# ON THE OPTIMAL CONTROLLABILITY TIME FOR LINEAR HYPERBOLIC SYSTEMS WITH TIME-DEPENDENT COEFFICIENTS 

JEAN-MICHEL CORON AND HOAI-MINH NGUYEN


#### Abstract

The optimal time for the controllability of linear hyperbolic systems in one dimensional space with one-side controls has been obtained recently for time-independent coefficients in our previous works. In this paper, we consider linear hyperbolic systems with time-varying zeroorder terms. We show the possibility that the optimal time for the null-controllability becomes significantly larger than the one of the time-invariant setting even when the zero-order term is indefinitely differentiable. When the analyticity with respect to time is imposed for the zero-order term, we also establish that the optimal time is the same as in the time-independent setting.


Key words: hyperbolic systems, controllability, optimal time, time-varying coefficients, analytic coefficients in time, unique continuation principle, well-posedness of hyperbolic systems.

Mathematics Subject Classification: 93C20, 35Q93, 35L50, 47A55.

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## 1. Introduction and statement of the main results

Hyperbolic systems in one dimensional space are frequently used in the modeling of many systems such as traffic flow [2], heat exchangers [51], fluids in open channels [28, 22, 29, 24], and phase transition [25]. Many other interesting examples can be found in [6] and the references therein. The optimal time for the controllability of hyperbolic systems in one dimensional space with one-side controls has been derived recently for time-independent coefficients [17, 20]. In this paper, we consider hyperbolic systems with time-varying zero-order terms. It is known that these systems are controllable in some positive time. In this paper, we show the possibility that the optimal time for the null-controllability becomes significantly larger than the one of the timeinvariant setting even when the zero-order term is indefinitely differentiable. When the analyticity with respect to time is imposed for the zero-order term, we also establish that the optimal time is the same as in the time-independent setting. The first result is quite surprising since the zero-order

[^0]term does not interfere with the characteristic flows of the system. The later result complement to the first one can be then viewed as an extension of a well-known controllability property of linear differential equations: if a linear control system is controllable in some positive time and is analytic, then it is controllable in any time greater than the optimal time, which is 0.

Let us first briefly discuss known results for the time-independent coefficients to underline the phenomena. Consider the system

$$
\begin{equation*}
\partial_{t} u(t, x)=\Sigma(x) \partial_{x} u(t, x)+C(x) u(t, x) \text { for }(t, x) \in \mathbb{R}_{+} \times(0,1) . \tag{1.1}
\end{equation*}
$$

Here $u=\left(u_{1}, \cdots, u_{n}\right)^{\top}: \mathbb{R}_{+} \times(0,1) \rightarrow \mathbb{R}^{n}(n \geq 2), \Sigma$ and $C$ are $(n \times n)$ real, matrix-valued functions defined in $[0,1]$. We assume that, for every $x \in[0,1]$, the matrix $\Sigma(x)$ is diagonalizable with $m \geq 1$ distinct positive eigenvalues and $k=n-m \geq 1$ distinct negative eigenvalues. Using Riemann coordinates, one might assume that $\Sigma(x)$ is of the form

$$
\begin{equation*}
\Sigma(x)=\operatorname{diag}\left(-\lambda_{1}(x), \cdots,-\lambda_{k}(x), \lambda_{k+1}(x), \cdots, \lambda_{n}(x)\right), \tag{1.2}
\end{equation*}
$$

where

$$
\begin{equation*}
-\lambda_{1}(x)<\cdots<-\lambda_{k}(x)<0<\lambda_{k+1}(x)<\cdots<\lambda_{k+m}(x) . \tag{1.3}
\end{equation*}
$$

In what follows, we assume that

$$
\begin{equation*}
\lambda_{i} \text { is of class } C^{2} \text { on }[0,1] \text { for } 1 \leq i \leq n(=k+m) \tag{1.4}
\end{equation*}
$$

and denote

$$
u_{-}=\left(u_{1}, \cdots, u_{k}\right)^{\top} \text { and } u_{+}=\left(u_{k+1}, \cdots, u_{k+m}\right)^{\top} .
$$

We are interested in the following type of boundary conditions and boundary controls. The boundary conditions at $x=0$ are given by

$$
\begin{equation*}
u_{-}(t, 0)=B u_{+}(t, 0) \text { for } t \geq 0 \tag{1.5}
\end{equation*}
$$

for some $(k \times m)$ real constant matrix $B$, and at $x=1$
$u_{+}(t, 1)$ is controlled for $t \geq 0$.
Let us recall that the control system (1.1), (1.5), and (1.6) is null-controllable (resp. exactly controllable) at time $T>0$ if, for every initial datum $u_{0}:(0,1) \rightarrow \mathbb{R}^{n}$ in $\left[L^{2}(0,1)\right]^{n}$ (resp. for every initial datum $u_{0}:(0,1) \rightarrow \mathbb{R}^{n}$ in $\left[L^{2}(0,1)\right]^{n}$ and for every (final) state $u_{T}:(0,1) \rightarrow \mathbb{R}^{n}$ in $\left.\left[L^{2}(0,1)\right]^{n}\right)$, there is a control $U:(0, T) \rightarrow \mathbb{R}^{m}$ in $\left[L^{2}(0, T)\right]^{m}$ such that the solution of (1.1), (1.5), and (1.6) (with $\left.u_{+}=U\right)$ satisfying $u(t=0, x)=u_{0}(x)$ vanishes (resp. reaches $u_{T}$ ) at the time $T$ : $u(t=T, \cdot)=0\left(\right.$ resp. $\left.u(t=T, \cdot)=u_{T}\right)$.

Throughout this paper, we consider broad solutions in $L^{2}$ with respect to $t$ and $x$ for an initial datum in $\left[L^{2}(0,1)\right]^{n}$ and a control in $\left[L^{2}(0, T)\right]^{m}$ (see, for example, [37, Section 3]). In particular, the solutions belong to $C\left([0, T] ;\left[L^{2}(0,1)\right]^{n}\right)$ and $C\left([0,1] ;\left[L^{2}(0, T)\right]^{n}\right)$. The well-posedness for broad solutions for system (1.1), (1.5), and (1.6) even when $\Sigma$ and $C$ depending also on $t$ is standard.

Set

$$
\begin{equation*}
\tau_{i}:=\int_{0}^{1} \frac{1}{\left|\lambda_{i}(\xi)\right|} d \xi \text { for } 1 \leq i \leq n \tag{1.7}
\end{equation*}
$$

The exact controllability, the null-controllability, and the boundary stabilization problem of hyperbolic system in one dimensional space have been widely investigated in the literature for almost half a century, see, e.g., [6] and the references therein. Concerning the exact controllability and the null-controllability related to (1.1), (1.5) and (1.6), the pioneer works date back to the ones of Rauch and Taylor 40] and Russell [42. In particular, it was shown, see [42, Theorem 3.2], that system (1.1), (1.5), and (1.6) is null-controllable for time $\tau_{k}+\tau_{k+1}$, and is exactly controllable at the same time if $k=m$ and $B$ is invertible. The extension of this result for quasilinear systems was initiated by Greenberg and Li [27] and Slemrod [43].

A recent efficient way in the study of the stabilisation and the controllability of system (1.1), (1.5), and (1.6) is via a backstepping approach. The backstepping approach for the control of partial differential equations was pioneered by Miroslav Krstic and his coauthors (see [35] for a concise introduction). The backstepping method is now frequently used for various control problems, modeling by partial differential equations in one dimensional space. For example, it has been used to stabilize the wave equations [34, 47, 44], the parabolic equations in [45, 46], nonlinear parabolic equations [50], and to obtain the null-controllability of the heat equation [16]. The standard backstepping approach relies on the Volterra transform of the second kind. It is worth noting that, in some situations, more general transformations have to be considered as for Korteweg-de Vries equations [8], Kuramoto-Sivashinsky equations [15], Schrödinger's equation [12], and hyperbolic equations with internal controls [53].

The use of backstepping approach for the hyperbolic system in one dimensional space was first proposed by Coron et al. 21] for $2 \times 2 \operatorname{system}(m=k=1)$. Later, this approach has been extended and now can be applied for general pairs $(m, k)$, see [23, 31, 4, 13, 17, 20, 32].

Set

$$
T_{o p t}:=\left\{\begin{array}{cl}
\max \left\{\tau_{1}+\tau_{m+1}, \ldots, \tau_{k}+\tau_{m+k}, \tau_{k+1}\right\} & \text { if } m \geq k  \tag{1.8}\\
\max \left\{\tau_{k+1-m}+\tau_{k+1}, \tau_{k+2-m}+\tau_{k+2}, \ldots, \tau_{k}+\tau_{k+m}\right\} & \text { if } m<k
\end{array}\right.
$$

Involving the backstepping technique, we established [17, 20] that the null-controllability holds at $T_{o p t}$ for generic $B$ and $C$, and the null-controllability holds for any $T>T_{o p t}$ under the condition $B \in \mathcal{B}$. Here

$$
\begin{equation*}
\mathcal{B}:=\left\{B \in \mathbb{R}^{k \times m} ; \text { such that }(1.10) \text { holds for } 1 \leq i \leq \min \{k, m-1\}\right\} \tag{1.9}
\end{equation*}
$$

where
(1.10) the $i \times i$ matrix formed from the last $i$ columns and the last $i$ rows of $B$ is invertible.

Roughly speaking, the condition $B \in \mathcal{B}$ allows us to implement $l$ controls corresponding to the fastest positive speeds to control $l$ components corresponding to the lowest negative speeds 11 . It is clear that $B \in \mathcal{B}$ for almost every $k \times m$ matrix $B$. It is worthy noting that the condition $T>T_{o p t}$ is necessary, see [17, Assertion 2) of Theorem 1.1]. The optimality of $T_{o p t}$ was established under the additional condition (1.10) being valid with $i=m$ when $k \geq m$, see [17, Proposition 1.6]. Our results improved the time to reach the null-controllability obtained previously. Similar conclusions hold for the exact controllability under the natural conditions $m \geq k$ and (1.10) for $1 \leq i \leq k$ (see [17, 20, 32]). When the system is homogeneous, i.e., $C \equiv 0$, we established that the null-controllability can be achieved via a time-independent feedback even for the quasilinear setting [18]. We also constructed Lyapunov functions which yield the null-controllability for such a system at the optimal time $T_{o p t}[19]$.

In this paper, we are interested in hyperbolic systems with time-dependent coefficients in one dimensional space. More precisely, instead of (1.1), (1.5), and (1.6), we deal with

$$
\begin{equation*}
\partial_{t} u(t, x)=\Sigma(x) \partial_{x} u(t, x)+C(t, x) u(t, x) \text { for }(t, x) \in \mathbb{R}_{+} \times(0,1) \tag{1.11}
\end{equation*}
$$

and (1.5), and (1.6).
The first result of the paper reveals that the optimal time for the null-controllability of system (1.11), (1.5), and (1.6) might be significantly larger than the one for the time-independent setting even when $\Sigma$ is constant and $C$ is indefinitely differentiable. More precisely, we have

[^1]Theorem 1.1. Let $k \geq 1, m \geq 2$, and $\Sigma$ be constant such that (1.3) holds. Assume that

$$
\begin{equation*}
B_{k, 1} \neq 0, \quad B_{k, \ell} \neq 0, \quad B_{k, j}=0 \text { for } 2 \leq j \leq m \text { with } j \neq \ell, \tag{1.12}
\end{equation*}
$$

for some $2 \leq \ell \leq m$. There exists $C \in C^{\infty}([0,+\infty) \times[0,1])$ such that for all $\varepsilon>0$, system (1.11), (1.5), and (1.6) is not null-controllable at time

$$
\begin{equation*}
T=\tau_{k}+\tau_{k+1}-\varepsilon . \tag{1.13}
\end{equation*}
$$

Remark 1.1. The definition of the null-controllability for system (1.11), (1.5), and (1.6) is similar to the one corresponding to (1.1), (1.5), and (1.6).
Remark 1.2. There are infinitely many matrices $B \in \mathcal{B}$ satisfying (1.12).
In a recent work, Coron et al. [14] establish the null-controllability of (1.11), (1.5), and (1.6) for time $\tau_{k}+\tau_{k+1}$ for all $k \times m$ matrices $B$. They also obtain stabilizing feedbacks and derive similar results when $\Sigma$ depends on $t$. Combining Theorem 1.1 and their results, one obtains the optimality for the time $\tau_{k}+\tau_{k+1}$ when $m \geq 2$ and $k \geq 1$, and for a large class of $B$.

The proof of Theorem 1.1 is based on constructing counter-examples for the associated observability inequality. The construction is inspired by the one given in the proof of [17, Assertion 2) of Theorem 1.1] but much more involved.

When the analyticity of $C$ with respect to time is imposed, the situation changes dramatically. To state our results in this direction, we first introduce some notations. For a non-empty interval $(a, b)$ of $\mathbb{R}$ and a Banach space $\mathcal{X}$, we denote

$$
\mathcal{H}((a, b) ; \mathcal{X})=\{\Phi:(a, b) \rightarrow \mathcal{X} ; \Phi \text { is analytic }\} .
$$

When the space $\mathcal{X}$ is clear, we simply call a $\Phi \in \mathcal{H}((a, b) ; \mathcal{X})$ that $\Phi$ is analytic in $(a, b)$. For $m \geq k$, set

$$
\begin{equation*}
\mathcal{B}_{e}:=\left\{B \in \mathbb{R}^{k \times m} ; \text { such that (1.10) holds for } 1 \leq i \leq k\right\} . \tag{1.14}
\end{equation*}
$$

Denote

$$
\begin{equation*}
T_{1}=\tau_{k}+\tau_{k+1} . \tag{1.15}
\end{equation*}
$$

Our main results for the analytic setting are the following two theorems. The first one on the null-controllability is:
Theorem 1.2. Let $k \geq m \geq 1$, and let $B \in \mathcal{B}$ be such that (1.10) holds for $i=m$. Assume that $C \in \mathcal{H}\left(I ;\left[L^{\infty}(0,1)\right]^{n \times n}\right)$ for some open interval I containing $\left[0, T_{1}\right]$. System (1.11), (1.5), and (1.6) starting from time 0 is null-controllable at any time $T>T_{\text {opt }}$.

The second one on the exact-controllability is:
Theorem 1.3. Let $m \geq k \geq 1$, and let $B \in \mathcal{B}_{e}$. Assume, for some open interval I containing $\left[T_{\text {opt }}-T_{1}, T_{\text {opt }}\right]$, that $C \in \mathcal{H}\left(I ;\left[L^{\infty}(0,1)\right]^{n \times n}\right)$ System (1.11), (1.5), and (1.6) starting from time 0 is exact-controllable at any time $T>T_{\text {opt }}$.

Except for the case where $m=1$ for which $T_{1}=T_{\text {opt }}$, Theorems 1.2 and 1.3 are new to our knowledge. Theorems 1.1 to 1.3 reveal the crucial role of the analytic assumption of the coefficients on the optimal controllability time. It is well-known that a linear control system modeled by differential equations is controllable in some time $T$ and is analytic, then it is controllable in any time greater than the optimal time, which is 0 , see, e.g., [10, Chapter 1] or [48, Chapter 3]. Theorems 1.2 and 1.3, which are complement to Theorem 1.1, can be thus viewed as an extension of this well-known result for linear hyperbolic systems in one dimensional space.

A related context to Theorem 1.3 is the one of the wave equation. For the wave equation with time varying, first and zero-order terms being analytic in time, it is known that the controllability
holds under a sharp geometric control condition, introduced in [5] (see also [40]). This can be obtained by combining the results in [5], on the propagation of singularities for the wave equation, and the unique continuation principle for the wave equations with coefficients analytic in time using Carleman's estimates due to Tataru-Hörmander-Robbiano-Zuily [49, 30, 41] (see also [36] for a discussion). Related results concerning the Schrödinger equation are due to Nalini Anantharaman, Matthieu Léautaud, and Fabricio Macià [3].

We now say a few words on the proof of Theorems 1.2 and 1.3. Theorem 1.3 is derived from Theorem 1.2 using our arguments in the proof of [20, Theorem 3]. The proof of Theorem 1.2 is inspired from the analysis in [20], in which we established similar result for the time-independent setting. The crucial part of the analysis is then to locate the essential, analytic nature of the system, the smoothness is not sufficient as shown previously in Theorem 1.1. This is done by exploring both the orignal system and its dual one. The proof also involves the theory of perturbations of analytic compact operators, see, e.g., 33. As a consequence of our analysis, we also obtain the unique continuation principle for hyperbolic systems for the optimal time in the analytic setting (see Proposition [3.3), which has its own interest. The strategy of the proof is described in more details at the beginning of Section 3,

The paper is organized as follows. Theorems 1.1 to 1.3 are given in Sections 2 to 4, respectively. In the appendix, we establish properties of hyperbolic systems used in Section 3. The situation is non-standard in the sense that the domains considered are not rectangle and the boundary conditions are involved. The analysis is delicate and has its own interest.

## 2. Analysis in the smooth setting - Proof of Theorem 1.1

The starting point of the proof of Theorem 1.1 is the equivalence between the null-controllability of system (1.11), (1.5), and (1.6) and its corresponding observability inequality. To this end, we first introduce some notations and recall this property.

Fix $T>0$ and define

$$
\begin{aligned}
\mathcal{F}_{T}: \quad\left[L^{2}(0, T)\right]^{m} & \rightarrow\left[L^{2}(0,1)\right]^{n} \\
\mathcal{F}_{T}(U) & \mapsto u(T, \cdot)
\end{aligned}
$$

where $u$ is the unique solution of system (1.11), (1.5), and (1.6) with $u_{+}(\cdot, 1)=U$ and with $u(0, \cdot)=0$. Denote

$$
\Sigma_{-}=\operatorname{diag}\left(-\lambda_{1}, \cdots,-\lambda_{k}\right) \quad \text { and } \quad \Sigma_{+}=\operatorname{diag}\left(\lambda_{k+1}, \cdots, \lambda_{k+m}\right) .
$$

As usual, we have
Lemma 2.1. Let $T>0$. We have, for $\varphi \in\left[L^{2}(0,1)\right]^{n}$,

$$
\mathcal{F}_{T}^{*}(\varphi)=\Sigma_{+}(1) v_{+}(\cdot, 1) \text { in }(0, T),
$$

where $v$ is the unique solution of the system

$$
\begin{equation*}
\partial_{t} v(t, x)=\Sigma(x) \partial_{x} v(t, x)+\left(\Sigma^{\prime}(x)-C^{\top}(t, x)\right) v(t, x) \text { for }(t, x) \in(0, T) \times(0,1) \tag{2.1}
\end{equation*}
$$

with, for $0<t<T$,

$$
\begin{gather*}
v_{-}(t, 1)=0  \tag{2.2}\\
\Sigma_{+}(0) v_{+}(t, 0)=-B^{\top} \Sigma_{-}(0) v_{-}(t, 0) \tag{2.3}
\end{gather*}
$$

and

$$
\begin{equation*}
v(t=T, \cdot)=\varphi \text { in }(0,1) \tag{2.4}
\end{equation*}
$$

The proof of Lemma 2.1 is standard and omitted, see, e.g., [20, the proof of Lemma 1] for a closely related context.

From Lemma 2.1, one derives the following characterization of the null-controllability of system (1.11), (1.5), and (1.6) in time $T$, whose proof is standard and omitted, see, e.g., 10, Section 2.3].

Lemma 2.2. Let $T>0$. System (1.11), (1.5), and (1.6) starting at time 0 is null-controllable in time $T$ if and only if there exists a positive constant $C_{T}$ such that

$$
\left\|v_{+}(\cdot, 1)\right\|_{L^{2}(0, T)} \geq C_{T}\|v(0, \cdot)\|_{L^{2}(0,1)},
$$

for all solutions $v$ of system (2.1), (2.2), and (2.3).
We are ready to give
Proof of Theorem 1.1. In what follows, we will assume that

$$
T \geq \max \left\{T_{o p t}, \tau_{k}+\tau_{k+\ell}\right\}
$$

where $T_{\text {opt }}$ is defined by (1.8); hence $\varepsilon$ is assumed to be sufficiently small (note that $\tau_{k}+\tau_{k+1}>$ $\max \left\{T_{o p t}, \tau_{k}+\tau_{k+\ell}\right\}$ since $\left.2 \leq \ell \leq m\right)$. We will consider the coefficient $C(t, x)$ satisfying the following structure:

$$
C_{i, j}(t, x)=\left\{\begin{array}{cl}
-\alpha(t, x) & \text { if }(i, j)=(k, k+\ell)  \tag{2.5}\\
-\beta(t, x) & \text { if }(i, j)=(k+1, k+\ell) \\
0 & \text { otherwise }
\end{array}\right.
$$

where $\alpha$ and $\beta$ are two smooth functions defined later.
Since $\Sigma$ is constant and $C$ satisfies (2.5), system (2.1) is equivalent to, for $(t, x) \in(0, T) \times(0,1)$,

$$
\begin{equation*}
\partial_{t} v_{j}(t, x)=\Sigma_{j, j} \partial_{x} v_{j}(t, x) \text { if } 1 \leq j \leq n \text { with } j \neq k+\ell \tag{2.6}
\end{equation*}
$$

and

$$
\begin{equation*}
\partial_{t} v_{k+\ell}(t, x)=\lambda_{k+\ell} \partial_{x} v_{k+\ell}(t, x)+\alpha(t, x) v_{k}(t, x)+\beta(t, x) v_{k+1}(t, x) \tag{2.7}
\end{equation*}
$$

( $\Sigma_{j, j}=-\lambda_{j}$ if $1 \leq j \leq k$ and $\Sigma_{j, j}=\lambda_{j}$ otherwise).
Under appropriate choices of $\alpha$ and $\beta$ determined later, we will construct a smooth solution $v$ of system (2.6) and (2.7) for which, for $t \in(0, T)$,

$$
\begin{gather*}
v_{-}(t, 1)=0  \tag{2.8}\\
\Sigma_{+}(0) v_{+}(t, 0)=-B^{\top} \Sigma_{-}(0) v_{-}(t, 0) \tag{2.9}
\end{gather*}
$$

and $v$ satisfies the following additional conditions:

$$
\begin{equation*}
v_{+}(\cdot, 1)=0 \quad \text { and } \quad v(0, \cdot) \not \equiv 0 \tag{2.10}
\end{equation*}
$$

By Lemma 2.2, the conclusion of Theorem 1.1 follows from this construction.
We now construct $\alpha$ and $\beta$. To this end, we first derive their constrains. From (2.7), we have, for $\tau_{k+\ell} \leq t+\tau_{k+\ell} \leq T$ and $0 \leq s \leq 1$,

$$
\begin{aligned}
\frac{d}{d s}\left(v_{k+\ell}\left(t+\tau_{k+\ell} s, 1-s\right)\right)=\tau_{k+\ell} \alpha\left(t+\tau_{k+\ell} s, 1\right. & -s) v_{k}\left(t+\tau_{k+\ell} s, 1-s\right) \\
& +\tau_{k+\ell} \beta\left(t+\tau_{k+\ell} s, 1-s\right) v_{k+1}\left(t+\tau_{k+\ell} s, 1-s\right)
\end{aligned}
$$

This implies, for $\tau_{k+\ell} \leq t+\tau_{k+\ell} \leq T$,

$$
\begin{align*}
v_{k+\ell}\left(t+\tau_{k+\ell}, 0\right)= & \int_{0}^{1} \tau_{k+\ell} \alpha\left(t+\tau_{k+\ell} s, 1-s\right) v_{k}\left(t+\tau_{k+\ell} s, 1-s\right) d s  \tag{2.11}\\
& \quad+\int_{0}^{1} \tau_{k+\ell} \beta\left(t+\tau_{k+\ell} s, 1-s\right) v_{k+1}\left(t+\tau_{k+\ell} s, 1-s\right) d s+v_{k+\ell}(t, 1)
\end{align*}
$$

It follows that, if $v_{k+\ell}(t, 1)=0$ for $\tau_{k+\ell} \leq t+\tau_{k+\ell} \leq T$, then

$$
\begin{align*}
& v_{k+\ell}\left(t+\tau_{k+\ell}, 0\right)=\int_{0}^{1} \tau_{k+\ell} \alpha\left(t+\tau_{k+\ell} s, 1-s\right) v_{k}\left(t+\tau_{k+\ell} s, 1-s\right) d s  \tag{2.12}\\
&+\int_{0}^{1} \tau_{k+\ell} \beta\left(t+\tau_{k+\ell} s, 1-s\right) v_{k+1}\left(t+\tau_{k+\ell} s, 1-s\right) d s
\end{align*}
$$

for $\tau_{k+\ell} \leq t+\tau_{k+\ell} \leq T$.
We will assume that, for $t \in(0, T)$,

$$
\begin{equation*}
\tau_{k+\ell} \alpha\left(t+\tau_{k+\ell} s, 1-s\right)=\widetilde{\alpha}\left(t+\tau_{k+\ell}\right) \text { for } s \in[0,1] \tag{2.13}
\end{equation*}
$$

and

$$
\begin{equation*}
\tau_{k+\ell} \beta\left(t+\tau_{k+\ell} s, 1-s\right)=\widetilde{\beta}\left(t+\tau_{k+\ell}\right) \text { for } s \in[0,1] \tag{2.14}
\end{equation*}
$$

for some functions $\widetilde{\alpha}$ and $\widetilde{\beta}$ constructed later; this implies that the LHS of (2.13) and (2.14) are constant with respect to $s \in[0,1]$. Given $\widetilde{\alpha}$ and $\widetilde{\beta}$ defined in $\mathbb{R}$, one can verify that (2.13) and (2.14) hold if

$$
\begin{equation*}
\alpha(t, x)=\tau_{k+\ell}^{-1} \widetilde{\alpha}\left(t+\tau_{k+\ell} x\right) \quad \text { and } \quad \beta(t, x)=\tau_{k+\ell}^{-1} \widetilde{\beta}\left(t+\tau_{k+\ell} x\right) \tag{2.15}
\end{equation*}
$$

Under conditions (2.13) and (2.14), by replacing first $s$ by $1-s$ and then $t+\tau_{k+\ell}$ by $t$, identity (2.12) can be then written as, for $t \in\left(\tau_{k+\ell}, T\right)$,

$$
\begin{equation*}
v_{k+\ell}(t, 0)=\widetilde{\alpha}(t) \int_{0}^{1} v_{k}\left(-\tau_{k+\ell} s+t, s\right) d s+\widetilde{\beta}(t) \int_{0}^{1} v_{k+1}\left(-\tau_{k+\ell} s+t, s\right) d s \tag{2.16}
\end{equation*}
$$

We write (2.9) as

$$
\begin{equation*}
v_{+}(t, 0)=-\Sigma_{+}^{-1} B^{\top} \Sigma_{-} v_{-}(t, 0) \tag{2.17}
\end{equation*}
$$

In what follows, we consider the solution $v$ satisfying

$$
\begin{equation*}
v_{1}(T, \cdot)=\cdots=v_{k-1}(T, \cdot)=v_{k+1}(T, \cdot)=\cdots=v_{k+m}(T, \cdot)=0 \tag{2.18}
\end{equation*}
$$

and

$$
\begin{equation*}
v_{k}(T, x)=0 \text { for } 0 \leq x \leq \frac{T-\tau_{k+1}}{\tau_{k}}<1 \text { since } T<\tau_{k}+\tau_{k+1} \tag{2.19}
\end{equation*}
$$

From the system of $v(2.6),(2.7),(2.8)$, and (2.9), the solution $v$ is then uniquely determined by $v_{k}(T, x)$ for $\frac{T-\tau_{k+1}}{\tau_{k}}<x \leq 1$.

Since, for $t \in(0, T)$,

$$
v_{1}(t, 1)=\cdots=v_{k-1}(t, 1)=0 \quad(\text { by }(2.8))
$$

and, for $(t, x) \in(0, T) \times(0,1)$,

$$
\partial_{t} v_{j}(t, x)=-\lambda_{j} \partial_{x} v_{j}(t, x) \text { for } 1 \leq j \leq k-1 \quad(\text { by (2.7) })
$$

it follows from (2.18) that, for $t \in(0, T)$,

$$
v_{1}(t, 0)=\cdots=v_{k-1}(t, 0)=0
$$

We then derive from (1.12) and (2.17) that, for $t \in(0, T)$,

$$
\begin{equation*}
v_{k+1}(t, 0)=\gamma_{k+1} v_{k}(t, 0) \quad \text { and } \quad v_{k+\ell}(t, 0)=\gamma_{k+\ell} v_{k}(t, 0), \tag{2.20}
\end{equation*}
$$

where

$$
\begin{equation*}
\gamma_{k+1}:=\lambda_{k+1}^{-1} \lambda_{k} B_{k, 1} \stackrel{(1.12)}{\neq} 0 \quad \text { and } \quad \gamma_{k+\ell}:=\lambda_{k+\ell}^{-1} \lambda_{k} B_{k, \ell} \stackrel{(1.12)}{\neq 0} 0 . \tag{2.21}
\end{equation*}
$$

Since

$$
\partial_{t} v_{k}(t, x) \stackrel{(2.6)}{=}-\lambda_{k} \partial_{x} v_{k}(t, x),
$$

$v_{k}(t, 1)=0$ for $t \in(0, T)$ by (2.8), and $T \geq \tau_{k}+\tau_{k+l}$, one has, for $t \in\left(\tau_{k+\ell}, \tau_{k+1}\right)$,

$$
\int_{0}^{1} v_{k}\left(-\tau_{k+\ell} s+t, s\right) d s=\int_{0}^{\gamma_{k}(t)} v_{k}\left(-\tau_{k+\ell} s+t, s\right) d s
$$

(see Figure 1 for the definition of $\gamma_{k}(t)$ ). This implies, by (2.6) applied with $i=k$, for $t \in$ $\left(\tau_{k+\ell}, \tau_{k+1}\right)$,

$$
\begin{equation*}
\int_{0}^{1} v_{k}\left(-\tau_{k+\ell} s+t, s\right) d s=\theta_{k} \int_{\tau_{k+\ell}}^{t} v_{k}(s, 0) d s, \tag{2.22}
\end{equation*}
$$

where

$$
\begin{equation*}
\theta_{k}=\frac{\gamma_{k}(t)}{t-\tau_{k+\ell}} \quad\left(=\frac{1}{\sqrt{1+\tau_{k+\ell}^{2}}} \frac{\sqrt{1+\tau_{k+\ell}^{2}}}{\tau_{k}+\tau_{k+\ell}}=\frac{1}{\tau_{k}+\tau_{k+\ell}}\right) \quad: \text { independent of } t \tag{2.23}
\end{equation*}
$$

Similarly, since

$$
\begin{aligned}
\partial_{t} v_{k+1}(t, x) & \stackrel{(\sqrt{2.6)}}{=} \lambda_{k+1} \partial_{x} v_{k+1}(t, x), \\
v_{k+1}(t, 1) & =0 \text { for } t \in(0, T)
\end{aligned}
$$

thanks to $v_{k}(t, 0) \stackrel{(2.6),(2.19)}{=} 0$ for $t \in\left(\tau_{k+1}, T\right)$ and $(2.20)$, and $v_{k+1}(T, \cdot) \stackrel{(2.18)}{=} 0$, we obtain, for $t \in\left(\tau_{k+\ell}, \tau_{k+1}\right)$,

$$
\int_{0}^{1} v_{k+1}\left(-\tau_{k+\ell} s+t, s\right) d s=\int_{0}^{\gamma_{k+1}(t)} v_{k+1}\left(-\tau_{k+\ell} s+t, s\right) d s
$$

(see Figure 1 for the definition of $\gamma_{k+1}(t)$ ).
This implies, by (2.6) applied with $i=k+1$, for $t \in\left(\tau_{k+\ell}, \tau_{k+1}\right)$,

$$
\begin{equation*}
\int_{0}^{1} v_{k+1}\left(-\tau_{k+\ell} s+t, s\right) d s=\theta_{k+1} \int_{t}^{\tau_{k+1}} v_{k+1}(s, 0) d s \tag{2.24}
\end{equation*}
$$

where

$$
\begin{equation*}
\theta_{k+1}=\frac{\gamma_{k+1}(t)}{\tau_{k+1}-t} \quad\left(=\frac{1}{\sqrt{1+\tau_{k+\ell}^{2}}} \frac{\sqrt{1+\tau_{k+\ell}^{2}}}{\tau_{k+1}-\tau_{k+\ell}}=\frac{1}{\tau_{k+1}-\tau_{k+\ell}}\right) \quad: \text { independent of } t \tag{2.25}
\end{equation*}
$$

Using (2.22) and (2.24), we derive from (2.16) that

$$
v_{k+\ell}(t, 0)=\widetilde{\alpha}(t) \theta_{k} \int_{\tau_{k+\ell}}^{t} v_{k}(s, 0) d s+\widetilde{\beta}(t) \theta_{k+1} \int_{t}^{\tau_{k+1}} v_{k+1}(s, 0) d s \quad \text { for } t \in\left(\tau_{k+\ell}, \tau_{k+1}\right)
$$

This implies, by (2.20),

$$
\begin{equation*}
v_{k+\ell}(t, 0)=\hat{\alpha}(t) \int_{\tau_{k+\ell}}^{t} v_{k+\ell}(s, 0) d s+\hat{\beta}(t) \int_{t}^{\tau_{k+1}} v_{k+\ell}(s, 0) d s \quad \text { for } t \in\left(\tau_{k+\ell}, \tau_{k+1}\right), \tag{2.26}
\end{equation*}
$$



Figure 1. On the definition of $\gamma_{k}$ and $\gamma_{k+1}$ for $t \in\left(\tau_{k+\ell}, \tau_{k+1}\right): \gamma_{k}(t)$ is the abscise of the intersection of the line passing $(0, t)$ and $\left(1, t-\tau_{k+\ell}\right)$, and the line passing $\left(0, \tau_{k+\ell}\right)$ and $\left(1, \tau_{k+\ell}+\tau_{k}\right) ; \gamma_{k+1}(t)$ is the abscise of the intersection of the line passing $(0, t)$ and $\left(1, t-\tau_{k+\ell}\right)$, and the line passing $\left(0, \tau_{k+1}\right)$ and $(1,0)$.
where

$$
\begin{equation*}
\hat{\alpha}=\gamma_{k+\ell}^{-1} \theta_{k} \widetilde{\alpha} \quad \text { and } \quad \hat{\beta}=\gamma_{k+1} \gamma_{k+\ell}^{-1} \theta_{k+1} \widetilde{\beta} . \tag{2.27}
\end{equation*}
$$

Since

$$
\tau_{k}+\tau_{k+1}-\varepsilon \stackrel{\sqrt{[1.13]}}{=} T,
$$

it follows that, at least if $\varepsilon>0$ is small enough so that $T>\tau_{k+\ell}$,

$$
I:=\left(\tau_{k+\ell}, \tau_{k+1}\right) \cap\left(T-\tau_{k}, T\right) \neq \emptyset .
$$

Fix $\varphi \in C_{\mathrm{c}}^{\infty}(\mathbb{R})$ such that

$$
\begin{equation*}
\operatorname{supp} \varphi \subset I \quad \text { and } \quad \int_{I} \varphi=1 \tag{2.28}
\end{equation*}
$$

Set

$$
\begin{equation*}
v_{k+\ell}(t, 0)=\varphi(t) \text { for } t \in\left(\tau_{k+\ell}, \tau_{k+1}\right) \quad \text { and } \quad \hat{\alpha}(t)=\hat{\beta}(t)=\varphi(t) \text { for } t \in \mathbb{R} \tag{2.29}
\end{equation*}
$$

One can check that (2.26) holds for this choice. From (2.20), we have

$$
\begin{equation*}
v_{k}(t, 0)=\gamma_{k+\ell}^{-1} \varphi(t) \quad \text { and } \quad v_{k+1}(t, 0)=\gamma_{k+1} \gamma_{k+\ell}^{-1} \varphi(t) \quad \text { for } t \in\left(\tau_{k+\ell}, \tau_{k+1}\right) \tag{2.30}
\end{equation*}
$$

We have just presented arguments for a choice of $\alpha$ and $\beta$, and a choice of $v(T, \cdot)$ so that (2.8), (2.9), and (2.10) hold. We now proceed in the opposite direction to rigorously establish this.

Consider $\alpha$ and $\beta$ defined by, for $(t, x) \in(0, T) \times(0,1)$,

$$
\begin{equation*}
\alpha(t, x)=\lambda_{k+\ell} \gamma_{k+\ell} \theta_{k}^{-1} \varphi\left(t+\tau_{k+\ell} x\right) \tag{2.31}
\end{equation*}
$$

and

$$
\begin{equation*}
\beta(t, x)=\lambda_{k+\ell} \gamma_{k+1}^{-1} \gamma_{k+\ell} \theta_{k+1}^{-1} \varphi\left(t+\tau_{k+\ell} x\right) \tag{2.32}
\end{equation*}
$$

as suggested by (2.15), (2.27), and (2.29), where $\varphi$ is determined as above.

Let $v(T, \cdot) \in C_{\mathrm{c}}^{\infty}(0,1)$ be such that (2.18) holds and $v_{k}(T, \cdot)$ is chosen such that

$$
v_{k}(t, 0)=\gamma_{k+\ell}^{-1} \varphi(t) \text { for } t \in\left(T-\tau_{k}, T\right)
$$

as suggested by (2.19) and (2.30). This implies, by (2.6) applied with $i=k$ and the fact $v_{k}(t, 1)=0$ for $t \in(0, T)$ (see (2.2) $)$,

$$
\begin{equation*}
v_{k}(t, 0)=\gamma_{k+\ell}^{-1} \varphi(t) \text { for } t \in(0, T) \tag{2.33}
\end{equation*}
$$

since $\operatorname{supp} \varphi \subset I \subset\left(T-\tau_{k}, T\right)$. One can check that (2.20) holds by the same arguments used to derive it as before. One can also check that (2.26) holds by (2.28). Using (2.20), one then obtains (2.16) for $t \in\left(\tau_{k+\ell}, \tau_{k+1}\right)$, which implies (2.12) for $t \in\left(0, \tau_{k+1}-\tau_{k+\ell}\right)$. From (2.12) being valid for $t \in\left(0, \tau_{k+1}-\tau_{k+\ell}\right)$, and (2.7) (see also (2.11)), we derive that

$$
\begin{equation*}
v_{k+\ell}(t, 1)=0 \text { for } t \in\left(0, \tau_{k+1}-\tau_{k+\ell}\right) \tag{2.34}
\end{equation*}
$$

Since $\hat{\alpha}=\hat{\beta}=0$ for $t \in\left(\tau_{k+1}, T\right)$, which implies $\widetilde{\alpha}=\widetilde{\beta}=0$ for $t \in\left(\tau_{k+1}, T\right)$, it follows from (2.11) (see also (2.16)) that

$$
\begin{equation*}
v_{k+\ell}\left(t+\tau_{k+\ell}, 0\right)=v_{k+\ell}(t, 1) \text { for } t \in\left(\tau_{k+1}-\tau_{k+\ell}, T-\tau_{k+\ell}\right) \tag{2.35}
\end{equation*}
$$

This implies, by (2.20) and (2.33),

$$
\begin{equation*}
v_{k+\ell}(t, 1)=0 \text { for } t \in\left(\tau_{k+1}-\tau_{k+\ell}, T-\tau_{k+\ell}\right) \tag{2.36}
\end{equation*}
$$

Similarly, since $v_{k+\ell}(T, \cdot)=0$, we derive from (2.31) and (2.32) ${ }^{2}$ that

$$
\begin{equation*}
v_{k+\ell}(t, 1)=0 \text { for } t \in\left(T-\tau_{k+\ell}, T\right) \tag{2.37}
\end{equation*}
$$

Combining (2.34), (2.36), and (2.37) yields

$$
\begin{equation*}
v_{k+\ell}(t, 1)=0 \text { for } t \in(0, T) \tag{2.38}
\end{equation*}
$$

From the choice of $v(T, \cdot)$ in (2.18), the property of $v$ given in (2.6), and the fact $v_{-}(t, 1)=0$ for $t \in(0, T)$, we have, for $1 \leq j \leq k-1$,

$$
v_{j}(t, 0)=0 \text { for } t \in[0, T]
$$

Since $B_{k, j}=0$ for $2 \leq j \leq m$ with $j \neq \ell$ by (1.12), it follows from (2.17) that, for $k+2 \leq j \leq k+m$ with $j \neq k+\ell$,

$$
\begin{equation*}
v_{j}(t, 0)=0 \text { for } t \in[0, T] \tag{2.39}
\end{equation*}
$$

We derive from (2.6), the choice of $v(T, \cdot)$ in (2.18), and (2.39) that, for $k+2 \leq j \leq k+m$ with $j \neq k+\ell$,

$$
\begin{equation*}
v_{j}(t, 1)=0 \text { for } t \in[0, T] \tag{2.40}
\end{equation*}
$$

From (2.33) (see also (2.19) ), we obtain that $v_{k}(t, 0)=0$ for $t \in\left(\tau_{k+1}, T\right)$. This implies, by (2.17) and (2.39) (see also (2.20)),

$$
v_{k+1}(t, 0)=0 \text { for } t \in\left(\tau_{k+1}, T\right)
$$

We derive that, by using (2.6) and (2.18),

$$
\begin{equation*}
v_{k+1}(t, 1)=0 \text { for } t \in[0, T] \tag{2.41}
\end{equation*}
$$

We have, by (2.20) and (2.33),

$$
v_{k+1}(t, 0)=\gamma_{k+1} \gamma_{k+\ell}^{-1} \varphi(t) \text { for } t \in(0, T)
$$

This implies, by (2.6),

$$
v_{k+1}(0, x)=\gamma_{k+1} \gamma_{k+\ell}^{-1} \varphi\left(\tau_{k+1} x\right) \text { for } x \in[0,1]
$$

[^2]We thus arrive, since $\operatorname{supp} \varphi \subset\left(0, \tau_{k+1}\right)$,

$$
\begin{equation*}
v(0, \cdot) \not \equiv 0 . \tag{2.42}
\end{equation*}
$$

From (2.38), (2.40), (2.41) and (2.42), we reach

$$
v_{+}(\cdot, 1)=0 \quad \text { and } \quad v(0, \cdot) \not \equiv 0 .
$$

The proof is complete.

## 3. Null-controllability in the analytic setting - Proof of Theorem 1.2

This section is devoted to the proof of Theorem 1.2. The proof is divided into three steps described below:

- Step 1: for each $\tau$, we characterize the space $H(\tau)\left(\subset\left[L^{2}(0,1)\right]^{n}\right)$, which is of finite dimension, for which one can steer any element in $H(\tau)^{\perp}{ }^{3}$ at time $\tau$ to 0 in time $T_{\text {opt }}$. (In particular, from this definition of $H(\tau)$, one cannot steer any element in $H(\tau) \backslash\{0\}$ at time $\tau$ to 0 in time $T_{\text {opt }}$.) Moreover, we show that $H(\cdot)$ is analytic in a neighborhood $I_{1}$ of $\left[0, T_{1}-T_{\text {opt }}\right]$ except for a discrete subset, which is removable 4 .
- Step 2: For each $\tau \in I_{1}$, we characterize the subspace $J(\tau)$ of $H(\tau)$ for which one can steer every element $\varphi$ in $J(\tau)$ from time $\tau$ to 0 in time $T_{\text {opt },+}$, i.e., in time $T_{\text {opt }}+\delta$ for all $\delta>0$. Let $M(\tau)$ be the orthogonal complement of $J(\tau)$ in $H(\tau)$. We also show that there exists a constant $\varepsilon_{0}$ such that, roughly speaking, the following property holds: if $\tau \in I_{1}$ and $\varphi \in M(\tau) \backslash\{0\}$, then one cannot steer $\varphi$ from time $\tau$ to 0 in time $T_{\text {opt }}+\varepsilon_{0}$.
- Step 3: We give the proof of Theorem 1.2 using Steps 1 and 2.

Let us make some comments on these three steps before proceeding them. Concerning Step 1, the fact that $H(\tau)$ is of finite dimension already appeared in our previous analysis [20]. Some necessary conditions on $H(\tau)$ are derived in [20] and the starting point of the analysis there is the backstepping technique. In this paper, the (complete) characterization of $H(\tau)$ is given and it plays a crucial role in our proof of Theorem 1.2. This characterization can be obtained by first applying the backstepping technique (and then by using similar ideas given here). However, this way requires a quite strong assumption on the analyticity of $C$ in the step of using backstepping technique (see Remark (3.4). To avoid it, we implement a new approach applied directly to the original system. The analysis is though strongly inspired/guided by our understanding in the form obtained via the backstepping. A part of technical points in this step is to establish the well-posedness of hyperbolic equations with unusual boundary conditions (the boundary condition of a component can be given both on the left at $x=0$ for some interval of time and on the right at $x=1$ for some other interval of time), and in a domain which is not necessary to be a rectangle in $x t$ plane. The analysis is interesting but delicate, and presented in the appendix. After characterizing $H(\cdot)$, the analyticity of $H(\cdot)$ is established by suitably applying the theory of perturbations of analytic compact operators, see, e.g., 33. These results are given in Proposition 3.1 in Section 3.1. Concerning Steps 1 and 2 , the characterizations of all states for which one can steer from time $\tau$ to 0 in time $T_{\text {opt }}$ or in time $T_{\text {opt },+}$ can be done for $C \in\left[L^{\infty}(I \times(0,1))\right]^{n \times n}$. The analyticity of $C$ is not required for this purpose. It is in the proof of the existence of $\varepsilon_{0}$, given in Step 2, that the analyticity of $C$ plays a crucial role. The analysis of Step 3 is also based on a technical lemma (Lemma 3.5). The approach proposed in this paper is quite robust and might be applied to other contexts.

[^3]The rest of this section containing four subsections is organized as follows. In the first section, we introduce notations and present preliminary results related to observability inequalities, which are the starting point of our analysis. Steps 1,2 , and 3 are then given in the second, the third, and the fourth subsection, respectively.
3.1. Preliminaries. Fix $\tau \in I$ and $T>0$ such that $[\tau, \tau+T] \subset I$. Define

$$
\begin{aligned}
\mathcal{F}_{\tau, T}:\left[L^{2}(\tau, \tau+T)\right]^{m} & \rightarrow \quad\left[L^{2}(0,1)\right]^{n} \\
U & \mapsto u(\tau+T, \cdot),
\end{aligned}
$$

where $u$ is the unique solution of the system

$$
\begin{gather*}
\partial_{t} u(t, x)=\Sigma(x) \partial_{x} u(t, x)+C(t, x) u(t, x) \text { for }(t, x) \in(\tau, \tau+T) \times(0,1),  \tag{3.1}\\
u_{-}(t, 0)=B u_{+}(t, 0) \text { for } t \in(\tau, \tau+T),  \tag{3.2}\\
u_{+}(t, 1)=U(t) \text { for } t \in(\tau, \tau+T),  \tag{3.3}\\
u(t=\tau, \cdot)=0 \text { in }(0,1) . \tag{3.4}
\end{gather*}
$$

Set, for $(t, x) \in I \times(0,1)$,

$$
\begin{equation*}
\mathbf{C}(t, x)=\Sigma^{\prime}(x)-C^{\boldsymbol{\top}}(t, x) . \tag{3.5}
\end{equation*}
$$

The following result provides the formula for the adjoint $\mathcal{F}_{\tau, T}^{*}$ of $\mathcal{F}_{\tau, T}$.
Lemma 3.1. We have, for $\varphi \in\left[L^{2}(0,1)\right]^{n}$,

$$
\mathcal{F}_{\tau, T}^{*}(\varphi)=\Sigma_{+}(1) v_{+}(\cdot, 1) \text { in }(\tau, \tau+T),
$$

where $v$ is the unique solution of the system

$$
\begin{equation*}
\partial_{t} v(t, x)=\Sigma(x) \partial_{x} v(t, x)+\mathbf{C}(t, x) v(t, x) \text { for }(t, x) \in(\tau, \tau+T) \times(0,1) \tag{3.6}
\end{equation*}
$$

with, for $0<t<T$,

$$
\begin{gather*}
v_{-}(t, 1)=0  \tag{3.7}\\
\Sigma_{+}(0) v_{+}(t, 0)=-B^{\top} \Sigma_{-}(0) v_{-}(t, 0) \tag{3.8}
\end{gather*}
$$

and

$$
\begin{equation*}
v(t=\tau+T, \cdot)=\varphi \text { in }(0,1) . \tag{3.9}
\end{equation*}
$$

The proof of Lemma 3.1 is quite standard and similar to the one of [20, Lemma 1]. The details are omitted.

Using the same method, we also obtain the following two results, see, e.g., the proof of [20, Lemma 2] for the analysis.

Lemma 3.2. Assume that $u$ is a solution of (3.1) -(3.3) such that $u_{+}(\cdot, 1)=0$ in $(\tau, \tau+T)$. Then, for $\varphi \in\left[L^{2}(0,1)\right]^{n}$, we have ${ }^{5}$

$$
\int_{0}^{1}\langle u(\tau+T, x), v(\tau+T, x)\rangle d x=\int_{0}^{1}\langle u(\tau, x), v(\tau, x)\rangle d x
$$

where $v$ is a solution of (3.6)-(3.9).

[^4]Lemma 3.3. Assume that $u$ is a solution of (3.1)-(3.3). Then

$$
\int_{0}^{1}\langle u(\tau+T, x), v(\tau+T, x)\rangle d x=\int_{0}^{1}\langle u(\tau, x), v(\tau, x)\rangle d x,
$$

where $v$ is a solution of (3.6) $-(3.8)$ satisfying $v_{+}(\cdot, 1)=0$.
Applying the Hilbert uniqueness method, see e.g. [10, Chapter 2] and [39], we have
Lemma 3.4. Let $E$ be a closed subspace of $\left[L^{2}(0,1)\right]^{n}$. System (3.1)-(3.3) is null controllable at the time $\tau+T$ for initial datum at time $\tau$ in $E$ if and only if, for some positive constant $C_{\tau, T}$,

$$
\begin{equation*}
\int_{\tau}^{\tau+T}\left|v_{+}(t, 1)\right|^{2} d t \geq C_{\tau, T} \int_{0}^{1}\left|\operatorname{Proj}_{E} v(\tau, x)\right|^{2} d x \quad \forall \varphi \in\left[L^{2}(0,1)\right]^{n} \tag{3.10}
\end{equation*}
$$

where $v$ is the solution of (3.6)-(3.9).
3.2. Characterization of states at time $\tau$ steered to 0 in time $T_{\text {opt }}$. In what follows in this section, we assume that $I=(\alpha, \beta)$ is an open bounded interval containing $\left[0, T_{1}\right]$ and set $I_{1}=\left(\alpha, \beta-T_{\text {opt }}\right)$.

We first characterize states which can be steered at time $\tau$ to 0 in time $T_{\text {opt }}$. The following proposition is the key result of this section and is the starting point of our analysis in the analytic setting.
Proposition 3.1. Let $k \geq m \geq 1$ and let $B \in \mathcal{B}$ be such that (1.10) holds for $i=m$. Assume that $C \in\left[L^{\infty}(I \times(0,1))\right]^{n \times n}$. There exist a compact operator $\mathcal{K}(\tau):\left[L^{2}(0,1)\right]^{n} \rightarrow\left[L^{2}(0,1)\right]^{n}$ and a continuous linear operator $\mathcal{L}(\tau):\left[L^{2}(0,1)\right]^{n} \rightarrow\left[L^{2}\left(0, T_{\text {opt }}-\tau_{k-m+1}\right)\right]^{m}$ defined for $\tau \in I_{1}$ such that they are uniformly bounded in $I_{1}$ and, with

$$
\begin{equation*}
H(\tau):=\left\{\varphi \in\left[L^{2}(0,1)\right]^{n} ; \varphi+\mathcal{K}(\tau) \varphi=0 \text { and } \mathcal{L}(\tau) \varphi=0\right\} \tag{3.11}
\end{equation*}
$$

the following two facts, concerning system (3.1)-(3.3), hold
i) one can steer $\varphi \in H(\tau)^{\perp}$ at time $\tau$ to 0 at time $\tau+T_{\text {opt }}$.
ii) one cannot steer any element $\varphi$ in $H(\tau) \backslash\{0\}$ at time $\tau$ to 0 at time $\tau+T_{\text {opt }}$.

Assume in addition that $C \in \mathcal{H}\left(I,\left[L^{\infty}(0,1)\right]^{n \times n}\right)$. Then $\mathcal{K}$ and $\mathcal{L}$ are analytic in $I_{1}$.
We also obtain an explicit characterization of the space $H(\tau)$ in Proposition 3.1 via the dual system. In fact, the characterization of $H(\tau)$ in (3.11) is proved using such a characterization. For the later use in the proof of Theorem 1.2, we state it in a slightly more general form:
Lemma 3.5. Let $k \geq m \geq 1$ and let $B \in \mathcal{B}$ be such that (1.10) holds for $i=m$. Let $\varphi \in\left[L^{2}(0,1)\right]^{n}$, $\tau \in I$, and $T \geq T_{\text {opt }}$ be such that $\tau+T \in I$. There exists a subspace $H(\tau, T)$ of $H(\tau)$ such that the following two facts, concerning system (3.1)-(3.3), hold
i) one can steer $\varphi \in H(\tau, T)^{\perp}$ at time $\tau$ to 0 at time $\tau+T$.
ii) one cannot steer any element $\varphi$ in $H(\tau, T) \backslash\{0\}$ at time $\tau$ to 0 at time $\tau+T$.

Moreover, $\varphi \in H(\tau, T)$ if and only if there exists a solution $v$ of the system

$$
\begin{equation*}
\partial_{t} v(t, x)=\Sigma(x) \partial_{x} v(t, x)+\mathbf{C}(t, x) v(t, x) \text { for }(t, x) \in(\tau, \tau+T) \times(0,1), \tag{3.12}
\end{equation*}
$$

with, for $\tau<t<\tau+T$,

$$
\begin{gather*}
v_{-}(t, 1)=0  \tag{3.13}\\
\Sigma_{+}(0) v_{+}(t, 0)=-B^{\top} \Sigma_{-}(0) v_{-}(t, 0)  \tag{3.14}\\
v_{+}(t, 1)=0 \tag{3.15}
\end{gather*}
$$



Figure 2. Geometry of the setting considered in the proof of Proposition 3.1 when $\Sigma$ is constant. The boundary conditions imposed at $x=0$ for $w_{j}$ with $k-m+2 \leq j \leq k$ are given on the left, and the boundary conditions imposed at $x=1$ and $t=0$ are given on the right.
and

$$
\begin{equation*}
v(\tau, \cdot)=\varphi . \tag{3.16}
\end{equation*}
$$

Remark 3.1. Assume that the assumptions in Theorem 1.3 hold. Let $\tau \in I$ and $T>T_{\text {opt }}$. Assume that $\tau+T_{1} \in I$. We later prove that $H(\tau, T)=\{0\}$ (see Proposition (3.3) which is the unique continuation principle corresponding to (3.12)-(3.15).

As a consequence of Proposition 3.1 and the theory of analytic compact operators, see, e.g., [33], we can prove

Lemma 3.6. Assume that $C \in \mathcal{H}\left(I,\left[L^{\infty}(0,1)\right]^{n \times n}\right)$. Then $H(\tau)$ is analytic in $I_{1}$ except for a discrete set, which is removable ${ }^{6}$.

The proofs of Proposition [3.1, Lemmas 3.5 and 3.6 are given in the next three subsections, respectively.

Before entering the details of the proof, we introduce some notations on the characteristic flows which are used several times later. Extend $\lambda_{i}$ in $\mathbb{R}$ with $1 \leq i \leq k+m$ by $\lambda_{i}(0)$ for $x<0$ and $\lambda_{i}(1)$ for $x>1$. For $(s, \xi) \in[0,+\infty) \times[0,1]$, define $x_{i}(t, s, \xi)$ for $t \in \mathbb{R}$ by

$$
\begin{equation*}
\frac{d}{d t} x_{i}(t, s, \xi)=\lambda_{i}\left(x_{i}(t, s, \xi)\right) \text { and } x_{i}(s, s, \xi)=\xi \text { if } 1 \leq i \leq k \tag{3.17}
\end{equation*}
$$

and

$$
\begin{equation*}
\frac{d}{d t} x_{i}(t, s, \xi)=-\lambda_{i}\left(x_{i}(t, s, \xi)\right) \text { and } x_{i}(s, s, \xi)=\xi \text { if } k+1 \leq i \leq k+m \tag{3.18}
\end{equation*}
$$

3.2.1. Proof of Proposition 3.1. Fix $\tau \in I_{1}$. Let $v$ be a solution of the system

$$
\begin{equation*}
\partial_{t} v(t, x)=\Sigma(x) \partial_{x} v(t, x)+\mathbf{C}(t+\tau, x) v(t, x) \text { for }(t, x) \in\left(0, T_{\text {opt }}\right) \times(0,1), \tag{3.19}
\end{equation*}
$$

[^5]with, for $0<t<T_{\text {opt }}$,
\[

$$
\begin{gather*}
v_{-}(t, 1)=0  \tag{3.20}\\
\Sigma_{+}(0) v_{+}(t, 0)=-B^{\top} \Sigma_{-}(0) v_{-}(t, 0) \tag{3.21}
\end{gather*}
$$
\]

such that

$$
\begin{equation*}
v_{+}(t, 1)=0 \text { for } t \in\left(0, T_{\text {opt }}\right) . \tag{3.22}
\end{equation*}
$$

Recall that $\mathbf{C}$ is defined in (3.5).
The proof is now divided into two steps:

- Step 1: We give a characterization of $v(0, \cdot)$ where $v$ is a solution of (3.19)-(3.21) satisfying (3.22).
- Step 2: We establish assertions i) and ii).

Step 1 is the key part of the proof. The operators $\mathcal{K}(\tau)$ and $\mathcal{L}(\tau)$ will be introduced in Step 1. The proof of Step 2 is quite standard after Step 1 and the results in Section 3.1,

We now proceed with Steps 1 and 2.

- Step 1: For $1 \leq i \leq k \leq j \leq k+m$, we denote, for a vector $v \in \mathbb{R}^{k+m}$,

$$
v_{-, \geq i}=\left(v_{i}, \cdots, v_{k}\right)
$$

and

$$
v_{<i, \geq j}=\left(v_{1}, \ldots, v_{i-1}, v_{j}, \cdots, v_{k+m}\right) .
$$

Using condition (1.10) with $i=1$, one can write the last equation of (3.21) in an equivalent form:

$$
\begin{equation*}
v_{-, \geq k}(t, 0)=Q_{k} v_{<k, \geq k+m}(t, 0), \tag{3.23}
\end{equation*}
$$

for some $1 \times k$ matrix $Q_{k}$.
Using condition (1.10) with $i=2$, one can write the last two equations of (3.21) in an equivalent form:

$$
\begin{equation*}
v_{-, \geq k-1}(t, 0)=Q_{k-1} v_{<k-1, \geq k+m-1}(t, 0), \tag{3.24}
\end{equation*}
$$

for some $2 \times k$ matrix $Q_{k-1}$.
Using condition (1.10) with $i=m-1$, one can write the last $(m-1)$ equations of (3.21) in an equivalent form:

$$
\begin{equation*}
v_{-, \geq k-m+2}(t, 0)=Q_{k-m+2} v_{<k-m+2, \geq k+2}(t, 0), \tag{3.25}
\end{equation*}
$$

for some $(m-1) \times k$ matrix $Q_{k-m+2}$.
Using condition (1.10) with $i=m$, one can write the last $m$ equations of (3.21) in an equivalent form:

$$
\begin{equation*}
v_{-, \geq k-m+1}(t, 0)=Q_{k-m+1} v_{<k-m+1, \geq k+1}(t, 0), \tag{3.26}
\end{equation*}
$$

for some $m \times k$ matrix $Q_{k-m+1}$.
Let $\Omega$ be the region of points $(t, x) \in(0,+\infty) \times(0,1)$ such that in the $x t$-plane they are below the characteristic flow of $v_{k-m+1}$ passing the point $\left(1, T_{\text {opt }}\right)$, see Figure 2,

Given $f \in\left[L^{2}\left(0, T_{\text {opt }}\right)\right]^{n}$ and $g \in\left[L^{2}(0,1)\right]^{m}$, we consider the system

$$
\begin{gather*}
w_{t}(t, x)=\Sigma(x) \partial_{x} w(t, x)+\mathbf{C}(t+\tau, x) w(t, x) \text { for }(t, x) \in \Omega,  \tag{3.27}\\
w(\cdot, 1)=f \text { in }\left(0, T_{\text {opt }}\right)  \tag{3.28}\\
w_{+}(0, \cdot)=g \text { in }(0,1), \tag{3.29}
\end{gather*}
$$

$$
w_{-, \geq k-1}(t, 0)=Q_{k-1} w_{<k-1, \geq k+m-1}(t, 0) \text { for } t \in\left(T_{o p t}-\tau_{k-1}, T_{o p t}-\tau_{k-2}\right),
$$

$$
w_{-, \geq k-m+2}(t, 0)=Q_{k-m+2} w_{<k-m+2, \geq k+2}(t, 0) \text { for } t \in\left(T_{\text {opt }}-\tau_{k-m+2}, T_{\text {opt }}-\tau_{k-m+1}\right),
$$

(see Figure [2).
For $\tau \in I_{1}$, define

$$
\begin{array}{ccc}
\mathcal{T}(\tau):\left[L^{2}(0,1)\right]^{n} \times\left[L^{2}(0,1)\right]^{m} & \rightarrow & {\left[L^{2}(\Omega)\right]^{n}}  \tag{3.33}\\
(f, g) & \mapsto & w,
\end{array}
$$

where $w$ is the (broad) solution of (3.27)-(3.32) (see Definition A. 1 for the definition of broad solutions and Theorem A. 1 for their existence and uniqueness, both in the appendix).

We claim that
$v$ is a solution of (3.19)-(3.21) satisfying (3.22) if and only if $v(0, \cdot) \in H(\tau)$ defined in (3.11), where $\mathcal{K}(\tau)$ and $\mathcal{L}(\tau)$ are determined below.

We now introduce $\mathcal{K}$ and $\mathcal{L}$. Set $w=\mathcal{T}(\tau)\left(0, v_{+}(0, \cdot)\right)$. By noting that $v=w$,

- $D K$ ) the operator $\mathcal{K}(\tau)$ is determined/summarized (the details are given below) by
$D K_{m}$ ) the $m$ equations of system (3.21) imposed for $w$ in $\left(0, \tau_{k+m}\right)$, $\left.D K_{m-1}\right)$ the first $m-1$ equations of system (3.21) imposed for $w$ in $\left(\tau_{k+m}, \tau_{k+m-1}\right)$,
$\left.D K_{1}\right)$ the first equation of system (3.21) imposed for $w$ in $\left(\tau_{k+2}, \tau_{k+1}\right)$, (these above conditions are on $\left.v_{+}(0, \cdot)\right)$, and
$\left.D K_{-}\right) v_{-}(0, \cdot)=w_{-}(0, \cdot)$.
- DL) $\mathcal{L}(\tau)$ is defined by (3.21) in $\left(0, T_{\text {opt }}-\tau_{k-m+1}\right)$.

Let us explain how to define the operators $\mathcal{K}(\tau)$ and $\mathcal{L}(\tau)$ from these conditions. To this end, we first introduce some notations. For $x \in[0,1]$ and $1 \leq j \leq k+m$, let $\tau(j, x) \in[0,+\infty)$ be such that

$$
x_{j}(\tau(j, x), 0, x)=0 \text { for } k+1 \leq j \leq k+m,
$$

and

$$
x_{j}(\tau(j, x), 0, x)=1 \text { for } 1 \leq j \leq k
$$

(see Figure (3). Recall that $x_{j}(t, s, \xi)$ is defined in (3.17) and (3.18).
We now consider $\mathcal{K}(\tau)$ and first deal with the conditions $\left.\left.D K_{m}\right), \ldots, D K_{1}\right)$. The condition $\left.D K_{m}\right)$ can be understood as follows. We have, for $1 \leq j \leq m$,

$$
\frac{d}{d t} w_{k+j}\left(t, x_{k+j}(t, 0, x)\right)=\left(\mathbf{C}\left(t+\tau, x_{k+j}(t, 0, x)\right) w\left(t, x_{k+j}(t, 0, x)\right)\right)_{k+j}
$$

Integrating from 0 to $\tau(k+j, x)$ yields, for $1 \leq j \leq m$ and for $x \in\left(0, x_{k+j}\left(0, \tau_{k+m}, 0\right)\right.$,

$$
w_{k+j}(0, x)=w_{k+j}(\tau(k+j, x), 0)-\int_{0}^{\tau(k+j, x)}\left(\mathbf{C}\left(t+\tau, x_{k+j}(t, 0, x)\right) w\left(t, x_{k+j}(t, 0, x)\right)\right)_{k+j} d t
$$



Figure 3. $\Sigma$ is constant; a) the definition of $x_{i}(\cdot, 0,1), x_{j}(\cdot, 0,1), x_{j}\left(0, \tau_{i}, 0\right)$, and $\tau_{j}(x)$ for $k+1 \leq j<i \leq k+m$; b) the definition of $x_{\ell}(\cdot, 0,0)$ and $\tau(\ell, x)$ for $1 \leq \ell \leq k$.

Using the $m$ equations of system (3.21), one has, for $1 \leq j \leq m$ and for $x \in\left(0, x_{k+j}\left(0, \tau_{k+m}, 0\right)\right)$,

$$
\begin{align*}
& w_{k+j}(0, x)=-\left(\Sigma_{+}(0)^{-1} B^{\top} \Sigma_{-}(0) w_{-}(\tau(k+j, x), 0)\right)_{j}  \tag{3.35}\\
&-\int_{0}^{\tau(k+j, x)}\left(\mathbf{C}\left(t+\tau, x_{k+j}(t, 0, x)\right) w\left(t, x_{k+j}(t, 0, x)\right)\right)_{k+j} d t .
\end{align*}
$$

Similarly, the condition $\left.D K_{m-1}\right)$ can be written as, for $1 \leq j \leq m-1$ and for $x \in\left(x_{k+j}\left(0, \tau_{k+m}, 0\right), x_{k+j}\left(0, \tau_{k+m-1}, 0\right)\right)$,

$$
\begin{align*}
& w_{k+j}(0, x)=-\left(\Sigma_{+}(0)^{-1} B^{\top} \Sigma_{-}(0) w_{-}(\tau(k+j, x), 0)\right)_{j}  \tag{3.36}\\
&-\int_{0}^{\tau(k+j, x)}\left(\mathbf{C}\left(t+\tau, x_{k+j}(t, 0, x)\right) w\left(t, x_{k+j}(t, 0, x)\right)\right)_{k+j} d t
\end{align*}
$$

$\ldots$, and the condition $\left.D K_{1}\right)$ can be written as, for $x \in\left(x_{k+1}\left(0, \tau_{k+2}, 0\right), x_{k+1}\left(0, \tau_{k+1}, 0\right)\right)=$ $\left(x_{k+1}\left(0, \tau_{k+2}, 0\right), 1\right)$,

$$
\begin{align*}
& w_{k+1}(0, x)=-\left(\Sigma_{+}(0)^{-1} B^{\top} \Sigma_{-}(0) w_{-}(\tau(k+1, x), 0)\right)_{1}  \tag{3.37}\\
&-\int_{0}^{\tau(k+1, x)}\left(\mathbf{C}\left(t+\tau, x_{k+1}(t, 0, x)\right) w\left(t, x_{k+1}(t, 0, x)\right)\right)_{k+1} d t .
\end{align*}
$$

We now deal with the condition $D K_{-}$). We have, for $1 \leq j \leq k$,

$$
\frac{d}{d t} w_{j}\left(t, x_{j}(t, 0, x)\right)=\left(\mathbf{C}\left(t+\tau, x_{j}(t, 0, x)\right) w_{j}\left(t, x_{j}(t, 0, x)\right)\right)_{j}
$$

Integrating from 0 to $\tau(j, x)$ yields, for $1 \leq j \leq k$ and for $x \in(0,1)$,

$$
w_{j}(0, x)=w_{j}(\tau(j, x), 1)-\int_{0}^{\tau(j, x)}\left(\mathbf{C}\left(t+\tau, x_{j}(t, 0, x)\right) w\left(t, x_{j}(t, 0, x)\right)\right)_{j} d t
$$

Since $f=0$, it follows that, for $1 \leq j \leq k$ and $x \in(0,1)$,

$$
\begin{equation*}
w_{j}(0, x)=-\int_{0}^{\tau(j, x)}\left(\mathbf{C}\left(t+\tau, x_{j}(t, 0, x)\right) w\left(t, x_{j}(t, 0, x)\right)\right)_{j} d t \tag{3.38}
\end{equation*}
$$

The operator $\mathcal{K}(\tau)$ is then defined via (3.35)-(3.38), with $v_{+}(0, \cdot)=\varphi_{+}$and $w=\mathcal{T}\left(\tau, 0, v_{+}(0, \cdot)\right)$ as follows:

- for $1 \leq j \leq m$ and $x \in(0,1)$,

$$
\begin{align*}
&(\mathcal{K}(\tau)(\varphi)(x))_{k+j}=\left(\Sigma_{+}(0)^{-1} B^{\top} \Sigma_{-}(0) w_{-}(\tau(k+j, x), 0)\right)_{j}  \tag{3.39}\\
&+\int_{0}^{\tau(k+j, x)}\left(\mathbf{C}\left(t+\tau, x_{k+j}(t, 0, x)\right) w\left(t, x_{k+j}(t, 0, x)\right)\right)_{k+j} d t
\end{align*}
$$

- for $1 \leq j \leq k$ and $x \in(0,1)$,

$$
\begin{equation*}
(\mathcal{K}(\tau)(\varphi)(x))_{j}=\int_{0}^{\tau(j, x)}\left(\mathbf{C}\left(t+\tau, x_{j}(t, 0, x)\right) w\left(t, x_{j}(t, 0, x)\right)\right)_{j} d t \tag{3.40}
\end{equation*}
$$

Using Proposition A. 1 in the appendix, one can derive that $\mathcal{K}(\tau)$ is uniformly bounded in $I_{1}$ and is analytic in $I_{1}$ if $C \in \mathcal{H}\left(I,\left[L^{\infty}(0,1)\right]^{n \times n}\right)$.

The definition and the properties of $\mathcal{L}(\tau)$ follow from $D L$, with $v_{+}(0, \cdot)=\varphi_{+}$and $w=$ $\mathcal{T}\left(\tau, 0, v_{+}(0, \cdot)\right)$ as follows:

$$
\mathcal{L}(\tau)(\varphi)=\Sigma_{+}(0) w_{+}(t, 0)+B^{\top} \Sigma_{-}(0) w_{-}(t, 0) \text { in }\left(0, T_{\text {opt }}-\tau_{k-m+1}\right) .
$$

It is clear that $H(\tau) \subset\left\{\varphi \in\left[L^{2}(0,1)\right]^{n} ; \varphi+\mathcal{K}(\tau) \varphi=0\right.$ and $\left.\mathcal{L}(\varphi)=0\right\}$. It remains to prove that

$$
\begin{equation*}
\left\{\varphi \in\left[L^{2}(0,1)\right]^{n} ; \varphi+\mathcal{K}(\tau) \varphi=0 \text { and } \mathcal{L}(\tau) \varphi=0\right\} \subset H(\tau) \tag{3.41}
\end{equation*}
$$

To this end, we introduce another operator $\hat{\mathcal{T}}$ related to $\mathcal{T}$. Consider the system, for $(f, g) \in$ $\left[L^{2}(0,1)\right]^{n} \times\left[L^{2}(0,1)\right]^{m}$,

$$
\begin{gather*}
\partial_{t} \hat{w}(t, x)=\Sigma(x) \partial_{x} \hat{w}(t, x)+\mathbf{C}(t+\tau, x) \hat{w}(t, x) \text { for }(t, x) \in\left(0, T_{\text {opt }}\right) \times(0,1),  \tag{3.42}\\
\hat{w}(\cdot, 1)=f \text { in }\left(0, T_{\text {opt }}\right),  \tag{3.43}\\
\hat{w}_{+}(0, x)=g(x) \text { in }(0,1),  \tag{3.44}\\
\hat{w}_{i}\left(T_{o p t}, \cdot\right)=0 \text { in }(0,1), \text { for } 1 \leq i \leq k-m,  \tag{3.45}\\
\hat{w}_{-, \geq k}(t, 0)=Q_{k} \hat{w}_{<k, \geq k+m}(t, 0) \text { for } t \in\left(T_{\text {opt }}-\tau_{k}, T_{\text {opt }}-\tau_{k-1}\right),  \tag{3.46}\\
\hat{w}_{-, \geq k-1}(t, 0)=Q_{k-1} \hat{w}_{<k-1, \geq k+m-1}(t, 0) \text { for } t \in\left(T_{\text {opt }}-\tau_{k-1}, T_{\text {opt }}-\tau_{k-2}\right), \tag{3.47}
\end{gather*}
$$

$$
\begin{gather*}
\hat{w}_{-, \geq k-m+2}(t, 0)=Q_{k-m+2} \hat{w}_{<k-m+2, \geq k+2}(t, 0) \text { for } t \in\left(T_{\text {opt }}-\tau_{k-m+2}, T_{\text {opt }}-\tau_{k-m+1}\right),  \tag{3.48}\\
\hat{w}_{-, \geq k-m+1}(t, 0)=Q_{k-m+1} \hat{w}_{<k-m+1, \geq k+1}(t, 0) \text { for } t \in\left(T_{\text {opt }}-\tau_{k-m+1}, T_{\text {opt }}\right) \tag{3.49}
\end{gather*}
$$

(it is at this stage that the condition (1.10) with $i=m$ is required!).
For $\tau \in I_{1}$, define

$$
\begin{array}{rlc}
\hat{\mathcal{T}}(\tau):\left[L^{2}(0,1)\right]^{n} \times\left[L^{2}(0,1)\right]^{m} & \rightarrow & {\left[L^{2}\left(\left(0, T_{\text {opt }}\right) \times(0,1)\right)\right]^{n}}  \tag{3.50}\\
(f, g) & \mapsto & \hat{w}
\end{array}
$$

where $\hat{w}$ is the unique broad solution of (3.42)-(3.49) (see Theorem A.2 in the appendix for the existence and uniqueness of broad solutions; the definition of broad solutions is similar to Definition A.11).

It is clear that

$$
\begin{equation*}
\mathcal{T}(\tau)(0, g) \text { is the restriction of } \hat{\mathcal{T}}(\tau)(0, g) \text { in } \Omega \text { for } g \in\left[L^{2}(0,1)\right]^{m} . \tag{3.51}
\end{equation*}
$$

Fix

$$
\begin{equation*}
\varphi_{0} \in\left\{\varphi \in\left[L^{2}(0,1)\right]^{n} ; \varphi+\mathcal{K}(\tau) \varphi=0 \text { and } \mathcal{L}(\tau) \varphi=0\right\} \tag{3.52}
\end{equation*}
$$

Denote

$$
w=\mathcal{T}(\tau)\left(0, \varphi_{0,+}\right) \quad \text { and } \quad \hat{w}=\hat{\mathcal{T}}(\tau)\left(0, \varphi_{0,+}\right)
$$

Then, by (3.51),

$$
\begin{equation*}
\hat{w}=w \text { in } \Omega \tag{3.53}
\end{equation*}
$$

Since $\varphi+\mathcal{K}(\tau)(\varphi)=0$ (see also the condition $D K_{-}$)), we have

$$
w(0, \cdot)=\varphi_{0} \text { in }(0,1) .
$$

Since $\mathcal{L}(\tau)(\varphi)=0$, we obtain

$$
\begin{equation*}
\Sigma_{+}(0) \hat{w}_{+}(t, 0)=-B^{\top} \Sigma_{-}(0) \hat{w}_{-}(t, 0) \text { for } t \in\left(0, T_{\text {opt }}-\tau_{k-m+1}\right) . \tag{3.54}
\end{equation*}
$$

On the other hand, by the definition of $\hat{\mathcal{T}}$ (in particular, condition (3.49)), one has,

$$
\begin{equation*}
\Sigma_{+}(0) \hat{w}_{+}(t, 0)=-B^{\top} \Sigma_{-}(0) \hat{w}_{-}(t, 0) \text { for } t \in\left(T_{o p t}-\tau_{k-m+1}, T_{o p t}\right) \tag{3.55}
\end{equation*}
$$

Combining (3.54) and (3.55) yields

$$
\begin{equation*}
\Sigma_{+}(0) \hat{w}_{+}(t, 0)=-B^{\top} \Sigma_{-}(0) \hat{w}_{-}(t, 0) \text { for } t \in\left(0, T_{o p t}\right) \tag{3.56}
\end{equation*}
$$

Thus $\hat{w}$ is a solution of (3.19)-(3.21) satisfying (3.22) with $\hat{w}(0, \cdot)=w(0, \cdot)=\varphi_{0}$.

- Step 2: We derive i) and ii). We begin with assertion ii). Let $\varphi \in H(\tau) \backslash\{0\}$ be arbitrary. By Step 1, there exists a solution $v$ of (3.19)-(3.21) such that

$$
\begin{equation*}
v_{+}(\cdot, 1)=0 \text { in }\left(0, T_{\text {opt }}\right) \quad \text { and } \quad v(0, \cdot)=\varphi \text { in }(0,1) \tag{3.57}
\end{equation*}
$$

Set

$$
v^{(\tau)}(t, x)=v(t-\tau, x) \text { for }(t, x) \in\left(\tau, \tau+T_{\text {opt }}\right) .
$$

Let $w$ be a solution of (3.1) - (3.3) ${ }^{7}$ with $T=T_{o p t}$, in which $u$ is replaced by $w$, with $w(\tau, \cdot)=$ $v^{(\tau)}(\tau, \cdot)=\varphi$. By Lemma 3.3, we have

$$
\int_{0}^{1}\left\langle w\left(T_{o p t}+\tau, x\right), v^{(\tau)}\left(T_{o p t}+\tau, x\right)\right\rangle d x=\int_{0}^{1}\left\langle w(\tau, x), v^{(\tau)}(\tau, x)\right\rangle d x=\int_{0}^{1}|\varphi|^{2} \neq 0 .
$$

Therefore, one cannot steer $\varphi$ from time $\tau$ to 0 at time $\tau+T_{o p t}$.
We next establish assertion i) by a contradiction argument. Assume that this is not true. By Lemma 3.4 with $E=H(\tau)^{\perp}$, there exists a sequence of solutions $\left(v_{N}\right)$ of (3.19)-(3.21) such that

$$
\begin{equation*}
\lim _{N \rightarrow+\infty}\left\|v_{N,+}(\cdot, 1)\right\|_{L^{2}\left(0, T_{o p t}\right)}=0 \quad \text { and } \quad\left\|\operatorname{Proj}_{H(\tau)^{\perp}} v_{N}(0, \cdot)\right\|_{L^{2}(0,1)}=1 \tag{3.58}
\end{equation*}
$$

Set

$$
\varphi_{N}=\operatorname{Prof}_{H(\tau)} v_{N}(0, \cdot) \in H(\tau) \subset\left[L^{2}(0,1)\right]^{n}
$$

Define

$$
w_{N}=\mathcal{T}(\tau)\left(0, \varphi_{N,+}\right) \text { in } \Omega \quad \text { and } \quad \hat{w}_{N}=\hat{\mathcal{T}}(\tau)\left(0, \varphi_{N,+}\right) \text { in }\left(0, T_{\text {opt }}\right) \times(0,1) .
$$

[^6]Since $\varphi_{N} \in H(\tau)$, it follows from the definition of $\mathcal{K}(\tau)$ that

$$
w_{N}(0, \cdot)=\varphi_{N} \text { in }(0,1)
$$

We derive from (3.51) that

$$
\hat{w}_{N}(0, \cdot)=\varphi_{N} \text { in }(0,1) .
$$

Replacing $v_{N}$ by $v_{N}-\hat{w}_{N}$ if necessary, without loss of generality, one can assume in addition that $v_{N}(0, \cdot) \in H(\tau)^{\perp}$, which yields in particular that $\left\|v_{N}(0, \cdot)\right\|_{L^{2}(0,1)}=\left\|\operatorname{Proj}_{H(\tau)} v_{N}(0, \cdot)\right\|_{L^{2}(0,1)}=1$. This will be assumed from now on.

Consider $f_{N} \in\left[L^{2}\left(0, T_{\text {opt }}\right)\right]^{n}$ defined by

$$
f_{N}=v_{N}(\cdot, 1)
$$

Since $v_{N,-}(\cdot, 1)=0$ in $\left(0, T_{\text {opt }}\right)$ and $\lim _{N \rightarrow+\infty}\left\|v_{N,+}(\cdot, 1)\right\|_{L^{2}\left(0, T_{o p t}\right)}=0$, it follows that

$$
\begin{equation*}
\lim _{N \rightarrow+\infty} f_{N}=0 \text { in }\left[L^{2}\left(0, T_{o p t}\right)\right]^{n} \tag{3.59}
\end{equation*}
$$

Set, in $\Omega$,

$$
u_{N}=\mathcal{T}(\tau)\left(f_{N}, v_{N,+}(0, \cdot)\right) .
$$

Then

$$
\begin{equation*}
u_{N}=v_{N} \text { in } \Omega . \tag{3.60}
\end{equation*}
$$

Since $v_{N}(0, \cdot)+\mathcal{K}(\tau) v_{N}(0, \cdot)=0,\left\|v_{N}(0, \cdot)\right\|_{L^{2}(0,1)}=1$, and $\mathcal{K}(\tau)$ is compact, it follows that, for some subsequence, $v_{N_{k}}(0, \cdot) \rightarrow \varphi$ in $\left[L^{2}(0,1)\right]^{n}$ and hence $\varphi \in H(\tau)^{\perp}$ by (3.59) and the continuity of $\mathcal{T}(\tau)$ (see Proposition A. 1 in the appendix). Set $u=\mathcal{T}(\tau)\left(0, \varphi_{+}\right)$. Since, by (3.60),

$$
\Sigma_{+}(0) u_{N,+}(t, 0)=-B^{\top} \Sigma_{-}(0) u_{N,+}(t, 0) \text { for } t \in\left(0, T_{o p t}\right)
$$

and

$$
v_{N}(0, \cdot)=\mathcal{T}(\tau)\left(f_{N}, v_{N,+}(0, \cdot)\right) \text { in }(0,1),
$$

we derive from (3.59), (3.60) and the continuity of $\mathcal{T}(\tau)$ (see Proposition A. 1 in the appendix) that

$$
\Sigma_{+}(0) u_{+}(t, 0)=-B^{\top} \Sigma_{-}(0) u_{+}(t, 0) \text { for } t \in\left(0, T_{o p t}\right),
$$

and

$$
\varphi(x)=\mathcal{T}(\tau)\left(0, \varphi_{+}\right)(0, x) \text { for } x \in(0,1)
$$

This implies that $\varphi \in H(\tau)$. It follows that $\varphi=0$ since $\varphi \in H(\tau)^{\perp}$. We deduce that

$$
0=\left\|\operatorname{Proj}_{H(\tau)^{\perp}} \varphi\right\|=\lim _{k \rightarrow+\infty}\left\|\operatorname{Proj}_{H(\tau)^{\perp}} v_{N_{k}}(0, \cdot)\right\|=1 .
$$

We have a contradiction. Assertion i) is proved.
The proof is complete.
3.2.2. Proof of Lemma 3.5. The proof of Lemma 3.5 follows from the one of Proposition 3.1 by replacing $T_{\text {opt }}$ by $T$. One just notes here that $H(\tau, T)$ is the set of $v(0, \cdot)$ where $v$ is a solution of (3.19)-(3.21) satisfying (3.22) with $T_{\text {opt }}$ replaced by $T$. The details of the proof are omitted.
3.2.3. Proof of Lemma 3.6. Set, for $\tau \in I_{1}$,

$$
\begin{equation*}
E(\tau)=\left\{\varphi \in\left[L^{2}(0,1)\right]^{n} ; \varphi+\mathcal{K}(\tau) \varphi=0\right\}, \tag{3.61}
\end{equation*}
$$

and

$$
\begin{equation*}
\text { let } P(\tau) \text { be the generalized eigenspace of } \mathcal{K}(\tau) \text { with respect to the eigenvalue }-1 \text {. } \tag{3.62}
\end{equation*}
$$

From (3.11), we have, for $\tau \in I_{1}$,

$$
\begin{equation*}
H(\tau)=E(\tau) \cap\left\{\varphi \in\left[L^{2}(0,1)\right]^{n} ; \mathcal{L}(\tau) \varphi=0\right\} \tag{3.63}
\end{equation*}
$$

Applying the theory of the perturbation of analytic compact operators, see e.g. [33], one derives that

$$
\begin{equation*}
P(\tau) \text { is analytic in } I_{1} \text { except for a discrete subset, which is removable. } \tag{3.64}
\end{equation*}
$$

Indeed, since $K(\tau)$ is compact, it follows that the eigenvalue -1 of $K(\tau)$ is isolated for each $\tau \in I_{1}$. From [33, Section 3 of Chapter 7] (see also [33, Section 3 of Chapter 2]), for each $\tau$ there is a $\gamma>0(\gamma$ depends on $\tau)$ such that the sum of the eigenprojections for all the eigenvalues of $\mathcal{K}(\tau)$ lying inside $\{z \in \mathbb{C}:|z+1|<\gamma\}$ is analytic. We now can apply the theory of the perturbation of analytic operators in a finite dimensional space, see [33, Chapter 2], involving the theory of algebraic functions, see e.g. [1, Section 2 of Chapter 8], to derive (3.64).

We have

$$
E(\tau) \stackrel{(3.61)}{=}\left\{\varphi \in\left[L^{2}(0,1)\right]^{n} ; \varphi+\mathcal{K}(\tau) \varphi=0\right\} \stackrel{(3.62)}{=}\{\varphi \in P(\tau) ; \varphi+\mathcal{K}(\tau) \varphi=0\} .
$$

It follows that

$$
\begin{equation*}
H(\tau)=\{\varphi \in P(\tau) ; \varphi+\mathcal{K}(\tau) \varphi=0 \text { and } \mathcal{L}(\tau) \varphi=0\} \tag{3.65}
\end{equation*}
$$

We now can use the theory of the perturbation of the null-space of analytic matrices. Applying [26, Theorem S6.1 on page 388-389] and using (3.64), we derive that $8^{8}$

$$
\begin{equation*}
H(\tau) \text { is analytic in } I_{1} \text { except for a discrete subset, which is removable. } \tag{3.66}
\end{equation*}
$$

The proof is complete.
3.3. Characterization of states at time $\tau$ steered to 0 in time $T_{o p t,+}$. Fix $\gamma_{0}>0$ such that $\left[0, T_{1}\right] \subset\left(\alpha+\gamma_{0}, \beta-\gamma_{0}\right)$. Set

$$
\begin{equation*}
I_{2}=\left(\alpha+\gamma_{0}, \beta-\gamma_{0}-T_{o p t}\right) \tag{3.67}
\end{equation*}
$$

Given $0<\varepsilon<\gamma_{0}$ and $\tau \in I_{2}$, consider the system, for $V \in\left[L^{2}(0, \varepsilon)\right]^{m}$,

$$
\left\{\begin{array}{c}
\partial_{t} v(t, x)=\Sigma(x) \partial_{x} v(t, x)+C(t+\tau, x) v(t, x) \text { for }(t, x) \in(0, \varepsilon) \times(0,1)  \tag{3.68}\\
v_{-}(t, 0)=B v_{+}(t, 0) \text { for } t \in(0, \varepsilon) \\
v_{+}(t, 1)=V(t) \text { for } t \in(0, \varepsilon) \\
v(0, \cdot)=0 \text { in }[0,1]
\end{array}\right.
$$

[^7]Define ${ }^{9}$

$$
\begin{aligned}
\mathcal{T}_{\tau, \varepsilon}^{c}:\left[L^{2}(0, \varepsilon)\right]^{m} & \rightarrow\left[L^{2}(0,1)\right]^{n} \\
V & \mapsto y(\varepsilon, \cdot)
\end{aligned}
$$

where $v$ is the solution of (3.68). Consider two subsets $Y_{\tau, \varepsilon}$ and $A_{\tau, \varepsilon}{ }^{10}$ of $\left[L^{2}(0,1)\right]^{n}$ defined by

$$
\begin{equation*}
Y_{\tau, \varepsilon}=\mathcal{T}_{\tau, \varepsilon}^{c}\left\{\left[L^{2}(0, \varepsilon)\right]^{m}\right\} \quad \text { and } \quad A_{\tau, \varepsilon}=\operatorname{Proj}_{H(\tau+\varepsilon)}\left\{Y_{\tau, \varepsilon}\right\} \tag{3.69}
\end{equation*}
$$

Given $0<\varepsilon<\gamma_{0}$ and $\tau \in I_{2}$, we also define ${ }^{11}$

$$
\begin{aligned}
\mathcal{T}_{\tau, \varepsilon}^{I}:\left[L^{2}(0,1)\right]^{n} & \rightarrow\left[L^{2}(0,1)\right]^{n} \\
\varphi & \mapsto
\end{aligned}
$$

where $w$ is the solution of

$$
\left\{\begin{array}{c}
\partial_{t} w(t, x)=\Sigma(x) \partial_{x} w(t, x)+C(t+\tau, x) w(t, x) \text { for }(t, x) \in(0, \varepsilon) \times(0,1)  \tag{3.70}\\
w_{-}(t, 0)=B w_{+}(t, 0) \text { for } t \in(0, \varepsilon) \\
w_{+}(t, 1)=0 \text { for } t \in(0, \varepsilon) \\
w(0, \cdot)=\varphi \text { in }[0,1]
\end{array}\right.
$$

Set, for $0<\varepsilon<\gamma_{0}$ and for $\tau \in I_{2}$,

$$
\begin{equation*}
J(\tau, \varepsilon):=\left\{\varphi \in H(\tau) ; \operatorname{Proj}_{H(\tau+\varepsilon)} \mathcal{T}_{\tau, \varepsilon}^{I}(\varphi) \in A_{\tau, \varepsilon}\right\} \tag{3.71}
\end{equation*}
$$

The motivation for the definition of $\mathcal{T}_{\tau, \varepsilon}^{c}$ and $\mathcal{T}_{\tau, \varepsilon}^{I}$ is:
Lemma 3.7. Let $0<\varepsilon<\gamma_{0}$ and $\tau \in I_{2}$. Then $J(\tau, \varepsilon)$ is the space of (functions) states in $H(\tau)$ such that one can steer them from time $\tau$ to 0 at time $\tau+T_{\text {opt }}+\varepsilon$. As a consequence, for $\tau \in I_{2}$,

$$
\begin{equation*}
J\left(\tau, \varepsilon^{\prime}\right) \subset J(\tau, \varepsilon) \text { for } 0<\varepsilon^{\prime}<\varepsilon<\gamma_{0}, \tag{3.72}
\end{equation*}
$$

and the limit $J(\tau)$ of $J(\tau, \varepsilon)$ as $\varepsilon \rightarrow 0_{+}$exists.
Remark 3.2. The monotone property of $J(\tau, \varepsilon)$ with respect to $\varepsilon$ given in (3.72) will play a role in our analysis.
Remark 3.3. The analyticity of $C$ in $I$ is not required in Lemma 3.7,
Proof of Lemma 3.7. Given $\varphi \in J(\tau, \varepsilon)$, by the definition of $J(\tau, \varepsilon)$, there exists $\hat{V} \in\left[L^{2}(0, \varepsilon)\right]^{m}$ such that

$$
\operatorname{Proj}_{H(\tau+\varepsilon)} \hat{w}(\varepsilon, \cdot)=0,
$$

where $\hat{w}$ defined in $(0, \varepsilon) \times(0,1)$ is the solution of the system

$$
\left\{\begin{array}{c}
\partial_{t} \hat{w}(t, x)=\Sigma(x) \partial_{x} \hat{w}(t, x)+C(t+\tau, x) \hat{w}(t, x) \text { for }(t, x) \in(0, \varepsilon) \times(0,1)  \tag{3.73}\\
\hat{w}_{-}(t, 0)=B \hat{w}_{+}(t, 0) \text { for } t \in(0, \varepsilon) \\
\hat{w}_{+}(t, 1)=\hat{V} \text { for } t \in(0, \varepsilon) \\
\hat{w}(0, \cdot)=\varphi \text { in }[0,1]
\end{array}\right.
$$

It follows that, by the characterization of $H(\tau+\varepsilon)$, there exists $\widetilde{V} \in\left[L^{2}\left(\varepsilon, T_{\text {opt }}+\varepsilon\right)\right]^{m}$ such that

$$
\widetilde{w}\left(T_{\text {opt }}+\varepsilon, \cdot\right)=0 \text { in }(0,1),
$$

[^8]where $\widetilde{w}$ defined in $\left(\varepsilon, T_{\text {opt }}+\varepsilon\right) \times(0,1)$ is the solution of the system
\[

\left\{$$
\begin{array}{c}
\partial_{t} \widetilde{w}(t, x)=\Sigma(x) \partial_{x} \widetilde{w}(t, x)+C(t+\tau, x) \widetilde{w}(t, x) \text { for }(t, x) \in\left(\varepsilon, T_{\text {opt }}+\varepsilon\right) \times(0,1),  \tag{3.74}\\
\widetilde{w}_{-}(t, 0)=B \widetilde{w}_{+}(t, 0) \text { for } t \in\left(\varepsilon, T_{\text {opt }}+\varepsilon\right), \\
\widetilde{w}_{+}(t, 1)=\widetilde{V} \text { for } t \in\left(\varepsilon, T_{\text {opt }}+\varepsilon\right), \\
\widetilde{w}(\varepsilon, \cdot)=\hat{w}(\varepsilon, \cdot) \text { in }[0,1] .
\end{array}
$$\right.
\]

Let $w$ be defined in $\left(0, T_{\text {opt }}+\varepsilon\right) \times(0,1)$ by $\hat{w}$ in $(0, \varepsilon) \times(0,1)$ and by $\widetilde{w}$ in $\left(\varepsilon, T_{o p t}+\varepsilon\right) \times(0,1)$. Set

$$
\mathbf{w}(t, x)=w(t-\tau, x) \text { in }\left(\tau, \tau+T_{\text {opt }}+\varepsilon\right) \times(0,1) .
$$

Then $\mathbf{w}$ is a solution starting from $\varphi$ at time $\tau$ and arriving at 0 at time $\tau+T_{\text {opt }}+\varepsilon$, i.e.,

$$
\left\{\begin{array}{c}
\partial_{t} \mathbf{w}(t, x)=\Sigma(x) \partial_{x} \mathbf{w}(t, x)+C(t, x) \mathbf{w}(t, x) \text { for }(t, x) \in\left(\tau, \tau+T_{\text {opt }}+\varepsilon\right) \times(0,1),  \tag{3.75}\\
\mathbf{w}_{-}(t, 0)=B \mathbf{w}_{+}(t, 0) \text { for } t \in\left(\tau, \tau+T_{\text {opt }}+\varepsilon\right) \\
\mathbf{w}(\tau, \cdot)=\varphi \text { and } \mathbf{w}\left(\tau+T_{\text {opt }}+\varepsilon, \cdot\right)=0 \text { in }[0,1] .
\end{array}\right.
$$

We have thus proved that one can steer $\varphi \in J(\tau, \varepsilon)$ at time $\tau$ to 0 at time $\tau+T_{\text {opt }}+\varepsilon$.
Conversely, let $\varphi \in H(\tau)$ be such that one can steer $\varphi$ at time $\tau$ to 0 at time $\tau+T_{\text {opt }}+\varepsilon$ using a control $W \in\left[L^{2}\left(\tau, \tau+T_{\text {opt }}+\varepsilon\right)\right]^{m}$. Let $\mathbf{w}$ be the corresponding solution, and set $w(t, x)=\mathbf{w}(t+\tau, x)$ in $\left(0, T_{\text {opt }}+\varepsilon\right) \times(0,1)$. Since $\mathbf{w}(\tau+\varepsilon, \cdot)$ is steered from time $\tau+\varepsilon$ to 0 at time $\tau+T_{\text {opt }}+\varepsilon$, it follows from the characterization of $H(\tau+\varepsilon)$ that

$$
\operatorname{Proj}_{H(\tau+\varepsilon)} \mathbf{w}(\tau+\varepsilon, \cdot)=0
$$

In other words,

$$
\operatorname{Proj}_{H(\tau+\varepsilon)} w(\varepsilon, \cdot)=0
$$

This yields that $\varphi \in J(\tau, \varepsilon)$.
We thus proved that $J(\tau, \varepsilon)$ is the space of (functions) states in $H(\tau)$ such that one can steer them from time $\tau$ to 0 at time $\tau+T_{\text {opt }}+\varepsilon$. The other conclusions of Lemma 3.7 are direct consequences of this fact and the details of the proof are omitted.

Concerning $A_{\tau, \varepsilon}$, we have
Lemma 3.8. Let $0<\varepsilon<\gamma_{0}$. Assume that $C$ is analytic in I. We have

$$
A_{\tau, \varepsilon} \text { is analytic in } I_{2} \text { except for a discret set, which is removable. }
$$

Recall that $A_{\tau, \varepsilon}$ is defined in (3.69).
Proof. Denote

$$
l=\max _{\substack{\tau \in I_{2} \\ H \text { is continuous at } \tau+\varepsilon}} \operatorname{dim} A_{\tau, \varepsilon}<+\infty .
$$

Fix $\tau_{0} \in I_{2}$ such that $\operatorname{dim} A_{\tau_{0}, \varepsilon}=l$ and fix $\xi_{1}, \cdots, \xi_{l} \in\left[L^{2}(0, \varepsilon)\right]^{m}$ such that

$$
\left\{\operatorname{Proj}_{H\left(\tau_{0}+\varepsilon\right)} \mathcal{T}_{\tau_{0}, \varepsilon}^{c}\left(\xi_{j}\right) ; 1 \leq j \leq l\right\} \text { is an orthogonal basis of } A_{\tau_{0}, \varepsilon}
$$

Since, for fixed $\varepsilon, \mathcal{T}_{,, \varepsilon}^{c}$ is analytic in $I_{2}$ and $H(\cdot+\varepsilon)$ is analytic in $I_{2}$ except for a discrete subset which is removable, it follows that

$$
\begin{equation*}
\operatorname{dim} \operatorname{span}\left\{\operatorname{Proj}_{H(\tau+\varepsilon)} \mathcal{T}_{\tau, \varepsilon}^{c}\left(\xi_{j}\right) ; 1 \leq j \leq l\right\}=l \text { in } I_{2} \text { except for a discrete subset. } \tag{3.76}
\end{equation*}
$$

This in turn implies, by the property of $l$,

$$
\begin{equation*}
A(\tau, \varepsilon)=\operatorname{span}\left\{\operatorname{Proj}_{H(\tau+\varepsilon)} \mathcal{T}_{\tau, \varepsilon}^{c}\left(\xi_{j}\right) ; 1 \leq j \leq l\right\} \text { in } I_{2} \text { except for a discrete subset. } \tag{3.77}
\end{equation*}
$$

Combining (3.76) and (3.77) yields the conclusion.
Let

$$
\begin{equation*}
M(\tau) \text { be the orthogonal complement of } J(\tau) \text { in } H(\tau) \tag{3.78}
\end{equation*}
$$

It is clear that for each $\tau \in I_{1}$, there exists some $\varepsilon_{\tau}>0$ such that one cannot steer any $\varphi \in$ $M(\tau) \backslash\{0\}$ at time $\tau$ to 0 at time $\tau+T_{\text {opt }}+\varepsilon_{\tau}$. The constant $\varepsilon_{\tau}$ can be chosen independently of $\varphi \in M(\tau) \backslash\{0\}$, for example, one can take $\varepsilon_{\tau}$ so that $J(\tau, \varepsilon)=J(\tau)$ for $0 \leq \varepsilon \leq \varepsilon_{\tau} / 2$. The analyticity of $C$ is not required for this purpose. Nevertheless, when the analyticity of $C$ in $I$ is imposed, one can obtain a uniform lower bound for $\varepsilon_{\tau}$ for $\tau \in I_{2}$ in a sense which will be precise now. The uniform lower bound of $\varepsilon_{\tau}$ will play a crucial role in our proof of Theorem 1.2, To establish this property, for $0<\varepsilon<\gamma_{0}$ and $\tau \in I_{2}$, we first write $J(\tau, \varepsilon)$ under the form

$$
\begin{equation*}
J(\tau, \varepsilon)=\left\{\varphi \in H(\tau) ; \operatorname{Proj}_{A_{\tau, \varepsilon}} \operatorname{Proj}_{H(\tau+\varepsilon)} \mathcal{T}_{\tau, \varepsilon}^{I}(\varphi)-\operatorname{Proj}_{H(\tau+\varepsilon)} \mathcal{T}_{\tau, \varepsilon}^{I}(\varphi)=0\right\} \tag{3.79}
\end{equation*}
$$

Since the operator

$$
\operatorname{Proj}_{A,, \varepsilon} \operatorname{Proj}_{H(\cdot+\varepsilon)} \mathcal{T}_{\cdot, \varepsilon}^{I}-\operatorname{Proj}_{H(\cdot+\varepsilon)} \mathcal{T}_{\cdot, \varepsilon}^{I} \text { is analytic in } I_{2},
$$

except for a discret subset, which is removable,
one has, as in the proof of Lemma 3.6,

$$
J(\cdot, \varepsilon) \text { is analytic in } I_{2} \text { except for a discret set, which is removable. }
$$

We derive that for each $n \in \mathbb{N}$ with $1 / n<\gamma_{0}$, there exists a discrete subset $D_{n}$ of $I_{2}$ such that

$$
J(\tau, 1 / n) \text { is analytic in } I_{2} \text { except for a discrete set } D_{n} \text {, which is removable } 12 .
$$

As a consequence, one has

$$
\begin{equation*}
\operatorname{dim} J(\cdot, 1 / n) \text { is constant in } I_{2} \backslash D_{n} \tag{3.80}
\end{equation*}
$$

Set

$$
\begin{equation*}
D=\bigcup_{n \in \mathbb{N} ; 1 / n<\gamma_{0}} D_{n} \tag{3.81}
\end{equation*}
$$

and fix $\tau_{0} \in I_{2} \backslash D$. There exists $0<\varepsilon_{0}<\gamma_{0}$ such that

$$
J\left(\varepsilon, \tau_{0}\right)=J\left(\tau_{0}\right) \text { for } 0<\varepsilon<\varepsilon_{0}
$$

It follows from Lemma 3.7 and (3.80) that, for $0<\varepsilon<\varepsilon_{0}$ and $\tau, \tau^{\prime} \in I_{2} \backslash D$, one has

$$
\begin{equation*}
J(\tau, \varepsilon)=J(\tau) \quad \text { and } \quad \operatorname{dim} J(\tau)=\operatorname{dim} J\left(\tau^{\prime}\right) \tag{3.82}
\end{equation*}
$$

We thus proved
Lemma 3.9. There exists a discrete set $D{ }^{13}$ and $0<\varepsilon_{0}<\gamma_{0}$ such that

$$
\operatorname{dim} M(\tau)=\operatorname{dim} M\left(\tau^{\prime}\right) \text { for } \tau, \tau^{\prime} \in I_{2} \backslash D
$$

and one cannot steer any $v \in M(\tau) \backslash\{0\}$ from time $\tau$ to 0 at time $\tau+T_{\text {opt }}+\varepsilon_{0}$ for $\tau \in I_{2} \backslash D$.
We now summarize the results which have been derived in this section:
Proposition 3.2. There exist an orthogonal decomposition of $H(\tau)$ via $H(\tau)=J(\tau) \otimes M(\tau)$ for $\tau \in I_{1}$, a discrete subset $D$ of $I_{2}$, and a constant $\varepsilon_{0}>0$ such that the following four properties hold:

[^9]i) For $\varphi \in J(\tau)$, one can steer $v$ at time $\tau$ to 0 at time $\tau+T_{o p t}+\delta$ for all $\delta>0$.
ii) For $\varphi \in M(\tau) \backslash\{0\}$, there exists $\varepsilon_{\tau}>0$ such that one cannot steer $\varphi$ at time $\tau$ to 0 at time $\tau+T_{o p t}+\delta$ for $0<\delta<\varepsilon_{\tau}$.
iii)
$$
\operatorname{dim} M(\tau)=\operatorname{dim} M\left(\tau^{\prime}\right) \text { for } \tau, \tau^{\prime} \in I_{2} \backslash D
$$
iv) For $\tau \in I_{2} \backslash D$, and $\varphi \in M(\tau) \backslash\{0\}$, one cannot steer $\varphi$ at time $\tau$ to 0 at time $\tau+T_{o p t}+\varepsilon_{0}$.

Proposition 3.2 also gives the characterization of states which can be steered at time $\tau$ to 0 at time $\tau+T_{o p t}+\delta$ for all $\delta>0$. Indeed, one has, for $\tau \in I_{1}$,

- For $v \in H(\tau) \cup J(\tau)$, one can steer $v$ at time $\tau$ to 0 at time $\tau+T_{o p t}+\delta$ for all $\delta>0$.
- For $v \in M(\tau) \backslash\{0\}$, there exists $\varepsilon_{\tau}>0$ such that one cannot steer $v$ at time $\tau$ to 0 at time $\tau+T_{o p t}+\delta$ for $0<\delta<\varepsilon_{\tau}$.
3.4. Null-controllability in time $T_{o p t,+}$ - Proof of Theorem $\mathbf{1 . 2}$. We first assume that $0 \notin D$. We will prove that $M(0)=\{0\}$ by contradiction, and the conclusion follows from Proposition 3.2, Assume that there exists $\varphi \in M(0) \backslash\{0\}$. Since $M(0) \subset H\left(0, T_{o p t}+\varepsilon_{0}\right)$ by assertion iv) of Proposition 3.2, it follows from Lemma 3.5 that there exists a solution $v^{(0)}$ of the system

$$
\begin{equation*}
\partial_{t} v^{(0)}(t, x)=\Sigma(x) \partial_{x} v^{(0)}(t, x)+\mathbf{C}(t, x) v^{(0)}(t, x) \text { for }(t, x) \in\left(0, T_{o p t}+\varepsilon_{0}\right) \times(0,1) \tag{3.83}
\end{equation*}
$$

with, for $t \in\left(0, T_{o p t}+\varepsilon_{0}\right)$,

$$
\begin{gather*}
v^{(0)}(t, 1)=0  \tag{3.84}\\
\Sigma_{+}(0) v_{+}^{(0)}(t, 0)=-B^{\top} \Sigma_{-}(0) v_{-}^{(0)}(t, 0)  \tag{3.85}\\
v^{(0)}(t=0, \cdot)=\varphi \text { in }(0,1) \tag{3.86}
\end{gather*}
$$

Fix $t_{1} \in\left(\varepsilon_{0} / 3, \varepsilon_{0} / 2\right) \backslash D$ (recall that $D$ is discrete). By Lemma 3.5, one has

$$
v^{(0)}\left(t_{1}, \cdot\right) \in H\left(t_{1}, T_{o p t}+\varepsilon_{0}-t_{1}\right)
$$

This in turn implies that, since $H\left(t_{1}, T_{o p t}+\varepsilon_{0}-t_{1}\right)=M\left(t_{1}\right)=H\left(t_{1}, T_{o p t}+\varepsilon_{0}\right)$ by assertion iv) of Proposition 3.2.

$$
v^{(0)}\left(t_{1}, \cdot\right) \in H\left(t_{1}, T_{o p t}+\varepsilon_{0}\right)
$$

By Lemma 3.5 again, there exists a solution $v^{(1)}$ of the system

$$
\begin{equation*}
\partial_{t} v^{(1)}(t, x)=\Sigma(x) \partial_{x} v^{(1)}(t, x)+\mathbf{C}(t, x) v^{(1)}(t, x) \text { for }(t, x) \in\left(t_{1}, t_{1}+T_{o p t}+\varepsilon_{0}\right) \times(0,1) \tag{3.87}
\end{equation*}
$$ with, for $t \in\left(t_{1}, t_{1}+T_{o p t}+\varepsilon_{0}\right)$,

$$
\begin{gather*}
v^{(1)}(t, 1)=0  \tag{3.88}\\
\Sigma_{+}(0) v_{+}^{(1)}(t, 0)=-B^{\top} \Sigma_{-}(0) v_{-}^{(1)}(t, 0)  \tag{3.89}\\
v^{(1)}\left(t=t_{1}, \cdot\right)=v^{(0)}\left(t_{1}, \cdot\right) \text { in }(0,1) \tag{3.90}
\end{gather*}
$$

Consider the solution $v$ of system (3.6)-(3.8) for the time interval $\left(0, t_{1}+T_{o p t}+\varepsilon_{0}\right)$ with $v\left(t_{1}+\right.$ $\left.T_{o p t}+\varepsilon_{0}, \cdot\right)=v^{(1)}\left(t_{1}+T_{o p t}+\varepsilon_{0}, \cdot\right)$ (backward system). One can check that

$$
v(t, \cdot)=v^{(1)}(t, \cdot) \text { for } t \in\left(t_{1}, t_{1}+T_{o p t}+\varepsilon_{0}\right)
$$

and, since $v^{(1)}\left(t_{1}, \cdot\right)=v^{(0)}\left(t_{1}, \cdot\right)$,

$$
v(t, \cdot)=v^{(0)}(t, \cdot) \text { for } t \in\left(0, t_{1}\right)
$$

For notational ease, we will denote this $v$ by $v^{(1)}$. We thus proved that there exists a solution $v^{(1)}$ of (3.6)-(3.8) such that

$$
v^{(1)}(\cdot, 1)=0 \text { in }\left(0, t_{1}+T_{o p t}+\varepsilon_{0}\right),
$$

and

$$
v^{(1)}(0, \cdot)=\varphi \text { in }(0,1)
$$

Continuing this process, there exist $0=t_{0}<t_{1}<\cdots<t_{N-1} \leq T_{1}-T_{\text {opt }}<t_{N}<\beta-T_{\text {opt }}$ and a family of $v^{(\ell)}$ with $1 \leq \ell \leq N$ such that $t_{\ell} \in I \backslash D$,

$$
\begin{equation*}
\partial_{t} v^{(\ell)}(t, x)=\Sigma(x) \partial_{x} v^{(\ell)}(t, x)+\mathbf{C}(t, x) v^{(\ell)}(t, x) \text { for }(t, x) \in\left(0, t_{\ell}+T_{o p t}+\varepsilon_{0}\right) \times(0,1) \tag{3.91}
\end{equation*}
$$

with, for $t \in\left(0, t_{\ell}+T_{\text {opt }}+\varepsilon_{0}\right)$,

$$
\begin{gather*}
v^{(\ell)}(t, 1)=0  \tag{3.92}\\
\Sigma_{+}(0) v_{+}^{(\ell)}(t, 0)=-B^{\top} \Sigma_{-}(0) v_{-}^{(\ell)}(t, 0),  \tag{3.93}\\
v^{(\ell)}(t=0, \cdot)=\varphi(\cdot) \text { in }(0,1), \tag{3.94}
\end{gather*}
$$

and

$$
\varepsilon_{0} / 3 \leq t_{\ell}-t_{\ell-1} \leq \varepsilon_{0} / 2
$$

This implies, by Lemma [3.5, that one cannot steer $\varphi$ from time 0 to 0 at time $T_{1}$. We have a contradiction since the system is null-controllable at the time $T_{1}$. The conclusion follows in the case $0 \in I_{2} \backslash D$.

The proof in the general case can be derived from the previous case by noting that, using the same arguments, one has

$$
M\left(\tau_{0}\right)=\{0\} \text { for } \tau_{0} \in I_{2} \backslash D \text { and } \tau_{0} \text { is close to } 0
$$

The details are omitted.
The proof is complete.
The proof of Theorem 1.2 also yields the following unique continuation principle:
Proposition 3.3. Let $k \geq m \geq 1$ and let $B \in \mathcal{B}$ be such that (1.10) holds for $i=m$. Assume that $C_{1} \in \mathcal{H}\left(I,\left[L^{\infty}(0,1)\right]^{n \times n}\right)$. Let $\tau \in I$ and $T>T_{\text {opt }}$. Assume that $\tau+T_{1} \in I$. Let $v$ be a solution of system

$$
\begin{equation*}
\partial_{t} v(t, x)=\Sigma(x) \partial_{x} v(t, x)+C_{1}(t, x) v(t, x) \text { for }(t, x) \in(\tau, \tau+T) \times(0,1) \tag{3.95}
\end{equation*}
$$

with, for $\tau<t<\tau+T$,

$$
\begin{gather*}
v_{-}(t, 1)=0  \tag{3.96}\\
\Sigma_{+}(0) v_{+}(t, 0)=-B^{\top} \Sigma_{-}(0) v_{-}(t, 0),  \tag{3.97}\\
v_{+}(t, 1)=0 . \tag{3.98}
\end{gather*}
$$

Then $v=0$.
Recall that $T_{1}=\tau_{k}+\tau_{k+1}$, see (1.15).
Proof. The conclusion of (3.3) follows from the proof of Theorem 1.1 applied to $C(t, x)$ defined by $\Sigma^{\prime}(x)-C(t, x)^{\top}=C_{1}(t, x)$.

The unique continuation result stated in Proposition 3.3 can be seen as a variant of the unique continuation principle for the wave equations whose first and zero-order terms are analytic in time due to Tataru-Hörmander-Robbiano-Zuily. Our strategy was mentioned at the beginning of Section 3. We do not know if such a unique continuation principle can be proved using Carleman's estimate as in the wave setting. It is worth noting that if this is possible then the analyticity of $C_{1}$ in time must be taken into account by Theorem [1.1. More importantly the conditions $B \in \mathcal{B}$ and (1.10) holding for $i=m$ have to be essentially used in the proof process since it is known that the unique continuation does not hold without this assumption even in the case $C_{1} \equiv 0$. The advantage of Carleman's estimate might be that the analyticity of $C_{1}$ is only required for a neighborhood of $\left[0, T_{\text {opt }}\right]$ instead of $\left[0, T_{1}\right]$.

Remark 3.4. It is natural to compare the direct approach here with the one involving the backstepping technique. In the time-invariant setting, both approaches yield the same result since (1.10) with $i=m$ is not imposed to establish the compactness of $\mathcal{K}(\tau)$ (see Step 1 of the proof of Proposition 3.1). Nevertheless, (equivalent) control-forms obtained from the backstepping approach are easier to handle/understand. The analysis in this paper is strongly inspired/guided by such control-forms. In the time-varying setting, one might derive the same conclusion under the assumption that $C$ is analytic in $\mathbb{R}$ and its holomorphic extension in $\{z \in \mathbb{C} ;|\Im(z)|<\gamma\}$ is bounded for some $\gamma>0$. This quite strong assumption on the analyticity of $C$ comes from the construction of the kernel in the step of using backstepping and might not be necessary.

## 4. Exact controllability in the analytic setting - Proof of Theorem 1.3

Theorem 1.3 can be derived from Theorem [1.2, as in the proof of [20, Theorem 3]. For the convenience of the reader, we reproduce the proof.

We first consider the case $m=k$. Let $T>T_{\text {opt }}$ be such that $T \in I$. Set

$$
\widetilde{w}(t, x)=w(T-t, x) \text { for } t \in(0, T), x \in(0,1)
$$

Then

$$
\widetilde{w}_{-}(t, 0)=\widetilde{B}^{-1} \widetilde{w}_{+}(t, 0),
$$

with $\widetilde{w}_{-}(t, \cdot)=\left(w_{2 k}, \ldots, w_{k+1}\right)^{\top}(T-t, \cdot)$, and $\widetilde{w}_{+}(t, \cdot)=\left(w_{k}, \ldots, w_{1}\right)^{\top}(T-t, \cdot)$, and $\widetilde{B}_{i j}=B_{p q}$ with $p=k-i$ and $q=k-j$. Note that the $i \times i$ matrix formed from the first $i$ columns and rows of $\widetilde{B}$ is invertible. Using the Gaussian elimination method, one can find $(k \times k)$ matrices $T_{1}, \ldots, T_{N}$ such that

$$
T_{N} \ldots T_{1} \widetilde{B}=U
$$

where $U$ is a $(k \times k)$ upper triangular matrix, and $T_{i}(1 \leq i \leq N)$ is the matrix given by the operation which replaces a row $p$ by itself plus a multiple of a row $q$ for some $1 \leq q<p \leq N$. It follows that

$$
\widetilde{B}^{-1}=U^{-1} T_{N} \ldots T_{1} .
$$

One can check that $U^{-1}$ is an invertible, upper triangular matrix, and $T_{N} \ldots T_{1}$ is an invertible, lower triangular matrix. It follows that the $i \times i$ matrix formed from the last $i$ columns and rows of $\widetilde{B}^{-1}$ is the product of the matrix formed from the last $i$ columns and rows of $U^{-1}$ and the matrix formed from the last $i$ columns and rows of $T_{N} \ldots T_{1}$. Therefore, $\widetilde{B}^{-1} \in \mathcal{B}$. One can also check that the exact controllability of the system for $w(\cdot, \cdot)$ at the time $T$ from time 0 is equivalent to the null-controllability of the system for $\widetilde{w}(\cdot, \cdot)$ at the same time from time 0 . The conclusion of Theorem 1.3 now follows from Theorem 1.2 by noting that $C(\cdot-T, \cdot)$ is analytic in a neighborhood of $\left[0, T_{1}\right]$.

The case $m>k$ can be obtained from the case $m=k$ as follows. Consider $\hat{w}(\cdot, \cdot)$ the solution of the system

$$
\partial_{t} \hat{w}(t, x)=\hat{\Sigma}(x) \partial_{x} \hat{w}(t, x)+\hat{C}(t, x) \hat{w}(t, x),
$$

$$
\hat{w}_{-}(t, 0)=\hat{B} \hat{w}_{+}(t, 0), \quad \text { and } \quad \hat{w}_{+}(t, 1) \text { are controls. }
$$

Here

$$
\hat{\Sigma}=\operatorname{diag}\left(-\hat{\lambda}_{1}, \ldots,-\hat{\lambda}_{m}, \hat{\lambda}_{m+1}, \ldots \hat{\lambda}_{2 m}\right)
$$

with $\hat{\lambda}_{j}=-(1+m-k-j) \varepsilon^{-1}$ for $1 \leq j \leq m-k$ with positive small $\varepsilon, \hat{\lambda}_{j}=\lambda_{j-(m-k)}$ if $m-k+1 \leq j \leq m$, and $\hat{\lambda}_{j+m}=\lambda_{j+k}$ for $1 \leq j \leq m$,

$$
\hat{C}(t, x)=\left(\begin{array}{cc}
0_{m-k, m-k} & 0_{m-k, n} \\
0_{n, m-k} & C(t, x)
\end{array}\right)
$$

and

$$
\hat{B}=\left(\begin{array}{cc}
I_{m-k} & 0_{m-k, m} \\
0_{m-k, m} & B
\end{array}\right)
$$

where $I_{\ell}$ denotes the identity matrix of size $\ell \times \ell$ for $\ell \geq 1$. Here $0_{i, j}$ denotes the zero matrix of size $i \times j$ for $i, j, \ell \geq 1$. Then the exact controllability of $w$ at the time $T$ from time 0 can be derived from the exact controllability of $\hat{w}$ at the same time from time 0 . One then can deduce the conclusion of Theorem 1.3 from the case $m=k$ using Theorem 1.2 by noting that the optimal time for the system of $\hat{w}$ converges to the optimal time for the system of $w$ as $\varepsilon \rightarrow 0_{+}$.

## Appendix A. Hyperbolic systems in non-rectangle domains

In this section, we give the meaning of broad solutions used to define $\mathcal{T}(\tau)$ and $\hat{\mathcal{T}}(\tau)$ and study their well-posedness. We also establish the boundedness and the analyticity of $\mathcal{T}(\tau)$ under appropriate assumptions. The key point of the analysis is to find suitable weighted norms in order to apply the fixed point arguments. This matter is subtle (see Remark A.3). In this section, we assume that $k \geq m \geq 1$ although the arguments are quite robust and also work for the case $m>k \geq 1$ under appropriate modifications.

Let $F \in\left[L^{\infty}(\Omega)\right]^{n \times n},(f, g) \in\left[L^{2}\left(0, T_{\text {opt }}\right)\right]^{n} \times\left[L^{2}(0,1)\right]^{m}$, and $\gamma \in\left[L^{2}(\Omega)\right]^{n}$. We first deal with the following system, which is slightly more general than the system (3.27)-(3.32):

$$
\begin{gather*}
\partial_{t} w(t, x)=\Sigma(x) \partial_{x} w(t, x)+F(t, x) w(t, x)+\gamma(t, x) \text { for }(t, x) \in \Omega,  \tag{A.1}\\
w(\cdot, 1)=f \text { in }\left(0, T_{\text {opt }}\right),  \tag{A.2}\\
w_{+}(0, \cdot)=g \text { in }(0,1),  \tag{A.3}\\
w_{-, \geq k}(t, 0)=Q_{k} w_{<k, \geq k+m}(t, 0) \text { for } t \in\left(T_{\text {opt }}-\tau_{k}, T_{\text {opt }}-\tau_{k-1}\right),  \tag{A.4}\\
w_{-, \geq k-1}(t, 0)=Q_{k-1} w_{<k-1, \geq k+m-1}(t, 0) \text { for } t \in\left(T_{o p t}-\tau_{k-1}, T_{o p t}-\tau_{k-2}\right),  \tag{A.5}\\
w_{-, \geq k-m+2}(t, 0)=Q_{k-m+2} w_{<k-m+2, \geq k+2}(t, 0) \text { for } t \in\left(T_{\text {opt }}-\tau_{k-m+2}, T_{\text {opt }}-\tau_{k-m+1}\right) . \tag{A.6}
\end{gather*}
$$

Given a subset $O$ of $\mathbb{R}^{2}$ and a point $(t, x) \in \mathbb{R}^{2}$, we denote

$$
O_{t}=\{y \in \mathbb{R} ;(t, y) \in O\} \quad \text { and } \quad O_{x}=\{s \in \mathbb{R} ;(s, x) \in O\} .
$$

We next give the definition of the broad solutions of system (A.1)- (A.6).

Definition A.1. Let $F \in\left[L^{\infty}(\Omega)\right]^{n \times n},(f, g) \in\left[L^{2}\left(0, T_{\text {opt }}\right)\right]^{n} \times\left[L^{2}(0,1)\right]^{m}$, and $\gamma \in\left[L^{2}(\Omega)\right]^{n}$. A vector-valued function $w \in \mathcal{Y}:=\left[L^{2}(\Omega)\right]^{n} \cap C\left(\left[0, T_{\text {opt }}\right] ;\left[L^{2}\left(\Omega_{t}\right)\right]^{n}\right) \cap C\left([0,1] ;\left[L^{2}\left(\Omega_{x}\right)\right]^{n}\right){ }^{14}$ is called a broad solution of (A.1)-(A.6) if for almost $\left(t_{1}, \xi_{1}\right) \in \Omega$, the following conditions hold

1. for $1 \leq j \leq k-m+1$,

$$
\begin{equation*}
w_{j}\left(t_{1}, \xi_{1}\right)=\int_{t}^{t_{1}}\left(F\left(s, x_{j}\left(s, t_{1}, \xi_{1}\right)\right) w\left(s, x_{j}\left(s, t_{1}, \xi_{1}\right)\right)\right)_{j} d s+\int_{t}^{t_{1}} \gamma_{j}\left(s, x_{j}\left(s, t_{1}, \xi_{1}\right)\right) d s+f_{j}(t) \tag{A.7}
\end{equation*}
$$

where $t$ is such that $x_{j}\left(t, t_{1}, \xi_{1}\right)=1$;
2. for $k-m+2 \leq j \leq k$,

$$
\begin{equation*}
w_{j}\left(t_{1}, \xi_{1}\right)=\int_{t}^{t_{1}}\left(F\left(s, x_{j}\left(s, t_{1}, \xi_{1}\right)\right) w\left(s, x_{j}\left(s, t_{1}, \xi_{1}\right)\right)\right)_{j} d s+\int_{t}^{t_{1}} \gamma_{j}\left(s, x_{j}\left(s, t_{1}, \xi_{1}\right)\right) d s+f_{j}(t), \tag{A.8}
\end{equation*}
$$ if $t \in\left(0, T_{\text {opt }}\right)$ where $t$ is such that $x_{j}\left(t, t_{1}, \xi_{1}\right)=1$, otherwise,

$$
\begin{align*}
w_{j}\left(t_{1}, \xi_{1}\right)=\int_{\hat{t}}^{t_{1}}\left(F\left(s, x_{j}\left(s, t_{1}, \xi_{1}\right)\right)\right. & \left.w\left(s, x_{j}\left(s, t_{1}, \xi_{1}\right)\right)\right)_{j} d s  \tag{A.9}\\
& +\int_{\hat{t}}^{t_{1}} \gamma_{j}\left(s, x_{j}\left(s, t_{1}, \xi_{1}\right)\right) d s+\left(Q_{l} w_{<l, \geq l+m}(\hat{t}, 0)\right)_{j-l+1}
\end{align*}
$$

if $\hat{t} \in\left(T_{\text {opt }}-\tau_{l}, T_{\text {opt }}-\tau_{l-1}\right)$ where $\hat{t}$ is such that $x_{j}\left(\hat{t}, t_{1}, \xi_{1}\right)=0$.
3. for $k+1 \leq j \leq k+m$,

$$
\begin{equation*}
w_{j}\left(t_{1}, \xi_{1}\right)=\int_{t}^{t_{1}}\left(F\left(s, x_{j}\left(s, t_{1}, \xi_{1}\right)\right) w\left(s, x_{j}\left(s, t_{1}, \xi_{1}\right)\right)\right)_{j} d s+\int_{t}^{t_{1}} \gamma_{j}\left(s, x_{j}\left(s, t_{1}, \xi_{1}\right)\right) d s+f_{j}(t) \tag{A.10}
\end{equation*}
$$

if $t \in\left(0, T_{\text {opt }}\right)$ where $t$ is such that $x_{j}\left(t, t_{1}, \xi_{1}\right)=1$, otherwise
(A.11)

$$
w_{j}\left(t_{1}, \xi_{1}\right)=\int_{0}^{t_{1}}\left(F\left(s, x_{j}\left(s, t_{1}, \xi_{1}\right)\right) w\left(s, x_{j}\left(s, t_{1}, \xi_{1}\right)\right)\right)_{j} d s+\int_{0}^{t_{1}} \gamma_{j}\left(s, x_{j}\left(s, t_{1}, \xi_{1}\right)\right) d s+g_{j-k}(\eta)
$$

where $\eta \in(0,1)$ is such that $x_{j}\left(0, t_{1}, \xi_{1}\right)=\eta$.
Recall that the characteristic flow $x_{j}$ with $1 \leq j \leq k+m$ is defined in (3.17) and (3.18).
In this definition, the term $Q_{l} w_{<l,>l+m}(\hat{t}, 0)$ in ( $\left.\mathbf{A} .9\right)$ is required to be replaced by the corresponding expression in the RHS of (A.7), or (A.8), or (A.10), or (A.11) with $(\hat{t}, 0)$ standing for $\left(t_{1}, \xi_{1}\right)$.

The well-posedness of broad solutions of (A.1)-(A.6) is given in the following.
Theorem A.1. Let $F \in\left[L^{\infty}(\Omega)\right]^{n \times n},(f, g) \in\left[L^{2}\left(0, T_{\text {opt }}\right)\right]^{n} \times\left[L^{2}(0,1)\right]^{m}$, and $\gamma \in\left[L^{2}(\Omega)\right]^{n}$. There exists a unique broad solution $w \in \mathcal{Y}$ of (A.1)-(A.6). Moreover,

$$
\begin{equation*}
\|w\|_{\mathcal{Y}} \leq C\left(\|f\|_{L^{2}\left(0, T_{o p t}\right)}+\|g\|_{L^{2}(0,1)}+\|\gamma\|_{L^{2}(\Omega)}\right) \tag{A.12}
\end{equation*}
$$

for some positive constant $C$ depending on an upper bound of $\|F\|_{L^{\infty}(\Omega)}$ and $\Sigma$.
Here we denote

$$
\|w\|_{\mathcal{Y}}=\max \left\{\sup _{x \in[0,1]}\|w\|_{L^{2}\left(\Omega_{x}\right)}, \sup _{t \in\left[0, T_{\text {opt }}\right]}\|w\|_{L^{2}\left(\Omega_{t}\right)} ; 1 \leq i \leq n\right\} .
$$

[^10]

Figure 4. Geometry of $\Omega_{\ell}$ and $\Gamma_{\ell}$ with $k-m+1 \leq \ell \leq k$ for a constant $\Sigma$.
Remark A.1. The analysis of Theorem A.1 can be easily extended to cover the case where source terms in $L^{2}$ are added in (A.4)-(A.6).

Before giving the proof of Theorem A.1, let us introduce some notations. For $k-m+1 \leq \ell \leq$ $k-1$, let $\Omega_{\ell}$ be the region of $\Omega$ between the characteristic curves of $x_{\ell}$ and $x_{\ell+1}$ both passing the point ( $T_{\text {opt }}, 1$ ) in the $x t$-plane. We also denote $\Omega_{k}$ the region of $\Omega$ below the characteristic curve of $x_{k}$ passing the point $\left(T_{o p t}, 1\right)$ in the $x t$-plane. Let $\Gamma_{\ell}$ with $k-m+1 \leq \ell \leq k$ be the boundary part of $\Omega_{\ell}$ formed by the characteristic curve of $x_{\ell}$ passing the point $\left(T_{o p t}, 1\right)$. See Figure 4 ,

The proof of Theorem A. 1 is based on two lemmas below. The first one is the following.
Lemma A.1. Let $F \in\left[L^{\infty}\left(\Omega_{k}\right)\right]^{n \times n},(f, g) \in\left[L^{2}\left(0, T_{o p t}\right)\right]^{n} \times\left[L^{2}(0,1)\right]^{m}$, and $\gamma \in\left[L^{2}\left(\Omega_{k}\right)\right]^{n}$. There exists a unique board solution $w \in \mathcal{Y}_{k}:=\left[L^{2}\left(\Omega_{k}\right)\right]^{n} \cap C\left(\left[0, T_{o p t}\right] ;\left[L^{2}\left(\Omega_{k, t}\right)\right]^{n}\right) \cap C\left([0,1] ;\left[L^{2}\left(\Omega_{k, x}\right)\right]^{n}\right)$ of the system

$$
\begin{gather*}
\partial_{t} w(t, x)=\Sigma(x) \partial_{x} w(t, x)+F(t, x) w(t, x)+\gamma(t, x) \text { for }(t, x) \in \Omega_{k},  \tag{A.13}\\
w(\cdot, 1)=f \text { in }\left(0, T_{\text {opt }}\right),  \tag{A.14}\\
w_{+}(0, \cdot)=g \text { in }(0,1) . \tag{A.15}
\end{gather*}
$$

Moreover,

$$
\begin{equation*}
\|w\|_{y_{k}} \leq C\left(\|f\|_{L^{2}\left(0, T_{o p t}\right)}+\|g\|_{L^{2}(0,1)}+\|\gamma\|_{L^{2}\left(\Omega_{k}\right)}\right) \tag{A.16}
\end{equation*}
$$

for some positive constant $C$ depending only on an upper bound of $\|F\|_{L^{\infty}\left(\Omega_{k}\right)}$ and $\Sigma$.
Here we denote

$$
\|w\|_{\mathcal{Y}_{k}}=\max \left\{\sup _{x \in[0,1]}\|w\|_{L^{2}\left(\Omega_{k, x}\right)}, \sup _{t \in\left[0, T_{o p t}\right]}\|w\|_{L^{2}\left(\Omega_{k, t}\right)} ; 1 \leq i \leq n\right\} .
$$

The broad solutions considered in Lemma A. 1 are defined similarly as the one of Definition A. 1 as follows:

Definition A.2. Let $F \in\left[L^{\infty}\left(\Omega_{k}\right)\right]^{n \times n}$, and $(f, g) \in\left[L^{2}\left(0, T_{\text {opt }}\right)\right]^{n} \times\left[L^{2}(0,1)\right]^{m}$, and $\gamma \in\left[L^{2}\left(\Omega_{k}\right)\right]^{n}$. A vector-valued function $w \in \mathcal{Y}_{k}$ is called a broad solution of (A.13)-(A.15) if for almost $\left(t_{1}, \xi_{1}\right) \in$ $\Omega_{k}$, the following conditions hold

1. for $1 \leq j \leq k$,

$$
\begin{equation*}
w_{j}\left(t_{1}, \xi_{1}\right)=\int_{t}^{t_{1}}\left(F\left(s, x_{j}\left(s, t_{1}, \xi_{1}\right)\right) w\left(s, x_{j}\left(s, t_{1}, \xi_{1}\right)\right)\right)_{j} d s+\int_{t}^{t_{1}} \gamma_{j}\left(s, x_{j}\left(s, t_{1}, \xi_{1}\right)\right) d s+f_{j}(t) \tag{A.17}
\end{equation*}
$$

where $t$ is such that $x_{j}\left(t, t_{1}, \xi_{1}\right)=1$;
2. for $k+1 \leq j \leq k+m$,

$$
\begin{equation*}
w_{j}\left(t_{1}, \xi_{1}\right)=\int_{t}^{t_{1}}\left(F\left(s, x_{j}\left(s, t_{1}, \xi_{1}\right)\right) w\left(s, x_{j}\left(s, t_{1}, \xi_{1}\right)\right)\right)_{j} d s+\int_{t}^{t_{1}} \gamma_{j}\left(s, x_{j}\left(s, t_{1}, \xi_{1}\right)\right) d s+f_{j}(t) \tag{A.18}
\end{equation*}
$$

if $t \in\left(0, T_{\text {opt }}\right)$ where $t$ is such that $x_{j}\left(t, t_{1}, \xi_{1}\right)=1$, otherwise

$$
\begin{equation*}
w_{j}\left(t_{1}, \xi_{1}\right)=\int_{0}^{t_{1}}\left(F\left(s, x_{j}\left(s, t_{1}, \xi_{1}\right)\right) w\left(s, x_{j}\left(s, t_{1}, \xi_{1}\right)\right)\right)_{j} d s+\int_{0}^{t_{1}} \gamma_{j}\left(s, x_{j}\left(s, t_{1}, \xi_{1}\right)\right) d s+g_{j-k}(\eta) \tag{A.19}
\end{equation*}
$$

where $\eta \in(0,1)$ is such that $x_{j}\left(0, t_{1}, \xi_{1}\right)=\eta$.
Proof of Lemma A.1. For $v \in\left[L^{2}\left(\Omega_{k}\right)\right]^{n}$, set

$$
T_{k}(v)(t, x)=e^{L x} v(t, x) \text { for }(t, x) \in \Omega_{k},
$$

where $L$ is a large positive constant determined later.
We now introduce

$$
\|v\|_{\Omega_{k}}:=\max \left\{\sup _{x \in[0,1]}\left\|\left(T_{k} v\right)_{i}\right\|_{L^{2}\left(\Omega_{k, x}\right)}, \sup _{t \in\left[0, T_{\text {opt }}\right]}\left\|(T v)_{i}\right\|_{L^{2}\left(\Omega_{k, t}\right)} ; 1 \leq i \leq n\right\}
$$

One can check that $\mathcal{Y}_{k}$ equipped with the norm $\|\cdot\|_{\Omega_{k}}$ is a Banach space. It is also clear that $\|\cdot\|_{\Omega_{k}}$ is equivalent to $\|\cdot\|_{\mathcal{Y}_{k}}$.

The proof is now based on a fixed point argument. To this end, define $\mathcal{F}_{k}$ from $\mathcal{Y}_{k}$ into itself as follows: for $v \in \mathcal{Y}_{k}$, and for $\left(t_{1}, \xi_{1}\right) \in \Omega_{k}$ and $1 \leq j \leq k+m$ :

$$
\begin{equation*}
\left(\mathcal{F}_{k}(v)\right)_{j}\left(t_{1}, \xi_{1}\right) \text { is the RHS of (A.17), or (A.18), or (A.19) } \tag{A.20}
\end{equation*}
$$ under the corresponding conditions.

We claim that, for $L$ large enough, $\mathcal{F}_{k}$ is a contraction mapping from $\mathcal{Y}_{k}$ equipped with the norm $\|\cdot\|_{\Omega_{k}}$ into itself; and the conclusion follows then.

For $v \in \mathcal{Y}_{k}$, one can check that $\mathcal{F}(v) \in \mathcal{Y}_{k}$.
Let $v, w \in \mathcal{Y}_{k}$ be arbitrary. Fix $\xi_{1} \in[0,1]$. Let $1 \leq j \leq k$. We have for $\left(t_{1}, \xi_{1}\right) \in \Omega_{k}$, by (A.17),

$$
\begin{equation*}
\mathcal{F}(v)_{j}\left(t_{1}, \xi_{1}\right)-\mathcal{F}(w)_{j}\left(t_{1}, \xi_{1}\right)=\int_{t}^{t_{1}}\left(F\left(s, x_{j}\left(s, t_{1}, \xi_{1}\right)\right)(v-w)\left(s, x_{j}\left(s, t_{1}, \xi_{1}\right)\right)\right)_{j} d s \tag{A.21}
\end{equation*}
$$

where $t=t\left(t_{1}, \xi_{1}\right)$ is such that $x_{j}\left(t, t_{1}, \xi_{1}\right)=1$. This implies

$$
\begin{aligned}
& \int_{\Omega_{k, \xi_{1}}} e^{2 L \xi_{1}}\left|\mathcal{F}(v)_{j}\left(t_{1}, \xi_{1}\right)-\mathcal{F}(w)_{j}\left(t_{1}, \xi_{1}\right)\right|^{2} d t_{1} \\
& \\
& \leq C \int_{\Omega_{k, \xi_{1}}} \operatorname{sign}\left(t-t_{1}\right) \int_{t_{1}}^{t} e^{2 L \xi_{1}}|v-w|^{2}\left(s, x_{j}\left(s, t_{1}, \xi_{1}\right)\right) d s d t_{1}
\end{aligned}
$$

where $\operatorname{sign}(\theta)=1$ if $\theta>0$ and -1 if $\theta<0$. Here and in what follows in this proof, $C$ denotes a positive constant which depends only on an upper bound of $\|F\|_{L^{\infty}\left(\Omega_{k}\right)}$ and $\Sigma$, and can change from one place to another.

Since

$$
e^{2 L \xi_{1}}|v-w|^{2}\left(s, x_{j}\left(s, t_{1}, \xi_{1}\right)\right)=e^{2 L\left(\xi_{1}-x_{j}\left(s, t_{1}, \xi_{1}\right)\right)} e^{2 L x_{j}\left(s, t_{1}, \xi_{1}\right)}|v-w|^{2}\left(s, x_{j}\left(s, t_{1}, \xi_{1}\right)\right),
$$

and, for $s$ between $t_{1}$ and $t$,

$$
\xi_{1}-x_{j}\left(s, t_{1}, \xi_{1}\right) \leq 0
$$

by a change of variables $x=x_{j}\left(s, t_{1}, \xi_{1}\right) \sqrt{15}$, one obtains, for $1 \leq j \leq k$,

$$
\begin{align*}
\int_{\Omega_{k, \xi_{1}}} e^{2 L \xi_{1}} \mid \mathcal{F}(v)_{j}\left(t_{1}, \xi_{1}\right)- & \left.\mathcal{F}(w)_{j}\left(t_{1}, \xi_{1}\right)\right|^{2} d t_{1}  \tag{A.22}\\
& \leq C \int_{\Omega_{k} ; x \geq \xi_{1}} e^{2 L\left(\xi_{1}-x\right)} e^{2 L x}|v-w|^{2}(s, x) d s d x \leq \frac{C}{L}\|v-w\|_{\Omega_{k}}^{2}
\end{align*}
$$

We next consider $k+1 \leq j \leq k+m$. Using (A.18) and (A.19), similar to (A.22) for $1 \leq j \leq k$, we also reach (A.22) for $k+1 \leq j \leq k+m$. Combining this with (A.22) for $1 \leq j \leq k$ yields

$$
\begin{equation*}
\int_{\Omega_{k, \xi_{1}}} e^{2 L \xi_{1}}\left|\mathcal{F}(v)\left(t_{1}, \xi_{1}\right)-\mathcal{F}(w)\left(t_{1}, \xi_{1}\right)\right|^{2} d t_{1} \leq \frac{C}{L}\|v-w\|_{\Omega_{k}}^{2} \tag{A.23}
\end{equation*}
$$

Fix $t_{1} \in\left[0, T_{\text {opt }}\right]$. Let $1 \leq j \leq k$. From (A.21), we obtain, for $\left(t_{1}, \xi_{1}\right) \in \Omega_{k}$,

$$
\begin{aligned}
& \int_{\Omega_{k, t_{1}}} e^{2 L \xi_{1}}\left|\mathcal{F}(v)_{j}\left(t_{1}, \xi_{1}\right)-\mathcal{F}(w)_{j}\left(t_{1}, \xi_{1}\right)\right|^{2} d \xi_{1} \\
& \leq C \int_{\Omega_{k, t_{1}}} \operatorname{sign}\left(t-t_{1}\right) \int_{t_{1}}^{t} e^{2 L \xi_{1}}|v-w|^{2}\left(s, x_{j}\left(s, t_{1}, \xi_{1}\right)\right) d s d t_{1}
\end{aligned}
$$

Similar to (A.22), we obtain, for $1 \leq j \leq k$,

$$
\begin{align*}
& \int_{\Omega_{k, t_{1}}} e^{2 L \xi_{1}}\left|\mathcal{F}(v)_{j}\left(t_{1}, \xi_{1}\right)-\mathcal{F}(w)_{j}\left(t_{1}, \xi_{1}\right)\right|^{2} d \xi_{1}  \tag{A.24}\\
& \leq C \int_{\Omega_{k} ; x \geq \xi_{1}} e^{2 L\left(\xi_{1}-x\right)} e^{2 L x}|v-w|^{2}(s, x) d s d t_{1} \leq \frac{C}{L}\|v-w\|_{\Omega_{k}}^{2}
\end{align*}
$$

Using (A.18) and (A.19), similar to (A.24) for $1 \leq j \leq k$, we also reach (A.24) for $k+1 \leq j \leq k+m$. Combining this with (A.24) for $1 \leq j \leq k$ yields

$$
\begin{equation*}
\int_{\Omega_{t_{1}}} e^{-2 L \xi_{1}}\left|\mathcal{F}(v)\left(t_{1}, \xi_{1}\right)-\mathcal{F}(w)\left(t_{1}, \xi_{1}\right)\right|^{2} d \xi_{1} \leq \frac{C}{L}\|v-w\|_{\Omega_{k}}^{2} \tag{A.25}
\end{equation*}
$$

The claim now follows from (A.23) and (A.25). The proof is complete.
The second lemma used in the proof of Theorem A. 1 is the following.
Lemma A.2. Let $k-m+1 \leq \ell \leq k-1, F \in\left[L^{\infty}\left(\Omega_{\ell}\right)\right]^{n \times n}, \gamma \in\left[L^{2}\left(\Omega_{\ell}\right)\right]^{n}$, and $h_{j} \in L^{2}\left(\Gamma_{\ell+1}\right)$ for $1 \leq j \leq k+m$ and $j \neq \ell+1$. There exists a unique board solution $w \in \mathcal{Y}_{\ell}:=\left[L^{2}\left(\Omega_{\ell}\right)\right]^{n} \cap$ $C\left(\left[0, T_{\text {opt } t}\right] ;\left[L^{2}\left(\Omega_{\ell, t}\right)\right]^{n}\right) \cap C\left([0,1] ;\left[L^{2}\left(\Omega_{\ell, x}\right)\right]^{n}\right)$ of the system

$$
\begin{gather*}
\partial_{t} w(t, x)=\Sigma(x) \partial_{x} w(t, x)+F(t, x) w(t, x)+\gamma(t, x) \text { for }(t, x) \in \Omega_{\ell},  \tag{A.26}\\
w_{j}=h_{j} \text { on } \Gamma_{\ell+1}, \text { for } 1 \leq j \leq k+m \text { and } j \neq \ell+1, \tag{A.27}
\end{gather*}
$$

[^11]\[

$$
\begin{equation*}
w_{-, \geq \ell+1}(0, \cdot)=Q_{\ell+1} w_{<\ell+1, \geq k+\ell+1} \text { for } t \in\left(T_{\text {opt }}-\tau_{\ell+1}, T_{\text {opt }}-\tau_{\ell}\right) . \tag{A.28}
\end{equation*}
$$

\]

Moreover,

$$
\|w\|_{\mathcal{Y}_{\ell}} \leq C\left(\sum_{1 \leq j \leq k+m ; j \neq \ell+1}\left\|h_{j}\right\|_{L^{2}\left(\Gamma_{\ell}\right)}+\|\gamma\|_{L^{2}\left(\Omega_{\ell}\right)}\right)
$$

for some positive constant $C$ depending only on an upper bound of $\|F\|_{L^{\infty}\left(\Omega_{\ell}\right)}$ and $\Sigma$.
Remark A.2. The analysis of Lemma A. 2 can be easily extended to cover the case where source terms in $L^{2}$ are added in (A.28).

The broad solutions considered in Lemma A.2, which are in the same spirit of the ones in Lemma A.1, are defined as follows:

Definition A.3. Let $k-m+1 \leq \ell \leq k-1, F \in\left[L^{\infty}\left(\Omega_{\ell}\right)\right]^{n \times n}, \gamma \in\left[L^{2}\left(\Omega_{\ell}\right)\right]^{n}$, and $h_{j} \in L^{2}\left(\Gamma_{\ell+1}\right)$ for $1 \leq j \leq k+m$ and $j \neq \ell+1$. A vector-valued function $w \in \mathcal{Y}_{\ell}$ is called a broad solution of (A.26)-(A.28) if for almost $\left(t_{1}, \xi_{1}\right) \in \Omega_{\ell}$, the following conditions hold

1. for $1 \leq j \leq \ell$ and for $k+1 \leq j \leq k+m$,

$$
\begin{equation*}
w_{j}\left(t_{1}, \xi_{1}\right)=\int_{t}^{t_{1}}\left(F\left(s, x_{j}\left(s, t_{1}, \xi_{1}\right)\right) w\left(s, x_{j}\left(s, t_{1}, \xi_{1}\right)\right)\right)_{j} d s+\int_{t}^{t_{1}} \gamma_{j}\left(s, x_{j}\left(s, t_{1}, \xi_{1}\right)\right) d s+h_{j}(t) \tag{A.29}
\end{equation*}
$$

where $t$ is such that $x_{j}\left(t, t_{1}, \xi_{1}\right) \in \Gamma_{\ell+1}$;
2. for $\ell+1 \leq j \leq k$,

$$
\begin{align*}
& w_{j}\left(t_{1}, \xi_{1}\right)=\int_{\hat{t}}^{t_{1}}\left(F\left(s, x_{j}\left(s, t_{1}, \xi_{1}\right)\right) w\left(s, x_{j}\left(s, t_{1}, \xi_{1}\right)\right)\right)_{j} d s  \tag{A.30}\\
& \quad+\int_{\hat{t}}^{t_{1}} \gamma_{j}\left(s, x_{j}\left(s, t_{1}, \xi_{1}\right)\right) d s+\left(Q_{\ell+1} w_{<\ell+1, \geq \ell+m+1}\right)_{j-\ell}(\hat{t}, 0)
\end{align*}
$$

if $\hat{t} \in\left(T_{\text {opt }}-\tau_{\ell+1}, T_{\text {opt }}-\tau_{\ell}\right)$ where $\hat{t}$ is such that $x_{j}\left(\hat{t}, t_{1}, \xi_{1}\right)=0$, otherwise,

$$
\begin{equation*}
w_{j}\left(t_{1}, \xi_{1}\right)=\int_{t}^{t_{1}}\left(F\left(s, x_{j}\left(s, t_{1}, \xi_{1}\right)\right) w\left(s, x_{j}\left(s, t_{1}, \xi_{1}\right)\right)\right)_{j} d s+\int_{t}^{t_{1}} \gamma_{j}\left(s, x_{j}\left(s, t_{1}, \xi_{1}\right)\right) d s+h_{j}(t) \tag{A.31}
\end{equation*}
$$

where $t$ is such that $x_{j}\left(t, t_{1}, \xi_{1}\right) \in \Gamma_{\ell+1}$.
As in Definition A.1, the term $Q_{\ell+1} w_{<\ell+1, \geq \ell+m+1}(\hat{t}, 0)$ in (A.30) is required to be replaced by the corresponding expression in the RHS of (A.30) or (A.31) with $(\hat{t}, 0)$ standing for $\left(t_{1}, \xi_{1}\right)$.

Proof of Lemma A.2. The key part of the proof is to introduce an appropriate weighted norm, which is adapted to the geometry and the boundary conditions considered, for which the fixed point argument works (see Remark A. 3 for comments on this point).

We begin with the case where $\Sigma$ is constant. For $1 \leq j \leq k+m$, let $\vec{v}_{j}$ be the unit vector parallel to the characteristic curve of $x_{j}$ directed to the boundary for which the boundary condition for $v_{j}$ is given $\left(\vec{v}_{j}\right.$ is parallel to $\left(1, \Sigma_{j j}\right)^{\top}$ in the $x t$-plane $)$. Set

$$
G_{1}=\left\{\vec{v}_{j} ; 1 \leq j \leq \ell, k+1 \leq j \leq k+m\right\} \quad \text { and } \quad G_{2}=\left\{\vec{v}_{j} ; \ell+1 \leq j \leq k\right\} .
$$

Here are some useful observations. There exist two non-zero vectors $\vec{u}_{1}$ and $\vec{u}_{2}$ such that
a1) $G_{1} \cup G_{2} \cup\left\{\vec{u}_{1}\right\}$ lies strictly on one side of the line containing $\vec{u}_{2}$
a2) $G_{1}$ is a subset of the open, solid, cone centered at the origin and formed by $\vec{u}_{1}$ and $\vec{u}_{2}$, i.e., in the set $\left\{s_{1} \vec{u}_{1}+s_{2} \vec{u}_{2} ; s_{1}, s_{2}>0\right\}$.


Figure 5. Geometry of $\vec{v}_{j}$ for $1 \leq j \leq n$, and $\vec{u}_{1}$ and $\vec{u}_{2}$ for $\Omega_{\ell}$ when $\Sigma$ is constant.
a3) $G_{2}$ is a subset of the open, solid, cone centered at the origin and formed by $\vec{u}_{1}$ and $-\vec{u}_{2}$, i.e., in the set $\left\{s_{1} \vec{u}_{1}-s_{2} \vec{u}_{2} ; s_{1}, s_{2}>0\right\}$.
(For example, one can choose $\vec{u}_{1}=(0,-1)^{\top}$ and $\overrightarrow{u_{2}}$ is close to $\vec{v}_{\ell}$ but with a larger slope in the $x t$-plane, see Figure 5.

We are ready to introduce the weighted norm used. For $v \in\left[L^{2}\left(\Omega_{\ell}\right)\right]^{n}$, set

$$
\begin{equation*}
T_{\ell}(v)(t, x)=e^{L y_{1}(t, x)} v(t, x) \text { for }(t, x) \in \Omega_{\ell}, \tag{A.32}
\end{equation*}
$$

where $y_{1}(t, x)$ is the first component of $\left(y_{1}, y_{2}\right)(t, x)$ which is the coordinate of $(t, x)$ corresponding to the basis $\vec{u}_{1}$ and $\vec{u}_{2}$ (in the $x t$-plane).

We now introduce

$$
\begin{equation*}
\|v\|_{\Omega_{\ell}}:=\max \left\{\sup _{x \in[0,1]}\left\|\left(T_{\ell} v\right)_{i}\right\|_{L^{2}\left(\Omega_{\ell, x}\right)}, \sup _{t \in\left[0, T_{\text {opt }}\right]}\left\|\left(T_{\ell} v\right)_{i}\right\|_{L^{2}\left(\Omega_{\ell, t}\right)} ; 1 \leq i \leq n\right\} . \tag{A.33}
\end{equation*}
$$

One can check that $\mathcal{Y}_{\ell}$ equipped with the norm $\|\cdot\|_{\Omega_{\ell}}$ is a Banach space. It is also clear that $\|\cdot\|_{\Omega_{\ell}}$ is equivalent to $\|\cdot\|_{\mathcal{Y}_{\ell}}$.

The proof is now based on a fixed point argument as in the one of LemmaA.1. To this end, define $\mathcal{F}_{\ell}$ from $\mathcal{Y}_{\ell}$ equipped with the norm $\|\cdot\|_{\Omega_{\ell}}$ into itself as follows: for $v \in \mathcal{Y}_{\ell}$ and for $\left(t_{1}, \xi_{1}\right) \in \Omega_{\ell}$ :

$$
\begin{equation*}
\left(\mathcal{F}_{\ell}(v)\right)_{i}\left(t_{1}, \xi_{1}\right) \text { is the RHS of (А.29), or (А.30), or (A.31) } \tag{А.34}
\end{equation*}
$$

under the corresponding conditions.
Fix $\xi_{1} \in[0,1]$. Let $1 \leq j \leq \ell$ or $k+1 \leq j \leq k+m$. We have, for $\left(t_{1}, \xi_{1}\right) \in \Omega_{\ell}$, by (A.29),

$$
\begin{equation*}
\mathcal{F}(v)_{j}\left(t_{1}, \xi_{1}\right)-\mathcal{F}(w)_{j}\left(t_{1}, \xi_{1}\right)=\int_{t}^{t_{1}}\left(F\left(s, x_{j}\left(s, t_{1}, \xi_{1}\right)\right)(v-w)\left(s, x_{j}\left(s, t_{1}, \xi_{1}\right)\right)\right)_{j} d s \tag{A.35}
\end{equation*}
$$

where $t$ is such that $x_{j}\left(t, t_{1}, \xi_{1}\right) \in \Gamma_{\ell+1}$. This implies

$$
\begin{align*}
& \int_{\Omega_{\ell, \xi_{1}}} e^{2 L y_{1}\left(t_{1}, \xi_{1}\right)}\left|\mathcal{F}(v)_{j}\left(t_{1}, \xi_{1}\right)-\mathcal{F}(w)_{j}\left(t_{1}, \xi_{1}\right)\right|^{2} d t_{1}  \tag{A.36}\\
& \quad \leq C \int_{\Omega_{\ell, \xi_{1}}} \operatorname{sign}\left(t-t_{1}\right) \int_{t_{1}}^{t} e^{2 L y_{1}\left(t_{1}, \xi_{1}\right)}|v-w|^{2}\left(s, x_{j}\left(s, t_{1}, \xi_{1}\right)\right) d s d t_{1} .
\end{align*}
$$

Here and in what follows in this proof, $C$ (resp. c) denotes a positive constant which depends only on an upper bound of $\|F\|_{L^{\infty}\left(\Omega_{k}\right)}$ and $\Sigma$ (resp. $\Sigma$ ), and can change from one place to another.

We have

$$
\begin{align*}
e^{2 L y_{1}\left(t_{1}, \xi_{1}\right)}|v-w|^{2} & \left(s, x_{j}\left(s, t_{1}, \xi_{1}\right)\right)  \tag{A.37}\\
& =e^{2 L\left(y_{1}\left(t_{1}, \xi_{1}\right)-y_{1}\left(s, x_{j}\left(s, t_{1}, \xi_{1}\right)\right)\right)} e^{2 L y_{1}\left(s, x_{j}\left(s, t_{1}, \xi_{1}\right)\right)}|v-w|^{2}\left(s, x_{j}\left(s, t_{1}, \xi_{1}\right)\right)
\end{align*}
$$

and, for $s$ between $t_{1}$ and $t$,

$$
\begin{equation*}
\left.y_{1}\left(t_{1}, \xi_{1}\right)-y_{1}\left(s, x_{j}\left(s, t_{1}, \xi_{1}\right)\right) \leq-c\left|\xi_{1}-x_{j}\left(s, t_{1}, \xi_{1}\right)\right| \text { by a2 }\right) \text { and the definition of } G_{1} \tag{A.38}
\end{equation*}
$$

Making a change of variables $x=x_{j}\left(s, t_{1}, \xi_{1}\right)$, we derive from (A.36) that, for $1 \leq j \leq \ell$ or $k+1 \leq j \leq k+m$,

$$
\begin{align*}
& \int_{\Omega_{\ell, \xi_{1}}} e^{2 L y_{1}\left(t_{1}, \xi_{1}\right)}\left|\mathcal{F}(v)_{j}\left(t_{1}, \xi_{1}\right)-\mathcal{F}(w)_{j}\left(t_{1}, \xi_{1}\right)\right|^{2} d t_{1}  \tag{A.39}\\
& \quad \leq C \int_{\Omega_{\ell}} e^{-c L\left|\xi_{1}-x\right|} e^{2 L y_{1}(s, x)}|v-w|^{2}(s, x) d s d x \leq \frac{C}{L}\|v-w\|_{\Omega_{\ell}}^{2}
\end{align*}
$$

We next deal with $\ell+1 \leq j \leq k$. Set

$$
\Omega_{\ell, \xi_{1}, 1}=\left\{t_{1} \in\left[0, T_{\text {opt }}\right] ;(\widehat{\text { A. } 30)} \text { holds }\} \quad \text { and } \quad \Omega_{\ell, \xi_{1}, 2}=\left\{t_{1} \in\left[0, T_{\text {opt }}\right] ; \text { (A.31) holds }\right\} .\right.
$$

We have, by (A.30), for $t_{1} \in \Omega_{\ell, \xi_{1}, 1}$,
(A.40) $\quad \mathcal{F}(v)_{j}\left(t_{1}, \xi_{1}\right)-\mathcal{F}(w)_{j}\left(t_{1}, x_{1}\right)$

$$
=\int_{\hat{t}}^{t_{1}}\left(F\left(s, x_{j}\left(s, t_{1}, \xi_{1}\right)\right)(v-w)\left(s, x_{j}\left(s, t_{1}, \xi_{1}\right)\right)\right)_{j} d s+\left(Q_{\ell+1}(v-w)_{<\ell+1, \geq \ell+m+1}\right)_{j-\ell}(\hat{t}, 0)
$$

where $\hat{t}=\hat{t}\left(t_{1}, \xi_{1}\right)$ is such that $x_{j}\left(\hat{t}, t_{1}, \xi_{1}\right)=0$.
We next estimate

$$
\int_{\Omega_{\ell, \xi_{1}, 1}} \operatorname{sign}\left(\hat{t}-t_{1}\right) \int_{t_{1}}^{\hat{t}} e^{2 L y_{1}\left(t_{1}, \xi_{1}\right)}|v-w|^{2}\left(s, x_{j}\left(s, t_{1}, \xi_{1}\right)\right) d s d t_{1}
$$

We have, for $s$ between $t_{1}$ and $\hat{t}$,

$$
\begin{equation*}
\left.y_{1}\left(t_{1}, \xi_{1}\right)-y_{1}\left(s, x_{j}\left(s, t_{1}, \xi_{1}\right)\right) \leq-c\left|\xi_{1}-x_{j}\left(s, t_{1}, \xi_{1}\right)\right| \text { by a3 }\right) \text { and the definition of } G_{2} \tag{A.41}
\end{equation*}
$$

Making a change of variables $x=x_{j}\left(s, t_{1}, \xi_{1}\right)$, we derive from (A.37) that

$$
\begin{aligned}
& \int_{\Omega_{\ell, \xi_{1}, 1}} \operatorname{sign}\left(\hat{t}-t_{1}\right) \int_{t_{1}}^{\hat{t}} e^{2 L y_{1}\left(t_{1}, \xi_{1}\right)}|v-w|^{2}\left(s, x_{j}\left(s, t_{1}, \xi_{1}\right)\right) d s d t_{1} \\
& \quad \leq C \int_{\Omega_{\ell}} e^{-c L\left|\xi_{1}-x\right|} e^{-2 L y_{1}(s, x)}|v-w|^{2}(s, x) d s d x
\end{aligned}
$$

This implies

$$
\begin{equation*}
\int_{\Omega_{\ell, \xi_{1}}} \operatorname{sign}\left(\hat{t}-t_{1}\right) \int_{t_{1}}^{\hat{t}} e^{2 L y_{1}\left(t_{1}, \xi_{1}\right)}|v-w|^{2}\left(s, x_{j}\left(s, t_{1}, \xi_{1}\right)\right) d s d t_{1} \leq \frac{C}{L}\|v-w\|_{\Omega_{\ell}}^{2} \tag{A.42}
\end{equation*}
$$

By (A.39), we also have

$$
\begin{equation*}
\int_{\Omega_{\ell, 0}} e^{2 L y_{1}(\hat{t}, 0)}\left|Q_{\ell+1}(v-w)_{<\ell+1, \geq \ell+m+1}(\hat{t}, 0)\right|^{2} d \hat{t} \leq \frac{C}{L}\|v-w\|_{\Omega_{\ell}}^{2} \tag{A.43}
\end{equation*}
$$

Using (A.41), and making a change of variable $\hat{t}=\hat{t}\left(t_{1}, \xi_{1}\right)$, we derive that

$$
\begin{align*}
& \int_{\Omega_{\ell, \xi_{1}}} e^{L y_{1}\left(t_{1}, \xi_{1}\right)}\left|Q_{\ell+1}(v-w)_{<\ell+1, \geq \ell+m+1}\left(\hat{t}\left(t_{1}, \xi_{1}\right), 0\right)\right|^{2} d t_{1}  \tag{A.44}\\
& \leq C \int_{\Omega_{\ell, 0}} e^{2 L y_{1}(\hat{t}, 0)}\left|Q_{\ell+1}(v-w)_{<\ell+1, \geq \ell+m+1}(\hat{t}, 0)\right|^{2} d \hat{t}
\end{align*}
$$

Combining (A.30), (A.42), (A.43), and (A.44) yields, for $\ell+1 \leq j \leq k$,

$$
\begin{equation*}
\int_{\Omega_{\ell, \xi_{1}, 1}} e^{2 L y_{1}\left(t_{1}, \xi_{1}\right)}\left|\mathcal{F}(v)_{j}\left(t_{1}, \xi_{1}\right)-\mathcal{F}(w)_{j}\left(t_{1}, x_{1}\right)\right|^{2} d t_{1} \leq \frac{C}{L}\|v-w\|_{\Omega_{\ell}}^{2} \tag{A.45}
\end{equation*}
$$

Using similar arguments, we also obtain, for $\ell+1 \leq j \leq k$,

$$
\begin{equation*}
\int_{\Omega_{\ell, \xi_{1}, 2}} e^{2 L y_{1}\left(t_{1}, \xi_{1}\right)}\left|\mathcal{F}(v)_{j}\left(t_{1}, \xi_{1}\right)-\mathcal{F}(w)_{j}\left(t_{1}, x_{1}\right)\right|^{2} d t_{1} \leq \frac{C}{L}\|v-w\|_{\Omega_{\ell}}^{2} \tag{A.46}
\end{equation*}
$$

We derive from (A.45) and (A.46) that

$$
\begin{equation*}
\int_{\Omega_{\ell, \xi_{1}}} e^{2 L y_{1}\left(t_{1}, \xi_{1}\right)}\left|\mathcal{F}(v)_{j}\left(t_{1}, \xi_{1}\right)-\mathcal{F}(w)_{j}\left(t_{1}, x_{1}\right)\right|^{2} d t_{1} \leq \frac{C}{L}\|v-w\|_{\Omega_{\ell}}^{2} \tag{A.47}
\end{equation*}
$$

From (A.39) and (A.47), we obtain

$$
\begin{equation*}
\int_{\Omega_{\xi_{1}}} e^{2 L y_{1}\left(t_{1}, \xi_{1}\right)}\left|\mathcal{F}(v)\left(t_{1}, \xi_{1}\right)-\mathcal{F}(w)\left(t_{1}, x_{1}\right)\right|^{2} d t_{1} \leq \frac{C}{L}\|v-w\|_{\Omega_{\ell}}^{2} \tag{A.48}
\end{equation*}
$$

For $t_{1} \in\left(0, T_{\text {opt }}\right)$, by the same approach used to derive (A.48), we also have

$$
\begin{equation*}
\int_{\Omega_{t_{1}}} e^{2 L y_{1}\left(t_{1}, \xi_{1}\right)}\left|\mathcal{F}(v)\left(t_{1}, \xi_{1}\right)-\mathcal{F}(w)\left(t_{1}, x_{1}\right)\right|^{2} d \xi_{1} \leq \frac{C}{L}\|v-w\|_{\Omega_{\ell}}^{2} \tag{A.49}
\end{equation*}
$$

The conclusion in the case where $\Sigma$ is constant now follows from (A.48) and (A.49).
We next make necessary modifications to derive the conclusion in the general case. The idea is to find a replacement for $y_{1}(t, x)$ which is increasing when one follows the characteristic flows directed to the boundary for which the boundary conditions are imposed. To this end, for $1 \leq j \leq k+m$, let $\vec{v}_{j}=\vec{v}_{j}(t, x)$ be the unit vector tangent to the characteristic curve of $x_{j}$ at the point $(t, x)$ directed to the boundary where the boundary condition for $v_{j}$ is given. The vector $\vec{v}_{j}(t, x)$ is parallel to $\left(1, \Sigma_{j j}(x)\right)^{\top}$ in the $x t$-plane and so that one can choose it independent of $t$ and in fact we will do. We will denote it by $\vec{v}_{j}(x)$ from now on. Set

$$
G_{1}(x)=\left\{\vec{v}_{j}(x) ; 1 \leq j \leq \ell, k+1 \leq j \leq k+m\right\} \quad \text { and } \quad G_{2}(x)=\left\{\vec{v}_{j}(x) ; \ell+1 \leq j \leq k\right\} .
$$

Let $\varphi(x)$ be such that

$$
\vec{v}_{\ell}(x) \text { is parallel to and has the same direction with }(\varphi(x), 1)^{\top} \text {. }
$$

Set, in the $x t$-plane,

$$
\vec{u}_{1}(x)=(0,-1)^{\top},
$$

and

$$
\vec{u}_{2}(x)=(\varphi(x)-\varepsilon, 1)^{\top},
$$

where $\varepsilon$ is a constant which is positive and sufficiently small, the smallness of $\varepsilon$ is independent of $x$, such that, $\varphi(x)>2 \varepsilon$, and
a1) $G_{1}(x) \cup G_{2}(x) \cup\left\{\vec{u}_{1}(x)\right\}$ lies on one side of the line containing $\vec{u}_{2}(x)$
a2) $G_{1}(x)$ is a subset of the open solid cone centered at the origin and formed by $\vec{u}_{1}(x)$ and $\vec{u}_{2}(x)$, i.e., in the set $\left\{s_{1} \vec{u}_{1}(x)+s_{2} \vec{u}_{2}(x) ; s_{1}, s_{2}>0\right\}$.
a3) $G_{2}(x)$ is a subset of the open solid cone centered at the origin and formed by $\vec{u}_{1}(x)$ and $-\vec{u}_{2}(x)$, i.e., in the set $\left\{s_{1} \vec{u}_{1}(x)-s_{2} \vec{u}_{2}(x) ; s_{1}, s_{2}>0\right\}$.
Fix such a positive constant $\varepsilon$. For a point $\left(x_{0}, t_{0}\right) \in \Omega_{\ell}$, let $(x(s), t(s))$ for $s \in[\alpha, \beta] \subset \mathbb{R}$ be a (piecewise) $C^{1}$ regular curve ${ }^{16}$ in $\bar{\Omega}_{\ell}$ (in the $x t$-plane) starting from ( 0,0 ) and arriving at ( $x_{0}, t_{0}$ ). We first claim that

$$
\begin{equation*}
\int_{\alpha}^{\beta} y_{1}\left(x^{\prime}(s), t^{\prime}(s), x(s), t(s)\right)\left|\left(x^{\prime}(s), t^{\prime}(s)\right)\right| d s \text { depends on }\left(t_{0}, x_{0}\right) \tag{A.50}
\end{equation*}
$$

but is independent of the curve and the parametrization.
Here $y_{1}\left(t^{\prime}(s), x^{\prime}(s), t(s), x(s)\right)$ is the first coordinate of the vector $\left(t^{\prime}(s), x^{\prime}(s)\right) /\left|\left(t^{\prime}(s), x^{\prime}(s)\right)\right|$ in the bases $\vec{u}_{1}(t(s), x(s))$ and $\vec{u}_{2}(t(s), x(s))$.

We now establish the claim. For notational ease, we assume that $\left|\left(t^{\prime}(s), x^{\prime}(s)\right)\right|=1$. We first compute $y_{1}\left(t^{\prime}(s), x^{\prime}(s), t(s), x(s)\right)$. Let $a$ and $b$ in $\mathbb{R}$ be such that

$$
\left(x^{\prime}(s), t^{\prime}(s)\right)=a(0,-1)+b(\varphi(x(s))-\varepsilon, 1) .
$$

We have

$$
a=-t^{\prime}(s)+\frac{x^{\prime}(s)}{\varphi(x(s))-\varepsilon} \quad \text { and } \quad b=\frac{x^{\prime}(s)}{\varphi(x(s))-\varepsilon} .
$$

Thus

$$
y_{1}\left(t^{\prime}(s), x^{\prime}(s), t(s), x(s)\right)=-t^{\prime}(s)+\frac{x^{\prime}(s)}{\varphi(x(s))-\varepsilon} .
$$

It follows that

$$
\begin{equation*}
\int_{\alpha}^{\beta} y_{1}\left(x^{\prime}(s), t^{\prime}(s), x(s), t(s)\right) d s=-t_{0}+\Phi\left(x_{0}\right) \tag{A.51}
\end{equation*}
$$

where

$$
\Phi(\xi)=\int_{0}^{\xi} \frac{1}{\varphi(s)+\varepsilon} d s \text { for } \xi \in[0,1]
$$

The claim is proved.
Define

$$
\begin{array}{cccc}
Y_{1} & : \Omega_{\ell} & \rightarrow & \mathbb{R} \\
& (t, x) & \mapsto & -t+\Phi(x) .
\end{array}
$$

The proof in the general case follows as in the constant case with $T_{\ell}$ now defined by

$$
\begin{equation*}
T_{\ell}(v)(t, x)=e^{L Y_{1}(t, x)} v(t, x) \tag{A.52}
\end{equation*}
$$

One just notes that (A.38) and (A.41) hold with $y_{1}$ replaced by $Y_{1}$. Indeed, one has

$$
\begin{aligned}
& Y_{1}\left(s, x_{j}\left(s, t_{1}, \xi_{1}\right)\right)-Y_{1}\left(t_{1}, \xi_{1}\right)=\int_{t_{1}}^{s} y_{1}\left(\partial_{\theta} x_{j}\left(\theta, t_{1}, \xi_{1}\right), x_{j}\left(\theta, t_{1}, \xi_{1}\right)\right)\left|\partial_{\theta} x_{j}\left(\theta, t_{1}, \xi_{1}\right)\right| d \theta \\
& \quad \geq C \operatorname{sign}\left(s-t_{1}\right) \int_{t_{1}}^{s} y_{1}\left(\vec{v}_{j}\left(x_{j}\left(\theta, t_{1}, \xi_{1}\right)\right), x_{j}\left(\theta, t_{1}, \xi_{1}\right)\right) d \theta \geq C\left|t_{1}-s\right| \geq C\left|x_{j}\left(s, t_{1}, \xi_{1}\right)-\xi_{1}\right|
\end{aligned}
$$

The details are omitted.
We are ready to give

[^12]Proof of Theorem A.1. We first prove the uniqueness. Assume that $f=0, g=0$, and $\gamma=0$. Then the restriction of $w$ into $\Omega_{k}$ is 0 by Lemma A.1. It follows that the restriction of $w$ into $\Omega_{k-1}=0$ by Lemma A.2, $\ldots$, the restriction of $w$ into $\Omega_{k-m+1}=0$ by Lemma A.2. Therefore, $w=0$ in $\Omega$.

To establish the existence, we proceed as follows. Let $w^{(k)}$ be the unique broad solution in $\Omega_{k}$ corresponding to $(f, g)$, let $w^{(k-1)}$ be the unique broad solution in $\Omega_{k-1}$ where the data on $\Gamma_{k}$ come from $w^{(k)}, \ldots$, let $w^{(k-m+1)}$ be the unique broad solution in $\Omega_{k-m+1}$ where the data on $\Gamma_{k-m+2}$ come from $w^{(k-m+2)}{ }^{17}$. The corresponding solution is obtained by gluing these solutions together. The proof is complete.

Remark A.3. The introduction of appropriate weighted norms plays a crucial role in the proof of the well-posedness of broad solutions considered so far in this section, in particular in the proof of Lemma A.2. The introduction of weighted norms in order to be able to apply the fixed point argument used in establishing the well-posedness of hyperbolic system is not new. The standard one is $e^{-L t}$ where $L$ is a large positive number, see e.g. [38, (1.18), p. 78] or [7] (3.36), p. 50], while the weight $e^{-L x}$ is used in [52, 11 to prove exponential stability; see also [9, $V$ defined in Section 3.2] for the Euler equations of incompressible fluids. In [17], we used the weight $e^{-L_{1} x-L_{2} t}$ where $L_{1}$ and $L_{2}$ are two large positive numbers with $L_{2}$ being much larger than $L_{1}$. The introduction of $e^{-L_{1} x}$ in the weight is to handle the non-local term from the boundary condition imposed on the right (at $x=1$ ) considered there. In these settings, $t$-direction has a privileged role. In the settings considered in this section, the domain is not a rectangle with respect to $t$ and $x$, and the boundary conditions are quite complicated. Therefore, the time direction and the space direction play almost the same role here. In the setting of Lemma A. 1 , the privileged direction is $x$-direction so the weighted norm is chosen of the form $e^{L x}$. In Lemma A.2, the new weighted norm introduced in (A.33) with $T_{\ell}$ given by (A.32) or (A.52) adapts the geometry and the boundary conditions, imposed in a nontrivial way. It is interesting to note that $Y_{1}$ is a non-linear function of $t$ and $x$. The analysis here is inspired by [17] (see also [18]).

As a consequence of Theorem A.1, we can prove
Proposition A.1. Let $C \in\left[L^{\infty}(I \times(0,1))\right]^{n \times n}$. Define, for $\tau \in I_{1}$,

$$
\begin{array}{cc}
\mathcal{T}(\tau):\left[L^{2}\left(0, T_{\text {opt }}\right)\right]^{n} \times\left[L^{2}(0,1)\right]^{m} & \rightarrow \mathcal{Y}  \tag{A.53}\\
(f, g) & \mapsto w
\end{array}
$$

where $w$ is the solution of (3.27)-(3.32). Then $\mathcal{T}(\tau)$ is uniformly bounded in $I_{1}$. Assume in addition that $C \in \mathcal{H}\left(I,\left[L^{\infty}(0,1)\right]^{n \times n}\right)$. Then $\mathcal{T}(\tau, \cdot, \cdot)$ is analytic in $I_{1}$.
Proof. By Theorem A.1, for each $(f, g) \in\left[L^{2}(0,1)\right]^{n} \times\left[L^{2}(0,1)\right]^{m}$, there exists a unique broad solution $w \in \mathcal{Y}$ of (3.27)-(3.32). Hence $\mathcal{T}(\tau)$ is well-defined. The uniform boundedness of $\mathcal{T}$ is also a direct consequence of Theorem A.1, in particular of (A.12).

We next deal with the analyticity of $\mathcal{T}$ and thus assume that $C \in \mathcal{H}\left(I,\left[L^{\infty}(0,1)\right]^{n \times n}\right)$. Fix $\tau_{0}$ in a sufficiently small neighborhood of $I_{1}$ (in the complex plane). We will prove that $\mathcal{T}$ is differentiable at $\tau_{0}$ in the complex sense. For notational ease, we will assume that $\tau_{0}=0$.

Fix $(f, g) \in\left[L^{2}\left(0, T_{\text {opt }}\right)\right]^{n} \times\left[L^{2}(0,1)\right]^{n}$. Set $w^{(\tau)}=\mathcal{T}(\tau)(f, g)$ in $\Omega$ for $\tau$ in a small neighborhood (in the complex plane) of 0 and let $v \in \mathcal{Y}$ be the unique broad solution of the system

$$
\begin{gather*}
v_{t}(t, x)=\Sigma(x) \partial_{x} v(t, x)+\mathbf{C}(t, x) v(t, x)+\mathbf{C}_{\tau}(t, x) w^{(0)}(t, x) \text { for }(t, x) \in \Omega  \tag{A.54}\\
v(\cdot, 1)=0 \text { in }\left(0, T_{o p t}\right) \tag{A.55}
\end{gather*}
$$

[^13]\[

$$
\begin{gather*}
v_{+}(0, \cdot)=0 \text { in }(0,1),  \tag{A.56}\\
v_{-, \geq k}(t, 0)=Q_{k} v_{<k, \geq k+m}(t, 0) \text { for } t \in\left(T_{\text {opt }}-\tau_{k}, T_{o p t}-\tau_{k-1}\right),  \tag{A.57}\\
v_{-, \geq k-1}(t, 0)=Q_{k-1} v_{<k-1, \geq k+m-1}(t, 0) \text { for } t \in\left(T_{\text {opt }}-\tau_{k-1}, T_{\text {opt }}-\tau_{k-2}\right), \tag{A.58}
\end{gather*}
$$
\]

$$
\begin{equation*}
v_{-, \geq k-m+2}(t, 0)=Q_{k-m+2} v_{<k-m+2, \geq k+2}(t, 0) \text { for } t \in\left(T_{o p t}-\tau_{k-m+2}, T_{o p t}-\tau_{k-m+1}\right) \tag{A.59}
\end{equation*}
$$

Here $\mathbf{C}_{\tau}(\tau, x)$ denotes the derivative of $\mathbf{C}(\tau, x)$ with respect to $\tau$ in the complex sense. The existence and uniqueness of $v$ follows from Theorem A.1.

We claim that
the derivative of $\mathcal{T}$ at 0 is given by $\mathcal{T}_{1}$ where $\mathcal{T}_{1}(f, g)=v$ in $\Omega$
(the derivative of $\mathcal{T}$ is considered in the complex sense). To this end, for $\tau$ in a small neighborhood (in the complex plane) of 0 but not 0 , we consider $d w \in \mathcal{Y}$ defined by

$$
d w:=\frac{1}{\tau}\left(w^{(\tau)}-w^{(0)}-\tau v\right) \text { in } \Omega .
$$

Then $d w \in \mathcal{Y}$ is a broad solution of the system

$$
\begin{align*}
\partial_{t} d w(t, x)=\Sigma(x) \partial_{x} d w( & t, x)+\mathbf{C}(t, x) d w(t, x)  \tag{A.61}\\
& +\frac{1}{\tau}(\mathbf{C}(t+\tau, x)-\mathbf{C}(t, x)) w^{(\tau)}(t, x)-\mathbf{C}_{\tau}(t, x) w^{(0)}(t, x) \text { in } \Omega,
\end{align*}
$$

and (A.55)-(A.59) with $v$ replaced by $d w$. We derive from Theorem A. 1 that

$$
\begin{equation*}
\|d w\|_{\mathcal{Y}} \leq C\left(\left\|w^{(\tau)}\right\|_{L^{2}(\Omega)}+\left\|w^{(0)}\right\|_{L^{2}(\Omega)}\right) \leq C\left(\|f\|_{L^{2}\left(0, T_{o p t}\right)}+\|g\|_{L^{2}(0,1)}\right) \tag{A.62}
\end{equation*}
$$

Using the definition of $d w$, we can write the last two terms in (A.61) under the form

$$
\begin{align*}
& \frac{1}{\tau}(\mathbf{C}(t+\tau, x)-\mathbf{C}(t, x))\left(w^{(0)}+\tau d w+\tau v\right)-\mathbf{C}_{\tau}(t, x) w^{(0)}(t, x)  \tag{A.63}\\
= & \frac{1}{\tau}\left(\mathbf{C}(t+\tau, x)-\mathbf{C}(t, x)-\tau \mathbf{C}_{\tau}(t, x)\right) w^{(0)}(t, x)+\frac{1}{\tau}(\mathbf{C}(t+\tau, x)-\mathbf{C}(t, x))(\tau d w+\tau v) .
\end{align*}
$$

Note that the $L^{2}(\Omega)$-norm of the RHS of (A.63) is bounded by

$$
C|\tau|\left(\left\|w^{(0)}\right\|_{L^{2}(\Omega)}+\|d w\|_{L^{2}(\Omega)}+\|v\|_{L^{2}(\Omega)}\right)
$$

Applying Theorem A. 1 again, we derive from (A.62) that

$$
\begin{equation*}
\|d w\|_{\mathcal{Y}} \leq C|\tau|\left(\left\|w^{(0)}\right\|_{L^{2}(\Omega)}+\|v\|_{L^{2}(\Omega)}+\|f\|_{L^{2}\left(0, T_{o p t}\right)}+\|g\|_{L^{2}(0,1)}\right) \tag{A.64}
\end{equation*}
$$

By noting that

$$
\left\|w^{(0)}\right\|_{L^{2}(\Omega)}+\|v\|_{L^{2}(\Omega)} \leq C\left(\|f\|_{L^{2}\left(0, T_{o p t}\right)}+\|g\|_{L^{2}(0,1)}\right)
$$

claim (A.60) follows from (A.64). The proof is complete.
Remark A.4. Let $C \in\left[L^{\infty}(I \times(0,1))\right]^{n \times n}$. One can prove that $\mathcal{T}(\tau)$ is strongly continuous, i.e., $\mathcal{T}(\tau)(f, g) \rightarrow \mathcal{T}\left(\tau_{0}\right)(f, g)$ in $\mathcal{Y}$ as $\tau \rightarrow \tau_{0}$ in $I_{1}$ for all $(f, g) \in\left[L^{2}\left(0, T_{\text {opt }}\right)\right]^{n} \times\left[L^{2}(0,1)\right]^{m}$. Indeed, let assume that $\tau_{0}=0$ for notational ease. Set $w^{(\tau)}=\mathcal{T}(\tau)(f, g)$ in $\Omega$ for $\tau \in I_{1}$ and for $(f, g) \in\left[L^{2}\left(0, T_{\text {opt }}\right)\right]^{n} \times\left[L^{2}(0,1)\right]^{m}$. Denote $\delta w=w^{(\tau)}-w^{(0)}$ in $\Omega$. We have, in $\Omega$

$$
\partial_{t} \delta w(t, x)=\Sigma(x) \partial_{x} \delta w(t, x)+\mathbf{C}(t+\tau, x) \delta w(t, x)+(\mathbf{C}(t+\tau, x)-\mathbf{C}(t, x)) w^{(0)}(t, x)
$$

and $\delta w$ satisfies the same boundary conditions as $d w$. Applying Theorem A.1, one has

$$
\|\delta w\|_{\mathcal{Y}} \leq C\|g\|_{L^{2}(\Omega)}
$$

where $g(t, x)=(\mathbf{C}(t+\tau, x)-\mathbf{C}(t, x)) w^{(0)}(t, x)$. Since $\|g\|_{L^{2}(\Omega)} \rightarrow 0$ as $\tau \rightarrow 0$, the conclusion follows.

We next discuss the broad solutions used in the definition of $\hat{\mathcal{T}}(\tau)$ and their well-posedness. Let $F \in\left[L^{\infty}\left(\left(0, T_{o p t}\right) \times(0,1)\right)\right]^{n \times n},(f, g) \in\left[L^{2}\left(0, T_{o p t}\right)\right]^{n} \times\left[L^{2}(0,1)\right]^{m}$, and let $q \in\left[L^{2}(0,1)\right]^{k-m} 18$. Consider the system

$$
\begin{equation*}
\partial_{t} \hat{w}(t, x)=\Sigma(x) \partial_{x} \hat{w}(t, x)+F(t, x) \hat{w}(t, x)+\gamma(t, x) \text { for }(t, x) \in \Omega \tag{A.65}
\end{equation*}
$$

$$
\begin{gather*}
\hat{w}(\cdot, 1)=f \text { in }\left(0, T_{o p t}\right)  \tag{A.66}\\
\hat{w}_{+}(0, \cdot)=g \text { in }(0,1)  \tag{A.67}\\
\hat{w}_{j}\left(T_{o p t}, \cdot\right)=q_{j} \text { in }(0,1), \text { for } 1 \leq j \leq k-m  \tag{A.68}\\
\hat{w}_{-, \geq k}(t, 0)=Q_{k} \hat{w}_{<k, \geq k+m}(t, 0) \text { for } t \in\left(T_{o p t}-\tau_{k}, T_{o p t}-\tau_{k-1}\right)  \tag{A.69}\\
\hat{w}_{-, \geq k-1}(t, 0)=Q_{k-1} \hat{w}_{<k-1, \geq k+m-1}(t, 0) \text { for } t \in\left(T_{o p t}-\tau_{k-1}, T_{o p t}-\tau_{k-2}\right), \tag{A.70}
\end{gather*}
$$

$$
\begin{gather*}
\hat{w}_{-, \geq k-m+2}(t, 0)=Q_{k-m+2} \hat{w}_{<k-m+2, \geq k+2}(t, 0) \text { for } t \in\left(T_{o p t}-\tau_{k-m+2}, T_{o p t}-\tau_{k-m+1}\right)  \tag{A.71}\\
\hat{w}_{-, \geq k-m+1}(t, 0)=Q_{k-m+1} \hat{w}_{<k-m+1, \geq k+1}(t, 0) \text { for } t \in\left(T_{o p t}-\tau_{k-m+1}, T_{o p t}\right) \tag{A.72}
\end{gather*}
$$

We have the following result, which implies the well-posedness of $\hat{\mathcal{T}}(\tau)$.
Theorem A.2. Let $F \in\left[L^{\infty}\left(\left(0, T_{o p t}\right) \times(0,1)\right)\right]^{n \times n},(f, g) \in\left[L^{2}\left(0, T_{o p t}\right)\right]^{n} \times\left[L^{2}(0,1)\right]^{m}, q \in$ $\left[L^{2}(0,1)\right]^{k-m} \sqrt[19]{19}$, and $\gamma \in\left[L^{2}(\Omega)\right]^{n}$. There exists a unique broad solution

$$
\hat{w} \in \hat{\mathcal{Y}}:=\left[L^{2}\left(\left(0, T_{o p t}\right) \times(0,1)\right)\right]^{n} \cap C\left(\left[0, T_{o p t}\right] ;\left[L^{2}(0,1)\right]^{n}\right) \cap C\left([0,1] ;\left[L^{2}\left(0, T_{o p t}\right)\right]^{n}\right)
$$

of (A.65)-(A.72). Moreover,

$$
\|\hat{w}\|_{\hat{\mathcal{Y}}} \leq C\left(\|f\|_{L^{2}\left(0, T_{o p t}\right)}+\|g\|_{L^{2}(0,1)}+\|q\|_{L^{2}(0,1)}+\|\gamma\|_{L^{2}\left(\left(0, T_{o p t}\right) \times(0,1)\right)}\right)
$$

for some positive constant $C$ depending only on an upper bound of $\|F\|_{L^{\infty}\left(\Omega_{\ell}\right)}$ and $\Sigma$.
Here we denote

$$
\|\hat{w}\|_{\hat{\mathcal{Y}}}=\max \left\{\sup _{x \in[0,1]}\|\hat{w}\|_{L^{2}\left(0, T_{o p t}\right)}, \sup _{t \in\left[0, T_{o p t}\right]}\|\hat{w}\|_{L^{2}(0,1)} ; 1 \leq i \leq n\right\}
$$

Remark A.5. The analysis of Theorem A.2 can be extended to cover the case where source terms in $L^{2}$ are added in (A.69)- (A.72).

The definition of broad solutions $\hat{w} \in \hat{\mathcal{Y}}$ of A.65) (A.72) is similar to the one given in Definition A. 1 and left to the reader. The proof of (A.2) is similar to the one of Theorem A.1. Nevertheless, in addition to Lemmas A.1 and A.2, we also use the following.

[^14]Lemma A.3. Set $\Omega_{k-m}=\left[0, T_{\text {opt }}\right] \times(0,1) \backslash \Omega$. Let $F \in\left[L^{\infty}\left(\Omega_{k-m}\right)\right]^{n \times n} \gamma \in\left[L^{2}\left(\Omega_{k-m}\right)\right]^{n}$, $h_{j} \in L^{2}\left(\Gamma_{k-m+1}\right)$ for $1 \leq j \leq k+m$ and $j \neq k-m+1$, and let $q_{j} \in L^{2}\left(\Gamma_{k-m}\right)$ for $1 \leq j \leq k-m$ where $\Gamma_{k-m}=\left\{T_{\text {opt }}\right\} \times(0,1)$. There exists a unique broad solution $w \in \mathcal{Y}_{k-m}:=\left[L^{2}\left(\Omega_{k-m}\right)\right]^{n} \cap$ $C\left(\left[0, T_{\text {opt }}\right] ;\left[L^{2}\left(\Omega_{k-m, t}\right)\right]^{n}\right) \cap C\left([0,1] ;\left[L^{2}\left(\Omega_{k-m, x}\right)\right]^{n}\right)$ of the system

$$
\begin{gather*}
\partial_{t} w(t, x)=\Sigma(x) \partial_{x} w(t, x)+F(t, x) w(t, x)+\gamma(t, x) \text { for }(t, x) \in \Omega_{k-m}  \tag{A.73}\\
w_{j}=h_{j} \text { on } \Gamma_{k-m+1}, \text { for } 1 \leq j \leq k+m \text { and } j \neq k-m+1,  \tag{A.74}\\
w_{j}=q_{j} \text { on } \Gamma_{k-m}, \text { for } 1 \leq j \leq k-m,  \tag{A.75}\\
w_{-, \geq k-m+1}(0, \cdot)=Q_{k-m+1} w_{<k-m+1, \geq k+1} \text { for } t \in\left(T_{\text {opt }}-\tau_{k-m+1}, T_{\text {opt }}\right) . \tag{A.76}
\end{gather*}
$$

Moreover,

$$
\|w\| y_{\ell} \leq C\left(\sum_{1 \leq j \leq k+m ; j \neq k-m+1}\left\|h_{j}\right\|_{L^{2}\left(\Gamma_{k-m+1}\right)}+\sum_{1 \leq j \leq k-m}\left\|q_{j}\right\|_{L^{2}\left(\Gamma_{k-m}\right)}+\|\gamma\|_{L^{2}\left(\Omega_{k-m}\right)}\right)
$$

for some positive constant $C$ depending only on an upper bound of $\|F\|_{L^{\infty}\left(\Omega_{k-m}\right)}$ and $\Sigma$.
Remark A.6. The analysis of Lemma A. 3 can be extended to cover the case where source terms in $L^{2}$ are added in (A.76).
Proof. The proof of Lemma A. 3 is similar to the one of Lemma A.2. We just mention here how to define $G_{1}, G_{2}$ and determine $\overrightarrow{u_{1}}$ and $\vec{u}_{2}$ in the general case ( $\Sigma$ is not required to be constant). For $1 \leq j \leq k+m$, let $\vec{v}_{j}=\vec{v}_{j}(t, x)$ be the unit vector tangent to the characteristic curve of $x_{j}$ at the point $(t, x)$ directed to the boundary where the boundary condition for $v_{j}$ is given. The vector $\vec{v}_{j}(t, x)$ is parallel to $\left(1, \Sigma_{j j}(x)\right)^{\top}$ in the $x t$-plane and so that we can choose it independent of $t$ and in fact we will do. We denote it by $\vec{v}_{j}(x)$ from now on. Set
$G_{1}(x)=\left\{\vec{v}_{j}(x) ; 1 \leq j \leq k-m, k+1 \leq j \leq k+m\right\} \quad$ and $\quad G_{2}(x)=\left\{\vec{v}_{j}(x) ; k-m+1 \leq j \leq k\right\}$.
Let $\varphi(x)$ be such that

$$
\vec{v}_{1}(x) \text { is parallel to and has the same direction with }(\varphi(x), 1)^{\top} \text {. }
$$

Set, in the $x t$-plane,

$$
\vec{u}_{1}(x)=(0,-1)^{\top},
$$

and

$$
\vec{u}_{2}(x)=(\varphi(x)-\varepsilon, 1)^{\top} \text { if } k>m \text {, otherwise } \overrightarrow{u_{2}}=(1,0)^{\top},
$$

where $\varepsilon$ is positive and sufficiently small, the smallness of $\varepsilon$ is independent of $x$, such that, $\varphi(x)>2 \varepsilon$ (the choice of $\varepsilon$ is irrelevant when $k=m$ ), and
a1) $G_{1}(x) \cup G_{2}(x) \cup\left\{\vec{u}_{1}(x)\right\}$ lies on one side of the line containing $\vec{u}_{2}(x)$.
a2) $G_{1}(x)$ is a subset of the open solid cone centered at the origin and formed by $\vec{u}_{1}(x)$ and $\vec{u}_{2}(x)$.
a3) $G_{2}(x)$ is a subset of the open solid cone centered at the origin and formed by $\vec{u}_{1}(x)$ and $-\vec{u}_{2}(x)$.
The rest of the proof is then almost unchanged and left to the reader.
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(J.-M. Coron) Sorbonne Université

Université Paris-Diderot SPC, CNRS, INRIA,
Laboratoire Jacques-Louis Lions, équipe Cage, Paris, France
Email address: coron@ann.jussieu.fr
(H.-M. Nguyen) Department of Mathematics

EPFL SB CAMA
Station 8 CH-1015 Lausanne, Switzerland
Email address: hoai-minh.nguyen@epfl.ch


[^0]:    Date: 03/03/2021.

[^1]:    ${ }^{1}$ The $i$ direction $(1 \leq i \leq n)$ is called positive (resp. negative) if $\Sigma_{i i}$ is positive (resp. negative).

[^2]:    ${ }^{2}$ Since $\varphi(t)=0$ for $t>\tau_{k+1}$, (2.31) and (2.32) imply that $\alpha=\beta=0$ in the region of $(t, x)$ which is below the characteristic flow of $v_{k+l}$ passing $(0, T)$ in the $x t$-plane.

[^3]:    ${ }^{3}$ Here and in what follows, for a closed subspace $E$ of $\left[L^{2}(0,1)\right]^{n}$, we denote $\operatorname{Prof}_{E}$ the projection to $E$, and $E^{\perp}$ its orthogonal complement, both with respect to the standard $L^{2}(0,1)$-scalar product.
    ${ }^{4}$ The analyticity of $H(\tau)$ is understood via the analyticity of the mapping $\operatorname{Prof}_{H(\tau)}$. This convention is used throughout the paper.

[^4]:    ${ }^{5}$ The notation $\langle\cdot, \cdot\rangle$ stands for the Euclidean scalar product in $\mathbb{R}^{\ell}$ for $\ell \geq 1$.

[^5]:    ${ }^{6}$ The analyticity of $H(\tau)$ means the analyticity of $\operatorname{Proj}_{H(\cdot)}$.

[^6]:    ${ }^{7}$ Condition(3.3) means that $w_{+}(t, 1) \in\left[L^{2}(\tau, \tau+T)\right]^{m}$.

[^7]:    ${ }^{8}$ One way to apply the theory of the perturbation of the null-space of analytic matrices can be done as follows. One can first locally choose an analytic orthogonal basis $\left\{\varphi_{1}(\tau), \cdots, \varphi_{\ell}(\tau)\right\}$ of $P(\tau)$. We then represent the operator $I d+\mathcal{K}(\tau)$ (where $I d$ denotes the identity map) in this basis after noting that it is an application from $P(\tau)$ into $P(\tau)$. We also represent $\mathcal{L}(\tau)$ using the set $\left\{\mathcal{L}(\tau)\left(\varphi_{1}(\tau)\right), \ldots, \mathcal{L}(\tau)\left(\varphi_{\ell}(\tau)\right)\right\}$ for which the dimension of the span space is constant outside a discrete set, which is removable.

[^8]:    ${ }^{9}$ The sub-index $c$ means that controls are used.
    ${ }^{10}$ The letter $A$ means the attainability.
    ${ }^{11}$ The sub-index $I$ means that initial data are considered.

[^9]:    ${ }^{12}$ Replacing $\gamma_{0}$ by $\gamma_{0} / 2$ if necessary, one can even assume that $D_{n}$ is finite.
    ${ }^{13}$ The set mentioned here is the union of the set $D$ given in (3.81) and the set of $\tau \in I_{2}$ such that $\operatorname{dim} H(\tau)$ is constant, which is discrete. For notational ease, we still use the same notation $D$.

[^10]:    ${ }^{14} \mathrm{~A}$ function $\varphi \in L^{2}(\Omega)$ is said to be in $C\left(\left[0, T_{o p t}\right] ; L^{2}\left(\Omega_{t}\right)\right)$ if $\left(t_{n}\right) \subset\left[0, T_{\text {opt }}\right]$ converging to $t$ then $\lim _{n \rightarrow+\infty}\left(\left\|f\left(t_{n}, \cdot\right)-f(t, \cdot)\right\|_{L^{2}\left(\Omega_{t_{n}} \cap \Omega_{t}\right)}+\left\|f\left(t_{n}, \cdot\right)\right\|_{L^{2}\left(\Omega_{t_{n}} \backslash \Omega_{t}\right)}+\|f(t, \cdot)\|_{L^{2}\left(\Omega_{t} \backslash \Omega_{t_{n}}\right)}\right)=0$. Similar meaning is used for $C\left([0,1] ; L^{2}\left(\Omega_{x}\right)\right)$.

[^11]:    ${ }^{15} x_{j}$ is continuously differentiable with respect to $s, t_{1}, \xi_{1}$ when $x_{j}\left(s, t_{1}, \xi_{1}\right)$ is in $\bar{\Omega}$ since $\Sigma$ is of class $C^{2}$.

[^12]:    ${ }^{16}$ Regularity means that $\left(x^{\prime}(s), t^{\prime}(s)\right) \neq(0,0)$ for $s \in[\alpha, \beta]$ such that $\left(x^{\prime}(s), t^{\prime}(s)\right)$ is well-defined.

[^13]:    ${ }^{17}$ The data coming from $w^{(k)}$ on $\Gamma_{k}, \ldots, w^{(k-m+2)}$ on $\Gamma_{k-m+2}$ are given by the RHS of (A.17)-A.19) in Definition A. 2 for $\left(t_{1}, \xi_{1}\right) \in \Gamma_{k}$, and A.29)-A.31) in Definition A.3 for $\left(t_{1}, \xi_{1}\right) \in \Gamma_{\ell}$ with $\ell=k-1, \ldots, k-m+1$, respectively.

[^14]:    ${ }^{18} q$ is irrelevant when $k=m$.
    ${ }^{19} q$ is irrelevant when $k=m$.

