence in ℓ. It is clear that the conclusion holds for $\ell = 1$. Indeed, if $q_1 \neq 0$ then since $e^{2q_1 t}$, 0, and $e^{-2q_1 t}$ are independent, the conclusion follows. Otherwise, $q_1 = 0$. Since $M_{1,1}$ is real and $a_1 \in i\mathbb{R}$, we have

$$g(t) = 2|a_1|^2 N_{1,1}.$$

The conclusion in the case $\ell = 1$ follows since $N_{1,1} \neq 0$.

Assume that the conclusion holds for $\ell \geq 1$, we prove that the conclusion holds for $\ell + 1$. Without loss of generality, one might assume that

$$0 \leq q_1 < q_2 < \cdots < q_\ell < q_{\ell+1}. \tag{4.5}$$

We will prove (4.4) for $\ell + 1$ by contradiction. Assume that there exist a_j and $q_j \geq 0$ with $1 \leq j \leq \ell + 1$, M_{j_1,j_2}, $N_{j_1,j_2} \in \mathbb{C}$ with $1 \leq j_1, j_2 \leq \ell + 1$ such that (4.1), (4.2), and (4.5) hold, and, for all $t \in \mathbb{R}_+$,

$$\sum_{j_1=1}^{\ell+1}\sum_{j_2=1}^{\ell+1}\Big(a_{j_1}a_{j_2}M_{j_1,j_2}e^{-i(q_{j_1}+q_{j_2})t} + \bar{a}_{j_1}\bar{a}_{j_2}\bar{M}_{j_1,j_2}e^{i(q_{j_1}+q_{j_2})t} + 2a_{j_1}\bar{a}_{j_2}N_{j_1,j_2}e^{-i(q_{j_1}-q_{j_2})}\Big) = 0. \tag{4.6}$$

Since the function $e^{-2iq_{\ell+1}t}$ defined in $\mathbb{R}_+$ does not belong to the space

$$\operatorname{span}\Big(\Big\{e^{-it(q_{j_1}+q_{j_2})}; 1 \leq j_1 \leq \ell+1; 1 \leq j_2 \leq \ell\Big\},$$
$$\Big\{e^{it(q_{j_1}+q_{j_2})}; 1 \leq j_1 \leq \ell+1; 1 \leq j_2 \leq \ell+1\Big\},$$
$$\Big\{e^{-it(q_{j_1}-q_{j_2})}; 1 \leq j_1 \leq \ell+1; 1 \leq j_2 \leq \ell+1\Big\}\Big),$$

for $t \in \mathbb{R}_+$ by (4.5), we have

$$a_{\ell+1}^2 M_{\ell+1,\ell+1} = 0.$$

This yields, since $M_{\ell+1,\ell+1} \neq 0$,

$$a_{\ell+1} = 0.$$

It follows from (4.6) that

$$\sum_{j_1=1}^{\ell}\sum_{j_2=1}^{\ell}\Big(a_{j_1}a_{j_2}M_{j_1,j_2}e^{-i(q_{j_1}+q_{j_2})t} + \bar{a}_{j_1}\bar{a}_{j_2}\bar{M}_{j_1,j_2}e^{i(q_{j_1}+q_{j_2})t} + 2a_{j_1}\bar{a}_{j_2}N_{j_1,j_2}e^{-i(q_{j_1}-q_{j_2})}\Big) = 0. \tag{4.7}$$

We now can use the assumption on the recurrence to obtain a contradiction. The proof of (4.4) is complete. □

Using Lemma 4.1 and the theory of quasi-periodic functions, see e.g. [1], we can derive the following useful result for the proof of Theorem 1.1.

Proposition 4.1. *Let $\ell \in \mathbb{N}_*$, $a_j \in \mathbb{C}$, $q_j \geq 0$ for $1 \leq j \leq \ell$, and $M_{j_1,j_2}, N_{j_1,j_2} \in \mathbb{C}$ with $1 \leq j_1, j_2 \leq \ell$. Assume that (4.1) holds and denote g by (4.3). For all $0 < \gamma_1 < \gamma_2$ there exist $\gamma_0 > 0$ and $\tau_0 > 0$ depending only on γ_1, γ_2, ℓ, q_j, M_{j_1,j_2}, and N_{j_1,j_2} such that if*

$$\gamma_1 \leq \sum_{j=1}^{\ell}|a_j|^2 \leq \gamma_2, \tag{4.8}$$

then

$$\|g\|_{L^2(\tau,2\tau)} \geq \gamma_0 \text{ for all } \tau \geq \tau_0. \tag{4.9}$$

Proof. Instead of (4.9), it suffices to prove

$$\|g\|_{L^\infty(\tau,2\tau)} \geq \gamma_0 \text{ for } \tau \geq \tau_0 \tag{4.10}$$

by contradiction since $|g'(t)| \leq C$ in $\mathbb{R}$. Assume that for all $n \in \mathbb{N}_*$ there exist $(a_{j,n})_{j=1}^{\ell} \subset \mathbb{C}$ and $(t_n) \subset \mathbb{R}$ such that $\gamma_1 \leq \sum_{j=1}^{\ell} |a_{j,n}|^2 \leq \gamma_2$, $t_n \geq n$, and

$$\|g_n\|_{L^\infty(t_n,2t_n)} \leq 1/n, \tag{4.11}$$

where g_n is defined in (4.3) where a_{j_1} and a_{j_2} are replaced by $a_{j_1,n}$ and $a_{j_2,n}$. Without loss of generality, one might assume that

$$\lim_{n\to+\infty} a_{j,n} = a_j \in \mathbb{C}$$

and $\gamma_1 \leq \sum_{j=1}^{N} |a_j|^2 \leq \gamma_2$. Consider g defined by (4.3) with these a_j. We have

$$\lim_{n\to+\infty} \|g_n - g\|_{L^\infty(\mathbb{R})} = 0. \tag{4.12}$$

Since g is an almost-periodic function with respect to t (see e.g. [1, Corollary on page 38]), it follows from the definition of almost-periodic functions, see e.g. [1, Section 44 on pages 32 and 33], that for every $\varepsilon > 0$, there exists $\mathcal{L}_\varepsilon > 0$ such that every interval $(\alpha, \alpha + \mathcal{L}_\varepsilon)$ containing a number $\tau(\varepsilon, \alpha)$ for which it holds

$$|g(t + \tau(\varepsilon, \alpha)) - g(t)| \leq \varepsilon \text{ for all } t \in \mathbb{R}. \tag{4.13}$$

The proof is now divided into two cases.

Case 1: $\liminf_{\varepsilon\to 0} \mathcal{L}_\varepsilon < +\infty$. Denote $\mathcal{L}_0 = \liminf_{\varepsilon\to 0} \mathcal{L}_\varepsilon$. We claim that g is T-periodic for some period $T \leq \mathcal{L}_0 + 1$. Indeed, by (4.13) applied with $\alpha = 1/2$, there exists a sequence $(\tau_n) \subset (1/2, \mathcal{L}_0 + 1)$ such that, for large n,

$$|g(t + \tau_n) - g(t)| \leq 1/n \text{ for all } t \in \mathbb{R}.$$

By choosing $T = \liminf_{n\to+\infty} \tau_n$, we have

$$g(t + T) = g(t) \text{ for all } t \in \mathbb{R}.$$

The claim is proved.

Since g is T-periodic, we have

$$\|g\|_{L^\infty(t_n,t_n+T+1)} = \|g\|_{L^\infty(0,T+1)} \text{ for } n \in \mathbb{N}_*,$$

and since g is analytic and $g \neq 0$ by Lemma 4.1, we obtain

$$\|g\|_{L^\infty(0,T+1)} > 0.$$

This contradicts (4.11) and (4.12). The proof of Case 1 is complete.

Case 2: $\lim_{\varepsilon\to 0} L_\varepsilon = +\infty$. Set

$$\rho = \|g\|_{L^\infty(0,1)}. \tag{4.14}$$

It follows from Lemma 4.1 that g is not identically equal to 0. Since g is analytic, we derive that

$$\rho > 0. \tag{4.15}$$

Let $n_0 \ge 2$ be such that

$$\|g_n - g\|_{L^\infty(\mathbb{R})} < \rho/4, \quad \|g_n\|_{L^\infty(t_n,2t_n)} < \rho/4 \text{ for } n \ge n_0.$$

Such an n_0 exists by (4.11), (4.12), and (4.15). We have, for $n \ge n_0$,

$$\|g\|_{L^\infty(t_n,2t_n)} \le \|g_n - g\|_{L^\infty(t_n,2t_n)} + \|g_n\|_{L^\infty(t_n,2t_n)} \le \rho/4 + \rho/4 = \rho/2. \tag{4.16}$$

Fix $0 < \varepsilon < \rho/4$ and fix $n \ge n_0$ such that $1 \le \mathcal{L}_\varepsilon \le t_n/2$. Such a number n exists since $t_n \ge n$. It follows from the definition of $\tau(\varepsilon, t_n)$ that

$$\tau(\varepsilon, t_n) \in (t_n, t_n + \mathcal{L}_\varepsilon) \subset (t_n, 3t_n/2), \tag{4.17}$$

and

$$\big|g\big(t + \tau(\varepsilon, t_n)\big) - g(t)\big| \le \varepsilon \text{ for all } t \in \mathbb{R}. \tag{4.18}$$

This yields

$$\|g\|_{L^\infty(t_n,2t_n)} \overset{(4.17)}{\ge} \|g\|_{L^\infty(\tau(\varepsilon,t_n),\tau(\varepsilon,t_n)+1)} \overset{(4.18)}{\ge} \|g\|_{L^\infty(0,1)} - \varepsilon \ge \rho - \rho/4 = 3\rho/4. \tag{4.19}$$

Combining (4.16) and (4.19) yields a contradiction since $\rho > 0$ by (4.15). The proof of Case 2 is complete. □

5. An upper bound for the decay rate - Proof of Theorem 1.1

This section containing two subsections is devoted to the proof of Theorem 1.1. The main ingredient is given in the first section and the proof is presented in the second one.

5.1. A key lemma

In this section, we prove

Lemma 5.1. *Let $L \in \mathcal{N}$. Assume that* $\dim \mathcal{M} = 1$ *or* (1.14) *holds. There exist $\varepsilon_0 > 0$, $C > 0$, and $T_0 > 0$ depending only on L such that for all (real) $u_0 \in L^2(0, L)$ with $\|u_0\|_{L^2(0,L)} \le \varepsilon_0$, the unique solution $u \in C\big([0, +\infty); L^2(0, L)\big) \cap L^2_{\mathrm{loc}}\big([0, +\infty); H^1(0, L)\big)$ of system* (1.1) *satisfies*

$$\|u(T, \cdot)\|_{L^2(0,L)} \le \|u_0\|_{L^2(0,L)}\Big(1 - C\|u_0\|^2_{L^2(0,L)}\Big) \textit{ for } T \ge T_0. \tag{5.1}$$

Proof. We first collect several known facts. Let $T_1 > 0$ be such that

$$\|v_x(\cdot, 0)\|_{L^2(0,t)} \geq \frac{1}{2}\|v(0, \cdot)\|_{L^2(0,L)} \text{ for } t \geq T_1, \tag{5.2}$$

for all solutions $v \in C\big([0, +\infty); L^2(0, L)\big) \cap L^2_{\text{loc}}\big([0, +\infty); H^1(0, L)\big)$ of the system

$$\begin{cases} v_t(t, x) + v_x(t, x) + v_{xxx}(t, x) = 0 & \text{in } (0, +\infty) \times (0, L), \\ v(t, x = 0) = v(t, x = L) = v_x(t, x = L) = 0 & \text{in } (0, +\infty), \end{cases} \tag{5.3}$$

with $v(0, \cdot) \in L^2(0, L)$ satisfying the condition

$$v(0, \cdot) \perp \mathcal{M}$$

(the orthogonality is considered with respect to $L^2(0, L)$-scalar product). The existence of such a constant T_1 follows from [18].

There exist two positive constants ε_0 and c_1 such that if $\|u_0\|_{L^2(0,L)} \leq \varepsilon_0$, then

$$\|u\|_{C([0,T_1];L^2(0,L))} + \|u\|_{L^2((0,T_1);H^1(0,L))} \leq c_1\|u_0\|_{L^2(0,L)} \tag{5.4}$$

(see e.g., [9, Proposition 14]).

There is a positive constant c_2 such that if $\widetilde{u}_0 \in L^2(0, L)$, $\widetilde{f} \in L^1\big((0, T_1); L^2(0, L)\big)$, and $\widetilde{y} \in C\big([0, T_1); L^2(0, L)\big) \cap L^2\big([0, T_1); H^1(0, L)\big)$ is the unique solution of the system

$$\begin{cases} \widetilde{u}_t(t, x) + \widetilde{u}_x(t, x) + \widetilde{u}_{xxx}(t, x) = \widetilde{f} & \text{in } (0, T_1) \times (0, L), \\ \widetilde{u}(t, x = 0) = \widetilde{u}(t, x = L) = \widetilde{u}_x(t, x = L) = 0 & \text{in } (0, T_1), \\ \widetilde{u}(t = 0, \cdot) = \widetilde{u}_0 & \text{in } (0, L), \end{cases} \tag{5.5}$$

then

$$\|\widetilde{u}_x(\cdot, 0)\|_{L^2(0,T_1)} + \|\widetilde{u}\|_{C([0,T_1];L^2(0,L))} + \|\widetilde{u}\|_{L^2((0,T_1);H^1(0,L))} \leq c_2\Big(\|\widetilde{u}_0\|_{L^2(0,L)} + \|\widetilde{f}\|_{L^1((0,T_1);L^2(0,L))}\Big). \tag{5.6}$$

There exists a positive constant c_3 depending only on L such that, for *all* $T > 0$,

$$\|\xi\xi_x\|_{L^1((0,T);L^2(0,L))} \leq c_3\|\xi\|^2_{L^2((0,T);H^1(0,L))} \tag{5.7}$$

(the constant c_3 is independent of T).

We now decompose u_0 into two parts:

$$u_0 = u_{0,1} + u_{0,2} \text{ in } (0, L), \tag{5.8}$$

where

$$u_{0,1} = \text{Projection}_{\mathcal{M}} u_0$$

with respect to $L^2(0, L)$-scalar product.

The proof is now divided into two cases, with $0 < \varepsilon = \|u_0\|_{L^2(0,L)} < \varepsilon_0$ (the conclusion is clear if $\varepsilon = 0$),

- Case 1: $\|u_{0,2}\|_{L^2(0,L)} \geq \beta\varepsilon^2 = \beta\|u_0\|^2_{L^2(0,L)}$,
- Case 2: $\|u_{0,2}\|_{L^2(0,L)} < \beta\varepsilon^2 = \beta\|u_0\|^2_{L^2(0,L)}$,

where

$$\beta = 4c_1^2 c_2 c_3. \tag{5.9}$$

Case 1: Assume that

$$\|u_{0,2}\|_{L^2(0,L)} \geq \beta\varepsilon^2 = \beta\|u_0\|^2_{L^2(0,L)}. \tag{5.10}$$

Let $\hat{u} \in C\big([0, T_1); L^2(0, L)\big) \cap L^2\big([0, T_1); H^1(0, L)\big)$ be the unique solution of

$$\begin{cases} \hat{u}_t(t, x) + \hat{u}_x(t, x) + \hat{u}_{xxx}(t, x) = 0 & \text{in } (0, T_1) \times (0, L), \\ \hat{u}(t, 0) = \hat{u}(t, L) = \hat{u}_x(t, L) = 0 & \text{in } (0, T_1), \\ \hat{u}(0, \cdot) = u_0 & \text{in } (0, L). \end{cases} \tag{5.11}$$

Then

$$\|(\hat{u} - u)_x(\cdot, 0)\|_{L^2(0,T_1)} \overset{(5.6)}{\leq} c_2\|uu_x\|_{L^1((0,T_1);L^2(0,L))} \overset{(5.4),(5.7)}{\leq} c_1^2 c_2 c_3 \varepsilon^2. \tag{5.12}$$

Let $\hat{u}_j \in C\big([0, T_1); L^2(0, L)\big) \cap L^2\big([0, T_1); H^1(0, L)\big)$ with $j = 1, 2$ be the unique solution of

$$\begin{cases} \hat{u}_{j,t}(t, x) + \hat{u}_{j,x}(t, x) + \hat{u}_{j,xxx}(t, x) = 0 & \text{for } t \in (0, T),\ x \in (0, L), \\ \hat{u}_j(t, 0) = \hat{u}_j(t, L) = \hat{u}_{j,x}(t, L) = 0 & \text{for } t \in (0, T), \\ \hat{u}_j(0, \cdot) = u_{0,j} & \text{in } (0, L). \end{cases} \tag{5.13}$$

Then

$$\hat{u} = \hat{u}_1 + \hat{u}_2 \text{ in } [0, T_1] \times [0, L].$$

We have

$$\hat{u}_{1,x}(\cdot, 0) = 0 \text{ in } [0, T_1], \tag{5.14}$$

and, by the choice of T_1 via (5.2),

$$\|\hat{u}_{2,x}(\cdot,0)\|_{L^2(0,T_1)} \ge \frac{1}{2}\|\hat{u}_2(0,\cdot)\|_{L^2(0,L)} = \frac{1}{2}\|u_{0,2}\|_{L^2(0,L)}. \tag{5.15}$$

It follows from (5.10) that

$$\|\hat{u}_x(\cdot,0)\|_{L^2(0,T_1)} \ge \frac{1}{2}\beta\varepsilon^2. \tag{5.16}$$

From (5.12) and (5.16), we obtain

$$\begin{aligned}\|u_x(\cdot,0)\|_{L^2(0,T_1)} &\ge \|\hat{u}_x(\cdot,0)\|_{L^2(0,T_1)} - \|(u-\hat{u})_x(\cdot,0)\|_{L^2(0,T_1)}\\ &\ge \left(\frac{1}{2}\beta - c_1^2c_2c_3\right)\varepsilon^2 \overset{(5.9)}{\ge} c_1^2c_2c_3\varepsilon^2.\end{aligned}$$

In other words,

$$\|u_x(\cdot,0)\|_{L^2(0,T_1)} \ge c_1^2c_2c_3\|u_0\|^2_{L^2(0,L)}. \tag{5.17}$$

Case 2: Assume that

$$\|u_{0,2}\|_{L^2(0,L)} < \beta\varepsilon^2 = \beta\|u_0\|^2_{L^2(0,L)}. \tag{5.18}$$

Since

$$\|u_{0,1}\|^2_{L^2(0,L)} + \|u_{0,2}\|^2_{L^2(0,L)} = \|u_0\|^2_{L^2(0,L)} = \varepsilon^2,$$

by considering ε sufficiently small, one can assume that

$$\|u_{0,1}\|_{L^2(0,L)} \ge \varepsilon/2.$$

Let $\alpha_m \in \mathbb{C}$ $(1 \le m \le n_L)$ be such that

$$\frac{1}{\varepsilon}u_{0,1} = \Re\left\{\sum_{m=1}^{n_L}\alpha_m\Psi_m(0,x)\right\}. \tag{5.19}$$

Since $u_{0,1} \in \mathcal{M}$, such a family of $(\alpha_m)_{m=1}^{n_L}$ exists. Since

$$1/2 \le \|\frac{1}{\varepsilon}u_{0,1}\|_{L^2(0,L)} \le 1$$

and $\Big(\Psi_m(0,\cdot)\Big)$ is orthogonal in $L^2(0,L)$ (with respect to the complex field), one can assume in addition that

$$0 < \gamma_1 \le \sum_{m=1}^{n_L}|\alpha_m|^2 \le \gamma_2,$$

for some constants γ_1, γ_2 depending only on L. Moreover, since $\Psi_m(0, x) \in i\mathbb{R}$ for $x \in [0, L]$ (by (2.31)) if $p_m = 0$ (see e.g. (2.31)), one can also assume that $a_m \in i\mathbb{R}$ if $p_m = 0$.

Let $\gamma_0 > 0$ and $\tau_0 > 0$ be the constants given in Proposition 4.1 with $\ell = n_L$, γ_1 and γ_2 determined above, $q_m = p_m$ given by (1.11),

$$M_{m_1,m_2} = \frac{1}{8}\varphi'_{m_1,m_2}(0) \quad \text{and} \quad N_{m_1,m_2} = \frac{1}{8}\phi'_{m_1,m_2}(0), \tag{5.20}$$

where φ_{m_1,m_2} and ϕ_{m_1,m_2} are defined in Proposition 2.1 and Proposition 2.2, respectively; in the case the definition of φ_{m_1,m_2} and ϕ_{m_1,m_2} in Proposition 2.1 and Proposition 2.2 are not unique, we fix a choice of φ_{m_1,m_2} and ϕ_{m_1,m_2}.

By Proposition 3.1, we have

$$M_{m,m} \neq 0,$$

and

$$(M_{m,m} \text{ is real and } N_{m,m} \neq 0) \text{ if } p_m = 0.$$

Then, by Proposition 4.1, for all $a_j \in \mathbb{C}$ $(1 \le j \le N)$ satisfying $\gamma_1 \le \sum_{j=1}^{N} |a_j|^2 \le \gamma_2$, it holds

$$\|g\|_{L^2(\tau,2\tau)} \ge \gamma_0 \text{ for all } \tau \ge \tau_0, \tag{5.21}$$

where

$$g(t) = \sum_{m_1=1}^{n_L} \sum_{m_2=1}^{n_L} \Big(a_{m_1} a_{m_2} M_{m_1,m_2} e^{-i(p_{m_1}+p_{m_2})t} + \bar{a}_{m_1}\bar{a}_{m_2}\bar{M}_{m_1,m_2} e^{i(p_{m_1}+p_{m_2})t} + 2a_{m_1}\bar{a}_{m_2} N_{m_1,m_2} e^{-i(p_{m_1}-p_{m_2})} \Big). \tag{5.22}$$

Define

$$A = \beta + 2\sum_{m_1=1}^{n_L}\sum_{m_2=1}^{n_L} \|\varphi_{m_1,m_2}\|_{L^2(0,L)} + 2\sum_{m_1=1}^{n_L}\sum_{m_2=1}^{n_L} \|\phi_{m_1,m_2}\|_{L^2(0,L)}, \tag{5.23}$$

and set

$$c_4 = 1/(2A). \tag{5.24}$$

Let $T_2 \ge 2\tau_0$ be such that

$$\|y_x(\cdot, 0)\|_{L^2(T_2/2,T_2)} \le c_4\gamma_0 \|y(0, \cdot)\|_{L^2(0,L)}, \tag{5.25}$$

for all solutions $y \in C\big([0, +\infty); L^2(0, L)\big) \cap L^2_{\text{loc}}\big([0, +\infty); H^1(0, L)\big)$ of

$$\begin{cases} y_t(t,x) + y_x(t,x) + y_{xxx}(t,x) = 0 & \text{for } t \in (0,+\infty),\ x \in (0,L), \\ y(t,0) = y(t,L) = y_x(t,L) = 0 & \text{for } t \in (0,+\infty), \end{cases} \tag{5.26}$$

with $y(0,\cdot) \in L^2(0,L)$. Note that T_2 is independent of $y(0,\cdot)$. The existence of T_2 can be proved by decomposing $y(0,\cdot) = y_1(0,\cdot) + y_2(0,\cdot)$ with $y_1(0,\cdot) \in \mathcal{M}$, and noting that (5.25) holds for the solution with initial data being $y_2(0,\cdot)$ since the solution is exponential decay, and the contribution for $y_x(\cdot,0)$ from the solution with initial data is $y_1(0,\cdot)$ is 0.

Let $\widetilde{u}_1,\ \widetilde{u}_2 \in C\big([0,+\infty); L^2(0,L)\big) \cap L^2_{\text{loc}}\big([0,+\infty); H^1(0,L)\big)$ be the unique solution of

$$\begin{cases} \widetilde{u}_{1,t}(t,x) + \widetilde{u}_{1,x}(t,x) + \widetilde{u}_{1,xxx}(t,x) = 0 & \text{for } t \in (0,+\infty),\ x \in (0,L), \\ \widetilde{u}_1(t,0) = \widetilde{u}_1(t,L) = \widetilde{u}_{1,x}(t,L) = 0 & \text{for } t \in (0,+\infty), \\ \widetilde{u}_1(0,\cdot) = \dfrac{1}{\varepsilon} u_{0,1} & \text{in } [0,L], \end{cases} \tag{5.27}$$

and

$$\begin{cases} \widetilde{u}_{2,t}(t,x) + \widetilde{u}_{2,x}(t,x) + \widetilde{u}_{2,xxx}(t,x) + \widetilde{u}_1 \widetilde{u}_{1,x} = 0 & \text{for } t \in (0,+\infty),\ x \in (0,L), \\ \widetilde{u}_2(t,0) = \widetilde{u}_2(t,L) = \widetilde{u}_{2,x}(t,L) = 0 & \text{for } t \in (0,+\infty), \\ \widetilde{u}_2(0,\cdot) = \dfrac{1}{\varepsilon^2} u_{0,2} & \text{in } [0,L]. \end{cases} \tag{5.28}$$

Set

$$V(t,x) = \sum_{m=1}^{n_L} \alpha_m \Psi_m(t,x) \quad \text{and} \quad U(t,x) = \Re V(t,x).$$

We have

$$\begin{cases} U(t,x) + U_x(t,x) + U_{xxx}(t,x) = 0 & \text{for } t \in (0,+\infty),\ x \in (0,L), \\ U(t,x=0) = U(t,x=L) = U_x(t,x=L) = 0 & \text{for } t \in (0,+\infty), \\ U(t=0,\cdot) = \dfrac{1}{\varepsilon} u_{0,1} & \text{in } [0,L]. \end{cases} \tag{5.29}$$

This implies

$$\widetilde{u}_1 = U \text{ in } (0,+\infty) \times (0,L).$$

Define

$$V_1(t,x) = \sum_{m_1=1}^{n_L} \sum_{m_2=1}^{n_L} \alpha_{m_1} \alpha_{m_2} \varphi_{m_1,m_2}(x) e^{-i(p_{m_1}+p_{m_2})t}, \tag{5.30}$$

and

$$V_2(t,x)=\sum_{m_1=1}^{n_L}\sum_{m_2=1}^{n_L}\alpha_{m_1}\bar{\alpha}_{m_2}\phi_{m_1,m_2}(x)e^{-i(p_{m_1}-p_{m_2})t}. \tag{5.31}$$

Then, by the construction of φ_{m_1,m_2},

$$\begin{cases} V_{1,t}(t,x)+V_{1,x}(t,x)+V_{1,xxx}(t,x)+\big(V(t,x)V(t,x)\big)_x=0 & \text{for } t\in(0,+\infty),\ x\in(0,L),\\ V_1(t,0)=V_1(t,L)=V_{1,x}(t,L)=0 & \text{for } t\in(0,+\infty), \end{cases} \tag{5.32}$$

and, by the construction of ϕ_{m_1,m_2},

$$\begin{cases} V_{2,t}(t,x)+V_{2,x}(t,x)+V_{2,xxx}(t,x)+\big(|V(t,x)|^2\big)_x=0 & \text{for } t\in(0,+\infty),\ x\in(0,L),\\ V_2(t,0)=V_2(t,L)=V_{2,x}(t,L)=0 & \text{for } t\in(0,+\infty). \end{cases} \tag{5.33}$$

Set

$$W=\frac{1}{8}\Big(V_1+\bar{V}_1+2V_2\Big) \text{ in } (0,+\infty)\times(0,L). \tag{5.34}$$

It follows from (5.22) that $W_x(t,0)=g(t)$ in $\mathbb{R}_+$ and hence, by (5.21),

$$\|W_x(t,0)\|_{L^2(\tau,2\tau)}\geq\gamma_0 \text{ for all } \tau\geq\tau_0. \tag{5.35}$$

Since

$$\big(V(t,x)V(t,x)\big)_x+\overline{\big(V(t,x)V(t,x)\big)_x}+2\big(|V(t,x)|^2\big)_x=8U(t,x)U_x(t,x),$$

we derive from (5.34) that

$$\begin{cases} W_t(t,x)+W_x(t,x)+W_{xxx}(t,x)+U(t,x)U_x(t,x)=0 & \text{for } t\in(0,+\infty),\ x\in(0,L),\\ W(t,0)=W(t,L)=W_x(t,L)=0 & \text{for } t\in(0,+\infty). \end{cases} \tag{5.36}$$

Let $\widetilde{W}\in C\big([0,+\infty);L^2(0,L)\big)\cap L^2_{\mathrm{loc}}\big([0,+\infty);H^1(0,L)\big)$ be the unique solution of

$$\begin{cases} \widetilde{W}_t(t,x)+\widetilde{W}_x(t,x)+\widetilde{W}_{xxx}(t,x)=0 & \text{for } t\in(0,+\infty),\ x\in(0,L),\\ \widetilde{W}(t,0)=\widetilde{W}(t,L)=\widetilde{W}_x(t,L)=0 & \text{for } t\in(0,+\infty),\\ \widetilde{W}(0,\cdot)=\widetilde{u}_2(0,\cdot)-W(0,\cdot). \end{cases} \tag{5.37}$$

Then

$$\widetilde{u}_2=\widetilde{W}+W \text{ in } (0,+\infty)\times(0,L). \tag{5.38}$$

We have

$$\|\widetilde{u}_{2,x}(\cdot,0)\|_{L^2(T_2/2,T_2)}\overset{(5.38)}{\geq}\|W_x(\cdot,0)\|_{L^2(T_2/2,T_2)}-\|\widetilde{W}_x(\cdot,0)\|_{L^2(T_2/2,T_2)}, \tag{5.39}$$

$$\|\widetilde{W}_x(\cdot,0)\|_{L^2(T_2/2,T_2)} \overset{(5.25)}{\leq} c_4\gamma_0\|\widetilde{W}(0,\cdot)\|_{L^2(0,L)}, \tag{5.40}$$

and, since $T_2 \geq \tau_0$,

$$\|W_x(\cdot,0)\|_{L^2(T_2/2,T_2)} \overset{(5.35)}{\geq} \gamma_0. \tag{5.41}$$

Since, by (5.30), (5.31), and (5.34)

$$8W(0,x) = \sum_{m_1=1}^{n_L}\sum_{m_2=1}^{n_L} \alpha_{m_1}\alpha_{m_2}\varphi_{m_1,m_2}(x)$$

$$+ \sum_{m_1=1}^{n_L}\sum_{m_2=1}^{n_L} \bar{\alpha}_{m_1}\bar{\alpha}_{m_2}\bar{\varphi}_{m_1,m_2}(x) + 2\sum_{m_1=1}^{n_L}\sum_{m_2=1}^{n_L} \alpha_{m_1}\bar{\alpha}_{m_2}\phi_{m_1,m_2}(x),$$

it follows that

$$\|W(0,\cdot)\|_{L^2(0,L)} \leq 2\sum_{m_1=1}^{n_L}\sum_{m_2=1}^{n_L} \|\varphi_{m_1,m_2}\|_{L^2(0,L)} + 2\sum_{m_1=1}^{n_L}\sum_{m_2=1}^{n_L} \|\phi_{m_1,m_2}\|_{L^2(0,L)}. \tag{5.42}$$

By the definition of A in (5.23), we obtain from (5.18) and (5.42) that

$$A \geq \|\widetilde{u}_2(0,\cdot)\|_{L^2(0,L)} + \|W(0,\cdot)\|_{L^2(0,L)} \overset{(5.38)}{\geq} \|\widetilde{W}(0,\cdot)\|_{L^2(0,L)}. \tag{5.43}$$

Combining (5.39), (5.40), (5.41), and (5.43) yields

$$\|\widetilde{u}_{2,x}(\cdot,0)\|_{L^2(T_2/2,T_2)} \geq \gamma_0 - c_4\gamma_0 A.$$

Since $c_4 = 1/(2A)$ by (5.24), we obtain

$$\|\widetilde{u}_{2,x}(\cdot,0)\|_{L^2(T_2/2,T_2)} \geq \gamma_0/2. \tag{5.44}$$

Set

$$u_d = \varepsilon\widetilde{u}_1 + \varepsilon^2\widetilde{u}_2 - u \text{ in } (0,+\infty)\times(0,L),$$

and

$$f_d = uu_x - \varepsilon^2\widetilde{u}_1\widetilde{u}_{1,x} \text{ in } (0,+\infty)\times(0,L). \tag{5.45}$$

We have, by (5.27) and (5.28),

$$\begin{cases} u_{d,t}(t,x) + u_{d,x}(t,x) + u_{d,xxx}(t,x) = f_d(t,x) & \text{for } t\in(0,+\infty),\ x\in(0,L),\\ u_d(t,x=0) = u_d(t,x=L) = u_d(t,x=L) = 0 & \text{for } t\in(0,+\infty),\\ u_d(t=0,\cdot) = 0 & \text{in } (0,L). \end{cases} \tag{5.46}$$

It is clear that

$$\|f_d\|_{L^1((0,T_2);L^2(0,L))} \le C\varepsilon^2,$$

where C is a positive constant depending only on T_2 and L. It follows that

$$\|u_d\|_{C([0,T_2];L^2(0,L))} + \|u_d\|_{L^2((0,T_2);H^1(0,L))} \le C\varepsilon^2.$$

This in turn implies that

$$\|f_d\|_{L^1((0,T_2);L^2(0,L))} \le C\varepsilon^3,$$

and therefore,

$$\|u_{d,x}(\cdot,0)\|_{L^2(0,T_2)} \le C\varepsilon^3. \tag{5.47}$$

Combining (5.44) and (5.47), and noting that $\widetilde{u}_{1,x}(t,0) = 0$ yield

$$\|u_x(\cdot,0)\|_{L^2(T_2/2,T_2)} \ge C\varepsilon^2. \tag{5.48}$$

The analysis of Step 2 is complete.

The conclusion now follows from Case 1 where one obtains (5.17) and Case 2 where one obtains (5.48) by choosing $T_0 = \max\{T_1, T_2\}$ and using (1.5). The proof is complete. □

We are ready to give

5.2. *Proof of Theorem 1.1*

By Lemma 5.1, we have

$$\|u(T_2,\cdot)\|_{L^2(0,L)} \le \|u(0,\cdot)\|_{L^2(0,L)}\Big(1 - C\|u(0,\cdot)\|^2_{L^2(0,L)}\Big).$$

This yields, with $\|u(0,\cdot)\|_{L^2(0,L)} = \varepsilon > 0$ and p being the largest integer less than $1/(2C\varepsilon^2)$,

$$\|u(pT_2,\cdot)\|_{L^2(0,L)} \le \frac{1}{2}\|u(0,\cdot)\|_{L^2(0,L)}.$$

Here we also used (1.5). Using (1.5) again, it follows that, for $T \ge C/\|u(0,\cdot)\|^2_{L^2(0,L)}$,

$$\|u(T,\cdot)\|_{L^2(0,L)} \le \frac{1}{2}\|u(0,\cdot)\|_{L^2(0,L)}.$$

This implies, by recurrence, that

$$\|u(T,\cdot)\|_{L^2(0,L)} \le 2^{-n}\|u_0\|_{L^2(0,L)} \text{ for } T \ge C\sum_{p=0}^{n-1} 2^{2p}/\|u(0,\cdot)\|^2_{L^2(0,L)}$$

since $\|u(t,\cdot)\|_{L^2(0,L)}$ is a non-increasing function with respect to t. In particular, we obtain, since $\|u(t,\cdot)\|_{L^2(0,L)}$ is a non-increasing function with respect to t,

$$\|u(t,\cdot)\|_{L^2(0,L)} \le C/t^{1/2}. \tag{5.49}$$

The proof is complete. □

6. A lower bound for the decay rate - Proof of Proposition 1.1

Fix $1 \le m \le n_L$ and $\alpha_m \in \mathbb{C}$ with $|\alpha_m| = 1$ such that

$$\Re(\alpha_m \varphi_{m,m}(x)) \text{ is not identically equal to 0 in } [0, L].$$

Let $\widetilde{u}_1 \in C\big([0,+\infty); L^2(0,L)\big) \cap L^2_{\rm loc}\big([0,+\infty); H^1(0,L)\big)$ be the unique solution of

$$\begin{cases} \widetilde{u}_{1,t}(t,x) + \widetilde{u}_{1,x}(t,x) + \widetilde{u}_{1,xxx}(t,x) = 0 & \text{for } t \in (0,+\infty),\ x \in (0,L), \\ \widetilde{u}_1(t,x=0) = \widetilde{u}_1(t,x=L) = \widetilde{u}_{1,x}(t,x=L) = 0 & \text{for } t \in (0,+\infty), \\ \widetilde{u}_1(0,\cdot) = \Re(\alpha_m \varphi_{m,m}). \end{cases} \tag{6.1}$$

Set

$$V_1(t,x) = \alpha_m^2 \varphi_{m,m}(x) e^{-2ip_m t}, \tag{6.2}$$

$$V_2(t,x) = |\alpha_m|^2 \phi_{m,m}(x), \tag{6.3}$$

and denote

$$\widetilde{u}_2 = \frac{1}{8}\Big(V_1 + \bar{V}_1 + 2V_2\Big) \text{ in } (0,+\infty) \times (0,L). \tag{6.4}$$

Since $\phi_{m,m}$ is real by 3) of Proposition 2.2, it follows that V_2 is real and hence so is $\widetilde{u}_2$.

As in the proof of Lemma 5.1, we have

$$\widetilde{u}_1(t,x) = \Re\Big(\alpha_m \varphi_{m,m}(x) e^{-ip_m t}\Big),$$

and

$$\begin{cases} \widetilde{u}_{2,t}(t,x) + \widetilde{u}_{2,x}(t,x) + \widetilde{u}_{2,xxx}(t,x) + \widetilde{u}_1(t,x)\widetilde{u}_{1,x}(t,x) = 0 & \text{for } t \in (0,+\infty),\ x \in (0,L), \\ \widetilde{u}_2(t,x=0) = \widetilde{u}_2(t,x=L) = \widetilde{u}_{2,x}(t,x=L) = 0 & \text{for } t \in (0,+\infty). \end{cases}$$

Let $u \in C\big([0,+\infty); L^2(0,L)\big) \cap L^2_{\rm loc}\big([0,+\infty); H^1(0,L)\big)$ be a (real) solution of (1.1) with

$$\|u(0,\cdot)\|_{L^2(0,L)} \le \Gamma\varepsilon,$$

where

$$\Gamma := \sup_t \|\widetilde{u}_1(t,\cdot)\|_{L^2(0,L)} + 1.$$

Set

$$\widetilde{u}_2(t,x) = W(t,x) \text{ in } (0,+\infty)\times(0,L),$$
$$u_d = \varepsilon\widetilde{u}_1 + \varepsilon^2\widetilde{u}_2 - u \text{ in } (0,+\infty)\times(0,L),$$
$$f_d = uu_x - \varepsilon^2\widetilde{u}_1\widetilde{u}_{1,x} \text{ in } (0,+\infty)\times(0,L).$$

We have

$$\begin{cases} u_{d,t}(t,x) + u_{d,x}(t,x) + u_{d,xxx}(t,x) = f_d(t,x) & \text{for } t\in(0,+\infty),\ x\in(0,L), \\ u_d(t,x=0) = u_d(t,x=L) = u_d(t,x=L) = 0 & \text{for } t\in(0,+\infty), \\ u_d(t=0,\cdot) = 0 & \text{in } (0,L). \end{cases} \tag{6.5}$$

Denote

$$g_d = \varepsilon^3(\widetilde{u}_1\widetilde{u}_{2,x} + \widetilde{u}_2\widetilde{u}_{1,x}) + \varepsilon^4\widetilde{u}_2\widetilde{u}_{2,x}. \tag{6.6}$$

We write f_d under the form

$$\begin{aligned} f_d =&(u - \varepsilon\widetilde{u}_1 - \varepsilon^2\widetilde{u}_2)u_x + (\varepsilon\widetilde{u}_1 + \varepsilon^2\widetilde{u}_2)(u - \varepsilon\widetilde{u}_1 - \varepsilon^2\widetilde{u}_2)_x + g_d \\ =& - u_d u_x - (\varepsilon\widetilde{u}_1 + \varepsilon^2\widetilde{u}_2)u_{d,x} + g_d. \end{aligned}$$

Multiplying the equation of u_d with u_d (which is real), integrating by parts in $(1,t)\times(0,L)$, and using the form of f_d just above give

$$\begin{aligned} \int_0^L |u_d(t,x)|^2\,dx \le \int_0^L |u_d(1,x)|^2\,dx + 2\int_1^t\int_0^L |u_d|^2|u_x|\,dx\,ds \\ + \int_1^t\int_0^L (\varepsilon|\widetilde{u}_1| + \varepsilon^2|\widetilde{u}_2|)_x|u_d|^2\,dx\,ds + 2\int_1^t\int_0^L |g_d||u_d|. \end{aligned} \tag{6.7}$$

Since

$$\int_0^L |u(t,x)|^2\,dx \le \int_0^L |u(0,x)|^2 \le C\varepsilon \text{ for } t\ge 0,$$

and the effect of the regularity, one has

$$|u(t,x)| + |u_x(t,x)| \le C\varepsilon \text{ for } t\ge 1,\ x\in[0,L]. \tag{6.8}$$

Let a be a (small) positive constant defined later (the smallness of a depending only on L). Let $t_0 \in [1, a/\varepsilon]$ be such that

$$\int_0^L |u_d(t_0, x)|^2 \, dx = \max_{t \in [1, a/\varepsilon]} \int_0^L |u_d(t, x)|^2 \, dx.$$

Combining (6.7) with $t = t_0$ and (6.8) yields

$$\int_0^L |u_d(t_0, x)|^2 \, dx \leq \int_0^L |u_d(1, x)|^2 \, dx + Ca \int_0^L |u_d(t_0, x)|^2 \, dx + \int_1^{a/\varepsilon} \int_0^L \varepsilon^{-1} |g_d|^2 \, dx.$$

This implies, if a is sufficiently small,

$$\int_0^L |u_d(t_0, \cdot)|^2 \, dx \, dx \leq C \int_0^L |u_d(1, x)|^2 \, dx + C\varepsilon^4$$

by (6.6).

On the other hand, one has

$$\int_0^L |u_d(t, \cdot)|^2 \, dx \, dx \leq C \int_0^L |u_d(0, x)|^2 \, dx + C\varepsilon^4 \text{ for } t \in [0, 1].$$

We have just proved that, for a sufficiently small,

$$\sup_{t \in [0, a/\varepsilon]} \|u_d(t, \cdot)\|_{L^2(0,L)} \leq C\Big(\|u_d(0, \cdot)\|_{L^2(0,L)} + \varepsilon^2 \Big).$$

Continuing this process, we obtain

$$\sup_{t \in [0, an/\varepsilon]} \|u_d(t, \cdot)\|_{L^2(0,L)} \leq C^n \|u_d(0, \cdot)\|_{L^2(0,L)} + \sum_{k=1}^{n} C^k \varepsilon^2. \tag{6.9}$$

We now consider u with

$$u(0, \cdot) = \varepsilon \widetilde{u}_1(0, \cdot) + \varepsilon_2 \widetilde{u}_2(0, \cdot).$$

Thus

$$u_d(0, \cdot) = 0. \tag{6.10}$$

Fix $\gamma > 0$ such that

$$\inf_{t \in \mathbb{R}} \int_0^L |\widetilde{u}_1(t,x)|^2 \, dx \geq 4\gamma. \tag{6.11}$$

With n being the largest integer number such that $C^{n+1} \leq \gamma \varepsilon^{-1}$ (we assume now and later on that $C \geq 2$), we derive from (6.9) and (6.10) that

$$\sup_{t \in [0, an/\varepsilon]} \|u_d(t, \cdot)\|_{L^2(0,L)} \leq \gamma \varepsilon.$$

Since

$$u_d = \varepsilon \widetilde{u}_1 + \varepsilon^2 \widetilde{u}_2 - u,$$

by the choice of γ, we have, for ε sufficiently small,

$$\|u(an/\varepsilon, \cdot)\|_{L^2(0,L)} \geq \gamma \varepsilon.$$

We deduce that, with $\tau = an/\varepsilon \sim \varepsilon^{-1} \ln \varepsilon^{-1}$ (hence $\varepsilon^{-1} \sim \tau / \ln \tau$),

$$\|u(\tau, \cdot)\|_{L^2(0,L)} \geq \gamma \varepsilon \geq C\gamma \ln \tau / \tau.$$

The proof is complete. □

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Appendix A. Proof of Lemma 2.2

Let $L \in \mathcal{N}$ and $z \in \mathcal{P}_L$. Then, from [18], $z = p_m$ for some $1 \leq m \leq n_L$ and

$$\lambda_j = \eta_{j,m}.$$

One can then check that $\det Q = 0$. On the other hand, if $z \neq \pm 2/(3\sqrt{3})$ and $\det Q(z) = 0$, it follows that there exists $(a_1, a_2, a_3) \in \mathbb{C}^3 \setminus \{0\}$ such that the function ξ defined by

$$\xi(x) = \sum_{j=1}^{3} a_j e^{\lambda_j(z)x}$$

satisfies

$$\xi(0) = \xi(L) = \xi'(L) = 0.$$

Since $\xi''' + \xi' = iz\xi$, by an integration by parts, one has

$$\xi'(0) = 0 \text{ if } z \text{ is real.}$$

Hence, from [18], if $z \in \mathbb{R} \setminus \{\pm 2/(3\sqrt{3})\}$ and $\det Q(z) = 0$, then $L \in \mathcal{N}$ and $z = p_m$ for some $1 \le m \le n_L$. We finally note that, $\{\pm 2/(3\sqrt{3})\} \cap \mathcal{P}_L = \emptyset$ for all $L \in \mathcal{N}$ since, for $k \ge l \ge 1$,

$$0 \le \frac{(2k+l)(k-l)(2l+k)}{3\sqrt{3}(k^2+kl+l^2)^{3/2}} = \frac{(2k+l)(k^2+kl-2l^2)}{3\sqrt{3}(k^2+kl+l^2)^{3/2}} < \frac{(2k+l)}{3\sqrt{3}(k^2+kl+l^2)^{1/2}} < \frac{2}{3\sqrt{3}}.$$

The proof is complete. □

Appendix B. Scilab program for checking $s(k,l) \neq 0$

```
clc
N=2000;
t=100;
a=0;
b=0;
for k=2:N
    for l=1:k-1
        h1 = 3 * (k*k + l*l + k*l);
        h0= 2 * (2*k + l)* (2*l + k)*(k-l);
        p =  poly([h0 -h1 0 1],'x','c');
        r = roots(p);
        c=exp(4 * %pi * %i * (k-l)/3);
        a1= c* exp(2 * %i * %pi *r(1) /3)
                        + exp(-2 * %i * %pi * r(1)/3);
        a2 = c* exp(2 *  %i * %pi *r(2) /3)
                        +  exp(-2 *  %i *%pi * r(2)/3);
        a3=c* exp(2 * %i * %pi *r(3) /3)
                        + exp(-2 * %i * %pi * r(3)/3);
        s  = r(1)* (r(3) - r(2))* a1
                        + r(2)* (r(1) - r(3))* a2
                        + r(3)* (r(2) - r(1))* a3;
        if abs(s) < t then t=abs(s); a=k; b=l;
        end
end
end
disp(a, b, t);
```

The outcome is $t = 0.0000164$, $a = 736$, and $b = 611$. This means

$$\min\Big\{|s(k,l)|;\, 1 \le l < k \le 2000\Big\} = t = 0.0000164$$

and

$$s(736, 611) = t.$$

Appendix C. A range of k and l for which $s(k, l) \neq 0$

In this section, we prove

Proposition C1. *There exists a constant $C \geq 1$ such that for $k, l \in \mathbb{N}_*$ satisfying the conditions that $k > l$, $k + l$ is odd, and*

$$C(k - l)^2 \leq k,$$

we have

$$s(k, l) \neq 0.$$

Proof. Instead of proving $s(k, l) \neq 0$, we will prove the following equivalent fact, by (3.22),

$$S(k, l) \neq 0,$$

where

$$S(k, l) = \sum_{j=1}^{3} \lambda_j (\lambda_{j+2} - \lambda_{j+1}) \Big(e^{2\eta_2 L} e^{\lambda_j L} + e^{-\lambda_j L} \Big), \tag{C1}$$

where λ_j are the solutions of

$$\lambda^3 + \lambda = 2ip$$

with $p = p(k, l)$, and $\eta_2 = \eta_2(k, l) = 2\pi i(k - l)/(3L)$.

We have

$$k^2 + kl + l^2 = 3k^2 + 3k(l - k) + (l - k)^2,$$

which implies

$$L = 2\pi k \left(1 + \frac{l - k}{k} + \frac{(l - k)^2}{3k^2} \right)^{1/2}.$$

Using the fact, for small x,

$$(1 + x)^{1/2} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + O(x^3),$$

we derive that

$$L = 2\pi k\left(1 + \frac{l-k}{2k} + \frac{(l-k)^2}{24k^2} + O\big((k-l)^3/k^3\big)\right) \tag{C2}$$

$$= \pi\left(k + l + \frac{(l-k)^2}{12k} + O\big((k-l)^3/k^2\big)\right). \tag{C3}$$

Here and in what follows for a number a, we denote $O(a)$ the quantity which is bounded by $C|a|$ for some positive constant C independent of k and l.

Set

$$M = M(k,l) = (2k+l)(2l+k)(k-l).$$

Then

$$M = k^2\left(3 + \frac{l-k}{k}\right)\left(3 + \frac{2(l-k)}{k}\right)(k-l) = 9k^2(k-l)\left(1 + \frac{l-k}{k} + O\big((k-l)^2/k^2\big)\right). \tag{C4}$$

We have

$$p = \left(\frac{2\pi}{3L}\right)^3 M \overset{(C2),(C4)}{=} \frac{1}{3k}(k-l)\Big(1 - \frac{l-k}{2k} + O\big((k-l)^2/k^2\big)\Big) \tag{C5}$$

$$= \frac{1}{3k}(k-l)\Big(1 + O\big((k-l)/k\big)\Big). \tag{C6}$$

From the definition of λ_j, we obtain

$$\lambda_1 = -i - ip + \frac{3ip^2}{2} + O(p^3), \quad \lambda_2 = 2ip + O(p^3), \quad \lambda_3 = i - ip - \frac{3ip^2}{2} + O(p^3). \tag{C7}$$

It follows from (C1) that

$$\begin{aligned} S(k,l) = {} & (-i - ip + 3ip^2/2)(i - 3ip - 3ip^2/2)\Big(e^{-\lambda_1 L} + e^{2\eta_2 L}e^{\lambda_1 L}\Big) \\ & + 2ip(-2i + 3ip^2)\Big(e^{-\lambda_2 L} + e^{2\eta_2 L}e^{\lambda_2 L}\Big) \\ & + (i - ip - 3ip^2/2)(i + 3ip - 3ip^2/2)\Big(e^{-\lambda_3 L} + e^{2\eta_2 L}e^{\lambda_3 L}\Big) + O(p^3). \end{aligned} \tag{C8}$$

We have

$$\begin{aligned} \lambda_1 L - \lambda_3 L & \overset{(C7)}{=} (-2i + 3ip^2)L + O(p^3 L) \\ & \overset{(C3),(C6)}{=} -2(k+l)\pi i - \frac{\pi i(l-k)^2}{6k} + 3ip^2 L + O(p^3 L) \\ & \overset{(C2),(C6)}{=} -2(k+l)\pi i - \frac{3}{4}ip^2 L + 3ip^2 L + O(p^3 L) \end{aligned}$$

$$= -2(k+l)\pi i + \frac{9}{4} i p^2 L + O(p^3 L). \tag{C9}$$

Here in the third identity, we used the fact that

$$p^2 L = \frac{2\pi}{9} \frac{(k-l)^2}{k} + O((k-l)^3)/k^2 = \frac{2\pi}{9} \frac{(k-l)^2}{k} + O(p^3 L).$$

It follows from (C9) that

$$e^{-\lambda_3 L} + e^{2\eta_2 L} e^{\lambda_3 L} = e^{-\lambda_1 L + \frac{9}{4} i p^2 L + O(p^3 L)} + e^{2\eta_2 L} e^{\lambda_1 L - \frac{9}{4} i p^2 L + O(p^3 L)}$$
$$= e^{-\lambda_1 L}(1 + \frac{9}{4} i p^2 L) + e^{2\eta_2 L} e^{\lambda_1 L}(1 - \frac{9}{4} i p^2 L) + O(p^4 L^2).$$

Since

$$(-i - ip + 3ip^2/2)(i - 3ip - 3ip^2/2) + (i - ip - 3ip^2/2)(i + 3ip - 3ip^2/2)$$
$$= -4p + 6p^3 + O(p^4),$$

and

$$(i - ip - 3ip^2/2)(i + 3ip - 3ip^2/2) = -1 - 2p + O(p^2),$$

we derive from (C8) that

$$S(k,l) = (-4p + 6p^3)\Big(e^{-\lambda_1 L} + e^{2\eta_2 L} e^{\lambda_1 L}\Big)$$
$$- \frac{9}{4} i p^2 L (1 + 2p)\Big(e^{-\lambda_1 L} - e^{2\eta_2 L} e^{\lambda_1 L}\Big) + (4p - 6p^3)\Big(e^{-\lambda_2 L} + e^{2\eta_2 L} e^{\lambda_2 L}\Big) + O(p^4 L^2). \tag{C10}$$

Here we also used the fact $p^2 = O(p^3 L)$. Note that

$$\lambda_1 L \overset{(C7)}{=} -iL - ipL + \frac{3ip^2 L}{2} + O(p^3 L) \overset{(C3),(C2),(C6)}{=} -\pi(k+l)i - \eta_2 L + O(p^2 L),$$

and

$$\lambda_2 L \overset{(C2),(C6)}{=} 2\eta_2 L + O\big((k-l)^2/k\big) = 2\eta_2 L - \frac{9ip^3 L}{4} + O(p^4 L^2).$$

From (C10), we obtain

$$
\begin{aligned}
S(k,l) =& 2(-4p+6p^3)e^{\eta_2 L}e^{\pi(k+l)i} + (4p-6p^3)\Big(e^{-2\eta_2 L} + e^{4\eta_2 L}\Big) \\
&+ O(p^4L^2) \quad (\text{since } e^{3\eta_2 L} = 1) \\
=& 2(-4p+6p^3)e^{\eta_2 L}\Big(e^{\pi(k+l)i} - 1\Big) + O(p^4L^2) \quad (\text{since } e^{3\eta_2 L} = 1).
\end{aligned}
$$

The proof is complete. □

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