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Critical values in an anisotropic percolation on $\mathbb{Z}^2$

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Love and freedom. . .

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Abstract

In this thesis, we consider an anisotropic finite-range bond percolation model on $\mathbb{Z}^2$. On each horizontal layer $\{(x,i) : x \in \mathbb{Z}\}$ for $i \in \mathbb{Z}$, we have edges $\langle (x,i),(y,i)\rangle$ for $1 \leq |x-y| \leq N$ with $N \in \mathbb{N}$. There are also vertical edges connecting two nearest neighbor vertices on distinct layers $\langle (x,i),(x,i+1)\rangle$ for $x, i \in \mathbb{Z}$. On this graph, we consider the following anisotropic percolation model: horizontal edges are open with probability $\lambda/(2N)$ with $\lambda \geq 1$, while vertical edges are open with probability ϵ to be suitably tuned as N grows to infinity. This question is motivated by a result on the analogous layered ferromagnetic Ising model at mean field critical temperature (Fontes et al. (2015)).

We first deal with the critical case when $\lambda = 1$. If $\epsilon = \kappa N^{-2/5}$, we see a phase transition in κ: positive and finite constants C_1, C_2 exist so that there is no percolation if $\kappa < C_1$ while percolation occurs for $\kappa > C_2$. The derivation of the scaling limit is inspired by works on the long range contact process (Mueller and Tribe (1995)). The proof relies on the analysis of the scaling limit of the critical branching random walk that dominates the growth process restricted to each horizontal layer and a careful analysis of the true horizontal growth process, which is interesting by itself. A renormalization argument is used for the percolative regime.

We then deal with the supercritical case when $\lambda > 1$. If $\epsilon = e^{-\kappa N}$, we can also see a phase transition in κ. The horizontal and vertical edges can be discovered through subordinate process in each regime. The proof is based on the analysis of supercritical branching random walk but several levels of attritions are introduced to make sure the independent structure. The comparison between our original percolation and the percolation on the inhomogeneous square lattice is used in the renormalization scheme.

Keywords: *Percolation, renormalization argument, branching random walk, critical scaling.*

Résumé

Dans cette thèse, nous considérons une percolation par arêtes anisotrope et de portée finie sur $\mathbb{Z}^2$. Sur chaque couche horizontale $\{(x,i) : x \in \mathbb{Z}\}$, avec $i \in \mathbb{Z}$, nous avons des arêtes $\langle (x,i),(y,i)\rangle$ pour tous $x, y \in \mathbb{Z}$ vérifiant $1 \le |x-y| \le N$, pour $N \in \mathbb{N}^*$. Il existe également des arêtes verticales reliant les sommets avec leurs deux voisins directs sur les couches verticales $\langle (x,i),(x,i+1)\rangle$ pour $x, i \in \mathbb{Z}$. Sur ce graphe, nous considérons le modèle de percolation anisotrope suivant : les arêtes horizontales sont ouvertes avec probabilité $\lambda/(2N)$ avec $\lambda \ge 1$, alors que les arêtes verticales sont ouvertes avec probabilité $\epsilon = \epsilon(N)$, qui sera convenablement choisi lorsque N tend vers l'infini. Ce modèle est motivé par un résultat sur le modèle d'Ising à couches analogues dans un champ moyen à température critique (Fontes et al. (2015)).
Nous étudions dans un premier temps le cas critique où $\lambda = 1$. Pour $\epsilon = \kappa N^{-2/5}$, nous montrons l'existence d'une transition de phase en κ : il existe alors une constante $C_1 > 0$ telle que la percolation n'a pas lieu pour $\kappa < C_1$ et une constante C_2, à partir de laquelle la percolation se produit ($\kappa > C_2$). La preuve repose sur l'analyse de la limite d'échelle de la marche aléatoire branchante critique dominant le processus de croissance restreint à chaque couche horizontale et une analyse minutieuse du véritable processus de croissance horizontal, qui est intéressant en soi. Un argument de renormalisation est utilisé pour le régime percolatif et la méthode utilisée pour dériver la limite d'échelle est inspirée des travaux de Mueller and Tribe (1995) sur le processus de contact de longue portée.
Nous étudions aussi le cas surcritique avec $\lambda > 1$. Les arêtes horizontales et verticales peuvent être découvertes grâce à des processus subordonnés dans chaque régime. Nous montrons, dans le ca où $\epsilon = e^{-\kappa N}$, l'existence d'une transition de phase en κ. La preuve est basée sur l'analyse de la marche aléatoire branchante surcritique munie de plusieurs niveaux d'attrition assurant une structure indépendante. La comparaison entre notre percolation originale et la percolation inhomogène sur $\mathbb{Z}^2$ est utilisée dans le schéma de renormalisation.

Mots-clefs : *Percolation, argument de renormalisation, marche aléatoire de branchement, l'échelle critique.*

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1 Introduction

This thesis concerns an anisotropic percolation model and we will study some critical values of this model. Our anisotropic percolation is an extension of classical percolation. Percolation is a simple probabilistic model to study the behaviour of a certain *fluid* flowing in a random porous *medium*. In the classical (bond) percolation, the *medium* is modelled by a graph $\mathbb{L}^2$ on $\mathbb{Z}^2$ with edges $\langle x, y\rangle$ connecting $x, y \in \mathbb{Z}^2$ that satisfy $\|x-y\|_1 = |x_1 - y_1| + |x_2 - y_2| = 1$, and the random structure equipped in this *medium* is to make the passing probability of the *fluid* along each edge be $p \in [0,1]$. In higher dimensions, the medium can be modelled by a graph $\mathbb{L}^d, d \geq 2$ on $\mathbb{Z}^d$ with edges $\langle x, y\rangle$ connecting $x, y \in \mathbb{Z}^d$ that satisfy $\|x-y\|_1 = 1$.

More generally, the *medium* can be an (infinite) graph G with a set of countable edges E and a set of vertices V. Percolation can be either bond type (like the example above) or site type. In the bond percolation, each edge $e \in E$ can be open or closed. We can encode a configuration function $\omega : E \to \{0,1\}$, where $\omega(e) = 1$ denotes the edge to be open, meaning that the fluid passes through the bond e, and 0 if the edge is closed. The open edges form open clusters which are random subgraph of G. *Percolation* occurs when there is an infinite open cluster containing the origin for bond percolation. In the site percolation on $G = (V, E)$, each vertex $v \in V$ can be open or closed. We can then also define a state function $\omega : V \to \{0,1\}$ with the same meanings of 1 and 0 to be open and closed. v and v' are connected if there exists a finite path from v to v': there is a sequence of sites $v_1 = v, \cdots, v_n = v'$ so that $\|v_i - v_{i-1}\| = 1$, $1 < i \leq n$ and $\omega(v_i) = 1$ for $1 \leq i \leq n$. This partitions the vertex set into fully connected subsets. Percolation occurs if the connected subset containing the origin is infinite.

The oldest bond percolation model can be traced back to the Bernoulli percolation introduced by Broadbent and Hammersley (1957). In this model, a random environment is imposed on the *medium*. Each edge is open with probability p, i.e. $\omega(e), e \in E$ are independent Bernoulli random variables with parameter p. Let $C \subset G$ be the collection of open edges and vertices connected by the open edges containing the origin. Mathematicians are interested in studying the connectivity structure of this random subgraph C.

A fundamental problem is for which values of p can we observe an infinite open cluster (containing 0) as p increases on a multidimensional graph, e.g. $\mathbb{Z}^d$. A principal quantity in the translation invariant system is the percolation probability, being the probability that there is an infinite cluster C containing the origin,

$$\theta(p) = \mathbf{P}_p(|C| = \infty).$$

In fact, if $\theta(p) > 0$, $\mathbf{P}_p(\exists$ a unique infinite cluster$) = 1$.

On $\mathbb{L}^d$, the uniqueness of infinite cluster was first shown by Aizenman et al. (1987) and then Burton and Keane (1989) used a beautiful trifurcation argument to prove this result. To see $\theta(p) = 0$ when p is small, one simply observes that the probability that the origin is connected to a site of L^1-distance n is less than $(2dp)^n$ since there are less than $(2d)^n$ self avoiding paths of length n. With the help of Peierl's argument (Hammersley (1959)), one can also find that as long as p sufficiently close to 1, $\theta(p) > 0$. It is easily seen $\theta(p)$ is monotone in p as in Figure 1.1. This shows an existence of *phase transition* in p: for values p below a certain threshold, the connected component containing 0 is finite and once p is above the threshold, this connected component containing 0 is infinite with positive probability.

We define

$$p_c(G) = \sup\{p : \theta(p) = 0\}.$$

By monotonicity, p_c is critical in the following sense

$$\theta(p) \begin{cases} = 0 & \text{if } p < p_c, \\ > 0 & \text{if } p > p_c. \end{cases}$$

When $G = \mathbb{L}^1$, it is of no interest, since whenever $p < 1$, there are infinitely many close edges almost surely, and hence no infinite open cluster. This implies $p_c(\mathbb{L}^1) = 1$. When $d \geq 2$, we expect that the percolation probability behaves in the following manner $\theta(p_c) = 0$. The behaviour of θ at criticality p_c is unknown in general and we even do not know $\theta(p_c)$ on $\mathbb{L}^d$ with $3 \leq d \leq 10$ (ref. Fitzner and van der Hofstad (2017)). But we can give some explicit answers for bond percolation when $d = 2$.

For Bernoulli percolation on the square lattice $\mathbb{Z}^2$, the lower bound of $p_c(\mathbb{Z}^2)$ was given by Harris (1960) who showed that $\theta(1/2) = 0$. In 1980, Kesten (1980) showed that the critical probability on $\mathbb{Z}^2$ is exactly $1/2$ based on the Russo-Seymour-Welsh argument (Russo (1978) and Seymour and Welsh (1978)). Lots of *good* properties hold in the Bernoulli percolation on $\mathbb{Z}^d$ which is invariant under rotation $\pi/2$. First of all, the connection probability p is the same regardless of the directions. This is what we call homogeneity. Most importantly, $\mathbb{L}^2$ has a self-duality. Any configuration ω on $\mathbb{Z}^2$ is associated with a dual configuration ω^* on the dual square lattice $(\mathbb{Z}^2)^* = \mathbb{Z}^2 + \left(\frac{1}{2}, \frac{1}{2}\right)$. The configuration function

$$\omega^*(e^*) := 1 - \omega(e).$$

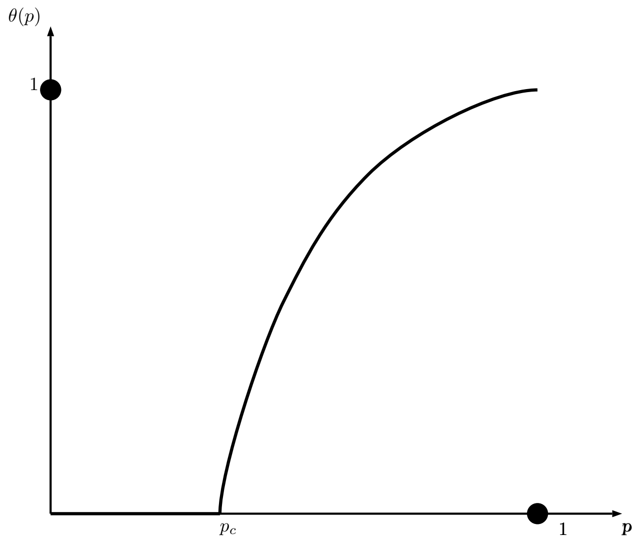

Figure 1.1 – Percolation probability

Heuristically, the dual edge e^* is open if the primal edge e is close. We can see that $\omega^*(e^*)$ are independent Bernoulli random variables with parameter $p^* := 1 - p$. From here, it is natural to guess that p_c is the value when $p = p^* = 1/2$. It is natural to consider the behaviour of the percolation system at criticality. In general, it is very difficult to calculate the exact values of p_c except in several special cases. Since the eighties, thanks to the new tools like discrete complex analysis, the explicit values for percolation and Ising model on the planar graph were calculated from a new perspective.

Planar Bernoulli percolation and its criticality have been thoroughly investigated since the eighties because of the good structure of square lattice or triangular lattice and the homogeneity of Bernoulli percolation. The conformal field theory suggested that at criticality, the planar Bernoulli percolation is conformal invariant. In 2001, Smirnov (2001) gave an example of conformal invariant property by showing Cardy's formula for critical planar percolation on the triangular lattice. Thanks to Cardy's formula proved by Smirnov and *Schramm-Loewner evolution* proposed by Schramm (1999), mathematicians are able to describe the scaling limit of the interface between open and closed sites. The tool *discrete complex analysis* exploited by Smirnov (2010) was used to prove many deep and beautiful results about the interface on percolation, Ising model and generalized random cluster models.

However, mathematicians started to wonder what happens if we get rid of the good conditions such as the opening probability of an edge p_e is not a constant but depends on $e \in \mathbf{E}$. Suppose the opening probability of edge $e \in E$ is p_e. When $p_e \not\equiv p$ but depends on the choice of e which means that the model is inhomogeneous, we are interested in the connectivity properties in this case. For instance, even though $\theta\left(\frac{1}{2}\right) = 0$, what happens if one vertical (or horizontal) line on $\mathbb{Z}^2$ breaks the homogeneity and there is a very strong preference to percolate on this line?

One can think of a percolation model on $\mathbb{Z}^2$ but for a fixed point $x \in \mathbb{Z}$,

$$p_{\langle y,z\rangle} = \begin{cases} p_x & \text{if } y = (v,x), z = (v+1,x), \text{ for } v \in \mathbb{Z}, \\ p & \text{otherwise.} \end{cases}$$

It is clear that if $p_x = 1$, then percolation occurs for any p. When $p_x < 1$, it was shown by Madras et al. (1994) that percolation does not occur when $p < p_c$ and by Barsky et al. (1991) percolation occurs when $p > p_c$. As long as $p_x < 1$, it does not change the result of Harris.

Theorem 1.1 (Zhang (1994))**.** *On the square lattice $\mathbb{Z}^2$, for any $0 \le p_x < 1$, there is no percolation at $p = p_c = \frac{1}{2}$, i.e. $\theta\left(\frac{1}{2}\right) = 0$.*

A similar model called Brochette percolation was introduced by Duminil-Copin et al. (2018). Here, instead of along a certain vertical column $E(x \times \mathbb{Z})$, the configuration is inhomogeneous on a collections of vertical columns $E_{ver}(\Lambda \times \mathbb{Z}) = \{\langle (x_1,x_2),(x_1,x_2+1)\rangle : x_1 \in \Lambda, x_2 \in \mathbb{Z}\}, \Lambda \subset \mathbb{Z}$. The configuration $\{\omega(e), e \in E(\mathbb{Z}^2)\}$ with distribution $\mathbf{P}^{\Lambda}_{p,q}$ are then independent Bernoulli random variables with parameters

$$p_e = \begin{cases} p & \text{if } e \in E_{ver}(\Lambda \times \mathbb{Z}) \\ q & \text{if } e \notin E_{ver}(\Lambda \times \mathbb{Z}). \end{cases}$$

Suppose the columns of Λ are chosen randomly following a product probability measure ν_ρ, under which measure, column $x \times \mathbb{Z}, x \in \mathbb{Z}$ is selected with probability $\rho \in [0,1]$. It was shown that a small sub-criticality on q will not influence the existence of percolation if $p > 1/2$.

Theorem 1.2 (Duminil-Copin et al. (2018))**.** *For any $\varepsilon \in (0,1/2]$ and $\rho > 0$, there exists $\delta > 0$ so that*

$$\mathbf{P}^{\Lambda}_{p_c+\varepsilon, p_c-\delta}(|C| = \infty) > 0, \quad \nu_\rho - \textit{ almost surely.}$$

This extends the result of Kesten (1982) about percolation on inhomogeneous square lattice. The inhomogeneous percolation considered by Kesten is special case of Brochette percolation when $\Lambda = \mathbb{Z}$, then q is the opening probability of horizontal edges and p is the opening probability of vertical edges. It was shown that $\theta(p,q) > 0$ if $p + q > 1$ and $p + q = 1$ is called informally 'the critical surface'. We emphasize that to find the critical value p in homogeneous percolation or the critical relationship between p and q in inhomogeneous percolation is a very sophisticated work.

In fact, anisotropy does exist in many fields of nature science. Liquid crystals are anisotropic liquids and anisotropic magnet may occur in the plasma. However, the *good* property of nearest-neighbour transition may not correspond to the nature. This issue arose in the Ising model and a model was proposed by Kac et al. (1964) where the interaction between two spins in a magnetic field was finite-range rather than only nearest-neighbour. This area seems separate from percolation, but in fact, these two models can be coupled in Fortuin-Kasteleyn percolation (introduced in detail later). Hence it is very natural to consider the finite range interaction in percolation.

This was discussed by a series papers Kac et al. (1964) about Ising model when the distribution of energy follows a finite-range law. To be more precise, consider a one-dimensional Kac-Ising model as in Cassandro et al. (1993) which is a one-dimensional spin system with values ± 1. The spin at site $x \in \mathbb{Z}$ is σ_x. The system can be of infinite volume with $x \in \mathbb{Z}$ or of finite volume $x \in \Lambda \subset \mathbb{Z}$. The configuration space is then $\{-1,+1\}^{\mathbb{Z}}$ or $\{-1,+1\}^{\Lambda}$ respectively. A Kac potential is a $L^1(\mathbb{R})$ function $J_\gamma(r), \gamma > 0, r \in \mathbb{R}$ such that $J_\gamma(r) = \gamma J(\gamma r)$ satisfies the following conditions:

- $J(r) = 0$ for $|r| \geq 1$
- $J(r) > 0$ for $|r| < 1$
- $J(r) = J(-r)$
- $J(r)$ is continuous in $[-1,1]$ and $J'(r)$ is bounded in $(-1,1)$
- $\int J(r)dr = 1$.

The Gibbs measure with Kac potential on finite graph Λ with free boundary condition is

$$\mathbf{P}_\gamma^\Lambda(\sigma) = \frac{1}{Z_\gamma^\Lambda} \exp(-H_\gamma^\Lambda(\sigma)),$$

where Z_γ^Λ is the normalization factor and the Hamiltonian

$$H_\gamma^\Lambda(\sigma) = -\beta \sum_{x \neq y \text{ and } x,y \in \Lambda} J_\gamma(|x-y|)\sigma_x \sigma_y. \tag{1.1}$$

Suppose now we discard one more *good* property of percolation on $\mathbb{L}^2$ that the connection of edges is no longer nearest-neighbour. Our model will be a finite-range and inhomogeneous percolation on $\mathbb{Z}^2$. The transition inside each horizontal layer $\mathbb{Z} \times i, i \in \mathbb{Z}$ follows a finite-range law similar to that of Kac et al. (1964) and the transition between nearest layers $\mathbb{Z} \times i$ and $\mathbb{Z}^d \times \{i \pm 1\}$ is different from the horizontal layer. Intuitively speaking, we can imagine layers of materials, the energy transitions inside the layer and between layers are different. Besides, we allow the interaction inside a layer within a finite range.

Our aim is to find the critical relation between horizontal transition and vertical transition, then get a much more complex relation than 'the critical surface' : $p + q = 1$.

Outline of the thesis

The thesis is structured as follows. Chapter 2 states the main results of this thesis and introduces several important prerequisite tools and areas in the proof including stochastic partial differential equations (SPDEs), the martingale problem (MP) and the renormalization argument.

Chapter 3 and 4 study the finite-range anisotropic percolation on $\mathbb{Z}^2$ when the horizontal transition is critical. Chapter 3 shows the weak convergence of the horizontal movements to a certain SPDE. An envelope process which dominates the horizontal process is introduced and its weak convergence is also proved in Chapter 3. The relation between the two weak limits is also discussed by a MP perspective.

After getting the weak limits of the envelope process and horizontal process, we then in Chapter 4 use the former one to find the *upper* bound behaviour of the critical relationship, above which, percolation does occur. We then use the latter one and renormalization argument to find the *lower* bound behaviour of the critical relationship, below which, we cannot observe a percolation.

Chapter 5 investigates the case when horizontal transition is supercritical. The critical relationship between transition inside a layer and transition between layers is shown by renormalization argument and a meticulous treatment with the supercritical branching random walk.

2 Results and Prerequisites

In this chapter, we first state our model including the anisotropic percolation we consider, the horizontal process and the envelope process. We then review some tools that we will use in the proof, such as SPDEs, weak convergence, martingale problems and renormalization (or block) arguments.

2.1 The anisotropic percolation on $\mathbb{Z}^2$

We have seen the importance of inhomogeneity in percolation, hence we consider a percolation model that is anisotropic in two ways:

- The horizontal interaction is finite-range but the vertical interaction is nearest-neighbour
- The connection probabilities along horizontal edges and vertical edges are different.

For this, we let $\mathbb{Z}^2 = (V, E)$ be the graph with vertex set $V = \{v = (x, i) : x \in \mathbb{Z}, i \in \mathbb{Z}\}$ and edge set

$$E = \{e = \langle v_1, v_2 \rangle : v_k = (x_k, i_k), k = 1,2; \text{ either } x_1 = x_2, |i_1 - i_2| = 1 \text{ or } i_1 = i_2, 1 \leq |x_1 - x_2| \leq N\}.$$

The edges are assigned in two senses: horizontal sense and vertical sense. Horizontally, we can draw edge between two points within distance N. Vertically, we can draw edge between two points within distance 1. The edge set can be then partitioned into two disjoint subsets $E = E_h \cup E_v$. E_h is the set of horizontal edges s.t. $E_h = \{e = \langle v_1, v_2 \rangle : i_1 = i_2\}$ and E_v is the set of vertical edges s.t. $E_v = \{e = \langle v_1, v_2 \rangle : x_1 = x_2\}$ (here (x_k, i_k) corresponds to $v_k, k = 1,2$). The opening probability of each edge p_e is inhomogeneous in the following sense. Each horizontal edge is open with probability $\lambda/(2N), \lambda > 0$ and each vertical edge is open with probability $\epsilon(N)$, and they are all independent of each other.

$$p_e = \begin{cases} \frac{\lambda}{2N} & \text{if } e \in E_h \\ \epsilon(N) & \text{if } e \in E_v. \end{cases} \tag{2.1}$$

Horizontally, the opening probability $\frac{\lambda}{2N}$ corresponds to the Kac-Ising model introduced in Chapter 1 when the Kac potential J is uniform in $[-1,1]$, the scaling factor $\gamma = \frac{1}{N}$ and the inverse temperature $\beta = \lambda$. Two models can be coupled by the random-cluster measure introduced later.

Remark. *Notice that the λ here does not have to be constant. The only thing that matters here is that the expected number of edges connecting a certain site is bigger, equal or smaller than the one which corresponds to supercritical, critical and subcritical branching random walk. This criticality of the expected number of edges connecting with a certain site also relates to a more general setting of J in the Kac potential when $\int J(r)dr > 1, = 1$ or < 1.*

Our main purpose is to find the criticality of $\epsilon(N)$ as N tends to infinity. $\lambda = 1$ corresponds to the critical horizontal mechanism and $\lambda > 1$ corresponds to supercritical horizontal mechanism. Heuristically speaking, if we only focus on one horizontal layer $\mathbb{Z} \times i$, $\lambda = 1$ means that the expected number of open horizontal edges connected from (x,i) is 1. We will deal with the critical case when $\lambda = 1$ in Chapter 3 and 4 which is the main concern, then deal with the case of $\lambda > 1$ in Chapter 5, but the case when $\lambda < 1$ is of no interest since percolation never appears if ϵ is small enough (depending on λ but being uniform in N).

As was discussed in Chapter 1, one motivation of the finite-range comes from a series of works Kac et al. (1964) and Kac and Helfand (1963), where the finite-range Kac-Ising model was introduced. Addition of inhomogeneity to the Kac-Ising model was raised in Fontes et al. (2015), where the authors investigated the existence of phase transition for an anisotropic Ising spin system on the square lattice $\mathbb{Z}^2$. On each horizontal layer, $\{(x,i) : x \in \mathbb{Z}\}$, the $\{-1,+1\}$-valued spins $\sigma(x,i)$ interact through a ferromagnetic Kac potential at the mean field critical temperature. The distribution of configuration σ on finite interval $\Lambda_L = [-L, L]$ is

$$\mathbf{P}_\gamma^{\Lambda_L,\pm}(\sigma) = \frac{1}{Z}\exp\left(-H_\gamma^{\Lambda_L,\pm}(\sigma)\right),$$

where $\pm$ correspond to $+1$ and -1 boundary conditions. $H_\gamma^{\Lambda_L,\pm}(\sigma)$ is the Hamiltonian of the configuration σ as in (1.1),

$$H(\sigma) = -\sum_{x,y\in\Lambda_L} J_\gamma(x,y)\sigma(x,i)\sigma(y,i), \sum_{y\neq x} J_\gamma(x,y) = 1,$$

where $J_\gamma(x,y) = c_\gamma \gamma J(\gamma(x-y))$ and one assumes $J(r), r \in \mathbb{R}$ to be smooth and symmetric with support in $[-1,1]$, $J(0) > 0$, $\int J(r)dr = 1$ and moreover c_γ is the normalization constant ($c_\gamma \to 1$ as $\gamma \to 0$). To this one adds a small nearest-neighbour vertical interaction to $H(\sigma)$,

$$H(\sigma) = -\sum_{(x,i)\in\Lambda_L^2} J_\gamma(x,y)\sigma(x,i)\sigma(y,i) - \epsilon\sigma(x,i)\sigma(x,i+1).$$

The authors proved that given any $\epsilon > 0$, for all $\gamma > 0$ small $\mu_\gamma^+ \neq \mu_\gamma^-$, where $\mu_\gamma^+, \mu_\gamma^-$ denote the Dobrushin-Lanford-Ruelle (DLR) measures obtained as thermodynamic limits of the Gibbs measures with $+1$, respectively -1 boundary conditions (as $L \to \infty$). The authors conjectured that if $\epsilon = \epsilon(\gamma) = \kappa\gamma^{2/3}$, we shall see a different behaviour while varying κ.

The bridge between Ising model and percolation is connected through a big class of model called Fortuin-Kasteleyn percolation (also called random-cluster model). The conjecture in anisotropic Ising model above can help us to guess the vertical interaction $\epsilon(N)$ in our anisotropic percolation model (2.1). Conversely, the results in percolation can open a vision in random cluster model which covers Ising and Potts model through FKG comparison theorem.

A *random-cluster measure* on $G = (V, E)$, a sub-graph of $\mathbb{Z}^d$ with boundary condition $\xi \in \{0,1\}^{E(\mathbb{Z}^d)\setminus E}$ and two parameters: $p \in (0,1)$ and $q > 0$ is defined on $\{0,1\}^E$ such that

$$\phi^{\xi}_{p,q}(\omega) = \frac{1}{Z^{\xi}_{p,q}} \left\{ \prod_{e\in E} p^{\omega(e)}(1-p)^{1-\omega(e)} \right\} q^{k_\xi(\omega)},$$

where $k(\omega)$ is the number of open clusters including isolated vertices in $\omega \cup \xi$ and $Z^{\xi}_{p,q}$ is the normalization constant. The shape parameter q indicates a *favour* of clusters. $q \le 1$ favours fewer clusters and $q > 1$ favours more clusters.

Two extremal cases $\xi = 0, 1$ of special importance are the boundary configurations with all edges close or all edges open respectively. $\xi = 0$ denotes the free boundary condition when $\forall e \in E(\mathbb{Z}^d)\setminus E, \xi(e) = 0$. $\xi = 1$ denotes the wired boundary condition which refer to the fact that all the edges in $E(\mathbb{Z}^d)\setminus E$ are connected together and only contribute 1 in $k_\xi(\omega)$.

To find the appearance of percolation in random-cluster model, which covers the appearance of percolation in percolation model and appearance of phase transition in Ising model by coupling, we need to define the limit measure as the sub-graph tends to $\mathbb{Z}^d$.

Definition 2.1. *Let $p \in [0,1]$ and $q \in (0,\infty)$. A probability measure ϕ on $(\Omega, \mathscr{F})$ is called a limit-random-cluster measure with parameters p and q if for some boundary condition $\xi \in \Omega$, there exists a sequence $(\Lambda_n, n = 1, 2, \cdots)$ of boxes such that $\Lambda_n \uparrow \mathbb{Z}^d$ as $n \to \infty$, and*

$$\phi^{\xi}_{\Lambda_n,p,q} \Rightarrow \phi \text{ as } n \to \infty.$$

The set of all such limit measures ϕ is denoted by $\mathscr{W}_{p,q}$.

It was shown (Grimmett (2009)) that for either $b \in \{0,1\}$, the weak limits

$$\phi^b_{p,q} = \lim_{n\to\infty} \phi^b_{\Lambda_n,p,q}$$

exist and are independent of the choice of $(\Lambda_n)_{n\ge 1}$.

There is a second way to construct the infinite-volume random-cluster measure based on Dobrushin-Lanford-Ruelle (DLR) Gibbs states.

Definition 2.2. *Let $p \in [0,1]$ and $q \in (0,\infty)$. A probability measure ϕ on $(\Omega, \mathscr{F})$ is called DLR* *random-cluster measure with parameters p and q if for all* $A \in \mathscr{F}$ *and finite boxes* $\Lambda \subset \mathbb{Z}^d$,

$$\phi(A \mid \mathscr{T}_\Lambda) = \phi^{\xi}_{\Lambda,p,q}(A) \text{ for } \phi - a.e.\xi,$$

where $\mathscr{T}_\Lambda$ *is the tail* σ*-field generated by the states of edges on* $E(\mathbb{Z}^d)\backslash E(\Lambda)$.

The special case $q = 1$ corresponds to percolation, $q = 2$ corresponds to Ising model and $q \in \{2,3,\cdots\}$ corresponds to general Potts model with q local states. Here we give an example to show how the random cluster model with $q = 2$ is coupled to the Ising model (Edwards and Sokal (1988)).

Proposition 2.1. *Suppose* $\omega \sim \phi_{p,2}$. *Sample an i.i.d. family of* ± 1 *random variables* $(\sigma_\varphi)_\varphi$ *following Bernoulli*$(1/2)$ *induced by the connected component* φ *of* ω. *Set* $\sigma_x = \sigma_\varphi$ *for any* $x \in \varphi$ *and* $\beta = -\frac{1}{2}\log(1-p)$, *then* $\sigma \sim \mathbf{P}^G_\beta$, *which is the Ising measure with free boundary condition on* G.

Proof. Denote $\omega \sim \sigma$ if $\omega_{\langle x,y\rangle} = 1$ implies $\sigma_x = \sigma_y$. The joint probability measure of ω and σ is

$$\mathbf{P}((\omega,\sigma)) = \frac{1}{Z_{p,2}}\left(\frac{p}{1-p}\right)^{|\omega|} 2^{k(\omega)}2^{-k(\omega)}\mathbf{I}_{\omega\sim\sigma}.$$

Then sum over ω,

$$\begin{aligned}
\mathbf{P}(\sigma) &= \frac{1}{Z_{p,2}}\sum_{\omega\sim\sigma}\left(\frac{p}{1-p}\right)^{|\omega|}\\
&= \frac{(1-p)^{-|E|}}{Z_{p,2}}\sum_{\omega\sim\sigma} p^{|\omega|}(1-p)^{|E|-\omega}, \text{ let } E(\sigma) = \{e = xy : \sigma_x \neq \sigma_y\}\\
&= \frac{(1-p)^{|E(\sigma)|}}{Z_{p,2}(1-p)^{|E|}}\underbrace{\sum_{\omega\in\{0,1\}^{E\backslash E(\sigma)}} p^{|\omega|}(1-p)^{|E\backslash E(\sigma)|-|\omega|}}_{=1}\\
&= \frac{e^{\beta|E|}}{Z_{p,2}(1-p)^{|E|}}e^{-\beta H(\sigma)} \text{ since } H(\sigma) = -|E| + 2|E(\sigma)|.
\end{aligned}$$

□

The existence of phase transition in anisotropic Ising model can be formulated in terms of existence of percolation for random-cluster measure with shape parameter $q = 2$ and edge probabilities of $\{\langle v_1, v_2\rangle \in E : v_1 = (x_1, i_1), v_2 = (x_2, i_2)\}$ to be

$$p(\langle v_1, v_2\rangle) = 1 - e^{-J_\gamma(x,y)}\mathbf{I}_{\{\langle v_1,v_2\rangle\in E_h\}} - e^{-\epsilon(\gamma)}\mathbf{I}_{\{\langle v_1,v_2\rangle\in E_v\}}.$$

Fontes et al. (2015) has shown that when $\epsilon > 0$ is a constant, there exists a phase transition. To compare the measure ϕ_{p_1,q_2} and ϕ_{p_2,q_2}, we need some basic definitions on the relation between these two measures. The configuration space $\Omega = \{0,1\}^E$ is partially ordered: $\omega_1 \leq \omega_2$

if $\omega_1(e) \le \omega_2(e), \forall e \in E$. A random variable $X : \Omega \to \mathbb{R}$ is called *increasing* if $X(\omega_1) \le X(\omega_2)$ when $\omega_1 \le \omega_2$. For two probability measures μ_1, μ_2 on Ω, we say that μ_1 is stochastically smaller than μ_2, noted as $\mu_1 \le_{\mathrm{st}} \mu_2$ if $\mathbf{E}_{\mu_1}(X) \le \mathbf{E}_{\mu_2}(X)$ for all increasing random variables X on Ω.

Theorem 2.1 (Grimmett (2009) Chapter 3)**.** *Comparison inequalities:*

- $\phi_{p_1,q_1} \le_{st} \phi_{p_2,q_2}$ *if* $q_1 \ge q_2, q_1 \ge 1$ *and* $p_1 \le p_2$.
- $\phi_{p_1,q_1} \ge_{st} \phi_{p_2,q_2}$ *if* $q_1 \ge q_2, q_1 \ge 1$ *and* $\frac{p_1}{q_1(1-p_1)} \ge \frac{p_2}{q_2(1-p_2)}$.

The probability of percolation for $q = 2$ is bounded from above by that when $q = 1$. As a consequence, if there is no percolation when $q = 1$, we can conclude that there is no phase transition for Ising model.

We suppose our $\epsilon(N) = \kappa N^{-b}, b > 0$ and our aim in Chapter 3 and 4 is to find the critical value of b when $\lambda = 1$ and show the different behaviours while κ varies.

We first consider the behaviour at each single horizontal layer. For simplicity, we will consider the horizontal behaviour on layer 0. With respect to layer 0, we denote $\mathscr{C}_{\mathbf{0}}^0$ as the cluster containing $\mathbf{0} = (0,0)$:

$$\mathscr{C}_{\mathbf{0}}^0 = \{x : (0,0) = \mathbf{0} \to (x,0) \text{ with all the edges along the path in } \mathbb{Z} \times \{0\}\},$$

where $v_1 \to v_2$ means there is an open path from vertex v_1 to v_2. We can speak of generations on each horizontal layer. $x \in \mathscr{C}_{\mathbf{0}}^0$ is of k-th generation if the shortest open path from $(0,0)$ to $(x,0)$ is of length k. That means there are vertices $v_1', \cdots, v_k'$ such that $v_1' = (0,0)$, $v_k' = (x,0)$ and for any $1 \le i \le k-1$, $\langle v_i', v_{i+1}' \rangle \in E_h$ is open. Use $\tilde{G}_k^0(x) \in \{0,1\}$ to indicate if the site $(x,0)$ can be reached from $\mathbf{0}$ at k-th generations, i.e. $\tilde{G}_k^0(x) = 1$ indicates that $(x,0)$ is visited at generation k. The sites of $\{\tilde{G}_k^0\}_{k\ge 0}$ form a process very close to a branching random walk starting from 0. The difference between $\{\tilde{G}_k^0\}_{k\ge 0}$ and a critical branching random walk is the domain of the state function. Denote $\{G_k^0\}_{k\ge 0}$ as the critical branching random walk. At each time n, particles at each occupied sites branch following Binomial$(2N, 1/(2N))$ and the offsprings move to its $2N$ neighbours uniformly. The criticality means that the expected number of offspring of each particle is one. Note that it is different from the model in Lalley (2009) discussed in the next paragraph. The state function $G_k^0(x) \in \mathbb{Z}_+$ since it counts the number of particles at an occupied site. However, the process $\{\tilde{G}_k^0\}_{k\ge 0}$ only tells if the site is occupied or not, hence $\tilde{G}_k^0(x) \in \{0,1\}$. In Chapter 3, we will show that these two processes are not too different on an appropriate time scale. This motivates us to consider the asymptotic density on each horizontal layer and use it to derive the cumulated occupied sites over generations. But the introduction of generations will cause a problem in the percolation system if we only consider the branching random walk. Because we are interested in percolation, the vertical connections should be considered only once over the generations. Therefore, the true process we are considering is a branching random walk with *attrition*. Attrition is the rule that if any

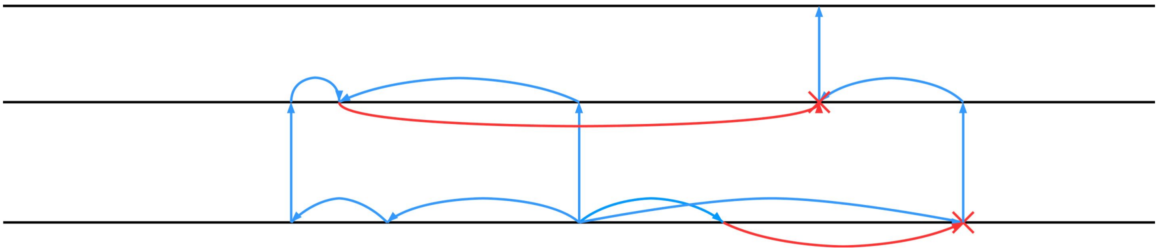

Figure 2.1 – Attrition sites noted by ×

site has been visited during the propagation, it cannot be visited again. Figure 2.1 helps to explain the problem occurrs because of generations. For example on the mid layer, since the vertical interaction only happens once, if it has already generated an vertical arrow upward after the 1 generation left movement from its right neighbour (blue leftward arrow from the right-most site), then the red rightward arrow from the left-most site after 2 generations should not be counted again.

The way of dealing with horizontal propagation is motivated by the work of Lalley (2009) on the scaling limit of spatial epidemics on the one-dimensional lattice $\mathbb{Z}$ to Dawson-Watanabe process with killings. There are two processes (the *SIR* epidemics and the *SIS* expidemics) considered in Lalley (2009). At each site i, there is a fixed population (or village) of N individuals and each of them can be either susceptible, infected or recovered (in the *SIR* epidemics). The model runs in discrete time; an infected individual recovers after a unit of time; in the *SIR* epidemics, infected individuals recover and are immune from infection, while in the *SIS* epidemics, infected individuals become again susceptible after recoverty. An infected individual may transmit the infection to randomly selected (susceptible) individuals in the same or in the neighbouring villages. Denote $p_N(i,j)$ as the transmission probability between any infected particle at site i and any susceptible particle at site $j = i+e$, where $e = 0$ or ± 1. For any pair (x_i, u_j) of infected and susceptible individuals located at i and j respectively with $|i-j| \leq 1$, the transmission probability is taken as

$$p_N(i,j) = \frac{1}{3N},$$

which makes it asymptotically critical as $N \to \infty$. The evolution of this dynamics can be studied with the help of a branching random walk envelope: any individual at site x and time t lives for one unit and reproduces, placing a random number of individuals at a nearest site (village) y with $|y-x| \leq 1$, where the random number is of law Binomial$(N, 1/(3N))$. An infected individual will stay infected for one unit of time then get recovered and cannot get infected again (immune). The individuals are categorized into Susceptible, Infected and Recovered (SIR) among which the number of infected and recovered (immune) individuals at site $x \in \mathbb{Z}$ and time $n \in \mathbb{N}$ are denoted by $Y_n^N(x)$ and $R_n^N(x)$. Lalley (2009) studied the scaling limit (space factor $N^{\beta/2}$ and time factor N^{β}) of this system, namely

$$X^N(t,x) := \frac{Y^N_{[N^\beta t]}(\sqrt{N^\beta}x)}{\sqrt{N^\beta}} \text{ for } x \in \mathbb{Z}/\sqrt{N^\beta}.$$

Initially, the support $\mathrm{supp}(X^N(0,\cdot)) \subset J$, where J is compact. As $N \to \infty$,

$$X^N(t,x) \Rightarrow X(t,x)$$

where $X(t,x)$ is the density of a Dawson-Watanabe process X_t with initial density $X(0,x)$ and killing rate $\theta(t,x)$ depending on the choice of β. When $\beta < 2/5$, $\theta(t,x) = 0$ and when $\beta = 2/5$,

$$\theta(t,x) = \int_0^t X(s,x)ds.$$

Heuristically speaking, when $\beta < 2/5$, the cumulation of immune individuals over a time period $[tN^\beta]$ is negligible. However, when $\beta = 2/5$, the cumulated immune individuals will make a significant contribution to the deduction of infected individuals.

To study the scaling limit of our process on horizontal level, we first need to perform space and time rescaling on the approximate density. First, we have to scale the space with N, then the movement of the edges from x will have a uniform displacement on $(x+[-1,1]) \cap (\mathbb{Z}/N)\backslash\{0\}$. Then, to get the weak convergence, we will renormalize the space and time with N^α and $N^{2\alpha}$ respectively. The state of the process at time $n \in \mathbb{Z}^+$ is given by $\hat{\xi}_n(\cdot) : \mathbb{Z}/N^{1+\alpha} \to \{0,1\}$. $\hat{\xi}_n(x) = 0$ indicates that the site x is vacant and $\hat{\xi}_n(x) = 1$ indicates that the site x is occupied. Two sites are neighbors in the scaled space, denoted by $y \sim x$ if $|x-y| \le N^{-\alpha}$ (or $j \sim i$ if $|j-i| \le N$ in the unrescaled space). We are going to use the idea in Mueller and Tribe (1995) to derive the asymptotic approximate density of

$$(A\hat{\xi})(x) = \frac{1}{2N^\alpha}\sum_{y\sim x}\hat{\xi}(y)$$

and study its limit after the above mentioned time change.

Since we are considering the existence of percolation, to consider the infinite cluster containing $(0,0)$ is equivalent to consider $2\lfloor N^{2\alpha}\rfloor$ equally spaced particles on $\{-\lfloor N^{1+\alpha}\rfloor,\dots,0,\dots,\lfloor N^{1+\alpha}\rfloor\}$ (so the distance between particles in $\mathbb{Z}$ is of order $N^{1-\alpha}$). Indeed, if we denote $[-r,r]_N = [-r,r]\cap\mathbb{Z}/N^{1+\alpha}$ as the rescaled discrete interval, to show percolation we may take an initial condition $\hat{\xi}_0$ with finite support, such that $A(\hat{\xi}_0)(x) = 1$ for $x \in [-1,1]_r$ and whose linear interpolation tends (as $N \to \infty$) to a continuous function f with compact support such that $f(x) = 1$ for $x \in [-1,1]$. For simplicity, we may take f to vanish outside $[-1-\delta, 1+\delta]$ for some $\delta > 0$ fixed, and linear in $[-1-\delta, 1]$ and $[1, 1+\delta]$.

The method in Lalley (2009) is to calculate the log-likelihood function with respect to a branching envelope with known asymptotic density. However, we do not have the log-likelihood function in our case. A more standard argument is to show the weak convergence of the rescaled continuous-time particle system by verifying the tightness criteria as Ethier and Kurtz (2009) like in Cox et al. (2000), Durrett and Perkins (1999) and Mueller and Tribe (1995). We will mainly refer to the way of Mueller and Tribe (1995) dealing with long-range contact process and long-range voter model and adapt it to our discrete (space) model to get the asymptotic SPDE. Our strategy on the horizontal layer is to derive the asymptotic density of the branching random walk without attrition dominating the true system, where the states are denoted by $\xi(x)$. We call this process without attrition the envelope process and the state function

$$\xi_n(x) : \mathbb{Z}/N^{1+\alpha} \to \mathbb{Z}_+.$$

The mechanism of this envelope process is as follows. The number of particles at site x will increase by 1 if one of its neighbours branches following Binomial$(2N, 1/(2N))$ and then chooses x uniformly among the $2N$ neighbours. The number of particles at site x after $n+1$ steps is

$$\xi_{n+1}(x) = \sum_{y\sim x} \sum_{w=1}^{\xi_n(y)} \eta_{n+1}^{w}(y,x),$$

where $(\eta_{n+1}^{w}(y,x))_{w,n,y,x}$ is an i.i.d. sequence with distribution Bernoulli$(1/(2N))$.

The horizontal process $\hat{\xi}$ is dominated by this envelope process in two senses: $\hat{\xi}_n(\cdot)$ does not allow multiple particles at one site and any site visited before cannot be visited again. At the end of Section 3.1, we will show that the probability of multiple particles at one site is very small, of order $O(N^{2(\alpha-1)})$ which is negligible when $\alpha < 1$.

Section 3.1 considers the asymptotic behaviour (as $N \to \infty$) of the approximate density function of the dominating envelope process

$$A(\xi_{\lfloor tN^{2\alpha}\rfloor})(x) = \frac{1}{2N^{\alpha}} \sum_{y\sim x} \xi_{\lfloor tN^{2\alpha}\rfloor}(y)$$

extended to $\mathbb{R}$ as the linear interpolation of its values on $\mathbb{Z}/N^{1+\alpha}$. This is made precise in Theorem 2.2 below.

Remark. *The same interpolation is used when considering the approximate density of the process* $\hat{\xi}$.

Setting $\mathrm{e}_\lambda(x) = \mathrm{e}^{\lambda|x|}$ for $\lambda \in \mathbb{R}$, we define

$$\mathscr{C} = \left\{f : \mathbb{R} \to [0,\infty) \text{ continuous with } |f(x)e_\lambda(x)| \to 0 \text{ as } |x| \to \infty, \forall \lambda < 0\right\},$$

to which we give the topology induced by the norms $(\|\cdot\|_\lambda, \lambda < 0)$, where

$$\|f\|_\lambda = \sup_x |f(x)e_\lambda(x)|.$$

In the following convergences (Theorem 2.2 and Theorem 2.3), we consider the law of $A(\xi)$ or $A(\hat{\xi})$ in the space $D([0,\infty), \mathscr{C})$, the space of $\mathscr{C}$-valued paths equipped with Skorohod topology.

Theorem 2.2. *Assume that as $N \to \infty$, $A(\xi_0)$ converges in $\mathscr{C}$ to a continuous function f with compact support. For any $\alpha > 0$, as $N \to \infty$, $A(\xi_{\lfloor tN^{2\alpha} \rfloor})(x)$ converges in law to $u_t(x)$, which is the solution to one dimensional Dawson-Watanabe process:*

$$\begin{cases} \frac{\partial u_t}{\partial t} = \frac{1}{6}\Delta u_t + \sqrt{u_t}\dot{W}(t,\cdot) \\ u_0 = f, \end{cases} \tag{2.2}$$

where Δ is the Laplacian operator acting in the spatial coordinates and $\dot{W}$ is the space-time white noise.

Regarding the real horizontal process, we have to deduct the attritions from the envelope process in two ways: the state at each site can only be occupied or vacant and any site can only be visited once (refer Figure 2.1). The state function is then

$$\hat{\xi}_n(x) : \mathbb{Z}/N^{1+\alpha} \to \{0,1\}.$$

The mechanism can be expressed as

$$\hat{\xi}_{k+1}(x) = \begin{cases} 1 & \text{if } \sum_{j\le k} \hat{\xi}_j(x) = 0 \text{ and } \sum_{y\in\mathscr{N}_k(x)} \eta_{k+1}(y,x) \ge 1, \\ 0 & \text{otherwise,} \end{cases}$$

where $\mathscr{N}_k(x) = \{y \sim x : \hat{\xi}_k(y) = 1\}$ having cardinality $N_k(x) = \sum_{y\sim x} \hat{\xi}_k(y)$ and $(\eta_{k+1}(y,x))_{k,y,x}$ is an i.i.d. sequence with distribution Bernoulli$(1/(2N))$.

Theorem 2.3. *Assume that as $N \to \infty$, $A(\hat{\xi}_0)$ converges in $\mathscr{C}$ to a continuous function f with compact support. When $\alpha = 1/5$, as $N \to \infty$, $A(\hat{\xi}_{\lfloor tN^{2\alpha} \rfloor})(x)$ converges in law to $\hat{u}_t(x)$, which is the unique in law solution to the following SPDE*

$$\begin{cases} \frac{\partial \hat{u}_t}{\partial t} = \frac{1}{6}\Delta \hat{u}_t - \hat{u}_t \int_0^t \hat{u}_s ds + \sqrt{\hat{u}_t}\dot{W}(t,\cdot) \\ \hat{u}_0 = f, \end{cases} \tag{2.3}$$

where $\dot{W}$ is the space-time white noise.

We will show Theorem 2.2 and Theorem 2.3 in Chapter 3. After giving some prerequisite knowledge about the solutions to SPDEs, we will discuss the renormalization argument in the later part of this Chapter. When adding the vertical interactions and showing percolation, by a renormalization argument, we can reduce our layered system to an oriented percolation model. We can define a site in the renormalized space (or a block in the primary space) as open if its corresponding block has a certain amount of cumulated density, since we have already taken into account the attrition in the real system. After building the renormalization argument, we are able to use the criteria in Durrett (1995) to determine the existence of percolation. The main result in the critical horizontal case i.e. when $\lambda = 1$ in (2.1) is the following.

Theorem 2.4. *When $\lambda = 1$, the critical values of the scaling and interaction factors are $b = 2\alpha = 2/5$. That is there exist positive constants C_1 and C_2 not depending on N such that for $\kappa < C_1$, there is no percolation and for $\kappa > C_2$, there is a percolation, where κN^{-b} is the opening probability of vertical edges.*

The critical value $\alpha = 1/5$ can be guessed by standard coupling as in Lalley (2009). Initially, there are $2\lfloor N^{2\alpha} \rfloor$ particles with at most one on each site and they are distributed uniformly on $2\lfloor N^{1+\alpha} \rfloor$ sites in $[-1,1]_N$. In the beginning, there are $O(N^{\alpha-1})$ particles at each site on average or we can say that every site in $[-1,1]_N$ has a chance of $O(N^{\alpha-1})$ to hold one particle. These particles are partitioned into red particles and blue particles according to whether they are alive or dead respectively. Initially, all the particles are red. Each individual produces offsprings at their neighbourhoods (with in distance N or distance $N^{-\alpha}$ in the renormalized space $[-1,1]_N$) following a Binomial distribution $\text{Bin}(2N, 1/(2N))$. In other word, each particle will independently follows a law of critical branching random walk. Since the branching random walk is critical, i.e. the expectation of offspring is exactly 1, the fact that there are $O(N^{\alpha-1})$ particles on each site will not change too much during the propagation. The branching random walk will finally become extinct as we know from Athreya and Ney (2004). When we introduce vertical connection as in Figure 2.1, the system contains a phenomenon of attrition that one site can be visited at most once. The red particles are killed by attrition and become blue particles. More precisely, the offsprings of blue particles are still blue and if a site x has been visited in the past, then the offsprings of red particles that are produced at x become blue. The critical branching random walk will last for $O(N^{2\alpha})$ time units (generations) (ref. Athreya and Ney (2004)). Up to extinction, the chance of dying for any particle caused by attrition become $O(N^{3\alpha-1})$. Over this period of $O(N^{2\alpha})$ time units, the cumulated blue particles is of magnitude $O(N^{5\alpha-1})$. Hence if $\alpha = 1/5$, the total attrition is $O(1)$. The rigorous proof of Theorem 2.4 is by using renormalization argument in Chapter 4.

The similar question about the critical exponent as in Theorem 2.4 arises when $\lambda > 1$.

Theorem 2.5. *Suppose $\lambda > 1$ in (2.1), then the critical vertical interaction is $\epsilon(N) = e^{-\kappa N}$. That is there exist positive constants C_1 and C_2 such that when $\kappa < C_1$, there is a percolation and when $\kappa > C_2$, there is no percolation.*

The proof of Theorem 2.5 is based on properties of supercritical branching random walk and renormalization argument. We will show them in Chapter 5. Before showing the results, in the remaining part of this chapter, we will give a short introduction about the tools we use in the proof.

2.2 Stochastic comparison

In many proofs, we use the terminology stochastically dominance. In this part, we will give some background on stochastic comparison and the idea of coupling.

Definition 2.3. *Suppose X and Y are two real-valued random variables on $(\Omega, \mathscr{F}, \mathbf{P})$. X is stochastically dominated by Y noted by $X \preceq Y$ if*

$$\mathbf{P}(X \geq x) \leq \mathbf{P}(Y \geq x) \text{ for any } x \in \mathbb{R}.$$

For example, if $X \sim \text{Binomial}(n_1, p)$ and $Y \sim \text{Binomial}(n_2, p)$ where X, Y are independent and $n_1 \leq n_2$, it is easy to see that $X \preceq Y$.

Lemma 2.1. *$X \preceq Y$ if and only if there is a coupling $(\tilde{X}, \tilde{Y})$ of X, Y such that*

$$\mathbf{P}^{(\tilde{X},\tilde{Y})}(\tilde{X} \leq \tilde{Y}) = 1.$$

The idea of stochastic dominance can be extended to the interacting particle system (ref. Chapter II of Liggett (1985)). Suppose X is a compact metric space with a partial order. For example, X can be $\{0,1\}^S$ or $\{0,1,\cdots\}^S$, where S is a finite or countable set. Let $\mathscr{M}$ denote the the class of monotone functions on X such that for $f \in \mathscr{M}$, $f(\eta) \leq f(\eta')$ when $\eta \leq \eta'$.

Definition 2.4. *Suppose μ_1, μ_2 are two probability measures on X. $\mu_1 \preceq \mu_2$ if*

$$\int f d\mu_1 \leq \int f d\mu_2$$

for any $f \in \mathscr{M}$.

A Feller process $\{\eta_t\}_{t\geq 0}$ on X with semigroup S_t is monotone or attractive if one of the following two conditions hold:

a) $f \in \mathscr{M}$ implies $S_t f \in \mathscr{M}$ for all $t \geq 0$.

b) $\mu_1 \preceq \mu_2$ implies $\mu_1 S_t \preceq \mu_2 S_t$ for all $t \geq 0$.

Using the same notations as in the Introduction, let $\{G_k\}_{k\geq 0}$ be the envelope process on $X = \{0,1,\cdots\}^{\mathbb{Z}}$. $G_k(x) \in \mathbb{Z}^+$ is the number of particles at site $x \in \mathbb{Z}$ and at generation $k \in \mathbb{Z}_+$. $\tilde{G}_k(x) \in \{0,1\}$ is to indicate if the site x is occupied by particles or not. Hence $\tilde{G}_k \preceq G_k$ given the same initial condition. Now we start to construct the true horizontal process $\{\hat{G}_k\}_{k\geq 0}$ on X. Particles are partitioned into red (alive) and blue (dead). G_k and $\hat{G}_k$ are both the collections of red particles at generation k. For G_k, each particle produces offsprings at their neighbourhood within distance N following a Binomial distribution $\text{Bin}(2N, 1/(2N))$. However for $\hat{G}_k$, after each generation, multiple (red or blue) particles at site x, if any will emerge into one (red or blue) particle and if there are blue particles, the emerged particle is blue. This is the first level of attrition. The particles in $\hat{G}_k$ can be killed by the second level of attrition following the rule: if site x at generation k has been visited by red particles in the previous generations, then the offsprings of red particles that are produced at x become blue. The offsprings of each particle inherit their colors. Given the same initial condition, $\hat{G}_k$ is stochastically dominated by G_k

because of these two attritions, merging and killing. Notice the initial condition allows the existence of blue particles.

If we replace the state space S from $\mathbb{Z}$ to the rescaled space $\mathbb{Z}/N^{1+\alpha}$, this stochastic comparison can be observed from the inductive construction of $\hat{\xi}_n$.

2.3 Stochastic partial differential equations

As we can see, the asymptotic densities in Theorem 2.2 and Theorem 2.3 are solutions to some SPDEs with $\dot{W}$ being the space-time white noise. In this section, we will mainly refer Walsh (1986) and the lecture series *An Introductory Course on Stochastic Partial Differential Equations* given at EPFL by Marta Sanz-Solé in 2016 to give a rigorous meaning to the SPDEs.

Consider the following equation

$$\mathscr{L}u(x) = b(u(x)) + \sigma(u(x))\dot{W}(x), x \in O \subset \mathbb{R}^{1+d} \tag{2.4}$$

satisfying certain initial conditions and (or) boundary conditions. Here $b, \sigma : \mathbb{R} \to \mathbb{R}$, $\mathscr{L}$ is a linear differential operators and $\dot{W}$ is a random turbulence. In our case, we only consider when $O = [0, T) \times D$ (T can be infinity) which is any subset of $\mathbb{R}_+ \times \mathbb{R}^d$ and $\mathscr{L} = \frac{\partial}{\partial t} - c\Delta, c \in \mathbb{R}_+$ is the heat operator. We first want to make clear the meaning of the random turbulence $\dot{W}$.

Suppose G is the Green function of $\mathscr{L}$ such that

$$\mathscr{L}G = \delta_{\{0\}}.$$

For example, the Green function for $\mathscr{L} = \frac{\partial}{\partial t} - \Delta$ when $d = 1$ is the heat kernel

$$G(t, x) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}}.$$

Then the unique solution to

$$\mathscr{L}u = f,$$

when f is a distribution with compact support, is given by convolution $u = G * f$. This helps us to state the mild solution to (2.4) is of the form (ref. Chapter 3 of Walsh (1986))

$$u(t, x) = I_0(t, x) + \int_0^t \int_{\mathbb{R}^d} G(t-s, x-y) b(u(s, y)) ds dy + \int_0^t \int_{\mathbb{R}^d} G(t-s, x-y) W(ds, dy), \tag{2.5}$$

where

$$I_0(t, x) = \int_0^t \int_{\mathbb{R}^d} G(t, x-y) u_0(y) dy$$

is the initial data. Our second objective is to give a meaning to the stochastic integral, i.e. the last term in (2.5) which we will use to represent some quantities. For simplicity, we let $b = 0$ in the following argument.

2.3.1 White noise

Let ν be a σ-finite measure on $\mathbb{R}^k$ and $\mathscr{B}_f(\mathbb{R}^k) = \{B \in \mathscr{B}(\mathbb{R}^k) : \nu(B) < \infty\}$ be the σ-algebra generated by finite Borel sets.

Definition 2.5. *A white noise defined on probability space* $(\Omega, \mathscr{F}, \mathbf{P})$ *and based on* ν *is a Gaussian random field* $\{W(A) : A \in \mathscr{B}_f(\mathbb{R}^k)\}$ *measurable with respect to* $\mathscr{F}$ *with mean* $\mathbf{E}[W(A)] = 0$ *and covariance* $\mathbf{E}[W(A)W(B)] = \nu(A \cap B)$.

Suppose $k = 1$ and $\nu(dx) = \mathbf{I}_{\mathbb{R}_+}(x)dx$, then define white noise $\{W(A) : A \in \mathscr{B}_f(\mathbb{R})\}$ based on ν. Notice that

$$B_t = W([0,t]), t \geq 0$$

is Brownian motion.

Proposition 2.2. *White noise on* $\mathbb{R}^k$ *has the following properties.*

1. *If* $A \cap B = \varnothing$, *then* $W(A)$ *and* $W(B)$ *are independent.*
2. *If* $A \cap B = \varnothing$, *then* $W(A \cup B) = W(A) + W(B)$.
3. *If* $A_n \uparrow A$ *with* $A, A_n \in \mathscr{B}_f(\mathbb{R}^k), \forall n$, *then* $W(A_n) \to W(A)$ *in* $L^2(\Omega, \mathscr{F}, \mathbf{P})$.

Notice that these properties are very similar to those of Brownian motion when we replace A or B by time interval $[s,t]$ on $\mathbb{R}_+$.

For $A \in \mathscr{B}_f(\mathbb{R}^k)$, set

$$W(\mathbf{I}_A) = W(A).$$

Then for any simple function $h = \sum_{i=1}^n c_i \mathbf{I}_{A_i}$ where $c_i \in \mathbb{R}$ and $A_1, \cdots, A_n$ are disjoint, we can define

$$W(h) := \sum_{i=1}^n c_i W(A_i).$$

From this, for any $h \in L^2(\mathbb{R}^k, \nu)$, we can define $W(h)$ in the terms of an integral:

$$W(h) = \int_{\mathbb{R}^k} h(x) W(dx).$$

Proposition 2.3 (Wiener's isometry)**.** *The Gaussian random field* $\{W(h), h \in L^2(\mathbb{R}^k, \nu)\}$ *are with mean* $\mathbf{E}[W(h)] = 0$ *and covariance* $\mathbf{E}[W(h)W(g)] = \langle h, g\rangle_{L^2(\mathbb{R}^k,\nu)}$.

Notice that $\{W(h), h \in L^2(\mathbb{R}^k, \nu)\}$ can be extended to a stochastic process with index $h \in H$, where H is a real separable Hilbert space with inner product $\langle \cdot, \cdot \rangle_H$.

Definition 2.6. $\{W(h), h \in H\}$ *defined on* $(\Omega, \mathscr{F}, \mathbf{P})$ *is an isonormal Gaussian process on* H *if for any* $h, g \in H$

1. $W(h) \sim \mathcal{N}\left(0, \|h\|_H^2\right)$.

2. $\mathbf{E}[W(h)W(g)] = \langle h, g\rangle_H$.

If $H = L^2(\mathbb{R}_+ \times \mathbb{R}^d, dt \times dx)$, then

$$\mathbf{E}[W(\phi)W(\psi)] = \int_0^\infty dt \int_{\mathbb{R}^d} \phi(t,x)\psi(t,x)dx = \langle \phi, \psi\rangle_H.$$

The Gaussian process induced by $L^2(\mathbb{R}_+ \times \mathbb{R}^d, dt \times dx)$ is called *space-time white noise in* $\mathbb{R}^d$.

2.3.2 Walsh stochastic integral

The Walsh stochastic integral integrates a class of *predictable processes* with respect to a *martingale measure.*

Definition 2.7. *Predictable processes are defined in the following way.*

- *Denote* $\mathscr{B}_b(\mathbb{R}^d)$ *the set of bounded Borel sets in* $\mathbb{R}^d$. $(\Omega, \mathscr{F}, \mathbf{P})$ *is a probability space with increasing and complete filtration* $(\mathscr{F}_t)_{t\geq 0}$ *The* σ*-field in* $[0,T]\times\mathbb{R}^d\times\Omega$ *generated by*

$$\{(r,t]\times B\times F : B \in \mathscr{B}_b(\mathbb{R}^d), F \in \mathscr{F}_r\}$$

  is called the predictable σ*-field.*

- *A process measurable with respect to predictable* σ*-field is called predictable process.*

Definition 2.8. *A stochastic process* $\{M_t(A), A \in \mathscr{B}_b(\mathbb{R}^d), t \geq 0\}$ *with values in* $L^2(\Omega)$ *is a martingale measure if*

1. $M_0(A) = 0, \forall A \in \mathscr{B}_b(\mathbb{R}^d)$.

2. *For any* $t > 0$, $M_t(\cdot)$ *is a* σ*-finite* $L^2(\Omega)$*-valued measure in* $\mathscr{B}_b(\mathbb{R}^d)$.

3. *For any* $A \in \mathscr{B}_b(\mathbb{R}^d)$, $(M_t(A), t > 0)$ *is a martingale.*

The covariance functional of M is defined as

$$Q_M([0,t]\times A\times B) = \langle M(A), M(B)\rangle_t,$$

for any $A, B \in \mathscr{B}_b(\mathbb{R}^d)$. $\langle\cdot,\cdot\rangle$ denotes the quadratic covariation of the martingale.

Let $(W(\phi), \phi \in L^2(\mathbb{R}_+ \times \mathbb{R}^d, dt\times dx))$ be the space-time white noise. For $A \in \mathscr{B}_b(\mathbb{R}^d)$ and $t \geq 0$, set

$$M_t(A) := W(\mathbf{I}_{[0,t]\times A}) = W([0,t]\times A).$$

$\{M_t(A), t \geq 0, A \in \mathscr{B}_b(\mathbb{R}^d)\}$ is then a martingale measure. The covariance functional is

$$Q_M([0,t] \times A \times B) = t|A \cap B|.$$

For elementary function $g : \mathbb{R}_+ \times \mathbb{R}^d \times \Omega \to \mathbb{R}$ of the form

$$g(s,x,\omega) = \mathbf{I}_{(r,u]}(s) \times \mathbf{I}_A(x) \times X(\omega),$$

with $0 \leq r < u$, $A \in \mathscr{B}_B(\mathbb{R}^d)$ and X is a bounded random variable in $\mathscr{F}_r$.

The stochastic integral of g with respect to a martingale measure is

$$g \cdot M : \mathbb{R}_+ \times \mathscr{B}_b(\mathbb{R}^d) \times \Omega \to \mathbb{R}$$

defined as

$$(g \cdot M)_t(B) = X(\omega)\left(M_{t \wedge u}(A \cap B) - M_{t \wedge r}(A \cap B)\right),$$

where $t \geq 0$ and $B \in \mathscr{B}_b(\mathbb{R}^d)$. It has variance

$$\begin{aligned}\mathbf{E}[(g \cdot M)_t(B)^2] &= \mathbf{E}\left[\int_0^t \int_{A\cap B} \int_{A \cap B} g(s,x)g(s,y)Q_M(ds,dx,dy)\right] \\ &= \mathbf{E}[X^2](u \wedge t - r \wedge t)|A \cap B|^2.\end{aligned}$$

Definition 2.9. *A martingale measure M is worthy if there exists a σ-finite measure K_M on $\mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d$ satisfying:*

1. *$K_M([0,t] \times \cdot \times \cdot)$ is symmetric and non-negative definite.*
2. *For any $A, B \in \mathscr{B}_b(\mathbb{R}^d)$, $\{K_M([0,t] \times A \times B), t \geq 0\}$ is predictable.*
3. *For any $T > 0$, $\mathbf{E}[K_M([0,t] \times E_n \times E_n)] < \infty \forall n$, where $\{E_n, n \in \mathbb{N}\}$ is a partition of $\mathbb{R}^d$.*
4. *For any $A, B \in \mathscr{B}_b(\mathbb{R}^d)$ and $t \geq 0$,*

$$Q_M([0,t] \times A \times B) \leq K_M([0,t] \times A \times B), a.s.$$

We then use this *dominating measure* to define the norm $\|\cdot\|_{+,M}$ s.t.

$$\|g\|_{+,M}^2 = \mathbf{E}\left[\int_0^\infty \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |g(s,x)| \cdot |g(s,y)| K_M(ds,dx,dy)\right],$$

and the space

$$\mathscr{P}_{+,M} = \{g : \mathbb{R}_+ \times \mathbb{R}^d, g \text{ predictable }, \|g\|_{+,M} < \infty\}.$$

Theorem 2.6 (Walsh theorem)**.** *If M is worthy, $(\mathscr{P}_{+,M}, \|\cdot\|_{+,M})$ is a Banach space and the set of elementary functions $\mathscr{E}$ is dense in $\mathscr{P}_{+,M}$.*

Thanks to Walsh theorem, the stochastic integral of elementary functions can be extended to any $g \in \mathscr{P}_{+,M}$. We use the following notation for integral:

$$(g\cdot M)_t = (g\cdot M)_t(\mathbb{R}^d) = \int_0^t \int_{\mathbb{R}^d} g(s,x)M(ds,dx).$$

2.4 Martingale problems

We will represent the solutions to the SPDEs in Theorem 2.2 and Theorem 2.3 with a martingale problem approach. In this section, we will give a short introduction to the martingale problem.

We start with the Lévy's characterization of Brownian motion.

Theorem 2.7. *Let M be a continuous local martingale with quadratic variation $\langle M\rangle_t = t$. Then M is a standard Brownian motion.*

As a result of Theorem 2.7 and the Doob decomposition, we can characterize Brownian motion in a *martingale problem* way.

Corollary 2.1. *Suppose M is a continuous local martingale and $M^2(t) - t$ is also a continuous local martingale, then M is a standard Brownian motion.*

This characterization can be extended to any Markov process. Let $\{X(t), t\geq 0\}$ be a time-homogeneous Markov process on $(\Omega, \mathscr{F}, \mathbf{P})$ with values in E. For bounded function f, define the semi-group $\{T(t), t\geq 0\}$ of $\{X(t), t\geq 0\}$

$$T(t)f(x) = \mathbf{E}[f(X(s+t)) \mid X(s) = x].$$

The (infinitesimal) generator of $\{X(t), t\geq 0\}$ is a linear operator $\mathscr{A}$ acting on a class of suitable functions $f : E\to\mathbb{R}$ such that

$$\mathscr{A}f(x) = \lim_{t\downarrow 0}\frac{T(t)f(x)-f(x)}{t} = \lim_{t\downarrow 0}\frac{\mathbf{E}[f(X(t)) \mid X(0)=x]}{t}.$$

The set of functions f for which the limit above exist is the domain of $\mathscr{A}$ denoted by $D(\mathscr{A})$. When $E=\mathbb{R}^d$, the set of twice differentiable functions with compact support $C_c^2(\mathbb{R}^d)\subset D(\mathscr{A})$.

Definition 2.10. *X is a solution to the martingale problem for linear generator $\mathscr{A}$ if and only if X is $\{\mathscr{F}_t\}$-adapted and*

$$f(X(t)) - f(X(0)) - \int_0^t \mathscr{A}f(X(s))ds$$

is an $\{\mathscr{F}_t\}$-martingale for any $f\in D(\mathscr{A})$.

For a diffusion process on $\mathbb{R}^d$,

$$dX(t) = b(X(t))dt + \sigma(X(t))dB(t), \tag{2.6}$$

where B is a m-dimensional standard Brownian motion and

$$b : \mathbb{R}^d \to \mathbb{R}^d, \sigma : \mathbb{R}^d \to \mathbb{R}^{d\times m}.$$

Denote $a(x) = \sigma(x)\sigma^T(x) \in \mathbb{R}^{d\times d}$, then the generator of X is

$$\mathscr{A} f(x) = \frac{1}{2}\sum_{1\le i,j\le d} a_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j} f(x) + \sum_{1\le i\le d} b_i(x)\frac{\partial}{\partial x_i} f(x), \text{ for } f \in C_c^2(\mathbb{R}^d)$$

which corresponds to the drift term of $f(X(t))$ by Itô formula and as a result of which

$$f(X(t)) - f(X(0)) - \int_0^t \mathscr{A} f(X(s))ds$$

is a martingale. Notice that to characterize the solution to (2.6), one needs to describe the drift term $\int_0^t b(X(s))ds$ and the noise term $\int_0^t \sigma(X(s))dB(s)$. For the noise term, besides the martingale property caused by $\{B(t), t \ge 0\}$, one also has to characterize the quadratic variation of the noise as in the Lévy characterization. The generator $\mathscr{A}$ of X can catch these features and equivalently if

$$M(t) = X(t) - X(0) - \int_0^t b(X(s))ds \text{ and } M_i(t)M_j(t) - \int_0^t a_{ij}(X(s))ds$$

are continuous local martingales, then X is a solution to the martingale problem for $\mathscr{A}$.

To characterize the solutions to (2.2) and (2.3), we also need to maintain the martingale property and the quadratic variation of the noise term. $\{u_t(x), t \in \mathbb{R}_+, x \in \mathbb{R}\}$ is a solution to (2.2) if for any $\phi \in C_c^2(\mathbb{R})$ we have that

$$m_t(\phi) = \int \phi(x)u_t(x)dx - \int \phi(x)f(x) - \int_0^t \int \frac{1}{6}\Delta\phi(x)u_s(x)dxds$$

is a continuous martingale and

$$m_t^2(\phi) - \int_0^t \int \phi^2(x)u_s(x)dxds$$

is also a continuous local martingale. The solution to (2.3) is characterized in a similar way, but one more term is added. For any $\phi \in C_c^2(\mathbb{R})$,

$$\hat{m}_t(\phi) = \int \phi(x)\hat{u}_t(xdx) - \int \phi(x)f(x)dx - \int_0^t \int \frac{1}{6}\Delta\phi(x)\hat{u}_s(x)dxds - \int_0^t \int_0^s \int \phi(x)\hat{u}_s(x)\hat{u}_r(x)dxdrds$$

is a continuous local martingale. The last term above is to eliminate the phenomenon of $\times$ in Figure 2.1.

$$\hat{m}_t^2(\phi) - \int_0^t \int \phi^2(x)\hat{u}_s(x)dxds$$

is also a continuous local martingale.

2.5 Renormalization argument

In this part, we will introduce a different aspect of probability theory which is not related to any stochastic integral but plays a very import role in the field of percolation. The block argument is a tool to investigate the behaviour of percolation at criticality and in the supercritical case by renormalizing the lattice on $\mathbb{Z}^d$. Instead of dealing with every site on $\mathbb{Z}^d$, the idea is to treat the vertices on a renormalized lattice space, where every vertex represents a large block of $\mathbb{Z}^d$ sites. These blocks are disjoint with each other. In the renormalized lattice, we can define a block to be *good* if the paths inside this block satisfy certain conditions. We can then compare the percolation on the primitive space and the classic oriented percolation on the renormalized space. The introduction of oriented percolation and the comparison argument in this section mainly refers to Durrett (1995).

Let

$$\mathscr{L}_0 = \{(m,n) \in \mathbb{Z} \times \mathbb{Z}_+ : m+n \text{ is even}\}.$$

$\mathscr{L}_0$ is made into a graph by drawing oriented edges from (m,n) to $(m+1,n+1)$ and to $(m-1,n+1)$. Random variables $\omega(m,n) \in \{0,1\}$ whether indicates site (m,n) is *open* ($\omega(m,n)=1$) or *closed* ($\omega(m,n)=0$). We say that there is an *open* path from (m,n) to (m',n') if there exists a sequence of points $x_n = m, \cdots, x_{n'} = m'$ so that $|x_l - x_{l-1}| = 1, n < l \le n'$ and $\omega(x_l, l) = 1$ for $n \le l \le n'$. This *open* path is denoted as $(m,n) \to (m',n')$. Figure 2.2 shows an example of the oriented percolation on $\mathscr{L}_0$.

In the settings above, the state random variables $\omega(m,n)$ are independent. However, in most cases, it is very difficult to define completely disjoint blocks but the dependency within finite blocks does not matter too much. To use the comparison argument, we need a more general set-up by introducing dependencies between $\omega(m,n)$.

Definition 2.11. *$\omega(m,n)$ are M-dependent with density at least $1-\gamma$ if the sequence of sites $(m_i,n_i), 1 \le i \le I$ with $\|(m_i,n_i)-(m_j,n_j)\|_\infty > M$ for $i \ne j$ satisfies*

$$\mathbf{P}(\omega(m_i,n_i) = 0 \textit{ for } 1 \le i \le I) \le \gamma^I \ \forall I.$$

To be consistent with the notations in section 2.1, given an initial condition $G_0 \subset 2\mathbb{Z} = \{m : (m,0) \in \mathscr{L}_0\}$, we define a process

$$G_n := \{m' : (m,0) \to (m',n) \text{ for some } m \in G_0\}$$

which are the *open* sites at time (generation) n. We denote G_n^0 as the process when the initial condition $G_0 = \{0\}$ and let

$$\mathscr{C}_0 = \{(m,n) : (0,0) \to (m,n)\}$$

be all the *open* sites that can be connected by an *open* path from $(0,0)$ i.e. $\mathscr{C}_0 = \bigcup_{n\in\mathbb{N}} G_n^0 \times \{n\}$. $\mathscr{C}_0$ is called the *open* cluster containing the origin. Percolation occurs when $|\mathscr{C}_0| = \infty$ i.e. the *open* cluster containing zero is infinite. One main result in oriented percolation says the

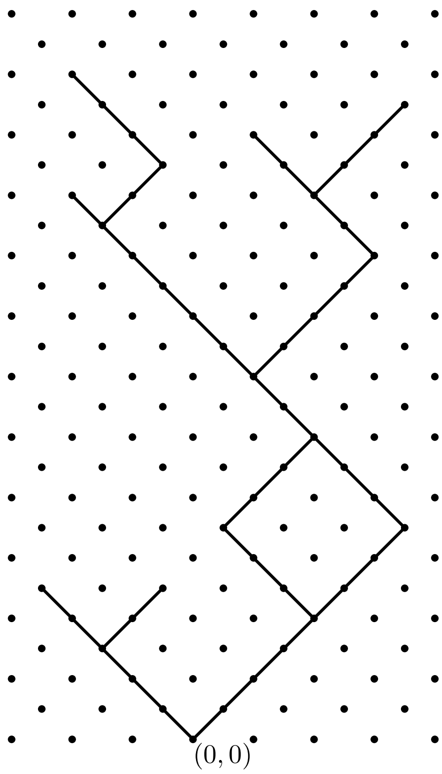

Figure 2.2 – Oriented percolation on $\mathscr{L}_0$

relation between $\{|\mathscr{C}_0| = \infty\}$ and the density γ.

Theorem 2.8 (Durrett (1995)). *If $\gamma < 6^{-4(2M+1)^2}$, then*

$$\mathbf{P}(|\mathscr{C}_0| < \infty) \leq 55\gamma^{1/(2M+1)^2} \leq \frac{1}{20}.$$

The idea to prove Theorem 2.8 is heuristically simple. Since the size of $\mathscr{C}_0$ is finite, there is a contour of *closed* sites surrounding $\mathscr{C}_0$ and prevents it from percolation. The probability of the occurence of such a *closed* contour of length n can be bounded by $\varepsilon(\gamma)^n$ where $\varepsilon(\gamma)$ tends to zero as $\gamma \to 0$. Moreover, the number of such contours of length n is bounded by 3^n since any site has at most 3 edges except the potential edge connected in the *open* cluster. Then

$$\mathbf{P}(|\mathscr{C}|_0 < \infty) \leq \sum_{n=4}^{\infty} 3^{n-1}\varepsilon(\gamma)^n.$$

Some meticulous work should be done to deal with the dependencies and estimate $\varepsilon(\gamma)$.

Comparison argument

The example in Durrett (1995) can help to make the comparison argument clear which is quite similar as we will do in our renormalization argument. Suppose there is a translation invariant finite-range process $\xi_t : \mathbb{Z}^d \to \{0,1\}$.

Example (Contact model). *Suppose $(\xi_t)_{t\geq 0}$ is a continuous time Markov process on $\{0,1\}^{\mathbb{Z}^d}$ with generator*

$$\mathscr{A} f(\xi) = \sum_{x\in\mathbb{Z}^d} c(x,\xi)\left(f(\xi^x) - f(\xi)\right),$$

where

$$\xi^x(y) = \begin{cases} \xi(y) & \text{if } y \neq x \\ 1-\xi(y) & \text{if } y = x \end{cases}$$

denotes the switch of state at $x \in \mathbb{Z}^d$ and $c(x,\xi)$ is the transition rate.

When the transition rate

$$c(x,\xi) = \begin{cases} \lambda \sum_{y\sim x} \xi(y) & \text{if } \xi(x) = 0 \\ 1 & \text{if } \xi(x) = 1, \end{cases}$$

$(\xi_t)_{t\geq 0}$ is called contact process with rate $\lambda > 0$.

We now assume this process $(\xi_t)_{t\geq 0}$ satisfies some *good* property (defined soon) for example when the contact process has a supercritical rate $\lambda > \lambda_c$ and compare it with the oriented percolation. The *good* collection of ξ is defined as

$$G_\xi = \left\{\left|\xi_0\left([-L,L]^d\right)\right| \geq K, \operatorname{supp}(\xi_t) \subset [-4L,4L]^d \text{ for } t \in [0,T] \text{ and } \left|\xi_T\left(\sigma_{\pm 2Le_1}[-L,L]^d\right)\right| \geq K\right\}$$

where $\sigma_y \xi(x) = \xi(x+y)$ is the translation of ξ by y. Suppose $\mathbf{P}(G_\xi) \geq 1-\gamma$. In other words, G_ξ means that if ξ has a pile of K 1's in $[-L,L]^d$ at time 0, then with probability more than $1-\gamma$, ξ has a pile of at least K 1's in $[-L,L]^d + 2Le_1$ and $[-L,L]^d - 2Le_1$ at time T. Figure 2.3 shows a graphical representation of this *good* property.

Let $M = 4$ and the space-time regions

$$R_{m,n} = (2mLe_1, nT) + [-4L,4L]^d \times [-T,T].$$

correspond to points $(m,n) \in \mathscr{L}_0$. Under this renormalization, $R_{m,n}$ and $R_{m',n'}$ are disjoint if $n \neq n'$ $\forall m,m'$ or $n = n'$ but $|m-m'| > 4$. If the *good* event G_ξ happens in $R_{m,n}$, then we draw an arrow from (m,n) to $(m+1,n+1)$ and $(m-1,n+1)$. More formally, let

$$X_n = \left\{m : (m,n) \in \mathscr{L}_0, \left|\xi_{nT}\left(\sigma_{2mLe_1}[-L,L]^d\right)\right| \geq K \text{ and } \xi \text{ is supported in } R_{m,n}\right\}.$$

We can then define random variables $\omega(m,n)$ on $\mathscr{L}_0$ so that $\{X_n\}$ dominates an M-dependent oriented percolation with density at least $1-\gamma$. This means that if $\gamma < 6^{-4\cdot 9^2}$, then $\{\omega(m,n)\}$

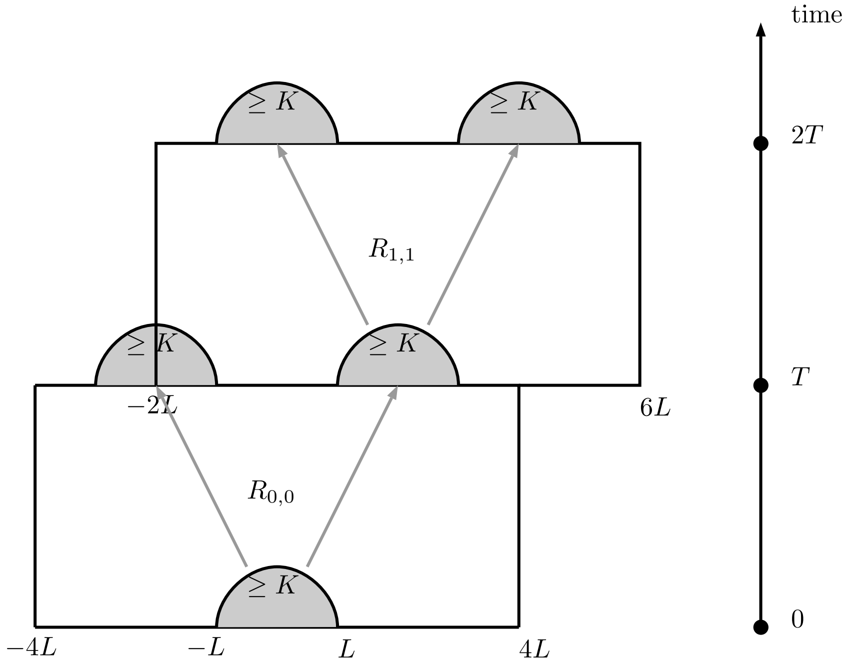

Figure 2.3 – Explanation of *good* property

percolates in $\mathscr{L}_0$ and hence ξ survives with a cluster of infinite connecting 1's

3 Weak Convergence of Our Systems

This chapter is mainly to show Theorem 2.2, the weak limit of the envelope process and Theorem 2.3, the weak limit of the true horizontal process. These two proofs refer the idea in Mueller and Tribe (1995) by first defining the martingale problem of the discrete system, doing several moments estimations and showing the final tightness criteria.

3.1 The envelope process

Since the proof of our main theorem involves the scaling limit of the process $\hat{\xi}$, as in Lalley (2009) we are led to consider first the situation of the corresponding branching random walk, as our 'envelope process', and we first study its scaling limit in Theorem 2.2. This result should probably be contained in the literature, even if not stated exactly as convenient for the consideration of our true model in Section 3.2; the ideas are contained in Mueller and Tribe (1995).

Before studying the asymptotic behaviour of the process, we first study that of an envelope process. In this section, we consider the state function $\xi_n(\cdot) : \mathbb{Z}/N^{1+\alpha} \to \mathbb{Z}_+$. The mechanism of this envelope process is as follows. The number of particles at site x will increase by 1 if one of its neighbours branches following Binomial$(2N, 1/(2N))$ and then chooses x uniformly among the $2N$ neighbours. It can be written as

$$\xi_{n+1}(x) = \sum_{y\sim x} \sum_{w=1}^{\xi_n(y)} \eta_{n+1}^w(y,x),$$

where $(\eta_{n+1}^w(y,x))_{w,n,y,x}$ is an i.i.d. sequence of random variables with distribution Bernoulli$(1/(2N))$. The horizontal process $\hat{\xi}_n(\cdot) : \mathbb{Z}/N^{1+\alpha} \to \{0,1\}$ analysed in Section 3.2 is dominated by this envelope process in two senses: $\hat{\xi}_n(\cdot)$ does not allow multiple particles at one site and any visited site cannot be visited again. At the end of this section, we will show that the probability of multiple particles at one site i.e. $\xi_n(x) > 1$ is quite small, of order $O(N^{2(\alpha-1)})$ which is negligible when $\alpha < 1$.

The main result of this section is that the asymptotic density of the dominating envelope

$$A(\xi_{\lfloor tN^{2\alpha}\rfloor})(x) = \frac{1}{2N^\alpha}\sum_{y\sim x}\xi_{\lfloor tN^{2\alpha}\rfloor}(y)$$

with initial condition $A(\xi_0)$ converging to f in $\mathscr{C}$ follows the following SPDE (recall Theorem 2.2):

$$\begin{cases} \frac{\partial u_t}{\partial t} = \frac{1}{6}\Delta u_t + \sqrt{u_t}\dot{W}(t,\cdot) \\ u_0 = f, \end{cases}$$

where f is continuous and compactly supported, and $\dot{W}$ is the space-time white noise. We have already introduced the necessary ingredients like SPDE and space-time white noise in Section 2.3.

The idea of the proof is to write the mechanism as a martingale problem, then introduce a Green function representation (see (3.7)) to simplify the approximate density. The tightness criteria in Ethier and Kurtz (2009) can be applied to get the weak convergence. We will follow the blueprint of Mueller and Tribe (1995) to show the tightness.

Before starting the proof, we first explain the notation used in the following sections. For f, g functions on our discrete space $\mathbb{Z}/N^{1+\alpha}$, we write, whenever meaningful

$$(f,g) = \frac{1}{N^{1+\alpha}}\sum_x f(x)g(x)$$

Similarly, for f, g defined in $\mathbb{R}$, we write

$$(f,g) = \int fg\,dx.$$

Define the discrete measure generated by ξ_n as

$$\nu_n^N = \frac{1}{N^{2\alpha}}\sum_x \xi_n(x)\delta_x$$

and for a function f and measure ν, we write

$$(f,\nu) = \int f\,d\nu$$

for the integral whenever it is well defined. In Lemma 3.6, we will see that for any test function f which is bounded with compact support

$$(f, A\xi_n) - (f,\nu_n^N) \to 0 \text{ in } L^2.$$

We define the amplitude of a function around a neighbourhood as

$$D(f,\delta)(x) = \sup\{|f(y) - f(x)| : |y - x| \leq \delta\}. \tag{3.1}$$

3.1.1 Martingale Problem

Suppose $\xi_0(x)$ is deterministic and with a finite (depending on N) support. Rewriting the mechanism of $\xi_n(x)$, we have

$$\begin{aligned}
\xi_{n+1}(x) &= \sum_{y\sim x}\sum_{w=1}^{\xi_n(y)}\left(\eta_{n+1}^w(y,x) - \frac{1}{2N}\right) + \frac{1}{2N}\sum_{y\sim x}\xi_n(y) \\
&= \sum_{y\sim x}\sum_{w=1}^{\xi_n(y)}\left(\eta_{n+1}^w(y,x) - \frac{1}{2N}\right) + \frac{1}{2N}\sum_{y\sim x}(\xi_n(y) - \xi_n(x)) + \xi_n(x).
\end{aligned} \tag{3.2}$$

The first term will contribute to the space-time white noise part and the second term will contribute to the Laplacian in the SPDE.

Take a discrete test function $\phi^N(k,x)$ for $x \in \mathbb{Z}/N^{1+\alpha}$ ($\phi_k^N(x) = \phi^N(k,x)$). $\phi^N(k,x) : \mathbb{N} \times \mathbb{Z}/N^{1+\alpha}$ satisfying the following conditions:

$$\begin{aligned}
&\sum_{k=1}^{\lfloor TN^{2\alpha}\rfloor}(|\phi_k^N - \phi_{k-1}^N|, 1) < \infty, \\
&\frac{1}{\lfloor TN^{2\alpha}\rfloor}\sum_{k=1}^{\lfloor TN^{2\alpha}\rfloor}(|\phi_k^N| + |\phi_k^N|^2, 1) < \infty.
\end{aligned} \tag{3.3}$$

Summation by parts and (3.2) give

$$\begin{aligned}
(\nu_n^N, \phi_n^N) &= \frac{1}{N^{2\alpha}}\sum_x \xi_n(x)\phi_n^N(x) \\
&= \frac{1}{N^{2\alpha}}\sum_x \xi_n(x)(\phi_n^N(x) - \phi_{n-1}^N(x)) + \frac{1}{N^{2\alpha}}\sum_x \xi_n(x)\phi_{n-1}^N(x) \\
&= (\nu_n^N, \phi_n^N - \phi_{n-1}^N) + (\nu_{n-1}^N, \phi_{n-1}^N) + \frac{1}{N^{2\alpha}}\sum_x \frac{1}{2N}\sum_{y\sim x}(\xi_{n-1}(y) - \xi_{n-1}(x))\phi_{n-1}^N(x) \\
&\quad + \frac{1}{N^{2\alpha}}\sum_x\sum_{y\sim x}\sum_{w=1}^{\xi_{n-1}(y)}\phi_{n-1}^N(x)\left(\eta_n^w(y,x) - \frac{1}{2N}\right),
\end{aligned}$$

where we use the decomposition (3.2) in the last equality. Denote $\Delta_D f(x) = \frac{N^{2\alpha}}{2N}\sum_{y\sim x}(f(y) -$

$f(x))$. Summing by parts again, we can obtain

$$
\begin{aligned}
(\nu_n^N,\phi_n^N)-(\nu_{n-1}^N,\phi_{n-1}^N) &= (\nu_n^N,\phi_n^N-\phi_{n-1}^N)+(\nu_{n-1}^N,N^{-2\alpha}\Delta_D\phi_{n-1}^N)+\\
&\quad+\frac{1}{N^{2\alpha}}\sum_x\sum_{y\sim x}\sum_{w=1}^{\xi_{n-1}(y)}\phi_{n-1}^N(x)\left(\eta_n^w(y,x)-\frac{1}{2N}\right)\\
&= (\nu_n^N,\phi_n^N-\phi_{n-1}^N)+(\nu_{n-1}^N,N^{-2\alpha}\Delta_D\phi_{n-1}^N)+d_n(\phi^N),
\end{aligned}
$$

where

$$
d_n(\phi^N)=\frac{1}{N^{2\alpha}}\sum_x\sum_{y\sim x}\sum_{w=1}^{\xi_{n-1}(y)}\phi_{n-1}^N(x)\left(\eta_n^w(y,x)-\frac{1}{2N}\right). \tag{3.4}
$$

Summing up n from 1 to m, we get a semimartingale decomposition

$$
(\nu_m^N,\phi_m^N)-(A\xi_0,\phi_0^N)=(\nu_m^N,\phi_m^N-\phi_{m-1}^N)+\sum_{i=1}^{m-1}(\nu_i^N,\phi_i^N-\phi_{i-1}^N+N^{-2\alpha}\Delta_D\phi_i)+M_m(\phi^N), \tag{3.5}
$$

where we use the identity $(\nu^N,N^{-2\alpha}\Delta_D\phi^N+\phi^N)=(A\xi,\phi^N)$.

$M_m(\phi^N)=\sum_{k=1}^m d_k(\phi^N)$ is a martingale with square variation

$$
\begin{aligned}
\langle M(\phi^N)\rangle_m &= \sum_{k=1}^m \mathbf{E}_{k-1}d_k^2\\
&= \sum_{k=1}^m \frac{1}{2N^{1+4\alpha}}\sum_x\sum_{y\sim x}\xi_{k-1}(y)(\phi_{k-1}^N)^2(x)\left(1-\frac{1}{2N}\right)\\
&\le \sum_{k=1}^m \frac{\|\phi_{k-1}^N\|_0}{2N^{1+4\alpha}}\sum_x\sum_{y\sim x}\xi_{k-1}(y)\phi_{k-1}^N(x)\\
&= \sum_{k=1}^m \frac{\|\phi_{k-1}^N\|_0}{N^{2\alpha}}(A\xi_{k-1},\phi_{k-1}^N).
\end{aligned} \tag{3.6}
$$

For any $x\in\mathbb{Z}/N^{1+\alpha}$, let $\psi_i^z(x)\ge 0$ be the solution to

$$
\begin{cases}
\psi_i^z-\psi_{i-1}^z=N^{-2\alpha}\Delta_D\psi_{i-1}^z\\
\psi_0^z(x)=\frac{N^\alpha}{2}\mathbf{I}(x\sim z).
\end{cases}
$$

The solution of this equation is $\psi_n^z=N^{1+\alpha}\mathbf{P}(S_{n+1}=x-z)$, where $S_n=\sum_{i=1}^n Y_i$, with (Y_i) i.i.d. uniformly distributed on $\{i/N^{1+\alpha},|i|\le N\}$. Δ_D can be seen as the generator of this symmetric random walk S_n with steps of variance $\frac{c_3}{3}N^{-2\alpha}$ and $\mathbf{E}[Y^4]=\frac{c_4}{5N^{4\alpha}}$, where $c_3(N),c_4(N)\to 1$. $\psi_t^z(x)$ behaves asymptotically as $p(\frac{c_3 t}{3N^{2\alpha}},z-x)$, where $p(t,x)$ is the Brownian transition probability density.

We apply (3.5) with test function $\phi_k^N=\psi_{n-k}$ for $k\le n-1$, so that the first drift term vanishes and $(\nu_n^N,\phi_n)=(\nu_n^N,\psi_0^x)=A(\xi_n)(x)$. Thus we obtain an approximation

$$
A(\xi_n)(x)=(\nu_0,\psi_n^x)+M_n(\psi_{n-\cdot}^x), \tag{3.7}
$$

where $M_n(\psi^x_{n-\cdot}) = \sum_{k=1}^n d_k(\psi^x_{n-k})$ and $d_k(\phi)$ is as in (3.4). Proving tightness of $A(\xi_{\lfloor tN^{2\alpha}\rfloor})$ is equivalent to proving that of $M_{\lfloor tN^{2\alpha}\rfloor}$. We first show some estimations on ψ_n and the moments of $A(\xi_n)$, which will be used in the proof of tightness.

3.1.2 Estimations for showing tightness

Estimations of characteristic function

First, we need some bounds on the distribution function of S_t. Recall that $S_t = \sum_{i=1}^t Y_i$, with Y_i's i.i.d. uniformly distributed on $\{i/N^{1+\alpha}, |i| \le N\}$ and $p(t,x)$ is the transition probability of the standard Brownian motion.

Lemma 3.1. *There exists m, such that for $N \ge m$ and any $t \in \mathbb{N}$,*

$$\left| N^{1+\alpha}\mathbf{P}(S_t = y) - p\left(\frac{c_3 t}{3N^{2\alpha}}, y\right)\right| \le CN^{\alpha} t^{-\frac{3}{2}}, \tag{3.8}$$

where c_3 is the constant in Section 3.1.1 that tends to 1 *as $N \to \infty$.*

The proof follows from Mueller and Tribe (1995) and Bhattacharya and Rao (1986), and the inversion formula of characteristic function in Durrett (2019).

Proof. $\mathbf{E}e^{iuS_t} = \rho^t(u)$, where $\rho(u) = \mathbf{E}e^{iuY}$.

$$\begin{aligned}
\rho(u) &= \mathbf{E}e^{iuY} \\
&= \frac{1}{2N}\left(\sum_{k=1}^N e^{\frac{iuk}{N^{1+\alpha}}} + e^{\frac{-iuk}{N^{1+\alpha}}}\right) \\
&= \frac{1}{N}\left(\sum_{k=1}^N \cos\left(\frac{uk}{N^{1+\alpha}}\right)\right) \\
&= 1 - \frac{c_3 u^2}{2!\cdot 3N^{2\alpha}} + \frac{c_4 u^4}{4!\cdot 5N^{4\alpha}} r,\ |r| \le 1
\end{aligned}$$

This directly gives that for $u \le N^{\alpha}/2$,

$$|\rho(u)| \le \exp\left(-\frac{c_3 u^2}{12N^{2\alpha}}\right),$$

and for $u \ge N^{\alpha}/2$, $|\rho(u)| \le 23/24$ for N large.

Moreover, with the help of Theorem 8.5 of Bhattacharya and Rao (1986), for $u \le N^{\alpha}/2$,

$$\left|\rho^t(u) - \exp\left(-\frac{c_3 t u^2}{6N^{2\alpha}}\right)\right| \le Ct^{-1}\exp\left(-\frac{c_3 t u^2}{6N^{2\alpha}}\right).$$

Follow the inversion formula Durrett (2019),

$$N^{1+\alpha}\mathbf{P}(S_t=y)=\frac{1}{2\pi}\int_{-\pi N^{1+\alpha}}^{\pi N^{1+\alpha}}\mathrm{e}^{iuy}\rho^t(u)du,$$

$$p\left(\frac{c_3t}{3N^{2\alpha}},y\right)=\frac{1}{2\pi}\int\mathrm{e}^{iuy}\mathrm{e}^{-\frac{tu^2}{6N^{2\alpha}}}du.$$

The difference satisfies

$$\begin{aligned}|N^{1+\alpha}\mathbf{P}(S_t=y)-p\left(\frac{c_3t}{3N^{2\alpha}},y\right)|&\le\frac{1}{\pi}\int_{\pi N^{1+\alpha}}^{\infty}\mathrm{e}^{-\frac{c_3tu^2}{6N^{2\alpha}}}du+\frac{1}{\pi}\int_0^{\pi N^{1+\alpha}}|\rho^t(u)-\mathrm{e}^{-\frac{c_3tu^2}{6N^{2\alpha}}}|du\\&\le\frac{1}{\pi}\int_{\pi N^{1+\alpha}}^{\infty}\mathrm{e}^{-\frac{c_3tu^2}{6N^{2\alpha}}}du+\frac{1}{\pi}\int_{N^\alpha/2}^{\pi N^{1+\alpha}}|\rho^t(u)|+\mathrm{e}^{-\frac{c_3tu^2}{6N^{2\alpha}}}du\\&\quad+\frac{1}{\pi}\int_0^{N^\alpha/2}\left|\rho^t(u)-\mathrm{e}^{-\frac{c_3tu^2}{6N^{2\alpha}}}\right|du\\&\le\frac{1}{\pi}\int_{N^\alpha/2}^{\infty}\mathrm{e}^{-\frac{c_3tu^2}{6N^{2\alpha}}}du+N^{1+\alpha}\left(\frac{23}{24}\right)^t+\frac{1}{\pi t}\int_0^{N^\alpha/2}\mathrm{e}^{-\frac{c_3tu^2}{6N^{2\alpha}}}du\\&\le Ct^{-1}N^\alpha e^{-\frac{c_3t}{24}}+N^{1+\alpha}\mathrm{e}^{-\frac{t}{24}}+CN^\alpha t^{-\frac{3}{2}}.\end{aligned}$$

Therefore, we get the bound

$$\left|N^{1+\alpha}\mathbf{P}(S_t=y)-p\left(\frac{c_3t}{3N^{2\alpha}},y\right)\right|\le C(N^{1+\alpha}\mathrm{e}^{-\frac{t}{24}}+N^\alpha t^{-\frac{3}{2}}).$$

Because of (a) in the next lemma and $p(t,x)\le t^{-1/2}$, we have

$$\left|N^{1+\alpha}\mathbf{P}(S_t=y)-p\left(\frac{c_3t}{3N^{2\alpha}},y\right)\right|\le CN^\alpha t^{-\frac{3}{2}}.$$

□

Estimations of ψ_n^z

With the help of Lemma 3.1, we can get the estimations on ψ_n^z.

Lemma 3.2. *We have the following estimates on ψ_n^z:*

(a) $(\psi_k^z,1)=1$, $\|\psi_k^z\|_0\le CN^\alpha,\forall k\ge 0$.

(b) $(e_\lambda,\psi^z_{\lfloor tN^{2\alpha}\rfloor})\le C(\lambda,T)e_\lambda(z)$ *for* $0\le t\le T$.

(c) $\|\psi_k^z\|_\lambda\le C(\lambda)e_\lambda(z)N^\alpha k^{-\frac{1}{2}}$.

(d) *For* $|x-y|\le 1$,
$\|\psi_k^x-\psi_k^y\|_\lambda\le C(\lambda)e^{\frac{C(\lambda)k}{N^{2\alpha}}}e_\lambda(x)\left(|x-y|^{\frac{1}{2}}k^{-\frac{1}{2}}N^\alpha+N^{\frac{\alpha}{2}}k^{-\frac{3}{4}}\right)$.

(e) $\|\psi_k^y-\psi_l^y\|_\lambda\le C(\lambda)e^{\frac{C(\lambda)k}{N^{2\alpha}}}e_\lambda(y)N^{\frac{\alpha}{2}}\left(|k-l|^{\frac{1}{2}}l^{-\frac{3}{4}}+k^{-\frac{3}{4}}\right)$.

Proof. (a)

$$(\psi_k^z, 1) = \frac{1}{N^{1+\alpha}} \sum_x \frac{N^\alpha}{2} \mathbf{I}(x \sim z) = \frac{1}{2N} \sum_x \mathbf{I}(x \sim z) = 1.$$

The second statement is because $\mathbf{P}(X_k = y) \le c/N$ for any y.

(b)

$$\begin{aligned}
(\mathrm{e}_\lambda, \psi_t^z) &= \frac{1}{N^{1+\alpha}} \sum_x \mathrm{e}_\lambda(x) \psi_t^z(x) \\
&= \sum_x \mathrm{e}_\lambda(x) \mathbf{P}(S_{t+1} = x - z) \\
&\le 2\mathrm{e}_\lambda(z) \sum_x \mathrm{e}^{\lambda x} \mathbf{P}(S_{t+1} = x) \\
&\le 2\mathrm{e}_\lambda(z) (\mathbf{E}\mathrm{e}^{\lambda Y})^{t+1} \\
&\le 2\mathrm{e}_\lambda(z) \left(1 + \frac{\lambda^2}{3N^{2\alpha}}\right)^{t+1} \\
&\le 2\mathrm{e}_\lambda(z) \exp\left(\frac{\lambda^2(t+1)}{3N^{2\alpha}}\right).
\end{aligned}$$

(c) By (3.8) and $p(k, y) \le Ck^{-1/2}$, we have

$$N^{1+\alpha} \mathbf{P}(S_k = y) \le C\left(N^\alpha k^{-\frac{1}{2}} + N^\alpha k^{-\frac{3}{2}}\right),$$

then,

$$\psi_k^0(y) \le C\left(N^\alpha (k+1)^{-\frac{1}{2}} + N^\alpha (k+1)^{-\frac{3}{2}}\right).$$

Therefore

$$\|\psi_k^z\|_\lambda \le C(\lambda, T) \mathrm{e}_\lambda(z) N^\alpha k^{-\frac{1}{2}}.$$

(d) For $|x| \ge 1$,

$$\begin{aligned}
\mathbf{P}(S_k = x) &\le N^{-(1+\alpha)} \mathbf{P}(S_k \ge |x| - 1) \\
&\le N^{-(1+\alpha)} \exp(-u(|x| - 1)) \mathbf{E} \exp(uX_k) \\
&\le N^{-(1+\alpha)} \exp(-u(|x| - 1)) \exp\left(\frac{u^2 k}{6N^{2\alpha}}\right).
\end{aligned}$$

Hence, for $|x - z| \ge 1$,

$$\psi_k^x(z) \le \exp(-u|x - z|) \exp\left(\frac{u^2 k}{6N^{2\alpha}}\right).$$

This gives that for $|x-z|\geq 2$,

$$\psi_k^x(z)+\psi_k^y(z)\leq \exp(-2\lambda|x-z|)\exp\left(\frac{2\lambda^2 k}{3N^{2\alpha}}\right).$$

From (3.8) and $|p(t,x)-p(t,y)|\leq Ct^{-1}|x-y|$, we have

$$\|\psi_k^x-\psi_k^y\|_0\leq C\left(|x-y|N^{2\alpha}k^{-1}+N^{\alpha}k^{-\frac{3}{2}}\right).$$

So,

$$\begin{aligned}\|\psi_k^x-\psi_k^y\|_\lambda\leq &\sup_{|x-z|<2} C(\lambda)\|\psi_k^x-\psi_k^y\|_0 \mathrm{e}_\lambda(z)\\ &+\sup_{|x-z|\geq 2}\min\left(\|\psi_k^x-\psi_k^y\|_0,\mathrm{e}^{\frac{C(\lambda)k}{N^{2\alpha}}}\exp(-2\lambda|x-z|)\right)\mathrm{e}_\lambda(z)\\ &\leq C(\lambda)\mathrm{e}^{\frac{C(\lambda)k}{N^{2\alpha}}}\mathrm{e}_\lambda(x)\left(\|\psi_k^x-\psi_k^y\|_0+\|\psi_k^x-\psi_k^y\|_0^{\frac{1}{2}}\right)\\ &\leq C(\lambda)\mathrm{e}^{C(\lambda)k/N^{2\alpha}}\mathrm{e}_\lambda(x)\left(|x-y|^{\frac{1}{2}}k^{-\frac{1}{2}}N^{\alpha}+N^{\frac{\alpha}{2}}k^{-\frac{3}{4}}\right).\end{aligned}$$

(e) By using $|p(t,y)-p(s,y)|\leq C|t-s|s^{-3/2}$ and (3.8), we have

$$\|\psi_k^y-\psi_l^y\|_0\leq C\left(|k-l|l^{-\frac{3}{2}}N^{\alpha}+N^{\alpha}k^{-\frac{3}{2}}+N^{\alpha}l^{-\frac{3}{2}}\right).$$

Similarly as the argument in (d), we can get

$$\|\psi_k^y-\psi_l^y\|_\lambda\leq C(\lambda)\mathrm{e}^{\frac{C(\lambda)k}{N^{2\alpha}}}\mathrm{e}_\lambda(y)N^{\frac{\alpha}{2}}\left(|k-l|^{\frac{1}{2}}l^{-\frac{3}{4}}+k^{-\frac{3}{4}}\right).$$

□

Moment estimations

Recall the Burkholder-Davis-Gundy (BDG) inequality for discrete martingale (see Beiglböck and Siorpaes (2015)):

$$\mathbf{E}(\sup_{1\leq i\leq t}|M_i|^p)\leq C(p)\mathbf{E}\langle M\rangle_t^{\frac{p}{2}},$$

where $\langle M\rangle_t=\sum_{k=1}^t \mathbf{E}_{k-1}(d_k^2), d_k=M_k-M_{k-1}$ and $1<p<\infty$. The notation $\mathbf{E}_{k-1}(\cdot)$ means conditional expectation given $\mathscr{F}_{k-1}$, where $\mathscr{F}_k=\sigma\left(\xi_j, j\leq k\right)$.

We will use the discrete Gronwall inequality several times in the proof of moment estimations.

Lemma 3.3 (discrete Gronwall inequality)**.** *Let* $\{y_n\}_{n\geq 0},\{f_n\}_{n\geq 0}$ *and* $\{g_n\}_{n\geq 0}$ *be nonnegative* *sequences such that*

$$y_n \le f_n + \sum_{k=0}^{n-1} g_k y_k, \text{ for } n \ge 0. \tag{3.9}$$

Then,

$$y_n \le f_n + \sum_{k=0}^{n-1} f_k g_k \exp\left(\sum_{j=k+1}^{n-1} g_j\right). \tag{3.10}$$

Proof. Define

$$G_n := \prod_{j=0}^{n-1}(1+g_j),$$

then following mathematical induction, we have

$$G_n = 1 + \sum_{j=0}^{n-1} g_j G_j,$$

for $n \ge 0$ and more generally for $0 \le k \le n-1$,

$$G_n = G_k + \sum_{j=k}^{n-1} g_j G_j. \tag{3.11}$$

Based on (3.11), we give the exact solution when the inequality in (3.9) is replaced by equality. Suppose nonnegative sequence $\{x_n\}_{n\ge 0}$ satisfies

$$x_n = f_n + \sum_{k=0}^{n-1} g_k x_k.$$

Then,

$$x_n = f_n + \sum_{k=0}^{n-1} f_k g_k \prod_{j=k+1}^{n-1}(1+g_j) = f_n + \sum_{k=0}^{n-1} f_k g_k \frac{G_n}{G_{k+1}}. \tag{3.12}$$

Simple proof is as follows. It is obvious that (3.12) holds when $n = 0$. Suppose that (3.12) holds for any $0 \le k \le m-1$, then

$$\begin{aligned}
x_m &= f_m + \sum_{k=0}^{m-1} g_k x_k \\
&= f_m + \sum_{k=0}^{m-1} g_k \left(f_k + \sum_{j=0}^{k-1} f_j g_j \frac{G_k}{G_{j+1}}\right) \\
&= f_m + \sum_{k=0}^{m-1} g_k f_k + \sum_{k=0}^{m-1}\sum_{j=0}^{k-1} g_k f_j g_j \frac{G_k}{G_{j+1}} \\
&= f_m + \sum_{k=0}^{m-1} f_k g_k \frac{G_m}{G_{k+1}},
\end{aligned}$$

where we use (3.11) in the last equality. (3.12) follows by induction. If $\{y_n\}_{n\ge 0}$ satisfies (3.9), then by (3.12),

$$y_n \leq f_n + \sum_{k=0}^{n-1} f_k g_k \prod_{j=k+1}^{n-1} (1+g_j).$$

We can conclude Lemma 3.3 by the fact that

$$1+g_j \leq \exp(g_j)$$

for any $j \geq 0$. □

We then have the following moment estimations.

Lemma 3.4. *Suppose the initial condition $A(\xi_0)$ whose linear interpolation converges in $\mathscr{C}$ to f that is continuous and compact supported, then for $T \geq 0$, $p \geq 2$, $\lambda > 0$*

(a) $\mathbf{E}\Big(\sup_{k\leq \lfloor tN^{2\alpha}\rfloor} (\nu_k^N, e_{-\lambda})^p\Big) \leq C(\lambda, p, f, T)$.

(b) $(\nu_0^N, \psi_t^z)^p \leq C(\lambda, p, f) e_{\lambda p}(z)$.

(c) $\|\mathbf{E}(A^p(\xi_{\lfloor tN^{2\alpha}\rfloor}))\|_{-\lambda p} \leq C(\lambda, p, f, T)$ *for* $t \leq T$.

Proof. (a) Plugging $\phi_i^N = \mathrm{e}_{-\lambda}$ into (3.5) gives

$$(\nu_n^N, \mathrm{e}_{-\lambda}) = (A\xi_0, \mathrm{e}_{-\lambda}) + \sum_{i=1}^{n-1} (\nu_i^N, N^{-2\alpha}\Delta_D \mathrm{e}_{-\lambda}) + M_n(\mathrm{e}_{-\lambda}).$$

Since $\Delta_D \mathrm{e}_{-\lambda} \leq C(\lambda)\mathrm{e}_{-\lambda}$, thanks to Hölder inequality, we have

$$\begin{aligned}
\mathbf{E}\Big(\sup_{k\leq \lfloor tN^{2\alpha}\rfloor} (\nu_k^N, \mathrm{e}_{-\lambda})^p\Big) &\leq C(\lambda,p,f) + C(\lambda,p)\mathbf{E}\left(\sum_{i=1}^{\lfloor tN^{2\alpha}\rfloor -1} (\nu_i^N, N^{-2\alpha}\mathrm{e}_{-\lambda})\right)^p + C(p)\mathbf{E}\sup_{k\leq \lfloor tN^{2\alpha}\rfloor} |M_s(\mathrm{e}_{-\lambda})|^p \\
&\leq C(\lambda,p,f) + C(\lambda,p)t^{p-1}N^{-2\alpha}\sum_{k=1}^{\lfloor tN^{2\alpha}\rfloor -1} \mathbf{E}(\nu_k^N, \mathrm{e}_{-\lambda})^p + C(p)\mathbf{E}\langle M(\mathrm{e}_{-\lambda})\rangle_{\lfloor tN^{2\alpha}\rfloor}^{\frac{p}{2}}
\end{aligned}$$

The square variation in the last term satisfies

$$\begin{aligned}
\langle M(\mathrm{e}_{-\lambda})\rangle_{\lfloor tN^{2\alpha}\rfloor} &\leq \frac{C(\lambda)}{N^{2\alpha}} \sum_{k=1}^{\lfloor tN^{2\alpha}\rfloor} (\nu_{k-1}^N, \mathrm{e}_{-2\lambda}) \\
&\leq C(\lambda)\frac{1}{N^{2\alpha}} \sum_{k=1}^{\lfloor tN^{2\alpha}\rfloor} 1 + (\nu_{k-1}^N, \mathrm{e}_{-\lambda})^2.
\end{aligned}$$

Use Hölder inequality again, we have

$$
\begin{aligned}
\mathbf{E}\Big(\sup_{k\le\lfloor tN^{2\alpha}\rfloor}(\nu_k^N,\mathrm{e}_{-\lambda})^p\Big) &\le C(\lambda,p,f)+C(\lambda,p)\,t^{p-1}N^{-2\alpha}\sum_{k=1}^{\lfloor tN^{2\alpha}\rfloor-1}\mathbf{E}(\nu_k^N,\mathrm{e}_{-\lambda})^p\\
&\quad+C(p,\lambda)\mathbf{E}\left(\frac{1}{N^{2\alpha}}\sum_{k=1}^{\lfloor tN^{2\alpha}\rfloor}(\nu_{k-1}^N,e_{-\lambda})^2\right)^{\frac{p}{2}}\\
&\le C(\lambda,p,f)+C(\lambda,p)\,T^{p-1}N^{-2\alpha}\sum_{k=1}^{\lfloor tN^{2\alpha}\rfloor-1}\mathbf{E}(\nu_k^N,\mathrm{e}_{-\lambda})^p\\
&\quad+C(\lambda,p)\,T^{\frac{p}{2}-1}N^{-2\alpha}\sum_{k=1}^{\lfloor tN^{2\alpha}\rfloor-1}\mathbf{E}(\nu_k^N,e_{-\lambda})^p\\
&\le C(\lambda,p,f,T)+C(\lambda,p,f,T)N^{-2\alpha}\sum_{k=1}^{\lfloor tN^{2\alpha}\rfloor-1}\mathbf{E}(\nu_k^N,e_{-\lambda})^p
\end{aligned}
$$

The discrete Gronwall's inequality (Lemma 3.3) concludes part (a)

$$
\mathbf{E}\Big(\sup_{k\le\lfloor tN^{2\alpha}\rfloor}(\nu_k^N,\mathrm{e}_{-\lambda})^p\Big)\le C(\lambda,p,f,T).
$$

(b) Let $\overline{\psi}_t^z(x)=N^{1+\alpha}\mathbf{P}(X_t=x-z)$. Since $\psi_t^z(x)=\frac{1}{2N}\sum_{y\sim x}\overline{\psi}_t^z(y)$, $(\nu_n,\psi_t^z)=(A(\xi_n),\overline{\psi}_t^z)$. It is directly by using (b) of Lemma 3.2 since

$$
\begin{aligned}
(\nu_0,\psi_t^z)^p &= (A\xi_0,\overline{\psi}_t^z)^p\\
&\le \|A(\xi_0)\|_{-\lambda}^p(\mathrm{e}_\lambda,\overline{\psi}_t^z)^p\\
&\le C(\lambda,p,f)\mathrm{e}_{\lambda p}(z)
\end{aligned}
$$

(c) By (3.7) and (b),

$$
\|\mathbf{E}(A^p(\xi_{\lfloor tN^{2\alpha}\rfloor}))\|_{-\lambda p}\le C(\lambda,p,f,T)+C(p)\|\mathbf{E}|M_{\lfloor tN^{2\alpha}\rfloor}(\psi_{\lfloor tN^{2\alpha}\rfloor-\cdot})|^p\|_{-\lambda p}.
$$

For the second term above, by BDG inequality,

$$\begin{aligned}
\mathbf{E}\left|M_{\lfloor tN^{2\alpha}\rfloor}(\psi_{\lfloor tN^{2\alpha}\rfloor-\cdot})\right|^p &\le \mathbf{E}\langle M(\psi_{\lfloor tN^{2\alpha}\rfloor-\cdot})\rangle_{\lfloor tN^{2\alpha}\rfloor}^{\frac{p}{2}} \\
&\le \mathbf{E}\left(\sum_{k=1}^{\lfloor tN^{2\alpha}\rfloor}\frac{\|\psi_{\lfloor tN^{2\alpha}\rfloor-k+1}\|}{N^{2\alpha}}(A\xi_{k-1},\psi_{\lfloor tN^{2\alpha}\rfloor-k+1})\right)^{\frac{p}{2}} \\
&\le \mathbf{E}\left(\sum_{k=1}^{\lfloor tN^{2\alpha}\rfloor}\frac{|\lfloor tN^{2\alpha}\rfloor-k+1|^{-\frac{1}{2}}}{N^{\alpha}}(A\xi_{k-1},\psi_{\lfloor tN^{2\alpha}\rfloor-k+1})\right)^{p/2} \quad \text{(Lemma 3.2 (c))} \\
&\le C(p,T)\sum_{k=1}^{\lfloor tN^{2\alpha}\rfloor}\frac{|\lfloor tN^{2\alpha}\rfloor-k+1|^{-\frac{1}{2}}}{N^{\alpha}}\left(\mathbf{E}A^{\frac{p}{2}}(\xi_{k-1}),\psi_{\lfloor tN^{2\alpha}\rfloor-k+1}\right) \quad \text{(Lemma 3.2 (a))} \\
&\le C(p,T)\sum_{k=1}^{\lfloor tN^{2\alpha}\rfloor}\frac{|\lfloor tN^{2\alpha}\rfloor-k+1|^{-\frac{1}{2}}}{N^{\alpha}}\|\mathbf{E}A^{\frac{p}{2}}(\xi_{k-1})\|_{-\lambda p}\left(\mathrm{e}_{\lambda p},\psi_{\lfloor tN^{2\alpha}\rfloor-k+1}\right) \\
&\le C(p,T,\lambda)\mathrm{e}_{\lambda p}(z)\sum_{k=1}^{\lfloor tN^{2\alpha}\rfloor}\frac{|\lfloor tN^{2\alpha}\rfloor-k+1|^{-\frac{1}{2}}}{N^{\alpha}}\|\mathbf{E}A^{p}(\xi_{k-1})+1\|_{-\lambda p} \quad \text{(Lemma 3.2(b))} \\
&\le C(p,T,\lambda)\mathrm{e}_{\lambda p}(z)\left(1+\sum_{k=1}^{\lfloor tN^{2\alpha}\rfloor}\frac{|\lfloor tN^{2\alpha}\rfloor-k+1|^{-\frac{1}{2}}}{N^{\alpha}}\|\mathbf{E}(A^{p}(\xi_{k-1}))\|_{-\lambda p}\right).
\end{aligned}$$

This gives that

$$\|\mathbf{E}(A^p(\xi_{\lfloor tN^{2\alpha}\rfloor}))\|_{-\lambda p} \le C(\lambda,p,f,T)\left(1+\sum_{k=1}^{\lfloor tN^{2\alpha}\rfloor}\frac{|\lfloor tN^{2\alpha}\rfloor-k+1|^{-\frac{1}{2}}}{N^{\alpha}}\|\mathbf{E}(A^{p}(\xi_{k-1}))\|_{-\lambda p}\right).$$

The discrete Gronwall inequality (Lemma 3.3) completes this proof.

□

3.1.3 Tightness

In this section, we assume an initial condition so that the linear interpolation of $A(\xi_0)$ converges to f in $\mathscr{C}$. To get the centred approximated density, by (3.7), let

$$\hat{A}(\xi_n)(x) = A(\xi_n)(x) - (\nu_0^N, \psi_n^x).$$

Lemma 3.5. *For* $0 \le s \le t \le T$, $x, y \in \mathbb{Z}N^{-(1+\alpha)}$, $|t-s| \le 1$, $|x-y| \le 1$, $\lambda > 0$ *and* $p \ge 2$,

$$\mathbf{E}|\hat{A}(\xi_{\lfloor tN^{2\alpha}\rfloor})(x) - \hat{A}(\xi_{\lfloor sN^{2\alpha}\rfloor})(y)|^p \le C(\lambda,p,f,T)e_{\lambda p}(x)\left(|x-y|^{\frac{p}{4}} + |t-s|^{\frac{p}{4}} + N^{-\frac{\alpha p}{2}}\right). \qquad (3.13)$$

Proof. We decompose this difference into space difference $\hat{A}(\xi_{\lfloor tN^{2\alpha}\rfloor})(x) - \hat{A}(\xi_{\lfloor tN^{2\alpha}\rfloor})(y)$ and time difference $\hat{A}(\xi_{\lfloor tN^{2\alpha}\rfloor})(y) - \hat{A}(\xi_{\lfloor sN^{2\alpha}\rfloor})(y)$. First, we deal with the space difference. The Burkholder-Davis-Gundy (BDG) inequality (discrete version recalled in Section 3.1.2) gives that

$$\mathbf{E}|\hat{A}(\xi_{\lfloor tN^{2\alpha}\rfloor})(x)-\hat{A}(\xi_{\lfloor tN^{2\alpha}\rfloor})(y)|^p \le \mathbf{E}\langle M(\psi^x_{\lfloor tN^{2\alpha}\rfloor-\cdot}-\psi^y_{\lfloor tN^{2\alpha}\rfloor-\cdot})\rangle^{\frac{p}{2}}_{\lfloor tN^{2\alpha}\rfloor}.$$

The constants $C(\lambda,p,f,T)$ in the following proof are generic constants. With a similar argument as in (3.6),

$$\begin{aligned}&\langle M(\psi^x_{\lfloor tN^{2\alpha}\rfloor-\cdot}-\psi^y_{\lfloor tN^{2\alpha}\rfloor-\cdot})\rangle_{\lfloor tN^{2\alpha}\rfloor}\\ &\le \sum_{k=1}^{\lfloor tN^{2\alpha}\rfloor}\frac{\|\psi^x_{\lfloor tN^{2\alpha}\rfloor-k+1}-\psi^y_{\lfloor tN^{2\alpha}\rfloor-k+1}\|_\lambda}{N^{2\alpha}}\left(A\xi_{k-1}e_{-\lambda},\psi^x_{\lfloor tN^{2\alpha}\rfloor-k+1}+\psi^y_{\lfloor tN^{2\alpha}\rfloor-k+1}\right)\end{aligned}$$

By Lemma 3.2 (d),

$$\begin{aligned}&\langle M(\psi^x_{\lfloor tN^{2\alpha}\rfloor-\cdot}-\psi^y_{\lfloor tN^{2\alpha}\rfloor-\cdot})\rangle_{\lfloor tN^{2\alpha}\rfloor}\\ &\le \sum_{k=1}^{\lfloor tN^{2\alpha}\rfloor}\left((\lfloor tN^{2\alpha}\rfloor-k+1)^{-\frac{1}{2}}N^{-\alpha}|x-y|^{\frac{1}{2}}+N^{-\frac{3\alpha}{2}}k^{-\frac{3}{4}}\right)\cdot\\ &\qquad\qquad\left(A\xi_{k-1}e_{-\lambda},\psi^x_{\lfloor tN^{2\alpha}\rfloor-k+1}+\psi^y_{\lfloor tN^{2\alpha}\rfloor-k+1}\right).\end{aligned}$$

By using Lemma 3.2 (b) and Lemma 3.4 (c),

$$\begin{aligned}&\mathbf{E}|\hat{A}(\xi_{\lfloor tN^{2\alpha}\rfloor})(x)-\hat{A}(\xi_{\lfloor tN^{2\alpha}\rfloor})(y)|^p\\ &\le C(\lambda,p,f,T)\mathrm{e}_{\lambda p}(x)\left(\sum_{k=1}^{\lfloor tN^{2\alpha}\rfloor}(\lfloor tN^{2\alpha}\rfloor-k+1)^{-\frac{1}{2}}N^{-\alpha}|x-y|^{\frac{1}{2}}+N^{-\frac{3\alpha}{2}}k^{-\frac{3}{4}}\right)^{\frac{p}{2}}.\end{aligned}$$

It is easily seen that

$$\mathbf{E}|\hat{A}(\xi_{\lfloor tN^{2\alpha}\rfloor})(x)-\hat{A}(\xi_{\lfloor tN^{2\alpha}\rfloor})(y)|^p \le C(\lambda,p,f,T)\mathrm{e}_{\lambda p}(x)\left(|x-y|^{\frac{p}{4}}+N^{-\frac{\alpha p}{2}}\right). \tag{3.14}$$

For the time difference,

$$\begin{aligned}
&\hat{A}(\xi_{\lfloor tN^{2\alpha}\rfloor})(y) - \hat{A}(\xi_{\lfloor sN^{2\alpha}\rfloor})(y) \\
&= M_{\lfloor tN^{2\alpha}\rfloor}(\psi^y_{\lfloor tN^{2\alpha}\rfloor - \cdot}) - M_{\lfloor sN^{2\alpha}\rfloor}(\psi^y_{\lfloor sN^{2\alpha}\rfloor - \cdot}) \\
&= \frac{1}{N^{2\alpha}} \sum_{k=1}^{\lfloor tN^{2\alpha}\rfloor} \sum_x \sum_{z\sim x} \sum_{w=1}^{\xi_{k-1}(z)} \psi^y_{\lfloor tN^{2\alpha}\rfloor - k+1}(x) \left(\eta^w_k(z,x) - \frac{1}{2N}\right) \\
&\quad - \frac{1}{N^{2\alpha}} \sum_{k=1}^{\lfloor sN^{2\alpha}\rfloor} \sum_x \sum_{z\sim x} \sum_{w=1}^{\xi_{k-1}(z)} \psi^y_{\lfloor sN^{2\alpha}\rfloor - k+1}(x) \left(\eta^w_k(z,x) - \frac{1}{2N}\right) \\
&= \frac{1}{N^{2\alpha}} \sum_{k=1}^{\lfloor sN^{2\alpha}\rfloor} \sum_x \sum_{z\sim x} \sum_{w=1}^{\xi_{k-1}(z)} (\psi^y_{\lfloor tN^{2\alpha}\rfloor - k+1}(x) - \psi^y_{\lfloor sN^{2\alpha}\rfloor - k+1}(x)) \left(\eta^w_k(z,x) - \frac{1}{2N}\right) \\
&\quad + \frac{1}{N^{2\alpha}} \sum_{k=\lfloor sN^{2\alpha}\rfloor+1}^{\lfloor tN^{2\alpha}\rfloor} \sum_x \sum_{z\sim x} \sum_{w=1}^{\xi_{k-1}(z)} \psi^y_{\lfloor tN^{2\alpha}\rfloor - k+1}(x) \left(\eta^w_k(z,x) - \frac{1}{2N}\right) \\
&= M^{(1)}_{\lfloor sN^{2\alpha}\rfloor} + \left(M^{(2)}_{\lfloor tN^{2\alpha}\rfloor} - M^{(2)}_{\lfloor sN^{2\alpha}\rfloor}\right),
\end{aligned}$$

where the two martingales are

$$M^{(1)}_{\lfloor sN^{2\alpha}\rfloor} = \frac{1}{N^{2\alpha}} \sum_{k=1}^{\lfloor sN^{2\alpha}\rfloor} \sum_x \sum_{z\sim x} \sum_{w=1}^{\xi_{k-1}(z)} (\psi^y_{\lfloor tN^{2\alpha}\rfloor - k+1}(x) - \psi^y_{\lfloor sN^{2\alpha}\rfloor - k+1}(x)) \left(\eta^w_k(z,x) - \frac{1}{2N}\right)$$

and

$$M^{(2)}_{\lfloor sN^{2\alpha}\rfloor} = \frac{1}{N^{2\alpha}} \sum_{k=1}^{\lfloor sN^{2\alpha}\rfloor} \sum_x \sum_{z\sim x} \sum_{w=1}^{\xi_{k-1}(z)} \psi^y_{\lfloor tN^{2\alpha}\rfloor - k+1}(x) \left(\eta^w_k(z,x) - \frac{1}{2N}\right).$$

For $M^{(1)}_{\lfloor sN^{2\alpha}\rfloor}$, we use the similar argument as (3.6), and get

$$\begin{aligned}
\langle M^{(1)}\rangle_{\lfloor sN^{2\alpha}\rfloor} &\le \sum_{k=1}^{\lfloor sN^{2\alpha}\rfloor} \frac{\|\psi^y_{\lfloor tN^{2\alpha}\rfloor - k+1} - \psi^y_{\lfloor sN^{2\alpha}\rfloor - k+1}\|_\lambda}{N^{2\alpha}} \left(A\xi_{k-1}e_{-\lambda}, \psi^y_{\lfloor tN^{2\alpha}\rfloor - k+1} + \psi^y_{\lfloor tN^{2\alpha}\rfloor - k+1}\right) \\
&\le \sum_{k=1}^{\lfloor sN^{2\alpha}\rfloor} \left(N^{-\frac{\alpha}{2}} (\lfloor sN^{2\alpha}\rfloor - k+1)^{-\frac{3}{4}} |t-s|^{\frac{1}{2}} + N^{-\frac{3\alpha}{2}} (\lfloor sN^{2\alpha}\rfloor - k+1)^{-\frac{3}{4}}\right) \\
&\qquad \cdot \left(A\xi_{k-1}e_{-\lambda}, \psi^y_{\lfloor tN^{2\alpha}\rfloor - k+1} + \psi^y_{\lfloor sN^{2\alpha}\rfloor - k+1}\right).
\end{aligned}$$

By BDG inequality,

$$\begin{aligned}
\mathbf{E}\left|M^{(1)}_{\lfloor sN^{2\alpha}\rfloor}\right|^p &\le \mathbf{E}\langle M^{(1)}\rangle^{\frac{p}{2}}_{\lfloor sN^{2\alpha}\rfloor} \\
&\le C(p) \sum_{k=1}^{\lfloor sN^{2\alpha}\rfloor} \left(N^{-\frac{\alpha}{2}} (\lfloor sN^{2\alpha}\rfloor - k+1)^{-\frac{3}{4}} |t-s|^{\frac{1}{2}} + N^{-\frac{3\alpha}{2}} (\lfloor sN^{2\alpha}\rfloor - k+1)^{-\frac{3}{4}}\right)^{\frac{p}{2}} \\
&\qquad \cdot \mathbf{E}\left(A\xi_{k-1}e_{-\lambda}, \psi^y_{\lfloor tN^{2\alpha}\rfloor - k+1} + \psi^y_{\lfloor sN^{2\alpha}\rfloor - k+1}\right)^{\frac{p}{2}}.
\end{aligned}$$

Writing $e_{-\frac{\lambda p}{2}} = e_{-\frac{3\lambda p}{2}} e_{\lambda p}$ and implementing Lemma 3.4 (c) give

$$\|\mathbf{E}(A^{p/2}(\xi_{k-1}))\|_{-\frac{3\lambda p}{2}} \leq C(\lambda, p, f, T).$$

Together with Lemma 3.2 (b), we get

$$\begin{aligned}\mathbf{E}\left(A\xi_{k-1}e_{-\lambda}, \psi^y_{\lfloor tN^{2\alpha}\rfloor-k+1} + \psi^y_{\lfloor sN^{2\alpha}\rfloor-k+1}\right)^{\frac{p}{2}} &\leq \mathbf{E}\left((A^{p/2}(\xi_{k-1}))e_{-\frac{\lambda p}{2}}, \psi^y_{\lfloor tN^{2\alpha}\rfloor-k+1} + \psi^y_{\lfloor sN^{2\alpha}\rfloor-k+1}\right)\\ &\leq C(\lambda, p, T)e_{\lambda p}(y).\end{aligned}$$

Hence,

$$\begin{aligned}\mathbf{E}\left|M^{(1)}_{\lfloor sN^{2\alpha}\rfloor}\right|^p &\leq C(\lambda, p, f, T)\mathrm{e}_{\lambda p}(y)\left(\sum_{k=1}^{\lfloor sN^{2\alpha}\rfloor} N^{-\frac{\alpha}{2}}(\lfloor sN^{2\alpha}\rfloor - k+1)^{-\frac{3}{4}}|t-s|^{\frac{1}{2}} + N^{-\frac{3\alpha}{2}}(\lfloor tN^{2\alpha}\rfloor - k+1)^{-\frac{3}{4}}\right)^{\frac{p}{2}}\\ &\leq C(\lambda, p, f, T)\mathrm{e}_{\lambda p}(y)\left(|t-s|^{\frac{p}{4}} + N^{-\frac{\alpha p}{2}}\right),\end{aligned} \tag{3.15}$$

where the second inequality is because of the fact that

$$\sum_{k=1}^{\lfloor sN^{2\alpha}\rfloor}(\lfloor sN^{2\alpha}\rfloor - k+1)^{-\frac{3}{4}} \leq C(T)N^{\frac{\alpha}{2}}.$$

For $M^{(2)}_{\lfloor tN^{2\alpha}\rfloor} - M^{(2)}_{\lfloor sN^{2\alpha}\rfloor}$,

$$\mathbf{E}\left|M^{(2)}_{\lfloor tN^{2\alpha}\rfloor} - M^{(2)}_{\lfloor sN^{2\alpha}\rfloor}\right|^p \leq \mathbf{E}\left(\langle M^{(2)}\rangle_{\lfloor tN^{2\alpha}\rfloor} - \langle M^{(2)}\rangle_{\lfloor sN^{2\alpha}\rfloor}\right)^{\frac{p}{2}}.$$

Similar as the argument in (3.6),

$$\begin{aligned}\langle M^{(2)}\rangle_{\lfloor tN^{2\alpha}\rfloor} - \langle M^{(2)}\rangle_{\lfloor sN^{2\alpha}\rfloor} &\leq \sum_{k=\lfloor sN^{2\alpha}\rfloor+1}^{\lfloor tN^{2\alpha}\rfloor} \frac{\|\psi^y_{\lfloor tN^{2\alpha}\rfloor-k+1}\|_\lambda}{N^{2\alpha}}(A\xi_{k-1}e_{-\lambda}, \psi_{\lfloor tN^{2\alpha}\rfloor-k+1})\\ &\leq \sum_{k=\lfloor sN^{2\alpha}\rfloor+1}^{\lfloor tN^{2\alpha}\rfloor}\left(N^{-\alpha}(\lfloor tN^{2\alpha}\rfloor - k+1)^{-\frac{1}{2}}\right)(A\xi_{k-1}e_{-\lambda}, \psi^y_{\lfloor tN^{2\alpha}\rfloor-k+1}).\end{aligned}$$

Thanks again to Lemma 3.2 (b) and Lemma 3.4 (c),

$$\begin{aligned}\mathbf{E}\left|M^{(2)}_{\lfloor tN^{2\alpha}\rfloor} - M^{(2)}_{\lfloor sN^{2\alpha}\rfloor}\right|^p &\leq C(\lambda, p, f, T)\mathrm{e}_{\lambda p}(y)\left(\sum_{k=\lfloor sN^{2\alpha}\rfloor+1}^{\lfloor tN^{2\alpha}\rfloor} N^{-\alpha}(\lfloor tN^{2\alpha}\rfloor - k+1)^{-\frac{1}{2}}\right)^{\frac{p}{2}}\\ &\leq C(\lambda, p, f, T)\mathrm{e}_{\lambda p}(y)|t-s|^{p/4}.\end{aligned} \tag{3.16}$$

Summarising (3.14)(3.15)(3.16), we can get (3.13). □

Tightness of $\left\{A(\xi_{\lfloor tN^{2\alpha}\rfloor}), N \geq 1\right\}$ follows from Lemma 3.5.

Lemma 3.6. *For* $\phi : \mathbb{Z}/N^{1+\alpha} \to [0,\infty)$ *and* $\lambda > 0$,

$$|(\nu_k^N, \phi) - (A(\xi_k), \phi)| \le \|D(\phi, N^{-\alpha})\|_\lambda (\nu_k^N, e_{-\lambda}).$$

with $D(\phi, N^{-\alpha})$ *as defined in (3.1).*

Proof.

$$\begin{aligned}
(A(\xi_k), \phi) &= \frac{1}{N^{1+\alpha}} \sum_x A(\xi_k)(x)\phi(x) \\
&= \frac{1}{2N^{1+2\alpha}} \sum_x \sum_{y\sim x} \xi_k(y)\phi(x) \\
&= \frac{1}{2N^{1+2\alpha}} \sum_x \sum_{y\sim x} \xi_k(y)(\phi(x) - \phi(y)) + (\nu_k^N, \phi).
\end{aligned}$$

Therefore,

$$\begin{aligned}
|(\nu_k^N, \phi) - (A(\xi_k), \phi)| &\le \frac{1}{2N^{1+2\alpha}} \sum_x \sum_{y\sim x} \xi_k(y) D(\phi, N^{-\alpha})(y) \\
&= (\nu_k^N, D(\phi, N^{-\alpha})) \\
&\le \|D(\phi, N^{-\alpha})\|_\lambda (\nu_k^N, \mathrm{e}_{-\lambda}).
\end{aligned}$$

□

Lemma 3.6 together with (a) of Lemma 3.4 give that

$$\mathbf{E}\left(\sup_{1\le k\le \lfloor tN^{2\alpha} \rfloor} \|(\nu_k^N, \phi) - (A(\xi_k), \phi)\|^p \right) \le C(\lambda, p, f, T) \|D(\phi, N^{-\alpha})\|_\lambda^p.$$

This implies the tightness of $\left\{(\nu^N_{\lfloor tN^{2\alpha}\rfloor}, \phi), N \ge 1\right\}$ for each such test function and therefore the tightness of $\left\{\nu^N_{\lfloor tN^{2\alpha}\rfloor}, N \ge 1\right\}$ as a measure-valued process under vague topology. Hence, with probability one, for all $T > 0, \lambda > 0$, and test function $\phi \in C_0^2(\mathbb{R})$ with compact support, we can have a subsequence of $A(\xi^N_{\lfloor tN^{2\alpha}\rfloor})$ and $\nu^N_{\lfloor tN^{2\alpha}\rfloor}$ such that

$$\begin{aligned}
\sup_{t\le T} \|A(\xi^N_{\lfloor tN^{2\alpha}\rfloor}) - u_t\|_{-\lambda} \to 0, \text{ as } N \to \infty, \\
\sup_{t\le T} \left| \int \phi(x) \nu^N_{\lfloor tN^{2\alpha}\rfloor}(dx) - \int \phi(x) \nu_t(dx) \right| \to 0, \text{ as } N \to \infty,
\end{aligned}$$

From Lemma 3.6, we can see that ν_t is absolutely continuous with density $\nu_t(dx) = u_t(x)dx$. By substituting $\phi_k^N = \phi \in C_0^2(\mathbb{R})$ as the test function with compact support in the decomposition

(3.5), we can see that

$$M^N_{\lfloor tN^{2\alpha}\rfloor}(\phi) = (\nu^N_{\lfloor tN^{2\alpha}\rfloor}, \phi) - (A(\xi_0), \phi) - \sum_{k=1}^{\lfloor tN^{2\alpha}\rfloor} \frac{1}{N^{2\alpha}}(\nu^N_k, \Delta_D \phi)$$

is a martingale and every term on the right-hand side converges almost surely by Lemma 3.5. Hence $M^N_{\lfloor tN^{2\alpha}\rfloor}(\phi)$ converges to a local martingale

$$\begin{aligned} m_t(\phi) &= \int \phi(x)\nu_t(dx) - \int \phi(x)\nu_0(dx) - \int_0^t \int \frac{1}{6}\Delta\phi(x)\nu_s(dx)ds \\ &= \int \phi(x)u_t(x)dx - \int \phi(x)f(x)dx - \int_0^t \int \frac{1}{6}\Delta\phi(x)u_s(x)dxds, \end{aligned} \tag{3.17}$$

which is continuous since every term on the right-hand side is continuous. Moreover, from (3.6),

$$(M^N_{\lfloor tN^{2\alpha}\rfloor})^2 - \sum_{k=1}^{\lfloor tN^{2\alpha}\rfloor} \frac{1}{N^{2\alpha}}(A(\xi_{k-1}), \phi^2)\left(1 - \frac{1}{2N}\right)$$

is a martingale. As $N \to \infty$,

$$m_t^2(\phi) - \int_0^t \int \phi^2(x)u_s(x)dxds \tag{3.18}$$

is also a continuous local martingale. (3.17) and (3.18) prove that any subsequential weak limit $\nu_t(dx) = u_t(x)dx$ solves (2.2). The uniqueness follows from Theorem 5.7.1 of Dawson (1993) which finish the proof of Theorem 2.2.

3.1.4 Multiple particles at one site

In this subsection, we show the probability of multiple particles at one site is negligible in the branching envelope. Then, the state function can be reduced from its number of particles to that it is occupied or vacant. We will first show a property (see Lemma 3.7) that refers to the weak limit of the envelope process. This is then used to deal with the discrete process.

Let X_t denote the total mass of this system at time t, that is

$$X_t = (\nu_t, 1) = \int u_t(x)dx.$$

We then have that

$$X_t = \int_0^t \int \sqrt{u_s(x)}W(ds, dx),$$

and therefore its quadratic variation is $\langle X\rangle_t = \int_0^t \int u_s(x)dxds$. Hence

$$\begin{cases} X_t &= \int_0^t \sqrt{X_s}dB_s, \\ X_0 &= 2. \end{cases}$$

For $k \geq 1$, let T_k denote the stopping time given by

$$T_k = \inf\left\{t > 0 : X_t \geq 2^k \text{ or } \int_0^t X_s ds \geq 2^{2k}\right\} = T'_k \wedge T''_k,$$

where

$$T'_k = \inf\left\{t > 0 : X_t \geq 2^k\right\},\ T''_k = \inf\left\{\int_0^t X_s ds \geq 2^{2k}\right\}.$$

Lemma 3.7. *For a fixed initial condition f which is continuous and compact supported satisfying $f(x) = 1$ for $x \in [-1,1]$, there exists constant C so that*

$$\mathbf{P}(T_k < \infty) \leq C2^{-k}, k \in \mathbb{N}.$$

Proof. $\mathbf{P}(T_k < \infty)$ can be decomposed as

$$\begin{aligned} \mathbf{P}(T_k < \infty) &= \mathbf{P}\left(T'_k < \infty, T'_k < T''_k\right) + \mathbf{P}\left(T''_k < \infty, T''_k < T'_k\right) \\ &\leq \mathbf{P}\left(T'_k < \infty\right) + \mathbf{P}\left(T''_k < T'_k\right). \end{aligned} \tag{3.19}$$

If we denote $H_0 = \inf\{t > 0 : X_t = 0\}$ as the first hitting time of zero then $\mathbf{P}(T'_k < \infty) = \mathbf{P}(T'_k < H_0)$, hence the first term on the right hand side of (3.19) is simply

$$\mathbf{P}\left(T'_k < \infty\right) < \frac{X_0}{2^k}.$$

The event $\{T''_k < T'_k\} \subset \cup_{j=1}^{k-1} A_j$, where

$$A_j = \left\{T'_j < \infty, \int_{T'_j}^{T'_{j+1}} X_u du \geq \frac{6 \cdot 2^{2k}}{\pi(k-j)^2}\right\}.$$

By using the Markov property of X_t,

$$\mathbf{P}(A_j) \leq \mathbf{P}(T'_j < \infty)\mathbf{P}\left(\int_{T'_j}^{T'_{j+1}} X_u du \geq \frac{6 \cdot 2^{2k}}{\pi(k-j)^2}\right). \tag{3.20}$$

The total mass satisfies

$$X_t - X_s = \int_s^t \sqrt{X_u} dB_u,$$

then

$$\mathbf{E}\left((X_t - X_s)^2 | X_s\right) = \mathbf{E}\left(\int_s^t X_u du \middle| X_s\right).$$

Hence we can get

$$\mathbf{E}\int_{T'_j}^{T'_{j+1}} X_u du \le 2^{2j}.$$

From this, the second term in (3.20) can be bounded by using Markov inequality

$$\mathbf{P}\left(\int_{T'_j}^{T'_{j+1}} X_u du \ge \frac{6 \cdot 2^{2k}}{\pi(k-j)^2}\right) \le \frac{C(k-j)^2}{2^{2(k-j)}}.$$

Therefore,

$$\mathbf{P}(A_j) \le \frac{1}{2^k} \cdot \frac{CX_0(k-j)^2}{2^{k-j}}.$$

After plugging in $\mathbf{P}(T_k < \infty) \le \mathbf{P}(T'_k < \infty) + \sum_{j=1}^{k-1} \mathbf{P}(A_j)$, we have that,

$$\mathbf{P}(T_k < \infty) \le \frac{C}{2^k}.$$

□

Corollary 3.1. *For any $t > 0$,*

$$\int u_t(x)dx \text{ and } \int_0^t \int u_s(x)dxds$$

are finite with probability one.

Then for the discrete state function, we have:

Lemma 3.8. *For any $k \in \mathbb{N}$ and any $x \in \mathbb{Z}/N^{1+\alpha}$,*

$$\mathbf{P}(\xi_k(x) > 1 \mid \mathscr{F}_{k-1}) < (A\xi_{k-1}(x))^2 N^{2\alpha-2},$$

where $\{\mathscr{F}_k\}_{k\ge0}$ is the natural filtration $\mathscr{F}_k = \sigma\left(\{\xi_j, 0 \le j \le k\}\right)$.

Proof. In the discrete system, we have

$$\xi_k(x) = \sum_{y\sim x} \sum_{w=1}^{\xi_{k-1}(y)} \eta_k^w(y,x).$$

Denote $N' = \sum_{y\sim x} \xi_{k-1}(y) = 2N^{\alpha}(A\xi_{k-1}(x))$. Given $\{\xi_{k-1}(y), y \in \mathbb{Z}/N^{1+\alpha}\}$, we have

$$\xi_k(x) \overset{d}{=} \text{Binomial}(N', 1/(2N)).$$

Note $\mathbf{P}_{k-1}(\cdot)$ as conditional probability given $\mathscr{F}_{k-1}$.

$$\begin{aligned}
\mathbf{P}_{k-1}(\xi_k(x) \geq 2) &= 1 - \mathbf{P}_{k-1}(\xi_k(x) = 0) - \mathbf{P}_{k-1}(\xi_k(x) = 1) \\
&= 1 - \left(1 - \frac{1}{2N}\right)^{N'} - N' \frac{1}{2N}\left(1 - \frac{1}{2N}\right)^{N'-1} \\
&\leq \left(\frac{N'}{2N}\right)^2 \\
&= N^{2\alpha-2}(A\xi_{k-1}(x))^2.
\end{aligned}$$

The notation $\mathbf{P}_{k-1}(\cdot)$ means conditional probability given $\mathscr{F}_{k-1}$. $\square$

Since the branching envelope dominates the true horizontal process, this property will also hold for the true horizontal process.

3.2 The true horizontal process

The true process we consider is dominated by the branching random walk in the preceding section (ref. Figure 2.1 in Chapter 2), which means that at each time step, the particles will move and reproduce following the mechanism of ξ_k. But if the site x has been occupied by some particle before, then it cannot be occupied again. We denote $\{\hat{\xi}_k(x) \in \{0,1\}, k \in \mathbb{Z}_+, x \in \mathbb{Z}/N^{1+\alpha}\}$ as the mechanism of the true process. It can be expressed as

$$\hat{\xi}_{k+1}(x) = \begin{cases} 1 & \text{if } \sum_{j\leq k} \hat{\xi}_j(x) = 0 \text{ and } \sum_{y\in\mathcal{N}_k(x)} \eta_{k+1}(y,x) \geq 1, \\ 0 & \text{otherwise,} \end{cases}$$

where $\mathcal{N}_k(x) = \{y \sim x : \hat{\xi}_k(y) = 1\}$ having cardinality $N_k(x) = \sum_{y\sim x} \hat{\xi}_k(y)$ and $(\eta_{k+1}(y,x))_{k,y,x}$ is an i.i.d. sequence with distribution Bernoulli$(1/(2N))$. $\hat{\xi}_{k+1}(x)$ can be rewritten as

$$\begin{aligned}
\hat{\xi}_{k+1}(x) &= \mathbf{I}_{\{\sum_{y\in\mathcal{N}_k(x)} \eta_{k+1}(y,x)\geq 1\}}\left(1 - \sum_{j\leq k} \hat{\xi}_j(x)\right) \\
&= \underbrace{\sum_{y\in\mathcal{N}_k(x)} \eta_{k+1}(y,x)\left(1 - \sum_{j\leq k} \hat{\xi}_j(x)\right)}_{\text{main}} + \underbrace{\left(\mathbf{I}_{\{\sum_{y\in\mathcal{N}_k(x)} \eta_{k+1}(y,x)\geq 1\}} - \sum_{y\in\mathcal{N}_k(x)} \eta_{k+1}(y,x)\right)\left(1 - \sum_{j\leq k} \hat{\xi}_j(x)\right)}_{\text{error}} \\
&= \left(\sum_{y\in\mathcal{N}_k(x)} \eta_{k+1}(y,x)\right)\left(1 - \sum_{j\leq k} \hat{\xi}_j(x)\right) + E_k(x).
\end{aligned} \tag{3.21}$$

The main goal of this section is to describe the limiting behaviour of the true horizontal process (Theorem 2.3). Recall the SPDE that will be shown to govern the true horizontal process (2.3) is

$$\begin{cases} \frac{\partial \hat{u}_t}{\partial t} = \frac{1}{6}\Delta \hat{u}_t - \hat{u}_t \int_0^t \hat{u}_s ds + \sqrt{\hat{u}_t}\dot{W}(t,\cdot) \\ \hat{u}_0 = f, \end{cases}$$

where f is continuous and compact supported, and $\dot{W}$ is the space-time white noise. Again, the meaning of solution to this SPDE and space-time white noise is introduced in Section 2.3.

Remark. *For the initial condition, as $N \to \infty$, $A(\hat{\xi}_0)$ converges to a continuous function f with compact support. Throughout the proof, we frequently choose f such that $f(x) = 1$ for $x \in [-r, r], r > 0$.*

The proof is given in the next two subsections: we first prove the tightness and that any weak limit satisfies (2.3), and in Subsection 3.2.2 we prove the uniqueness.

3.2.1 Limit behaviour of the rescaled horizontal process

Similarly to (3.2) and by (3.21),

$$\begin{aligned}
\hat{\xi}_{k+1}(x) &= \left(\frac{1}{2N}\sum_{y\sim x}\hat{\xi}_k(y) + \sum_{y\in\mathcal{N}_k(x)}\left(\eta_{k+1}(y,x) - \frac{1}{2N}\right)\right)\left(1 - \sum_{j\le k}\hat{\xi}_j(x)\right) + E_k(x) \\
&= \hat{\xi}_k(x) + \frac{1}{2N}\sum_{y\sim x}(\hat{\xi}_k(y) - \hat{\xi}_k(x)) + \sum_{y\in\mathcal{N}_k(x)}\left(\eta_{k+1}(y,x) - \frac{1}{2N}\right) \\
&\quad - \frac{1}{2N}\sum_{y\sim x}\sum_{j\le k}\hat{\xi}_k(y)\hat{\xi}_j(x) - \sum_{y\in\mathcal{N}_k(x)}\left(\eta_{k+1}(y,x) - \frac{1}{2N}\right)\sum_{j\le k}\hat{\xi}_j(x) + E_k(x) \\
&= \hat{\xi}_k(x) + \frac{1}{2N}\sum_{y\sim x}(\hat{\xi}_k(y) - \hat{\xi}_k(x)) + \sum_{y\in\mathcal{N}_k(x)}\left(\eta_{k+1}(y,x) - \frac{1}{2N}\right) \\
&\quad - \frac{1}{N^{1-\alpha}}A(\hat{\xi}_k(x))\sum_{j\le k}\hat{\xi}_j(x) - \sum_{y\in\mathcal{N}_k(x)}\left(\eta_{k+1}(y,x) - \frac{1}{2N}\right)\sum_{j\le k}\hat{\xi}_j(x) + E_k(x).
\end{aligned}$$

Denote $\hat{\nu}_k^N = \frac{1}{N^{2\alpha}}\sum_x \hat{\xi}_k(x)\delta_x$ as the measure generated by $\hat{\xi}_k$. Choose test function ϕ^N satisfying (3.3) and sum by parts,

$$\begin{aligned}
(\hat{\nu}_k^N, \phi_k^N) - (\hat{\nu}_{k-1}^N, \phi_{k-1}^N) &= (\hat{\nu}_k^N, \phi_k^N - \phi_{k-1}^N) + (\hat{\nu}_{k-1}^N, N^{-2\alpha}\Delta_D\phi_{k-1}^N) + d_k^{(1)}(\phi^N) \\
&\quad - \frac{1}{N^{1-\alpha}}\sum_{j\le k-1}(A(\hat{\xi}_{k-1})\phi_{k-1}^N, \hat{\nu}_j^N) - d_k^{(2)}(\phi) + E_k(\phi^N),
\end{aligned}$$

with the error term

$$E_k(\phi^N) = \frac{1}{N^{2\alpha}}\sum_x E_k(x)\phi_k^N(x),$$

and martingale terms

$$d_k^{(1)}(\phi^N) = \frac{1}{N^{2\alpha}} \sum_x \phi_{k-1}^N(x) \sum_{y \in \mathcal{N}_{k-1}(x)} \left(\eta_k(y,x) - \frac{1}{2N} \right),$$
$$d_k^{(2)}(\phi^N) = \frac{1}{N^{2\alpha}} \sum_x \phi_{k-1}^N(x) \sum_{y \in \mathcal{N}_{k-1}(x)} \left(\eta_k(y,x) - \frac{1}{2N} \right) \sum_{j \le k-1} \hat{\xi}_j(x).$$

Summing k from 1 to n, we can get a semimartingale decomposition

$$\begin{aligned}(\hat{v}_n^N, \phi_n^N) - (A(\hat{\xi}_0), \phi_0) = (\hat{v}_n^N, \phi_n^N - \phi_{n-1}^N) + \sum_{k=1}^{n-1} (\hat{v}_k^N, \phi_k^N - \phi_{k-1}^N + N^{-2\alpha} \Delta_D \phi^N) \\ - \sum_{k=1}^{n-1} \sum_{j \le k} \frac{1}{N^{1-\alpha}} (A(\hat{\xi}_k)\phi_k^N, \hat{v}_j^N) + \hat{M}_n(\phi^N) + \sum_{k=1}^{n} E_k(\phi^N),\end{aligned} \tag{3.22}$$

where the martingale $\hat{M}_n(\phi^N) = M_n^{(1)}(\phi^N) - M_n^{(2)}(\phi^N) = \sum_{k=1}^n \left(d_k^{(1)}(\phi^N) - d_k^{(2)}(\phi^N) \right)$ has square variation

$$\begin{aligned}\langle \hat{M}(\phi^N) \rangle_n &= \sum_{k=1}^n \mathbf{E}_{k-1}(d_k^{(1)}(\phi^N) - d_k^{(2)}(\phi^N))^2 \\ &= \sum_{k=1}^n \frac{1}{2N^{1+4\alpha}} \sum_x \sum_{y \sim x} \hat{\xi}_{k-1}(y) \left(1 - \left(\sum_{j \le k-1} \hat{\xi}_j(x) \right)^2 \right) (\phi_{k-1}^N)^2(x) \left(1 - \frac{1}{2N} \right) \\ &\le \sum_{k=1}^n \frac{\|\phi_{k-1}^N\|_0}{2N^{1+4\alpha}} \sum_x \sum_{y \sim x} \hat{\xi}_{k-1}(y) \left(1 - \sum_{j \le k-1} \hat{\xi}_j(x) \right) \phi_{k-1}^N(x) \\ &= \sum_{k=1}^n \frac{\|\phi_{k-1}^N\|_0}{N^{2\alpha}} (A(\hat{\xi}_{k-1}), \phi) - \frac{\|\phi\|_0}{N^{1+\alpha}} \sum_{k=1}^n \sum_{j \le k-1} (A(\hat{\xi}_{k-1})\phi_{k-1}^N, \hat{v}_j^N),\end{aligned} \tag{3.23}$$

where we use the fact that $\sum_{j \le k} \hat{\xi}_j(x) \in \{0,1\}$ to get the first inequality. We first show that the error term in (3.22) is negligible.

Lemma 3.9. *When $\alpha < 1/3$, for $t \le T$, the cumulative error term over time $\lfloor tN^{2\alpha} \rfloor$*

$$\frac{1}{N^{2\alpha}} \sum_{k=1}^{\lfloor tN^{2\alpha} \rfloor} \sum_x E_k(x) \phi_k^N(x) \to 0 \text{ in } L^2 \text{ as } N \to \infty.$$

The test function $\phi_k^N(x)$ is chosen as the discrete approximation of $\phi(t,x) : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ by taking $\phi^N(k,x) = \phi\left(\frac{k}{N^{2\alpha}}, x\right)$ for $x \in \mathbb{Z}/N^{1+\alpha}$, where ϕ is compact supported and twice differentiable in t and x.

Proof. By Hölder inequality,

$$\begin{aligned}\mathbf{E}\left|\frac{1}{N^{2\alpha}}\sum_{k=1}^{\lfloor tN^{2\alpha}\rfloor}\sum_x E_k(x)\phi_k^N(x)\right|^2 &\le \frac{t}{N^{2\alpha}}\sum_{k=1}^{\lfloor tN^{2\alpha}\rfloor}\mathbf{E}\left(\sum_x E_k(x)\phi_k^N(x)\right)^2\\ &\le \frac{T}{N^{2\alpha}}\sum_{k=1}^{\lfloor tN^{2\alpha}\rfloor}\sum_x \mathbf{E}\left(\mathbf{I}_{\{\sum_{y\in\mathcal{N}_k(x)}\eta_{k+1}(y,x)\ge 1\}}-\sum_{y\in\mathcal{N}_k(x)}\eta_{k+1}(y,x)\right)^2(\phi_k^N)^2(x),\end{aligned}$$

where in the second inequality, we used the facts that $\left|1-\sum_{j\le k}\hat{\xi}_j(x)\right|\le 1$ and, given $\mathscr{F}_k$,

$$\mathbf{I}_{\{\sum_{y\in\mathcal{N}_k(x)}\eta_{k+1}(y,x)\ge 1\}}-\sum_{y\in\mathcal{N}_k(x)}\eta_{k+1}(y,x),\, x\in\mathbb{Z}/N^{1+\alpha}$$

are conditionally independent. Following similar reason as Lemma 3.8,

$$\mathbf{E}\left(\mathbf{I}_{\{\sum_{y\in\mathcal{N}_k(x)}\eta_{k+1}(y,x)\ge 1\}}-\sum_{y\in\mathcal{N}_k(x)}\eta_{k+1}(y,x)\right)^2\le\frac{\mathbf{E}[N_k(x)^2]}{4N^2}.$$

$N_k(x)=\sum_{y\sim x}\hat{\xi}_k(y)$ can be written as $2N^\alpha A\hat{\xi}_k(x)$. Hence

$$\begin{aligned}\mathbf{E}\left|\frac{1}{N^{2\alpha}}\sum_{k=1}^{\lfloor tN^{2\alpha}\rfloor}\sum_x E_k(x)\phi_k^N(x)\right|^2 &\le \frac{T}{N^2}\sum_{k=1}^{\lfloor tN^{2\alpha}\rfloor}\sum_x\mathbf{E}(A\hat{\xi}_k(x))^2(\phi_k^N)^2(x)\\ &\le\frac{C(\lambda,f,T)}{N^{1-3\alpha}}\sum_{k=1}^{\lfloor tN^{2\alpha}\rfloor}\frac{1}{N^{2\alpha}}(\phi_k^N,e_\lambda)^2 \text{ (Lemma 3.4 (c)).}\end{aligned}$$

The result follows by using the properties of test functions in the assumption. □

Choosing $\phi_k^N=\psi_{n-k}$ as in Section 3.1.1, we can obtain

$$A(\hat{\xi}_n)(x)=(\hat{v}_0^N,\psi_n^x)-\sum_{k=1}^n\sum_{j\le k-1}\frac{1}{N^{1-\alpha}}\left(A(\hat{\xi}_{k-1})\psi_{n-k}^x,\hat{v}_j^N\right)+\hat{M}_n(\psi_{n-\cdot}^x)+\sum_{k=1}^n E_k(\psi_{n-k}^x). \quad (3.24)$$

Since $\hat{\xi}_k(x)$ is dominated by $\xi_k(x)$, the estimations in Lemma 3.4 also hold for $\hat{\xi}_k(x)$. As in Section 3.1.3, we will use the estimations in Lemma 3.2 and Lemma 3.4 to get the tightness of $A(\hat{\xi}_{\lfloor tN^{2\alpha}\rfloor})(x)$. We assume that the linear interpolation of $A(\hat{\xi}_0)$ converges to f under in $\mathscr{C}$ as $N\to\infty$ and let the centred approximated density be

$$\hat{A}(\hat{\xi}_k)(x)=A(\hat{\xi}_k)(x)-(\hat{v}_0^N,\psi_k^x).$$

Lemma 3.10. *When* $\alpha=1/5$, *for* $0\le s\le t\le T$, $x,y\in\mathbb{Z}/N^{1+\alpha}$, $|t-s|\le 1$, $|x-y|\le 1$, $\lambda>0$ *and* $p\ge 2$,

$$\mathbf{E}|\hat{A}(\hat{\xi}_{\lfloor tN^{2\alpha}\rfloor})(x)-\hat{A}(\hat{\xi}_{\lfloor sN^{2\alpha}\rfloor})(y)|^p\le C(\lambda,p,f,T)e_{\lambda p}(x)\left(|x-y|^{\frac{p}{4}}+|t-s|^{\frac{p}{4}}+N^{-\frac{\alpha p}{2}}\right). \quad (3.25)$$

Proof. We first deal with the error term and the remaining terms will be shown as in the proof of Lemma 3.5, where we decompose this difference into space and time differences.

The error term is

$$\frac{1}{N^{2\alpha}}\sum_{k=1}^{\lfloor tN^{2\alpha}\rfloor}\sum_{x'}\psi^z_{\lfloor tN^{2\alpha}\rfloor-k}(x')E_k(x'),\ \text{for } z=x \text{ or } y,$$

where

$$E_k(x')=\mathbf{I}_{\{\sum_{y\in\mathcal{N}_k(x')}\eta_{k+1}(y,x)\geq 1\}}-\sum_{y\in\mathcal{N}_k(x')}\eta_{k+1}(y,x).$$

We can decompose $E_k(x')=E_k^{(1)}(x')+E_k^{(2)}(x')$, where

$$E_k^{(1)}(x')=\mathbf{E}[E_k(x')\mid\mathscr{F}_k],$$

satisfying

$$\left|E_k^{(1)}(x)\right|\leq\left|1-\left(1-\frac{1}{2N}\right)^{N_k(x')}-\frac{N_k(x')}{2N}\right|\leq\frac{N_k(x')^2}{4N^2},$$

and

$$E_k^{(2)}(x)=E_k(x')-\mathbf{E}[E_k(x')\mid\mathscr{F}_k].$$

With respect to the first term $E_k^{(1)}$, we have

$$\begin{aligned}
&\mathbf{E}\left|\frac{1}{N^{2\alpha}}\sum_{k=1}^{\lfloor tN^{2\alpha}\rfloor}\sum_{x'}\psi^z_{\lfloor tN^{2\alpha}\rfloor-k}(x')E_k^{(1)}(x')\right|^p\\
&\leq\frac{C(p,T)}{N^{2\alpha}}\sum_{k=1}^{\lfloor tN^{2\alpha}\rfloor}\mathbf{E}\left(\sum_{x'}\psi^z_{\lfloor tN^{2\alpha}\rfloor-k}(x')E_k^{(1)}(x')\right)^p\\
&\leq\frac{C(p,T)}{N^{2\alpha+(2-2\alpha)p}}\sum_{k=1}^{\lfloor tN^{2\alpha}\rfloor}\mathbf{E}\left(\sum_{x'}(A\hat{\xi}_k(x'))^2\psi^z_{\lfloor tN^{2\alpha}\rfloor-k}(x')\right)^p\\
&\leq\frac{C(p,T)}{N^{2\alpha+(2-2\alpha)p}}\sum_{k=1}^{\lfloor tN^{2\alpha}\rfloor}\mathbf{E}\left(\sum_{x}(A\hat{\xi}_k(x'))^{2p}\psi_{\lfloor tN^{2\alpha}\rfloor-k}(x')\right)\cdot\left(\sum_{x'}\psi^z_{\lfloor tN^{2\alpha}\rfloor-k}(x')\right)^{p-1}\\
&\leq\frac{C(p,T)N^{(1+\alpha)(p-1)}}{N^{2\alpha+(2-2\alpha)p}}\sum_{k=1}^{\lfloor tN^{2\alpha}\rfloor}\mathbf{E}\left(\sum_{x'}(A\hat{\xi}_k(x'))^{2p}\psi^z_{\lfloor tN^{2\alpha}\rfloor-k}(x')\right)\ \text{(Lemma 3.2 (a))}\\
&\leq\frac{C(\lambda,p,f,T)N^{(1+\alpha)(p-1)}}{N^{2\alpha+(2-2\alpha)p}}\sum_{k=1}^{\lfloor tN^{2\alpha}\rfloor}\left(\sum_{x'}\psi^z_{\lfloor tN^{2\alpha}\rfloor-k}(x')e_{\lambda p}(x')\right)\ \text{(Lemma 3.4 (c))}\\
&\leq C(\lambda,p,f,T)e_{\lambda p}(z)N^{-(1-3\alpha)p}\ \text{(Lemma 3.2 (b))}.
\end{aligned}$$

Moreover,

$$M_n^{(2)}=\frac{1}{N^{2\alpha}}\sum_{k=1}^{n}\sum_{x'}\psi_{\lfloor tN^{2\alpha}\rfloor-k}(x')E_k^{(2)}(x'),n\leq\lfloor tN^{2\alpha}\rfloor.$$

is a martingale. Hence,

$$\langle M^{(2)}\rangle_{\lfloor tN^{2\alpha}\rfloor} \le \frac{C}{N^{2+2\alpha}} \sum_{k=1}^{\lfloor tN^{2\alpha}\rfloor} \sum_{x'} (A\hat{\xi}_k(x'))^2 (\psi^z_{\lfloor tN^{2\alpha}\rfloor - k}(x'))^2.$$

By BDG inequality, we have

$$\begin{aligned}
&\mathbf{E}\left|\frac{1}{N^{2\alpha}} \sum_{k=1}^{\lfloor tN^{2\alpha}\rfloor} \sum_{x'} \psi^z_{\lfloor tN^{2\alpha}\rfloor-k}(x') E_k^{(2)}(x')\right|^p \\
&\le \frac{C(p,T)}{N^{(1+\alpha)p-2\alpha(p/2-1)}} \sum_{k=1}^{\lfloor tN^{2\alpha}\rfloor} \mathbf{E}\left(\sum_{x'} (A\hat{\xi}_k(x'))^2 (\psi^z_{\lfloor tN^{2\alpha}\rfloor-k}(x'))^2\right)^{\frac{p}{2}} \\
&\le \frac{C(\lambda,p,T)e_{\lambda p}(z)}{N^{p+2\alpha}} \sum_{k=1}^{\lfloor tN^{2\alpha}\rfloor} N^{\frac{\alpha p}{2}} (\lfloor tN^{2\alpha}\rfloor - k)^{-\frac{p}{4}} \mathbf{E}\left(\sum_{x'} (A\hat{\xi}_k(x'))^2 \psi^z_{\lfloor tN^{2\alpha}\rfloor-k}(x') e_{-2\lambda}(x')\right)^{\frac{p}{2}} \quad \text{(Lemma 3.2 (c))} \\
&\le \frac{C(\lambda,p,T)e_{\lambda p}(z)}{N^{p+2\alpha}} \sum_{k=1}^{\lfloor tN^{2\alpha}\rfloor} N^{\frac{\alpha p}{2}} (\lfloor tN^{2\alpha}\rfloor - k)^{-\frac{p}{4}} \mathbf{E}\left(\sum_{x'} (A\hat{\xi}_k(x'))^p e_{-\lambda p}(x') \psi^z_{\lfloor tN^{2\alpha}\rfloor-k}(x')\right)\left(\sum_{x'} \psi^z_{\lfloor tN^{2\alpha}\rfloor-k}(x')\right)^{\frac{p}{2}-1} \\
&\le \frac{C(\lambda,p,f,T)e_{\lambda p}(z)N^{(1+\alpha)\frac{p}{2}}}{N^{p+2\alpha}} \sum_{k=1}^{\lfloor tN^{2\alpha}\rfloor} N^{\frac{\alpha p}{2}} (\lfloor tN^{2\alpha}\rfloor - k)^{-\frac{p}{4}} \quad \text{(Lemma 3.2 (a), (b) and Lemma 3.4 (c))} \\
&\le C(\lambda,p,f,T)e_{\lambda p}(z) N^{-\frac{1-\alpha}{2}p},
\end{aligned}$$

where the third inequality is from the fact that $(\psi^z_k, 1) = 1$ and Lemma 3.4 (c). To get the estimation of space difference, first we need to deal with

$$\begin{aligned}
&\mathbf{E}\left|\sum_{k=1}^{\lfloor tN^{2\alpha}\rfloor} \sum_{j\le k-1} \frac{1}{N^{1-\alpha}} \left(A(\hat{\xi}_{k-1})(\psi^x_{\lfloor tN^{2\alpha}\rfloor-k} - \psi^y_{\lfloor tN^{2\alpha}\rfloor-k}), \hat{v}^N_j\right)\right|^p \\
&\le C(\lambda,p,f,T)\mathrm{e}_{\lambda p}(x)\left(\frac{1}{N^{1-\alpha}} \sum_{k=1}^{\lfloor tN^{2\alpha}\rfloor} \sum_{j\le k-1} (\lfloor tN^{2\alpha}\rfloor - k+1)^{-\frac{1}{2}} N^{\alpha}|x-y|^{\frac{1}{2}} + N^{\frac{\alpha}{2}}(\lfloor tN^{2\alpha}\rfloor - k+1)^{-\frac{3}{4}}\right)^p \\
&\le C(\lambda,p,f,T)\mathrm{e}_{\lambda p}(x)\left(N^{(5\alpha-1)p}|x-y|^{\frac{p}{2}} + N^{(4\alpha-1)p}\right) \\
&\le C(\lambda,p,f,T)\mathrm{e}_{\lambda p}(x)\left(|x-y|^{\frac{p}{2}} + N^{-\alpha p}\right),
\end{aligned}$$

where the last inequality is because of $\alpha = 1/5$. Next, we will use BDG inequality to estimate

$$\mathbf{E}\left|M^{(2)}_{\lfloor tN^{2\alpha}\rfloor}(\psi^x_{\lfloor tN^{2\alpha}\rfloor-\cdot} - \psi^y_{\lfloor tN^{2\alpha}\rfloor\cdot})\right|^p \le \mathbf{E}\langle M^{(2)}(\psi^x_{\lfloor tN^{2\alpha}\rfloor-\cdot} - \psi^y_{\lfloor tN^{2\alpha}\rfloor\cdot})\rangle^{\frac{p}{2}}_{\lfloor tN^{2\alpha}\rfloor}.$$

As the argument in (3.23),

$$
\begin{aligned}
&\langle M^{(2)}(\psi^x_{\lfloor tN^{2\alpha}\rfloor-\cdot}-\psi^y_{\lfloor tN^{2\alpha}\rfloor-\cdot})\rangle_{\lfloor tN^{2\alpha}\rfloor}\\
\le&\frac{1}{N^{1+\alpha}}\sum_{k=1}^{\lfloor tN^{2\alpha}\rfloor}\sum_{j\le k-1}\|\psi^x_{\lfloor tN^{2\alpha}\rfloor-k+1}-\psi^y_{\lfloor tN^{2\alpha}\rfloor-k+1}\|_\lambda(A(\hat{\xi}_{k-1})e_{-\lambda}(\psi^x_{\lfloor tN^{2\alpha}\rfloor-k+1}+\psi^y_{\lfloor tN^{2\alpha}\rfloor-k+1}),\hat{v}^N_j)\\
\le&\sum_{k=1}^{\lfloor tN^{2\alpha}\rfloor}\sum_{j\le k-1}\left(N^{-1}|x-y|^{\frac12}(\lfloor tN^{2\alpha}\rfloor-k+1)^{-\frac12}k+N^{\frac{\alpha}{2}}(\lfloor tN^{2\alpha}\rfloor-k+1)^{-\frac34}k\right)\\
&\qquad\qquad\qquad\qquad\cdot(A(\hat{\xi}_{k-1})e_{-\lambda}(\psi^x_{\lfloor tN^{2\alpha}\rfloor-k+1}+\psi^y_{\lfloor tN^{2\alpha}\rfloor-k+1})),\hat{v}^N_j)
\end{aligned}
$$

Using (b), (c), (d) of Lemma 3.2 and (a), (c) of Lemma 3.4,

$$
\begin{aligned}
(A(\hat{\xi}_{k-1})e_{-\lambda}\psi^x_{\lfloor tN^{2\alpha}\rfloor-k+1}),\hat{v}^N_j)&\le\|A^p(\hat{\xi}_{k-1})\|^{\frac1p}_{-\lambda p}(\psi^x_{\lfloor tN^{2\alpha}\rfloor-k+1},\hat{v}^N_j)\\
&\le\|A^p(\hat{\xi}_{k-1})\|^{\frac1p}_{-\lambda p}\sup_{1\le j\le\lfloor tN^{2\alpha}\rfloor}(\mathrm{e}_{-\lambda},\hat{v}^N_j)\|\psi^x_{\lfloor tN^{2\alpha}\rfloor-k+1}\|_\lambda\\
&\le\|A^p(\hat{\xi}_{k-1})\|^{\frac1p}_{-\lambda p}\sup_{1\le j\le\lfloor tN^{2\alpha}\rfloor}(\mathrm{e}_{-\lambda},\hat{v}^N_j)\mathrm{e}_\lambda(x)N^\alpha(\lfloor tN^{2\alpha}\rfloor-k+1)^{-\frac12}.
\end{aligned}
\tag{3.26}
$$

Therefore, by using the fact that $\alpha=1/5$,

$$
\begin{aligned}
&\mathbf{E}\left|M^{(2)}_{\lfloor tN^{2\alpha}\rfloor}(\psi^x_{\lfloor tN^{2\alpha}\rfloor-\cdot}-\psi^y_{\lfloor tN^{2\alpha}\rfloor\cdot})\right|^p\\
\le&C(\lambda,p,f,T)\mathrm{e}_{\lambda p}(x)\cdot\left(\sum_{k=1}^{\lfloor tN^{2\alpha}\rfloor}N^{\alpha-1}|x-y|^{\frac12}(\lfloor tN^{2\alpha}\rfloor-k+1)^{-1}k+N^{\frac{\alpha}{2}-1}(\lfloor tN^{2\alpha}\rfloor-k+1)^{-\frac54}k\right)^{\frac p2}\\
\le&C(\lambda,p,f,T)\mathrm{e}_{\lambda p}(x)\left(N^{\frac{5\alpha-1}{2}p}|x-y|^{\frac p4}+N^{\frac{2\alpha-1}{2}p}\right)\\
\le&C(\lambda,p,f,T)\mathrm{e}_{\lambda p}(x)\left(|x-y|^{\frac p4}+N^{-\frac{3\alpha}{2}p}\right).
\end{aligned}
$$

Similarly, for the time difference, we first deal with the drift term

$$
\begin{aligned}
&\sum_{k=1}^{\lfloor tN^{2\alpha}\rfloor}\sum_{j\le k-1}\frac{1}{N^{1-\alpha}}\left(A(\hat{\xi}_{k-1})\psi^y_{\lfloor tN^{2\alpha}\rfloor-k+1},\hat{v}^N_j\right)-\sum_{k=1}^{\lfloor sN^{2\alpha}\rfloor}\sum_{j\le k-1}\frac{1}{N^{1-\alpha}}\left(A(\hat{\xi}_{k-1})\psi^y_{\lfloor sN^{2\alpha}\rfloor-k+1},\hat{v}^N_j\right)\\
=&\sum_{k=1}^{\lfloor sN^{2\alpha}\rfloor}\sum_{j\le k-1}\frac{1}{N^{1-\alpha}}\left(A(\hat{\xi}_{k-1})\left(\psi^y_{\lfloor tN^{2\alpha}\rfloor-k+1}-\psi^y_{\lfloor sN^{2\alpha}\rfloor-k+1}\right),\hat{v}^N_j\right)\\
&+\sum_{k=\lfloor sN^{2\alpha}\rfloor+1}^{\lfloor tN^{2\alpha}\rfloor}\sum_{j\le k-1}\frac{1}{N^{1-\alpha}}\left(A(\hat{\xi}_{k-1})\psi^y_{\lfloor tN^{2\alpha}\rfloor-k+1},\hat{v}^N_j\right).
\end{aligned}
$$

By (b), (e) of Lemma 3.2, (c) of Lemma 3.4 and the fact that $\alpha=1/5$, the p-th moment of the first term above can be bounded by

$$
\begin{aligned}
&\mathbf{E}\left|\sum_{k=1}^{\lfloor sN^{2\alpha}\rfloor}\sum_{j\le k-1}\frac{1}{N^{1-\alpha}}\left(A(\hat{\xi}_{k-1})\left(\psi^y_{\lfloor tN^{2\alpha}\rfloor-k+1}-\psi^y_{\lfloor sN^{2\alpha}\rfloor-k+1}\right),\hat{v}^N_j\right)\right|\\
&\le C(\lambda,p,f,T)\mathrm{e}_{\lambda p}(y)\left(\sum_{k=1}^{\lfloor sN^{2\alpha}\rfloor}N^{\frac{5}{2}\alpha-1}|t-s|^{\frac{1}{2}}(\lfloor sN^{2\alpha}\rfloor-k+1)^{-\frac{3}{4}}k+N^{\frac{3}{2}\alpha-1}(\lfloor sN^{2\alpha}\rfloor-k+1)^{-\frac{3}{4}}k\right)^p\\
&\le C(\lambda,p,f,T)\mathrm{e}_{\lambda p}(y)\left(N^{(5\alpha-1)p}|t-s|^{\frac{p}{2}}+N^{(4\alpha-1)p}\right)\\
&\le C(\lambda,p,f,T)\mathrm{e}_{\lambda p}(y)\left(|t-s|^{\frac{p}{2}}+N^{-\alpha p}\right).
\end{aligned}
$$

By (b), (c) of Lemma 3.2, (c) of Lemma 3.4 and the fact that $\alpha=1/5$, the p-th moment of the second term above can be bounded by

$$
\begin{aligned}
&\mathbf{E}\left|\sum_{k=\lfloor sN^{2\alpha}\rfloor+1}^{\lfloor tN^{2\alpha}\rfloor}\sum_{j\le k-1}\frac{1}{N^{1-\alpha}}\left(A(\hat{\xi}_{k-1})\psi^y_{\lfloor tN^{2\alpha}\rfloor-k+1},\hat{v}^N_j\right)\right|^p\\
&\le C(\lambda,p,f,T)\mathrm{e}_{\lambda p}(y)\left(\sum_{k=\lfloor sN^{2\alpha}\rfloor+1}^{\lfloor tN^{2\alpha}\rfloor}N^{2\alpha-1}(\lfloor tN^{2\alpha}\rfloor-k+1)^{-\frac{1}{2}}k\right)^p\\
&\le C(\lambda,p,f,T)\mathrm{e}_{\lambda p}(y)\left(N^{(5\alpha-1)p}|t-s|^{\frac{p}{2}}\right)\\
&\le C(\lambda,p,f,T)\mathrm{e}_{\lambda p}(y)|t-s|^{\frac{p}{2}}.
\end{aligned}
$$

To deal with the part of $M^{(2)}_{\lfloor tN^{2\alpha}\rfloor}(\psi^y_{\lfloor tN^{2\alpha}\rfloor-\cdot})-M^{(2)}_{\lfloor sN^{2\alpha}\rfloor}(\psi^y_{\lfloor sN^{2\alpha}\rfloor-\cdot})$, we can separate it into two parts and use BDG inequality.

The first part is $M^{(2)}_{\lfloor sN^{2\alpha}\rfloor}(\psi^y_{\lfloor tN^{2\alpha}\rfloor-\cdot}-\psi^y_{\lfloor sN^{2\alpha}\rfloor-\cdot})$ with quadratic variation

$$
\begin{aligned}
&\langle M^{(2)}(\psi^y_{\lfloor tN^{2\alpha}\rfloor-\cdot}-\psi^y_{\lfloor sN^{2\alpha}\rfloor-\cdot})\rangle_{\lfloor sN^{2\alpha}\rfloor}\\
&\le\sum_{k=1}^{\lfloor sN^{2\alpha}\rfloor}\sum_{j\le k-1}\frac{\|\psi^y_{\lfloor tN^{2\alpha}\rfloor-k+1}-\psi^y_{\lfloor sN^{2\alpha}\rfloor-k+1}\|_\lambda}{N^{1+\alpha}}\left(A(\hat{\xi}_{k-1})e_{-\lambda}(\psi^y_{\lfloor tN^{2\alpha}\rfloor-k+1}+\psi^y_{\lfloor sN^{2\alpha}\rfloor-k+1}),\hat{v}^N_j\right)\\
&\le\sum_{k=1}^{\lfloor sN^{2\alpha}\rfloor}\left(N^{\frac{\alpha}{2}-1}|t-s|^{\frac{1}{2}}(\lfloor sN^{2\alpha}\rfloor-k+1)^{-\frac{3}{4}}k+N^{-1-\frac{\alpha}{2}}(\lfloor sN^{2\alpha}\rfloor-k+1)^{-\frac{3}{4}}k\right)\\
&\qquad\cdot\left(A(\hat{\xi}_{k-1})e_{-\lambda}(\psi^y_{\lfloor tN^{2\alpha}\rfloor-k+1}+\psi^y_{\lfloor sN^{2\alpha}\rfloor-k+1}),\hat{v}^N_j\right).
\end{aligned}
$$

Inequality (3.26) gives us

$$
\begin{aligned}
&\mathbf{E}\left|M^{(2)}_{\lfloor sN^{2\alpha}\rfloor}(\psi^y_{\lfloor tN^{2\alpha}\rfloor-\cdot}-\psi^y_{\lfloor sN^{2\alpha}\rfloor-\cdot})\right|^p \\
&\le C(\lambda,p,f,T)\mathrm{e}_{\lambda p}(y)\cdot\left(\sum_{k=1}^{\lfloor sN^{2\alpha}\rfloor} N^{\frac{3}{2}\alpha-1}|t-s|^{\frac{1}{2}}(\lfloor sN^{2\alpha}\rfloor-k+1)^{-\frac{5}{4}}k+N^{\frac{\alpha}{2}-1}(\lfloor sN^{2\alpha}\rfloor-k+1)^{-\frac{5}{4}}k\right)^{\frac{p}{2}} \\
&\le C(\lambda,p,f,T)\mathrm{e}_{\lambda p}(y)\left(N^{\frac{3.5\alpha-1}{2}p}|t-s|^{\frac{p}{4}}+N^{\frac{2.5\alpha-1}{2}p}\right) \\
&\le C(\lambda,p,f,T)\mathrm{e}_{\lambda p}(y)\left(N^{-\frac{3\alpha}{4}p}|t-s|^{\frac{p}{4}}+N^{-\frac{5\alpha}{4}p}\right).
\end{aligned}
$$

The second part is $M^{(2)}_{\lfloor tN^{2\alpha}\rfloor}(\psi^y_{\lfloor tN^{2\alpha}\rfloor-\cdot})-M^{(2)}_{\lfloor sN^{2\alpha}\rfloor}(\psi^y_{\lfloor tN^{2\alpha}\rfloor-\cdot})$ with quadratic variation

$$
\begin{aligned}
&\langle M^{(2)}(\psi^y_{\lfloor tN^{2\alpha}\rfloor}-\cdot)\rangle_{\lfloor tN^{2\alpha}\rfloor}-\langle M^{(2)}(\psi^y_{\lfloor tN^{2\alpha}\rfloor}-\cdot)\rangle_{\lfloor sN^{2\alpha}\rfloor} \\
&\le \sum_{k=\lfloor sN^{2\alpha}\rfloor+1}^{\lfloor tN^{2\alpha}\rfloor}\sum_{j\le k-1}\frac{\|\psi^y_{\lfloor tN^{2\alpha}\rfloor-k+1}\|_\lambda}{N^{1+\alpha}}\left(A(\hat{\xi}_{k-1})e_{-\lambda}\psi^y_{\lfloor tN^{2\alpha}\rfloor-k+1},\hat{v}^N_j\right) \\
&\le \sum_{k=\lfloor sN^{2\alpha}\rfloor}^{\lfloor tN^{2\alpha}\rfloor}\sum_{j\le k-1}N^{-1}(\lfloor tN^{2\alpha}\rfloor-k+1)^{-\frac{1}{2}}\left(A(\hat{\xi}_{k-1})e_{-\lambda}\psi^y_{\lfloor tN^{2\alpha}\rfloor-k+1},\hat{v}^N_j\right).
\end{aligned}
$$

Inequality (3.26) again gives us

$$
\begin{aligned}
&\mathbf{E}\left|M^{(2)}_{\lfloor tN^{2\alpha}\rfloor}(\psi^y_{\lfloor tN^{2\alpha}\rfloor-\cdot})-M^{(2)}_{\lfloor sN^{2\alpha}\rfloor}(\psi^y_{\lfloor tN^{2\alpha}\rfloor-\cdot})\right|^p \\
&\le C(\lambda,p,f,T)\mathrm{e}_{\lambda p}(y)\left(\sum_{k=\lfloor sN^{2\alpha}\rfloor+1}^{\lfloor tN^{2\alpha}\rfloor}N^{\alpha-1}(\lfloor tN^{2\alpha}\rfloor-k+1)^{-1}k\right)^{\frac{p}{2}} \\
&\le C(\lambda,p,f,T)\mathrm{e}_{\lambda p}(y)\left(N^{\frac{5\alpha-1}{2}p}|t-s|^{\frac{p}{2}}\right) \\
&\le C(\lambda,p,f,T)\mathrm{e}_{\lambda p}(y)|t-s|^{\frac{p}{2}}.
\end{aligned}
$$

Combining with Lemma 3.5, we get (3.25). □

The tightness of $A(\hat{\xi}_{\lfloor tN^{2\alpha}\rfloor})$ follows from Lemma 3.10, which means that we can find a subsequence with a limit $\hat{u}_t$. Since the true process is dominated by the branching envelope, we easily see that Lemma 3.6 also holds for the true horizontal process. This implies the tightness of $\hat{v}^N_{\lfloor tN^{2\alpha}\rfloor}$ under vague topology. Let $\hat{v}_t$ be a weak limit. By substituting $\phi_k=\phi\in C^2_0(\mathbb{R})$ in the semimartingale decomposition (3.22) and Lemma 3.9, if $\alpha=1/5$, we can see that

$$
\hat{M}^N_{\lfloor tN^{2\alpha}\rfloor}(\phi)=(\hat{v}^N_{\lfloor tN^{2\alpha}\rfloor},\phi)-(A(\hat{\xi}_0),\phi)-\sum_{k=1}^{\lfloor tN^{2\alpha}\rfloor}(\hat{v}^N_k,\Delta_D\phi)-\sum_{k=1}^{\lfloor tN^{2\alpha}\rfloor}\sum_{j\le k-1}\frac{1}{N^{1-\alpha}}(A(\hat{\xi}_{k-1}\phi),\hat{v}^N_j)+O(N^{-2/5})
$$

is a martingale and every term on the right-hand side converges almost surely by Lemma 3.10.

Hence $\hat{M}^N_{\lfloor tN^{2\alpha}\rfloor}(\phi)$ converges to a local martingale

$$\begin{aligned}\hat{m}_t(\phi) &= (\hat{v}_t,\phi)-(\hat{v}_0,\phi)-\frac{1}{6}\int_0^t(\hat{v}_s,\Delta\phi)ds-\int_0^t\left(\hat{v}_s,\int_0^s\hat{u}_r\phi\right)ds\\ &=\int\phi(x)\hat{u}_t(x)dx-\int\phi(x)f(x)dx-\frac{1}{6}\int_0^t\int\Delta\phi(x)\hat{u}_s(x)dxds-\int_0^t\int_0^s\int\phi(x)\hat{u}_s(x)\hat{u}_r(x)dxdrds,\end{aligned} \tag{3.27}$$

which is continuous since every term on the right-hand side is continuous. Moreover, from (3.23),

$$(\hat{M}^N_{\lfloor tN^{2\alpha}\rfloor})^2-\sum_{k=1}^{\lfloor tN^{2\alpha}\rfloor}\frac{1}{N^{2\alpha}}(A(\hat{\xi}_{k-1}),\phi^2)\left(1-\frac{1}{2N}\right)-\sum_{k=1}^{\lfloor tN^{2\alpha}\rfloor}\sum_{j\le k-1}\frac{1}{N^{1+\alpha}}(A(\hat{\xi}_{k-1})\phi^2,\hat{v}^N_j)\left(1-\frac{1}{2N}\right)$$

is a martingale. As $N\to\infty$,

$$\hat{m}_t^2(\phi)-\int_0^t\int\phi^2(x)\hat{u}_s(x)dxds \tag{3.28}$$

is also a continuous local martingale. (3.27) and (3.28) prove that any subsequential weak limit $\hat{v}_t(dx)=\hat{u}_t(x)dx$ solves (2.3).

3.2.2 Girsanov transformation. Proof of the uniqueness in Theorem 2.3

As is discussed in Section 3.1.3, the envelope measure v_t solves the martingale problem: $\forall\phi\in C_0^2(\mathbb{R})$ test function twice differentiable with compact support, the process

$$m_t(\phi)=(v_t,\phi)-(v_0,\phi)-\frac{1}{6}\int_0^t(v_s,\Delta\phi)ds$$

is a continuous local martingale with quadratic variation process

$$\langle m(\phi)\rangle_t=\int_0^t(v_s,\phi^2)ds.$$

From this, we know that

$$e^{-(v_t,\phi)}-e^{-(v_0,\phi)}-\int_0^t e^{-(v_s,\phi)}\left(v_s,-\frac{1}{6}\Delta\phi+\phi^2\right)ds$$

is a continuous local martingale. Using the duality method in Section 4.4 of Ethier and Kurtz (2009), we can choose triplet $(h,0,0)$ on the space $\mathscr{M}_F\times C_0^2$, where $\mathscr{M}_F$ is the collection of finite Borel measures and $h(\cdot,\cdot)$ is defined as

$$h(v,\phi)=e^{-(v,\phi)}.$$

Then

$$\mathbf{E}h(v_t, \phi) = h(v_0, u_t^*),$$

where u_s^* is the solution to the deterministic equation

$$\begin{cases} \frac{\partial u_t^*}{\partial t} = \frac{1}{6}\Delta u_t^* - (u_t^*)^2 \\ u_0^* = \phi. \end{cases} \tag{3.29}$$

$\{u_t^*\}_{t\geq 0}$ is the dual process of the solution to the martingale problem. The existence of solution to (3.29) gives the uniqueness of $\{v_t\}_{t\geq 0}$.

Let $m(ds, dx)$ be the orthogonal martingale measure attached to $u_t(\cdot)$, which means that $m(ds, dx)$ is of intensity

$$v((0, t] \times A) = \int_A \int_0^t u_s(x) ds dx,$$

for any Borel measurable set $A \subset \mathbb{R}$. Then the Radon-Nykodym derivative of the true process with respect to the envelope is

$$\left.\frac{d\mathbf{Q}}{d\mathbf{P}}\right|_t = \exp\left\{-\int\int_0^t \theta(s,x) m(ds,dx) - \frac{1}{2}\int_0^t (u_s, \theta(s,\cdot)^2) ds\right\}, \tag{3.30}$$

where the drift term

$$\theta(s, x) = \int_0^s u_r(x) dr.$$

The uniqueness of $\{\hat{v}_t\}_{t\geq 0}$ follows directly from the uniqueness of $\{v_t\}_{t\geq 0}$. This completes the proof of Theorem 2.3.

4 Existence of Percolation

In the last chapter, we have shown that $\alpha = 1/5$ (in the sense of Theorem 2.3) is a critical exponent for the horizontal process. The envelope process on each horizontal layer follows the law with asymptotic approximate density given by the solution of (2.2). In the anisotropic percolation model, the horizontal movement has an attrition compared to the envelope process. The attrition comes from two parts:

- The envelope process is allowed to have multiple particles at each site. However, in the true mechanism, we only consider if a site is occupied or not, hence the configuration at each site can only take values in 0 or 1. Fortunately, the probability of multiple particles is negligible when $\alpha = 1/5$ (as we showed in Corollary 3.8).
- As was explained in the Introduction, the vertical interaction should be only considered once for any site in the anisotropic percolation. When we consider the horizontal movement, any site that has been visited before cannot be visited again. Under the critical exponent $\alpha = 1/5$, this attrition becomes significant and leads to the part

$$-\hat{u}_t \int_0^t \hat{u}_s ds$$

  in the asymptotic approximate density which makes the true horizontal process non-Markovian.

In this chapter we prove Theorem 2.4, by investigating the occurrence (or not) of percolation when on each layer we have the true model, and the vertical bonds between neighbouring sites are open with probability $p_v = \kappa N^{-2/5}$, all independently.

4.1 The case $\kappa < C_1$

As we have discussed in the Introduction, the occupied sites at each layer follow a true horizontal process with attrition whose asymptotic approximate density follows the SPDE

(2.3). Here we abuse the notation by denoting $\mathscr{C}_x^i$ as the cluster starting from x at layer i in the rescaled space $\mathbb{Z}/N^{6/5} \times \mathbb{Z}$. The main theorem to show in this subsection is as follows.

Theorem 4.1. *For the true horizontal process with attrition, there exists a constant L such that the cumulated number of occupied sites (or the cluster size) starting from zero satisfies*

$$\mathbf{E}[|\mathscr{C}_0^0|] \le LN^{2/5}.$$

Before proving the main theorem, let us show how it implies that there is no percolation when $\kappa < C_1$ for C_1 small enough.

Corollary 4.1. *Let $p_v = \kappa N^{-2/5}$ denote the probability of a vertical edge being open. There exists C_1 such that for $\kappa < C_1$, there is no percolation in the anisotropic percolation system for all N large.*

Proof of Corollary 4.1. Recall that the horizontal edges, i.e. edges between (x,i) and (y,i) for some i and $x \sim y$, are open with probability $1/(2N)$, while the vertical ones between (x,i) and (x,j) for some x and $|j-i| = 1$ are open with probability p_v, all independently. We say that there is a path from (x,i) to (y,k) denoted by $(x,i) \to (y,k)$ if there is n and $x_j, i_j, 1 \le j \le n$ so that $(x_1,i_1) = (x,i)$, $(x_n,i_n) = (y,k)$ and $\forall 1 \le j \le n-1$, the edge between (x_j,i_j) and (x_{j+1},i_{j+1}) is open.

We want to explore all sites that are connected to $(0,0)$, i.e. that can be reached by an open path from $(0,0)$. Once an open path reaches layer i, it can continue through vertical neighbours at layers $i \pm 1$, moving upward or downward; we can count the number of connected sites with a certain number of vertical movements from layer 0 rather than its layer number.

After n movements which contain m vertical movements (upward or downward), there is a collection of open paths from the origin $(x_0,i_0) = (0,0) \to (x_n,i_n)$. Let $I_v \subset 1,2,\cdots,n$ be the set of vertical movements such that $|I_v| = m$ and $\forall k \in I_v, |i_k - i_{k-1}| = 1, x_k = x_{k-1}$. For $k \in \{1,2,\cdots,n\}\backslash I_v$ i.e. the horizontal movement indices, $i_k = i_{k-1}, x_k \sim x_{k-1}$. Denote $\mathscr{S}_m$ as the collection of points which are the ends of these paths from $(0,0)$ to (x_n,i_n) (with the last edge being vertical edge) from the origin after m vertical movements (but can last for n total movements given $n \ge m$).

These sites $(x_0,i_0),\cdots,(x_n,i_n)$ are possibly to be distributed on different layers. In the development of $\{\mathscr{S}_m\}_{m\ge 0}$, we consider the horizontal movements and vertical movements separately at each time. More precisely (ref. Figure 4.1), we start with $(0,0)$, and following the law $\mathscr{C}_0^0$ we produce connected sites at layer 0. In the first vertical movement, these sites at layer 0 can connect to sites at layer ± 1. Before the second vertical movement, these connected sites at layers ± 1 will produce an horizontal cluster following the law of $\mathscr{C}_0^0$ at its layer, which will then connect to sites at layers ± 2 and 0. $\mathscr{S}_m$ can be constructed inductively by considering the total number of horizontal connected sites and then their vertical movements.

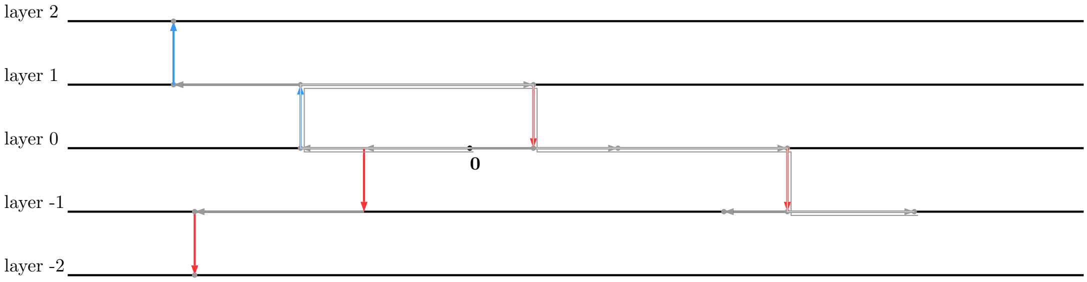

Figure 4.1 – Movement of $\mathscr{S}_m$

Due to attrition, in the horizontal connection we only consider a site to be occupied or not, rather than the number of particles at each site, the cardinality $\{|\mathscr{S}_m|\}_{m\geq 0}$ is stochastically dominated by a branching process $\{Z_m\}_{m\geq 0}$ following the law

$$Z_{m+1} = \sum_{i=1}^{Z_m} Y_{m,i}, \text{ where } Y_{m,i} \overset{\text{i.i.d.}}{\sim} \text{Binomial}(2\mathcal{N}_{m,i}, p_v) \text{ for } 1 \leq i \leq Z_m,$$

$$\mathcal{N}_{m,i} \text{ is independent of } Z_1, \cdots, Z_m \text{ for each } i, m \text{ and } \mathcal{N}_{m,i} \overset{\text{i.i.d.}}{\sim} |\mathscr{C}_0^0|.$$

Theorem 4.1 gives the upper bound of $\mathbf{E}[|\mathscr{C}|_0^0] = \mathbf{E}[\mathcal{N}_{m,i}]$ for each i, m and $p_v = \kappa N^{-2/5}$. When κ is small enough to make $2\kappa L < 1$, $\{Z_m\}_{m\geq 0}$ is a sub-critical branching process which will die out (ref. Theorem A.5.1 of Athreya and Ney (2004)). Once Z_m dies out, there is no percolation. Therefore, there exists positive constant $C_1 = (2L)^{-1}$ such that for $\kappa < C_1$, there is no percolations in this layered system. □

We now move to the proof of Theorem 4.1. The idea is to find the stopping time when the integrated mass can surpass the level $O(N^{2/5})$ noted as T'' ($\tilde{T}''_{\cdot}$ for the envelope process and $\hat{T}''_{\cdot}$ for the true horizontal process) or the total mass can reach level $O(N^{1/5})$ noted as T' ($\tilde{T}'_{\cdot}$ and $\hat{T}'_{\cdot}$ respectively). It is difficult for the envelope to reach $T' \wedge T''$ when the integrated mass is $O(N^{2/5})$ and so does the true horizontal process. Lemma 4.1 is to find the bound of $\mathbf{P}(T' \wedge T'' < \infty)$ for the envelope process, which is easier to do by similar analysis as in Section 3.1.4.

Conditioned on the above stopping time for the envelope process, the estimation of $\mathbf{P}(T' \wedge T'' < \infty)$ for the true horizontal process help us to get the probability of the minor event afterward.

For this we will need two inequalities (Lemma 4.1 and Proposition 4.1 below) which concern the following hitting times for the branching envelope and for the true horizontal process:

$$\tilde{T}_k = \inf\left\{ n : \sum_x \xi_n(x) \geq 2^k \text{ or } \sum_{i=1}^{n} \sum_x \xi_i(x) \geq 2^{2k} \right\},$$

which is just the discrete version of the hitting time T_k in Section 3.1.4, and

$$\hat{T}_k = \inf\left\{ n : \sum_x \hat{\xi}_n(x) \geq 2^k \text{ or } \sum_{i=1}^n \sum_x \hat{\xi}_i(x) \geq 2^{2k} \right\} = \hat{T}'_k \wedge \hat{T}''_k,$$

where

$$\hat{T}'_k = \inf\{n : \sum_x \hat{\xi}_n(x) \geq 2^k\}, \hat{T}''_k = \inf\left\{ n : \sum_{i=0}^n \sum_x \hat{\xi}_i(x) \geq 2^{2k} \right\}.$$

Lemma 4.1. *Suppose $\xi_0(x) = 1$ for $x = 0$ and $\xi_0(x) = 0$ otherwise, then we have*

$$\mathbf{P}(\tilde{T}_k < \infty) < C2^{-k}.$$

Proposition 4.1. *Let integer k_0 be defined by $2^{k_0} \leq N^{2/5} < 2^{k_0+1}$. There exists M_1 large such that for any $k = k_0 + \log_2 M_1 + r, r \geq 0$, we have*

$$\mathbf{P}(\hat{T}_k < \infty \mid \hat{T}_{k_0+\log_2 M_1} < \infty) \leq \frac{C}{8^r}.$$

Postponing the proofs of these estimates, we first see how they allow us to conclude the proof of Theorem 4.1.

Proof of Theorem 4.1. The proof is given in the following steps. As we can see in the proof of Theorem 2.3, the attrition part is negligible when $\alpha < 1/5$ becomes significant when $\alpha = 1/5$. Because of attractiveness with respect to the whole population, we only need to consider the attrition once the total mass is of order $O(N^{2/5})$. So we consider a process that dominates the horizontal process, which follows the pure branching random walk before the total mass reaches $M_1 N^{2/5}$ for some M_1 large and includes the attrition part after that. We are first interested in the crossing time of $\sum_x \xi_n(x)$ over level $M_1 N^{2/5}$.

The dominating process that we consider in this subsection follows $\{\xi_k(x)\}$ before $\tilde{T}_{k_0+\log_2 M_1}$ and follows $\{\hat{\xi}_k(x)\}$ after $\tilde{T}_{k_0+\log_2 M_1}$. The reason of separating the time is as follows. The size of cluster containing the origin satisfies

$$\begin{aligned}
\mathbf{E}[|\mathscr{C}_0^0|] &\leq \sum_{k=0}^{\infty} 2^{2(k+1)} \mathbf{P}(\hat{T}_k < \infty) \\
&\leq \sum_{k=0}^{k_0+\log_2 M_1} 2^{2(k+1)} \mathbf{P}(\tilde{T}_k < \infty) + \sum_{k \geq k_0+\log_2 M_1} 2^{2(k+1)} \mathbf{P}(\hat{T}_{k+1} < \infty) \\
&\leq \sum_{k=0}^{k_0+\log_2 M_1} 2^{2(k+1)} C2^{-k} + \sum_{k \geq k_0+\log_2 M_1} 2^{2(k+1)} \mathbf{P}(\hat{T}_{k+1} < \infty) \\
&\leq 8CM_1 N^{2/5} + \sum_{k \geq k_0+\log_2 M_1} 2^{2(k+1)} \mathbf{P}(\hat{T}_{k+1} < \infty).
\end{aligned}$$

The third inequality is by Lemma 4.1 and the fourth inequality is due to the fact that $2^{k_0} \le N^{2/5}$. The last work is to bound the second term in the last inequality. By Proposition 4.1, the size of cluster containing zero can be bounded by

$$\begin{aligned}\mathbf{E}[|\mathscr{C}_0^0|] &\le 8CM_1N^{2/5} + \sum_{k\ge k_0+\log_2 M_1} 2^{2k+2}C2^{-k_0-\log_2 M_1}8^{-(k-k_0-\log_2 M_1)} \\ &\le 8CM_1N^{2/5} + 8CM_1N^{2/5}.\end{aligned}$$

This finishes the proof of Theorem 4.1. □

In the following part of this subsection, we will show Lemma 4.1 and Proposition 4.1.

Proof of Lemma 4.1. The proof is similar as in Lemma 3.7. It is followed by replacing the corresponding part in the proof of Lemma 3.7 that

$$\mathbf{E}\left((X_t - X_s)^2 \mid X_s\right) = \mathbf{E}\left(\int_s^t X_u du \middle| X_s\right)$$

into the fact that

$$\mathbf{E}\left(\left(\sum_x \xi_n(x) - \sum_x \xi_m(x)\right)\middle|\mathscr{F}_m\right) = \left(1 - \frac{1}{2N}\right)\mathbf{E}\left(\sum_{j=m}^{n-1}\sum_x \xi_j(x)\middle|\mathscr{F}_m\right).$$

Then we can use the similar martingale technique as the proof of Lemma 3.7 and the fact that

$$\sum_x \xi_{n+1}(x) = \sum_x \xi_n(x) + \sum_x \sum_{y\sim x}\sum_{w=1}^{\xi_n(y)}\left(\eta_{n+1}^w(y,x) - \frac{1}{2N}\right) \tag{4.1}$$

is a martingale. Denote the discrete mass as

$$\tilde{X}_n = \sum_x \xi_n(x).$$

The desired probability can be decomposed as

$$\begin{aligned}\mathbf{P}(\tilde{T}_k < \infty) &= \mathbf{P}(\tilde{T}_k' < \infty, \tilde{T}_k' < \tilde{T}_k'') + \mathbf{P}(\tilde{T}_k'' < \infty, \tilde{T}_k'' < \tilde{T}_k') \\ &\le \mathbf{P}(\tilde{T}_k' < \infty) + \mathbf{P}(\tilde{T}_k'' < T_k'),\end{aligned}$$

where as written above

$$\tilde{T}_k' = \inf\{n : \sum_x \xi_n(x) \ge 2^k\},\ \tilde{T}_k'' = \inf\left\{n : \sum_{i=0}^n \sum_x \xi_i(x) \ge 2^{2k}\right\}.$$

Denote $\tilde{H}_0 = \inf\{n : \tilde{X}_n = 0\}$ as the first hitting time of zero, then $\mathbf{P}(\tilde{T}_k' < \infty) = \mathbf{P}(\tilde{T}_k' < \tilde{H}_0)$, and this is simply

$$\mathbf{P}(\tilde{T}'_k < \infty) \le \frac{\tilde{X}_0}{2^k} = \frac{1}{2^k}.$$

The event $\{\tilde{T}''_k < \tilde{T}'_k\} \subset \bigcup_{j=1}^{k-1} A_j$, where

$$A_j = \left\{ \tilde{T}'_j < \infty, \sum_{i=\tilde{T}'_j+1}^{\tilde{T}'_{j+1}} \tilde{X}_i \ge \frac{6 \cdot 2^{2k}}{\pi(k-j)^2} \right\}.$$

For $m < n$, we have

$$\begin{aligned}
&\mathbf{E}\left[(\tilde{X}_n - \tilde{X}_m)^2 \mid \tilde{X}_m\right] \\
&= \mathbf{E}\left[\left(\sum_{i=m}^{n-1} (\tilde{X}_{i+1} - \tilde{X}_i)\right)^2 \mid \tilde{X}_m\right] \\
&= \mathbf{E}\left[\sum_{i=m}^{n-1} (\tilde{X}_{i+1} - \tilde{X}_i)^2 \mid \tilde{X}_m\right] \\
&= \left(1 - \frac{1}{2N}\right) \mathbf{E}\left[\sum_{i=m}^{n-1} \tilde{X}_i \mid \tilde{X}_m\right],
\end{aligned}$$

where the second equality is by the Markov property of $\{\tilde{X}_k\}_{k\ge0}$. Letting $n = \tilde{T}'_{j+1}$ and $m = \tilde{T}'_j$ gives that

$$\mathbf{P}(A_j) \le \frac{(2^{j+2} - 2^j)^2 \cdot \pi(k-j)^2}{2^{2k}},$$

by the Markov property of $\{\tilde{X}_k\}_{k\ge0}$ conditioned on $\tilde{T}'_j$. Therefore,

$$\begin{aligned}
\mathbf{P}(\tilde{T}_k < \infty) &\le \frac{1}{2^k} + \sum_{j=1}^{k-1} \mathbf{P}(A_j) \\
&\le \frac{C}{2^k}.
\end{aligned}$$

□

The above proof immediately yields

Corollary 4.2. *Given a stopping time T with respect to the natural filtration of the $\{\xi_n(x), n \ge 0, x \in \mathbb{Z}/N^{6/5}\}$, the stopping time $T(k) = \inf\{n \ge T : \sum_x \xi_n(x) \ge 2^k$ or $\sum_{m=0}^n \sum_x \xi_m(x) \ge 2^{2k}\}$ satisfies*

$$\mathbf{P}(T(k) < \infty | \mathscr{F}_T) \le C2^{-r}$$

on the set

$$\left\{ \tilde{X}_T \le 2^{k-r}, \sum_{m=0}^{T} \tilde{X}_m \le 2^{2k}/2 \right\}$$

for universal finite C (uniform in N) and integer r.

To show Proposition 4.1, we need two properties of the branching processes: on the large deviations and the next one is on the population size of the critical branching process.

Lemma 4.2. *For a sequence of random variables $Y_i \overset{i.i.d.}{\sim}$ Binomial$(2N, 1/(2N))$, $i = 1, \dots, n$, we have for $a > \frac{1}{2}$,*

$$\mathbf{P}\left(\sum_{i=1}^{n}(Y_i - 1) \geq n^a\right) \leq e^{-cn^{a\wedge(2a-1)}},$$

for universal constant $c > 0$.

Proof. The proof follows from large deviation technique.

$$\begin{aligned}\mathbf{P}\left(\sum_{i=1}^{n}(Y_i - 1) \geq n^a\right) &= \mathbf{P}\left(e^{t\sum_{i=1}^{n} Y_i} \geq e^{(n+n^a)t}\right)\\ &\leq \exp\left(-(n+n^a)t + 2Nn\log\left(1+\frac{e^t-1}{2N}\right)\right), \forall t \in \mathbb{R},\end{aligned}$$

by the Markov inequality. The right hand side reaches the minimum if t satisfies

$$\frac{ne^t}{1+\frac{e^t-1}{2N}} = n + n^a.$$

From this, $t = \log(1+n^{a-1}) - \log(1-\frac{n^{a-1}}{2N-1})$. If $\frac{1}{2} < a < 1$, $t \approx n^{a-1}$ and hence

$$-(n+n^a)t + 2Nn\log\left(1+\frac{e^t-1}{2N}\right) \leq -cn^{2a-1}.$$

But for $a \geq 1$, $e^t \approx n^{a-1}$ and

$$-(n+n^a)t + 2Nn\log\left(1+\frac{e^t-1}{2N}\right) \leq -cn^a,$$

for universal constant $c > 0$.

□

Lemma 4.3. *Denote the critical binomial branching process as $\{Z_n^N\}_{n\geq 0}$ with $Z_0^N = 1$ and*

$$Z_{n+1}^N = \sum_{i=1}^{Z_n^N} Y_i^{n+1},$$

where $(Y_i^{n+1})_{n,i}$ is an i.i.d. sequence with distribution Binomial$(2N, 1/(2N))$.

(i) Given any $T > 1$,

$$\mathbf{P}\left(\frac{2Z^N_{\lfloor tN^{2/5}\rfloor}}{tN^{2/5}} > x \mid Z^N_{\lfloor tN^{2/5}\rfloor} > 0\right) \to e^{-x}$$

as $N \to \infty$ uniformly in $t \in [1/T, T]$.

(ii) $(tN^{2/5}) \cdot \mathbf{P}(Z^N_{\lfloor tN^{2/5}\rfloor} > 0) \to 2$ *as $N \to \infty$ uniformly in $t \in [1/T, T]$.*

Proof. We follow the argument in I.10 of Harris (2002). The moment generating function of Z^N_n, $f^N_n(t) = \mathbf{E}[t^{Z^N_n}]$ is given by

$$f^N_{n+1}(t) = f^N(f^N_n(t)), f^N(t) = f^N_1(t) = \left(1 + \frac{t-1}{2N}\right)^{2N}$$

with $\sup_N (f^N)'''(1) < \infty$. $f^N(e^{i\theta})$ is the characteristic function of Z^N_1. Suppose $\theta_0 \in (0,\pi)$, then for any $s \in \mathbb{C}$ such that $|s| < 1$ or $|s| = 1$ but $s \neq 1$ and $-\theta_0 \le \arg s \le \theta_0$, $|f^N(s)| < 1$. For such $s \in \mathbb{C}$ and any $n \ge 1$, $|f^N_n(s)| < 1$. Moreover,

$$1 - f^N(s) = 1 - s - \frac{(f^N)''(1)}{2}(1-s)^2 + \frac{(f^N)'''(1)}{6}(1-s)^3 - e(s), \tag{4.2}$$

where $e(s) = o(1-s)^3$ since $(f^N)'''(1)$ is bounded for any N. Denote $a = \frac{(f^N)''(1)}{2} = \frac{1}{2}$ and $b = \frac{(f^N)'''(1)}{6}$. From (4.2), since $\sup_N (f^N)'''(1) < \infty$,

$$\begin{aligned}
\frac{1}{1-f^N_j(s)} &= \frac{1}{1-f^N(f^N_{j-1}(s))} \\
&= \frac{1}{1-f^N_{j-1}(s) - a(1-f^N_{j-1}(s))^2 + b(s-f^N_{j-1}(s))^3 - e(f^N_{j-1}(s))} \\
&= \frac{1}{1-f^N_{j-1}(s)} + a + (a^2-b)(1-f^N_{j-1}(s)) + \frac{e(f^N_{j-1}(s))}{(1-f^N_{j-1}(s))^2} + e'(f^N_{j-1}(s)),
\end{aligned}$$

where $e'(s) = O(1-s)^2$. Summing j from 1 to n gives

$$\frac{1}{1-f^N_n(s)} = \frac{1}{1-s} + na + (a^2-b)\sum_{j=0}^{n-1}(1-f^N_j(s)) + \sum_{j=0}^{n-1}\frac{e(f^N_j(s))}{(1-f^N_j(s))^2} + e'(f^N_j(s)).$$

We can observe that $1 - f^N_n(s) = O(1/n)$ uniformly in s. Hence

$$\frac{1}{1-f^N_n(s)} = \frac{1}{1-s} + \frac{n(f^N)''(s)}{2} + O(\log n). \tag{4.3}$$

The s is chosen from $S = \{s \in \mathbb{C} : |s| < 1 \text{ or } |s| = 1 \text{ but } s \neq 1, -\theta_0 \le \arg s \le \theta_0\}$ and (4.3) holds for any N. Letting $s = 0$ gives the conclusion (ii) since

$$(f^N)''(1) = \frac{2N(2N-1)}{(2N)^2} \to 1 \text{ as } N \to \infty.$$

The characteristic function of $2Z_n^N/(n(f^N)''(1))$ given $Z_n^N > 0$ is

$$\begin{aligned}\mathbf{E}\left[e^{it\cdot\frac{2Z_n^N}{n(f^N)''(1)}} \mid Z_n^N > 0\right] &= \frac{\mathbf{E}\left[e^{it\cdot\frac{2Z_n^N}{n(f^N)''(1)}}\right]}{\mathbf{P}(Z_n^N > 0)} \\ &= \frac{f_n^N\left(e^{\frac{2it}{n(f^N)''(1)}}\right) - 1}{1 - f_n^N(0)} + 1.\end{aligned}$$

When n is large enough, $e^{\frac{2it}{n(f^N)''(1)}} \in S$. By (4.3),

$$\mathbf{E}\left[e^{it\cdot\frac{2Z_n^N}{n(f^N)''(1)}} \mid Z_n^N > 0\right] = 1 - \frac{\frac{n(f^N)''(1)}{2} + O(\log n)}{\frac{1}{1-e^{\frac{2it}{n(f^N)''(1)}}} + \frac{n(f^N)''(1)}{2} + O(\log n)}.$$

As $n \to \infty$,

$$\mathbf{E}\left[e^{it\cdot\frac{2Z_n^N}{n(f^N)''(1)}} \mid Z_n^N > 0\right] \to \frac{1}{1-it},$$

which is the characteristic function of Exp(1). This concludes (i).

□

With the help of these two properties, we can prove Proposition 4.1 used in the proof of Theorem 4.1.

We first show how Proposition 4.1 follows from the following.

Proposition 4.2. *Suppose $\hat{\xi}_0(x) = 1$ for $x = 0$ and $\hat{\xi}_0(x) = 0$ otherwise, and $\delta > 0$, then there exists M_3 sufficiently large that*

$$\mathbf{E}\left(\hat{X}_{M_3 N^{2/5}}\right) < \delta,$$

for all N, where $\hat{X}_n = \sum_x \hat{\xi}_n(x)$ is the discrete mass of the true horizontal process.

Remark. *To summarize the notations about the total mass, $\{X_t\}_{t\geq 0}$ denotes the total mass of the envelope process in continuous time given the initial condition f to be continuous, $f(x) = 1$ for $x \in [-1,1]$ and compact supported. $\{\tilde{X}_n\}_{n\geq 0}$ and $\{\hat{X}_n\}_{n\geq 0}$ represent the total mass of the envelope process and the true horizontal process in discrete time given the initial condition to be $\mathbf{I}_{\{0\}}$.*

This Proposition will be proven later.

Proof of Proposition 4.1. We note that it is sufficient to show that (with M_1 chosen sufficiently large)

$$\mathbf{P}(\hat{T}_{k+1} < \infty | \hat{T}_k < \infty) < \frac{1}{8}$$

for $k \geq k_0$. Lemma 4.2 shows that outside probability $e^{-2^{k/3}}$, we have that $\hat{X}_{\hat{T}_k} < 2^k + 2^{2k/3}$ and $\sum_{m=0}^{\hat{T}_k} \hat{X}_m < 2^{2k} + 2^{2k/3}$.

Let B_1 be the event that one of these two bounds fails (so $\mathbf{P}(B_1) \leq e^{-2^{k/3}} < 1/32$ supposing that M_1 is sufficiently large). We fix $\delta > 0$ (to be specified when needed) and let M_3 correspond to δ in Proposition 4.2. Let B_2 be the event that $\sup_{\hat{T}_k \leq i \leq \hat{T}_k + M_3 N^{2/5}} \hat{X}_i \geq M_3 2^k$. So by the martingale properties of the envelope process $\mathbf{P}(B_2 \backslash B_1) < \frac{2}{M_3} < 1/32$ supposing, as we may have that M_3 is sufficiently large. We note that on the complement of $B_1 \cup B_2$, $\sum_{i=0}^{\hat{T}_k + M_3 N^{2/5}} \hat{X}_i \leq 2^{2k} + 2^{2k/3} + M_3 N^{2/5} M_3 2^k < 2^{2(k+1)}/2$ if M_1 is chosen so that $M_1 > 8M_3^2$ and N is sufficiently large. Next we have by Proposition 4.2 and obvious monotonicity

$$\mathbf{E}(\hat{X}_{\hat{T}_k + M_3 N^{2/5}} \mathbf{I}_{(B_1 \cup B_2)^c}) < (2^k + 2^{2k/3})\delta$$

and so by the Markov's inequality, the event $B_3 = \{\hat{X}_{\hat{T}_k + M_3 N^{2/5}} \mathbf{I}_{(B_1 \cup B_2)^c} \geq 2\sqrt{\delta} 2^k\}$ has probability bounded by $\sqrt{\delta} < 1/32$ if δ was fixed sufficiently small. Finally we can apply Corollary 4.2 to see that $B_4 = \{\hat{T}_{k+1} < \infty\} \backslash (B_1 \cup B_2 \cup B_3)$ has probability $\mathbf{P}(B_4) < 1/32$ (again supposing δ to have been fixed sufficiently small). $\square$

In the proof above, we have that at time $\hat{T}_k$, there are around 2^k particles. For the process starting from each single one, we want to show that after $M_3 N^{2/5}$ steps, some killing property can help to reduce the quantity to be δ small. It remains to prove Proposition 4.2.

Proof of Proposition 4.2. We suppose that $\hat{X}$ is coupled with a envelope process Z so that $\hat{X}_n$ is dominated by Z_n for each n. We suppose that $\delta > 0$ is fixed. We wish to partition $Z_{N^{2/5}} \neq 0$ into sets B_1, B_2, and B_3 to show with M_3 fixed large that for each $k = 1, 2, 3$, $\mathbf{E}(\hat{X}_{M_3 N^{2/5}} \mathbf{I}_{B_k}) < \delta/4$. Let

$$\sigma = \inf\{n \geq N^{2/5}/2 : \hat{X}_n \leq \epsilon N^{2/5}\}$$

where ϵ is a small positive constant which remains to be fully specified. Let $B_1 = \{\sigma \leq N^{2/5}\}$. Then by the Strong Markov property applied at σ and the martingale property for the envelope of the process

$$\begin{aligned}
\mathbf{E}(\hat{X}_{M_3 N^{2/5}} \mathbf{I}_{B_1}) &\leq \mathbf{P}(\hat{X}_{N^{2/5}} \mathbf{I}_{B_1} \neq 0) \mathbf{E}\left[\mathbf{E}(\hat{X}_{M_3 N^{2/5}} \mathbf{I}_{B_1} | \mathscr{F}_\sigma)\right] \\
&\leq \mathbf{P}(\hat{X}_{N^{2/5}} \mathbf{I}_{B_1} \neq 0) \epsilon N^{2/5} \\
&\leq \delta/4,
\end{aligned}$$

if ϵ was chosen sufficiently small by Lemma 4.3.

We next consider $B_2 = \{R_{N^{2/5}} < 1/\epsilon\}$ where $R_{N^{2/5}}$ is the maximal absolute displacement from 0 of the critical branching random walk Z by time $N^{2/5}$, i.e.

$$R_n = \max_{m \le n} \left\{ x \in \mathbb{Z}/N^{6/5} : \xi_m(x) \neq 0 \right\}. \tag{4.4}$$

Theorem 1.1 of Kesten (1995) showed that

$$\mathbf{P}(R_{N^{2/5}} \ge z \mid Z_{N^{2/5}/2} > 0) \le C_a z^{-a}, \text{ for any } a > 0, \tag{4.5}$$

(It is easy to check C_a is uniform over N). Thus since $Z_{N^{2/5}}/N^{2/5}$ conditioned on being nonzero is uniformly integrable (again using Lemma 4.3), we have that (again supposing that ϵ is fixed small)

$$\mathbf{E}(\hat{X}_{M_3 N^{2/5}} \mathbf{I}_{B_2}) \le \mathbf{E}(Z_{M_3 N^{2/5}} \mathbf{I}_{B_2}) \ < \ \delta/4.$$

Finally we treat the complement B_3. On the complement of $B_1 \cup B_2$ we can find an interval with length 2ϵ contained in $(-1/\epsilon, 1/\epsilon)$, which we denote as $(x-\epsilon, x+\epsilon)$ such that

$$\sum_{y \in (x-\epsilon, x+\epsilon)} \sum_{j=N^{2/5}/2}^{N^{2/5}} \hat{\xi}_j(y) \ge \epsilon^3 N^{4/5}. \tag{4.6}$$

$(B_1 \cup B_2)^c$ make sure that we have sufficient number of visited sites in $(x-\epsilon, x+\epsilon)$. Denote

$$V = \left\{ y \in (x-\epsilon, x+\epsilon) : \exists\ N^{2/5}/2 \le j \le N^{2/5}, \hat{\xi}_j(y) = 1 \right\}$$

as the set of visited sites between $N^{2/5}/2$ and $N^{2/5}$. Without loss of generality, we assume that $x = 0$. For $y \in (-\epsilon, \epsilon)$, consider a random walk $\{S_i\}_{i \ge 0}$ starting from $S_0 = y$ and each step it moves to one of its neighbourhood z ($|y - z| \le N^{-1/5}$) with probability $1/(2N)$. Observe that $\mathbf{E}|S_i - S_{i-1}|^2 \approx \frac{1}{3N^{2/5}}$. Let $\tau_{2\epsilon} = \inf\{i > 0 : S_i \in [-2\epsilon, 2\epsilon]^c\}$.

$$3\epsilon^2 N^{2/5} \le \mathbf{E}[\tau_{2\epsilon}] \le 36\epsilon^2 N^{2/5}.$$

Since there are $2\epsilon N^{6/5}$ sites in $(-\epsilon, \epsilon)$. Hence for any $z \in (-\epsilon, \epsilon)$, there are positive constants $c_1 < c_2$ such that

$$c_1 \epsilon N^{-4/5} \le \mathbf{P}^y(S_i \text{ hits } z \text{ before } \tau_{2\epsilon}) \le c_2 \epsilon N^{-4/5}. \tag{4.7}$$

Let

$$N_V = \sum_{i=1}^{\tau_{2\epsilon}} \mathbf{I}_{S_i \in V}.$$

Then by (4.6) and (4.7),

$$\begin{aligned} \mathbf{E}[N_V] &= \sum_{z \in V} \mathbf{P}^y(S_i \text{ hits } z \text{ before } \tau_{2\epsilon}) \\ &\ge \epsilon^3 N^{4/5} \cdot c_1 \epsilon N^{-4/5} \\ &= c_1 \epsilon^4. \end{aligned}$$

Moreover,

$$\begin{aligned}\mathbf{E}[N_V^2] &= \sum_{z,z'\in V} \mathbf{P}^y(S_i \text{ hits } z,z' \text{ before } \tau_{2\epsilon})\\ &\le \mathbf{E}[N_V] + \frac{c_2^2}{c_1^2}(\mathbf{E}[N_V])^2.\end{aligned}$$

By Lemma 4.4,

$$\begin{aligned}\mathbf{P}(N_V > 0) &\ge \frac{(\mathbf{E}[N_V])^2}{\mathbf{E}[N_V^2]}\\ &\ge \frac{1}{c_2^2 c_1^{-2} + (\mathbf{E}[N_V])^{-1}} \\ &\ge \frac{1}{c_2^2 c_1^{-2} + (c_1\epsilon^4)^{-1}}.\end{aligned} \tag{4.8}$$

For any $y \in (-\epsilon,\epsilon)$, let $\{S_i\}_{i\ge 0}$ be the random walk starting from y with the same law above. Define

$$\sigma_0 = \inf\{i > 0 : S_i \in [-\epsilon/2, \epsilon/2]\}, \sigma_0' = \inf\{i > \sigma_0 : S_i \in [-2\epsilon, 2\epsilon]^c\},$$

and inductively

$$\sigma_n = \inf\{i > \sigma_{n-1}' : S_i \in [-\epsilon/2, \epsilon/2]\}, \sigma_n' = \inf\{i > \sigma_n : S_i \in [-2\epsilon, 2\epsilon]^c\}.$$

We say that S_i visits n times to interval $[-\epsilon/2,\epsilon/2]$ in $M_3N^{2/5}$ steps if $\sigma_n < M_3N^{2/5} < \sigma_{n+1}$. Denote $N_{\epsilon/2}$ to be this number of times of visiting to $[-\epsilon/2,\epsilon/2]$ before $M_3N^{2/5}$.

Once a particle starting from $(-\epsilon,\epsilon)$ visits V, it is killed with probability $\mathbf{P}(N_V > 0)$. Hence, each time a particle visit the interval $(-\epsilon/2,\epsilon/2)$, it can survive with probability $1-\mathbf{P}(N_V > 0)$. If the particle visits R times to $(-\epsilon/2,\epsilon/2)$, the probability of surviving will be small. If we have a big time horizon $M_3N^{2/5}$, we can make sure that the particle can visit $(-\epsilon/2,\epsilon/2)$ more than R times.

The probability that $\{S_i\}_{i\ge 0}$ starting from y does not hit V until $M_3N^{2/5}$

$$\begin{aligned}&\mathbf{P}^y\left(S_i \notin V \text{ for any } 0 \le i \le M_3 - N^{2/5}\right)\\ &\le \mathbf{P}^y(N_{\epsilon/2} \le R) + \mathbf{P}\left(S_i \notin V \text{ for any } 0 \le i \le M_3N^{2/5} \mid N_{\epsilon/2} > R\right).\end{aligned}$$

By (4.8) and the Markov property of $\{S_i\}_{i\ge 0}$,

$$\begin{aligned}\mathbf{P}^y\left(S_i \notin V \text{ for any } 0 \le i \le M_3N^{2/5} \mid N_v \ge R\right) &\le \left(1 - \frac{1}{c_2^2 c_1^{-2} + (c_1\epsilon^4)^{-1}}\right)^R\\ &\le \delta/2,\end{aligned}$$

if R is chosen large enough compared to ϵ^{-4}. Notice that

$$\{N_{\epsilon/2} \leq R\} \subset \left\{ \sigma_0 + \sum_{n=1}^{R} (\sigma_n - \sigma_{n-1}) \geq M_3 N^{2/5} \right\}.$$

We can write $\sigma_n - \sigma_{n-1} = \sigma_n - \sigma'_{n-1} + \sigma'_{n-1} - \sigma_{n-1}$. The sequence of random variables $(\sigma_n - \sigma'_{n-1})_{1\leq n\leq R}$ follow the same distribution and so do $(\sigma'_{n-1} - \sigma_{n-1})_{1\leq n\leq R}$. Since $\mathbf{E}|S_i - S_{i-1}|^2 \approx \frac{1}{3N^{2/5}}$, we have $\mathbf{E}[\sigma_0] \leq \frac{3\epsilon^2 N^{2/5}}{4}$ and for any $1 \leq n \leq R$,

$$6\epsilon^2 N^{2/5} \leq \mathbf{E}[\sigma_n - \sigma'_{n-1}], \mathbf{E}[\sigma'_{n-1} - \sigma_{n-1}] \leq 16\epsilon^2 N^{2/5}.$$

Hence,

$$\begin{aligned}
\mathbf{P}^y(N_{\epsilon/2} \leq R) &\leq \mathbf{P}^y\left(\sigma_0 + \sum_{n=1}^{R} (\sigma_n - \sigma_{n-1}) \geq M_3 N^{2/5}\right) \\
&\leq \mathbf{P}^y\left(R(\sigma_1 - \sigma_0) + \sigma_0 \geq M_3 N^{2/5}\right) \\
&\leq \frac{33\epsilon^2 R}{M_3} \\
&\leq \delta/2,
\end{aligned}$$

if M_3 is chosen large compared to R. This concludes that

$$\mathbf{E}[\hat{X}_{M_3 N^{2/5}}] \leq \delta,$$

since the initial condition is $\mathbf{I}_{\{0\}}$. □

Lemma 4.4. *Suppose X is a non-negative random variable with finite second moment, then for any $c \in (0,1)$,*

$$\mathbf{P}(X > c\mathbf{E}[X]) \geq \frac{((1-c)\mathbf{E}[X])^2}{\mathbf{E}[X^2]}.$$

Proof. By Cauchy-Schwarz inequality,

$$\mathbf{E}[X\mathbf{I}_{X > c\mathbf{E}[X]}] \leq (\mathbf{E}[X^2])^{1/2}\mathbf{P}(X > c\mathbf{E}[X])^{1/2}.$$

Hence

$$\mathbf{P}(X > c\mathbf{E}[X]) \geq \frac{\left(\mathbf{E}[X\mathbf{I}_{X > c\mathbf{E}[X]}]\right)^2}{\mathbf{E}[X^2]}.$$

Since

$$\begin{aligned}
\mathbf{E}[X\mathbf{I}_{X > c\mathbf{E}[X]}] &= \mathbf{E}[X] - \mathbf{E}[\mathbf{I}_{X \leq c\mathbf{E}[X]}] \\
&\geq (1-c)\mathbf{E}[X],
\end{aligned}$$

which concludes the proof. □

Remark. *Notice that without considering the attrition, we can have the probability* $\mathbf{P}(T_k < \infty) \leq C2^{-k}$. *This is not enough in the proof of Theorem 4.1. However, for the proof we are helped by the attrition: sites that were visited cannot be visited again. Even in a very small killing zone* $(x-\epsilon, x+\epsilon)$ *in the proof above, many particles will be killed in a finite but large time period.*

4.2 The case $\kappa > C_2$

In this case, we will prove some properties of the true process, and then lead to an oriented percolation construction. The first step is to show that the difference between the solution to (2.3) and the solution to deterministic heat equation is quite small for small times. Suppose under $\mathbf{Q}$, $u(t,x)$ is the solution to (2.3) and under $\mathbf{P}$, $u(t,x)$ is the solution to (2.2). The Radon-Nykodym derivative of $\mathbf{Q}$ with respect to $\mathbf{P}$ is (3.30). Let the initial condition f be continuous, compactly supported and $f(x) = 1$ for $x \in [-1,1]$. We can regard the initial condition as $\mathbf{I}_{[-1,1]}$ plus some nonsignificant term. To show the existence of percolation, we need a lower bound for $u(t,x)$ (under $\mathbf{Q}$). Define the difference

$$N(t,x) = u(t,x) - G_t f(x),$$

with $G_t f(x) = \mathbf{E}[f(x + B_{t/3})]$, where $(B_t)_{t\geq 0}$ is a standard Brownian motion. By Lemma 4.2 of Shiga (1994) (also ref. Lemma 4 of Lalley (2009)),

$$\mathbf{P}\left(|N(t,x)| \geq \sqrt{\delta}\mathrm{e}^{-(\delta^5 - t)|x|} \text{ for some } t \leq \delta^5 \text{ and } x \in \mathbb{R}\right) \leq C_1 \delta^{-1/12} \exp(-C_2 \delta^{-1/4}).$$

The bound of the difference under $\mathbf{Q}$:

Lemma 4.5. *Denote*

$$A_\delta = \left\{|N(t,x)| \leq \sqrt{\delta} e^{-(\delta^5 - t)|x|} \textit{ for } \forall\, t \leq \delta^5 \textit{ and } \forall\, x \in \mathbb{R}\right\}.$$

If under $\mathbf{Q}$, $u(t,x)$ *is the solution to (2.3) given the initial condition* f *satisfying* $f(x) = 1$ *for* $x \in [-1,1]$, $f(x) = 0$ *for* $x \in [-1-\delta, 1+\delta]$ *and* f *is linear in the other parts, then*

$$\mathbf{Q}(A_\delta) \geq 1 - 3\delta^{7/2} \textit{ for all } \delta > 0 \textit{ small enough.}$$

Proof. $A_\delta \in \mathscr{F}_{\delta^5}$, hence

$$\begin{aligned}
\mathbf{Q}(A_\delta) &= \int_{A_\delta} \frac{d\mathbf{Q}}{d\mathbf{P}} d\mathbf{P} \\
&\geq (1 - \delta^{7/2}) \left(\int_{A_\delta \cap \left\{ \frac{d\mathbf{Q}}{d\mathbf{P}}|_{\mathscr{F}_{\delta^5}} \geq 1 - \delta^{7/2} \right\}} d\mathbf{P} \right)
\end{aligned}$$

Since for δ small enough, $\mathbf{P}(A_\delta) \geq 1-\delta^{7/2}$, we only need to show that

$$\mathbf{P}\left(\left.\frac{d\mathbf{Q}}{d\mathbf{P}}\right|_{\mathscr{F}_{\delta^5}} \geq 1-\delta^{7/2}|A_\delta\right) \geq 1-\delta^{7/2}.$$

By (3.30),

$$\mathbf{P}\left(\left.\frac{d\mathbf{Q}}{d\mathbf{P}}\right|_{\mathscr{F}_{\delta^5}} \geq 1-\delta^{7/2}|A_\delta\right) \geq \mathbf{P}\left(\left|\int_0^{\delta^5}\int \theta(t,x)m(dt,dx)+\frac{1}{2}\int_0^{\delta^5}(u_t,\theta(t,\cdot)^2)dt\right| \leq \delta^{7/2}|A_\delta\right).$$

By Chebyshev's inequality,

$$\begin{aligned}
&\mathbf{P}\left(\left|\int_0^{\delta^5}\int \theta(t,x)m(dt,dx)+\frac{1}{2}\int_0^{\delta^5}(u_t,\theta(t,\cdot)^2)dt\right| \geq \delta^{7/2}|A_\delta\right)\\
&\leq \frac{2}{\delta^7}\left\{\mathbf{E}\left(\int_0^{\delta^5}(u_t,\theta(t,\cdot)^2)dt|A_\delta\right)+\mathbf{E}\left(\left(\int_0^{\delta^5}(u_t,\theta(t,\cdot)^2)dt\right)^2|A_\delta\right)\right\}.
\end{aligned}$$

Given A_δ,

$$u(t,x) \leq \sqrt{\delta}\mathrm{e}^{-(\delta^5-t)|x|}+\sqrt{\frac{3}{2\pi t}}\int_{-2}^{2}\mathrm{e}^{-\frac{3|x-y|^2}{2t}}f(y)dy.$$

By Hölder inequality,

$$\begin{aligned}
\mathbf{E}\left(\int_0^{\delta^5}(u_t,\theta(t,\cdot)^2)dt|A_\delta\right) &\leq \mathbf{E}\left(\int\left(\int_0^{\delta^5}u(t,x)dt\right)^3dx|A_\delta\right)\\
&\leq \frac{1}{4}\delta^{10}\int\int_0^{\delta^5}\mathbf{E}(u(t,x)^3|A_\delta)dtdx\\
&\leq \frac{1}{4}\delta^{10}\left\{3\delta^{3/2}\int\int_0^{\delta^5}\mathrm{e}^{-3(\delta^5-t)|x|}+\frac{3\sqrt{3}}{\sqrt{2\pi t}}\int\int_0^{\delta^5}\int_{-2}^{2}\mathrm{e}^{-\frac{3|x-y|^2}{2t}}f(y)dydtdx\right\}\\
&\leq \frac{1}{2}\delta^{23/2}+3\delta^{15}.
\end{aligned}$$

Similarly,

$$\mathbf{E}\left(\left(\int_0^{\delta^5}(u_t,\theta(t,\cdot)^2)dt\right)^2|A_\delta\right) \leq C\delta^{23}.$$

Therefore,

$$\mathbf{P}\left(\left|\int_0^{\delta^5}\int \theta(t,x)m(dt,dx)+\frac{1}{2}\int_0^{\delta^5}(u_t,\theta(t,\cdot)^2)dt\right| \geq \delta^{7/2}|A_\delta\right) \leq \delta^{9/2},$$

and we have the expected result. □

The previous result helps to get a lower bound for the total density in a small time period which is our first desired property.

Corollary 4.3. *Let f be the function given as: $f(x) = 1$ for $x \in [-r, r]$ with $r \geq 1$, $f(x) = 0$ for $x \in [-r-\delta^{5/2}, r+\delta^{5/2}]^c$ and f is linear in the other parts, then there exists constants $L_1(r) < L_2(r) < \infty$,*

$$\mathbf{P}\left(\forall x \in \left[-r-2\delta^{5/2}, r+2\delta^{5/2}\right], L_1\delta^5 \leq \int_{\delta^5/2}^{\delta^5} \hat{u}_t(x)dt \leq L_2\delta^5\right) > 1-\delta^{7/2},$$

for all $\delta > 0$ small.

Proof. By Lemma 4.5, we know that out of probability $\delta^{7/2}$,

$$\left|\hat{u}_t(x) - G_t f(x)\right| \leq \sqrt{\delta}, \forall t \in [0, \delta^5].$$

For any $x \in \left[-r-2\delta^{5/2}, r+2\delta^{5/2}\right]$ and any $t \in \left[\delta^5/2, \delta^5\right]$, given that δ is small enough,

$$\begin{aligned} G_t f(x) &\geq G_{\delta^5}\mathbf{I}_{[-r,r]}(r+2\delta^{5/2}) \\ &\geq 2L_1(r) \end{aligned}$$

for some constant $L_1(r) > 0$. Hence, for any $x \in \left[-r-2\delta^{5/2}, r+2\delta^{5/2}\right]$ and any $t \in \left[\delta^5/2, \delta^5\right]$,

$$\hat{u}_t(x) \geq 2L_1.$$

The upper bound $L_2\delta^5$ follows from the same reason as the lower bound. □

In our original percolation model, the edges are not directed. However, it suffices to show percolation in the related model where the vertical edges are directed upward. For this we shall build a block argument, reducing the analysis to that of an oriented percolation model. Here, we keep the notation as in Durrett (1995). Let

$$\mathscr{L}_0 = \{(m, n) \in \mathbb{Z} \times \mathbb{Z}_+ : m + n \text{ is even}\}.$$

$\mathscr{L}_0$ is made into a graph by drawing oriented edge from (m, n) to $(m-1, n+1)$ or $(m+1, n+1)$. Random variables $\omega(m, n) \in \{0, 1\}$ are to indicate whether (m, n) is open ($\omega(m, n) = 1$) or close ($\omega(m, n) = 0$). We say that there is a path from (m, n) to (m', n') if there is a sequence of points $x_n = m, \ldots, x_{n'} = m'$ so that $|x_l - x_{l-1}| = 1$ for $n < l \leq n'$ and $\omega(x_l, l) = 1$ for $n \leq l \leq n'$. Let

$$\mathbf{C}_0 = \{(m, n) : (0, 0) \to (m, n)\}$$

be the cluster containing the origin.

The following steps are to construct the blocks which are considered as sites in the renormalized graph, to define when a renormalized site (block) is open and to define when an edge is open in the renormalized graph. We can then use the comparison theorem in Durrett (1995). The definition of renormalized sites being open demands a more refined treatment of the approximate density, i.e. one needs to look at a smaller scale, and for those purposes $N^{-3/10}$ is adequate.

Definition 4.1. *For a closed interval $I = [a,b]$, $\hat{\xi}$ is said to be (I,δ,N)-good if for the continuous function f satisfying $f(x) = 1$ for $x \in I$, $f(x) = 0$ for $x \in [a-\delta, b+\delta]^c$ and f is linear in the other parts,*

$$\sum_{x \in J} \hat{\xi}(x) = \lfloor f(iN^{-3/10})N^{1/10} \rfloor,$$

for any interval $J \subset I$ of the form $[iN^{-3/10}, (i+1)N^{-3/10}]$ but for $J \cap I^c \neq \varnothing$, $\sum_{x\in J} \hat{\xi}(x) = 0$.

Corollary 4.3 and Lemma 4.5 immediately give the following result for the discrete horizontal process.

Corollary 4.4. *There exists $\delta_0 > 0$ so that given $1 \le r \le 2$ and $0 < \delta < \delta_0$, if $\hat{\xi}_0$ is $([-r,r],\delta^{5/2},N)$-good on $\mathbb{Z}/N^{6/5}$, then for N large, outside of probability $5\delta^{7/2}$, for each $x \in [-r-2\delta^{5/2}, r+2\delta^{5/2}]$, $A\hat{\xi}_k(x) \ge L_1/2$ for each $\delta^5 N^{2/5}/2 \le k \le \delta^5 N^{2/5}$.*

Definition 4.2. *Suppose $\hat{\xi}_0$ is $([a,b],\delta^{5/2},N)$-good. Let a $[a,b]$-subordinated process on certain horizontal layer $\{\tilde{\xi}_k(x)\}_{0\le k\le \delta^5 N^{2/5}}$ be $\{\hat{\xi}_k(x)\}_{0\le k\le\delta^5 N^{2/5}}$ killed on $[a-1/2, b+1/2]^c$, i.e. for $0 \le k \le \delta^5 N^{2/5}$*

$$\tilde{\xi}_{k+1}(x) = \begin{cases} 1 & \text{if } \sum_{j\le k} \tilde{\xi}_j(x) = 0 \text{ and } \sum_{y\in\mathcal{N}_k(x)} \tilde{\eta}_{k+1}(y,x) \ge 1 \\ 0 & \text{otherwise,} \end{cases}$$

where $\mathcal{N}_k(x) = \{y \sim x : \tilde{\xi}_k(y) = 1\}$ and $\tilde{\eta}_{k+1}(y,x) = 0$ if $x \in [a-1/2, b+1/2]^c$ but over $x, y \in [a-1/2, b+1/2]$, $(\tilde{\eta}_{k+1}(y,x))_{k,y,x}$ is an i.i.d. sequence of random variables with distribution Bernoulli$(1/(2N))$.

Note that this killing property means that no new particles are generated outside $[a-1/2, b+1/2]$ and it is to guarantee an independent structure in the renormalization argument. The $[a,b]$ will not appear when we use the subordinated process since it will always be clear from the context.

Corollary 4.5. *There exists $\delta_0 > 0$ so that under the conditions of Corollary 4.4, for $0 < \delta < \delta_0$ N large, outside probability $6\delta^{7/2}$, for each $x \in [-r-2\delta^{5/2}, r+2\delta^{5/2}]$, $A\tilde{\xi}_k(x) \ge L_1/2$ for each $\delta^5 N^{2/5}/2 \le k \le \delta^5 N^{2/5}$.*

Proof. We suppose $\{\tilde{\xi}_n\}_{0\le n\le\delta^5 N^{2/5}}$ is coupled with a true process $\{\hat{\xi}_n\}_{0\le n\le\delta^5 N^{2/5}}$ and an envelope process $\{\xi_n\}_{0\le n\le\delta^5 N^{2/5}}$. For any starting site z such that $\tilde{\xi}_0(z) = 1$, let ξ_n^z be the envelope process with initial condition $\mathbf{I}_{\{z\}}$. For any $0 \le n \le \delta^5 N^{2/5}$, we have

$$\xi_n(x) = \sum_z \xi_n^z(x),$$

where the sum is over the initial condition that is $([-r,r],\delta^{5/2},N)$-good. The event

$$\left\{\exists x\in[-r-1/2,r+1/2] \text{ and } \delta^5 N^{2/5}/2\le k\le \delta^5 N^{2/5}:\tilde{\xi}_k(x)=0 \text{ but } \hat{\xi}_k(x)=1\right\} \tag{4.9}$$

has probability bounded by

$$\sum_z 2\mathbf{P}\left(R^z_{\delta^5 N^{2/5}}\ge 1/2 \mid \sum_x \xi^z_{\delta^5 N^{2/5}/2}(x)>0\right)\mathbf{P}\left(\sum_x \xi^z_{\delta^5 N^{2/5}/2}(x)>0\right),$$

where the sum is over the initial condition that is $([-r,r],\delta^{5/2},N)$-good and

$$R^z_n=\max_{m\le n}\{x\in\mathbb{Z}/N^{6/5}:\xi_m(x)\ne 0\}-z$$

is the maximal displacement of ξ^z at time n. By (ii) of Lemma 4.3 and Kesten's result (4.5), we have

$$\mathbf{P}(R^z_{\delta^5 N^{2/5}}>1/2)\le\frac{4}{\delta^5 N^{2/5}}\cdot C_a\delta^{5a}$$

for any $a>0$. Hence the probability of event (4.9) can be bounded by $8C_a\delta^{5(a-1)}$ and we can conclude the proof by choosing $a>2$. □

For our block argument the result above provides many sites at level 1 that are connected to sites occupied by $\hat{\xi}$ at level 0. This by itself is insufficient since we require these (level 1) sites to be $([-r-\delta^{5/2},r+\delta^{5/2}],\delta^{5/2},N)$-good. The following is an important step in this direction.

Lemma 4.6. *Let $\tilde{\xi}_0$ be as in Corollary 4.5 and J be a fixed interval of length $N^{-3/10}$ in $[-r-2\delta^{5/2},r+2\delta^{5/2}]$. Then the event that*

$$\min_{x\in[-r-2\delta^{5/2},r+2\delta^{5/2}]}\sum_{k=\delta^5 N^{2/5}/2}^{\delta^5 N^{2/5}} A\tilde{\xi}_k(x)\ge L_1\delta^5 N^{2/5}/4. \tag{4.10}$$

but

$$\sum_{x\in J}\sum_{k=\delta^5 N^{2/5}/2}^{\delta^5 N^{2/5}}\tilde{\xi}_k(x)\ <L_1\delta^5\sqrt{N}/32,$$

has probability less than $e^{-c\delta^5\sqrt{N}}$ for universal $c>0$.

Proof. Let y be the midpoint of J and let

$$\tau=\inf\left\{k\ge\delta^5 N^{2/5}/2:\sum_{j=\delta^5 N^{2/5}/2}^{k}A\tilde{\xi}_j(y)\ge L_1\delta^5 N^{2/5}/4\right\}.$$

For N large enough the event $\{\tau<\delta^5 N^{2/5}\}$ is contained in event that (4.10) happens. For the proof we note that for every z within $N^{-1/5}$ of y (the range of random walk $\{S_i\}$ starting from y), there are at least $N^{9/10}/2$ points of J in $[z-N^{-1/5},z+N^{-1/5}]$ and so while $\sum_{x\in J}\sum_{k=N^{2/5}\delta^5/2}^{N^{2/5}\delta^5}\tilde{\xi}_k(x)\ <L_1\delta^5\sqrt{N}/4$, each such (z,k) pair with $\tilde{\xi}_k(z)=1$ represents a prob-

ability

$$\frac{N^{9/10}/2 - L_1\delta^5\sqrt{N}/4}{2N}$$

of yielding a fresh occupied site for $\tilde{\xi}$ in J at time $k+1$. The result now follows from standard tail bound (Lemma 4.7 below) of Binomial$\left(\frac{L_1\delta^5 N^{3/5}}{4}, \frac{N^{9/10}/2-\delta^5\sqrt{N}/4}{2N}\right)$.

□

The following standard tail bounds of Binomial distributions are frequently used in the proof above and in the following.

Lemma 4.7. *Suppose $S \sim$ Binomial(n,p). Then there exists constant $c>0$ not depending on p so that*

$$\mathbf{P}\left(S \le \frac{np}{2}\right) \le e^{-cnp}.$$

Proof. S has mean np and variance $np(1-p)$. We have

$$\begin{aligned}
\mathbf{P}(S \le np/2) &= \mathbf{P}\left(\frac{S-np}{\sqrt{np(1-p)}} \le \frac{np}{2\sqrt{np(1-p)}}\right) \\
&= \mathbf{P}\left(-t\frac{S-np}{\sqrt{np(1-p)}} \ge -t\frac{np}{2\sqrt{np(1-p)}}\right) \\
&\le \exp\left(-\frac{npt}{2\sqrt{np(1-p)}}\right)\mathbf{E}\left[e^{-t\frac{S-np}{\sqrt{np(1-p)}}}\right] \\
&= \exp\left(\frac{npt}{2\sqrt{np(1-p)}}\right)\left(1+p\left(e^{-\frac{t}{\sqrt{np(1-p)}}}-1\right)\right)^n \\
&= \exp\left\{\frac{npt}{2\sqrt{np(1-p)}} + n\log\left(1+p\left(e^{-\frac{t}{\sqrt{np(1-p)}}}-1\right)\right)\right\}
\end{aligned}$$

By taking the derivative with respect to t, we find that the RHS of the inequality above attains its maximum when $t = t_0$ satisfying

$$\frac{p}{2} = \frac{e^{-\frac{t_0}{\sqrt{np(1-p)}}}}{1+p\left(e^{-\frac{t_0}{\sqrt{np(1-p)}}}-1\right)},$$

i.e. when

$$e^{-\frac{t_0}{\sqrt{np(1-p)}}} = \frac{p-p^2}{2-p^2}.$$

With this t_0, we have

$$\begin{aligned}
\mathbf{P}(S \le np/2) &\le \exp\left\{-\frac{np}{2}\log\left(\frac{p-p^2}{2-p^2}\right) + n\log\left(1+p\left(\frac{p-p^2}{2-p^2}-1\right)\right)\right\} \\
&\le \exp\left\{-\frac{np}{4}\left(\frac{p-p^2}{2-p^2}-1\right) + np\left(\frac{p-p^2}{2-p^2}-1\right)\right\} \\
&\le \exp\left\{-np\left(\frac{2-p}{2(2-p^2)}\right)\right\} \\
&\le \exp\left(-\frac{1}{4}np\right).
\end{aligned}$$

$\square$

Let $\{\tilde{\xi}^i_k\}_{0\le k\le \delta^5 N^{2/5}}$ be the subordinate process after killing at level $i \in \mathbb{N}$, where the initial configuration will be recursively defined as indicated at the end of Proposition 4.3 and the subordination effect indicated by the corresponding interval where the configuration is good.

With the same initial condition, $\{\tilde{\xi}^i_k\}_{0\le k\le \delta^5 N^{2/5}}$ follows the same distribution on any vertical level i. We will first discuss how the vertical connections behave between layer 0 and layer 1 as follows. Suppose $\tilde{\xi}^0_0$ is $([-r,r],\delta^{5/2},N)$-good. Until $\delta^5 N^{2/5}$ time steps, there is a certain amount of sites x's such that $\tilde{\xi}^0_k(x)=1$. The opening probability of a vertical edge is $\kappa N^{-2/5}$, in the following proposition, we will show that with large probability, the open vertical edge $\langle (x,0),(x,1)\rangle$ can make the initial profile at layer 1 be $([-r-\delta^{5/2}, r+\delta^{5/2}],\delta^{5/2},N])$-good.

Proposition 4.3. *Given $1\le r\le 2$ and $\delta<\delta_0$, there exists vertical connection constant C_2, so that for $\kappa > C_2$ and N large enough, if $\tilde{\xi}^0_0$ is $([-r,r],\delta^{5/2},N)$-good on $\mathbb{Z}/N^{6/5}\times\{0\}$, then outside of probability $6\delta^{7/2}$, on layer 1, $\tilde{\xi}^1_0$ is $([-r-\delta^{5/2}, r+\delta^{5/2}],\delta^{5/2},N)$-good on $\mathbb{Z}/N^{6/5}\times\{1\}$. $\tilde{\xi}^1_0(x)=1$ implies that $\tilde{\xi}^0_k(x)=1$ for some $k\in[\delta^5N^{2/5}/2,\delta^5N^{2/5}]$ and vertical edge $\langle (x,0),(x,1)\rangle$ is open.*

Remark. *This vertical connection constant C_2 means the threshold, κ above which, we can observe percolation in our original anisotropic percolation.*

Proof. By Corollary 4.5 and Lemma 4.6, outside of probability $6\delta^{7/2}$ (for N large enough), we have that for every interval $J=[iN^{-3/10},(i+1)N^{-3/10})$ contained in $[-r-2\delta^{5/2}, r+2\delta^{5/2}]$, we have

$$\sum_{x\in J}\sum_{k=\delta^5N^{2/5}/2}^{\delta^5N^{2/5}} \hat{\xi}_k(x) \;\ge\; L_1\delta^5\sqrt{N}/32.$$

We simply require that the vertical connection probability C_2 be greater than $64/(L_1\delta^5)$. Then by standard tail bound (Lemma 4.7) of Binomial$\left(L_1\delta^5\sqrt{N}/32, 64/(L_1\delta^5)N^{-2/5}\right)$, there exists universal $c>0$ so that outside probability $2e^{-cN^{1/10}}N^{3/5}$, for every such interval J, the number of $x\in J$ so that for some $k\in[\delta^5N^{2/5}/2,\delta^5N^{2/5}], \tilde{\xi}_k(x)=1$ and $\langle (x,0),(x,1)\rangle$ is open is greater than $N^{1/10}$. $\square$

Initially, $\tilde{\xi}_0^0$ is $([-1,1],\delta^{5/2},N)$-good. By Proposition 4.3, with probability $1-6\delta^{7/2}$, $\tilde{\xi}_0^1$ is $([-1-\delta^{5/2},1+\delta^{5/2}],\delta^{5/2},N)$-good. We can define recursively $\{\tilde{\xi}_k^i\}_{0\le k\le\delta^5N^{2/5}}$ for $0\le i\le\delta^{-5/2}$ (from here toward the end, we take $\delta<\delta_0$ and $\delta^{-5/2}\in\mathbb{N}$. By FKG inequality, with probability

$$(1-6\delta^{7/2})^{2\delta^{-5/2}}\ge 1-12\delta,$$

$\tilde{\xi}^{2\delta^{-5/2}}_{\delta^5N^{2/5}}$ is $([-3,3],\delta^{5/2},N)$-good. We then split and only consider the particles in two intervals $[-3,-1]$ and $[1,3]$. We run over two processes $\{\tilde{\xi}_k^i\}_{0\le k\le\delta^5N^{2/5}}, 2\delta^{-5/2}\le i\le 4\delta^{-5/2}$ starting from layer $2\delta^{-5/2}$ with initial conditions to be $([-3,-1],\delta^{5/2},N)$-good and $([1,3],\delta^{5/2},N)$-good. Recursively, given $\tilde{\xi}_0^{2n\delta^{-5/2}}$ is $(2m+[-1,1],\delta^{5/2},N)$-good, then outside of probability 12δ, $\tilde{\xi}_0^{2(n+1)\delta^{-5/2}}$ is $(2(m+1)+[-1,1],\delta^{5/2},N)$-good and $(2(m-1)+[-1,1],\delta^{5/2},N)$-good.

Note that the particles from $\tilde{\xi}_0^{2n\delta^{-5/2}}$ with initial conditions $(2(m-1)+[-1,1],\delta^{5/2},N)$-good and $(2(m+1)+[-1,1],\delta^{5/2},N)$-good will meet in $[-1,1]+2m$ at layer $2(n+1)\delta^{-5/2}$. We will only inherit the particles with lower m index, i.e. the particles from those with initial condition $(2(m-1)+[-1,1],\delta^{5/2},N)$-good.

Now we can do the renormalization. The renormalizaed regions are defined as

$$R_{m,n}=[-4,4]\times[0,2\delta^{-5/2}]+(2m,2n\delta^{-5/2})$$

and

$$I_m=[-1,1]+2m.$$

The renormalized site (m,n) corresponds to the block $R_{m,n}$. The random variables $\omega(m,n)\in\{0,1\}$ is to indicate that the renormalized block (site in the renormalized graph) is open or close. $\omega(m,n)=1$ if $\tilde{\xi}_0^{2n\delta^{-5/2}}$ is $(2m+[-1,1],\delta^{5/2},N)$-good in $R_{m,n}$ and we say that $R_{m,n}$ is good. The event that $\omega(m,n)$ is open or not is measurable with respect to the graphical representations in $R_{m,n}$ by the definition of $\{\tilde{\xi}_k\}_{0\le k\le\delta^5N^{2/5}}$ on a certain level. For an edge $e=\langle(m,n),(m+1,n+1)\rangle$ or $e=\langle(m,n),(m-1,n+1)\rangle$, denote $\psi(e)$ as the state of the edge. For $e=\langle(m,n),(m+1,n+1)\rangle$, $\psi(e)=1$ if (m,n) and $(m+1,n+1)$ are open sites in the renormalized graph. The definition of $\psi(e)$ for $e=\langle(m,n),(m-1,n+1)\rangle$ is similar. Let the probability of an edge being open in the renormalized graph be $\mathbf{P}(\psi(e)=1)=1-12\delta$ and $\mathbf{P}(\psi(e)=0)=12\delta$.

Therefore, the renormalized space is $\mathscr{L}_0=\{(m,n)\in\mathbb{Z}^2: m+n \text{ is even }, n\ge 0\}$ and make $\mathscr{L}_0$ into a graph $\mathscr{G}=(\mathscr{V},\mathscr{E})$ by drawing oriented edges from (m,n) to $(m\pm1,n+1)$. The percolation process $(\psi(e))_{e\in\mathscr{E}}$ is called d-dependent percolation with density p if for a sequence of vertices $v_i=(m_i,n_i), 1\le i\le I$ with $\|v_i-v_j\|_\infty>d, i\ne j$ connected by a sequence of edges $e_i, 1\le I-1$,

$$\mathbf{P}(\psi(e_i)=0, 1\le i\le I-1)\le(1-p)^{I-1}.$$

Proposition 4.4. *The percolation process $(\psi(e))_{e\in\mathscr{E}}$ is a 1-dependent oriented percolation with density $1-12\delta$.*

The initial condition is $\omega(0,0)=1$. By using the comparison argument Theorem 4.3 in Durrett

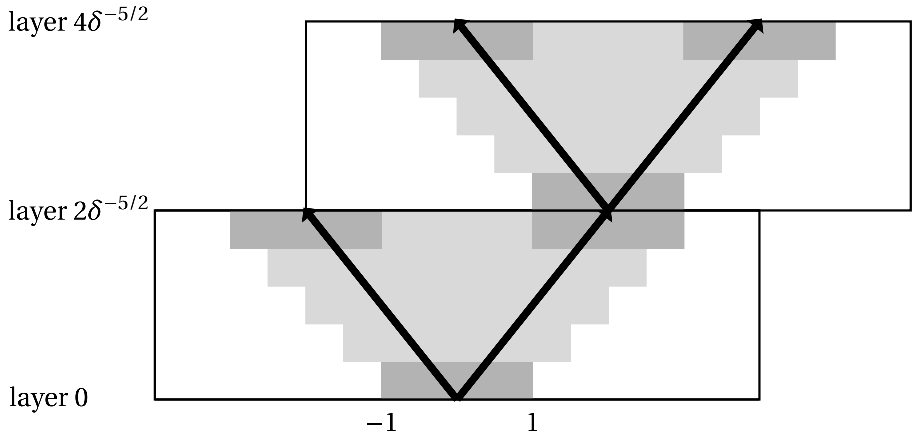

Figure 4.2 – Oriented percolation construction

(1995), we have the following result.

Theorem 4.2. *If there exists a percolation in the renormalized space $\mathscr{L}_0$ just defined, then there is a percolation in our anisotropic percolation process.*

The theorem of existence of percolation for d-dependent oriented percolation (Theorem 2.8) shows that if $12\delta < 6^{-4\cdot 9}$, there is a percolation.

Remark. *Figure 4.2 shows this renormalization construction.*

5 Supercritical Horizontal Movements

This chapter considers the anisotropic percolation on $\mathbb{Z}^2$ with supercritical horizontal movements. We continue to use the same notations as in the preceding three chapters. Each edge is open with probabilities (ref. (2.1)):

$$p_e = \begin{cases} \frac{\lambda}{2N} & \text{if } e \in E_h \\ \epsilon(N) & \text{if } e \in E_v. \end{cases}$$

We have shown Theorem 2.4 that the critical exponent of $\epsilon(N)$ when $\lambda = 1$ is $\epsilon(N) = \kappa N^{-2/5}$. This chapter aims to show the critical exponent of $\epsilon(N)$ when the horizontal movements are supercritical, i.e. $\lambda > 1$. Here we restate Theorem 2.5, the objective of this chapter.

Theorem 5.1 (Theorem 2.5)**.** *Suppose $\lambda > 1$ in (2.1), then the critical opening probability for vertical edges is $\epsilon(N) = e^{-\kappa N}$: there exist positive constants $C_1 < C_2$ such that when $\kappa < C_1$, there is a percolation and when $\kappa > C_2$, there is no percolation (for N large).*

To show this phase transition in κ, we follow a similar strategy as in Section 4.2 to show the existence of percolation when $\kappa < C_1$ for some $C_1 > 0$.Besides careful analysis dealing with the attritions in the previous chapters, we will introduce more levels of attritions to kill the true horizontal process to guarantee an independent structure. The case when $\kappa > C_2$ is easy to prove by direct estimation of the number of sites occupied by the envelope process in certain intervals. We do not need to rescale the space into $\mathbb{Z}/N^{6/5}$ in this chapter since we no longer need the scaling factor $\alpha = 1/5$ to obtain the convergence of the true horizontal process.

The horizontal edges can be reviewed as a discovery process along generations. With respect to layer 0, denote $\mathscr{C}_0^0$ as the cluster containing the origin,

$$\mathscr{C}_0^0 = \{x : (0,0) \to (x,0) \text{ with edges along the path in } \mathbb{Z} \times \{0\}\},$$

where $v_1 \to v_2$ means there is an open path from vertex v_1 to v_2. We can introduce generations on each horizontal layer. $x \in \mathscr{C}_0^0$ is of k-th generation if the shortest open path from $(0,0)$ to $(x,0)$ is of length k. There are vertices $v_1', \cdots, v_k'$ such that $v_1' = (0,0), v_k' = (x,0)$ and for any $1 \le i \le k-1$, $\langle v'_i, v'_{i+1}\rangle \in E_h$ is open. Let $\hat{\xi}(k,x) \in \{0,1\}$ denote the state of site $(x,0)$ at generation $k \in \mathbb{Z}_+$. $\hat{\xi}(k,x) = 1$ indicates that $(x,0)$ is visited at generation k. For $z \in \mathbb{Z}$, let $\{\hat{\xi}^z(k,x) \in \{0,1\}, k \in \mathbb{Z}_+, x \in \mathbb{Z}\}$ be derived through this discovery process starting from $z \in \mathbb{Z}$ on each horizontal layer:

$$\hat{\xi}^z(k+1,x) = \begin{cases} 1 & \text{if } \sum_{j\le k}\hat{\xi}^z(j,x) = 0 \text{ and } \sum_{y\in\mathcal{N}_k^z(x)}\eta^z(k+1,y,x), \\ 0 & \text{otherwise,} \end{cases}$$

where $\mathcal{N}_k^z(x) = \{y \sim x : \hat{\xi}^z(k,y) = 1\}$ be the available neighbourhood of x ($y \sim x$ means $1 \le |x-y| \le N$) at time k having cardinality $N_k^z(x) = \sum_{y\sim x}\hat{\xi}^z(k,y)$ and $(\eta^z(k,y,x))_{k,y,x}$ is an i.i.d. sequence with distribution Bernoulli$(\lambda/(2N))$ indicating the new edge linking y to x at time k. $\hat{\xi}^z$ is stochastically dominated by the supercritical branching random walk, which we call the envelope process, $\{\xi^z(k,x) \in \mathbb{N}, k \in \mathbb{Z}_+, x \in \mathbb{Z}\}$ defined as follows:

$$\xi^z(k+1,x) = \sum_{y\sim x}\sum_{w=1}^{\xi^z(k,y)}\eta_w^z(k+1,y,x),$$

where $\eta_w^z(k,y,x))_{w,k,y,x}$ is an i.i.d. sequence with distribution Bernoulli$(\lambda/(2N))$ indicating the new particle distributing from y to x at time k. By the definitions of $\hat{\xi}^z$ and ξ^z, we can observe the two levels of basic attritions:

- It is not allowed to have multiple particles at one site in the true horizontal process.
- Any site that has been visited before cannot be visited again by the true horizontal process.

We will introduce subordinate processes discovering horizontally and vertically based on the envelope process by excluding several levels of attritions. Outside of negligible probability, multiple particles are not allowed at one site. To give an independent structure, we need to introduce killing regions for the true horizontal process and the process killed by attritions is subordinated to the true horizontal process. After carefully analysing the probabilities of these attritions, we can build a block argument with independent structure based on the subordinate process. First, in the next section, we deal with the horizontal movements of the subordinated process.

5.1 Horizontal bond

In this section, we focus on one horizontal layer and construct the horizontal bond in the renormalized space. The block with the following form will be the site in the renormalized lattice.

$$H_i = [(iK-1)N, (iK+1)N] \cap \mathbb{Z},$$

where K is a large constant not depending on N to be fixed later.

Take the initial condition I_0 to be that there are εN occupied sites in $H_0 = [-N, N] \cap \mathbb{Z}$ (no matter where they are), where ε is a small value depending on λ and K to be chosen later. Our objective of this section is to show that on one layer, if there are εN in H_i, then after certain generations, there are also εN sites in H_{i+1}. Mathematically, we want to estimate the probability of the following event:

$$\sum_{z\in I_0}\sum_{x\in H_{i\pm1}} \hat{\xi}^z((k+1)n_0, x) \geq \varepsilon N \tag{5.1}$$

given the information of $\mathscr{F}^z(kn_0)$, where $\mathscr{F}^z(m) = \sigma\left(\hat{\xi}^z(k,\cdot), 0 \leq k \leq m\right)$ and

$$\sum_{z\in I_0}\sum_{x\in H_i} \hat{\xi}^z(kn_0, x) \geq \varepsilon N$$

with some $n_0 \in \mathbb{N}$ chosen later.

However this conditional probability (5.1) is very difficult to estimate and moreover, we cannot guarantee an independent structure only with the estimation of (5.1). We will analyse step by step by first showing that $\hat{\xi}^z$ is not too much different from ξ^z and then imposing some killing regions to ξ^z to give an independent structure.

It is enough to consider the case when $i = 0$, i.e. the propagation from H_0 to H_1. Since the subordinate process will be built on the envelope process. We now clarify some basic construction related to the branching random walk i.e. the envelope process. For a branching random walk $\{\xi^z(n,\cdot)\}_{n\geq0}$ starting from z, denote $\{T^z(n)\}_{n\geq0}$ as the vertex set that $\{\xi^z(n,\cdot)\}_{n\geq0}$ can reach, i.e.

$$T^z(n) = \{x : \xi^z(n,x) > 0\}.$$

Notice that for the envelope process, multiple particles can occupy one same site but we will show that this is of negligible probability. Any particle in $\{\xi^z(n,\cdot)\}_{n\geq0}$ can be recorded by an n-tuple $\mathbf{i}_n = (i_0, i_1, \cdots, i_n)$ with $i_k \in [0, 2N-1] \cap \mathbb{Z}$, where $i_0 = 0$ is the origin, and $\mathbf{i}_n$ is the offspring of $\mathbf{i}_{n-1}$ with i_n being the index of offspring in the n-th generation (ref. Figure 5.1). For the site with multiple particles, for example $(0,0,0,1)$, it is regarded as the offspring with ancestor with lower index. The red particle on the top visits the site visited by $(0,0,0)$, so it is killed. The event of occurrence of red sites is of negligible probability (shown later).

For any particle $(\mathbf{i}_n)_{n\geq0}$, at each step n, we first generate a random variable

$$N(\mathbf{i}_{n-1}) \sim \text{Binomial}(2N, 1/(2N))$$

indicating the number of offsprings of $\mathbf{i}_{n-1} = (i_0, \cdots, i_{n-1})$. Let S_{2N} be the permutation group of $([-N,N]\cap\mathbb{Z})\backslash\{0\}$ and $(\pi(1), \cdots, \pi(2N)) \in S_{2N}$ is a uniform random permutation. If $N(\mathbf{i}_{n-1}) > 0$, take

$$(X(i_0, \cdots, i_{n-1}, 0), \cdots, X(i_0, \cdots, i_{n-1}, N(\mathbf{i}_{n-1}) - 1)) = (\pi(1), \cdots, \pi(N(\mathbf{i}_{n-1})))$$

to be the placements of the $N(\mathbf{i}_{n-1})$ offsprings at generation n. In the following argument, we

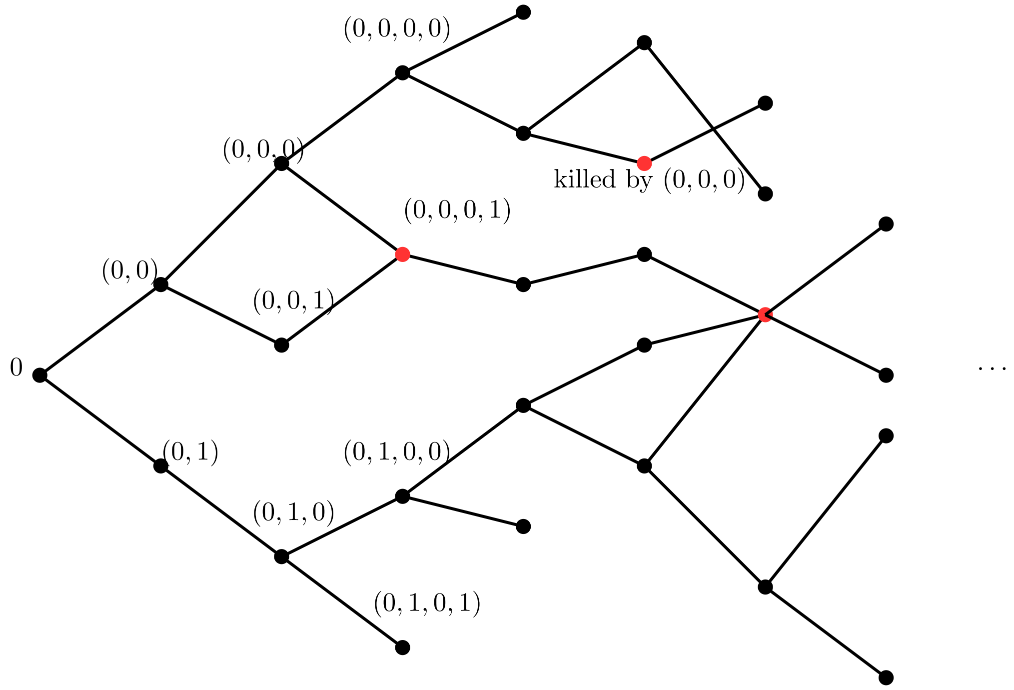

Figure 5.1 – Branching random walk

will only consider the particle $(\mathbf{i}_n)_{n\geq 0}$ satisfying

$$\forall n, i_n \in \{0, \cdots, N(\mathbf{i}_{n-1}) - 1\}.$$

The random path of $(\mathbf{i}_n)_{n\geq 0}$ starting from z can be formulated as

$$S^z(\mathbf{i}_n) = z + \sum_{k=1}^{n} X(\mathbf{i}_k),$$

where $(X(\mathbf{i}_k))_{k\geq 1}$ is an i.i.d. sequence with distribution

$$\mathbf{P}(X(\mathbf{i}_k) = j) = \frac{1}{2N}, \text{ for } j = -N, \cdots, -1, 1, \cdots, N. \tag{5.2}$$

We will use the following lemma frequently to estimate the density of particles in a certain interval.

Lemma 5.1. *For any $z \in I_0$, let*

$$N^z(n) = \sum_{x\in\mathbb{Z}} \xi^z(n, x)$$

be the total mass at generation n. Then,

$$\mathbf{E}[N^z(n)] = \lambda^n. \tag{5.3}$$

Moreover, for any $H \subset \mathbb{Z}$,

$$\mathbf{E}\left[\sum_{x\in H} \xi^z(n,x)\right] = \lambda^n \mathbf{P}\left(S^z(\mathbf{i}_n) \in H\right).$$

The second moment

$$\mathbf{E}[(N^z(n))^2] \leq \frac{\lambda}{\lambda-1}\lambda^{2n}. \tag{5.4}$$

Proof. $N^z(n)$ can be written as

$$N^z(n) = \sum_{j=1}^{N^z(n-1)} Y_j^n,$$

where $(Y_j^n)_{n,j}$ is an i.i.d. sequence with distribution Binomial$(2N, \lambda/(2N))$.

$$\begin{aligned}
\mathbf{E}[N^z(n)] &= \mathbf{E}\left[\mathbf{E}\left[\sum_{j=1}^{N^z(n-1)} Y_j^n \mid N^z(n-1)\right]\right] \\
&= \lambda \mathbf{E}[N^z(n-1)] \\
&= \lambda^n,
\end{aligned}$$

which concludes (5.3) by induction. Denote $\sigma^2 = \mathrm{Var}(Y_1^1) = \lambda(1-\lambda/(2N)) \leq \lambda$.

$$\begin{aligned}
\mathrm{Var}(N^z(n)) &= \mathbf{E}\left(\mathrm{Var}\left(\sum_{j=1}^{N^z(n-1)} Y_j^n \mid N^z(n-1)\right)\right) + \mathrm{Var}\left(\mathbf{E}\left(\sum_{j=1}^{N^z(n-1)} Y_j^n \mid N^z(n-1)\right)\right) \\
&= \lambda^{n-1}\sigma^2 + \lambda^2 \mathrm{Var}(N^z(n-1)) \\
&\leq \lambda^n + \lambda^2 \mathrm{Var}(N^z(n-1)) \\
&\leq \sum_{j=0}^{n-1} \lambda^{n+j} \\
&\leq \frac{1}{\lambda-1}\lambda^{2n}.
\end{aligned}$$

Hence,

$$\begin{aligned}
\mathbf{E}[(N^z(n))^2] &= \mathrm{Var}(N^z(n)) + \left(\mathbf{E}[N^z(n)]\right)^2 \\
&\leq \frac{1}{\lambda-1}\lambda^{2n} + \lambda^{2n} \\
&= \frac{\lambda}{\lambda-1}\lambda^{2n}.
\end{aligned}$$

This concludes (5.4). □

Denote $\{\xi_+^z(n,\cdot)\}_{n\geq 0}$ ($\{\xi_-^z(n,\cdot)\}_{n\geq 0}$) as the right (left) biased branching random walk. For the right biased branching random walk $\{\xi_+^z(n,\cdot)\}_{n\geq 0}$,

$$\xi_+^z(n+1,x) = \sum_{y\sim x} \sum_{w=1}^{\xi_+^z(n,y)} \eta_+^w(n+1,y,x),$$

where $(\eta_+^w(n,y,x))_{n,y,x}$ is an i.i.d. sequence with distribution:

$$\eta_+^w(n,y,x) \sim \text{Bernoulli}(\lambda/(2N)) \text{ for } y<x,$$
$$\eta_+^w(n,y,x) \sim \text{Bernoulli}(1/(2N)) \text{ for } y>x.$$

Intuitively speaking, any particle will give birth to its right side with probability $\frac{\lambda}{(\lambda+1)N}$ and will give birth to its left side with probability $\frac{1}{(\lambda+1)N}$. The left biased branching random walk can be defined similarly. $\{\xi_\pm^z(n,\cdot)\}_{n\geq 0}$ also have corresponding vertex set $\{T_\pm^z(n)\}_{n\geq 0}$ and random paths $\{S_\pm^z(\mathbf{i}_n)\}_{n\geq 0}$. By similar construction of $(S^z(\mathbf{i}_n))_{n\geq 0}$, the right biased random path is defined as

$$S_+^z(\mathbf{i}_n) = z + \sum_{k=1}^n X_+(\mathbf{i}_k),$$

where $(X_+(\mathbf{i}_k))_{k\geq 1}$ is an i.i.d. sequence with distribution

$$\mathbf{P}(X_+(\mathbf{i}_k)=j) = \begin{cases} \frac{\lambda}{(\lambda+1)N} & \text{for } j=1,\cdots,N \\ \frac{1}{(\lambda+1)N} & \text{for } j=-N,\cdots,-1. \end{cases} \tag{5.5}$$

The left biased random path can be defined similarly. For the biased branching random walk, we have similar results as Lemma 5.1 by identical reason.

Corollary 5.1. *For any $z\in I_0$, let*

$$N_\pm^z(n) = \sum_{x\in\mathbb{Z}} \xi_\pm^z(n,x)$$

be the total mass at generation n. Then,

$$\mathbf{E}[N_\pm^z(n)] = \left(\frac{\lambda+1}{2}\right)^n.$$

Moreover for any $H\subset\mathbb{Z}$,

$$\mathbf{E}\left[\sum_{x\in H}\xi_+^z(n,x)\right] = \left(\frac{\lambda+1}{2}\right)^n \mathbf{P}\left(S_+^z(\mathbf{i}_n)\in H\right).$$

The second moment

$$\mathbf{E}[(N_\pm^z(n))^2] \leq \frac{\lambda+1}{\lambda-1}\left(\frac{\lambda+1}{2}\right)^{2n}.$$

Based on the preceding settings, we can see that $\{\xi_\pm^z(n,\cdot)\}_{n\geq 0}$ are each stochastically dominated by $\{\xi^z(n,\cdot)\}_{n\geq 0}$. In the following argument, we will use $\{\xi_+^z(n,\cdot)\}_{n\geq 0}$ to construct the subordinate process dominated by the true horizontal process $\{\hat{\xi}^z(n,\cdot)\}_{n\geq 0}$. Similar argument holds for the left biased branching random walk.

Suppose initially, there are εN open sites in H_0 denoted by I_0 (no matter where they are in H_0). List the εN particles in I_0 as $z_1,\cdots,z_{\varepsilon N}$ (with ε suitably chosen to make εN an integer). We now start to build the attritions. We first want to bound the cumulative $N_+^z(n)$. For $z\in I_0$, denote

$$A_1^z(n_0) = \left\{ \sum_{k=1}^{n_0} N_+^z(k) \geq n_0 K e^{(\lambda+1)K} \right\}.$$

This upper bound is for the further use of the Azuma's inequality.

Lemma 5.2. *Let $n_0 = \frac{2K(\lambda+1)}{\lambda-1}$ (K suitably chosen to make n_0 an integer), then*

$$\mathbf{P}(A_1^z(n_0)) \leq \frac{1}{K^3},$$

for any $z \in I_0$.

Proof. Let

$$A_k = \left\{ \sum_{i=1}^{k} N_+^z(i) \geq n_0 K e^{(\lambda+1)K} \text{ but } \sum_{i=1}^{j} N_+^z(i) < n_0 K e^{(\lambda+1)K} \text{ for } j < k \right\}.$$

Since A_k are disjoint and $\sum_{i=1}^{k} N_+^z(i)$ is increasing in k,

$$\begin{aligned}
\mathbf{E}\left[\left(\sum_{i=1}^{n_0} N_+^z(i) \right)^2 \right] &\geq \sum_{k=1}^{n_0} \int_{A_k} \left(\sum_{i=1}^{n_0} N_+^z(i) \right)^2 d\mathbf{P} \\
&\geq \sum_{k=1}^{n_0} \int_{A_k} \left(\sum_{i=1}^{k} N_+^z(i) \right)^2 d\mathbf{P} \\
&\geq \left(n_0 K e^{(\lambda+1)K} \right)^2 \sum_{k=1}^{n_0} \mathbf{P}(A_k) \\
&\geq \left(n_0 K e^{(\lambda+1)K} \right)^2 \mathbf{P}\left(\sum_{i=1}^{n_0} N_+^z(i) \geq n_0 K e^{(\lambda+1)K} \right).
\end{aligned}$$

The second moment in Corollary 5.1 gives that

$$\begin{aligned}
\mathbf{E}\left[\left(\sum_{i=1}^{n_0} N_+^z(i) \right)^2 \right] &\leq 2 \sum_{i=1}^{n_0} \frac{\lambda+1}{\lambda-1} \left(\frac{\lambda+1}{2} \right)^{2i} \\
&\leq 2 \frac{\lambda+1}{\lambda-1} n_0 \left(\frac{\lambda+1}{2} \right)^{2n_0}.
\end{aligned}$$

Hence,

$$\begin{aligned}
\mathbf{P}\left(\sum_{i=1}^{n_0} N_+^z(i) \geq n_0 K e^{(\lambda+1)K} \right) &\leq \frac{2 \frac{\lambda+1}{\lambda-1} n_0 e^{2(\lambda+1)K}}{\left(n_0 K e^{(\lambda+1)K} \right)^2} \\
&\leq \frac{1}{K^3}.
\end{aligned}$$

□

The second attrition comes from multiple visits to one site (the red sites in Figure 5.1). If one site is visited more than once by $\{\xi_+^z(n,\cdot)\}_{n\geq 0}$, it means that there are two particles $(\mathbf{i}_n)_{n\geq 0}, (\mathbf{i}'_n)_{n\geq 0}$ starting from two sites $z, z' \in I_0$ whose paths are $\{S_+^z(\mathbf{i}_n)\}_{n\geq 0}$ and $\{S_+^{z'}(\mathbf{i}'_n)\}_{n\geq 0}$ and they meet with each other at certain moment (may not be the same time). This refers to:

- A site visited before is visited again.
- There are multiple particles at one site.

This is a global attrition over all $z_1, \cdots, z_\varepsilon$. Let

$$A_2(n_0) = \left\{\exists z, z' \in I_0, (z, (\mathbf{i}_n)_{n\geq 0}) \neq (z', (\mathbf{i}'_n)_{n\geq 0}) \text{ and } \exists 1 \leq k, k' \leq n_0, \text{ s.t. } S_+^z(\mathbf{i}_k) = S_+^{z'}(\mathbf{i}'_{k'})\right\},$$

where $(\mathbf{i}_n)_{n\geq 0}$ and $(\mathbf{n}')_{n\geq 0}$ are two particles starting from z and z'. z and z' in $A_2(n_0)$ may be identical and two paths starting from z coalesce later.

Lemma 5.3. *Let K be chosen large so that $K^{-1} < \lambda - 1$ and $n_0 = \frac{2K(\lambda+1)}{\lambda-1}$. Then,*

$$\mathbf{P}\left(A_2(n_0) \mid \bigcap_{z\in I_0} \left(A_1^z(n_0)\right)^c\right) \leq \varepsilon^2(\lambda+1)^2 K^2 e^{2(\lambda+1)K},$$

for ε small and N large enough.

Proof. Denote $A_+^z(m)$ as the total number of visited sites of $\{\xi_+^z(n,\cdot)\}_{0\leq n\leq n_0}$ until step $m \leq n_0$, i.e.

$$A_+^z(m) = \sum_{k\leq m}\sum_x \xi_+^z(k,x).$$

We have

$$\begin{aligned} A_+^z(n_0) &\leq n_0 K e^{(\lambda+1)K} \\ &\leq 2(\lambda+1)K^2 e^{(\lambda+1)K}, \end{aligned}$$

since $K^{-1} < \lambda - 1$. The initial condition has $|I_0| = \varepsilon N$, so

$$\sum_{z\in I_0} A_+^z(n_0) \leq 2\varepsilon(\lambda+1)K e^{(\lambda+1)K} N.$$

Hence,

$$\mathbf{P}(A_2(n_0)) \leq \varepsilon^2(\lambda+1)^2 K^2 e^{2(\lambda+1)K}.$$

□

Now we remove the paths belonging to the event $A_1^z(n_0)$ and $A_2(n_0)$ to get $\{\xi'^z(n,\cdot)\}_{0\le n\le n_0}$ defined strictly as follows,

$$\xi'^z_+(n+1,x)=\begin{cases}1 & \text{if } \sum_{z'\in I_0}\sum_{j\le n}\xi'^{z'}_+(j,x)=0 \text{ and } \sum_{y\in\mathcal{N}_n(x)}\eta'_+(n+1,y,x)\ge 1\\ 0 & \text{otherwise,}\end{cases}$$

where $\mathcal{N}_n(x)=\{y\sim x:\xi'^{z'}_+(n,y)=1 \text{ for some } z'\in I_0\}$ and

- $\eta'_+(j,y,x)=0$ for any y,x and $j\ge k$ if $\sum_{i=1}^k\sum_{i\in\mathbb{Z}}\xi'^z_+(k,x)\ge n_0Ke^{(\lambda+1)K}$ for some $k\le n_0$;
- otherwise, $(\eta'^z_+(n,y,x))_{n,y,x}$ is an i.i.d. sequence with distribution:

$$\begin{aligned}\eta'_+(n,y,x)&\sim \text{Bernoulli}(\lambda/(2N)) \text{ for } y<x,\\ \eta'_+(n,y,x)&\sim \text{Bernoulli}(1/(2N)) \text{ for } y>x.\end{aligned}$$

Denote $\{S'^z_+(\mathbf{i}_n)\}_{0\le n\le n_0}$ with $S'^z_+(\mathbf{i}_n)=z+\sum_{k=1}^n X'_+(\mathbf{i}_k)$ and $X'_+(\mathbf{i}_k)$ the same distribution as (5.5) be the random paths of $\{\xi'^z_+(n,\cdot)\}_{0\le n\le n_0}$. From the construction above, we can see that $\{\xi'^z_+(n,\cdot)\}_{0\le n\le n_0}$ is subordinated to the true horizontal process $\{\hat{\xi}^z(n,\cdot)\}_{0\le n\le n_0}$.

We now introduce the next level of attritions of $\{\xi'^z_+(k,\cdot)\}_{0\le k\le n_0}$.

(A1) The right-biased walks $\{S'^z_+(\mathbf{i}_k),0\le k\le n_0\},z\in I_0$ do not visit beyond the left bound $-\frac{1}{2}KN$.

(A2) The right-biased walks $\{S'^z_+(\mathbf{i}_k),0\le k\le n_0\},z\in I_0$ do not visit beyond the right bound $2KN$.

To summarise, let

$$A_3^z(n_0)=\left\{\exists 0\le k\le n_0, S'^z_+(\mathbf{i}_k)\in\left[-\frac{1}{2}KN,2KN\right]^c\right\}.$$

Lemma 5.4. *Let $n_0=\frac{2K(\lambda+1)}{\lambda-1}$, then there exists $c(\lambda)>0$, so that for any $z\in I_0$,*

$$\mathbf{P}(A_3^z(n_0))\le e^{-c(\lambda)K}.$$

for K and N large enough.

Proof. The proof follows from the reflection principle and the large deviation principle. Since

$$\mathbf{E}[X_+(\mathbf{i}_k)]=\frac{(\lambda-1)(N+1)}{2(\lambda+1)},$$

for any $z \in I_0$,

$$\mathbf{E}[S_+^z(\mathbf{i}_{n_0})] \in H_1 = [(K-1)N, (K+1)N].$$

Without loss of generality, we only need to show the bound for

$$\mathbf{P}\left(\exists j \le n_0, S_+^0(\mathbf{i}_j) > 2KN\right).$$

By reflection principle,

$$\begin{aligned}
\mathbf{P}\left(\exists j \le n_0, S_+^0(\mathbf{i}_j) > 2KN\right) &\le \frac{1}{1-\mathbf{P}(X_+(\mathbf{i}_1) \le 0)}\mathbf{P}\left(S_+^0(\mathbf{i}_{n_0}) > 2KN\right) \\
&= \frac{\lambda+1}{\lambda}\mathbf{P}(S_+^0(\mathbf{i}_{n_0}) > 2KN).
\end{aligned}$$

For $c > 0$,

$$\begin{aligned}
\mathbf{P}(S_+^0(\mathbf{i}_{n_0}) > 2KN) &= \mathbf{P}\left(S_+^0(\mathbf{i}_{n_0}) - \mathbf{E}[S_+^0(\mathbf{i}_{n_0})] > KN\right) \\
&\le \mathbf{E}\left[e^{c\left(S_+^0(\mathbf{i}_{n_0}) - \mathbf{E}[S_+^0(\mathbf{i}_{n_0})]\right)}\right]e^{-cKN} \\
&= e^{-cKN}\left(\mathbf{E}\left[e^{c(X_+(\mathbf{i}_1)-\mathbf{E}[X_+(\mathbf{i}_1)])}\right]\right)^{n_0},
\end{aligned}$$

since $(X_+(\mathbf{i}_k))_{1\le k\le n_0}$ is an i.i.d. sequence. If we pick $c = t/N$ with some $0 < t < 1$,

$$e^{c(X_+(\mathbf{i}_1)-\mathbf{E}[X_+(\mathbf{i}_1)])} \le 1 + c\left(X_+(\mathbf{i}_1) - \mathbf{E}[X_+(\mathbf{i}_1)]\right) + 2c^2\left(X_+(\mathbf{i}_1) - \mathbf{E}[X_+(\mathbf{i}_1)]\right)^2,$$

as $c|X_+(\mathbf{i}_1) - \mathbf{E}[X_+(\mathbf{i}_1)]| < 1$. Hence

$$\begin{aligned}
\mathbf{E}\left[e^{c(X_+(\mathbf{i}_1)-\mathbf{E}[X_+(\mathbf{i}_1)])}\right] &\le 1 + 2c^2\mathbf{E}\left[(X_+(\mathbf{i}_1) - \mathbf{E}[X_+(\mathbf{i}_1)])^2\right] \\
&\le e^{2c^2\mathbf{E}\left[(X_+(\mathbf{i}_1)-\mathbf{E}[X_+(\mathbf{i}_1)])^2\right]} \\
&\le e^{\frac{1}{6}t^2},
\end{aligned}$$

because

$$\begin{aligned}
\mathrm{Var}(X_+(\mathbf{i}_k)) &= \frac{(N+1)(2N+1)}{6} - \frac{(\lambda-1)^2(N+1)^2}{4(\lambda+1)^2} \\
&\approx \left(\frac{1}{3} - \frac{(\lambda-1)^2}{4(\lambda+1)^2}\right)N^2 \\
&\le \frac{1}{12}N^2,
\end{aligned}$$

as N large enough. Therefore,

$$\begin{aligned}
\mathbf{P}(S_+^0(\mathbf{i}_{n_0}) > 2KN) &\le e^{-tK+\frac{1}{6}t^2 n_0} \\
&= e^{-tK+\frac{(\lambda+1)K}{3(\lambda-1)}t^2} \\
&\le e^{-\frac{3(\lambda-1)}{4(\lambda+1)}K},
\end{aligned}$$

if we pick $t = \frac{3(\lambda-1)}{2(\lambda+1)}$. $\square$

Remark. *Notice that $A_2(n_0)$ is a global attrition over all initial site $z \in I_0$ and $A_1^z(n_0), A_3^z(n_0)$ are local attritions of the related process starting from $z \in I_0$.*

Now we remove further the paths belonging to $A_3^z(n_0)$ to get $\{\xi_+^{\prime\prime z}(n,\cdot)\}_{0\le n\le n_0}$ defined strictly as follows,

$$\xi_+^{\prime\prime z}(n+1,x) = \begin{cases} 1 & \text{if } \sum_{z'\in I_0}\sum_{j\le n}\xi_+^{\prime\prime z'}(j,x) = 0 \text{ and } \sum_{y\in\mathcal{N}_n(x)}\eta_+^{\prime\prime}(n+1,y,x) \ge 1 \\ 0 & \text{otherwise,} \end{cases}$$

where $\mathcal{N}_n(x) = \{y \sim x : \xi^{\prime\prime z'}(n,y) = 1 \text{ for some } z' \in I_0\}$ and

- $\eta_+^{\prime\prime}(n,y,x) = 0$ if $x \in \left[-\frac{KN}{2}, 2KN\right]^c$;
- $\eta_+^{\prime\prime}(j,y,x) = 0$ for any y, x and $j \ge k$ if $\sum_{x\in\mathbb{Z}}\xi_+^{\prime\prime z}(k,x) \ge Ke^{(\lambda+1)K}$ for some $k \le n_0$;
- otherwise, $(\eta_+^{\prime\prime}(n,y,x))_{n,y,x}$ is an i.i.d. sequence with distribution

$$\eta_+^{\prime\prime}(n,y,x) \sim \text{Bernoulli}(\lambda/(2N)) \text{ for } y < x,$$
$$\eta_+^{\prime\prime}(n,y,x) \sim \text{Bernoulli}(1/(2N)) \text{ for } y > x.$$

Let $\{S_+^{\prime\prime z}(\mathbf{i}_k)\}_{0\le k\le n_0}$ with $S_+^{\prime\prime z}(\mathbf{i}_n) = z + \sum_{k=1}^{n} X_+^{\prime\prime}(\mathbf{i}_k)$ and $X_+^{\prime\prime}(\mathbf{i}_k)$ the same distribution as (5.5) be the random path of $\{\xi_+^{\prime\prime z}(n,\cdot)\}_{0\le n\le n_0}$. It is easy to see that $\{\xi_+^{\prime\prime z}(n,\cdot)\}_{0\le n\le n_0}$ is subordinated to the true horizontal process $\{\hat{\xi}^z(n,\cdot)\}_{0\le n\le n_0}$. The events $A_1^z(n_0)$, $A_2(n_0)$ and $A_3^z(n_0)$ are shown in Figure 5.2 (in the figure $c_1(\lambda,K), c_2(\lambda,K) \le e^{-c(\lambda)K}$).

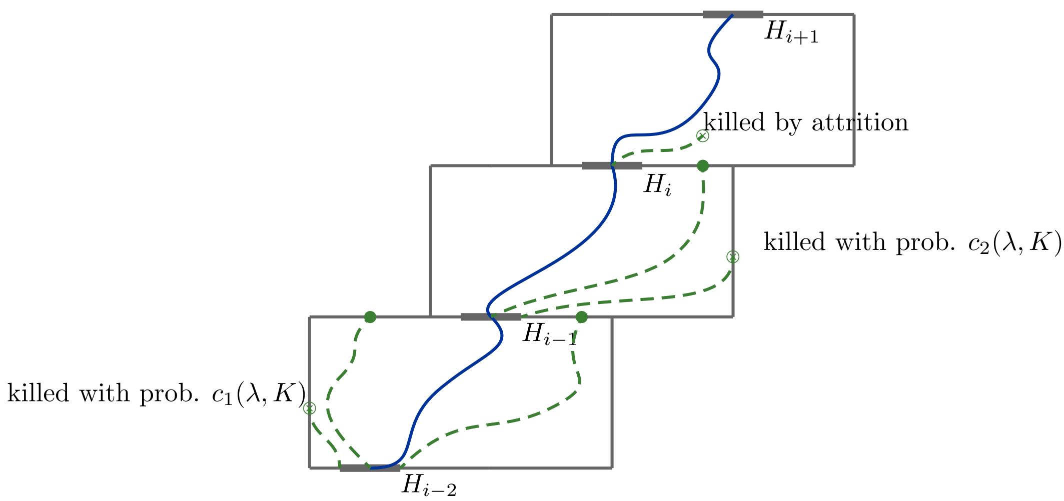

Figure 5.2 – Right-biased movements with attritions

Corollary 5.2. *For any $z \in I_0$, let*

$$N_+''^z(n_0) = \sum_x \mathbf{I}_{\{\xi_+''^z(n_0,x)>0\}},$$

where $n_0 = \frac{2K(\lambda+1)}{\lambda-1}$ and K is chosen so that $K^{-1} < \lambda - 1$, then

$$\mathbf{E}[N_+''^z(n_0)] \geq e^{(\lambda+1)K}\left(1 - \sqrt{\frac{\lambda+1}{\lambda-1}}\sqrt{\frac{1}{K^3} + \varepsilon^2(\lambda+1)^2K^2e^{2(\lambda+1)K} + e^{-c(\lambda)K}}\right),$$

for K and N large enough.

Proof. Suppose $N_+^z(n_0)$ is as in Lemma 5.1. Then by coupling,

$$N_+''^z(n_0) = \sum_{i=1}^{N_+^z(n_0)} Y_i,$$

$(Y_i)_{i\geq 1}$ is an i.i.d. sequence satisfying $Y_i = 0$ if one of the events $A_1^z(n_0), A_2(n_0), A_3^z(n_0)$ happens and 1 otherwise. Then

$$\begin{aligned}
\mathbf{E}[N_+''^z(n_0)] &= \mathbf{E}[N_+^z(n_0)] - \mathbf{E}[N_+^z(n_0)\mathbf{I}_{\{Y_1=0\}}] \\
&\geq \mathbf{E}[N_+^z(n_0)] - \sqrt{\mathbf{E}[(N_+^z(n_0))^2]}\sqrt{\mathbf{P}(Y_1=0)}(\text{ by Cauchy-Schwarz inequality}) \\
&\geq \left(\frac{\lambda+1}{2}\right)^{n_0} - \sqrt{\frac{\lambda+1}{\lambda-1}\left(\frac{\lambda+1}{2}\right)^{2n_0}}\sqrt{\mathbf{P}(A_1^z(n_0)) + \mathbf{P}(A_2(n_0)) + \mathbf{P}(A_3^z(n_0))} \\
&\geq e^{(\lambda+1)K}\left(1 - \sqrt{\frac{\lambda+1}{\lambda-1}}\sqrt{\frac{1}{K^3} + \varepsilon^2(\lambda+1)^2K^2e^{2(\lambda+1)K} + e^{-c(\lambda)K}}\right).
\end{aligned}$$

□

Remark. *Because we exclude the paths in $A_2(n_0)$ (a global attrition over all $z \in I_0$), the sequence $(N_+''^z(n_0))_{z\in I_0}$ are not independent. We can only have the lower bound rather than the exact value which depends on the order of initial particles $z \in I_0$.*

The number of open sites in H_1 at generation $n_0 = \frac{2K(\lambda+1)}{\lambda-1}$ moved from H_0 through the subordinate discovery process $\{\xi_+''^z(n,\cdot)\}_{0\leq n\leq n_0}$ is

$$N_+''(n_0, H_1) = \sum_{x\in H_1} \mathbf{I}_{\{\sum_z \xi_+''^z(n_0,x)>0\}}.$$

The main result of this section is the following lemma which shows that outside of small probability, the subordinate process $\{\xi_+''^z(n,\cdot)\}_{0\leq n\leq n_0}$ can transfer εN particles from H_i to H_{i+1}. Here it is enough to consider the case when $i = 0$.

Lemma 5.5. *Suppose there are εN occupied sites in $[-N,N]$ noted by I_0 and $n_0 = \frac{2K(\lambda+1)}{\lambda-1}$, then there exist $K>0$ large and $\varepsilon>0$ small enough so that*

$$\mathbf{P}(N''_+(n_0,H_1)\geq \varepsilon N)\geq 1-e^{-\frac{1}{(\lambda+1)K^4}\varepsilon N}.$$

Proof. As we have seen in the proof of Lemma 5.4 that

$$\mu = E[X_+(\mathbf{i}_k)] = \frac{(\lambda-1)(N+1)}{2(\lambda+1)}, 1\leq k\leq n_0.$$

Hence, we choose

$$n_0 = \frac{2K(\lambda+1)}{\lambda-1}.$$

For any $z\in I_0$, $\mathbf{E}\left[S^z_+(\mathbf{i}_{n_0})\right]\in H_1$.

Suppose the sites in I_0 are ranked as $z_1\leq z_2\leq\cdots\leq z_{\varepsilon N}$. $N''_+(n_0,H_1)$ can be coupled as

$$\begin{aligned} N''_+(n_0,H_1) &= \sum_{x\in H_1}\mathbf{I}_{\{\sum_z \xi''^z_+(n_0,x)>0\}}\\ &= \sum_{x\in H_1}\sum_{i=1}^{\varepsilon N} Y^i_x(n_0), \end{aligned}$$

where

$$Y^i_x(n_0) = \begin{cases} 1 & \text{if } Y^j_x(n_0)=0 \text{ for any } j\leq i \text{ and } \xi''^{z_i}_+(n_0,x)>0\\ 0 & \text{otherwise.}\end{cases}$$

Note that we cannot write

$$N''_+(n_0,H_1) = \sum_{z\in I_0}\sum_{x\in H_1}\mathbf{I}_{\{\xi''^z_+(n_0,x)>0\}},$$

which does not exclude the paths belonging to $A_2(n_0)$.

By Corollary 5.1, we have for any $1\leq i\leq \varepsilon N$,

$$\mathbf{E}\left[\sum_{x\in H_1} Y^i_x(n_0)\right] = \mathbf{E}[N''^{z_i}_+(n_0)]\mathbf{P}(S''^{z_i}_+(\mathbf{i}_{n_0})\in H_1).$$

By the local central limit theorem, the last probability above satisfies

$$\sqrt{n_0}\mathbf{P}(S''^{z_i}_+(\mathbf{i}_{n_0})\in H_1)\to\sqrt{\frac{1}{3}-\frac{(\lambda-1)^2}{4(\lambda+1)^2}}\int_{-1}^{1}\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}\,dx,$$

as n_0 tends to infinity. If n_0 is large enough (K large enough), we have

$$\mathbf{P}(S_+^{\prime\prime z_i}(\mathbf{i}_{n_0}) \in H_1) \geq \sqrt{\frac{1}{6n_0}}. \tag{5.6}$$

Hence by Corollary 5.2, for any $1 \leq i \leq \varepsilon N$,

$$\begin{aligned}\mathbf{E}\left[\sum_{x\in H_1} Y_x^i(n_0)\right] &\geq e^{(\lambda+1)K}\sqrt{\frac{\lambda-1}{12K(\lambda+1)}}\left(1-\sqrt{\frac{\lambda+1}{\lambda-1}}\sqrt{\frac{1}{K^3}+\varepsilon^2(\lambda+1)^2K^2e^{2(\lambda+1)K}+e^{-c(\lambda)K}}\right)\\ &\geq \sqrt{\frac{1}{(\lambda+1)K^2}}e^{(\lambda+1)K},\end{aligned} \tag{5.7}$$

if K is large and ε is small enough. By the definition of $A_1^{z_i}(n_0)$, we know for any $1 \leq i \leq \varepsilon N$,

$$\sum_{x\in H_1} Y_x^i(n_0) \leq Ke^{(\lambda+1)K}.$$

The probability we want to prove is

$$\begin{aligned}\mathbf{P}\left(N_+''(n_0, H_1) \leq \varepsilon N\right) &= \mathbf{P}\left(\sum_{i=1}^{\varepsilon N}\sum_{x\in H_1} Y_x^i(n_0) \leq \varepsilon N\right)\\ &= \mathbf{P}\left(\sum_{i=1}^{\varepsilon N}\left(\sum_{x\in H_1} Y_x^i(n_0) - \mathbf{E}\left[\sum_{x\in H_1} Y_x^i(n_0)\right]\right) \leq \varepsilon N - \sum_{i=1}^{\varepsilon N}\mathbf{E}\left[\sum_{x\in H_1} Y_x^i(n_0)\right]\right).\end{aligned}$$

For $1 \leq n \leq \varepsilon N$, take

$$M_+''(n) = \sum_{i=1}^{n}\left(\sum_{x\in H_1} Y_x^i(n_0) - \mathbf{E}\left[\sum_{x\in H_1} Y_x^i(n_0) \mid \mathscr{F}_+''(i-1)\right]\right),$$

where

$$\mathscr{F}_+''(i) = \sigma\left(\sum_{x\in H_1} Y_x^j(n_0), j = 1, \cdots, i\right).$$

We can see that $\{M_+''(n)\}_{1\leq n\leq \varepsilon N}$ is a martingale with $|M_+''(n) - M_+''(n-1)| \leq Ke^{(\lambda+1)K}$ by the definition of $A_1^{z_i}(n_0)$. Since (5.7) does not depend on the order of $1 \leq i \leq \varepsilon N$, (5.7) also holds for $\mathbf{E}\left[\sum_{x\in H_1} Y_x^i(n_0) \mid \mathscr{F}_+''(i-1)\right]$ for any $1 \leq i \leq \varepsilon N$.

By Azuma's inequality,

$$\begin{aligned}
\mathbf{P}\left(N''_+(n_0, H_1) \le \varepsilon N\right) &\le \mathbf{P}\left(M''_+(\varepsilon N) \le \varepsilon N \left(1 - \sqrt{\frac{1}{(\lambda+1)K^2}} e^{(\lambda+1)K}\right)\right) \\
&\le \exp\left(-\frac{\left(\sqrt{\frac{1}{\lambda+1}} \frac{1}{K} e^{(\lambda+1)K} - 1\right)^2 \varepsilon^2 N^2}{\varepsilon N K^2 e^{2(\lambda+1)K}}\right) \\
&= e^{-\frac{1}{(\lambda+1)K^4}\varepsilon N}.
\end{aligned}$$

This finishes the proof of Lemma 5.5. □

We now finish the construction of the first subordinate process on the horizontal layer. Let

$$H_m^n = [(mK-1)N, (mK+1)N] \times \{n\}.$$

Suppose on each horizontal layer n, the bonds discover the horizontal space $\mathbb{Z} \times \{n\}$ following $\xi''_+(k, \cdot)$. After every n_0 steps, $\xi''_+(k, \cdot)$ transfer εN particles from H_m^n to H_{m+1}^n. In the next section, we find another subordinate process to transfer the εN particles from H_m^n to H_m^{n+1}.

5.2 Vertical bond

To show the existence of percolation when $\kappa < C_1$, it is enough to show that for the subordinate process. The subordinate process in the horizontal sense is $\{\xi''^z_+(n, \cdot)\}_{n\ge 0}$ which is dominated by the true horizontal process $\{\hat{\xi}^z(n, \cdot)\}_{n\ge 0}$. Some attritions were introduced in the previous section to ensure an independent structure.

In the previous section, we have already seen how the horizontal subordinate process moves from H_m^n to H_{m+1}^n on a given layer. To investigate the existence of percolation, we also need to analyse the vertical movement from H_m^n to H_m^{n+1}. Several levels of attritions will be introduced to construct the vertical subordinate process which can also guarantee an independent structure.

Consider the square lattice with nearest-neighbour edges $\mathbb{L}^2$ and let $\mathbf{p} = (p_h, p_v) \in [0,1]^2$ be the opening probabilities for horizontal and vertical edges to be open. We will use the following theorem (Theorem 11.115 in Grimmett (1999)]) shown by Kesten.

Theorem 5.2 (Kesten). *Suppose* $\mathbf{p}$ *is such that* $0 < p_h, p_v < 1$. *We have that*

$$\theta(\mathbf{p}) = \begin{cases} = 0 & \text{if } p_h + p_v \le 1 \\ > 0 & \text{if } p_h + p_v > 1, \end{cases}$$

where $\theta(\mathbf{p})$ *is the percolation probability defined on page 2.*

Suppose there are εN open sites in H_m^n. We have shown (Lemma 5.5) that outside of probability $e^{-\frac{1}{(\lambda+1)K^4}\varepsilon N}$, the subordinate process will generate εN open sites in H_{m+1}^n. $1-e^{-\frac{1}{(\lambda+1)K^4}\varepsilon N}$ can be regarded as p_h. In this section, we will find p_v, which depends on N and κ. We will show that there exists $C_1 > 0$ so that when $\kappa < C_1$, $p_h + p_v > 1$, hence percolation occurs.

The vertical movements can be decomposed into two steps. The first step is to transfer M particles from H_m^n to H_m^{n+1}, and the second step is to reproduce these M particles to εN. The event to transfer M particles from H_m^n to H_m^{n+1} is simply

$$\begin{aligned}\mathbf{P}\left(\sum_{i=1}^{\varepsilon N} B_i \geq M\right) &= \sum_{k=M}^{\varepsilon N}\binom{\varepsilon N}{k} e^{-k\kappa N}(1-e^{-\kappa N})^{\varepsilon N-k} \\ &\geq e^{-M\kappa N},\end{aligned} \tag{5.8}$$

where $(B_i)_{1\leq i\leq \varepsilon N}$ is an i.i.d. sequence with distribution Bernoulli$(\epsilon(N)), \epsilon(N) = e^{-\kappa N}$. Let these M sites noted as I_m^{n+1} be the initial condition for the reproduction. The second step is to reproduce these M particles to εN particles in n' generations.

Instead of dealing with the discovery of sites $\hat{\xi}^z$ on layer n, as in the previous section, we construct a subordinate process step by step to finalise the reproduction and guarantee an independent structure.

Separate the horizontal reproduction at layer $n+1$ into $n_0' = \log_2\left(\frac{\varepsilon N}{M}\right)$ big steps. In each big step, we evaluate the probability of reproducing from $2^l M$ to $2^{l+1}M$ particles in H_m^{n+1}, where $0 \leq l \leq n_0'-1$. In one big step, pick $2^l M$ particles in H_m^{n+1}, then these particles reproduce following a subordinate process constructed below, with the number of steps $n_0'' = \frac{\lambda}{\lambda-1}K$. Let $n' = n_0' n_0''$ be the total number of steps.

Remark. *Notice that in the following argument, during each big step $1 \leq l \leq n_0'-1$, some of the particles are killed by attritions (introduced later) and so do their paths. We want to show that $2^l M$ will reproduce more than $2^{l+1}M$ particles with significant probability. The starting points $z_j, j = 1,\cdots,2^l M$ in the following are different in each big step $l = 1,\cdots,n_0'-1$. We abuse the notations $\{z_1,\cdots,z_{2^l M}\}$ for each big step. After each big step l, the initial condition is reset by randomly choosing $2^l M$ particles in H_m^{n+1} for the next big step.*

Without loss of generality, we consider the case when $m = n = 0$. Suppose initially, there are M open sites connected by vertical bonds to the εN open sites in H_0^0. Denote these M particles in H_0^1 as I_0^1. Similar as $A_1^z(n_0)$, the first attrition for the reproduction process is regarding the mass. During each l big step, let $z_1,\cdots,z_{2^l M}$ as the initial condition for the l-th big step and

$$\hat{N}^{z_j}((l+1)n_0'') = \sum_{x\in\mathbb{Z}} \hat{\xi}^{z_j}((l+1)n_0'', x).$$

For $z_1, \cdots, z_{2^l M}$, define

$$\tilde{A}_1^{z_j}(l) = \left\{ \sum_{k=ln_0''+1}^{(l+1)n_0''} \hat{N}^{z_j}((l+1)n_0'') \geq n_0'' K e^{\lambda K} \right\}, 0 \leq l \leq n_0' - 1, 1 \leq j \leq 2^l M.$$

Lemma 5.6. *Suppose $n_0'' = \frac{\lambda}{\lambda - 1} K$ (K suitably chosen to make n_0'' an integer), then*

$$\mathbf{P}(\tilde{A}_1^{z_j}(l)) \leq \frac{1}{K^3}.$$

Proof. It follows similarly as for Lemma 5.2. $\square$

Similarly, as in the previous section, the second attrition comes from the multiple visits to one site. This is a global attrition over the M initial open sites in I_0^1 and over the time period $n' = n_0' n_0''$. Let

$$\tilde{A}_2(n') = \left\{ \exists z, z' \in I_0^1, (z, (\mathbf{i}_n)_{n \geq 0}) \neq (z', (\mathbf{i}_n')_{n \geq 0}) \text{ and } \exists 1 \leq k, k' \leq n', S^z(\mathbf{i}_k) = S^{z'}(\mathbf{i}_{k'}') \right\},$$

where $(\mathbf{i}_n)_{n \geq 0}$ and $(\mathbf{i}_n')_{n \geq 0}$ are particles starting from $z, z' \in I_0^1$

Lemma 5.7. *Suppose $n_0' = \log_2(\varepsilon / M)$ and $n_0'' = \frac{\lambda}{\lambda - 1} K$ are chosen as above. Let K be chosen large so that $K^{-1} < \lambda - 1$ (with K suitably chosen to make n_0', n_0'' integers). If in each big step $1 \leq l \leq n_0'$, and any $z_j, 1 \leq j \leq 2^l M$,*

$$\sum_{k=ln_0''+1}^{(l+1)n_0''} \hat{N}^{z_j}((l+1)n_0'') \leq n_0'' K e^{\lambda K},$$

then

$$\mathbf{P}\left(\tilde{A}_2(n') \mid \bigcap_{l=0}^{n_0'-1} \bigcap_{j=1}^{2^l M} \left(\tilde{A}_1^{z_j}(l) \right)^c \right) \leq \varepsilon^2 \lambda^2 K^6 e^{2\lambda K}.$$

Proof. We can consider this probability in each big step. Denote

$$\tilde{A}^l(n_0'') = \sum_{i=ln_0''+1}^{(l+1)n_0''} \sum_{j=1}^{2^l M} \sum_x \hat{\xi}^{z_j}(i, x)$$

as the number of generated particles in l-th big step, $0 \leq l \leq n_0' - 1$. By the assumption on the cumulative mass,

$$\begin{aligned} \tilde{A}^l(n_0'') &\leq n_0'' 2^l M K e^{\lambda K} \\ &\leq \lambda 2^l M K^3 e^{\lambda K}. \end{aligned}$$

Summing over n_0' big steps gives

$$\begin{aligned}\sum_{l=0}^{n_0'-1} \tilde{A}^l(n_0'') &\le \sum_{l=0}^{n_0'-1} \lambda 2^l M K^3 e^{\lambda K}\\ &\le \lambda K^3 e^{\lambda K} \varepsilon N.\end{aligned}$$

Hence,

$$\mathbf{P}(\tilde{A}_2(n')) \le \varepsilon^2 \lambda^2 K^6 e^{2\lambda K}.$$

□

We now introduce the next level of attritions. In the following argument, when we consider $z_j, 1 \le j \le 2^l M$, that is to consider the initial $2^l M$ particles at big step l.

($\tilde{A}1$) $\{S^{z_j}(\mathbf{i}_k)\}_{l n_0'' < k \le (l+1) n_0''}, 1 \le j \le 2^l M$ do not move outside $\left[-\frac{1}{2}KN, \frac{1}{2}KN\right]$ in n_0'' steps. Let

($\tilde{A}2$) The number of particles generated after each n_0'' steps is less than $Ke^{\lambda K}$. Let

Let

$$\tilde{A}_3^{z_j}(l) = \left\{ -\frac{1}{2}KN \le \inf_{l n_0'' < k \le (l+1) n_0''} S^{z_j}(\mathbf{i}_k) \le \sup_{l n_0'' < k \le (l+1) n_0''} S^{z_j}(\mathbf{i}_k) \le \frac{1}{2}KN \right\}.$$

By similar reason as the proof of Lemma 5.4, we have the following result fo $\tilde{A}_3^{z_j}(l)$.

Lemma 5.8. *Under the conditions of Lemma 5.7, there exists $\tilde{c}(\lambda) > 0$, so that for any $1 \le j \le 2^l M$ in the l-th big step,*

$$\mathbf{P}(\tilde{A}_3^{z_j}(l)) \le e^{-\tilde{c}(\lambda) K},$$

for K and N large enough.

We remove the paths belonging to $A_2(n')$ and in each big step, the paths belonging to $A_1^{z_j}(l)$ and $A_3^{z_j}(l), 1 \le j \le 2^l M, 1 \le l \le n_0'$ to get $\{\tilde{\xi}^{z_j}(n, \cdot)\}_{l n_0'' < n \le (l+1) n_0''}$.

$$\tilde{\xi}^{z_j}(n+1, x) = \begin{cases} 1 & \text{if } \sum_{1 \le j \le 2^l M} \sum_{k \le n} \tilde{\xi}^{z_j}(k, x) = 0 \text{ and } \sum_{y \in \mathcal{N}_n(x)} \tilde{\eta}(n+1, y, x) \ge 1 \\ 0 & \text{otherwise,} \end{cases}$$

where $\mathcal{N}_n(x) = \left\{ y \sim x : \tilde{\xi}^{z_i}(n, y) = 1 \text{ for some } 1 \le i \le 2^l M \right\}$ and

- $\tilde{\eta}(n, y, x) = 0$ if $x \in [-KN/2, KN/2]^c$;
- $\tilde{\eta}(i, y, x) = 0$ for any y, x and $i \ge k$ if $\sum_x \tilde{\xi}^{z_j}(k, x) \ge K e^{\lambda}$ for some $k \le (l+1) n_0''$;
- otherwise, $(\tilde{\eta}(n, y, x))_{n,y,x}$ is an i.i.d. sequence with distribution Bernoulli($\lambda/(2N)$).

Let $\{\tilde{S}^{z_j}(\mathbf{i}_k)\}_{ln_0''<k\le(l+1)n_0''}$ with $\tilde{S}^{z_j}(\mathbf{i}_n) = z_j + \sum_{k=1}^n \tilde{X}(\mathbf{i}_k)$ and $\tilde{X}(\mathbf{i}_k)$ the same distribution as (5.2). We can observe that $\{\tilde{\tilde{\xi}}^{z_j}(n,\cdot)\}_{ln_0''<n\le(l+1)n_0''}$ is subordinated to the true horizontal process $\{\hat{\xi}^{z_j}(n,\cdot)\}_{ln_0''<n\le(l+1)n_0''}$.

Corollary 5.3. *Under the conditions of Lemma 5.7, let*

$$\tilde{N}^{z_j}(l) = \sum_x \mathbf{I}_{\{\tilde{\xi}^{z_j}((l+1)n_0'',x)>0\}},$$

for $1 \le j \le 2^l M$ *and* $0 \le l \le n_0'$*, then*

$$\mathbf{E}[\tilde{N}^{z_j}(l)] \ge e^{\lambda K}\left(1-\sqrt{\frac{\lambda}{\lambda-1}}\sqrt{\frac{1}{K^3}+\varepsilon^2\lambda^2K^6e^{2\lambda K}+e^{-\tilde{c}(\lambda)K}}\right),$$

for K and N large enough.

Proof. We follow the proof of Corollary 5.2 and have

$$\begin{aligned}\mathbf{E}[\tilde{N}^{z_j}(l)] &\ge \lambda^{n_0''}-\sqrt{\frac{\lambda}{\lambda-1}\lambda^{2n_0''}}\sqrt{\frac{1}{K^3}+\varepsilon^2\lambda^2K^6e^{2\lambda K}+e^{-\tilde{c}(\lambda)K}}\\ &\ge e^{\lambda K}\left(1-\sqrt{\frac{\lambda}{\lambda-1}}\sqrt{\frac{1}{K^3}+\varepsilon^2\lambda^2K^6e^{2\lambda K}+e^{-\tilde{c}(\lambda)K}}\right).\end{aligned}$$

□

The number of open sites in H_0^1 at generation $n' = n_0' n_0''$ reproduced by $\{\tilde{\tilde{\xi}}^{z_j}(n,\cdot)\}_{ln_0''<n\le(l+1)n_0''}, 1 \le j \le 2^l M, 0 \le l \le n_0' - 1$ is

$$\tilde{N}(n', H_0^1) = \sum_{x\in H_0^1} \mathbf{I}_{\left\{\sum_{j=1}^{2^{n_0'-1}M}\tilde{\xi}^{z_j}(n',x)>0\right\}},$$

which is the number of particles reproduced in the last big step $n_0' - 1$.

Lemma 5.9. *Suppose there are* εN *open sites in* H_0^0*, then there exist* $n' \in \mathbb{N}$*,* K, M *large so that*

$$\mathbf{P}(\tilde{N}(n', H_0^1) \ge \varepsilon N) \ge \left(1-2e^{-\frac{1}{2\lambda^2K^4}M}\right)e^{-\kappa MN}.$$

If we pick $M = 2\lambda^2K^4$*,*

$$\mathbf{P}(\tilde{N}(n', H_0^1) \ge \varepsilon N) \ge e^{-4\kappa\lambda^2K^4N}.$$

Proof. In the beginning of this section, we have seen (5.8) that outside of probability $e^{-M\kappa N}$, we have M initial particles in H_0^1. In each big step $1 \le l \le n_0' - 1$, where $n_0' = \log(\varepsilon N/M)$, rank the sites as $z_1 < \cdots < z_{2^l M}$. Note that these $2^l M$ open sites are different in each big step

$1 \le l \le n_0'$. $\tilde{N}((l+1)n_0'', H_0^1)$ can be coupled as

$$\begin{aligned}\tilde{N}((l+1)n_0'', H_0^1) &= \sum_{x \in H_0^1} \mathbf{I}_{\left\{\sum_{j=1}^{2^l M} \tilde{\xi}^{z_j}((l+1)n_0'', x) > 0\right\}} \\ &= \sum_{x \in H_0^1} \sum_{i=1}^{2^l M} \tilde{Y}_x^i((l+1)n_0''),\end{aligned}$$

where

$$\tilde{Y}_x^i = \begin{cases} 1 & \text{if } \tilde{Y}_x^j((l+1)n_0'') = 0 \text{ for any } j \le i \text{ and } \tilde{\xi}^{z_i}((l+1)n_0'', x) > 0 \\ 0 & \text{otherwise.} \end{cases}$$

By Lemma 5.1, we have for any $1 \le i \le 2^l M, 0 \le l \le n_0' - 1$,

$$\mathbf{E}\left[\sum_{x \in H_0^1} \tilde{Y}_x^i((l+1)n_0'')\right] = \mathbf{E}[\tilde{N}^{z_i}(l)]\mathbf{P}\left(\tilde{S}^{z_i}(\mathbf{i}_{(l+1)n_0''}) \in H_0^1\right).$$

With the same reason as (5.6), we have

$$P\left(\tilde{S}^{z_i}(\mathbf{i}_{(l+1)n_0''}) \in H_0^1\right) \ge \sqrt{\frac{1}{6n_0''}},$$

for n_0'' large enough (K large enough). Hence by Corollary 5.3,

$$\begin{aligned}\mathbf{E}\left[\sum_{x \in H_0^1} \tilde{Y}_x^i((l+1)n_0'')\right] &\ge e^{\lambda K}\sqrt{\frac{\lambda - 1}{6\lambda K}}\left(1 - \sqrt{\frac{\lambda}{\lambda - 1}}\sqrt{\frac{1}{K} + \varepsilon^2\lambda^2 K^6 e^{2\lambda K} + e^{-\tilde{c}(\lambda)K}}\right) \\ &\ge \sqrt{\frac{1}{2\lambda K^2}} e^{\lambda K},\end{aligned}$$

if K is large and ε is small enough. By the definition of $\tilde{A}_1^{z_i}(l)$, we know that for any $1 \le i \le 2^l M, 0 \le l \le n_0' - 1$,

$$\sum_{x \in H_0^1} \tilde{Y}_x^i((l+1)n_0'') \le K e^{\lambda K}.$$

For $1 \le i \le 2^l M$ in the i-th big step, take

$$\tilde{M}(i) = \sum_{j=1}^{i}\left(\sum_{x \in H_0^1} \tilde{Y}_x^j((l+1)n_0'') - \mathbf{E}\left[\sum_{x \in H_0^1} \tilde{Y}_x^j((l+1)n_0'') \mid \tilde{\mathscr{F}}(j-1)\right]\right),$$

where

$$\tilde{\mathscr{F}}(j) = \sigma\left(\sum_{x \in H_0^1} \tilde{Y}_x^k((l+1)n_0''), k = 1, \cdots, j\right).$$

Note that $\{\tilde{M}(i)\}_{1 \le i \le 2^l M}$ and $\{\tilde{\mathscr{F}}(i)\}_{1 \le i \le 2^l M}$ are different in each big step l. We can see that

$\{\tilde{M}(i)\}_{1\le i\le 2^l M}$ is a martingale with $|\tilde{M}(i)-\tilde{M}(i-1)|\le Ke^{\lambda K}$ by the definition of $\tilde{A}^{z_j}(l), 1\le j\le 2^l M$. Since the lower bound of $\mathbf{E}\left[\sum_{x\in H_0^1}\tilde{Y}_x^j((l+1)n_0'')\right]$ does not depend on the choice of $1\le j\le 2^l M$, this lower bound also holds for $\mathbf{E}\left[\sum_{x\in H_0^1}\tilde{Y}_x^j((l+1)n_0'')\mid\tilde{\mathscr{F}}(j-1)\right]$ for $1\le j\le 2^l M, 0\le l\le n_0'-1$.

By Azuma's inequality, for the l-th big step

$$\begin{aligned}\mathbf{P}\left(\sum_{i=1}^{2^l M}\sum_x \tilde{Y}_x^i((l+1)n_0'')\le 2^{l+1}M\right) &\le \mathbf{P}\left(\tilde{M}((l+1)n_0'')\le 2^l M\left(2-\sqrt{\frac{1}{2\lambda K^2}}e^{\lambda K}\right)\right)\\ &\le \exp\left(-\frac{\left(2-\sqrt{\frac{1}{2\lambda K^2}}e^{\lambda K}\right)^2(2^l M)^2}{2^l M K^2 e^{2\lambda K}}\right)\\ &\le e^{-\frac{1}{2\lambda K^4}2^l M}.\end{aligned}$$

Over the n_0' big steps, we have

$$\begin{aligned}\mathbf{P}(\tilde{N}(n',H_0^1)\ge\varepsilon N) &\ge \prod_{k=0}^{n_0'-1}\left(1-e^{-\frac{1}{2\lambda K^4}2^l M}\right)\\ &\ge 1-2e^{-\frac{1}{2\lambda^2 K^4}M}.\end{aligned}$$

Combining with (5.8) finishes the proof. □

We now finish the construction of the second subordinate process in the vertical sense. With the help of Lemma 5.5 and Lemma 5.9, we now show Theorem 2.5 by renormalization argument.

Proof of Theorem 2.5. We want to compare our original percolation system with the percolation on the inhomogeneous square lattice $(\mathbb{L}^2,\mathbf{p}=(p_h,p_v))$. Let $\mathscr{L}=\{(m,n)\in\mathbb{Z}\times\mathbb{Z}\}$. $\mathscr{L}$ is made into a graph $(\mathscr{V},\mathscr{E})$ by drawing oriented edges from (m,n) to $(m\pm1,n)$ or $(m,n\pm1)$. Random variables $\psi(e)\in\{0,1\}$ are to indicate the bonds e are open ($\psi(e)=1$) or closed ($\psi(e)=0$). We say that there is a path from (m,n) to (m',n') denoted by $(m,n)\to(m',n')$ if there is l and $(x_j,y_j), 1\le j\le l$ so that $(x_1,y_1)=(m,n),(x_l,y_l)=(m',n')$ and $\langle(x_{j-1},y_{j-1}),(x_j,y_j)\rangle\in\mathscr{E}$ for any $1\le j\le l$. Let

$$\mathbf{C}_0=\{(m,n):(0,0)\to(m,n)\}$$

be the cluster containing the origin.

Definition 5.1. $H_m^n=[(mK-1)N,(mK+1)N]\times\{n\}$ *is* (ε,N)*-good if there are* εN *open sites in* H_m^n*.*

The renormalized site $(m,n)\in\mathscr{L}$ corresponds to the block H_m^n. $\omega(m,n)=1$ if H_m^n is (ε,N)-good. We explore the edges in $\mathscr{L}$ via a discovery algorithm. Initially, H_0^0 is (ε,N)-good.

Horizontally, H_0^0 is connected to H_1^0 denoted by $H_0^0 \to H_1^0$ if the horizontal subordinate process $\{\xi_+^z(n,\cdot)\}_{0\le n\le n_0}$ (with εN sites randomly chosen in H_0^0) transfer εN particles to H_1^0. Similarly, we can define $H_0^0 \to H_{-1}^0$ if the horizontal subordinate process $\{\xi_-^z(n,\cdot)\}_{0\le n\le n_0}$ transfer εN particles from H_0^0 to H_{-1}^0. Inductively, we can define $H_m^n \to H_{m\pm1}^n$. For horizontal edges in the renormalized graph $e_h = \langle (m,n),(m\pm1,n)\rangle \in \mathscr{E}_h$, the state of edges $\psi(e_h) = 1$ if the horizontal subordinate process transfer εN particles from H_m^n to $H_{m\pm1}^n$ ($H_m^n \to H_{m\pm1}^n$) so that both H_m^n and $H_{m\pm1}^n$ are (ε, N)-good and $\psi(e_h) = 0$ otherwise. By Lemma 5.5, we have

$$p_h = \mathbf{P}(\psi(e_h) = 1) \ge 1 - e^{-\frac{1}{(\lambda+1)K^4}\varepsilon N}.$$

Vertically, H_0^0 is connected to H_0^1 denoted by $H_0^0 \to H_0^1$ if $M = 2\lambda^2 K^4$ (K suitably chosen to make M an integer) vertical edges in the original graph ($\langle (x,0),(x,1)\rangle, x \in H_0^0$) are open (with probability $e^{-\kappa N}$) and these M particles reproduce to εN particles in H_0^1 following the subordinate process $\{\tilde{\xi}^z(n,\cdot)\}_{0\le n\le n'}$ (the sites of these M particles do not matter). Similarly, we can define $H_0^0 \to H_0^{-1}$ and inductively we can define $H_m^n \to H_m^{n\pm1}$. For vertical edges in the renormalized graph $e_v = \langle (m,n),(m,n\pm1)\rangle \in \mathscr{E}_v$, the state of edge $\psi(e_v) = 1$ if first $M = 2\lambda K^4$ particles are transferred from H_m^n to $H_m^{n\pm1}$ and these M particles reproduce following $\{\tilde{\xi}^z(n,\cdot)\}_{0\le n\le n'}$ in $H_m^{n\pm1}$ so that both H_m^n and $H_m^{n\pm1}$ are (ε,N)-good and $\psi(e_v) = 0$ otherwise. By Lemma 5.9, we have

$$p_v = \mathbf{P}(\psi(e_v) = 1) \ge e^{-4\kappa\lambda^2 K^4 N}.$$

Notice that H_m^n can be (ε,N)-good with two possibilities, when H_{m-1}^n is (ε,N)-good and the horizontal subordinate process ξ_+'' transfer εN particles from H_{m-1}^n, when H_{m+1}^n is (ε,N)-good and the horizontal subordinate process ξ_-'' transfers εN particles from H_{m+1}^n or when H_m^{n-1} is (ε,N)-good and the open vertical edges transfer M particles from H_m^{n-1} to H_m^n and then reproduction following $\tilde{\xi}$ happens in H_m^n. In this case, we will only inherit the εN particles from H_{m-1}^n. This mechanism is shown in Figure 5.3.

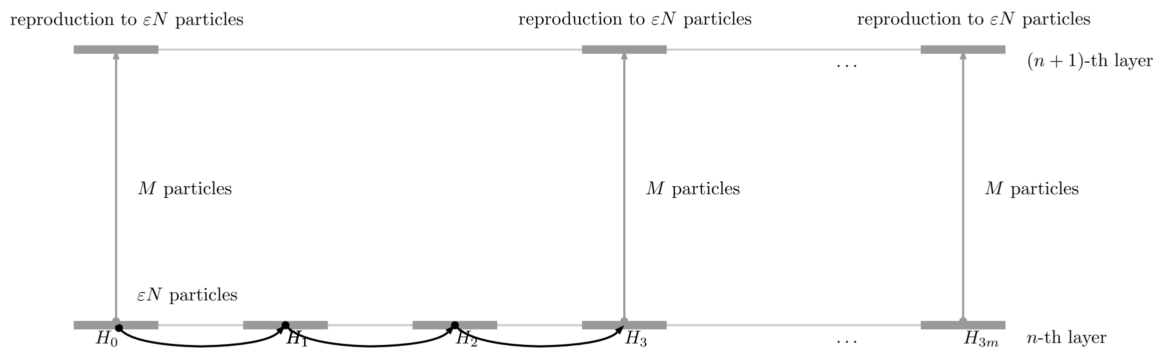

Figure 5.3 – Renormalized blocks H_{3m}^n

On each horizontal layer n, the processes starting from H_m^n and H_{m+1}^n, H_{m+2}^n are not independent. The random variables $(\psi(e_h))_{e_h\in\mathscr{E}_h}$ has joint law μ_h such that $\mu_h(\psi(e_h) = 1) = p_h$ and the connections between (m,n) and (m',n) follow the horizontal discovery algorithm above. By the definition of $A_3^z(n_0)$, (m,n) and (m',n) can be joined only if $|m-m'| \le 2$. However thanks to the argument of Liggett et al. (1997), we can find a product measure π_{ρ_h} on $\mathscr{E}_h$ with $\pi_{\rho_h}(\psi(e_h) = 1) = \rho_h$ and $(\psi(e_h))_{e_h \in \mathscr{E}_h}$ being independent so that π_{ρ_h} is dominated by μ_h.

Let $\mathbb{Z}_k$ be the graph with vertex set $\mathbb{Z}$ in which two vertices m, m' are joined by edge if $|m - m'| \leq k$. So $\mathbb{Z}_k(p)$ consists of k-dependent measures on subsets of $\mathbb{Z}$ of density at least p.

Theorem 5.3 (Theorem 1.5 of Liggett et al. (1997))**.** *Let $p, \alpha, r \in [0,1]$, let $k \geq 0$ and let $q = 1 - p$. Suppose that*

$$(1-\alpha)(1-r)^k \geq q$$

and

$$(1-\alpha)\alpha^k \geq q.$$

If $\mu \in \mathbb{Z}_k(p)$, then $\pi_{\alpha r} \preceq \mu$. In particular, if $q \leq k^k/(k+1)^{k+1}$, then $\pi_\rho \preceq \mu$, where

$$\rho = \left(1 - \frac{q^{1/(k+1)}}{k^{k/(k+1)}}\right)\left(1 - (qk)^{1/(k+1)}\right).$$

In our case $k = 2$ and $q \leq e^{-\frac{1}{(\lambda+1)K^4}\varepsilon N}$. By Theorem 5.3, the measure μ_h on each horizontal layer $\mathbb{Z} \times \{n\}$ stochastically dominates the product measure π_{ρ_h} with

$$\rho_h \geq 1 - e^{-\frac{1}{4(\lambda+1)K^4}\varepsilon N}.$$

Moreover,

$$\rho_h + p_v \geq 1 - e^{-\frac{1}{4(\lambda+1)K^4}\varepsilon N} + e^{-4\kappa\lambda^2 K^4 N}.$$

If we choose κ small enough compared to εK^{-8}, percolation occurs by Theorem 5.2.

The other direction is easy to show. Let $I_k = [kN, (k+1)N] \cap \mathbb{Z}, k \in \mathbb{Z}$. The probability that no edges generated from I_k (in 1 step) is

$$\begin{aligned} q_N &= \mathbf{P}\left(\sum_{z \in \mathbf{i}_k} N^z(1) = 0\right) \\ &\geq \left(1 - \frac{\lambda}{2N}\right)^{2N \cdot N} \\ &\geq e^{-\lambda N}. \end{aligned} \tag{5.9}$$

Let $\mathscr{N}_i$ be the total number of occupied sites at layer i.

$$\begin{aligned} \mathbf{E}[\mathscr{N}_0] &\leq \sum_{k=0} (k+1)N(1-q_N)^k q_N \\ &\leq \frac{N}{q_N} \\ &= Ne^{\lambda N}. \end{aligned} \tag{5.10}$$

If $\kappa > \lambda$, $\mathbf{E}[\mathscr{N}_i] \cdot e^{-\kappa N} < 1$ and there is not percolation. This concludes Theorem 2.5. $\square$

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