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Invariant Integrals on Topological Groups and Applications

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par

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"O frati," dissi, "che per cento milia
perigli siete giunti a l'occidente,
a questa tanto picciola vigilia

d'i nostri sensi ch'è del rimanente
non vogliate negar l'esperïenza,
di retro al sol, del mondo sanza gente.

Considerate la vostra semenza:
fatti non foste a viver come bruti,
ma per seguir virtute e canoscenza."

DANTE, *Inferno*,
canto XXVI, vv. 112-120

Abstract

We define and study a fixed-point property for linear representations of uniform topological groups on weakly complete proper cones in locally convex vector spaces. To this end, we translate this fixed-point property into a functional analysis framework using a new class of ordered vector spaces and functionals developed expressly for the occasion.

The specific case of uniform locally compact groups is investigated in-depth employing harmonic analysis theory.

The hereditary properties of the class of groups with the fixed-point property for cones are examined by applying the functional perspective in both the uniform topological case and the uniform locally compact case.

Finally, we consider applications to invariant Radon measures on locally compact spaces and operator algebras.

Keywords - topological group, uniform structure, ordered vector space, normed Riesz space, convex cone, amenability, fixed-point property.

Astratto

Definiamo e studiamo una proprietà del punto fisso per rappresentazioni lineari di gruppi topologici uniformi su coni propri debolmente completi in spazi vettoriali localmente convessi. A tal fine, traduciamo questa proprietà del punto fisso in termini di analisi funzionale utilizzando una nuova classe di spazi vettoriali ordinati e di funzionali sviluppate espressamente per l'occasione.

Il caso specifico dei gruppi localmente compatti uniformi viene studiato approfonditamente utilizzando la teoria dell' analisi armonica.

Le proprietà ereditarie della classe dei gruppi con la proprietà del punto fisso per i coni sono esaminate applicando metodi di analisi funzionale sia nel caso topologico uniforme che nel caso localmente compatto uniforme.

Infine, consideriamo applicazioni alle misure invarianti di Radon su spazi localmente compatti e alle algebre di operatori.

Parole chiave - gruppo topologico, struttura uniforme, spazio vettoriale ordinato, spazio normato di Riesz, cono convesso, amenabilità, proprietà del punto fisso.

Abstrakt

Wir definieren und untersuchen eine Fixpunkteigenschaft für lineare Darstellungen von uniformen topologischen Gruppen auf schwach vollständige Eigenkegeln in lokalkonvexen Vektorräumen. Zu diesem Zweck übersetzen wir diese Fixpunkteigenschaft in Begriffe der Funktionalanalysis unter Verwendung einer neuen Klasse von geordneten Vektorräumen und Funktionals, die ausdrücklich für diesen Anlass entwickelt wurden.

Der spezielle Fall von lokalkompakten uniformen Gruppen wird mit Hilfe der Theorie der harmonischen Analyse eingehend untersucht.

Die hereditären Eigenschaften der Klasse der Gruppen mit der Fixpunkteigenschaft für Kegel werden durch Anwendung von Methoden der Funktionalanalysis sowohl im uniformen topologischen Fall als auch im lokal kompakten uniformen Fall untersucht.

Schließlich betrachten wir Anwendungen an invarianten Radonmaßen auf lokalkompakte Räume und auf Operatoralgebren.

Stichwörter - topologische Gruppe, uniforme Struktur, geordneter Vektorraum, normierter Riesz-Raum, konvexer Kegel, Mittelbarkeit, Fixpunkteigenschaft.

Résumé

Nous définissons et étudions une propriété de point fixe pour les représentations linéaires de groupes topologiques uniformes sur des cônes propres faiblement complets dans des espaces vectoriels localement convexes. Pour cela, nous traduisons cette propriété de point fixe en termes d'analyse fonctionnelle en utilisant une nouvelle classe d'espaces vectoriels ordonnés et de fonctionelles développée expressément pour l'occasion.

Le cas spécifique des groupes uniformes localement compacts est étudié en détail en utilisant la théorie de l'analyse harmonique.

Les propriétés de stabilité de la classe des groupes ayant la propriété de point fixe pour les cônes sont examinées en appliquant des méthodes d'analyse fonctionnelle dans le cas topologique uniforme et dans le cas uniforme localement compact.

Enfin, nous considérons des applications aux mesures invariantes de Radon sur les espaces localement compacts et aux algèbres d'opérateurs.

Mots-clés - Groupe topologique, structure uniforme, espace vectoriel ordonné, espace de Riesz normé, cône convexe, moyennabilité, propriété de point fixe.

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Cuoira, October 15, 2021

Contents

Introduction

In [M17] Monod introduced a fixed-point property for discrete (abstract) groups acting on weakly complete proper convex cones in locally convex vector spaces. He provided different characterizations of this fixed-point property and studied the class of groups satisfying it, in particular, pointing out that not all discrete groups have the fixed-point property for cones, but those that display interesting functional and hereditary properties. One of the most exciting features of Monod's fixed-point property is that it is strictly related to the notion of amenability. Precisely, a discrete group satisfying this fixed-point property is supramenable, and hence amenable.

Our research work began from the following natural question: is Monod's fixed-point property for cones generalizable to topological groups?

Before we begin investigating this question, we should ask ourselves whether it is well-posed. The following two points convince us that this is the case:

(I) It is possible to define a good notion of amenability for topological groups;

(II) In 1976, Jenkins defined in [J76] a fixed-point property (called *property F*) for locally compact groups while studying locally compact groups of subexponential growth. It turned out that Monod and Jenkins' fixed-point properties are equivalent when considering discrete groups, see [M17, Subsection 10.C].

Therefore, we immersed ourselves in a world shaped by cones and groups acting on them, with the aim of developing a theory that could unify the works of Monod and Jenkins.

Regarding point (I), the theory of amenability has been an essential source of inspiration. Indeed, part of our research consists of generalising the main results of amenability theory for a distinct set of amenable groups (those with the fixed-point property for cones). Although amenability for topological groups is generally well-understood, when it comes to non-locally compact topological groups, the theory is not well-defined in the sense that there are no uniform notations and conventions. For this reason, Chapter 1 is dedicated to recalling and defining the theory of amenability in the generality required for our purposes (precisely Subsection 1.4.B).

Our first task is to look for as general as possible definition of the fixed-point property for cones since *what is essential is invisible to the eye*.[1] Indeed, in the property defined

[1] *Le Petit Prince* of Antoine de Saint-Exupéry.

in [M17] the discreteness of the groups annihilated many essential details. However, that is not the only motivation: by abstracting a property, we may appreciate its essence and thus better understand how to study it. Each component of the definition we are about to give is included for wholly unavoidable reasons, which will be discussed in depth in Chapter 5.

Definition (5.1.4)**.** Let G be a topological group and let $\mathscr{U}$ be a functionally invariant uniformity for G. We say that G has the **$\mathscr{U}$-fixed-point property for cones** if every representation of G on a non-empty weakly complete proper cone C in a (Hausdorff) locally convex vector space E which is locally bounded $(\mathscr{U}, \mathscr{U}_c)$-uniformly continuous and of cobounded type has a non-zero fixed-point.

Fixed points in compact subsets of vector spaces are often associated with invariant functionals, precisely invariant means, on function spaces. To mention only a few examples of this beautiful friendship we have: the Markov-Kakutani fixed-point Theorem and invariant means on bounded functions for abelian groups ([Bou81, IV §1 No.1 Théorème 1]); the Rickert-Day fixed-point Theorem and invariant means for amenable topological groups ([D61, Theorem 1]); the Ryll–Nardzewski fixed-point Theorem and invariant means on weakly almost periodic functions on topological groups ([G69, §3.1]) and also *the common fixed-point property on compacta* and extremely amenable topological groups ([M66, Theorem 1]). Bearing these examples in mind, it is reasonable to describe a fixed point in a cone through an invariant functional. The functional that fulfils this duty is called an invariant normalised integral.

Definition (4.1.4 & 4.1.8)**.** Let E be an ordered vector space and let G be a topological group. Suppose that G has a representation π on E by positive linear automorphisms and let $d \in E$ be a non-zero positive vector. An **invariant normalized integral** is a positive functional I defined on the vector space

$$(E, d) = \left\{ v \in E : \pm v \leq \sum_{j=1}^{n} \pi(g_j) d \text{ for some } g_1, ..., g_n \in G \right\}$$

such that $\mathrm{I}(d) = 1$ and $\mathrm{I}(\pi(g)v) = \mathrm{I}(v)$ for every $g \in G$ and $v \in E$. We say that G, or π, **has the invariant normalized integral property for** E if for every non-zero positive $d \in E$ there exists an invariant normalized integral on the space (E, d).

With this definition in mind, it is possible to give several characterizations of the aforementioned fixed-point property for cones.

Theorem (5.2.1)**.** *Let G be a topological group and let $\mathscr{U}$ be a functionally invariant uniformity for G. Then the following assertions are equivalent:*

a) the group G has the $\mathscr{U}$-fixed-point property for cones;

b) the group G has the invariant normalized integral property for $C_u^b(G, \mathscr{U})$;

c) *for every action γ of G on a compact space K such that $\mathcal{C}(K)$ is π_γ-invariant and for every non-zero positive $\phi \in \mathcal{C}(K)$ for which there is $k_0 \in K$ such that $\phi(k_0) \neq 0$ and the map $g \longmapsto gk_0$ is $\mathscr{U}$-uniformly continuous, there is an invariant normalized integral on $\mathcal{C}(K, \phi)$;*

d) *for every action γ of G on a uniform space $(X, \mathscr{U}_X)$ such that $\mathcal{C}_u^b(X, \mathscr{U}_X)$ is π_γ-invariant and for every non-zero positive $f \in \mathcal{C}_u^b(X, \mathscr{U}_X)$ for which there is $x_0 \in X$ such that $f(x_0) \neq 0$ and the map $g \longmapsto gx_0$ is $(\mathscr{U}, \mathscr{U}_X)$-uniformly continuous, there is an invariant normalized integral on $\big(\mathcal{C}_u^b(X, \mathscr{U}_X), f\big)$.*

Unfortunately, most of the tools and techniques used for handling convex compact sets are useless for handling convex cones. Working with cones is much more difficult (and less pleasant) than working with compacts. However, do not panic. Cones also bring gifts and not just coal. Indeed, a (proper) convex cone always defines a vector ordering on the vector space in which it lives, allowing us to apply the well-developed theory of ordered vector spaces. But this is still not enough for our goals. For this reason, part of our work focus on developing a particular class of ordered vector spaces and positive functionals on them, see Chapters 3 and 4. The theory produced in these two chapters opens up two different doors for us.

Firstly, we can give a functional characterization of the fixed-point property for cones. Note that similar characterizations were also studied and widely used in the case of amenability. See, for example, the work of Zimmer [Z84, Chapter 4].

Theorem (5.3.1)**.** *Let G be a topological group and let $\mathscr{U}$ be a functionally invariant uniformity for G. Then the following assertions are equivalent:*

a) *the group G has the $\mathscr{U}$-fixed-point property for cones;*

b) *every representation π of G on a normed Riesz space E such that E is G-dominated and π^* is locally bounded $(\mathscr{U}, \mathscr{U}_c^*)$-uniformly continuous admits an invariant normalized integral;*

c) *every representation π of G on a Banach lattice E such that E is asymptotically G-dominated and π^* is locally bounded $(\mathscr{U}, \mathscr{U}_c^*)$-uniformly continuous admits an invariant normalized integral.*

Secondly, we had the opportunity to understand better the invariant integral property and, hence, the fixed-point property for cones for locally compact groups. Inspired by the famous monograph of Greenleaf ([G69]), we were able to show that for a locally compact group, the invariant normalized integral properties on the classical Banach lattices are all equivalent.

Theorem (6.3.4)**.** *Let G be a locally compact group. If G has the invariant normalized integral property for one of the following Banach lattices*

$$L^\infty(G),\ \mathcal{C}^b(G),\ \mathcal{C}_{lu}^b(G),\ \mathcal{C}_{ru}^b(G) \text{ or } \mathcal{C}_u^b(G),$$

then G has the invariant normalized integral property for all the others.

This last theorem generalizes the famous result [G69, Theorem 2.2.1] stating the equivalence of invariant means for the above classical Banach spaces when considering amenable groups (see Remark 6.3.5).

In parallel with the investigation on the fixed-point property for cones, we were also interested in another functional property related to it, the so-called translate property. The translate property was originally a condition for non-zero positive bounded functions on groups which was first considered implicitly in a question posed by Greenleaf in [G69, §1.3.].[2] However, the first to name it and describe it in the form we know it today was Rosenblatt in his doctoral thesis ([R72]) while he was studying supramenability.

Paraphrasing, the original question posed by Greenleaf was: given a representation π of a group G on an ordered vector space E and a non-zero positive vector $d \in E$, does a non-zero invariant functional on the subspace $\operatorname{span}_{\mathbf{R}}(Gd)$ imply an invariant normalized integral one on the space (E, d)? The essential information to ensure a non-zero invariant functional on $\operatorname{span}_{\mathbf{R}}(Gd)$ is encoded in the following property of d, which is precisely the translate property.

Definition (4.2.1 & 4.2.7)**.** Let E be an ordered vector space and let G be a topological group. Suppose that G has a representation π on E by positive linear automorphisms. Then a non-zero positive vector $d \in E$ has the **translate property** if

$$\sum_{j=1}^{n} t_j \pi(g_j) d \geq 0 \quad \text{implies that} \quad \sum_{j=1}^{n} t_j \geq 0 \quad \text{for every } t_1, ..., t_n \in \mathbf{R} \text{ and } g_1, ..., g_n \in G.$$

We say that G **has the translate property for** E if every non-zero positive vector $d \in E$ has the translate property.

Therefore, Greenleaf's question can be rephrased in the following way: given a representation of a group G on an ordered vector space E and a non-zero positive vector $d \in E$, does the translate property for d imply the existence of an invariant normalized integral on (E, d)?

One of the first who studied this question was Rosenblatt, who was able to give a partial answer in his doctoral thesis ([R72, Corollary 1.3]) for discrete groups in the case $E = \ell^\infty(G)$ but only for vectors of the form $\mathbf{1}_A$, for some $A \subset G$. A complete answer for this case was finally given by Monod ([M17, Corollary 19] and [M17, Corollary 20]).

We studied this problem in Section 4.2, where we provided partial answers for topological groups and arbitrary ordered vector spaces (Propositions 4.2.20 and 4.2.21). A complete and exhaustive solution for locally compact groups is presented in Chapter 6. Indeed,

Theorem (6.3.8)**.** *Let G be a locally compact group. If G has the translate property for $C^b_{ru}(G)$, then G has the invariant normalized integral property for $C^b_{ru}(G)$.*

[2] Actually, we do not know if Greenleaf was the first to be interested in this problem.

Moreover, in the locally compact case, we answered Greenleaf's question not only for the space $\mathcal{C}^b_{ru}(G)$ but for all the classical ones.

Theorem (6.3.10)**.** *Let G be a locally compact group and let E be one of the following Banach lattices*

$$L^\infty(G),\ \mathcal{C}^b(G),\ \mathcal{C}^b_{ru}(G),\ \mathcal{C}^b_{lu}(G) \text{ or } \mathcal{C}^b_u(G).$$

Then G has the translate property for E if and only if G has the invariant normalized integral property for E.

Combining these last two theorems, we could also show that the translate properties on the classical Banach lattices are all equivalent.

Theorem (6.3.11)**.** *Let G be a locally compact group. If G has the translate property for one of the following Banach lattices*

$$L^\infty(G),\ \mathcal{C}^b(G),\ \mathcal{C}^b_{lu}(G),\ \mathcal{C}^b_{ru}(G) \text{ or } \mathcal{C}^b_u(G),$$

then G has the translate property for all the others.

Furthermore, we studied the relationship between means and the translate property in Section 4.2.B, where we showed that the translate property implies the existence of localized means on Banach lattices with an order unit (Theorem 4.2.11). In particular, the translate property for $\mathcal{C}^b_{ru}(G)$ implies amenability (Corollary 4.2.13).

The reason to spend so much time investigating the translate property can be understood by looking at Chapter 7.

In this chapter, we studied the hereditary property of the fixed-point property. First, we looked at topological groups, and we showed that the class of topological groups with the fixed-point property for cones is closed under continuous epimorphisms (Corollary 7.1.3), quotients (Corollary 7.1.5), open and dense subgroups (Propositions 7.1.6 and 7.1.10) and extensions by or of finite groups (Propositions 7.1.7 and 7.1.9). Nevertheless, the best results are in the locally compact case as here, the fixed-point property for cones is equivalent to the translate property. Indeed, working with the translate property is easier because we do not need control on the whole space (and its dual) but only on a *small* portion of it. Thanks to that, we could prove that the class of locally compact groups with the fixed-point property for cones is closed under closed subgroups (Theorem 7.2.1), directed limits (Theorem 7.2.2), Cartesian product with a group of subexponential growth (Theorem 7.1.18), extensions of compact groups (Proposition 7.1.8) and (discrete) central extensions (Theorem 7.2.7). Moreover, it contains the class of locally compact groups of subexponential growth (Corollary 7.2.3).

Finally, in the last (short) chapter, a couple of easy but interesting applications of the fixed-point property for cones are presented. In particular, the fixed-point property for cones is the correct property to use when studying problems where a non-zero invariant

Radon measure is required (Theorem 8.1.3). This will lead us to assert that each locally compact group with the fixed-point property for cones is unimodular (Corollary 8.2.1) and to slightly generalize a well-known theorem for finitely generated orderable groups (Theorem 8.3.1).

The last word: in this manuscript, there will be at least one error (mathematical or otherwise). We take full responsibility for all such mistakes. We have tried to remain as clear and simple as possible without ever losing the essence of maths.

"Perhaps we should discard the myth that mathematics is a rigorously deductive enterprise. It may be more deductive than other sciences, but hand-waving is intrinsic."

Desperately Seeking Mathematical Proof
M.B. NATHANSON [N09]

Organization

We give a brief overview of the manuscript structure.

The first chapters are dedicated to recalling essential basics. Chapter 1 regards the basics of topological groups and uniform structures, while Chapter 2 concerns cones and ordered vector spaces.

In Chapter 3 we develop the theory of dominating and asymptotically dominating spaces, and in Chapter 4 we deal with positive functionals on it. We introduce the concepts of invariant normalized integral and the translate property.

In Chapter 5 we state the fixed-point property for cones, and we begin to study it in the case of topological groups. In Chapter 6, we focus only on locally compact groups.

The goal of Chapter 7 is to understand the hereditary properties of the fixed-point property for cones and of the translate property.

Lastly, in Chapter 8, we give some applications of the fixed-point property for cones concerning invariant Radon measures.

The manuscript has two appendices. In the first, we discuss the problem of embedding an abstract cone into a vector space, while in the second, we see how the problems treated in the thesis could also be solved using an operator algebra approach.

General conventions and notations

The capital letter G always denote a group with some specified topology. An abstract group is nothing but a group endowed with the discrete topology. We consider only actions of groups and never anti-actions.

Every vector space is real. The notation E' and E^* are used for the topological dual, the set of all continuous linear functionals on E with respect to some given topology, and the algebraic dual, the set of all linear functionals on E, respectively. The completion of E with respect to some given uniformity, or some given topology, is written $\widehat{E}$.

Every locally compact topology and every locally convex topology is assumed to be Hausdorff.

Chapter 1

Groups, Topologies & Uniformities

The following is the first of two chapters dedicated to recapitulations. The goal here is to repeat basic concepts and results about uniform spaces, function spaces and (invariant) means.

We start by discussing general function spaces. Then we pass to uniform structures on sets and uniformly continuous functions. In particular, we are interested in uniform structures on topological groups. After, we repeat the notion of mean and use it to define uniform amenability. Finally, we conclude the chapter by giving different characterizations of amenability for uniform groups.

An essential prerequisite for this chapter is the theory of topological groups. Good references to refresh it are [Bou71, III §1 & §2], [HR63, Chapter Two] and [AT08, Chapter 3].

1.1 About function spaces

Let X be a set and let E be a real vector space. We define the set

$$E^X = \{f : X \longrightarrow E : f \text{ is a function}\}.$$

For two functions $f, h \in E^X$, their sum $f + h \in E^X$ is defined pointwise as the function given by

$$(f + h)(x) = f(x) + h(x) \quad \text{for every } x \in X.$$

We say that $f + h$ is the **pointwise sum**, or the **pointwise addition**, of f and h. Similarly, the multiplication between a function $f \in E^X$ and a scalar $\alpha \in \mathbf{R}$ is defined as the function $\alpha f \in E^X$ given pointwise by

$$(\alpha f)(x) = \alpha f(x) \quad \text{for every } x \in X.$$

We say that αf is the **pointwise scalar multiplication** between α and f.

By definition, the set E^X equipped with the pointwise addition and scalar multiplication is a vector space.

Definition 1.1.1. Let X be a set and let E be a vector space. A subset $\mathfrak{F}(X,E)$ of E^X is called a **vector-valued function space**, if it is a vector subspace of E^X under pointwise addition and scalar multiplication. If $E = \mathbf{R}$, then we only write $\mathfrak{F}(X)$ instead of $\mathfrak{F}(X,\mathbf{R})$,and we say that $\mathfrak{F}(X)$ is a **real function space**.

Suppose from now on that E is a locally convex vector space. Then a subset V of E is said **bounded**, if it is absorbed by every neighborhood of the origin, i.e., for every neighborhood of the origin U there is $\alpha > 0$ such that $V \subset \alpha U$. Here, $\alpha U = \{\alpha v : v \in U\}$. A function $f \in E^X$ is said a **bounded function**, if its image $im(f)$ is a bounded subset of E. If $\mathfrak{F}(X,E)$ is a vector-valued function space, then the set of all functions of $\mathfrak{F}(X,E)$ that are bounded is denoted by $\mathfrak{F}^b(X,E)$.

Proposition 1.1.2. *Let X be a set and let E be a locally convex vector space. Suppose that $\mathfrak{F}(X,E)$ is a vector-valued function space. Then, $\mathfrak{F}^b(X,E)$ is a vector subspace of $\mathfrak{F}(X,E)$.*

We recall that the sum of two sets A and B in a vector space E is defined as

$$A + B = \{v \in E : v = a + b \text{ for some } a \in A \text{ and } b \in B\}.$$

Proof of Proposition 1.1.2. Let f_1 and f_2 be two functions in $\mathfrak{F}^b(X,E)$. We want to show that their sum is still a bounded function. Let U be a neighborhood of the origin of E. We can suppose that U is a convex set, since E is a locally convex space. Then there are $\alpha_1 > 0$ and $\alpha_2 > 0$ such that $im(f_1) \subset \alpha_1 U$ and $im(f_2) \subset \alpha_2 U$. Set $\alpha = \alpha_1 + \alpha_2$. We claim that $im(f_1 + f_2) \subset \alpha U$. Indeed,

$$im(f_1 + f_2) \subset im(f_1) + im(f_2) \subset \alpha_1 U + \alpha_2 U.$$

Therefore, it is sufficient to show that $\alpha_1 U + \alpha_2 U \subset \alpha U$. Let $v \in \alpha_1 U + \alpha_2 U$. Then there are v_1 and v_2 in U such that $v = \alpha_1 v_1 + \alpha_2 v_2$. Thus,

$$v = \alpha_1 v_1 + \alpha_2 v_2 = (\alpha_1 + \alpha_2) \underbrace{\left(\frac{\alpha_1}{\alpha_1 + \alpha_2} v_1 + \frac{\alpha_2}{\alpha_1 + \alpha_2} v_2 \right)}_{\in U},$$

and so $v \in \alpha U$.

Now let $\lambda \in \mathbf{R}$ and $f \in \mathfrak{F}^b(X,E)$. We want to show that λf is still a bounded function. Let U be a neighborhood of the origin of E. We can suppose that U is balanced because of [Bou81, I §1 No.5 Proposition 4].[1] As f is bounded, there is $\alpha > 0$ such that $im(f) \subset \alpha U$. Set $\alpha_\lambda = \alpha|\lambda|$. We claim that $im(\lambda f) \subset \alpha_\lambda U$. Indeed,

$$im(\lambda f) = \lambda im(f) \subset \lambda \alpha U.$$

[1] Recall that a subset U of a vector space E is called **balanced** if $\alpha v \in U$ for all $v \in U$ and $\alpha \in \mathbf{R}$ such that $|\alpha| \leq 1$.

Hence, it is sufficient to show that $\lambda\alpha U \subset \alpha_\lambda U$. Let $v \in \lambda\alpha U$. Then there is $w \in U$ such that

$$v = \lambda\alpha w = \alpha|\lambda|\frac{\lambda}{|\lambda|}w = \alpha_\lambda \frac{\lambda}{|\lambda|}w.$$

But now $\frac{\lambda}{|\lambda|}w \in U$, as U is balanced. Thus, we can conclude that $v \in \alpha_\lambda U$. □

A direct application of the previous proposition shows that, for every set X and for every vector space E, the set $\ell^\infty(X, E)$ of all bounded functions from X to E is a vector space.

In the case where $(E, ||\cdot||_E)$ is a Banach space, saying that a function $f \in \mathfrak{F}(X, E)$ is bounded is equivalent to saying that the value $\sup_{x\in X} ||f(x)||_E$ is finite. Therefore, we can define the finite value

$$||f||_\infty = \sup_{x\in X} ||f(x)||_E$$

for every $f \in \mathfrak{F}^b(X, E)$. Then $||\cdot||_\infty$ is a norm on $\mathfrak{F}^b(X, E)$ that is called the **supremum norm** or the **uniform norm**.

1.2 Uniform structures on sets

This section recalls some definitions and results about uniform structures on sets and uniformly continuous functions. In particular, we discuss a unique uniform structure, namely the fine uniformity. Good references about uniformities are: [I64], [J87], [W70] and [Bou71, Chapitre II].

1.2.A. Definition of uniformity and first examples. Let X be a set and let A, B be two subsets of $X \times X$. We define the **inversion of a set** as

$$A^{-1} = \{(a_1, a_2) \in X \times X : (a_2, a_1) \in A\}$$

and the **composition of two sets** as

$$A \circ B = \{(a, b) \in X \times X : \exists x \in X \text{ s.t. } (a, x) \in A \text{ and } (x, b) \in B\}.$$

The **section of a point** $x \in X$ with respect to a subset $A \subset X \times X$ is given by

$$A[x] = \{y \in X : (x, y) \in A\},$$

while the **section of a set** with respect to $A \subset X \times X$ is defined as

$$A[S] = \bigcup_{x\in S} A[x].$$

Finally, the **diagonal** of $X \times X$ is the set $\Delta_X = \{(x, x) \in X \times X : x \in X\}$.

Definition 1.2.1. A **uniform structure** $\mathscr{U}$ for a set X is a collection of subsets of $X \times X$ with the following properties:

(U1) $X \times X \in \mathscr{U}$;

(U2) $\Delta_X \subset A$ for any $A \in \mathscr{U}$;

(U3) $A^{-1} \in \mathscr{U}$ for any $A \in \mathscr{U}$;

(U4) if $A, B \in \mathscr{U}$, then $A \cap B \in \mathscr{U}$;

(U5) if $A \in \mathscr{U}$ and $A \subset B$, then $B \in \mathscr{U}$;

(U6) for any $A \in \mathscr{U}$, there is $B \in \mathscr{U}$ such that $B \circ B \subset A$.

A set X together with a uniform structure $\mathscr{U}$ is called a **uniform space**.

If X is a set with a uniform structure $\mathscr{U}$, we say that $\mathscr{U}$ is a **uniformity** for X.

The elements of $\mathscr{U}$ are called **entourages** or **uniform entourages**. A **basis**, or a **fundamental system**, for $\mathscr{U}$ is any subset $\mathscr{B}$ of $\mathscr{U}$ such that every entourage of $\mathscr{U}$ contains an element of $\mathscr{B}$. A uniform structure $\mathscr{U}$ is said **separated**, or Hausdorff, if for any $x_1, x_2 \in X$ such that $x_1 \neq x_2$, there is an entourage $A \in \mathscr{U}$ such that $(x_1, x_2) \notin A$.

Example 1.2.2. (Examples of uniform structures) Let X be a set.

1) The **trivial uniformity** is the uniformity where the only entourage is $X \times X$, i.e., $\mathscr{U}_t = \{X \times X\}$. When X is equipped with the trivial uniformity, we call it a **trivial uniform space**.

2) The **discrete uniformity** is given by the collection $\mathscr{U}_d = \{A \subset X \times X : \Delta_X \subset A\}$. A set X equipped with $\mathscr{U}_d$ is called a **discrete uniform space**.

3) Suppose that X is a metric space and let d_X be its metric. For every non-zero $r \in \mathbf{R}$, define the set $A_r = \{(x_1, x_2) \in X \times X : d_X(x_1, x_2) < r\}$. Then the collection

$$\mathscr{U}_{d_X} = \{B \subset X \times X : \exists\, r > 0 \text{ s.t. } A_r \subset B\}$$

is a uniform structure on X. The uniform structure $\mathscr{U}_{d_X}$ is called the **metric uniform structure**, or the **metric uniformity**, associated to the metric space (X, d_X).

4) Suppose that X is a topological vector space. Let V be a neighborhood of the origin and define the set $A_V = \{(x_1, x_2) \in X \times X : x_2 - x_1 \in V\}$. Then the uniform structure $\mathscr{U}_c$ having as basis the sets A_V, where V runs over a neighborhood basis of the origin, is called the canonical uniform structure of the vector space X.

5) More generally, suppose that G is an abelian topological group. Then the **canonical uniform structure** $\mathscr{U}_c$ of G is the uniformity having as basis the sets

$$A_V = \left\{ (g_1, g_2) \in G \times G : g_1^{-1} g_2 \in V \right\},$$

where V runs over a neighborhood basis of the identity. This uniform structure is also called the **additive uniformity**, see [P13, Definition 2.4].

Clearly, the trivial uniformity is not Hausdorff, while the discrete uniformity is.

Definition 1.2.3. Let $(X, \mathscr{U})$ be a uniform space. The **topology induced by** $\mathscr{U}$ on X is the one where a neighborhood basis at a point $x \in X$ is given by the sets

$$\{A[x] : A \in \mathscr{U}\}.$$

If $(X, \mathscr{U})$ is a uniform space, then we write $\tau(\mathscr{U})$ for the topology induced by $\mathscr{U}$ and we call it the uniform topology of $(X, \mathscr{U})$. Note that $\tau(\mathscr{U})$ is Hausdorff if and only if the uniform structure $\mathscr{U}$ is Hausdorff ([J87, Proposition (8.8)]).

Example 1.2.4. (Examples of uniform topologies)

1) Let X be a set and $\mathscr{U}_t$ its trivial uniformity. The topology induced by $\mathscr{U}_t$ is the trivial one. In fact, $(X \times X)[x] = X$ for every $x \in X$.

2) Let X be a set together with its discrete uniform structure $\mathscr{U}_d$. Then the topology on X induced by $\mathscr{U}_d$ is the discrete topology. This is because $\Delta_X[x] = \{x\}$ for every $x \in X$.

3) Let (X, d_X) be a metric space and let $\mathscr{U}_{d_X}$ be the metric uniform structure associated. Then the topology induced by $\mathscr{U}_{d_X}$ is exactly the topology induced by the metric d_X.

A uniform structure induces not every topology on a set X. A necessary and sufficient condition to ensure that a topology comes from a uniform structure is asking that the topology is completely regular ([J87, Proposition (11.5)]). Precisely, every completely regular topology on a set is induced by a separated uniformity, and every topology induced by a separated uniformity is completely regular. Note that there are non-Hausdorff topologies that are induced by (non-separated) uniformities. An example is the trivial topology, as the trivial uniformity induces it.

Definition 1.2.5. Let (X, τ) be a topological space. Then X is said **uniformizable** if there is a uniform structure $\mathscr{U}$ on X inducing its topology.

Be aware that for a topology on a set X, different uniform structures may induce it. However, there are some cases where a topology is induced precisely by one uniform structure. An example is given by compact topological spaces, see [Bou71, II §4 No.1 Théorème 1].

1.2.B. Uniformly continuous functions. This subsection repeats the basic properties of uniformly continuous functions and vector spaces of uniformly continuous functions.

Definition 1.2.6. Let X and Y be two uniform spaces with uniform structures $\mathscr{U}_X$ and $\mathscr{U}_Y$, respectively. A function $f : X \longrightarrow Y$ is said **uniformly continuous** if for every entourage $A_Y \in \mathscr{U}_Y$ there is an entourage $A_X \in \mathscr{U}_X$ such that $(x_1, x_2) \in A_X$ implies $(f(x_1), f(x_2)) \in A_Y$. The function f is called a **uniform equivalence**, if it is bijective and both f and f^{-1} are uniformly continuous.

If we set $h = (f \times f)$, then f is uniformly continuous if and only if for every entourage A of $\mathscr{U}_Y$, we have that $h^{-1}(A) \in \mathscr{U}_X$.

Example 1.2.7. (Examples of uniformly continuous functions)

1) Every function from a discrete uniform space to a uniform space is uniformly continuous.

2) Let (X, d_X) and (Y, d_Y) be two metrics spaces and let $\mathscr{U}_{d_X}$ and $\mathscr{U}_{d_Y}$ be the respective metric uniform structures. Then a function f between $(X, \mathscr{U}_{d_X})$ and $(Y, \mathscr{U}_{d_Y})$ is uniformly continuous if and only if it is uniformly continuous as a function between the metric spaces (X, d_X) and (Y, d_Y), see [W70, Example 35.12-a)] for details.

3) Let $\mathbf{R}$ equipped with its metric uniformity coming from the Euclidean norm. Then the map $x \longmapsto x^2$ is continuous but not uniformly continuous.

Let X and Y be two uniform spaces with uniform structures $\mathscr{U}_X$ and $\mathscr{U}_Y$, respectively. If f is a map between X and Y, which is uniformly continuous for the respective uniform structures, we say that f is $(\mathscr{U}_X, \mathscr{U}_Y)$-uniformly continuous. In the case where Y is compact, we only say that f is $\mathscr{U}_X$-uniformly continuous, since there is only one uniform structure on Y, and hence no possible misunderstandings.

Write $\mathcal{C}_u((X, \mathscr{U}_X), (Y, \mathscr{U}_Y))$ for the **set of all uniformly continuous functions between** $(X, \mathscr{U}_X)$ **and** $(Y, \mathscr{U}_Y)$. If the uniform space Y is equal to $\mathbf{R}$ equipped with its canonical uniform structure, we only write $\mathcal{C}_u(X, \mathscr{U})$ instead of $\mathcal{C}_u((X, \mathscr{U}), (\mathbf{R}, \mathscr{U}_c))$. Similarly, we write $\mathcal{C}_u^b((X, \mathscr{U}_X), (Y, \mathscr{U}_Y))$ for the **set of all bounded uniformly continuous functions between** $(X, \mathscr{U}_X)$ **and** $(Y, \mathscr{U}_Y)$. As before, we write $\mathcal{C}_u^b(X, \mathscr{U})$ instead of $\mathcal{C}_u^b((X, \mathscr{U}), (\mathbf{R}, \mathscr{U}_c))$.

Proposition 1.2.8. *Let $(X, \mathscr{U}_X)$ be a uniform space and let E be a locally convex vector space. Then $\mathcal{C}_u((X, \mathscr{U}_X), (E, \mathscr{U}_c))$ and $\mathcal{C}_u^b((X, \mathscr{U}_X), (E, \mathscr{U}_c))$ are vector spaces when equipped with the pointwise addition and scalar multiplication.*

Proof. Let f_1, f_2 be two functions in $\mathcal{C}_u((X, \mathscr{U}_X), (E, \mathscr{U}_c))$. We want to show that the function $f_1 + f_2 \in \mathcal{C}_u((X, \mathscr{U}_X), (E, \mathscr{U}_c))$. Let $A \in \mathscr{U}_c$. We can suppose that A is of the form

$$A_V = \{(v_1, v_2) \in E \times E : v_2 - v_1 \in V\},$$

where V is a convex neighborhood of the origin of E. Then there are A_1 and A_2 in $\mathscr{U}_X$ such that $(x,y) \in A_1$ implies that $(f_1(x), f_1(y)) \in A_{\frac{1}{2}V}$, and $(x,y) \in A_2$ implies that $(f_2(x), f_2(y)) \in A_{\frac{1}{2}V}$. Here,

$$A_{\frac{1}{2}V} = \left\{ (v_1, v_2) \in E \times E : v_2 - v_1 \in \frac{1}{2}V \right\}.$$

Set $A_+ = A_1 \cap A_2 \in \mathscr{U}_X$. Therefore, if $(x,y) \in A_+$, then

$$(f_1 + f_2)(y) - (f_1 + f_2)(x) = \underbrace{f_1(x) - f_1(y)}_{\in \frac{1}{2}V} + \underbrace{f_2(x) - f_2(y)}_{\in \frac{1}{2}V} \in V.$$

This implies that $((f_1 + f_2)(x), (f_1 + f_2)(y)) \in A_V$, and hence that the sum $f_1 + f_2$ is in $\mathcal{C}_u\left((X, \mathscr{U}_X), (E, \mathscr{U}_c)\right)$ as wished.

Let now $\alpha \in \mathbf{R}$ be different from zero and $f \in \mathcal{C}_u\left((X, \mathscr{U}_X), (E, \mathscr{U}_c)\right)$. We want to show that αf is still in $\mathcal{C}_u\left((X, \mathscr{U}_X), (E, \mathscr{U}_c)\right)$. Let $A \in \mathscr{U}_c$. As before, we can suppose that A is of the form

$$A_V = \{(v_1, v_2) \in E \times E : v_2 - v_1 \in V\},$$

where V is a convex neighborhood of the origin of E. Then there is $A_\alpha \in \mathscr{U}_X$ such that $(x,y) \in A_\alpha$ implies $(f(x), f(y)) \in A_{\frac{1}{|\alpha|}V}$, where

$$A_{\frac{1}{|\alpha|}V} = \left\{ (v_1, v_2) \in E \times E : v_2 - v_1 \in \frac{1}{|\alpha|}V \right\}.$$

Therefore, if $(x,y) \in A_\alpha$, then

$$(\alpha f)(y) - (\alpha f)(x) = \alpha \underbrace{(f(x) - f(y))}_{\in \frac{1}{|\alpha|}V} \in V.$$

This implies that $((\alpha f)(x), (\alpha f)(y)) \in A_V$ showing that αf is in $\mathcal{C}_u\left((X, \mathscr{U}_X), (E, \mathscr{U}_c)\right)$.

We can hence conclude that $\mathcal{C}_u\left((X, \mathscr{U}_X), (E, \mathscr{U}_c)\right)$ is a vector space. Moreover, we have that the set $\mathcal{C}^b_u\left((X, \mathscr{U}_X), (E, \mathscr{U}_c)\right)$ is a vector subspace of $\mathcal{C}_u\left((X, \mathscr{U}_X), (E, \mathscr{U}_c)\right)$ by Proposition 1.1.2. □

Proposition 1.2.9. *Let X, Y and Z be uniform spaces and let $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ be uniformly continuous functions. Then the composition $f \circ g$ of f and g is a uniformly continuous function from X to Z.*

Proof. See [Bou71, II §2 No.1 Proposition 2]. □

We saw in point 3) of Example 1.2.7 that not every continuous function is uniformly continuous. However, the converse is true.

Proposition 1.2.10. *Let $(X, \mathscr{U}_X)$ and $(Y, \mathscr{U}_Y)$ be two uniform spaces and let $f : X \longrightarrow Y$ be a $(\mathscr{U}_X, \mathscr{U}_Y)$-uniformly continuous function. Then f is a continuous function with respect to the topologies $\tau(\mathscr{U}_X)$ and $\tau(\mathscr{U}_Y)$.*

Proof. See [Bou71, II §2 No.1 Proposition 1]. □

This last proposition implies, in particular, that the following inclusion

$$\mathcal{C}_u\Big((X, \mathscr{U}_X), (Y, \mathscr{U}_Y)\Big) \subset \mathcal{C}\Big(\left(X, \tau(\mathscr{U}_X)\right), \left(Y, \tau(\mathscr{U}_Y)\right)\Big)$$

holds for every uniform space $(X, \mathscr{U}_X)$ and $(Y, \mathscr{U}_Y)$.

Theorem 1.2.11. *Let $(X, \mathscr{U})$ be a uniform space and let $(E, ||\cdot||_E)$ be a Banach space. Then the function space $\mathcal{C}_u^b\left((X, \mathscr{U}), (E, \mathscr{U}_c)\right)$ equipped with the supremum norm $||\cdot||_\infty$ is a Banach space.*

Proof. Let $(f_n)_n$ be a Cauchy sequence in $\mathcal{C}_u^b\left((X, \mathscr{U}), (E, \mathscr{U}_c)\right)$ with respect to the supremum norm. We define the function f pointwise by $f(x) = \lim_n f_n(x)$ for every $x \in X$. We claim that $f \in \mathcal{C}_u^b\left((X, \mathscr{U}), (E, \mathscr{U}_c)\right)$, and that $(f_n)_n$ converges to f for the supremum norm.

First of all, note that f is a well-defined function. Indeed for every $x \in X$, the sequence $(f_n(x))_n$ is Cauchy for the $||\cdot||_E$-norm, and consequently, it converges by completeness of E. Thus, the limit $\lim_n f_n(x) = f(x)$ exists for every $x \in X$.

We proceed to show that $(f_n)_n$ converges to f for the supremum norm. Let $\epsilon > 0$. Then there is $n_0 \in \mathbf{N}$ such that

$$||f_n - f_m||_\infty = \sup_{x\in X} ||f_n(x) - f_m(x)||_E < \epsilon \quad \text{for every } n, m \geq n_0.$$

This implies that

$$||f(x) - f_n(x)||_E = ||\lim_m f_m(x) - f_n(x)||_E \leq \lim_m ||f_m(x) - f_n(x)||_E < \epsilon.$$

for every $n > n_0$ and $x \in X$. Therefore, we can conclude that

$$\begin{aligned} ||f - f_n||_\infty &= \sup_{x\in X} ||f(x) - f_n(x)||_E \\ &= \sup_{x\in X} ||\lim_m f_m(x) - f_n(x)||_E \\ &\leq \sup_{x\in X} \lim_m ||f_m(x) - f_n(x)||_E < \epsilon \end{aligned}$$

for every $n > n_0$.

Let's show that f is a bounded function. Let $\epsilon > 0$ and let $n_0 \in \mathbf{N}$ such that $||f - f_{n_0}||_\infty < \epsilon$. Then

$$||f||_\infty \leq ||f - f_{n_0}||_\infty + ||f_{n_0}||_\infty < \epsilon + ||f_{n_0}||_\infty$$

which proves that f is bounded.

It is left to check that f is uniformly continuous. Let $A \in \mathscr{U}_c$. We can suppose that A is of the form

$$A_\epsilon = \{(v_1, v_2) \in E \times E : ||v_2 - v_1||_E < \epsilon\}$$

for some $\epsilon > 0$. Now, there is $n_0 \in \mathbf{N}$ such that $||f - f_n||_\infty < \frac{\epsilon}{3}$ for every $n > n_0$. Fix $n_1 \in \mathbf{N}$ such that $n_1 > n_0$. Then there is $A_X \in \mathscr{U}$ such that $(x, y) \in A_X$ implies $(f_{n_1}(x), f_{n_1}(y)) \in A_{\frac{\epsilon}{3}}$, where $A_{\frac{\epsilon}{3}} = \left\{(v_1, v_2) \in E \times E : ||v_2 - v_1||_E < \frac{\epsilon}{3}\right\}$. We claim that $(f(x), f(y)) \in A_\epsilon$ for every $(x, y) \in A_X$. Indeed, given $(x, y) \in A_X$, we can compute that

$$\begin{aligned} ||f(x) - f(y)||_E &\leq ||f(x) - f_{n_1}(x)||_E + ||f_{n_1}(x) - f_{n_1}(y)||_E + ||f_{n_1}(y) - f(y)||_E \\ &\leq ||f - f_{n_1}||_\infty + ||f_{n_1}(x) - f_{n_1}(y)||_E + ||f_{n_1} - f||_\infty \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon. \end{aligned}$$

Therefore, $(f(x), f(y)) \in A_\epsilon$ as wished. □

Corollary 1.2.12. *Let X be a set and let $(E, ||\cdot||_E)$ be a Banach space. Then $\ell^\infty(X, E)$ equipped with the supremum norm is a Banach space.*

Proof. We have that $C_u^b((X, \mathscr{U}_d), (E, \mathscr{U}_c)) = \ell^\infty(X, E)$. Thus, we can conclude by Theorem 1.2.11. □

Suppose now that $(E, ||\cdot||_E)$ is a Banach algebra, i.e., a Banach space equipped with a multiplication such that $||v_1 v_2||_E \leq ||v_1||_E ||v_2||_E$ for every two vectors $v_1, v_2 \in E$. Then for a set X, we define the **pointwise multiplication** of two functions f_1 and f_2 in E^X as the function $f_1 f_2 \in E^X$ given pointwise by

$$(f_1 f_2)(x) = f_1(x) f_2(x) \qquad \text{for every } x \in X.$$

Theorem 1.2.13. *Let $(X, \mathscr{U})$ be a uniform space and let $(E, ||\cdot||_E)$ be a Banach algebra. Then $C_u^b((X, \mathscr{U}), (E, \mathscr{U}_c))$ equipped with the supremum norm is a Banach algebra under pointwise multiplication.*

Proof. We know already that $C_u^b((X, \mathscr{U}), (E, \mathscr{U}_c))$ equipped with the supremum norm is a Banach space by Theorem 1.2.11. Therefore, we only have to check that the space of uniformly continuous functions $C_u^b((X, \mathscr{U}), (E, \mathscr{U}_c))$ is closed under pointwise multiplication. Let f_1, f_2 be two non-zero functions in $C_u^b((X, \mathscr{U}), (E, \mathscr{U}_c))$ and let $A \in \mathscr{U}_c$. We can suppose that A is of the form

$$A_\epsilon = \{(v_1, v_2) \in E \times E : ||v_2 - v_1||_E < \epsilon\}$$

for some $\epsilon > 0$. Then there are $A_{f_1}, A_{f_2} \in \mathscr{U}$ such that for every $(x, y) \in A_{f_1}$, we have that

$$(f_1(x), f_1(y)) \in A_1 = \left\{(v_1, v_2) \in E \times E : ||v_2 - v_1||_E < \frac{\epsilon}{2||f_2||_\infty}\right\} \in \mathscr{U}_c,$$

and for every $(x,y) \in A_{f_2}$, we have that

$$(f_2(x), f_2(y)) \in A_2 = \left\{ (v_1, v_2) \in E \times E : ||v_2 - v_1||_E < \frac{\epsilon}{2||f_1||_\infty} \right\} \in \mathscr{U}_c.$$

Set $A_X = A_{f_1} \cap A_{f_2}$. Then

$$\begin{aligned} ||f_1(x)f_2(x) - f_1(y)f_2(y)||_E &= ||f_1(x)f_2(x) - f_1(y)f_2(x) + f_1(y)f_2(x) - f_1(y)f_2(y)||_E \\ &\leq ||f_2(x)||_E||f_1(x) - f_1(y)||_E + ||f_1(y)||_E||f_2(x) - f_2(y)||_E \\ &\leq ||f_2||_\infty \frac{\epsilon}{2||f_2||_\infty} + ||f_1||_\infty \frac{\epsilon}{2||f_1||_\infty} = \epsilon \end{aligned}$$

for every $(x,y) \in A_X$. This implies that $(f_1(x)f_2(x), f_1(y)f_2(y)) \in A_\epsilon$. Therefore, the pointwise product of f_1 with f_2 is in $\mathcal{C}_u^b((X,\mathscr{U}),(E,\mathscr{U}_c))$. So, $\mathcal{C}_u^b((X,\mathscr{U}),(E,\mathscr{U}_c))$ is an algebra. It is left to show that the supremum norm is submultiplicative to conclude that $\mathcal{C}_u^b((X,\mathscr{U}),(E,\mathscr{U}_c))$ is a Banach algebra. Let $f_1, f_2 \in \mathcal{C}_u^b((X,\mathscr{U}),(E,\mathscr{U}_c))$. Then

$$\begin{aligned} ||f_1 f_2||_\infty &= \sup_{x\in X} ||f_1(x)f_2(x)||_E \leq \sup_{x\in X} ||f_1(x)||_E||f_2(x)||_E \\ &\leq \sup_{x\in X} ||f_1(x)||_E \sup_{y\in X} ||f_2(y)||_E = ||f_1||_\infty ||f_2||_\infty \end{aligned}$$

as wished. □

Remark 1.2.14. Note that the boundness condition is a fundamental hypothesis to ensure the algebraic structure. For example, let $X = \mathbf{R}$ with the canonical uniformity given by its locally compact topology. Then the function $f(x) = x$ is uniformly continuous but $f(x)f(x) = x^2$ is not.

1.2.C. The fine uniformity. In the following subsection, we recall some facts about a particular uniform structure: the fine uniformity. Intuitively, the fine uniformity is the uniform structure that comes closest to a topology. This uniform structure is not so much discussed in the literature as studying it is similar to studying a topology. However, a good exposition of the fine uniformity can be founded in [W70, pp. 244-249].

Let X be a set with two uniform structures $\mathscr{U}_1$ and $\mathscr{U}_2$. We say that $\mathscr{U}_1$ is **finer** than $\mathscr{U}_2$ if the identity map $\mathrm{Id} : (X, \mathscr{U}_1) \longrightarrow (X, \mathscr{U}_2)$ is uniformly continuous. In other words, if every entourage of $\mathscr{U}_2$ is also an entourage of $\mathscr{U}_1$. If $\mathscr{U}_1$ is finer than $\mathscr{U}_2$, we write $\mathscr{U}_2 \subset \mathscr{U}_1$.

Theorem 1.2.15. *Let X be a uniformizable topological space. Then there is a finest uniformity on X, which induces its topology.*

Proof. See [W70, Theorem 36.12]. □

Definition 1.2.16. Let X be a uniformizable topological space. Then the finest uniformity on X, which induces its topology, is called the **fine uniformity** of X and it is denoted $\mathscr{F}$ or $\mathscr{F}_X$.

The following theorem is the reason why the fine uniformity is so close to being a topology.

Theorem 1.2.17. *Let $(X, \mathscr{U}_X)$ and $(Y, \mathscr{U}_Y)$ be uniform spaces. If the function $f : (X, \tau(\mathscr{U}_X)) \longrightarrow (Y, \tau(\mathscr{U}_Y))$ is continuous, then $f : (X, \mathscr{F}) \longrightarrow (Y, \mathscr{U}_Y)$ is uniformly continuous.*

Proof. See [W70, Theorem 36.18]. □

Therefore, the equality

$$\mathcal{C}_u\left((X, \mathscr{F}), (Y, \mathscr{U}_Y)\right) = \mathcal{C}\left((X, \tau), (Y, \tau(\mathscr{U}_Y))\right)$$

holds for every uniformizable topological space (X, τ) and every uniform space $(Y, \mathscr{U}_Y)$.

Corollary 1.2.18. *Every continuous function from a compact topological space to a uniform space is uniformly continuous.*

Proof. Let X be a compact topological space. By [Bou71, II §7 No.1 Théorème 1], there is only one uniform structure $\mathscr{U}_X$ on X. This means that $\mathscr{U}_X = \mathscr{F}$ by Theorem 1.2.15, and so we can conclude using Theorem 1.2.17. □

Corollary 1.2.19. *Let X be a topological uniformizable space and let $(E, ||\cdot||_E)$ be a Banach space. Then $\mathcal{C}^b(X, E)$ is a Banach algebra when equipped with the supremum norm and the pointwise multiplication.*

Proof. Let $\mathscr{F}$ be the fine uniform structure of X. Then

$$\mathcal{C}^b(X, E) = \mathcal{C}_u^b\left((X, \mathscr{F}), (E, \mathscr{U}_c)\right)$$

by Theorem 1.2.17. This last space is a Banach algebra when equipped with the supremum norm and the pointwise multiplication by Theorem 1.2.13. □

1.3 Uniform structures on groups

We focus on the case where X is a topological group. Precisely, we look at specific uniform structures, which show up naturally when the data of a topological group is given. Most of them were studied in [RD81]. Moreover, we explain how it is possible to describe different functions spaces on a topological group via spaces of uniformly continuous functions.

1.3.A. Spaces of functions on topological groups. Let G be a topological group, X a set and E a vector space. Suppose that the group G acts on X via the map

$$G \times X \longrightarrow X, \quad (g, x) \longmapsto \gamma(g)(x),$$

where $\gamma : G \longrightarrow \mathrm{Sym}(X)$. We define the **$\gamma$-representation** π_γ of G on E^X by the equation

$$\pi_\gamma(g)f(x) = f(\gamma(g)^{-1}(x)) \quad \text{for } x \in X,\ g \in G \text{ and } f \in E^X.$$

Consider the case $X = G$. Then there are, at least, two natural actions of G on itself. The first one is the **left-translation action** that is given by the map

$$L : G \times G \longrightarrow G, \quad (g, x) \longmapsto L(g)(x) = gx.$$

While the second one is the **right-translation action** that is given by the map

$$R : G \times G \longrightarrow G, \quad (g, x) \longmapsto R(g)(x) = xg^{-1}.$$

Therefore, we can introduce the respective representations. We define the **left-translation representation** π_L of G on E^G by

$$\pi_L(g)f(x) = f(L(g)^{-1}(x)) = f(g^{-1}x) \quad \text{for } g, x \in G \text{ and } f \in E^G.$$

Similarly, we define the **right-translation representation** π_R of G on E^G by

$$\pi_R(g)f(x) = f(R(g)^{-1}(x)) = f(xg) \quad \text{for } g, x \in G \text{ and } f \in E^G.$$

Suppose now that E is a locally convex vector space, and consider the sets $\mathcal{C}(G, E)$ of all **vector-valued continuous functions** from G to E and $\mathcal{C}^b(G, E)$ of all **bounded vector-valued continuous functions** from G to E. Then $\mathcal{C}(G, E)$ and $\mathcal{C}^b(G, E)$ are vector-valued function spaces with the pointwise addition and scalar multiplication by Proposition 1.2.8 and Theorem 1.2.17.

A function $f \in \mathcal{C}^b(G, E)$ is said to be a **bounded vector-valued right-uniformly continuous function** if for every neighborhood $V \subset E$ of the origin there is a neighborhood $U \subset G$ of the identity such that for every $a \in U$:

$$\pi_L(a)f(g) - f(g) \in V \quad \text{for every } g \in G.$$

We write $\mathcal{C}^b_{ru}(G, E)$ for the subset of all **bounded vector valued right-uniformly continuous functions** on G. Similarly, a function $f \in \mathcal{C}^b(G, E)$ is said to be a **bounded vector-valued left-uniformly continuous function** if for every neighborhood $V \subset E$ of the origin there is a neighborhood $U \subset G$ of the identity such that for every $a \in U$:

$$\pi_R(a)f(g) - f(g) \in V \quad \text{for every } g \in G.$$

We write $\mathcal{C}^b_{lu}(G,E)$ for the subset of all **bounded vector valued left-uniformly continuous functions** on G. We define the set of all **bounded vector-valued uniformly functions** on G as the intersection $\mathcal{C}^b_u(G,E) = \mathcal{C}^b_{ru}(G,E) \cap \mathcal{C}^b_{lu}(G,E)$. The sets $\mathcal{C}^b_{ru}(G,E)$, $\mathcal{C}^b_{lu}(G,E)$ and $\mathcal{C}^b_u(G,E)$ are vector subspaces of $\mathcal{C}^b(G,E)$.

Now, consider the case where $(E, ||\cdot||_E)$ is a Banach space. We can hence define on $\mathcal{C}^b(G,E)$ the supremum norm $||\cdot||_\infty$. A small computation shows that the set $\mathcal{C}^b_{ru}(G,E)$ is equal to the set of all $f \in \mathcal{C}^b(G,E)$ such that the orbital map

$$G \longrightarrow \mathcal{C}^b(G,E), \quad g \longmapsto \pi_L(g)f$$

is continuous with respect to the supremum norm on $\mathcal{C}^b(G,E)$. Similarly, the set $\mathcal{C}^b_{lu}(G,E)$ coincides with the set of all $f \in \mathcal{C}^b(G,E)$ such that the orbital map

$$G \longrightarrow \mathcal{C}^b(G,E), \quad g \longmapsto \pi_R(g)f$$

is continuous with respect to the supremum norm on $\mathcal{C}^b(G,E)$. In other words, $\mathcal{C}^b_{ru}(G,E)$ is the set of all the continuous vectors of the left-translation representation π_L and $\mathcal{C}^b_{lu}(G,E)$ is the set of all continuous vectors of the right-translation representation π_R.

If $E = \mathbf{R}$ with its Euclidean norm, then we write $\mathcal{C}^b(G)$ instead of $\mathcal{C}^b(G,\mathbf{R})$. Similarly for the spaces $\mathcal{C}^b_{ru}(G)$, $\mathcal{C}^b_{lu}(G)$ and $\mathcal{C}^b_u(G)$.

Unless otherwise specified, we always use the left-translation representation on all of the above function spaces, and we write gf instead of $\pi_L(g)f$ for $g \in G$ and $f \in \mathcal{C}^b(G,E)$.

1.3.B. The five standard uniform structures. We look at specific uniform structures for topological groups. The uniformities we are going to present are well known and have been intensely studied for their properties. Particularly good references for this section are [RD81] and [Bou71, Chapitre III §3].

A topological group together with a uniform structure that induces the group topology is called a **uniform group**.

We saw in the previous section that given a uniformizable topological space X, there might be more uniform structures on X that induce the topology. When X is a topological group, the choice for such uniformity is quite rich. We are particularly interested in five of them.

- **The fine uniformity $\mathscr{F}$.** The uniform structure given by Theorem 1.2.15.
- **The two-sided uniformity $\mathscr{R} \vee \mathscr{L}$.** A basis for this uniform structure is given by the sets

  $$A_U = \{(g_1, g_2) \in G \times G : g_2 \in g_1 U \cap U g_1\},$$

  where U runs over a neighborhood basis of the identity.

- **The right uniformity $\mathscr{R}$.** A basis for this uniform structure is given by the sets
$$A_U = \left\{(g_1, g_2) \in G \times G : g_1 g_2^{-1} \in U\right\},$$
where U runs over a neighborhood basis of the identity.

- **The left uniformity $\mathscr{L}$.** A basis for this uniform structure is given by the sets
$$A_U = \left\{(g_1, g_2) \in G \times G : g_1^{-1} g_2 \in U\right\},$$
where U runs over a neighborhood basis of the identity.

- **The Roelcke uniformity $\mathscr{R} \wedge \mathscr{L}$.** A basis for this uniform structure is given by the sets
$$A_U = \left\{(g_1, g_2) \in G \times G : g_1^{-1} g_2 \in U \text{ and } g_1 g_2^{-1} \in U\right\},$$
where U runs over a neighborhood basis of the identity.

Firstly, it is important to note that given a topological group G, it is always possible to define the uniformities $\mathscr{R} \vee \mathscr{L}$, $\mathscr{R}$, $\mathscr{L}$ and $\mathscr{R} \wedge \mathscr{L}$. Moreover, each of these uniform structures induces the topology of the group. Indeed, if $\mathscr{U} \in \{\mathscr{R} \vee \mathscr{L}, \mathscr{R}, \mathscr{L}, \mathscr{R} \wedge \mathscr{L}\}$ and A_U is a basis element of $\mathscr{U}$, then $A_U[g] = U$ for every $g \in G$. Therefore, every topological group is uniformizable.

By definition, we can see that $\mathscr{R}, \mathscr{L}, \mathscr{R} \wedge \mathscr{L} \subset \mathscr{R} \vee \mathscr{L}$ and that $\mathscr{R} \wedge \mathscr{L} \subset \mathscr{R}, \mathscr{L}$. Moreover, every of the above uniform structures is included in $\mathscr{F}$ by definition of this last. In general, all of these inclusions are strict, but there are cases where some of these uniform structures coincide.

Example 1.3.1. Let G be a topological group.

1) Suppose that G has the trivial topology. Then all the above uniform structures coincide as there are only two open sets.

2) Suppose that G is a discrete group. Then the five above uniformities coincide as a neighborhood basis of the identity is given by the set $\{e\}$.

3) Suppose that G is a compact group. Then the five above uniform structures coincide because there is only one uniformity on G that induces its topology ([Bou71, II §7 No.1 Théorème 1]).

4) Suppose that G is an abelian group. Then $\mathscr{R} = \mathscr{L}$ and so
$$\mathscr{R} \vee \mathscr{L} = \mathscr{R} = \mathscr{L} = \mathscr{R} \wedge \mathscr{L}.$$

However, in general, they are different from $\mathscr{F}$. Take, for example, $\mathbf{R}$ endowed with its locally compact topology. Then

$$\mathscr{R} \vee \mathscr{L} = \mathscr{R} = \mathscr{L} = \mathscr{R} \wedge \mathscr{L} = \mathscr{U}_{d_{\mathbf{R}}},$$

where $d_{\mathbf{R}}$ is the Euclidean metric on $\mathbf{R}$. But $\mathscr{U}_{d_{\mathbf{R}}} \subsetneq \mathscr{F}$. Indeed, $f(x) = x^2$ is $(\mathscr{F}, \mathscr{U}_c)$-uniformly continuous but not $(\mathscr{U}_{d_{\mathbf{R}}}, \mathscr{U}_c)$-uniformly continuous.

Note that there are even examples of locally compact groups where the right uniformity is different from the left uniformity, see [RD81, Example 2.13].

A topological group for which the left and the right uniform structure coincide is called a **SIN-group**. We refer to [RD81, pp. 40-41] for more information about SIN-groups.

Remark 1.3.2. The two-sided uniformity and the Roelcke uniformity are also called the upper and lower uniformities, respectively.

The Roelcke uniformity is so-called in honour of Walter Roelcke[2], who was the first to define and study it.

Let f be a uniformly continuous function between the uniform groups $(G_1, \mathscr{U}_1)$ and $(G_2, \mathscr{U}_2)$. Then we say that f is a **uniformly isomorphism** if f is a uniform equivalence and a group isomorphism.

For a group G with group operation $*$, the **opposite group** G^{op} of G is the group given by the same underlying set as G but with group operation $g_1 *_{op} g_2 = g_2 * g_1$ for every $g_1, g_2 \in G^{op}$. Moreover, $(G, *)$ is a topological group if and only if $(G^{op}, *_{op})$ is a topological group, and they are topologically isomorphic. See [RD81, 1.8] for details.

Proposition 1.3.3. *Let G be a topological group.*

a) Let $\mathscr{U} \in \{\mathscr{F}, \mathscr{R} \vee \mathscr{L}, \mathscr{R}, \mathscr{L}, \mathscr{R} \wedge \mathscr{L}\}$. Then the function

$$(G, \mathscr{U}) \longrightarrow (G, \mathscr{U}), \quad g \longmapsto agb$$

is uniformly continuous for every $a, b \in G$.

b) The inversion function

$$(G, \mathscr{R}) \longrightarrow (G^{op}, \mathscr{L}), \quad g \longmapsto g^{-1}$$

is a uniformly isomorphism.

c) Let $\mathscr{U} \in \{\mathscr{F}, \mathscr{R} \vee \mathscr{L}, \mathscr{R} \wedge \mathscr{L}\}$. Then the inversion function

$$(G, \mathscr{U}) \longrightarrow (G^{op}, \mathscr{U}), \quad g \longmapsto g^{-1}$$

is a uniformly isomorphism.

[2] Görliz 1928 - Krailling bei München 2005.

Proof. We start showing point a). Let $\mathscr{U} = \mathscr{F}$. Then for every $a, b \in G$ the functions $g \longmapsto agb$ and $g \longmapsto g^{-1}$ are continuous for the group topology by the definition of a topological group. Hence, they are uniformly continuous by Theorem 1.2.17. For $\mathscr{U} \in \{\mathscr{R} \vee \mathscr{L}, \mathscr{R}, \mathscr{L}, \mathscr{R} \wedge \mathscr{L}\}$ a proof can be founded in [RD81, Proposition 2.24].

The proof of point b) is given in [RD81, Lemma 2.21].

For point c). The map

$$G \longrightarrow G^{op}, \quad g \longmapsto g^{-1}$$

is a topological group isomorphism, see [RD81, 1.8]. Therefore, the statement is true for $\mathscr{U} = \mathscr{F}$, thanks to Theoreom 1.2.17, and $\mathscr{U} \in \{\mathscr{R} \vee \mathscr{L}\mathscr{R} \wedge \mathscr{L}\}$, thanks to [RD81, Proposition 2.25] □

In what follows, we identify different function spaces on topological groups with uniformly continuous function spaces on uniform groups. The idea is to build a bridge between the world of topological groups and the world of uniform groups.

Theorem 1.3.4. *Let G be a topological group and let E be a locally convex vector space. The following equality hold:*

a) $\mathcal{C}^b(G, E) = \mathcal{C}^b_u\Big((G, \mathscr{F}), (E, \mathscr{U}_c)\Big)$;

b) $\mathcal{C}^b_{ru}(G, E) = \mathcal{C}^b_u\Big((G, \mathscr{R}), (E, \mathscr{U}_c)\Big)$;

c) $\mathcal{C}^b_{lu}(G, E) = \mathcal{C}^b_u\Big((G, \mathscr{L}), (E, \mathscr{U}_c)\Big)$;

d) $\mathcal{C}^b_u(G, E) = \mathcal{C}^b_u\Big((G, \mathscr{L} \wedge \mathscr{R}), (E, \mathscr{U}_c)\Big)$.

Before discussing the proof of the above theorem, recall that every topological group admits a neighbourhood basis of the identity element composed by symmetric sets ([HR63, Theorem 4.6]). A subset U of a group G is called symmetric if $u^{-1} \in U$ for every $u \in U$.

Proof of Theorem 1.3.4. Point a) is only a consequence of Theorem 1.2.17.

We prove b) by double inclusion. Let $f \in \mathcal{C}^b_u\Big((G, \mathscr{R}), (E, \mathscr{U}_c)\Big)$ and we want to show that $f \in \mathcal{C}^b_{ru}(G, E)$, i.e., for every neighborhood of the origin $V \subset E$ there is a neighborhood of the identity $U \subset G$ such that for every $a \in U$: $\pi_L(a)f(g) - f(g) \in V$ for every $g \in G$. Therefore, let $V \subset E$ be a neighborhood of the origin and define the entourage

$$A_V = \{(v_1, v_2) \in E \times E : v_2 - v_1 \in V\} \in \mathscr{U}_c.$$

Then there is $A \in \mathscr{R}$ such that $(x, y) \in A$ implies that $(f(x), f(y)) \in A_V$. We can suppose that A is of the form

$$A_U = \Big\{(g_1, g_2) \in G \times G : g_1 g_2^{-1} \in U\Big\},$$

where $U \subset G$ is a symmetric neighborhood of the identity. Let now $g \in G$. Then for every $a \in U$ there is $x \in G$ such that $a^{-1}g = x$, and thus $xg^{-1} = a^{-1} \in U$. This implies that $(x, g) \in A_U$ and, consequently, that $(f(x), f(g)) \in A_V$. Hence,

$$\pi_L(a)f(g) - f(g) = f(a^{-1}g) - f(g) = f(x) - f(g) \in V$$

which shows the right-uniformly continuity of f. Let's prove the inverse inclusion. Let $f \in \mathcal{C}^b_{ru}(G, E)$ and let A be an entourage of E. We can suppose that A is of the form

$$A_V = \{(v_1, v_2) \in E \times E : v_2 - v_1 \in V\},$$

where $V \subset E$ is a neighborhood of the origin. Then there is $U \subset G$ a symmetric neighborhood of the identity such that for every $a \in U$: $\pi_L(a)f(g) - f(g) \in V$ for every $g \in G$. Define the entourage

$$A_U = \left\{(g_1, g_2) \in G \times G : g_1 g_2^{-1} \in U\right\} \in \mathscr{R}.$$

Now, suppose that $(x, y) \in A_U$. This means that there is $a^{-1} \in U$ such that $xy^{-1} = a^{-1}$, and hence that

$$f(x) - f(y) = f(a^{-1}y) - f(y) = \pi_L(a)f(y) - f(y) \in V.$$

We can conclude that $(f(x), f(y)) \in A_V$ and, therefore, that f is in $\mathcal{C}^b_u\Big((G, \mathscr{R}), (E, \mathscr{U}_c)\Big)$.

The proof of point c) is similar to the proof of point b).

To show that point d) is true, it is important to remark that a function is uniformly continuous for the Roelcke uniformity if it is uniformly continuous for both the right uniformity and the left uniformity. This is only because of the definition of the Roelcke uniform structure. Thus,

$$\begin{aligned}\mathcal{C}^b_u\Big((G, \mathscr{L} \wedge \mathscr{R}), (E, \mathscr{U}_c)\Big) &= \mathcal{C}^b_u\Big((G, \mathscr{R}), (E, \mathscr{U}_c)\Big) \cap \mathcal{C}^b_u\Big((G, \mathscr{L}), (E, \mathscr{U}_c)\Big) \\ &= \mathcal{C}^b_{ru}(G, E) \cap \mathcal{C}^b_{lu}(G, E) = \mathcal{C}^b_u(G, E)\end{aligned}$$

thanks to the points b) and c). □

We will switch freely between the interpretation of the above function spaces as continuous vectors or as uniformly continuous functions during this manuscript.

Corollary 1.3.5. *Let G be a topological group.*

a) If E is a locally convex vector spaces, then $\mathcal{C}^b_{ru}(G, E)$, $\mathcal{C}^b_{lu}(G, E)$ and $\mathcal{C}^b_u(G, E)$ are vector subspaces of $\mathcal{C}^b(G, E)$.

b) If E is a Banach space, then the vector spaces $\mathcal{C}^b(G, E)$, $\mathcal{C}^b_{ru}(G, E)$, $\mathcal{C}^b_{lu}(G, E)$ and $\mathcal{C}^b_u(G, E)$ are Banach spaces when equipped with the supremum norm.

c) *If E is a Banach algebra, then the vector spaces $\mathcal{C}^b(G,E)$, $\mathcal{C}^b_{ru}(G,E)$, $\mathcal{C}^b_{lu}(G,E)$ and $\mathcal{C}^b_u(G,E)$ are Banach algebras when equipped with the supremum norm and the pointwise multiplication.*

Proof. The proof is only a combination of Theorems 1.2.11, 1.3.4 and 1.2.13. □

Note that, in the case where E is a Banach space, the vector-valued function spaces $\mathcal{C}^b_{ru}(G,E)$, $\mathcal{C}^b_{lu}(G,E)$ and $\mathcal{C}^b_u(G,E)$ are closed in $\mathcal{C}^b(G,E)$ for the supremum norm. Moreover, $\mathcal{C}^b_u(G,E)$ is closed in both $\mathcal{C}^b_{ru}(G,E)$ and $\mathcal{C}^b_{lu}(G,E)$ also for the supremum norm.

1.4 Means on functions spaces

We begin the section by recalling the concept of mean. After, we introduce the notion of uniform amenability. A good reference about means on function spaces is [BJM89, Chapter 2].

1.4.A. About the notion of mean. Let X be a set and recall that the vector space $\ell^\infty(X)$ of all real bounded functions on X is a Banach space when equipped with the supremum norm.

Definition 1.4.1. Let E be a vector subspace of $\ell^\infty(X)$ which contains the constant functions. A **mean** on E is a linear functional $\mathrm{m} : E \longrightarrow \mathbf{R}$ such that $\mathrm{m}(\mathbf{1}_X) = 1$ and $||\mathrm{m}||_{op} = 1$.

Note that a mean m on a vector subspace E as above is always positive, i.e., $\mathrm{m}(f) \geq 0$ for every $f \in E$ such that $f(x) \geq 0$ for all $x \in X$. See point (2) of Example 2.2.6 for a proof of this fact.

Given E as in the definition above, the easier example of a mean is ev_x, the evaluation map at the point $x \in X$. Actually, every convex combination of evaluation maps is a mean, and, more generally, every convex combination of means is a mean. If we write $\mathcal{M}(E)$ for the **set of all means on** E, then an application of the Banach-Alaoglu Theorem ([AB99, Theorem 5.105]) shows the following proposition.

Proposition 1.4.2. *The set of all means $\mathcal{M}(E)$ is non-empty, convex and compact for the weak-* topology for every $E \subset \ell^\infty(X)$ vector subspace which contains the constant functions.*

Remark 1.4.3. If K is a compact space, then we can interpret every mean on $\mathcal{C}(K)$ as a probability measure on K by the Riesz Representation Theorem ([C13, Theorem 7.2.8]). In particular, the set of all probability measures $\mathrm{Prob}(K)$ on K, is compact for the weak-* topology.

Suppose now that there is a topological group G which acts on the set X via the action γ and consider the π_γ-representation of G on $\ell^\infty(X)$. Take a vector subspace E of $\ell^\infty(X)$ which contains the constant functions and which is π_γ-invariant. Then we say that a mean m on E is **invariant**, or **invariant for the representation** π_γ, if $\pi_\gamma^*(g)\mathrm{m} = \mathrm{m}$ for every $g \in G$, where π_γ^* is the adjoint of π_γ.

Unless otherwise specified, we always use the π_γ-representation induced by some action γ of G on a set X and we write gf instead of $\pi_\gamma(g)f$ for $g \in G$ and $f \in \ell^\infty(X)$.

Scholium 1.4.4. It is possible to generalize the notion of mean in two different ways. The first uses ordered vector spaces with an order unit, while the second uses unital algebras.

1.4.B. Uniform amenability. Consider the case where $X = G$ a topological group. Recall that G acts naturally on itself by the left and by the right. Therefore, there are two natural representations of G on $\ell^\infty(G)$. Namely, the left-translation representation π_L and the right-translation representation π_R.

Definition 1.4.5. Let G be a topological group. A **functionally invariant uniformity for** G is a uniform structure $\mathscr{U}$ for G which induces its topology and such that

$$\pi_L(g)f,\ \pi_R(g)f \in \mathcal{C}_u^b(G,\mathscr{U}) \quad \text{for every } g \in G \text{ and } f \in \mathcal{C}_u^b(G,\mathscr{U}).$$

We borrowed this last notion from Pachl ([P18]), who stated it for topological semigroups. This property distils the essence of what is needed from a uniform structure of G to define invariant functionals on subspaces of $\ell^\infty(G)$.

Lemma 1.4.6. *Let G be a topological group and let $(X,\mathscr{U})$ be a uniform space. Suppose that there is an action γ of G on $(X,\mathscr{U})$ and that the map*

$$(X,\mathscr{U}) \longrightarrow (X,\mathscr{U}), \quad x \longmapsto \gamma(g)(x) \quad \textit{is uniformly continuous for every } g \in G.$$

Then

$$\pi_\gamma(g)f \in \mathcal{C}_u^b(X,\mathscr{U}) \quad \textit{for every } g \in G \textit{ and } f \in \mathcal{C}_u^b(X,\mathscr{U}).$$

Proof. Let $g \in G$ and $f \in \mathcal{C}_u^b(X,\mathscr{U})$. We want to show that $\pi_\gamma(g)f$ is in $\mathcal{C}_u^b(X,\mathscr{U})$. Clearly, $\pi_\gamma(g)f$ is bounded. Therefore, we have only to show that it is uniformly continuous. To this aim, let

$$A_\epsilon = \{(r_1,r_2) \in \mathbf{R}\times\mathbf{R} : |r_2 - r_1| < \epsilon\} \in \mathscr{U}_c^{\mathbf{R}} \quad \text{for some } \epsilon > 0.$$

Then there is $A_f \in \mathscr{U}$ such that if $(x,y) \in A_f$, then $(f(x),f(y)) \in A_\epsilon$, and there is $A_g \in \mathscr{U}$ such that if $(x,y) \in A_g$, then $(\gamma(g)^{-1}(x),\gamma(g)^{-1}(y)) \in A_f$. Now, for every $(x,y) \in A_g$, we have that

$$(\pi_\gamma(g)f(x),\pi_\gamma(g)f(y)) = (f(\gamma(g)^{-1}(x)),f(\gamma(g)^{-1}(y))) \in A_\epsilon$$

as $(\gamma(g)^{-1}(x),\gamma(g)^{-1}(y)) \in A_f$. This shows that $\pi_\gamma(g)f$ is uniformly continuous. □

Corollary 1.4.7. *The five uniform structures*

$$\mathscr{F},\ \mathscr{R}\vee\mathscr{L},\ \mathscr{R},\ \mathscr{L}\ \text{ and }\ \mathscr{R}\wedge\mathscr{L}$$

are all functionally invariant uniformities for G.

Proof. The proof is a combination of Lemma 1.4.6 together with Proposition 1.3.3. □

Corollary 1.4.8. *Let G be a topological group and let K be a compact space. Suppose that G acts on K by homeomorphisms. Then $\pi_\gamma(g)\phi \in \mathcal{C}(K)$ for every $g \in G$ and $\phi \in \mathcal{C}(K)$.*

Proof. We can use Lemma 1.4.6 thanks to Corollary 1.2.18. □

We can finally state the definition of uniform amenability.

Definition 1.4.9. Let G be a topological group and let $\mathscr{U}$ be a functionally invariant uniformity for G. Then G is said $\mathscr{U}$**-amenable** if there is an invariant mean on $\mathcal{C}_u^b(G,\mathscr{U})$.

The above definition is quite general. However, the functionally invariant uniformities for which the notion of $\mathscr{U}$-amenability becomes interesting are essentially the ones introduced in Section 1.3.B, i.e., the uniform structures $\mathscr{F}$, $\mathscr{R}\vee\mathscr{L}$, $\mathscr{R}$, $\mathscr{L}$ and $\mathscr{R}\wedge\mathscr{L}$.

Let's analyze the notion of $\mathscr{U}$-amenability for these five uniformities. If G is $\mathscr{F}$-amenable, then G is also $\mathscr{U}$-amenable for every functionally invariant uniformity $\mathscr{U}$ for G by Theorem 1.2.15. If G is $\mathscr{R}\vee\mathscr{L}$-amenable, then G is also $\mathscr{U}$-amenable for $\mathscr{U} \in \{\mathscr{R},\mathscr{L},\mathscr{R}\wedge\mathscr{L}\}$, and if G is $\mathscr{U}$-amenable for $\mathscr{U} \in \{\mathscr{R},\mathscr{L}\}$, then it is also $\mathscr{R}\wedge\mathscr{L}$-amenable. Moreover, they are not equivalent as the following examples show.

Example 1.4.10. 1) Let $G = \mathrm{Sym}(\mathbf{N})$ be the group of all permutations of the natural numbers $\mathbf{N}$. Then there is only a Polish topology which makes G a topological group ([KR07, Theorem 1.11]). With respect to its unique Polish topology, we have that G is $\mathscr{R}$-amenable but not $\mathscr{F}$-amenable, see [GH17, Proposition 5.5].

2) Let $\mathcal{H}$ be a (complex) infinite-dimensional Hilbert space and let $U(\mathcal{H})$ be the group of unitary operators on $\mathcal{H}$. The group $U(\mathcal{H})$ considered with its strong operator topology is a topological group, see [H79, Section 1] for details. Then $U(\mathcal{H})$ is $\mathscr{R}$-amenable ([H73, II.2. Proposition 6]) but not $\mathscr{L}$-amenable ([P06, Example 3.6.3]). Moreover, this example shows that:

- $\mathscr{R}$-amenability doesn't imply $\mathscr{R}\vee\mathscr{L}$-amenability. Indeed, if it was the case, then $\mathscr{R}$-amenability will imply $\mathscr{L}$-amenability;
- $\mathscr{R}\wedge\mathscr{L}$-amenability doesn't imply $\mathscr{L}$-amenability as $U(\mathcal{H})$ is also $\mathscr{R}\wedge\mathscr{L}$-amenable.

3) In [C18], Carderi and Thom constructed a Polish group G which is $\mathscr{L}$-amenable but not $\mathscr{F}$-amenable. Precisely, the group constructed by Carderi and Thom is

$\mathscr{R}$-amenable ([C18, p. 259]) and SIN.[3] Therefore, $\mathscr{L}$-amenable. But it is not $\mathscr{F}$-amenable because it has the free group on two generators as a closed subgroup ([C18, Theorem 4.2] and [GH17, Corollary 4.6]).

If $\mathscr{R} \wedge \mathscr{L}$-amenability implies $\mathscr{R}$-amenability, $\mathscr{R}$-amenability implies $\mathscr{R} \vee \mathscr{L}$-amenability, and $\mathscr{R} \vee \mathscr{L}$-amenability implies $\mathscr{F}$-amenability are open questions.

Nevertheless, for locally compact groups, life is easier. In fact, all the amenabilities defined above coincide. Precisely, it was proved the following:

Theorem 1.4.11. *Let G be a locally compact group. If G has an invariant mean for one of the following spaces*

$$L^\infty(G),\ \mathcal{C}^b(G),\ \mathcal{C}^b_{lu}(G),\ \mathcal{C}^b_{ru}(G) \text{ or } \mathcal{C}^b_u(G),$$

then G has an invariant mean for all the others.

Proof. See [G69, Theorem 2.2.1]. □

In particular, if a locally compact group G is $\mathscr{U}$-amenable for a functionally invariant uniformity $\mathscr{U}$ which contains the Roelcke uniformity, then G is also $\mathscr{U}'$-amenable for every other functionally invariant uniformity $\mathscr{U}'$ for G.

Another exciting aspect is that every locally compact group admits a functionally invariant uniformity $\mathscr{U}$ for which it is $\mathscr{U}$-amenable.

Example 1.4.12 (M. Gheysens)**.** Let G be a locally compact group and denote $\widehat{G}$ its Alexandroff compactification. Then G naturally embeds in $\widehat{G}$. Now, consider on G the uniform structures $\mathscr{U}_A$ given by the restriction of the unique uniformity of $\widehat{G}$ to G. Then $\mathscr{U}_A$ is a functionally invariant uniformity for G which is called the **Alexandroff uniformity**. Peculiarity of this uniformity is that

$$\mathcal{C}^b_u(G, \mathscr{U}_A) = \left\{ f : G \longrightarrow \mathbf{R} : \lim_{g \to \infty} f(g) \text{ exists} \right\} = \mathcal{C}_0(G) \oplus \mathbf{R}\mathbf{1}_G.$$

Therefore, we can conclude that every locally compact group G is $\mathscr{U}_A$-amenable for the Alexandroff uniformity. An invariant mean for $\mathcal{C}^b_u(G, \mathscr{U}_A)$ can be explicitly given by $\mathrm{m}(f) = \lim_g f(g)$ for every $f \in \mathcal{C}^b_u(G, \mathscr{U}_A)$.

We proceed by giving a few characterizations of $\mathscr{U}$-amenability when $\mathscr{U}$ is a functionally invariant uniformity for G. The next theorem is well-known in the community, but it was impossible to find a formal proof. For this reason, we have decided to give it here. To this end, we need the following technical lemma that we will use in some specific case. However, it is helpful to have it stated in full generality for later purposes.

Let E be a dual vector space. We write $\mathscr{U}_c^*$ for the canonical uniformity of E with respect to the weak-* topology.

[3] This is because the projective limit of a countable family of SIN-groups is still a SIN-group.

Lemma 1.4.13. *Let $(G, \mathscr{U})$ be a uniform topological group which acts on a uniform space $(X, \mathscr{U}_X)$ via the action γ. Suppose that $\mathfrak{F}^b(X)$ is a π_γ-invariant function subspace of $\mathcal{C}_u^b(X, \mathscr{U}_X)$.*

a) The map

$$ev : (X, \mathscr{U}_X) \longrightarrow \left(\mathfrak{F}^b(X)^*, \mathscr{U}_c^*\right), \quad x \longmapsto ev(x) = ev_x$$

is bounded and uniformly continuous.

b) The map

$$\omega_\gamma : (G, \mathscr{U}) \longrightarrow \left(\mathfrak{F}^b(X)^*, \mathscr{U}_c^*\right), \quad g \longmapsto \omega_\gamma(g) = \pi_\gamma(g)^* ev_{x_0}$$

is bounded and uniformly continuous for every $x_0 \in X$ such that the orbital map

$$(G, \mathscr{U}) \longrightarrow (X, \mathscr{U}_X), \quad g \longmapsto \gamma(g)(x_0)$$

is uniformly continuous.

Proof. We start showing point a). Firstly, we prove that the map ev is $(\mathscr{U}_X, \mathscr{U}_c^*)$-uniformly continuous. Let $A \in \mathscr{U}_c^*$. We can suppose that A is of the form

$$A_V = \left\{(\psi_1, \psi_2) \in \mathfrak{F}^b(X)^* \times \mathfrak{F}^b(X)^* : \psi_2 - \psi_1 \in V\right\},$$

where $V = \{\psi \in \mathfrak{F}^b(X)^* : |\psi(f_j)| < \epsilon \text{ for every } j = 1, ..., n\}$ for $f_1, ..., f_n \in \mathfrak{F}^b(X)$ and $\epsilon > 0$. It is possible to make such assumption because a neighborhood basis at the origin for the weak-* topology is given by sets of the form of V, see [Bou81, II §6 No.2]. Note that every f_j's is also in $\mathcal{C}_u^b(X, \mathscr{U})$. Therefore, for every $j \in \{1, ..., n\}$ there is $A_j \in \mathscr{U}_X$ such that $(x, y) \in A_j$ implies $(f_j(x), f_j(y)) \in A_\epsilon$, where

$$A_\epsilon = \{(r_1, r_2) \in \mathbf{R} \times \mathbf{R} : |r_2 - r_1| < \epsilon\} \in \mathscr{U}_c^{\mathbf{R}}.$$

Set $A_X = \bigcap_{j=1}^n A_j \in \mathscr{U}_X$. Then $(x, y) \in A_X$ implies $(x, y) \in A_j$ for every $j = 1, ..., n$. Therefore,

$$|ev_x(f_j) - ev_y(f_j)| = |f_j(x) - f_j(y)| < \epsilon \quad \text{for every } j = 1, ..., n.$$

But this means that $ev_x - ev_y \in V$, and hence $(ev_x, ev_y) \in A_V$ as wished. It is left to show that ev is bounded for the weak-* topology, i.e., for every $V \subset \mathfrak{F}^b(X)^*$ neighborhood of the origin there is $\alpha > 0$ such that $im(ev) \subset \alpha V$. Thus, let V a basis neighborhood of the origin. As before, we can suppose that V is of the form

$$V = \left\{\psi \in \mathfrak{F}^b(X)^* : |\psi(f_j)| < \epsilon \text{ for all } j = 1, ..., n\right\},$$

where $f_1, ..., f_n \in \mathfrak{F}^b(X)$ and $\epsilon > 0$. Set $\alpha > \frac{\max_j ||f_j||_\infty}{\epsilon}$ and let $\psi \in im(ev)$. Then there is $x \in X$ such that $\psi = ev_x$, and so

$$|ev_x(f_j)| = |f_j(x)| \leq \max_j ||f_j||_\infty < \alpha\epsilon.$$

This implies that $im(ev) \subset \alpha V$.

For point b), write ω_γ as the composition of the maps

$$(G, \mathscr{U}) \longrightarrow (X, \mathscr{U}_X) \longrightarrow \left(\mathfrak{F}^b(X)^*, \mathscr{U}_c^*\right), \quad g \longmapsto gx_0 \longmapsto ev_{gx_0} = \pi_\gamma(g)^* ev_{x_0}.$$

As the composition of uniformly continuous maps is uniformly continuous by Proposition 1.2.9, ω_γ is $(\mathscr{U}, \mathscr{U}_c^*)$-uniformly continuous. Moreover, ω_γ is bounded, since $im(\omega_\gamma) \subset im(ev)$ and this last set is bounded. □

Theorem 1.4.14. *Let G be a topological group and let $\mathscr{U}$ be a functionally invariant uniformity for G. Then the following assertions are equivalent:*

a) there is an invariant mean on $\mathcal{C}_u^b(G, \mathscr{U})$, i.e., the group G is $\mathscr{U}$-amenable;

b) every affine action of G on a non-empty convex compact set K for which there is $k_0 \in K$ such that the map $g \longmapsto gk_0$ is $\mathscr{U}$-uniformly continuous has a fixed-point;

c) every action of G on a non-empty compact set K for which there is $k_0 \in K$ such that the map $g \longmapsto gk_0$ is $\mathscr{U}$-uniformly continuous and the space $\mathcal{C}(K)$ is π_γ-invariant has an invariant probability measure;

d) for every action γ of G on a uniform space $(X, \mathscr{U}_X)$ for which there is $x_0 \in X$ such that the map $g \longmapsto gx_0$ is $(\mathscr{U}, \mathscr{U}_X)$-uniformly continuous and the space $\mathcal{C}_u^b(X, \mathscr{U}_X)$ is π_γ-invariant there is an invariant mean on $\mathcal{C}_u^b(X, \mathscr{U}_X)$.

Proof. We start showing that a) implies b). Let K be a non-empty convex compact set and suppose that G acts on it by affine transformations. Let $k_0 \in K$ be the point for which the orbital action of G on K is $\mathscr{U}$-uniformly continuous. Define the linear operator

$$T : \mathcal{C}(K) \longrightarrow \mathcal{C}_u^b(G, \mathscr{U}), \quad \phi \longmapsto T(\phi)(g) = \phi(gk_0).$$

Then T is well-defined, equivariant and $T(\mathbf{1}_K) = \mathbf{1}_G$. By hypothesis, there is an invariant mean m on $\mathcal{C}_u^b(G, \mathscr{U})$. Now, the composition of T with m gives rise to an invariant mean $\overline{\mathrm{m}}$ defined on $\mathcal{C}(K)$. But an invariant mean on $\mathcal{C}(K)$ is nothing but an invariant probability measure on K. Let

$$\beta : \mathrm{Prob}(K) \longrightarrow K, \quad \mu \longmapsto \beta(\mu) = b_\mu$$

be the map which assign to every probability measure μ on K its barycenter b_μ. The map β is well-defined by [Bou63, IV §7 No.1 Proposition 1]. Moreover, β is linear and equivariant by definition, see [Bou63, IV §7 No.1 Définition 1]. It is straightforward to see that $b_{\overline{\mathrm{m}}}$ is a fixed-point for the action of G on K. Indeed,

$$gb_{\overline{\mathrm{m}}} = g\beta(\overline{\mathrm{m}}) = \beta(g\overline{\mathrm{m}}) = \beta(\overline{\mathrm{m}}) = b_{\overline{\mathrm{m}}} \qquad \text{for every } g \in G.$$

Let us show that b) implies c). Let K be a non-empty compact set and suppose that there is a point $k_0 \in K$ for which the orbital action of G on it is $\mathscr{U}$-uniformly continuous. The action of G on K induces an action of G on the non-empty convex compact set $\operatorname{Prob}(K)$. An application of Lemma 1.4.13 with $(X, \mathscr{U}_X) = (K, \mathscr{U}_K)$ and $\mathfrak{F}^b(X) = \mathcal{C}(K)$ shows that the map $g \longmapsto g\delta_{k_0}$ is $\mathscr{U}$-uniformly continuous. Here, δ_{k_0} is the Dirac measure at point $k_0 \in K$. Hence, we can apply the hypothesis of b) to find a fixed-point in $\operatorname{Prob}(K)$.

We prove that c) implies d). In fact, let $\mathcal{M}(\mathcal{C}_u^b(X, \mathscr{U}_X))$ be the set of means of $\mathcal{C}_u^b(X, \mathscr{U}_X)$ and let $x_0 \in X$ the point for which the group G acts orbitally $(\mathscr{U}, \mathscr{U}_X)$-uniformly continuous on $(X, \mathscr{U}_X)$. Then the orbital map $g \longmapsto gev_{x_0}$ is $\mathscr{U}_X$-uniformly continuous because of Lemma 1.4.13 with $\mathfrak{F}^b(X) = \mathcal{C}_u^b(X, \mathscr{U}_X)$. Therefore, there is an invariant probability measure on the compact set $\mathcal{M}(\mathcal{C}_u^b(X, \mathscr{U}_X))$. Using the barycenter map β, it is possible to find a fixed-point in $\mathcal{M}(\mathcal{C}_u^b(X, \mathscr{U}_X))$ as made before. But a fixed-point in $\mathcal{M}(\mathcal{C}_u^b(X, \mathscr{U}_X))$ is nothing but an invariant mean.

Finally d) implies a) only by applying the hypothesis to the case where $(X, \mathscr{U}_X) = (G, \mathscr{U})$. Note that, for $e \in G$, the map

$$(G, \mathscr{U}) \longrightarrow (G, \mathscr{U}), \quad g \longmapsto ge = g$$

is uniformly continuous. □

Scholium 1.4.15. It is natural to be interested in the hereditary properties of $\mathscr{U}$-amenability for one of the standard five uniform structures for topological groups. However, we decided not to discuss this topic here. For a clear and complete recapitulation of the hereditary properties for the uniform structures $\mathscr{F}$ and $\mathscr{R}$ for topological groups, we advise taking a look to [GH17]. For the uniform structure $\mathscr{L}$, we refer to the articles [JS20] and [P20].[4] For the case of locally compact groups where everything is equivalent, we refer to [P84, Chapter 3] and [P88, Chapter 3] for the specific case of Lie groups.

1.4.C. Amenability. In the present subsection, we focus on $\mathscr{R}$-amenability: the original and historical case for which the terms amenability was coined and studied.

Definition 1.4.16. We say that a topological group G is **amenable** if it is $\mathscr{R}$-amenable.

A particularity, and a strength, of this case, is that $\mathcal{C}_u^b(G, \mathscr{R}) = \mathcal{C}_{ru}^b(G)$ is the biggest vector subspace of $\ell^\infty(G)$ on which G acts continuously. This fact makes the difference between $\mathscr{R}$-amenability and general $\mathscr{U}$-amenability as continuity of the left-translation representation plays a tremendously important role.

[4] In both articles $\mathscr{L}$-amenability is called skew-amenability.

There is an infinite countable number of characterizations of amenability for topological groups (do not be scared; this number is even uncountable for locally compact groups). Fortunately, we are only interested in three of them. For a list of characterizations of amenability for topological groups, we refer to [L90, Theorem 3.2] and for locally compact groups to the books [P88] and [P84].

Theorem 1.4.17 (Day-Rickert fixed-point Theorem)**.** *Let G be a topological group. Then the following assertions are equivalent:*

a) every jointly continuous affine action of G on a non-empty convex compact set K has a fixed-point;

b) every orbitally continuous affine action of G on a non-empty convex compact set K has a fixed-point;

c) every affine action of G on a non-empty convex compact set K for which there is $k_0 \in K$ such that the map $g \longmapsto gk_0$ is $\mathscr{R}$-uniformly continuous has a fixed-point.

Note that each of these fixed-point properties implies amenability because of Theorem 1.4.14.

Proof of Theorem 1.4.17. We have that a) implies b) directly as every jointly continuous action is orbitally continuous.

Let's show that b) implies c). Consider the non-empty convex compact set $\mathcal{M}(\mathcal{C}_u^b(G, \mathscr{R}))$ and note that the action of G on it is orbitally continuous with respect to the weak-* topology. Therefore, we can apply b) to find an invariant mean. We can conclude using Theorem 1.4.14.

It is left to show that c) implies a). Therefore, suppose that G acts jointly continuous on a non-empty convex compact set K. Then the action on each orbit is $\mathscr{R}$-uniformly continuous by [G17, Lemma C.2.5] and [G17, Lemma C.2.4]. We can apply the hypothesis to find a fixed-point. □

As an application of the Day-Rickert fixed-point Theorem, we have the following famous characterization of amenability.

Corollary 1.4.18. *Let G be a topological group. Then the following assertions are equivalent:*

a) there is an invariant mean on $\mathcal{C}_{ru}^b(G)$, i.e., the group G is amenable;

b) every jointly continuous affine action of G on a non-empty convex compact set K has a fixed-point;

c) every jointly continuous affine action of G on a non-empty compact set K admits a non-zero invariant Radon measure.

Remark 1.4.19. Day showed Theorem 1.4.17 for discrete groups in [D61, Theorem 1]. In the same paper, he claimed to have a generalization of the fixed-point property for topological groups, see [D61, Section 4]. However, this generalization was characterizing the existence of invariant means on $\mathcal{C}^b(G)$ instead of on $\mathcal{C}^b_{ru}(G)$. He corrected this small error in [D64, Theorem 4], where he actually gave a proof of the equivalence of point a) and b) for $\mathscr{U} = \mathscr{F}$ of Theorem 1.4.14. Rickert formulated the correct generalization in [R67, Theorem 4.2], where he gave a proof of the fact that c) of Theorem 1.4.17 implies amenability. In his proof, he used secretly point a) to show that amenability implies the fixed-point property.

Chapter 2

Cones and Lattices

The chapter aims to recall definitions and results about cones and ordered vector spaces. Precisely: the relationship between cones and vector orderings, Riesz spaces and monotone norms.

In each section, we give a range of examples used in the chapters to come. The obsession with illustrating every definition and result with concrete examples also has the ambition to give the reader a panorama of how cones and ordered vector spaces are present in different domains of mathematics.

Basic knowledge in functional analysis is required. However, every functional analysis result we employ can be found in any standard textbooks, e.g., [R86], [C97] or [Bou81].

Standard references for ordered vector spaces are [M91], [S74] and the encyclopedic book [AB99]. Regarding convex cones in vector spaces, we recommend the book [AT07].

Note that an abstract approach to cones is exposed in Appendix A.

2.1 Cones in vector spaces

Let E be an abstract (real) vector space, i.e., a vector space considered without a topology.

Definition 2.1.1. A non-empty subset $C \subset E$ is called a **cone** if it satisfies the following two properties:

(C1) the set C is additive, i.e., $c_1 + c_2 \in C$ for every $c_1, c_2 \in C$;

(C2) the set C is positive homogeneous, i.e., $\alpha c \in C$ for every $c \in C$ and $\alpha \in \mathbf{R}_+$.

In the literature, there are many slightly different definitions of a cone. We decided to follow the approach of Bourbaki ([Bou81, II §2 No.4 Définition 3]). However, the definition of Bourbaki is more general than ours. In fact, a cone is automatically convex and pointed at the origin for us. This is not the case for Bourbaki. The distinction between the two notions is only due to the fact that we allow the scalar α of point $(C2)$ to be zero (cf. [Bou81, II §2 No.4 Proposition 10]).

If C is a cone in a vector space E, then

$$E_C = \{v \in E : v = c_1 - c_2 \text{ for } c_1, c_2 \in C\}$$

is called **the vector space generated by** C. A cone C in a vector space E is said **generating** if E_C is equal to E, i.e., for every $v \in E$ there are $c_1, c_2 \in C$ such that $v = c_1 - c_2$.

If $C \subset E$ is a cone, then the intersection $C \cap (-C)$ is a vector subspace of E. Actually, $C \cap (-C)$ is the biggest vector subspace of E contained in C ([Bou81, II §2 No.4 Corollaire 1]).

A cone C is said **proper** if $C \cap (-C) = \{0\}$.

Example 2.1.2. (Examples of cones)

1) Let $\mathbf{R}^n$ be the Euclidean space of dimension $n \in \mathbf{N}$. Consider the set given by

$$C = \left\{\mathbf{v} = (v_1, ..., v_n) \in \mathbf{R}^n : v_j \geq 0 \text{ for every } j = 1, ..., n\right\}.$$

Then C is a proper cone. Moreover, C is generating, since it is possible to write every $\mathbf{v} \in \mathbf{R}^n$ as $\mathbf{v} = \mathbf{v}_+ - \mathbf{v}_-$, where $\mathbf{v}_+ = (v_1^+, ..., v_n^+)$ and $\mathbf{v}_- = (v_1^-, ..., v_n^-)$ are given by

$$v_j^+ = \begin{cases} v_j & \text{if } v_j \geq 0 \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad v_j^- = \begin{cases} |v_j| & \text{if } v_j < 0 \\ 0 & \text{otherwise} \end{cases}$$

for every $j = 1, ..., n$.

2) Consider the Euclidean space $\mathbf{R}^n$ of dimension $n \in \mathbf{N}$, and define the **lexicographic cone** as

$$C = \left\{\mathbf{v} = (v_1, ..., v_n) \in \mathbf{R}^n : \exists k \in \{1, ..., n\} \text{ s.t. } v_j = 0 \text{ for } j \leq k \text{ and } v_{k+1} > 0\right\}.$$

Then C is proper and generating.

3) Let X be a set and let $\mathfrak{F}(X)$ be a real function space. Then we define the **cone of positive functions** as

$$C = \{f \in \mathfrak{F}(X) : f(x) \geq 0 \text{ for every } x \in X\}.$$

The cone C is proper, but it is not generating in general.

4) Let X be a set and consider the vector space $\mathbf{R}^X$ of all functions from X to $\mathbf{R}$. In this case, the cone of positive functions C is generating as we can write any $f \in \mathbf{R}^X$ as the difference $f = f_+ - f_-$, where

$$f_+(x) = \begin{cases} f(x) & \text{if } f(x) \,{}_{\mathbf{R}}{\geq} 0 \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad f_-(x) = \begin{cases} |f(x)| & \text{if } f(x) \leq_{\mathbf{R}} 0 \\ 0 & \text{otherwise} \end{cases}$$

are in C.

5) Let X be a set and C be a cone in a vector space E. Let $\mathfrak{F}(X, E)$ be a vector-valued function space. Then the set

$$\mathfrak{C} = \{f \in \mathfrak{F}(X, E) : f(x) \in C \text{ for all } x \in X\}$$

is the **cone of C-positive vector-valued functions**. If C is a proper cone, then $\mathfrak{C}$ is also a proper cone. However, in general $\mathfrak{C}$ is not generating even if C is.

6) Let C be a cone in a vector space E and let E^* be the algebraic dual of E. The **dual cone** of C, or the **polar cone** of C, is the cone defined by

$$C^* = \{\psi \in E^* : \psi(c) \geq 0 \text{ for all } c \in C\}\,.$$

In general, C^* is not generating but if C is proper, then C^* is also proper.

7) Let $\mathcal{A}$ be a (complex) C^*-algebra and let $\mathcal{A}_{sa}$ be the vector subspace of all self-adjoint elements of $\mathcal{A}$, i.e., $a \in \mathcal{A}_{sa}$ if and only if $a = a^*$. Define the set

$$C_{sa} = \{a \in \mathcal{A}_{sa} : a = b^*b \text{ for some } b \in \mathcal{A}\}\,.$$

Then C_{sa} is called the **C^*-cone of $\mathcal{A}$** or the **self-adjoint cone of $\mathcal{A}$**. It is a proper cone but it is not generating in $\mathcal{A}$. However, C_{sa} is a generating cone for $\mathcal{A}_{sa}$. We refer to [D77, Proposition 1.6.1] for more details.

The following lemma gives a useful and simple characterization of generating cones.

Lemma 2.1.3. *Let C be a cone in a vector space E. Then C is generating if and only if for every $v \in E$ there is $c \in C$ such that $c - v \in C$.*

Proof. Suppose that the cone C is generating and let $v \in E$. Then there are $c_1, c_2 \in C$ such that $v = c_1 - c_2$. This implies that $c_1 - v = c_2 \in C$.

Let now suppose that for every $v \in E$ there is $c \in C$ such that $c - v \in C$, and we want to show that C is generating. Let $v \in E$ and take $c \in C$ such that $c - v \in C$. Define $c_1 = c$ and $c_2 = c - v$. Then $v = c_1 - c_2$. □

This last lemma directly inspires the following definition.

Definition 2.1.4. Let C be a cone in a vector space E and let $u \in C$. We say that u is a C**-order unit** if for every $v \in E$ there is $\alpha \in \mathbf{R}_+$ such that $\alpha u - v \in C$.

If C is a cone in a vector space E and u is a C-order unit. We say that E admits a C-order unit and that u is the C-order unit of E.

Corollary 2.1.5. *Let C be a cone in a vector space E and suppose that E admits a C-order unit. Then the cone C is generating.*

Proof. Let u be the C-order unit of E. Then for every $v \in E$ there is $\alpha \in \mathbf{R}_+$ such that $\alpha u - v \in C$. We can conclude that C is generating by Lemma 2.1.3. □

Example 2.1.6. 1) Let X be a set and let $\mathfrak{F}^b(X)$ be a real function space of bounded functions. If $\mathfrak{F}^b(X)$ contains the characteristic function $\mathbf{1}_X$, then the cone of positive functions is generating.

2) Let K be a compact topological space and consider the vector space $\mathcal{C}(K)$ of all real continuous functions on K. Fix $k_0 \in K$ and define the cone

$$C = \{f \in \mathcal{C}(K) : f(k_0) = 0 \text{ and } f(k) \geq 0 \text{ for all } k \in K\}.$$

Then the cone C is proper but not generating. Indeed, suppose that C is generating and consider the vector $\mathbf{1}_K$. Then there are $f_1, f_2 \in C$ such that $\mathbf{1}_K = f_1 - f_2$. But this is a contradiction as $1_K(k_0) = 1 \neq 0 = f_1(k_0) - f_2(k_0)$.

Note that here $\mathbf{1}_K$ is an order unit for the cone of positive functions but not for the cone C. Indeed, $\mathbf{1}_K$ is not even in C.

3) Let $\mathcal{C}([0,2])$ be the vector space of all real continuous functions defined on the interval $[0,2]$. Define the cone

$$C = \{f \in \mathcal{C}([0,2]) : f(x) \geq 0 \text{ for every } x \in [0,1]\}.$$

Then the cone C is generating, as $\mathbf{1}_{[0,2]} \in C$. However, it is not proper. In fact,

$$C \cap (-C) = \{f \in \mathcal{C}([0,2]) : f(x) = 0 \text{ for every } x \in [0,1]\}.$$

4) Let $\mathbf{N}$ be the set of natural numbers, and let $c_{00}(\mathbf{N})$ be the vector space of finitely supported real functions defined on $\mathbf{N}$. Consider the cone of positive functions C of $c_{00}(\mathbf{N})$. Then C is generating but $c_{00}(\mathbf{N})$ admits no C-order unit. Indeed, suppose it is not the case. Then there is $u \in C$ which is a C-order unit of $c_{00}(\mathbf{N})$. Let $v \in c_{00}(\mathbf{N})$ such that $v \notin C$ and $\operatorname{supp}(v) \supsetneq \operatorname{supp}(u)$. Then there is no $\alpha \in \mathbf{R}_+$ such that $\alpha u - v \in C$.

2.2 Ordered vector spaces

Recall that an **order relation** $\leq$ on a set X is a binary relation with the following properties:

(O1) reflexivity, i.e., $x \leq x$ for all $x \in X$;

(O2) antisymmetry, i.e., if $x \leq y$ and $y \leq x$, then $x = y$ for all $x, y \in X$;

(O3) transitivity, i.e., if $x \leq y$ and $y \leq z$, then $x \leq z$ for all $x, y, z \in X$.

The pair $(X, \leq)$ is said a **partially ordered set**. The notation $x < y$, for $x, y \in X$, means that $x \leq y$ and $x \neq y$. When $x \leq y$, we also say that y **dominates** x and, in the case where $x < y$, that y **strictly dominates** x. Alternatively, we also use the symbols $\geq$ and $>$ for $\leq$ and $<$, respectively.

Let $(X, \leq)$ be a partially ordered set and let A be a non-empty subset of X. We say that A is **majorized by a point** $x \in X$, or that the point x is **majorizing** A, if $a \leq x$ for every $a \in A$. Similarly, we say that A is **majorized by a set** $B \subset X$, or that B is **majorizing** A, if for every $a \in A$ there is $b \in B$ such that $a \leq b$.

2.2.A. Vector ordering and vector spaces. Let E be a vector space and let $\leq$ be an order relation on E such that $(E, \leq)$ is a partially ordered set.

Definition 2.2.1. The order relation $\leq$ on E is called a **vector ordering** if it is compatible with the vector space structure. This means that $\leq$ satisfies, in addition, the following two axioms:

(V1) if $x \leq y$, then $x + z \leq y + z$ for every $x, y, z \in E$;

(V2) if $x \leq y$, then $\alpha x \leq \alpha y$ for every $x, y \in E$ and $\alpha \in \mathbf{R}_+$.

The pair $(E, \leq)$ is called an **ordered vector space**.

We only write E if the order $\leq$ is well-understood.

An element x of an ordered vector spaces E is called **positive** if $x \geq 0$. The set of all positive vectors of an ordered vector space is noted $E_+ = \{x \in E : x \geq 0\}$, and we refer to it as **the positive cone of** E. As the name suggests, E_+ is a proper cone in E.

Conversely, let C be a proper cone in a vector space E and define the binary relation

$$x \leq_C y \iff y - x \in C.$$

Then E equipped with $\leq_C$ becomes an ordered vector space with positive cone C. Actually, there is a one-to-one correspondence between vector orderings on vector spaces and proper cones.

Note that if the cone C is not proper, then the binary relation $\leq_C$ makes E a pre-order vector space, since the antisymmetry axiom is missing.

Example 2.2.2. (Examples of ordered vector spaces)

1) Let $\mathbf{R}^n$ be the Euclidean space of dimension $n \in \mathbf{N}$, and consider the cone C given in point 1) of Example 2.1.2. Then the corresponding ordering is given by

$$\mathbf{v} \leq \mathbf{w} \iff v_j \leq w_j \quad \text{for every } j = 1, ..., n,$$

where $\mathbf{v} = (v_1, ..., v_n)$ and $\mathbf{w} = (w_1, ..., w_n)$ are in $\mathbf{R}^n$.

2) Consider the vector space $\mathbf{R}^n$. We define the **lexicographic order** as

$$\mathbf{v} \leq \mathbf{w} \iff \exists k \in \{1, ..., n\} \text{ such that } v_j = w_j \text{ for all } j \leq k \text{ and } v_{k+1} < w_{k+1},$$

where $\mathbf{v} = (v_1, ..., v_n)$ and $\mathbf{w} = (w_1, ..., w_n)$ are in $\mathbf{R}^n$. The corresponding positive cone is the lexicographic cone.

3) Let X be a set and let $\mathfrak{F}(X)$ be a function space. We define the **pointwise order** as

$$f \leq g \iff f(x) \leq g(x) \quad \text{for every } x \in X,$$

where $f, g \in \mathfrak{F}(X)$. Then $(\mathfrak{F}(X), \leq)$ is an ordered vector space. The corresponding cone is the cone of positive functions.

4) Let X be a set and C be a proper cone in a vector space E. Let $\mathfrak{F}(X, E)$ be a vector-valued function space. We define the **C-pointwise order** as

$$f \leq g \iff f(x) \leq_C g(x) \quad \text{for every } x \in X,$$

where $f, g \in \mathfrak{F}(X, E)$ and $\leq_C$ is the vector ordering on E corresponding to C. Then $(\mathfrak{F}(X, E), \leq)$ is an ordered vector space. The corresponding cone is the cone of C-positive vector-valued functions.

5) Let E be an ordered vector space and let E^* be its algebraic dual. We define the **dual order** on E^* as

$$\psi_1 \leq \psi_2 \iff \psi_1(v) \leq \psi_2(v) \quad \text{for every } v \in E,$$

where $\psi_1, \psi_2 \in E^*$. Then $(E^*, \leq)$ is an ordered vector space with positive cone $(E^*)_+$, the dual cone of E_+.

6) Let E and V be two ordered vector spaces and consider the (abstract) tensor product $E \otimes V$. Then the set

$$C_\otimes = \left\{ \sum_{j=1}^{n} v_j \otimes w_j : v_j \in E_+ \text{ and } w_j \in V_+ \right\}$$

is a proper cone in $E \otimes V$. The cone $C_\otimes$ is called **the tensor cone of** $E \otimes V$ and $E \otimes V$ becomes an ordered vector space when equipped with the vector ordering given by $C_\otimes$. For more details see [ER70].

7) Let $\mathcal{A}$ be a (complex) C^*-algebra. Then the vector ordering corresponding to the C^*-cone of $\mathcal{A}$ makes $\mathcal{A}$ a (real) ordered vector space. We call this order the **C^*-order of $\mathcal{A}$**.

We can easily translate the notion of order unit for ordered vector spaces. Let E be an ordered vector space and let E_+ be its positive cone. Then a vector $u \in E_+$ is called an **order unit** if for every $v \in E$ there is $\alpha \in \mathbf{R}_+$ such that $v \leq \alpha u$.

Definition 2.2.3. An ordered vector space E is said **Archimedean** if $ny \leq x$ for all $n \in \mathbf{N}$ implies that $y \leq 0$, where $y \in E$ and $x \in E_+$.

A proper cone C in a vector space E is said Archimedean if the corresponding order is.

This property shall not be confused with the *Archimedean property* often used in number theory. For the real numbers with their standard order, the two definitions coincide, but this is generally not true. See [AB99, Example 8.3].

Example 2.2.4. The classical example of a non-Archimedean ordered vector space is $\mathbf{R}^n$, for $n \geq 2$, equipped with the lexicographic order. In fact, we have that $k\mathbf{e}_2 \leq \mathbf{e}_1$ for every $k \in \mathbf{N}$ but $\mathbf{e}_2 > 0$. Here, $\mathbf{e}_j$ is the vector with entry 1 at the j-th place and 0 everywhere else.

2.2.B. Operators between ordered vector spaces. The subsection introduces particular linear operators between ordered vector spaces, namely positive operators.

Definition 2.2.5. Let E and V be two ordered vector spaces and let $T : E \longrightarrow V$ be a linear operator. Then T is said a **positive operator** if it sends positive vectors to positive vectors, i.e., if $v \in E_+$, then $T(v) \in V_+$. The operator T is said a **strictly positive operator** if it sends non-zero positive vectors to non-zero positive vectors, i.e., if $v \in E_+$ and $v \neq 0$, then $T(v) \in V_+$ and $T(v) \neq 0$.

In other words, given two ordered vector spaces E and V with the respective positive cones C_E and C_V, a positive operator T between E and V is nothing but a linear map such that $T(C_E) \subset C_V$.

If the target space V is equal to $\mathbf{R}$ equipped with its natural order, then T is said a **functional**. If T is positive, then T is said a **positive functional**. Similarly, if T is strictly positive, then T is said a **strictly positive functional**. Note that a functional T is positive if and only if T is an element of the dual cone of E_+. The set of all positive functionals is also called the **cone of positive functionals** of E, and it is denoted E_+^*.

Example 2.2.6. (Examples of positive operators)

1) Let X be a set and let $\ell^\infty(X)$ be the space of all bounded real functions on X. Fix $f \in \ell^\infty(X)$ and define the linear operator

$$T_f : \ell^\infty(X) \longrightarrow \ell^\infty(X), \quad h \longmapsto T_f(h) = f \cdot h.$$

Then T_f is positive if and only if f is a positive function. Moreover, the operator T_f is strictly positive if and only if f is a strictly positive function such that $\operatorname{supp}(f) = X$.

2) Let X be a set and let E be a vector subspace of $\ell^\infty(X)$ which contains the constant functions. Then every mean m on E is a positive functional. Indeed, suppose it is not the case. Thus, there is a non-zero positive function $f \in E$ such that $\mathrm{m}(f) < 0$. We can suppose that $||f||_\infty = 1$. Then

$$\mathrm{m}\left(\mathbf{1}_X - f\right) = \mathrm{m}(\mathbf{1}_X) - \mathrm{m}(f) > 1.$$

But this is a contradiction with the fact that $\mathrm{m}(\mathbf{1}_X - f) \leq ||\mathbf{1}_X - f||_\infty \leq 1$. Therefore, m is positive.

3) Let $\mathbf{R}^\mathbf{N}$ be the ordered vector space of all real sequences. Then there are no strictly positive functionals on $\mathbf{R}^\mathbf{N}$, because every positive linear functional T on $\mathbf{R}^\mathbf{N}$ is of the form $T((a_n)_n) = \sum_{n=1}^{\infty} a_n b_n$, where $(a_n)_n \in \mathbf{R}^\mathbf{N}$ and $(b_n)_n \in c_{00}(\mathbf{N})$. We refer [AB99, Theorem 16.3] for details.

Positive functionals have interesting extension properties. The following theorem can be interpreted as an ordered vector space version of the famous Hahn-Banach Extension Theorem.

Theorem 2.2.7 (Kantorovich Theorem). *Let E be an ordered vector space and let $V \subset E$ be a vector subspace. Suppose that E is majorized by V. Then every positive functional $T : V \longrightarrow \mathbf{R}$ can be extended to a positive functional $\overline{T} : E \longrightarrow \mathbf{R}$.*

Proof. See [M91, Corollary 1.5.9]. □

It is possible to state the Kantorovich Theorem in a more general setting. See for example [AT07, Theorem 1.60]. However, the classical Kantorovich Theorem is enough for our future purposes.

2.3 Riesz spaces

The section introduces Riesz spaces, which are ordered vector spaces equipped with a particularly rich vector ordering. They are so named in honour of the Hungarian mathematician Frigyes Riesz[1], who first defined them in his paper [R29].

2.3.A. Definition and first examples. Let E be an ordered vector space and let $A \subset E$ be a non-empty subset.

Definition 2.3.1. A vector $v \in E$ is called the **supremum**, or **the least upper bound**, of A if

[1] Györ 1880 - Budapest 1956

a) v is an upper bound of A, i.e., $a \leq v$ for every $a \in A$;

b) v is the least upper bound of A, i.e., if $u \in E$ is an upper bound of A, then $v \leq u$.

If $v \in E$ is the supremum of a non-empty set A, then we write $v = \sup A$.

The definition of **infimum**, or **greatest lower bound**, is analogous. If v is the infimum of a non-empty set $A \subset E$, then we write $v = \inf A$.

For a finite set $\{v_1, ..., v_n\} \subset E$, we use the lattice notation

$$\sup\{v_1, ..., v_n\} = \bigvee_{j=1}^{n} v_j \quad \text{and} \quad \inf\{v_1, ..., v_n\} = \bigwedge_{j=1}^{n} v_j.$$

Definition 2.3.2. An ordered vector space E is said a **Riesz space** (or a **vector lattice**) if every two vectors of E have a supremum and an infimum in E.

A proper cone C in a vector space E is said a **lattice cone** if E equipped with the vector ordering given by C is a Riesz space.

In other words, an ordered vector space E is a Riesz space if and only if for every $v, w \in E$, the vectors $v \vee w$ and $v \wedge w$ exist and are in E.

For every vector v in a Riesz space E, we can define its **positive part** v_+ and its **negative part** v_- as follows: $v_+ = v \vee 0$ and $v_- = (-v) \vee 0$. This gives rise to a natural definition of absolute value. We define the **absolute value** of a vector $v \in E$ via the equation $|v| = v \vee (-v)$.

Let E be a Riesz space. Then the maps from $E \times E$ to E given by

$$(v, w) \longmapsto v \wedge w, \quad (v, w) \longmapsto v \vee w$$

and the maps from E to E given by

$$v \longmapsto |v|, \quad v \longmapsto v_+ \text{ and } v \longmapsto v_-$$

are called the **lattice operations on** E.

Example 2.3.3. (Examples of Riesz spaces)

1) The Euclidean space $\mathbf{R}^n$ with its standard order is a Riesz space. The lattice operations are given by

$$\mathbf{v} \vee \mathbf{w} = (\max(v_1, w_1), ..., \max(v_n, w_n)) \text{ and } \mathbf{v} \wedge \mathbf{w} = (\min(v_1, w_1), ..., \min(v_n, w_n)),$$

where $\mathbf{v} = (v_1, ..., v_n)$ and $\mathbf{w} = (w_1, ..., w_n)$ are in $\mathbf{R}^n$.

2) The Euclidean space $\mathbf{R}^n$ with the lexicographic order is a Riesz space because the lexicographic order is total.[2]

3) Let X be a set and let E be a Riesz space. For two functions f_1 and f_2 in E^X, we define their supremum $f \vee g$ and their infimum $f \wedge g$ pointwise by

$$(f \vee g)(x) = \max(f(x), g(x)) \quad \text{and} \quad (f \wedge g)(x) = \min(f(x), g(x)),$$

for every $x \in X$. Then E^X equipped with the pointwise order, supremum and infimum is a Riesz space. Similarly, the space $\ell^\infty(X, E)$ is also a Riesz space.

4) Let (Ω, Σ, μ) be a measurable space. Recall that two measurable functions are equivalent if they agree μ-almost everywhere. Let $L^0(\Omega, \mu)$ be the vector space of all equivalence classes of μ-measurable real functions. We define the **μ-almost everywhere pointwise order** on $L^0(\Omega, \mu)$ by

$$f \leq g \iff f(x) \leq g(x) \quad \text{for } \mu\text{-almost all } x \in \Omega.$$

If we define the supremum and the infimum of two measurable functions μ-almost everywhere pointwise, then $L^0(\Omega, \mu)$ is a Riesz space ([AB99, Theorem 4.27]).

Note that the positive cone of a Riesz space is always generating. Indeed, let E be a Riesz space. After Lemma 2.1.3, we only have to show that for every $v \in E$ there is $c \in E_+$ such that $c - v \in E_+$. The vector $|v| \in E_+$ accomplishes this task.

However, there are generating cones that are not lattice cones, as the following example shows.

Example 2.3.4. Let $\mathcal{C}^1((0,2))$ be the vector space of all real differentiable functions defined on $(0,2)$. Then $\mathcal{C}^1((0,2))$ equipped with the pointwise order is an ordered vector space. Consider the two differentiable functions $f(x) = 2 - x$ and $g(x) = x + 1$. We claim that $f \vee g$ doesn't exist in $\mathcal{C}^1((0,2))$. Suppose it is not the case. Then there is $h \in \mathcal{C}^1((0,2))$ such that $h = f \vee g$. Take the sequence of differentiable functions $(f_n)_{n \in \mathbf{N}}$ given by

$$f_n(x) = |1 - x|^{1 + \frac{1}{n}} + \left(1 + \frac{1}{n}\right).$$

Then

a) f_n is an upper bound of the set $\{f, g\}$ for every $n \in \mathbf{N}$;

b) $f_n \geq f_{n+1}$ for every $n \in \mathbf{N}$;

c) $(f_n)_n$ converges pointwise to the function $F(x) = |1 - x| + 1$.

[2] Recall that an order is said **total** if every two elements can be compared.

By definition of supremum, $f_n \geq h \geq F$ for every $n \in \mathbf{N}$. This means that $h = F$ by points b) and c). But this is a contradiction, as F is not in $\mathcal{C}^1((0,2))$.

However, the cone of positive functions is generating, since $\mathbf{1}_{(0,2)} \in \mathcal{C}^1((0,2))$.

The next lemma contains a (incomplete) collection of lattice identities for vectors in Riesz spaces. We decided to report only the identities used in this monograph. We refer to [AT07, Theorem 1.17], [M91, Theorem 1.1.1] and [S74, pp. 51-52] for more lattice identities.

Lemma 2.3.5 (Lattice identities). *Let v and w be vectors in a Riesz space E. Then the following identities hold:*

a) $v \vee w = -\left[(-v) \wedge (-w)\right]$ *and* $v \wedge w = -\left[(-v) \vee (-w)\right]$*;*

b) $\alpha(v \vee w) = (\alpha v) \vee (\alpha w)$ *and* $\alpha(v \wedge w) = (\alpha v) \wedge (\alpha w)$ *for every* $\alpha \in \mathbf{R}_+$*;*

c) $|\alpha v| = |\alpha||v|$ *for all* $\alpha \in \mathbf{R}$*;*

d) $|v + w| \leq |v| + |w|$ *and* $|v| - |w| \leq |v - w|$*;*

e) $v = v_+ - v_-$ *and* $v_+ \vee v_- = 0$*;*

f) $|v| = v_+ + v_-$*, and hence* $v = 0 \iff |v| = 0$*;*

g) $v \vee w = \frac{1}{2}\left(v + w + |v - w|\right)$*;*

h) $v \wedge w = \frac{1}{2}\left(v + w - |v - w|\right)$*;*

i) $v \vee w \leq |v| + |w|$ *and* $v, w \leq v \vee w$*;*

j) $v \wedge w \leq v, w$*.*

Proof. For a proof of points a) and b) see [AT07, Lemma 1.15], and for a proof of the points from c) to h) see [S74, Proposition 1.4 & Corollary 1]. Points i) and j) are consequences of the definition of supremum and minimum. □

From point a) of the above lemma, we can conclude that to check if an ordered vector space E is a Riesz space it is sufficient to show that for every two vectors $v, w \in E$ their supremum, or their infimum, exists, and belongs in E.

Note that points g) and h) are the maximum and minimum formulas for Riesz spaces.

Definition 2.3.6. Let X be a set and let E be a Riesz space. Let $\mathfrak{F}(X, E)$ be a vector-valued function space equipped with the pointwise order. Then for every $f, g \in \mathfrak{F}(X, E)$, we define their supremum $f \vee g$ and their infimum $f \wedge g$ pointwise by

$$(f \vee g)(x) = \max\left(f(x), g(x)\right) \quad \text{and} \quad (f \wedge g)(x) = \min\left(f(x), g(x)\right),$$

for $x \in X$. If $f \vee g$, or $f \wedge g$, is in $\mathfrak{F}(X, E)$ for every $f, g \in \mathfrak{F}(X, E)$, we say that $\mathfrak{F}(X, E)$ is a **Riesz vector-valued function space**.

Examples of Riesz vector-valued function spaces are given by E^X and $\ell^\infty(X, E)$ for a set X and a RIesz space E.

Actually, every Archimedean Riesz space can be seen as a Riesz function space on some set X ([L71, Chapter 7]).

Proposition 2.3.7. *Let $(X, \mathscr{U}_X)$ be a uniform space. Then $\mathcal{C}_u(X, \mathscr{U}_X)$ and $\mathcal{C}_u^b(X, \mathscr{U}_X)$ are Riesz function spaces.*

Proof. We already know that $\mathcal{C}_u(X, \mathscr{U}_X)$ and $\mathcal{C}_u^b(X, \mathscr{U}_X)$ are vector spaces by Propositions 1.2.8 and 1.1.2. Therefore, it suffices to show that the pointwise supremum and infimum of two uniformly continuous functions is still a uniformly continuous function. We give the proof only for the pointwise supremum. The one for the pointwise infimum is similar.

Let $f, g \in \mathcal{C}_u(X, \mathscr{U}_X)$, then the function $F = f \vee g$ is in $\mathbf{R}^X$ as this last space is a Riesz vector-valued function space. Thus, we want to show that $F = f \vee g$ is in $\mathcal{C}_u(X, \mathscr{U}_X)$. Let $A \in \mathscr{U}_{\mathbf{R}}$. We can suppose that A is of the form

$$A_\epsilon = \{(r_1, r_2) \in \mathbf{R} \times \mathbf{R} : |r_2 - r_1| < \epsilon\}$$

for $\epsilon > 0$. Then there are $A_f, A_g \in \mathscr{U}_X$ such that $(x, y) \in A_f$ implies $(f(x), f(y)) \in A_{\frac{\epsilon}{2}}$ and $(x, y) \in A_g$ implies $(g(x), g(y)) \in A_{\frac{\epsilon}{2}}$. Here, $A_{\frac{\epsilon}{2}} = \left\{(r_1, r_2) \in \mathbf{R} \times \mathbf{R} : |r_2 - r_1| < \frac{\epsilon}{2}\right\}$. Set $A_X = A_f \cap A_g \in \mathscr{U}_X$ and note that if $(x, y) \in A_X$, then

$$\begin{aligned}
|F(x) - F(y)| &= |(f \vee g)(x) - (f \vee g)(y)| \\
&= \left| \frac{1}{2}\Big(f(x) + g(x) + |f(x) - g(x)|\Big) - \frac{1}{2}\Big(f(y) + g(y) + |f(y) - g(y)|\Big) \right| \\
&\leq \frac{1}{2}\Big(|f(x) - f(y)| + |g(x) - g(y)| + \big|\, |f(x) - g(x)| - |f(y) - g(y)| \,\big| \Big) \\
&\leq \frac{1}{2}\Big(|f(x) - f(y)| + |g(x) - g(y)| + \big| f(x) - g(x) - (f(y) - g(y)) \big| \Big) \\
&\leq \frac{1}{2}\Big(|f(x) - f(y)| + |g(x) - g(y)| + |f(x) - f(y) + g(y) - g(x)| \Big) \\
&\leq |f(x) - f(y)| + |g(x) - g(y)| \\
&\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,
\end{aligned}$$

where we used point g) of Lemma 2.3.5 in the second equality. Therefore, $(F(x), F(y))$ is in A_ϵ, and hence F is uniformly continuous. This show that $\mathcal{C}_u(X, \mathscr{U}_X)$ is a Riesz space. For $\mathcal{C}_u^b(X, \mathscr{U}_X)$, we can use the same proof together with the fact that F is bounded by point i) of Lemma 2.3.5. Therefore, $\mathcal{C}_u^b(X, \mathscr{U}_X)$ is also a Riesz space. □

Corollary 2.3.8. *Let G be a topological group. Then the vector spaces*

$$\mathcal{C}^b(G),\ \mathcal{C}_{ru}^b(G),\ \mathcal{C}_{lu}^b(G) \ \text{ and } \ \mathcal{C}_u^b(G)$$

are all Riesz function spaces.

Proof. The proof is a combination of Proposition 2.3.7 and Theorem 1.3.4. □

Example 2.3.9. (More examples of Riesz spaces)

1) Let X be a uniformizable space. Then $\mathcal{C}^b(X) = \mathcal{C}^b_u(X, \mathscr{F})$ is a Riesz function space thanks to Proposition 2.3.7 and Theorem 1.2.17. More generally, $\mathcal{C}^b(X)$ is a Riesz function space for every, not a priori uniformizable, topological space X ([AB99, Corollary 2.29]).

2) Let G be a topological group and let $\mathscr{U}$ be a functionally invariant uniformity for G. Then $\mathcal{C}^b_u(G, \mathscr{U})$ is a Riesz function space by Proposition 2.3.7.

3) Let (Ω, Σ, μ) be a measurable space. For a function $f \in L^0(\Omega, \mu)$, we define the (possibly infinite) value

$$\mathcal{N}_p(f) = \int_\Omega |f(x)|^p d\mu(x),$$

where $p \in (0, +\infty)$ and the integral is taken in the sense of Lebesgue ([AB99, Definition 11.12]). For $p \in (0, +\infty)$, we say that a function $f \in L^0(\Omega, \mu)$ is p-**integrable** if $\mathcal{N}_p(f) < \infty$, and we write

$$L^p(\Omega, \mu) = \left\{ f \in L^0(\Omega, \mu) : \mathcal{N}_p(f) < \infty \right\}$$

for the set of all p-integrable functions defined on (Ω, Σ, μ). The set $L^p(\Omega, \mu)$ is actually a vector subspace of $L^0(\Omega, \mu)$. This is because of the monotonocity of the Lebesgue integral [AB99, Theorem 11.13 (3)] and because of the inequalities

$$|a + b|^p \leq (|a| + |b|)^p \leq (2|b|)^p \leq 2^p \left(|a|^p + |b|^p \right),$$

which are true for every $a, b \in \mathbf{R}$ such that $|a| \leq |b|$ and for every $p \in (0, +\infty)$. Moreover by points i) and j) of Lemma 2.3.5 and by the monotonocity of the Lebesgue integral, every $L^p(\Omega, \mu)$ is a Riesz space.

5) Let (Ω, Σ, μ) be a measurable space and let $f \in L^0(\Omega, \mu)$. Define the (possibly infinite) value

$$||f||_\infty = \inf \left\{ M \in \mathbf{R} : |f(x)| \leq M \text{ for } \mu\text{-almost all } x \in \Omega \right\},$$

where $\inf\{\emptyset\} = +\infty$. We say that a measurable function $f \in L^0(\Omega, \mu)$ is bounded if $||f||_\infty < \infty$, and we write

$$L^\infty(\Omega, \mu) = \left\{ f \in L^0(\Omega, \mu) : ||f||_\infty < \infty \right\}$$

for the set of all bounded measurable functions defined on (Ω, Σ, μ). It is easy to see that $L^\infty(\Omega, \mu)$ is a vector subspace of $L^0(\Omega, \mu)$. Moreover by points i) and j) of Lemma 2.3.5, $L^\infty(\Omega, \mu)$ is a Riesz space. Note that $|| \cdot ||_\infty$ is actually a norm on $L^\infty(\Omega, \mu)$ called the supremum norm.

Functionals give other examples of Riesz spaces. Indeed, let E be a Riesz space and recall that the cone of positive functionals E^*_+ of E is nothing but the polar cone of the positive cone of E.

Definition 2.3.10. The **order dual** E^*_r of E is the vector subspace of E^* generated by the cone of positive functionals, i.e.,

$$E^*_r = \{\psi_1 - \psi_2 : \psi_1, \psi_2 \in E^*_+\} \subset E^*.$$

Then E^*_r equipped with the order given by the cone E^*_+ is an ordered vector space. Moreover, this order turns E^*_r into a Riesz space.

Theorem 2.3.11 (Riesz)**.** *The order dual E^*_r of any Riesz space E is a Riesz space. The supremum $\psi_1 \vee \psi_2$ and the infimum $\psi_1 \wedge \psi_2$ of two elements $\psi_1, \psi_2 \in E^*_r$ are given pointwise by*

$$(\psi_1 \vee \psi_2)(v) = \sup\{\psi_1(w) + \psi_2(z) : w, z \in E_+ \text{ s.t. } w + z = v\}$$

and by

$$(\psi_1 \wedge \psi_2)(v) = \inf\{\psi_1(w) + \psi_2(z) : w, z \in E_+ \text{ s.t. } w + z = v\}$$

for $v \in E$. Moreover for $\psi \in E^*_r$, we have that

$$|\psi|(|v|) = \sup\{|\psi(w)| : |w| \leq v\} \quad \text{and} \quad |\psi(v)| \leq |\psi|(|v|) \quad \text{for every } v \in E.$$

Proof. See [AB99, Theorem 8.24]. □

Given a Riesz space E, the inclusions $E' \subset E^*_r \subset E^*$ hold, and E' is an ideal in E^*_r ([AB99, Theorem 8.48]).

2.3.B. Structure of a Riesz space. We recall some basic concepts about the intrinsic structure of Riesz spaces.

Definition 2.3.12. A vector subspace V of a Riesz space E is called a **Riesz subspace** if for every $v, w \in V$ their supremum, or their infimum, exists and is in V. A vector subspace V is called an **ideal** if for $v \in V$ and $w \in E$ the relation $|w| \leq |v|$ implies that $w \in V$.

Every Riesz subspace is itself a Riesz space. From point g) of Lemma 2.3.5, a vector subspace V of a Riesz space E is a Riesz subspace if and only if $v \in V$ implies that $|v| \in V$. Since an ideal is closed by taking absolute value, then every ideal is automatically a Riesz subspace.

Example 2.3.13. Let $\mathbf{R}$ with its natural locally compact topology and consider the Riesz space $\mathcal{C}(\mathbf{R})$ of all (possibly unbounded) real continuous functions on $\mathbf{R}$. Then

1) the vector subspace $\mathcal{C}^1(\mathbf{R})$ of all differentiable functions is neither a Riesz subspace nor an ideal. The proof of this fact is similar to the one made in Example 2.3.4;

2) the vector subspace $\mathbf{R}\mathbf{1}_\mathbf{R} = \{\alpha \mathbf{1}_\mathbf{R} : \alpha \in \mathbf{R}\}$ of all constant functions is a Riesz subspace of $\mathcal{C}(\mathbf{R})$ but not an ideal. This is because the pointwise order on $\mathbf{R}\mathbf{1}_\mathbf{R}$ is total, and because every non-constant bounded continuous function can be majorized by an element of $\mathbf{R}\mathbf{1}_\mathbf{R}$;

3) the vector subspace $\mathcal{C}^b(\mathbf{R})$ of all bounded continuous functions is an ideal. This can be proved using points i) and j) of Lemma 2.3.5. Consequently, $\mathcal{C}^b(\mathbf{R})$ is a Riesz subspace of $\mathcal{C}(\mathbf{R})$.

Definition 2.3.14. Let E be a Riesz space and let $A \subset E$ be a non-empty subset of E. We write $\mathcal{R}(A)$ for the **smallest Riesz subspace generated by** A**.** In other words, $\mathcal{R}(A)$ is the smallest Riesz subspace which includes A.

For a non-empty subset A of a Riesz space E, we write $A^\wedge$ for the collection of elements of E that can be written as infimum of elements of A, i.e., $a \in A^\wedge$ if there are $a_1, ..., a_n \in A$ such that $a = \bigwedge_{k=1}^n a_k$. Similarly, we define $A^\vee$ as the collection of elements of E that can be written as supremum of elements of A, i.e., $a \in A^\wedge$ if there are $a_1, ..., a_n \in A$ such that $a = \bigvee_{k=1}^n a_k$. We write $A^{\wedge\vee}$ for $\left(A^\wedge\right)^\vee$ and similarly for $A^{\vee\wedge}$. The equality $A^{\wedge\vee} = A^{\vee\wedge}$ holds thank to the infinite distributive law for lattices ([S74, Proposition 1.5]).

Lemma 2.3.15. *Let E be a Riesz space and let $A \subset E$ be a non-empty subset. Then*

$$\mathcal{R}(A) = (\mathrm{span}_\mathbf{R}(A))^{\vee\wedge} = (\mathrm{span}_\mathbf{R}(A))^{\wedge\vee}$$

Proof. See [AT07, Lemma 1.21]. □

We recall that a Riesz space E is said majorized by a subspace A if for every $v \in E$ there is $a \in A$ such that $v \leq a$. In this case, we call A a majorizing subspace of E.

Lemma 2.3.16. *Let E be a Riesz space and let A be a non-empty subset of E_+. Then $\mathcal{R}(A)$ is majorized by* $\mathrm{span}_\mathbf{R}(A)$.

Proof. Let $v \in \mathcal{R}(A)$. By Lemma 2.3.15, v is of the form

$$v = \bigwedge_{i=1}^m \left(\bigvee_{j=1}^n \left(\sum_{k=1}^{n_{(i,j)}} t_k^{(i,j)} a_k^{(i,j)} \right) \right),$$

where the $t_k^{(i,j)} \in \mathbf{R}$ and the $a_k^{(i,j)} \in A$ for every $k \in \{1, ..., n_{(i,j)}\}$ and for every pair $(i,j) \in \{1, ..., m\} \times \{1, ..., n\}$. Then

$$|v| = \left| \bigwedge_{i=1}^m \left(\bigvee_{j=1}^n \left(\sum_{k=1}^{n_{(i,j)}} t_k^{(i,j)} a_k^{(i,j)} \right) \right) \right| \leq \bigwedge_{i=1}^m \left(\bigvee_{j=1}^n \left(\sum_{k=1}^{n_{(i,j)}} \left|t_k^{(i,j)}\right| \left|a_k^{(i,j)}\right| \right) \right)$$
$$= \bigwedge_{i=1}^m \left(\bigvee_{j=1}^n \left(\sum_{k=1}^{n_{(i,j)}} \left|t_k^{(i,j)}\right| a_k^{(i,j)} \right) \right) \leq \bigwedge_{i=1}^m \left(\sum_{j=1}^n \left(\sum_{k=1}^{n_{(i,j)}} \left|t_k^{(i,j)}\right| a_k^{(i,j)} \right) \right),$$

where the last inequality is possible thanks to point i) of Lemma 2.3.5. But now we can fix $j_0 \in \{1, ..., m\}$ and use point j) of Lemma 2.3.5 to deduce that

$$\bigwedge_{i=1}^{m} \left(\sum_{j=1}^{n} \left(\sum_{k=1}^{n_{(i,j)}} \left| t_k^{(i,j)} \right| a_k^{(i,j)} \right) \right) \leq \sum_{j=1}^{n} \sum_{k=1}^{n_{(i_0,j)}} \left| t_k^{(i_0,j)} \right| a_k^{(i_0,j)}.$$

This last double sum belongs to $\operatorname{span}_{\mathbf{R}}(A)$, and so the lemma is proved. □

Definition 2.3.17. Let E be a Riesz space and let $A \subset E$ be a non-empty subset. We write

$$\mathcal{I}(A) = \left\{ v \in E : \text{ there are } a_1, ..., a_n \in A \text{ and } t_1, ..., t_n \in \mathbf{R} \text{ s.t. } |v| \leq \sum_{j=1}^{n} t_j |a_j| \right\}$$

for the **ideal generated by** A.

If $A = \{x\}$ is only a singleton, then we write E_x instead of $\mathcal{I}(\{x\})$, and we call it the **principal ideal generated by** x.

The ideal $\mathcal{I}(A)$ is the smallest ideal in E containing A.

Remark 2.3.18. It is also possible to define principal ideals in the context of ordered vector spaces. See [AT07, Section 2.7].

2.3.C. Positive operators between Riesz spaces.

We focus on a particular class of operators between Riesz spaces.

Definition 2.3.19. Let E and V be Riesz spaces and let $T : E \longrightarrow V$ be a linear operator. We say that T is a **Riesz operator**, or a **Riesz homomorphism**, if

$$T(v \vee w) = T(v) \vee T(w) \quad \text{and} \quad T(v \wedge w) = T(v) \wedge T(w) \quad \text{for all } v, w \in E.$$

Note that if T is a Riesz operator, then T is also a positive operator. In fact,

$$T(v) = T(v_+) = T(v) \vee 0 \geq 0 \quad \text{for every positive } v \in E.$$

However, a positive operator does not need to be a Riesz operator. For example, consider the positive operator

$$T : c_{00}(\mathbf{N}) \longrightarrow \mathbf{N}, \quad f \longmapsto T(f) = \sum_{j \in \mathbf{N}} f(j)$$

and the vectors $v = \delta_{-1} + \delta_0$ and $w = \delta_0 + \delta_1$. Then $T(v \vee w) = 2$ but $T(v) \vee T(w) = 1$.

Let $T : E \longrightarrow V$ be a positive operator between Riesz spaces. Then $|T(v)| \leq T(|v|)$. Indeed, $\pm v \leq |v|$ for every $v \in E$, and hence

$$\pm T(v) \leq |T(v)| \leq T(|v|).$$

The difference between positive operators and Riesz operators lies in this last inequality. The following result shows why.

Theorem 2.3.20. *Let $T : E \longrightarrow V$ be a linear operator between Riesz spaces. Then T is a Riesz operator if and only if for every $v \in E$ the identity $|T(v)| = T(|v|)$ holds.*

Proof. See [AB99, Theorem 9.15]. □

Example 2.3.21. (Examples of Riesz operators)

1) Let E be a Riesz space and let V be a Riesz subspace of E. Then the inclusion map

$$\iota : V \longrightarrow E, \quad v \longmapsto \iota(v) = v$$

is a Riesz operator.

2) Let X be a set and consider the Riesz space $\ell^\infty(X)$ of all bounded real functions on X. Let X_0 be a subset of X. Then the restriction map

$$\texttt{res} : \ell^\infty(X) \longrightarrow \ell^\infty(X_0), \quad f \longmapsto \texttt{res}(f) = f|_{X_0}$$

is a Riesz operator. Indeed, we can compute that

$$\texttt{res}\,(|f|) = (|f|)\,|_{X_0} = |f|_{X_0}| = |\texttt{res}(f)| \qquad \text{for every } f \in \ell^\infty(X).$$

3) Let (Ω, Σ, μ) be a measurable space and fix $p \in [1, +\infty]$. Then for every positive $\phi \in L^\infty(\Omega, \mu)$, the multiplication operator

$$T_\phi : L^p(\Omega, \mu) \longrightarrow L^p(\Omega, \mu), \quad f \longmapsto T_\phi(f) = \phi \cdot f$$

is a Riesz operator.

Definition 2.3.22. Let $T : E \longrightarrow V$ be a linear map between Riesz spaces. We say that T is a **Riesz isomorphism** if T is an onto and one-to-one Riesz homomorphism.

Two Riesz spaces E and V are said **Riesz isomorphic** if there exists a Riesz isomorphism $T : E \longrightarrow V$.

Theorem 2.3.23. *Let $T : E \longrightarrow V$ be an onto and one-to-one linear map between Riesz spaces. Then T is a Riesz isomorphism if and only if T and T^{-1} are positive operators.*

Proof. See [AB99, Theorem 9.17]. □

2.4 Normed Riesz spaces and Banach lattices

The section introduces monotone norms on Riesz spaces. Monotone norms are a particular class of norms behaving well with respect to the vector ordering. Note that so far, every ordered vector space was considered as an abstract vector space, i.e., without a topology on it.

2.4.A. Monotone norms on Riesz spaces. Let E be a Riesz space. A **monotone norm**, or a **lattice norm**, $||\cdot||$ on E is nothing but a norm such that $||v|| \leq ||w||$ for every $v, w \in E$ with $|v| \leq |w|$.

Definition 2.4.1. Let E be a Riesz space and let $||\cdot||$ be a monotone norm on E. Then we say that $(E, ||\cdot||)$ is a **normed Riesz space**. If, in addition, E is complete with respect to the norm $||\cdot||$, then we say that $(E, ||\cdot||)$ is a **Banach lattice**.

Intuitively, a normed Riesz space is a Riesz space equipped with a topology generated by a norm consistent with the Riesz space structure.

Example 2.4.2. (Examples of normed Riesz spaces)

1) The Euclidean space $\mathbf{R}^n$ with its standard order together with the Euclidean norm is a Banach lattice for every $n \in \mathbf{N}$. Moreover, every Banach lattices of finite dimension $n \in \mathbf{N}$ can be identified with $\mathbf{R}^n$. See [S74, Corollary 1 p. 70] and note that every Banach lattice is Archimedean;

2) Let (Ω, Σ, μ) be a measurable space. Fix $p \in [1, +\infty)$ and consider the Riesz space $L^p(\Omega, \mu)$ of all p-integrable functions. We define the L^p**-norm** on $L^p(\Omega, \mu)$ by
$$||f||_p = \mathcal{N}_p(f)^{1/p} \quad \text{for } f \in L^p(\Omega, \mu).$$
Then $L^p(\Omega, \mu)$ equipped with the L^p-norm is a Banach lattice by the Riesz-Fischer Theorem ([AB99, Theorem 13.5]).

3) Let (Ω, Σ, μ) be a measurable space and consider the Riesz space $L^\infty(\Omega, \mu)$ of all bounded measurable functions. Then $L^\infty(\Omega, \mu)$ equipped with the norm $||\cdot||_\infty$ is a Banach lattice by the Riesz-Fischer Theorem ([AB99, Theorem 13.5]).

4) Let $\mathcal{A}$ be a (complex) C^*-algebra equipped with its C^*-cone. Consider the vector subspace $\mathcal{A}_{sa}$ of all self-adjoint elements of $\mathcal{A}$. Then $\mathcal{A}_{sa}$ is a (real) Banach lattice when equipped with the C^*-order and the C^*-norm given by $\mathcal{A}$ if and only if $\mathcal{A}$ is commutative ([S51, Theorems 1 & 2]).

5) We call a Banach lattice $\mathcal{H}$ a **Hilbert lattice** if its underlying Banach space is a (complex) Hilbert space. Let $\mathcal{H}_1$ and $\mathcal{H}_2$ two Hilbert lattices. Then the Hilbert tensor product $\mathcal{H}_1 \otimes_{\mathrm{H}} \mathcal{H}_2$ equipped with the order given by the closure of the projective cone C_p is a Hilbert lattice ([S74, Example 3 p.275]). In particular, the space of Hilbert-Schmidt operators $\mathrm{HS}(\mathcal{H}) \cong \mathcal{H}' \otimes_{\mathrm{H}} \mathcal{H}$ of a Hilbert lattice $\mathcal{H}$ with the Hilbert-Schmidt norm and the vector ordering given by the tensor cone is a Hilbert lattice ([S74, Proposition 6.10 p. 273]).

6) Let G be a topological group and let $(E, ||\cdot||)$ be a Banach lattice. Suppose that G has a representation π on E by positive linear isometries. We claim that
$$E_c = \{v \in E : g \longmapsto gv \text{ is } ||\cdot||\text{-continuous}\}$$

is a Banach sublattice of E. Indeed, $(E_c, ||\cdot||)$ is a Banach space by [M01, Lemma 1.2.4]. Moreover, E_c is also an ideal in E because G acts on it by Riesz isomorphisms by Theorems 2.3.23 and 2.3.20. Therefore, we can conclude that $(E_c, ||\cdot||)$ is a Banach sublattice of E.

It is possible to generalise Proposition 2.3.7 using normed Riesz spaces. In fact, in the proof of this result, we tacitly used the fact that $\mathbf{R}$ with its Euclidean norm is a normed Riesz space.

Proposition 2.4.3. *Let $(X, \mathscr{U}_X)$ be a uniform space and let $(E, ||\cdot||_E)$ be a normed Riesz space. Then $\mathcal{C}_u((X, \mathscr{U}_X), (E, \mathscr{U}_c))$ and $\mathcal{C}_u^b((X, \mathscr{U}_X), (E, \mathscr{U}_c))$ are Riesz vector-valued function spaces.*

The proof is quite identical to the proof of Proposition 2.3.7. We repeat it only for the sake of completeness.

Proof of Proposition 2.4.3. We already know that the two sets $\mathcal{C}_u((X, \mathscr{U}_X), (E, \mathscr{U}_c))$ and $\mathcal{C}_u^b((X, \mathscr{U}_X), (E, \mathscr{U}_c))$ are vector spaces by Propositions 1.2.8 and 1.1.2. Therefore, it suffices to show that the pointwise supremum and infimum of two vector-valued uniformly continuous functions is still a vector-valued uniformly continuous function. We present the proof only for the pointwise supremum. The one for the pointwise infimum is similar.

Let $f, g \in \mathcal{C}_u((X, \mathscr{U}_X), (E, \mathscr{U}_c))$, then the function $F = f \vee g$ is in E^X as this last space is a Riesz vector-valued function space. Thus, we want to show that $F = f \vee g$ is in $\mathcal{C}_u((X, \mathscr{U}_X), (E, \mathscr{U}_c))$. Let $A \in \mathscr{U}_c$. We can suppose that A is of the form

$$A_\epsilon = \{(v_1, v_2) \in E \times E : ||v_2 - v_1||_E < \epsilon\}$$

for $\epsilon > 0$. Then there are $A_f, A_g \in \mathscr{U}_X$ such that $(x, y) \in A_f$ implies $(f(x), f(y)) \in A_{\frac{\epsilon}{2}}$ and $(x, y) \in A_g$ implies $(g(x), g(y)) \in A_{\frac{\epsilon}{2}}$. Here,

$$A_{\frac{\epsilon}{2}} = \left\{(v_1, v_2) \in E \times E : ||v_2 - v_1||_E < \frac{\epsilon}{2}\right\}.$$

Set $A_X = A_f \cap A_g \in \mathscr{U}_X$ and note that if $(x,y) \in A_X$, then

$$\begin{aligned}
||F(x) - F(y)||_E &= ||(f \vee g)(x) - (f \vee g)(y)||_E \\
&= \left|\left|\frac{1}{2}\Big(f(x) + g(x) + |f(x) - g(x)|\Big) - \frac{1}{2}\Big(f(y) + g(y) + |f(y) - g(y)|\Big)\right|\right|_E \\
&\leq \frac{1}{2}||f(x) - f(y)||_E + \frac{1}{2}||g(x) - g(y)||_E \\
&\qquad + \frac{1}{2}\Big|\Big|\, |f(x) - g(x)| - |f(y) - g(y)|\, \Big|\Big|_E \\
&\leq \frac{1}{2}||f(x) - f(y)||_E + \frac{1}{2}||g(x) - g(y)||_E \\
&\qquad + \frac{1}{2}||f(x) - g(x) - (f(y) - g(y))||_E \\
&\leq \frac{1}{2}||f(x) - f(y)||_E + \frac{1}{2}||g(x) - g(y)||_E \\
&\qquad + \frac{1}{2}||f(x) - f(y) + g(y) - g(x)||_E \\
&\leq ||f(x) - f(y)||_E + ||g(x) - g(y)||_E \\
&\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,
\end{aligned}$$

where we used point g) of Lemma 2.3.5 in the second equality. Therefore, $(F(x), F(y))$ is in A_ϵ, and hence F is uniformly continuous. This show that $\mathcal{C}_u\left((X, \mathscr{U}_X), (E, \mathscr{U}_c)\right)$ is a Riesz vector-valued function space. For $\mathcal{C}_u^b\left((X, \mathscr{U}_X), (E, \mathscr{U}_c)\right)$, we can use the same proof together with the fact that F is bounded by point i) of Lemma 2.3.5. We can conclude that $\mathcal{C}_u^b\left((X, \mathscr{U}_X), (E, \mathscr{U}_c)\right)$ is also a Riesz vector-valued function space. □

Corollary 2.4.4. *Let $(X, \mathscr{U}_X)$ be a uniform space and let $(E, ||\cdot||_E)$ be a Banach lattice. Then $\mathcal{C}_u^b((X, \mathscr{U}_X), (E, \mathscr{U}_c))$ is a Banach lattice when equipped with the supremum norm.*

Proof. By Proposition 2.4.3 and Theorem 1.2.11, we know that $\mathcal{C}_u^b((X, \mathscr{U}_X), (E, \mathscr{U}_c))$ is a Riesz vector-valued function space and a Banach space when equipped with the supremum norm. We only have to check that the supremum norm is monotone. Let $f, g \in \mathcal{C}_u^b\left((X, \mathscr{U}_X), (E, \mathscr{U}_c)\right)$ such that $|f| \leq |g|$. In particular, $|f|(x) \leq |g|(x)$ for every $x \in X$. But this implies that

$$||f||_\infty = \sup_{x \in X} ||f(x)||_E \leq \sup_{x \in X} ||g(x)||_E = ||g||_\infty$$

as wished. We can conclude that $\mathcal{C}_u^b\left((X, \mathscr{U}_X), (E, \mathscr{U}_c)\right)$ is a Banach lattice. □

Corollary 2.4.5. *Let G be a topological group and let E be a Banach lattice. Then the spaces*

$$\mathcal{C}^b(G, E),\ \mathcal{C}_{ru}^b(G, E),\ \mathcal{C}_{lu}^b(G, E)\ \text{ and }\ \mathcal{C}_u^b(G, E)$$

are all Banach lattices when equipped with the supremum norm.

Proof. The proof is a combination of Corollary 2.4.4 and of Theorem 1.3.4. □

Example 2.4.6. (More examples of normed Riesz spaces)

1) Let X be a set and let $(E, ||\cdot||)$ be a Banach lattice. Then the Riesz space $\ell^\infty(X, E) = \mathcal{C}_u^b((X, \mathscr{U}_d), (E, \mathscr{U}_c))$ equipped with the supremum norm is a Banach lattice by Corollary 2.4.4.

2) Let X be a uniformizable topological space and let E be a Banach lattice. Then the space $\mathcal{C}^b(X, E) = \mathcal{C}_u^b((X, \mathscr{F}), (E, \mathscr{U}_c))$ equipped with the supremum norm is a Banach lattice by Corollary 2.4.4 and Theorem 1.2.17. Moreover, the vector space $\mathcal{C}_{00}(X, E)$ of all compactly supported continuous functions from X to E is a normed Riesz space when equipped with the supremum norm as it is an ideal in $\mathcal{C}^b(X, E)$.

3) Let G be a topological group and let E be a Banach lattice. Then for every functionally invariant uniformity $\mathscr{U}$ for G the space $\mathcal{C}_u^b((G, \mathscr{U}), (E, \mathscr{U}_c))$ is a Banach lattice when equipped with the supremum norm by Corollary 2.4.4.

Other examples of normed Riesz spaces and Banach lattices can be founded using the following lemma.

Lemma 2.4.7. *The norm completion and the continuous dual of a normed Riesz space are Banach lattices.*

Proof. See [AB99, Lemma 9.4]. □

Example 2.4.8. Let X be a uniformizable topological space and let E be a Banach lattice. Then the vector space $\mathcal{C}_0(X, E)$ of all bounded continuous functions from X to E vanishing at infinity is a Banach lattice when equipped with the supremum norm. This is because $\mathcal{C}_{00}(X, E)$ is $||\cdot||_\infty$- dense in $\mathcal{C}_0(X, E)$ and Lemma 2.4.7.

Banach lattices encompass almost every usual vector space or, at least, almost every vector space we will use in this manuscript. However, it is possible to find Banach spaces that admit neither monotone norms nor orders consistent with their norm. In the following, we see a couple of examples.

Example 2.4.9. Let E be a vector space.

1) Suppose that E admits an order for which it is a non-Archimedean Riesz space. We claim that there is no norm on E which turns it into a normed Riesz space. Suppose this is not the case. Then there is a monotone norm $||\cdot||$ defined on E. As E is non-Archimedean, there are two non-zero vectors $v, w \in E$ such that $nv \leq w$ for every $n \in \mathbf{N}$ by [AT07, Exercice 5 p. 20]. Therefore,

$$n||v|| \leq ||w|| \quad \text{for every } n \in \mathbf{N}.$$

But this implies that $||v|| = 0$, and hence $v = 0$ which is a contradiction.

We can conclude that every normed Riesz space has to be Archimedean. In particular, there are no monotone norms on $\mathbf{R}^n$ with the lexicographic order.

2) The Lozanovsky Theorem ([AB85, Theorem 4.71]) says that a Banach lattice is reflexive, or it contains a Riesz isomorphic copy of ℓ^1 or c_0 . Let E be a non-reflexive Banach space without an isomorphic copy of ℓ^1 or c_0, then there is no order on E, which turns it into a Banach lattice. An example of such a space is the James space $\mathcal{J}$. A description of James space can be found in [M98, Section 4.5].

Remark 2.4.10. Some authors use the more general notion of normed ordered vector space, i.e., an ordered vector space equipped with a norm that respects the vector ordering. For example, the set $\mathcal{A}_{sa}$ of self-adjoint elements of a C^*-algebra $\mathcal{A}$ equipped with the C^*-norm of $\mathcal{A}$ and the order defined by its C^*-cone is an ordered Banach space. The majority of the results we prove in the following chapters remain valid for normed ordered vector spaces. However, we decided to work only with normed Riesz space as the theory for such vector spaces is quite rich and permits us to stay elegant and avoid unpleasant situations.

2.4.B. Operators on normed Riesz spaces. We present properties of positive linear operators between normed Riesz spaces.

Theorem 2.4.11 (Continuity of positive operators)**.** *Every positive operator between Banach lattices is continuous.*

Proof. See [AB99, Theorem 9.6]. □

The following example shows that the completeness assumption is necessary and can not be dropped.

Example 2.4.12. Let $c_{00}(\mathbf{N})$ be the normed Riesz space of all eventually zero real sequences equipped with the supremum norm. Consider the positive functional given by

$$T : c_{00}(\mathbf{N}) \longrightarrow \mathbf{R}, \quad f \longmapsto T(f) = \sum_{j\in\mathbf{N}} f(j).$$

On the one hand, T is a positive functional, as the vector order considered on $c_{00}(\mathbf{N})$ is the pointwise one. On the other hand, T is not continuous with respect to the $||\cdot||_\infty$-norm. Indeed, take the sequence $(f_n)_{n\in\mathbf{N}}$ given by

$$f_n(j) = \begin{cases} \frac{1}{n} & \text{if } j \leq n \\ 0 & \text{otherwise.} \end{cases}$$

Then $(f_n)_{n\in\mathbf{N}}$ converges to the zero function in $||\cdot||_\infty$-norm as $||f_n||_\infty = \frac{1}{n}$ for every $n \in \mathbf{N}$, but $T(f_n) = 1$ for every $n \in \mathbf{N}$.

Corollary 2.4.13. *Every two monotone norms which make a Riesz space into a Banach lattice are equivalent.*

Proof. Let E be a Riesz space and let $||\cdot||_1$ and $||\cdot||_2$ be two monotone norms which turn E into a Banach lattice. Consider the identity map

$$\mathrm{Id} : (E, ||\cdot||_1) \longrightarrow (E, ||\cdot||_2), \quad v \longmapsto \mathrm{Id}(v) = v.$$

Then Id is positive, and hence continuous by Theorem 2.4.11. This implies that the norm $||\cdot||_1$ is stronger than the norm $||\cdot||_2$. Switching $(E, ||\cdot||_1)$ with $(E, ||\cdot||_2)$, we have that the norms $||\cdot||_1$ and $||\cdot||_2$ are actually equivalent. □

Corollary 2.4.14. *Let E be a Banach lattice. Then the order dual and the continuous dual of E coincide, i.e., $E^*_r = E'$.*

Proof. See [AB99, Theorem 9.11]. □

With the help of this last result, we can understand in which case the inclusion $E^*_r \subset E^*$ is strict or not for Banach lattices. Indeed, suppose that E is a finite-dimensional Banach lattice, then the algebraic dual and the continuous dual of E coincide ([AB99, Theorem 5.21]). Therefore, we can conclude that $E' = E^*_r = E^*$. Let now suppose that E is an infinite-dimensional Banach lattice, then there is a discontinuous functional on E ([S97, 23.6-b]). By Corollary 2.4.14, we can conclude that the inclusion $E^*_r \subset E^*$ is strict.

Definition 2.4.15. Let E and V be two normed Riesz spaces. A linear map $T : E \longrightarrow V$ is said a **Riesz isometry** if it is a Riesz isomorphism and a linear isometry.

Two normed Riesz spaces E and V are said **Riesz isometric** if there is a Riesz isometry between them.

Lemma 2.4.16. *A Riesz isomorphism $T : E \longrightarrow V$ between normed Riesz spaces is a Riesz isometry if and only if $||T(v)||_V = ||v||_E$ for every positive $v \in E$.*

Proof. The *only if* part is a consequence of the definition of Riesz isometry. For the *if* part, suppose that $||T(v)||_V = ||v||_E$ for every positive $v \in E$. Then

$$||T(w)||_V = ||\ |T(w)|\ ||_V = ||T(|w|)||_V = ||\ |w|\ ||_E = ||w||_E \quad \text{for every } w \in E.$$

□

2.4.C. AM-spaces and AL-spaces. We spend this last subsection recalling two important classes of Banach lattices: the AM-spaces and the AL-spaces. They are both Banach lattices equipped with norms having unique features.

Definition 2.4.17. Let $||\cdot||$ be a monotone norm defined on a Riesz space E. Then $||\cdot||$ is said an

- **M-norm** if $||v \vee w|| = \max(||v||, ||w||)$ for every $v, w \in E_+$;
- **L-norm** if $||v + w|| = ||v|| + ||w||$ for every $v, w \in E_+$.

A Riesz space equipped with an M-norm (resp. an L-norm) is called an **M-space** (resp. an **L-space**) while a Banach lattice equipped with an M-norm (resp. an L-norm) is called an **AM-space** (resp. an **AL-space**).

Example 2.4.18. (Examples of AM-spaces and AL-spaces)

1) The one-dimensional Banach lattice $\mathbf{R}$ is both an AM-space and an AL-space. More generally, every Banach lattice, which is both an AM-space and an AL-space, is finite-dimensional ([AB99, Corollary 9.39]). We have a sort of converse for the class of reflexive Banach lattices. In fact, every reflexive Banach lattice, which is an AM-space or an AL-space, is finite-dimensional ([AB99, Theorem 9.38]).

2) Let $(E, ||\cdot||)$ be a Banach lattice. Then the principal ideal E_d generated by a vector $d \in E$ equipped with the norm

$$||v||_d = \inf\{\alpha \in \mathbf{R}_+ : |v| \leq \alpha |d|\},$$

where $v \in E_d$, is an AM-space with unit $|d|$ ([AB99, Theorem 9.28]). If d is an order unit for E, then we call the norm $||\cdot||_d$ the **order unit norm** of E. In this case, the norm $||\cdot||$ is equivalent to the order unit norm $||\cdot||_d$ by Theorem 2.4.11.

2) Let X be a set and let $\ell^\infty(X)$ be the Riesz space of all real bounded functions on X. Then $\ell^\infty(X)$ equipped with the supremum norm is an AM-space. Moreover, every Banach sublattice of $\ell^\infty(X)$ is an AM-space.

3) Let (Ω, Σ, μ) be a measurable space. Then the Riesz space $L^1(\Omega, \mu)$ equipped with the L^1-norm is an AL-space.

4) Let G be a locally compact group and write $\mathcal{M}(G)$ for the vector space of signed finite regular Borel measures on G. We define on it an order by

$$\mu \leq \lambda \quad \iff \quad \mu(A) \leq \lambda(A) \text{ for all } A \in \mathcal{B}(G).$$

Then the pair $(\mathcal{M}(G), \leq)$ is a Riesz space ([AT07, p. 22]). The Riesz space $\mathcal{M}(G)$ becomes an AL-space when equipped with the total variaton norm $||\cdot||_{\mathrm{TV}}$ ([AB99, 10.53 Theorem]). Recall that the total variation norm is defined by $||\mu||_{\mathrm{TV}} = |\mu|(G)$ for $\mu \in \mathcal{M}(G)$,

Surprisingly, AL-spaces and AM-spaces come always together. In fact, they are mutually dual.

Theorem 2.4.19. *A Banach space is an AM-space (resp. an AL-space) if and only if its dual is an AL-space (resp. an AM-space).*

Proof. See [AB85, Theorem 12.22 p. 188]. □

Being an AM-space or an AL-space is a pretty restrictive requirement, as the two following rigidity theorems show.

Note that the next theorem can be viewed as a Banach lattice version of the famous Gelfand-Naimark Representation Theorem for commutative Banach algebras ([K09, Theorem 2.2.7]).

Theorem 2.4.20. *Every AM-space with an order unit is Riesz isometric to* $(\mathcal{C}(K), ||\cdot||_\infty)$ *for some compact space* K.

Proof. See [AB99, Kakutani–Bohnenblust–M. Krein–S. Krein Theorem 9.32]. □

Theorem 2.4.21. *Every AL-space is Riesz isometric to* $\left(L^1(\Omega,\mu), ||\cdot||_1\right)$ *for some measurable space* (Ω,Σ,μ).

Proof. See [AB99, Kakutani Theorem 9.33]. □

Basically, AM-spaces are abstract versions of $\mathcal{C}(K)$-spaces for a compact set K, and AL-spaces are abstract versions of L^1-spaces.

One other important and interesting property of AM- and AL-spaces is that they have both the approximation property for Banach spaces ([S74, Theorem 2.4 p. 239]). This fact gives us more particular examples of Banach lattices.

Example 2.4.22. Let E be an AM-space or an AL-space.

1) The vector space $\mathcal{B}_0(E)$ of all **compact operators** defined on E is a Banach lattice when equipped with the operator norm and the vector ordering given by the closure of the tensor cone $C_\otimes$ ([S74, Corollary 2 p. 254]).

2) The vector space $\mathcal{N}(E)$ of all **nuclear operators** defined on E is a Banach lattice when equipped with the nuclear norm and the vector ordering given by the closure of the tensor cone $C_\otimes$ ([S74, Theorem 8.3 p. 281]).

We refer to [Ry02, Chapter 4] for more details about the approximation property for Banach spaces.

Chapter 3

Dominated Normed Lattices

The goal of the chapter is to introduce and study the notion of dominated ordered vector spaces.

Those kinds of vector spaces were already secretly present in the literature under the name of majorized vector spaces. Actually, a dominant vector space is nothing but a vector space majorized by the vector subspace generated by translates of a non-zero positive vector. The first to use them in the context of groups was Rosenblatt in his doctoral thesis [R72] when he was investigating supramenability. After, such notion has been resumed by Monod in [M17]. Both authors used it only in a specific case and without considering any topology on it.

The reason for studying this type of ordered vector spaces is that dominated ordered vector spaces are the natural domain of unbounded invariant functionals, in our case of invariant normalized integrals. Therefore, such spaces permit us to return to the comfort zone of bounded linear functionals.

In the first part of the chapter, we define dominated ordered vector spaces and then study their basic properties. After that, we introduce a class of norms defined on them (Sections 3.1 and 3.2). Speaking about dominated vector spaces implies that also the data of a group representation is given. Therefore, it is interesting to ask what are the continuous vectors with respect to this group representation. A universal answer is obviously not possible, but some partial ones are. Those are discussed in the last part of the chapter (Section 3.4).

We recommend Appendix B to an explanation of how unbounded invariant functionals and invariant normalized integrals are linked.

3.1 Dominated Riesz spaces

For instance, let E be an ordered vector space and let G be a topological group. An action of G on E is a representation of G on E by positive linear automorphisms, i.e., a group homomorphism $\pi : G \longrightarrow \mathrm{Aut}_L(E)_+$, where

$$\mathrm{Aut}_L(E)_+ = \{T : E \longrightarrow E : T \text{ is a bijective positive linear operator}\}.$$

We don't assume any continuity condition for the representation of G.

3.1.A. Dominated vector spaces. Suppose that G has a representation by positive linear automorphisms on an ordered vector space E.

Definition 3.1.1. A vector $d \in E$ G**-dominates** another vector $v \in E$, or v is G**-dominated** by d, if there are $g_1, ..., g_n \in G$ such that $\pm v \le \sum_{j=1}^{n} g_j d$. An ordered vector space E is G-dominated by d if d G-dominates every $v \in E$. In this case, d is called the G**-dominating element** of E.

Given a positive vector $d \in E$, we write (E, d) for the set of all vectors of E which are G-dominated by d. Therefore,

$$(E, d) = \left\{ v \in E : \pm v \le \sum_{j=1}^{n} g_j d \text{ for some } g_1, ..., g_n \in G \right\}.$$

Example 3.1.2. Let E be an ordered vector space and let G be a topological group. Suppose that G has a representation π on E by positive linear automorphisms.

1) If E admits an order unit u, then $(E, u) = E$.

2) For every non-zero positive vector $d \in E$, we have the inclusion $E_d \subset (E, d)$, where E_d is the principal ideal generated by d. We have equality if and only if d is fixed by the action of G on E given by π. In fact, if d is a G-fixed-point, then it is clear that $E_d = (E, d)$. Conversely, suppose that $(E, d) = E_d$. Then for every $g \in G$ there is a non-zero positive $\lambda_g \in \mathbf{R}$ such that $gd = \lambda_g d$. Let $||\cdot||_d$ be the Minkowski functional, or the gauge functional, associated to d ([AT07, p. 103]). Then

$$1 = ||d||_d = ||gd||_d = ||\lambda_g d||_d = \lambda_g ||d||_d = \lambda_g.$$

This means that $gd = d$.

3) Suppose that $E = \ell^\infty(G)$ and that $\pi = \pi_L$ the left-translation representation of G. Let δ_e be the function which is 1 at the identity element e of G and zero everywhere else. Then $(\ell^\infty(G), \delta_e) = c_{00}(G)$.

If E is a Riesz space and $d \in E$ is a positive vector, then (E, d) is the ideal generated by the set $\{gd : g \in G\}$. If the action of G on E is trivial, then $(E, d) = E_d$ the principal ideal generated by d.

Remark 3.1.3. The data that an ordered vector space E is G-dominated depends on the representation of G on the vector space. Therefore, it would be better to say that E is π-dominated, where π is the G-representation on E. However, we will do this only when the number of group representations could confuse.

Proposition 3.1.4. *For every positive $d \in E$, (E, d) is a vector subspace of E. If E is a Riesz space, then (E, d) is an ideal of E, and hence a Riesz subspace of E.*

Proof. Fix a positive vector $d \in E$ and let $v, w \in (E, d)$. We start showing that $v + w$ is in (E, d). Take $g_1, ..., g_n, h_1, ..., h_m \in G$ such that

$$\pm v \leq \sum_{j=1}^{n} g_j d \quad \text{and} \quad \pm w \leq \sum_{i=1}^{m} h_i d.$$

Then

$$\pm(v + w) \leq \sum_{j=1}^{n} g_j d + \sum_{i=1}^{m} h_i d \leq \sum_{k=1}^{n+m} x_k d,$$

where

$$x_k = \begin{cases} g_k & \text{for } k \in \{1, ..., n\} \\ h_{n-k} & \text{for } k \in \{n+1, ..., n+m\}. \end{cases}$$

The fact that $\alpha v \in (E, d)$ for every $\alpha \in \mathbf{R}$ and every $v \in (E, d)$ is direct by the definition of (E, d) and by (V2) of Definition 2.2.1.

Suppose now that E is a Riesz space, and we want to show that (E, d) is an ideal in E for every positive $d \in E$. To this aim, it suffices to show that (E, d) is closed by taking absolute value. Therefore, let $v \in (E, d)$. Then there are $g_1, ..., g_n \in G$ such that $\pm v \leq \sum_{j=1}^{n} g_j d$. But this implies that $|v| \leq \sum_{j=1}^{n} g_j d$, which shows that (E, d) is an ideal of E. □

Notation 3.1.5. Let $\mathfrak{F}(X)$ be a real function space on a set X and let f be a positive element of $\mathfrak{F}(X)$. Then we write $\mathfrak{F}(X, f)$ instead of $(\mathfrak{F}(X), f)$.

3.1.B. Dominated seminorms and norms. Suppose that G has a representation by positive linear automorphisms on an ordered vector space E.

Definition 3.1.6. For every positive vector $d \in E$, we define the possibly infinite value

$$p_d(v) = \inf \left\{ \sum_{j=1}^{n} t_j : \pm v \leq \sum_{j=1}^{n} t_j g_j d \ \text{ for some } \ t_1, ..., t_n \in \mathbf{R}_+ \ \text{ and } \ g_1, ..., g_n \in G \right\},$$

where $v \in E$.

The first thing to point out is that $p_d(v)$ is finite if and only if $v \in (E, d)$. Indeed, $(E, d) = \{v \in E : p_d(v) < \infty\}$. Therefore, we proceed to study p_d when restricted to (E, d).

Proposition 3.1.7. *For every positive vector $d \in E$, the map*

$$p_d : (E, d) \longrightarrow \mathbf{R}, \quad v \longmapsto p_d(v)$$

is absolutely homogeneous, sub-additive, G-invariant and monotone.

Proof. Firstly, let's show that p_d is positive homogeneous. Let $\alpha \in \mathbf{R}$ and $v \in (E, d)$. Then for every $\epsilon > 0$ there are $g_1, ..., g_n \in G$ and $t_1, ..., t_n \in \mathbf{R}_+$ such that

$$-\sum_{j=1}^{n} t_j g_j d \leq v \leq \sum_{j=1}^{n} t_j g_j d \quad \text{and} \quad \sum_{j=1}^{n} t_j \leq p_d(v) + \frac{\epsilon}{|\alpha|}.$$

Therefore,

$$-\sum_{j=1}^{n} |\alpha| t_j g_j d \leq -|\alpha| v \leq \alpha v \leq |\alpha| v \leq \sum_{j=1}^{n} |\alpha| t_j g_j d.$$

But now

$$\sum_{j=1}^{n} |\alpha| t_j = |\alpha| \sum_{j=1}^{n} t_j \leq |\alpha| \left(p_d(v) + \frac{\epsilon}{|\alpha|} \right) \leq |\alpha| p_d(v) + \epsilon.$$

Since ϵ was chosen arbitrarily, $p_d(\alpha v) = |\alpha| p_d(v)$.

We continue showing that p_d is sub-additive. Let $v, w \in (E, d)$ and $\epsilon > 0$. Then there are $g_1, ..., g_n, h_1, ..., h_m \in G$ and $t_1, ..., t_n, c_1, ..., c_m \in \mathbf{R}_+$ such that

$$\pm v \leq \sum_{j=1}^{n} t_j g_j d, \ \ \pm w \leq \sum_{i=1}^{m} c_i h_i d, \ \ \sum_{j=1}^{n} t_j \leq p_d(v) + \frac{\epsilon}{2} \quad \text{and} \quad \sum_{i=1}^{m} c_i \leq p_d(w) + \frac{\epsilon}{2}.$$

Thus,

$$-\sum_{k=1}^{n+m} b_k x_k d = -\left(\sum_{j=1}^{n} t_j g_j d + \sum_{i=1}^{m} c_i h_i d \right) \leq v + w \leq \sum_{j=1}^{n} t_j g_j d + \sum_{i=1}^{m} c_i h_i d = \sum_{k=1}^{n+m} b_k x_k d,$$

where the b_k's are given by

$$b_k = \begin{cases} t_k & \text{for } k \in \{1, ..., n\} \\ c_{n-k} & \text{for } k \in \{n+1, ..., n+m\} \end{cases}$$

and the x_k's are given by

$$x_k = \begin{cases} g_k & \text{for } k \in \{1, ..., n\} \\ h_{n-k} & \text{for } k \in \{n+1, ..., n+m\}. \end{cases}$$

But now

$$\sum_{k=1}^{n+m} b_k = \sum_{j=1}^{n} t_j + \sum_{i=1}^{m} c_i \leq p_d(v) + p_d(w) + \epsilon.$$

As ϵ was chosen arbitrarily, the map p_d is sub-additive.

The map p_d is G-invariant only because G acts linearly and positively on E.

It is only left to show that p_d is monotone, i.e., if $v, w \in (E, d)$ such that $v \leq w$, then $p_d(v) \leq p_d(w)$. Therefore, let $v, w \in (E, d)$ such that $v \leq w$. Then there are $g_1, ..., g_n \in G$ and $t_1, ..., t_n \in \mathbf{R}_+$ such that

$$-\left(\sum_{j=1}^{n} t_j g_j d\right) \leq w \leq \sum_{j=1}^{n} t_j g_j d.$$

This implies that

$$-\left(\sum_{j=1}^{n} t_j g_j d\right) \leq -w \leq v \leq w \leq \sum_{j=1}^{n} t_j g_j d,$$

which suffices to prove that $p_d(v) \leq p_d(w)$. □

Example 3.1.8. Let E be an ordered vector space and let G be a topological group. Suppose that G acts on E by positive linear automorphisms, and that d is a G-fixed-point. Then $(E, d) = E_d$ and p_d is only the Minkowski functional, or the gauge functional, of the principal ideal E_d. If E is a Banach lattice, then the pair $((E, d), p_d)$ is an AM-space ([AB99, Theorem 9.28]).

In general, p_d is not a norm on (E, d) as illustrated by the following example.

Example 3.1.9. Let E be a non-Archimedean Riesz space, e.g., $\mathbf{R}^2$ with the lexicographic order, and suppose that G has a representation on E by positive linear automorphisms. We claim that there is at least one positive vector $d \in E$ such that p_d is not a norm on (E, d). Indeed by [AT07, Exercice 5 p. 20], there are $x, y \in E_+$ such that $ny \leq x$ for all $n \in \mathbf{N}$ but $y > 0$. Consider the space (E, x). Then $y \in (E, x)$ and

$$0 \leq p_x(y) \leq \frac{1}{n} p_x(x) \quad \text{for every } n \in \mathbf{N}.$$

This implies that $p_x(y) = 0$. Therefore, the map p_x is not a norm on (E, x).

Therefore, a necessary condition to ensure that $p_d$can be a norm is that the Riesz space E is Archimedean. However, for every ordered vector space E and every positive vector $d \in E$, the map p_d is a seminorm on (E, d). Moreover, when E is a Riesz space, p_d is a **Riesz seminorm**, or a **lattice seminorm**.

Recall that a Riesz seminorm, or a lattice norm, on a Riesz space E is nothing but a seminorm p on E such that $p(x) \leq p(y)$ whenever $x, y \in E$ and $|x| \leq |y|$.

If E is a vector space and p is a seminorm on E, then we write $\tau(p)$ for the topology generated by the seminorm p.

Corollary 3.1.10. *Let E be an ordered vector space and suppose that G has a representation on E by positive linear automorphisms.*

a) The map p_d is a seminorm on (E, d) for every positive vector $d \in E$.

b) If E is a Riesz space, then the map p_d is a Riesz seminorm on (E, d) for every positive vector $d \in E$.

Proof. The proof is only a consequence of Proposition 3.1.7. □

Let E be a Riesz space. Then a topology τ on E is said **locally convex solid** if τ is a (possibly non-Hausdorff) locally convex linear topology for E, and if the lattice operations are uniformly continuous with respect to it. Then the space (E, τ) is called a **locally convex solid Riesz space**. Examples of locally convex solid Riesz spaces are given by Normed Riesz spaces. We refer to [AB99, Section 8.13] for more details about locally convex solid Riesz spaces.

Corollary 3.1.11. *Let E be a Riesz space, and suppose that G has a representation on E by positive linear automorphisms. Then for every positive vector $d \in E$, the pair $((E, d), \tau(p_d))$ is a locally convex solid Riesz space.*

Proof. The corollary is just a consequence of the fact that the topology generated by a Riesz seminorm is always locally convex solid ([AB99, Theorem 8.46]). □

Remark 3.1.12. Note that for every positive vector $d \in E$, the Riesz seminorm p_d is a norm on (E, d) if and only if the topology $\tau(p_d)$ is Hausdorff. Indeed, the topology $\tau(p_d)$ is Hausdorff if and only if the map

$$d_{p_d} : (E, d) \times (E, d) \longrightarrow \mathbf{R}_+, \quad (v, w) \longmapsto d_{p_d}(v, w) = p_d(v - w)$$

is a metric if and only if p_d is a norm. Therefore, if p_d is a norm, then the topology $\tau(p_d)$ is Hausdorff. Conversely, suppose that $\tau(p_d)$ is Hausdorff, then d_{p_d} is a metric. Hence, if $p_d(v) = 0$, then $d_{p_d}(v, 0) = p_d(v) = 0$. This implies that $v = 0$, and so the map p_d is a norm.

Bearing in mind this last remark and because working with Hausdorff topologies is always pleasant, we are interested in understanding when the map p_d is a norm. This happens when the ordered vector space E is supposed to be a normed Riesz space. Thereby, we ask something more from the representation of G on E.

Whenever we speak about a representation of a group G on a normed Riesz space E, we always mean a representation of G on E by positive linear isometries, i.e., a group homomorphism $\pi : G \longrightarrow \mathrm{Iso}_L(E)_+$, where

$$\mathrm{Iso}_L(E)_+ = \{T : E \longrightarrow E : T \text{ is a positive linear isometry}\}.$$

Be aware that some author asks that an isometry is only an affine operator which preserves the distance. For us, an isometry is also surjective.

Remark 3.1.13. Suppose that a topological group G acts by positive linear automorphisms on a Riesz space E. Then for each order-preserving norm $||\cdot||_{ub}$ defined on E for which the action of G on $(E, ||\cdot||_{ub})$ is uniformly bounded, i.e., there is $M > 0$ such that $\sup_{g\in G} ||gv||_{ub} < M$ for every $v \in E$, there is an order-preserving norm $||\cdot||_{is}$ on E for which the action of G on $(E, ||\cdot||_{is})$ is isometric. Indeed, it suffices to define a new norm as

$$||\cdot||_{is} := \sup_{g\in G} ||g\cdot||_{ub}.$$

Then $||\cdot||_{is}$ is order-preserving and equivalent to $||\cdot||_{ub}$ as

$$||\cdot||_{ub} \leq ||\cdot||_{is} \leq M||\cdot||_{ub},$$

where $M > 0$ is the uniformly bounded constant. Moreover, G acts by positive linear isometries on $(E, ||\cdot||_{is})$.

For our purpose, asking that the action of G on E is uniformly bounded would be enough. Anyway, there is no loss of generality by considering only isometric representations.

Proposition 3.1.14. *Let $(E, ||\cdot||)$ be a normed Riesz space and suppose that G has a representation on E by positive linear isometries. Let $d \in E$ be a non-zero positive vector. Then*

a) the inequality $||v|| \leq p_d(v)||d||$ holds for every $v \in (E, d)$;

b) for every $v, w \in (E, d)$ such that v is G-dominated by w and $w \geq 0$, we have that

$$p_d(v) \leq p_w(v)p_d(w).$$

Proof. We start showing point a). Let $\epsilon > 0$ and take $v \in (E, d)$. Then there are $g_1, ..., g_n \in G$, $t_1, ..., t_n \in \mathbf{R}_+$ such that $|v| \leq \sum_{j=1}^n t_j g_j d$ and $\sum_{j=1}^n t_j \leq p_d(v) + \frac{\epsilon}{||d||}$. Because the norm $||\cdot||$ is monotone we have that

$$||v|| \leq \sum_{j=1}^n t_j ||g_j d|| \leq \sum_{j=1}^n t_j ||d|| \leq p_d(v)||d|| + \epsilon.$$

As ϵ is arbitrary we are done.

Let's prove point b) . Let $\epsilon > 0$ be arbitrary and let $v, w \in (E, d)$ as in the hypothesis. Then there are $t_1, ..., t_n, c_1, ..., c_m \in \mathbf{R}_+$ and $g_1, ..., g_n, h_1, ..., h_m \in G$ such that

$$|v| \leq \sum_{j=1}^n t_j g_j w, \quad |w| \leq \sum_{k=1}^m c_k h_k d, \quad \sum_{j=1}^n t_j \leq p_w(v) + \epsilon \quad \text{and} \quad \sum_{k=1}^m c_k \leq p_d(w) + \epsilon.$$

Thus,

$$|v| \leq \sum_{j=1}^n t_j g_j w \leq \sum_{j=1}^n t_j g_j \sum_{k=1}^m c_k h_k d = \sum_{j=1}^n t_j \sum_{k=1}^m c_k g_j h_k d.$$

Taking the p_d-value of this last inequality, we conclude that

$$p_d(v) \leq \sum_{j=1}^{n} t_j \sum_{k=1}^{m} c_k \leq \sum_{j=1}^{n} t_j \left(p_d(w) + \epsilon\right) \leq \left(p_w(v) + \epsilon\right)\left(p_d(w) + \epsilon\right)$$
$$= p_w(v)p_d(w) + \epsilon p_w(v) + \epsilon p_d(w) + \epsilon^2.$$

As ϵ was chosen arbitrarily, we are done. □

Corollary 3.1.15. *The pair $((E,d), p_d)$ is a normed Riesz space for every normed Riesz space E and every non-zero positive vector $d \in E$. Moreover, the group G acts by positive linear isometries on it.*

Proof. It suffices to show that $p_d(v) = 0$ implies $v = 0$ for every $v \in (E,d)$. But if $p_d(v) = 0$, then $||v|| = 0$ by point a) of Proposition 3.1.14. Therefore, $v = 0$ as $||\cdot||$ is a norm.

The action of G on (E,d) is by isometries only because the p_d-norm is G-invariant by Proposition 3.1.7. □

Example 3.1.16. (Examples of dominated normed Riesz spaces)

1) Suppose that a group G has an action γ on some uniform space $(X, \mathscr{U})$ such that the Banach lattice $\mathcal{C}_u^b(X, \mathscr{U})$ is π_γ-invariant. Then

$$\left(\mathcal{C}_u^b((X,\mathscr{U}), \mathbf{1}_X), p_{\mathbf{1}_X}\right) = \left(\mathcal{C}_u^b(X,\mathscr{U}), ||\cdot||_\infty\right).$$

2) Let G be a discrete group and consider the left-translation representation of G on $\ell^1(G)$. Then

$$\left(\ell^1(G,\delta_e), p_{\delta_e}\right) = \left(c_{00}(G), ||\cdot||_1\right)$$

Let $(E, ||\cdot||)$ be a normed Riesz space and suppose that a topological group G has a representation by positive linear isometries on it. We know by point a) of Proposition 3.1.14 that the norm p_d is stronger than the restriction of the norm $||\cdot||$ on the vector subspace (E,d). In general, the two norms are not equivalent as illustrated by the following example.

Example 3.1.17. Let $G = \mathbf{Z}$ be the additive group of the integers and consider the space $c_{00}(\mathbf{Z}) = \ell^\infty(\mathbf{Z}, \delta_e)$. Take the sequence $(f_n)_n$ given by $f_n = \frac{1}{n}\sum_{j=1}^{n} \delta_n$. Then $(f_n)_n$ converges to the origin for the supremum norm but not for the p_{δ_e}-norm. Indeed,

$$||f_n||_\infty = \frac{1}{n}, \quad \text{while} \quad p_{\delta_e}(f_n) = 1 \quad \text{for every } n \in \mathbf{N}.$$

Scholium 3.1.18. It was natural to define dominated norms. In fact, there are a couple of examples where they were already considered. One is the functional proof of the existence of the Haar measure for every locally compact group ([Bou63, Chap. VII]). Here, dominating norms have been used on the vector space of compactly supported continuous functions, not as norms but as sub-additive maps with nice properties. Another example is given by Minkowski functionals defined on principal ideals in order complete Riesz spaces [AB99, Theorem 9.28].

3.2 Asymptotically dominated Banach lattices

For this section, we suppose that $(E, ||\cdot||)$ is a Banach lattice and that G acts on it by positive linear isometries.

We recall that a sequence $(t_j)_{j=1}^{\infty} \subset \mathbf{R}$ is called **summable** if $\sum_{j=1}^{\infty} t_j < \infty$ and **absolute summable** if $\sum_{j=1}^{\infty} |t_j| < \infty$. Note that given a positive vector $v \in E$, a positive summable sequence $(t_j)_{j=1}^{\infty} \subset \mathbf{R}$ and a sequence $(g_j)_{j=1}^{\infty} \subset G$, the sequence $(u_n)_{n=1}^{\infty} \subset E$ given by $u_n = \sum_{j=1}^{n} t_j g_j v$ converges with respect to the norm $||\cdot||$ because the infinite serie $\sum_{j=1}^{\infty} t_j g_j v$ converges absolutely in the Banach space E. In fact,

$$\sum_{j=1}^{\infty} ||t_j g_j v|| = \sum_{j=1}^{\infty} t_j ||g_j v|| = ||v|| \sum_{j=1}^{\infty} t_j,$$

which implies that $\lim_n u_n \in E$ by [M98, Theorem 1.3.9]. With an abuse of notation, we write $\sum_{j=1}^{\infty} t_j g_j d$ to mean the limit of the partial sum sequence $u_n = \sum_{j=1}^{n} t_j g_j d$ with respect to the norm $||\cdot||$.

Definition 3.2.1. A vector $d \in E$ **asymptotically** G**-dominates** another vector $v \in E$, or v is **asymptotically** G**-dominated** by d, if there is a summable positive sequence $(t_j)_{j=1}^{\infty} \subset \mathbf{R}$ and a sequence $(g_j)_{j=1}^{\infty} \subset G$ such that $|v| \leq \sum_{j=1}^{\infty} t_j g_j d$. A Banach lattice E is asymptotically G-dominated by d if for every $v \in E$ the vector d asymptotically G-dominates v. In this case, d is called the **asymptotically** G**-dominating element** of E.

Given a positive vector $d \in E$, we write $(E, d)_\infty$ for the set of all vectors of E which are asymptotically G-dominated by d. Clearly,

$$(E, d)_\infty = \left\{ v \in E : |v| \leq \sum_{j=1}^{\infty} t_j g_j d \ \text{ for a summable } \ (t_j)_{j=1}^{\infty} \subset \mathbf{R}_+ \text{ and } (g_j)_{j=1}^{\infty} \subset G \right\}.$$

Example 3.2.2. Let E be a Banach lattice and let G be a topological group. Suppose that G has a representation π on E by positive linear isometries.

1) The inclusions $(E, d) \subset (E, d)_\infty \subset E$ hold for every positive vector $d \in E$.

2) If E admits an order unit u, then $(E,u) = (E,u)_\infty = E$.

3) Suppose that $E = \ell^\infty(G)$ and that $\pi = \pi_L$ the left-translation representation of G. Let δ_e be the function which is 1 at the identity element e of G and zero everywhere else. Then $(\ell^\infty(G), \delta_e)_\infty = \ell^1(G)$.

Notation 3.2.3. Let $\mathfrak{F}(X)$ be a real Banach lattice function space on a set X and let f be a positive element of $\mathfrak{F}(X)$. We write $\mathfrak{F}(X,f)_\infty$ instead of $(\mathfrak{F}(X), f)_\infty$.

Proposition 3.2.4. *The set $(E,d)_\infty$ is an ideal of E for every positive vector $d \in E$. In particular, $(E,d)_\infty$ is Riesz subspace of E.*

Proof. We only need to show that $(E,d)_\infty$ is a vector subspace of E because its definition implies that it is closed by taking absolute value implying that it is an ideal in E. Let $v, w \in (E,d)_\infty$, then there are summable sequences $(t_j)_{j=1}^\infty, (a_i)_{i=1}^\infty \subset \mathbf{R}_+$ and $(g_j)_{j=1}^\infty, (h_i)_{i=1}^\infty \subset G$ such that $|v| \leq \sum_{j=1}^\infty t_j g_j d$ and $|w| \leq \sum_{i=1}^\infty a_i h_i d$. Therefore,

$$|\lambda v| \leq |\lambda| \sum_{j=1}^\infty t_j g_j d = \sum_{j=1}^\infty |\lambda| t_j g_j d \quad \text{for every } \lambda \in \mathbf{R}.$$

Note that the last equality is possible thanks to [M98, Proposition 1.3.7 (d)]. We can conclude that $\lambda v \in (E,d)_\infty$ as the sequence $(|\lambda| t_j)_{j=1}^\infty \subset \mathbf{R}_+$ is summable. Moreover,

$$|v+w| \leq |v| + |w| \leq \sum_{j=1}^\infty t_j g_j d + \sum_{i=1}^\infty a_i h_i d \leq \sum_{k=1}^\infty b_k x_k d,$$

where the b_k's and the x_k's are given by

$$b_k = \begin{cases} t_{\frac{k}{2}} & \text{if } k \text{ is even} \\ a_{\frac{k-1}{2}} & \text{otherwise} \end{cases} \quad \text{and} \quad x_k = \begin{cases} g_{\frac{k}{2}} & \text{if } k \text{ is even} \\ h_{\frac{k-1}{2}} & \text{otherwise.} \end{cases}$$

As the sum of two positive converging series converges, we have that $v + w \in (E,d)_\infty$. $\square$

Definition 3.2.5. For every positive vector $d \in E$, we define the possibly infinite value

$$p_d^\infty(v) = \inf\left\{ \sum_{j=1}^\infty t_j : |v| \leq \sum_{j=1}^\infty t_j g_j d \ \text{ for a summable } \ (t_j)_{j=1}^\infty \subset \mathbf{R}_+ \text{ and } (g_j)_{j=1}^\infty \subset G \right\},$$

where $v \in E$.

We can see that $p_d^\infty(v) < \infty$ if and only if $v \in (E,d)_\infty$, and hence

$$(E,d)_\infty = \{v \in E : p_d^\infty(v) < \infty\}.$$

Proposition 3.2.6. *Let $d \in E$ be a non-zero positive vector. Then*

a) the map

$$p_d^\infty : (E, d)_\infty \longrightarrow \mathbf{R}, \quad v \longmapsto p_d^\infty(v)$$

is absolutely homogeneous, sub-additive, G-invariant and monotone;

b) the inequality $||v|| \leq p_d^\infty(v)||d||$ holds for every $v \in (E, d)_\infty$;

c) for every $v, w \in (E, d)_\infty$ such that v is asymptotically G-dominated by w and $w \geq 0$, we have that

$$p_d^\infty(v) \leq p_w^\infty(v) p_d^\infty(w).$$

Proof. The proof of point a) is similar to the one of Proposition 3.1.7. Therefore, we don't repeat it here.

To prove points b) and c), we employ the same proof of Proposition 3.1.14. For b), we use in addition [M98, Proposition 1.3.7 (e)]. □

In particular, $\left((E, d)_\infty, p_d^\infty\right)$ is a normed Riesz space for every Banach lattice E and every non-zero positive vector $d \in E$. Moreover for every $v \in (E, d)$, the following inequalities hold:

$$\frac{||v||}{||d||} \leq p_d^\infty(v) \leq p_d(v).$$

Example 3.2.7. (Examples of asymptotically dominated normed Riesz spaces)

1) Suppose that a group G has an action γ on some uniform space $(X, \mathscr{U})$ such that the Banach lattice $\mathcal{C}_u^b(X, \mathscr{U})$ is π_γ-invariant. Then

$$\left(\mathcal{C}_u^b((X, \mathscr{U}), \mathbf{1}_X)_\infty, p_{\mathbf{1}_X}^\infty\right) = \left(\mathcal{C}_u^b(X, \mathscr{U}), ||\cdot||_\infty\right).$$

2) Let G be a discrete group and consider the left-translation representation of G on $\ell^1(G)$. Then

$$\left(\ell^1(G, \delta_e)_\infty, p_{\delta_e}^\infty\right) = \left(\ell^1(G), ||\cdot||_1\right).$$

The remainder of this section is devoted to showing that the spaces of the form $\left((E, d)_\infty, p_d^\infty\right)$, for E a Banach lattice and $d \in E$ a non-zero positive vector, are also Banach lattices.

We need first to solve some technical detail.

Lemma 3.2.8. *Let $(V, ||\cdot||_V)$ and $(E, ||\cdot||_E)$ be two normed vector spaces such that $V \subset E$ and $||\cdot||_E \leq ||\cdot||_V$ on V. Suppose that there is a sequence $(v_n)_n$ in V which converges to an element $v_1 \in V$ in $||\cdot||_V$-norm and to an element $v_2 \in E$ in $||\cdot||_E$-norm. Then $v_1 = v_2$.*

Proof. Let $\epsilon > 0$. Then there are $n_1, n_2 \in \mathbf{N}$ such that $||v_1 - v_n||_V < \frac{\epsilon}{2}$ for every $n > n_1$ and $||v_2 - v_n||_E < \frac{\epsilon}{2}$ for every $n > n_2$. Pick $N = \max(n_1, n_2)$ and compute that

$$\begin{aligned} ||v_1 - v_2||_E &= ||v_1 - v_N + v_N - v_2||_E \\ &\leq ||v_1 - v_N||_E + ||v_N - v_2||_E \\ &\leq ||v_1 - v_N||_V + ||v_N - v_2||_E \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

As ϵ was chosen arbitrarily, $v_1 = v_2$. □

Lemma 3.2.9. *Let $(E, ||\cdot||)$ be a Banach lattice and let $(x_k)_k$ and $(y_k)_k$ be sequences in E. Suppose that the two sequences $(x_k)_k$ and $(y_k)_k$ are absolutely summable, i.e.,*

$$\lim_n \sum_{k=1}^n ||x_k|| < +\infty \qquad \textit{and} \qquad \lim_n \sum_{k=1}^n ||y_k|| < +\infty,$$

and that for every $k \in \mathbf{N}$ the inequality $x_k \leq y_k$ holds. Then

$$\lim_n \sum_{k=1}^n x_k \leq \lim_n \sum_{k=1}^n y_k,$$

where the limit is taken with respect to the $||\cdot||$-norm.

Proof. Consider the sequence $(z_k)_k$ in E given by $z_k = y_k - x_k$. By hypothesis $(z_k)_k$ is positive, which means that it lives in the positive cone of E. Now,

$$\sum_{k=1}^n ||z_k|| \leq \sum_{k=1}^n \Big(||y_k|| + ||x_k|| \Big) = \sum_{k=1}^n ||y_k|| + \sum_{k=1}^n ||x_k|| \quad \text{for every } n \in \mathbf{N}.$$

Taking the limit on both sides, we have that $\lim_n \sum_k^n ||z_k|| < \infty$. As E is a Banach space, the limit $\lim_n \sum_k^n z_k$ with respect to the $||\cdot||$-norm exists and it is positive as the positive cone of a Banach lattice is always closed, see [AB99, Theorem 8.43 (1)]. This implies that

$$\lim_n \sum_{k=1}^n x_k \leq \lim_n \sum_{k=1}^n y_k,$$

where the two limits are taken with respect to the $||\cdot||$-norm. □

For a normed vector space $(E, ||\cdot||)$, write $\widehat{E}$ for the completion of E with respect to the uniformity given by the norm $||\cdot||$ and $\widehat{||\cdot||}$ for the completed norm of $\widehat{E}$. If $(v_n)_n$ is a sequence in E, write $||\cdot|| - \lim_n v_n$ to denote the limit of $(v_n)_n$ with respect to the $||\cdot||$-norm.

Recall that the completion $(\widehat{E}, \widehat{||\cdot||})$ of a normed Riesz space $(E, ||\cdot||)$ is a Banach lattice (Lemma 2.4.7) and that the lattice operations on a normed Riesz space are uniformly continuous, since the topology generated by the norm of a normed Riesz space is locally convex solid ([AB99, Theorem 8.41]).

Lemma 3.2.10. *Let $(E, ||\cdot||_E)$ be a Banach lattice and $(V, ||\cdot||_V)$ be a normed Riesz space. Let*

$$\iota : (V, ||\cdot||_V) \longrightarrow (E, ||\cdot||_E)$$

be an injective continuous Riesz homomorphism. Then the (unique) extension

$$\widehat{\iota} : (\widehat{V}, \widehat{||\cdot||}_V) \longrightarrow (E, ||\cdot||_E)$$

of ι is an injective continuous Riesz homomorphism.

Proof. We start showing that $\widehat{\iota}$ is a Riesz homomorphism, i.e., $\widehat{\iota}(|\widehat{v}|) = |\widehat{\iota}(\widehat{v})|$ for every $\widehat{v} \in \widehat{V}$. Let $\widehat{v} \in \widehat{V}$ and let $(v_n)_n$ be a sequence in V which converges to $\widehat{v}$ in $\widehat{||\cdot||}_V$-norm. Then the sequence $(\widehat{\iota}(v_n))_n$ converges to $\widehat{\iota}(\widehat{v})$ in $||\cdot||_E$-norm. Thus,

$$\begin{aligned}
\widehat{\iota}(|\widehat{v}|) &= \widehat{\iota}\Big(\Big|\widehat{||\cdot||}_V - \lim_n v_n\Big|\Big) = \widehat{\iota}\Big(\widehat{||\cdot||}_V - \lim_n |v_n|\Big) \\
&= ||\cdot||_E - \lim_n \widehat{\iota}(|v_n|) = ||\cdot||_E - \lim_n \iota(|v_n|) \\
&= ||\cdot||_E - \lim_n |\iota(v_n)| = \Big|||\cdot||_E - \lim_n \iota(v_n)\Big| \\
&= \Big|||\cdot||_E - \lim_n \widehat{\iota}(v_n)\Big| = \Big|\widehat{\iota}\Big(\widehat{||\cdot||}_V - \lim_n v_n\Big)\Big| = |\widehat{\iota}(\widehat{v})|.
\end{aligned}$$

It just remains to prove that $\widehat{\iota}$ is injective. Suppose it is not the case. Then there is a non-zero vector $\widehat{v} \in \widehat{V}$ such that $\widehat{\iota}(\widehat{v}) = 0$. We can suppose that $\widehat{v}$ is positive as $\widehat{\iota}$ is a Riesz homomorphism. By [L67, Theorem 60.4], there is a positive increasing sequence $(v_n)_n$ in V which converges to $\widehat{v}$ in $\widehat{||\cdot||}_V$-norm. This means that there is $n_0 \in \mathbf{N}$ such that $v_{n_0} \neq 0$. But now $0 < v_{n_0} \leq \widehat{v}$, and so $\widehat{\iota}(v_{n_0}) = \iota(v_{n_0}) = 0$ which is a contradiction. □

This last lemma is no longer true if we drop the monotonocity of the norm $||\cdot||_V$.

Example 3.2.11. Let $(E, ||\cdot||)$ be an infinite-dimensional Banach space and let T be a discontinuous linear functional on it. Define the map

$$||v||_T = ||v|| + |T(v)| \quad \text{for } v \in E.$$

Then $||\cdot||_T$ is a norm on E, which is strictly finer than the norm $||\cdot||$. Therefore, the identity map

$$\mathrm{Id} : (E, ||\cdot||') \longrightarrow (E, ||\cdot||)$$

is an injective continuous linear operator. We claim that the extension

$$\widehat{\mathrm{Id}} : (\widehat{E}, \widehat{||\cdot||_T}) \longrightarrow (E, ||\cdot||)$$

is not injective. Indeed, suppose that $\widehat{\mathrm{Id}}$ is injective. Then by the Closed Graph Theorem ([AB99, Theorem 5.20]) the inverse of $\widehat{\mathrm{Id}}$ has to be continuous. But this is not possible.

Corollary 3.2.12. *Let $d \in E$ be a non-zero positive vector. Then $\widehat{(E,d)}_\infty$ can be realized as a Riesz subspace of E.*

Proof. Consider the natural inclusion

$$\iota : (E,d)_\infty \longrightarrow E, \quad v \longmapsto \iota(v) = v.$$

By Lemma 3.2.10, we have that

$$\widehat{\iota} : \widehat{(E,d)}_\infty \longrightarrow E, \quad v \longmapsto \widehat{\iota}(v)$$

is an injective Riesz homomorphism. Therefore, we can realize $\widehat{(E,d)}_\infty$ as a Riesz subspace of E. □

Remark 3.2.13. Let $(E, ||\cdot||)$ be a Banach lattice and let $d \in E$ be a non-zero positive vector. Then Corollary 3.2.12 and Lemma 3.2.8 imply that

$$\widehat{p_d^\infty} - \lim_{n\in\mathbf{N}} \sum_{j=1}^{n} t_j g_j d = ||\cdot|| - \lim_{n\in\mathbf{N}} \sum_{j=1}^{n} t_j g_j d$$

for every absolutely convergent sequence $(t_j)_j \subset \mathbf{R}_+$ and every sequence $(g_j)_j \subset G$. This is because we can apply Lemma 3.2.8 to the sequence $(v_n)_n$ in $(E,d)_\infty$ given by $v_n = \sum_{j=1}^{n} t_j g_j d$.

We are finally ready to show the norm completeness of the space $\left((E,d)_\infty, p_d^\infty\right)$.

Theorem 3.2.14. *The pair $\left((E,d)_\infty, p_d^\infty\right)$ is a Banach lattice for every Banach lattice E and every non-zero positive vector $d \in E$. Moreover, the group G acts by positive linear isometries on it.*

Proof. We have to show that, for every sequence $(x_k)_k$ in $(E,d)_\infty$ such that $\lim_n \sum_k^n p_d^\infty(x_k) < \infty$, the limit $\lim_n \sum_k^n x_k$ with respect to the p_d^∞-norm exists in $(E,d)_\infty$.

First of all, note that $\sum_{k=1}^{n} \widehat{p}_d^\infty(x_k) = \sum_{k=1}^{n} p_d^\infty(x_k)$ for every $n \in \mathbf{N}$. Thus, we have that $\lim_n \sum_k^n \widehat{p}_d^\infty(x_k) < \infty$ which means that the limit $\ell = \lim_n \sum_k^n x_k$ exists in $\widehat{(E,d)}_\infty$ for the $\widehat{p}_d^\infty$-norm as the space $(\widehat{(E,d)}_\infty, \widehat{p}_d^\infty)$ is Banach.

We know that $\ell \in E$ by Corollary 3.2.12. We claim that ℓ is actually in $(E,d)_\infty$. Therefore, we have to show that there are sequences $(t_j)_j \subset \mathbf{R}_+$ and $(g_j)_j \subset G$ such that

$$|\ell| \leq \sum_{j=1}^{\infty} t_j g_j d = ||\cdot|| - \lim_n \sum_{j=1}^{n} t_j g_j d.$$

For every $k \in \mathbf{N}$ there are $(t_j^{(k)})_j \subset \mathbf{R}_+$ and $(g_j^{(k)}) \subset G$ such that

$$|x_k| \leq \sum_{j=1}^{\infty} t_j^{(k)} g_j^{(k)} d \qquad \text{and} \qquad \sum_{j=1}^{\infty} t_j^{(k)} \leq p_d^\infty(x_k) + 2^{-k}.$$

Set $y_k = \sum_j^\infty t_j^{(k)} g_j^{(k)} x_k$. Then $|x_k| \leq y_k$ for every $k \in \mathbf{N}$ and

$$\sum_{k=1}^{N} \widehat{p}_d^\infty(y_k) \leq \sum_{k=1}^{N} \sum_{j=1}^{\infty} t_j^{(k)} \leq \sum_{k=1}^{N} p_d^\infty(x_k) + 2^{-k}$$

for every $N \in \mathbf{N}$. Taking the limit on both sides of this last inequality, we have that

$$\lim_N \sum_{k=1}^{N} \widehat{p}_d^\infty(y_k) < \infty.$$

Therefore by Lemma 3.2.9,

$$|\ell| \leq \sum_{k=1}^{\infty} |x_k| \leq \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} t_j^{(k)} g_j^{(k)} d,$$

where all the limits are taken with respect to the $\widehat{p}_d^\infty$-norm. The last double sum converges also in $||\cdot||$-norm to the same limit, see Remark 3.2.13. This implies that $\ell \in (E,d)_\infty$. Now, it is easy to show that the sequence $(v_n)_n$ given by $v_n = \sum_k^n x_k$ converges to ℓ in p_d^∞-norm. Indeed, for every $\epsilon > 0$ there is $n_0 \in \mathbf{N}$ such that $\widehat{p}_d^\infty(\sum_k^n x_k - \ell) < \epsilon$ for every $n > n_0$ and so

$$p_d^\infty(v_n - \ell) = p_d^\infty\left(\sum_{k=1}^{n} x_k - \ell\right) = \widehat{p}_d^\infty\left(\sum_{k=1}^{n} x_k - \ell\right) < \epsilon$$

for every $n > n_0$. □

3.3 Relation between dominated and asymptotically dominated spaces

Let's shift our attention to the relationship between the spaces (E, d) and $(E, d)_\infty$. By definition, the inclusion $(E, d) \subset (E, d)_\infty$ holds for every positive $d \in E$. But it gets better. In fact, the two Riesz spaces have a topological friendship.

We recall that G is a topological group with a representation by positive linear isometries on a Banach lattice E.

Proposition 3.3.1. *Let $d \in E$ be a non-zero positive vector. Then (E, d) is dense in $(E, d)_\infty$ with respect to the p_d^∞-norm.*

To prove this last proposition, we need the following lemma, which is only a lattice version of the famous and standard result [AB99, Corollary 5.81]. The latter states that a vector subspace of a locally convex vector space is not dense exactly when there is a non-zero continuous linear functional that vanishes on it. We require an *ideal* version of it.

Lemma 3.3.2 (Non-density of ideals)**.** *Let E be a Banach lattice and let $V \subset E$ be an ideal. Then V is not dense in E if and only if there is a non-zero positive functional on E which vanishes on V.*

Proof. Suppose that there is a non-zero positive functional ψ vanishing on V. In particular, ψ is continuous by Theorem 2.4.11. Therefore, we can apply [AB99, Corollary 5.81] to conclude that V is not dense in E.

Suppose now that V is not dense in E. Then there is a non-zero continuous functional ψ on E which vanishes on V by [AB99, Corollary 5.81]. We claim that $|\psi|$ is the functional we are searching. Clearly, $|\psi|$ is a non-zero positive functional. We have only to show that it is zero on V. Let $|\psi| = \psi_+ + \psi_-$, where ψ_+ is the positive part of ψ and ψ_- is the negative part of ψ. We want to prove that the equality $\psi_+(v) = \psi_-(v) = 0$ holds for every positive vector $v \in V$. Actually, it suffices to prove that $\psi_+(v) = 0$. By Theorem 2.3.11,

$$\psi_+(v) = \sup \{\psi(w) : w \in E \text{ and } 0 \leq w \leq v\} = \sup \{\psi(w) : w \in V \text{ and } 0 \leq w \leq v\},$$

where the second equality is possible thanks to the fact that V is an ideal in E. As ψ_+ is continuous, we can conclude that ψ_+ vanishes on V. □

This yields:

Proof of Proposition 3.3.1. Let $d \in E$ be a non-zero positive vector and suppose that (E, d) is not dense in $(E, d)_\infty$ for the p_d^∞-norm. By Lemma 3.3.2, there exists a non-zero positive functional ψ which vanishes on (E, d). As ψ is non-zero, there is a non-zero positive vector $v \in (E,d)_\infty$ such that $\psi(v) > 0$. Now let $(t_j)_{j=1}^\infty \subset \mathbf{R}_+$ be a summable sequence and $(g_j)_{j=1}^\infty \subset G$ such that $v \leq \sum_{j=1}^\infty t_j g_j d$, and compute that

$$0 < \psi(v) \leq \psi\left(\sum_{j=1}^{\infty} t_j g_j d\right) = \sum_{j=1}^{\infty} t_j \psi(g_j d) = 0,$$

where the second-to-last equality is true thanks to [M98, Proposition 1.3.7 (d)]. But this is impossible. Thus, (E,d) is indeed dense in $(E,d)_\infty$ with respect to the p_d^∞-norm. □

Remark 3.3.3. Let $(E, ||\cdot||)$ be a Banach lattice and suppose that it is asymptotically G-dominated by some positive element $d \in E$. Then $E = (E,d)_\infty$. By Theorem 3.2.14, we know that $((E,d)_\infty, p_d^\infty)$ is a Banach lattice, and hence (E, p_d^∞) is also a Banach lattice. Now, two monotone norms that turn a Riesz space into a Banach lattice are equivalent by Corollary 2.4.13. This means that the original norm $||\cdot||$ of E is equivalent to the p_d^∞-norm. In particular, we can always suppose that a norm on an asymptotically G-dominated Banach lattice E is of the form p_d^∞, where d is the asymptotically G-dominating element of E.

Bearing in mind this last remark, we can formulate a converse of Proposition 3.3.1.

Proposition 3.3.4. *Let $(E, ||\cdot||)$ be a Banach lattice and suppose that G acts on it by positive linear isometries. Suppose in addition that E is asymptotically G-dominated by a positive element $d \in E$. Then there is a Riesz subspace D of E which is G-dominated by d and $||\cdot||$-dense in E.*

Proof. As discussed in Remark 3.3.3, the $||\cdot||$-norm is equivalent to the p_d^∞-norm on E. Set $D = (E,d)$. Then D is a Riesz subspace of E which is G-dominated by d and p_d^∞-norm dense in E by Proposition 3.3.1. As the p_d^∞-norm is equivalent to the $||\cdot||$-norm, D is also $||\cdot||$-norm dense in E concluding the proof. □

With the next theorem, we want to clarify once and for all the relationship between p_d and p_d^∞. As before, write $\widehat{(E,d)}$ for the completion of (E,d) with respect to the uniformity given by the p_d-norm and $\widehat{p}_d$ for the completed norm.

Theorem 3.3.5. *Let $d \in E$ be a non-zero positive vector. Then*

$$\left(\widehat{(E,d)}, \widehat{p}_d\right) = \left((E,d)_\infty, p_d^\infty\right).$$

Proof. Consider the natural inclusion

$$\iota : (E,d) \longrightarrow (E,d)_\infty, \quad v \longmapsto \iota(v) = v.$$

Note that ι is uniformly continuous as $p_d^\infty \leq p_d$, and that it is an injective Riesz homomorphism. Therefore by Lemma 3.2.10, the extension

$$\widehat{\iota} : \widehat{(E,d)} \longrightarrow (E,d)_\infty, \quad v \longmapsto \widehat{\iota}(v) = v$$

is an injective Riesz homomorphism.

Now, we want to show that $\widehat{\imath}$ is surjective. To this aim, we start proving that for every element $v_\infty \in (E,d)_\infty$ of the form $v_\infty = \sum_{j=1}^{\infty} t_j g_j d$, where $(t_j)_j \subset \mathbf{R}_+$ is an absolutely summable sequence and $(g_j)_j \subset G$, there is $\widehat{v} \in \widehat{(E,d)}$ such that $\widehat{\imath}(\widehat{v}) = v_\infty$. So, let v_∞ such a vector. Then, for every $n \in \mathbf{N}$, define the vector $S_n(v_\infty) = \sum_{j=1}^{n} t_j g_j d \in (E,d)$. We claim that the sequence $(S_n(v_\infty))_n$ converges to v_∞ in p_d^∞-norm. In fact, for every $\epsilon > 0$ there is $n_0 \in \mathbf{N}$ such that $\sum_{j=n_0}^{\infty} |t_j| < \epsilon$. This implies that

$$p_d^\infty(v_\infty - S_n(v_\infty)) \leq \sum_{j=n}^{\infty} |t_j| < \epsilon,$$

for every $n > n_0$. But at the same time, $(S_n(v_\infty))_n$ is also a Cauchy sequence with respect to the p_d-norm. Indeed, let $\epsilon > 0$ and $m \in \mathbf{N}$, then there is $n_0 \in \mathbf{N}$ such that $\sum_{j=n_0}^{\infty} |t_j| < \epsilon$, which implies that

$$p_d(S_{n+m}(v_\infty) - S_n(v_\infty)) \leq \sum_{j=n_0}^{n+m} |t_j| \leq \sum_{j=n_0}^{\infty} |t_j| < \epsilon,$$

for every $n > n_0$. Consequently, there is $\widehat{v} \in \widehat{(E,d)}$ such that $\lim_n S_n(v_\infty) = \widehat{v}$ in $\widehat{p}_d$-norm. Hence, we have that

$$\widehat{\imath}(\widehat{v}) = \widehat{\imath}(\,\widehat{p}_d - \lim_n S_n(v_\infty)) = p_d^\infty - \lim_n \widehat{\imath}(S_n(v_\infty)) = p_d^\infty - \lim_n S_n(v_\infty) = v_\infty.$$

Moreover, $v_\infty = \widehat{v}$ by Lemma 3.2.8.

Let's now take an arbitrary $v \in (E,d)_\infty$, and we show that it lies in the image of $\widehat{\imath}$. By Proposition 3.3.1, there is a sequence $(v_n)_n \subset (E,d)$ which converges to v in p_d^∞-norm. We claim that $(v_n)_n$ is a Cauchy sequence for the p_d-norm. Let $\epsilon > 0$ and $m \in \mathbf{N}$, then there is $n_0 \in \mathbf{N}$ such that $p_d^\infty(v_{n+m} - v_n) < \epsilon$ for every $n > n_0$. This means that there are $(t_j)_j \subset \mathbf{R}_+$ a summable sequence and $(g_j)_j \subset G$ such that $|v_{n+m} - v_n| \leq \sum_{j=1}^{\infty} t_j g_j d$ and $\sum_{j=1}^{\infty} t_j < \epsilon$. Define $v_\epsilon = \sum_{j=1}^{\infty} t_j g_j d$. As seen before, $v_\epsilon \in \widehat{(E,d)}$ and $\widehat{\imath}(v_\epsilon) = v_\epsilon$. Now, $\widehat{\imath}$ is a Riesz homomorphism which is injective and surjective on its image. By Theorem 2.3.23, $(\widehat{\imath})^{-1}$ is positive. Therefore, $|v_{n+m} - v_n| \leq v_\epsilon$ in $\widehat{(E,d)}$ and, consequently,

$$p_d(v_{n+m} - v_n) = \widehat{p}_d(v_{n+m} - v_n) \leq \widehat{p}_d(v_\epsilon) \leq \sum_{j=1}^{\infty} t_j < \epsilon,$$

for every $n > n_0$. This shows that $(v_n)_n$ is a Cauchy sequence for the p_d-norm. Thus, there is $\widehat{w} \in \widehat{(E,d)}$ such that $\lim_n v_n = \widehat{w}$ in $\widehat{p}_d$-norm. We can finally compute that

$$\widehat{\imath}(\widehat{w}) = \widehat{\imath}(\widehat{p}_d - \lim_n v_n) = p_d^\infty - \lim_n \widehat{\imath}(v_n) = p_d^\infty - \lim_n v_n = v.$$

As before, $v = \widehat{w}$ by Lemma 3.2.8.

Finally, we show that $\widehat{p}_d = p_d^\infty$. We already know that $p_d^\infty \leq \widehat{p}_d$ because of the fact that $p_d^\infty \leq p_d$. For the inverse inequality, take $v \in (E,d)_\infty$ and let $\epsilon > 0$ arbitrary. Then there are $(t_j)_j \subset \mathbf{R}_+$ a summable sequence and $(g_j)_j \subset G$ such that $|v| \leq \sum_{j=1}^\infty t_j g_j d$ and $\sum_{j=1}^\infty t_j \leq p_d^\infty(v) + \epsilon$. Set $v_\infty = \sum_{j=1}^\infty t_j g_j d$ and compute that

$$\widehat{p}_d(v) \leq \widehat{p}_d(v_\infty) = \lim_n p_d(S_n(v_\infty)) \leq \lim_n \sum_{j=1}^n t_j = \sum_{j=1}^\infty t_j \leq p_d^\infty(v) + \epsilon.$$

As ϵ was chosen arbitrarily, $\widehat{p}_d \leq p_d^\infty$. □

3.4 Continuous vectors for dominating norms

Let π be a representation of a topological group G on a locally convex vector space E by positive linear automorphisms. Then the representation π, or the action of the group G on E, is said **orbitally continuous** if for every $v \in E$ the map

$$G \longrightarrow E, \quad g \longmapsto \pi(g)v$$

is continuous. The representation π, or the action of the group G on E, is said **jointly continuous** if the map

$$G \times E \longrightarrow E, \quad (g,v) \longmapsto \pi(g)v$$

is continuous w.r.t. the product topology on $G \times E$.

In general, the two continuity notions are different, see [G17, Example C.2.6]. However, they coincide for a topological group acting by isometries on a normed space. In fact:

Lemma 3.4.1. *Suppose that G has a representation π on a normed space $(E, ||\cdot||)$ by positive linear isometries. Then the representation π is jointly continuous if and only if it is orbitally continuous.*

Proof. We only have to show that orbital continuity implies joint continuity. Therefore, let $(g_\alpha)_\alpha$ be a net in G converging to $e \in G$ and let $(v_n)_n$ be a sequence in E converging to some $v \in E$. Let $\epsilon > 0$. Then there is an α_0 and a $n_0 \in \mathbf{N}$ such that

$$||v_n - v|| < \frac{\epsilon}{2} \quad \text{and} \quad ||\pi(g_\alpha)v - v|| < \frac{\epsilon}{2} \quad \text{for every } \alpha \succ \alpha_0 \text{ and } n \geq n_0.$$

Therefore, we have the estimation

$$\begin{aligned} ||\pi(g_\alpha)v_n - v|| &\leq ||\pi(g_\alpha)v_n - \pi(g_\alpha)v|| + ||\pi(g_\alpha)v - v|| \\ &= ||v_n - v|| + ||\pi(g_\alpha)v - v|| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

This show that $\pi(g_\alpha)v_n$ converges to v proving the joint continuity. □

Consequently, we only say that G acts continuously on E or that the representation π of G on E is continuous.

It is a difficult task to understand if the action of G on (E, d) is orbitally, or jointly, continuous for the locally convex solid topology $\tau(p_d)$. For this reason, we only consider the case where E is a normed Riesz space.

Before continuing, there are two details to point out. Suppose we have a normed Riesz space E, a topological group G and a representation π of G on E by positive linear isometries. First, note that if we want to say something about the continuous vectors of the p_d-norm, we have to suppose that the representation π is continuous for the norm $||\cdot||$ because of point a) of Proposition 3.1.14. Secondly, note that if $v \in E$ is a continuous vector for a p_w-norm, where $w \geq 0$ and w G-dominates v, then v is also a continuous vector for the p_d-norm, where $d \geq 0$ and d G-dominates w. This last remark is explained more precisely in the next proposition.

Proposition 3.4.2. *Let π be a representation of a topological group G on a normed Riesz space E by positive linear isometries and let d be a non-zero positive vector of E. Let $v \in (E, d)$ and suppose that there is a non-zero positive $w \in (E, d)$ such that $v \in (E, w)$ and such that the map*

$$G \longrightarrow (E, w), \quad g \longmapsto \pi(g)v \quad \text{is } p_w\text{-continuous.}$$

Then the map

$$G \longrightarrow (E, d), \quad g \longmapsto \pi(g)v \quad \text{is } p_d\text{-continuous.}$$

Proof. Let $(g_\alpha)_\alpha$ be a net in G which converges to the identity element $e \in G$. Then the estimation

$$p_d(g_\alpha v - v) \leq p_w(g_\alpha v - v)p_d(w)$$

holds for every α by point b) of Proposition 3.1.14. This shows that the net $(g_\alpha v)_\alpha$ converges to v in p_d-norm as v is a continuous vector for the p_w-norm. □

Unfortunately, this is the best we can say for general continuous representation of G on a normed Riesz space E. From now on, we focus on a specific class of normed Riesz spaces and a specific representation of G.

Suppose to be in the following setting. Let G be a topological group that acts on a set X via the action γ. Then we consider real Riesz function spaces of bounded functions $\mathfrak{F}^b(X)$ on X which are π_γ-invariant and such that the representation π_γ of G on $\mathfrak{F}^b(X)$ is continuous with respect to the supremum norm.

It was not possible to show that for every non-zero positive function $f \in \mathfrak{F}^b(X)$, the restriction of π_γ to the subspace $\mathfrak{F}^b(X, f)$ is p_f-continuous. Nevertheless, this is true for a particular set of functions, precisely for those which dominate their support.

Definition 3.4.3. A non-zero positive function $f \in \mathfrak{F}^b(X)$ is said to be **support-dominating** if there are $g_1, ..., g_n \in G$ such that $\mathbf{1}_{\text{supp}(f)} \leq \sum_{j=1}^n g_j f$.

In general $\mathbf{1}_{\text{supp}(f)} \notin \mathfrak{F}^b(X)$. This depends on the space on which we are working and on the function f. For example, take $G = \mathbf{R}$ together with its natural locally compact topology and consider the left-translation representation of G on $\mathfrak{F}^b(G) = \mathcal{C}^b(G)$. Then every continuous compactly supported function f on G is support-dominating but the indicator function $\mathbf{1}_{\text{supp}(f)} \notin \mathcal{C}^b(G)$.

Lemma 3.4.4. *Let $f \in \mathfrak{F}^b(X)$ be a support-dominating function. Then the action of G on $\mathfrak{F}^b(X, f)$ is orbitally continuous with respect to the p_f-norm.*

Proof. Fix $\phi \in \mathfrak{F}^b(X, f)$ and let $(g_\alpha)_\alpha \subset G$ be a net such that $\lim_\alpha g_\alpha = e$. We want to show that $\lim_\alpha p_f(g_\alpha\phi - \phi) = 0$. Let $\epsilon > 0$ and note that

$$|g_\alpha\phi - \phi| \leq ||g_\alpha\phi - \phi||_\infty \left(g_\alpha \mathbf{1}_{\text{supp}(\phi)} + \mathbf{1}_{\text{supp}(\phi)}\right) \qquad \text{for every } \alpha.$$

As f is support-dominating, there are $g_1, ..., g_n \in G$ such that $\mathbf{1}_{\text{supp}(\phi)} \leq \sum_{j=1}^n g_j f$. Thus,

$$|g_\alpha\phi - \phi| \leq ||g_\alpha\phi - \phi||_\infty \left(g_\alpha \sum_{j=1}^n g_j f + \sum_{j=1}^n g_j f\right) \qquad \text{for every } \alpha.$$

As the representations of G on $\mathfrak{F}^b(X)$ is continuous for the supremum norm, there is α_0 such that $||g_\alpha\phi - \phi||_\infty < \frac{\epsilon}{2n}$ for every $\alpha \succeq \alpha_0$. We can conclude that

$$p_f(|g_\alpha\phi - \phi|) \leq ||g_\alpha\phi - \phi||_\infty p_f \left(g_\alpha \sum_{j=1}^n g_j f + \sum_{j=1}^n g_j f\right) \leq 2n||g_\alpha\phi - \phi||_\infty < \epsilon$$

for every $\alpha \succeq \alpha_0$. □

3.4.A. The case of compactly supported functions. We focus on representations on the set of compactly supported continuous functions. This case will be helpful later.

Let G be a topological group, and let X be a locally compact space X on which G acts via the action γ. Suppose that the vector space $\mathcal{C}_{00}(X)$ is π_γ-invariant and that the representation π_γ is $|| \cdot ||_\infty$-continuous.

Recall that the action of G on X is called **cocompact** if there is a compact subset K of X such that the image of K under the action of G covers X.

The following is a well-known lemma. We repeat the proof for the sake of completeness.

Lemma 3.4.5. *Let X be a locally compact topological space, and let K be a compact subset of X. Then there are a relatively compact subset $U \subset X$ and an open subset $V \subset X$ such that*

$$K \subset U \subset \overline{U} \subset V.$$

Proof. For every $k \in K$ take a relatively compact neighborhood of it, say U_k. Therefore, $K \subset \bigcup_{k\in K} U_k$. As K is compact, there are $k_1, ..., k_n \in K$ such that $K \subset \bigcup_{j=1}^n U_{k_j}$. Define $U = \bigcup_{j=1}^n U_{k_j}$. Because every U_{k_j} is relatively compact then so is U. As Hausdorff locally compact spaces are normal, there exists an open set $V \subset X$ such that $\overline{U} \subset V$. □

Proposition 3.4.6. *Let G be a topological group and let X be a locally compact space. Suppose that G acts on X. Then the following assertions are equivalent:*

a) the action of G on X is cocompact;

b) the space $\mathcal{C}_{00}(X)$ admits a support-dominating G-dominating element;

c) the space $\mathcal{C}_{00}(X)$ admits a G-dominating element.

Proof. We show that a) implies b). Suppose that the action of G on X is cocompact. Then there is $K \subset X$ a compact set such that $X = \bigcup_{g\in G} gK$. By Lemma 3.4.5, there are a relatively compact subset $U \subset X$ and an open subset $V \subset X$ such that

$$K \subset U \subset \overline{U} \subset V.$$

Now, we can use Uryshon Lemma ([R86, Lemma 2.12]) to find a positive $\psi \in \mathcal{C}_{00}(X)$ such that $\psi = 1$ on $\overline{U}$ and $\psi = 0$ on $X \setminus V$. We claim that ψ is a support-dominating G-dominating element of $\mathcal{C}_{00}(X)$. Indeed, let $\phi \in \mathcal{C}_{00}(X)$ and let $K' = \text{supp}(\phi)$. As the action of G on X is cocompact, there are $g_1, ..., g_n \in G$ such that

$$K' \subset \bigcup_{j=1}^n g_j K \subset \bigcup_{j=1}^n g_j \overline{U}.$$

This implies that

$$|\phi| \leq ||\phi||_\infty \mathbf{1}_{supp(\phi)} \leq \sum_{j=1}^n ||\phi||_\infty \mathbf{1}_{g_j\overline{U}} = \sum_{j=1}^n ||\phi||_\infty g_j \mathbf{1}_{\overline{U}} \leq \sum_{j=1}^n ||\phi||_\infty g_j \psi$$

which proves that ψ is a G-dominating element. The fact the ψ is also support-dominating is a consequence of the case where $\phi = \psi$ in the previous calculation.

Point b) implies point c) directly.

It is left to prove that c) implies a). Therefore, suppose that $\mathcal{C}_{00}(X)$ admits a G-dominating element, say ψ, and we want to show that the action of G on X is cocompact. To this aim, we have to prove that there is a compact set $K \subset G$ such that for every $x \in X$ there is $g \in G$ with $x \in gK$. Define $K = \text{supp}(\psi)$. Let now take $x \in X$. Then there is $K' \subset X$ a compact neighborhood of x as X is locally compact. By Urysohn lemma there exists $\phi \in \mathcal{C}_{00}(X)$ such that $\text{supp}(\phi) \subset K'$. So, there are $g_1, ..., g_n \in G$ such that $|\phi| \leq \sum_{j=1}^n g_j \psi$. But this implies that $K' \subset \bigcup_{j=1}^n g_j K$ showing that the action of G on X is cocompact. □

Remark 3.4.7. Actually, what we showed in the previous proposition is that there is K compact subset of X such that $X = \bigcup_{g \in G} gK$ if and only if there is a non-zero positive $\psi \in \mathcal{C}_{00}(X)$ such that $X = \bigcup_{g \in G} g\text{supp}(\psi)$ if and only if $\mathcal{C}_{00}(X)$ admits a G-dominating element. Moreover, every G-dominating element of $\mathcal{C}_{00}(X)$ is support-dominating.

With this remark in mind, we can state the following corollary.

Corollary 3.4.8. *Let G a topological group that acts on a locally compact space X. If the action of G on X is transitive, then every non-zero positive element of $\mathcal{C}_{00}(X)$ is G-dominating and support-dominating.*

Proof. Fix a non-zero positive function $\psi \in \mathcal{C}_{00}(X)$. If the action of G on X is transitive, then $X = \bigcup_{g \in G} g\text{supp}(\psi)$. Thus, ψ is a support-dominating G-dominating element by Remark 3.4.7. □

In particular, if a topological group G is acting transitively on a locally compact space X, then

$$\mathcal{C}_{00}(X, \phi) = \mathcal{C}_{00}(X, \psi) = \mathcal{C}_{00}(X) \quad \text{for every non-zero positive } \phi, \psi \in \mathcal{C}_{00}(X).$$

Moreover, the two norms p_ψ and p_ϕ are equivalent.

Proposition 3.4.9. *Let G be a topological group and suppose that it has a jointly continuous action γ on a locally compact space X. Then the π_γ representation of G on $\mathcal{C}_{00}(X)$ is continuous with respect to the supremum norm.*

Proof. Suppose it is not the case. Then there are $\epsilon > 0$, a non-zero $\psi \in \mathcal{C}_{00}(X)$ and a net $(g_\alpha)_\alpha$ in G which converges to $e \in G$ such that

$$||\pi_\gamma(g_\alpha)\psi - \psi||_\infty > \epsilon \quad \text{for every } \alpha.$$

This means that there is a net $(x_\alpha)_\alpha$ in X such that

$$|\pi_\gamma(g_\alpha)\psi(x_\alpha) - \psi(x_\alpha)| > \epsilon \quad \text{for every } \alpha.$$

As the net $(x_\alpha)_\alpha$ lies in the compact set $\text{supp}(\psi)$, there is a subnet $(x_\beta)_\beta$ converging to some $x \in \text{supp}(\psi)$. Therefore,

$$|\pi_\gamma(g_\alpha)\psi(x_\beta) - \psi(x_\beta)| > \epsilon \quad \text{for every } \alpha, \beta.$$

This implies that the net $(g_\alpha^{-1}, x_\beta)_{\alpha,\beta}$ in $G \times X$ is not converging to x for the product topology. However, this is a contradiction because the action of G on X is jointly continuous. Therefore, the π_γ representation of G on $\mathcal{C}_{00}(X)$ is continuous. □

Proposition 3.4.10. *Let G be a topological group and let X be a locally compact space. Suppose that G has an action γ on X, which is jointly continuous and cocompact. Then the π_γ representation of G on $\mathcal{C}_{00}(X)$ is continuous with respect to the p_ψ-norm, where ψ is a G-dominating element of $\mathcal{C}_{00}(X)$.*

Proof. We can apply Lemma 3.4.4 because the π_γ representation of G on $\mathcal{C}_{00}(X)$ is continuous by Proposition 3.4.9 and because ψ is support-dominating by Remark 3.4.7. □

Corollary 3.4.11. *Let G be a topological group which has a transitive jointly continuous action γ on a locally compact space X. Then the π_γ representation of G on $\mathcal{C}_{00}(X)$ is continuous for the p_ψ-norm for every non-zero positive $\psi \in \mathcal{C}_{00}(X)$.*

Proof. The proof is a combination of Proposition 3.4.10 and of Corollary 3.4.8. □

3.4.B. An application to nets. In general, convergence in $||\cdot||_\infty$-norm doesn't imply convergence in p_f-norm as the p_d-topology is stronger than the topology generated by the supremum norm (see example 3.1.17). However, the two convergences coincide if there are a few extra conditions.

Proposition 3.4.12. *Let $f \in \mathfrak{F}^b(X)$ be a non-zero positive support-dominating function and let $(\phi_\alpha)_\alpha$ by a net in $\mathfrak{F}^b(X, f)$ which converges in $||\cdot||_\infty$-norm to $\phi \in \mathfrak{F}^b(X, f)$ and which has decreasing supports, i.e., $supp(\phi_\alpha) \supset supp(\phi_{\alpha'})$ if $\alpha' \succeq \alpha$. Then $(\phi_\alpha)_\alpha$ converges to ϕ in p_f-norm.*

Proof. Let $\epsilon > 0$ and fix an α_0. Since f is a support-dominating function, there are $g_1, ..., g_n, y_1, ..., y_m \in G$ such that

$$\mathbf{1}_{\text{supp}(\phi)} \leq \sum_{j=1}^{n} g_j f \quad \text{and} \quad \mathbf{1}_{\text{supp}(\phi_{\alpha_0})} \leq \sum_{i=1}^{m} y_i f.$$

Now the inequality

$$|\phi - \phi_\alpha| \leq ||\phi - \phi_\alpha||_\infty \left(\mathbf{1}_{\text{supp}(\phi)} + \mathbf{1}_{\text{supp}(\phi_\alpha)}\right) \leq ||\phi - \phi_\alpha||_\infty \left(\sum_{j=1}^{n} g_j f + \sum_{i=1}^{m} y_i f\right)$$

holds for every $\alpha \succeq \alpha_0$. Taking α' such that $||\phi - \phi_\alpha||_\infty < \frac{\epsilon}{n+m}$ and such that $\alpha' \succeq \alpha_0$, we can conclude that

$$p_f(|\phi_\alpha - \phi|) \leq ||\phi - \phi_\alpha||_\infty p_f \left(\sum_{j=1}^{n} g_j f + \sum_{i=1}^{m} y_i f\right) < \epsilon \quad \text{for every } \alpha \succeq \alpha'.$$

□

Chapter 4

Positive Functionals on Dominated Spaces

The chapter is entirely dedicated to the investigation of positive linear functionals on dominated spaces. Here, dominated norms will play a central role in generating the suitable topology to work with the interested functionals.

The chapter is divided into two parts that deal with two different functional aspects of dominated Riesz spaces.

In the first part, we start characterizing the continuity and positivity of linear functionals on dominated ordered vector spaces. After that, we introduce and study the concept of normalized integral, a particular type of positive functional, which we will use in each of the following chapters. Finally, we define the invariant normalized integral property, and we give the first examples of group representations that have it.

In the second one, we define the concept of translate property and then explore its relationship with localized means and normalized integrals. In particular, we show that some translate property is powerful enough to provide amenability, and we give some first partial answers to Greenleaf's question. However, these answers to Greenleaf's question cover the discrete case totally.

4.1 The invariant normalized integral property

Let G be a topological group together with a representation by positive linear automorphisms on an ordered vector space E.

4.1.A. Continuity of positive functionals. We recall that a functional ψ on an ordered vector space E is said **uniformly bounded** on a set $V \subset E$ if there is $M > 0$ such that $\pm\psi(v) \leq M$ for every $v \in V$.

Proposition 4.1.1. *Let $d \in E$ be a non-zero positive vector, and let ψ be a positive functional defined on (E, d). Suppose that ψ is uniformly bounded on the set $\{gd : g \in G\}$. Then ψ is continuous with respect to the $\tau(p_d)$-topology.*

Proof. Let $(v_\alpha)_\alpha$ be a net in (E, d) which converges to some $v \in (E, d)$ for the $\tau(p_d)$-topology. Then there is a net $(\epsilon_\alpha)_\alpha$ in $\mathbf{R}$ converging to zero and there are $t_1^\alpha, ..., t_{n_\alpha}^\alpha \in \mathbf{R}$ and $g_1^\alpha, ..., g_{n_\alpha}^\alpha \in G$ such that

$$\pm(v_\alpha - v) \leq \sum_{j=1}^{n_\alpha} t_j^\alpha g_j^\alpha d \quad \text{and} \quad \sum_{j=1}^{n_\alpha} t_j^\alpha \leq \epsilon_\alpha \quad \text{for every } \alpha.$$

Set $M = \sup \{\psi(gd) : g \in G\}$. Then

$$\pm\psi(v_\alpha - v) \leq \sum_{j=1}^{n_\alpha} t_j^\alpha \psi(g_j^\alpha d) \leq M \sum_{j=1}^{n_\alpha} t_j^\alpha \leq M\epsilon_\alpha \quad \text{for every } \alpha.$$

Taking the limit of this last inequality, we have that

$$\lim_\alpha \pm\psi(v_\alpha - v) \leq M \lim_\alpha \epsilon_\alpha = 0.$$

This implies that the net $(\psi(v_\alpha))_\alpha$ converges to $\psi(v)$ showing the continuity of ψ. □

We can sharper this last result for representations on normed Riesz spaces. The following proposition is a generalization of the well-known result [R02, Proposition 1.1.2] for means.

Proposition 4.1.2. *Let G be a topological group which has a representation on a normed Riesz space E by positive linear isometries. Fix a non-zero positive vector $d \in E$, and let ψ be a linear functional defined on (E, d). Then*

a) if the functional ψ is positive and uniformly bounded on the set $\{gd : g \in G\}$, then ψ is continuous for the p_d-norm and

$$||\psi||_{op} \leq \sup \{\psi(gd) : g \in G\}.$$

In particular, if ψ is constant on $\{gd : g \in G\}$, then $||\psi||_{op} = \psi(d)$;

b) if the functional ψ is continuous for the p_d-norm, then

$$\sup \{\psi(gd) : g \in G\} \leq ||\psi||_{op} \leq \sup \{|\psi|(gd) : g \in G\}.$$

In particular, if ψ is positive, then $||\psi||_{op} = \sup \{\psi(gd) : g \in G\}$;

c) if the functional ψ is continuous for the p_d-norm and positively constant on the set $\{gd : g \in G\}$, then ψ is positive.

Proof. We start by showing point a). The fact that ψ is continuous for the p_d-norm is given by Proposition 4.1.1. Let now $v \in (E, d)$ and $M = \sup\{\psi(gd) : g \in G\}$. Then for every $\epsilon > 0$ there are $t_1, ..., t_n \in \mathbf{R}_+$ and $g_1, ..., g_n \in G$ such that

$$|v| \leq \sum_{j=1}^{n} t_j g_j d \quad \text{and} \quad \sum_{j=1}^{n} t_j \leq p_d(v) + \frac{\epsilon}{M}.$$

Therefore,

$$|\psi(v)| \leq \psi(|v|) \leq \psi\left(\sum_{j=1}^{n} t_j g_j d\right) = \sum_{j=1}^{n} t_j \psi(g_j d) \leq M \sum_{j=1}^{n} t_j \leq M p_d(v) + \epsilon.$$

As ϵ and v were chosen arbitrarily, we can conclude that $||\psi||_{op} \leq \sup\{\psi(gd) : g \in G\}$. If ψ is constant on the set $\{gd : g \in G\}$, then $\psi(d) \leq ||\psi||_{op}$ as $p_d(d) = 1$. This implies that $||\psi||_{op} = \psi(d)$.

Before proving point b), recall that if ψ is a continuous functional on a Riesz space, then $|\psi|$ is also a continuous functional with $||\psi||_{op} = ||\,|\psi|\,||_{op}$ by [AB99, Theorem 8.48]. Now the estimation

$$\psi(gd) \leq |\psi(gd)| \leq |\psi|(gd) \leq ||\psi||_{op} p_d(gd) = ||\psi||_{op} \quad \text{holds for every } g \in G.$$

Hence, we can conclude that $\sup\{\psi(gd) : g \in G\} \leq ||\psi||_{op}$. To prove the second inequality, set $M = \sup\{|\psi|(gd) : g \in G\}$, and note that this value is finite as $|\psi|$ is also continuous. Let $v \in (E, d)$. Then for every $\epsilon > 0$ there are $t_1, ..., t_n \in \mathbf{R}_+$ and $g_1, ..., g_n \in G$ such that

$$|v| \leq \sum_{j=1}^{n} t_j g_j d \quad \text{and} \quad \sum_{j=1}^{n} t_j \leq p_d(v) + \frac{\epsilon}{M}.$$

Thus,

$$|\psi|(v) \leq |\psi|(|v|) \leq |\psi|\left(\sum_{j=1}^{n} t_j g_j d\right) = \sum_{j=1}^{n} t_j |\psi|(g_j d) \leq M \sum_{j=1}^{n} t_j \leq M p_d(v) + \epsilon.$$

As ϵ and v were chosen arbitrarily, we can conclude that $||\psi||_{op} \leq \sup\{|\psi|(gd) : g \in G\}$.

It is left to show point c). In order to find a contradiction, suppose that ψ is not positive. Then there is a non-zero positive $v \in (E, d)$ such that $p_d(v) = 1$ and $\psi(v) < 0$. Consequently, for every $\epsilon > 0$ there are $t_1, ..., t_n \in \mathbf{R}_+$ and $g_1, ..., g_n \in G$ such that

$$v \leq \sum_{j=1}^{n} t_j g_j d \quad \text{and} \quad 1 \leq \sum_{j=1}^{n} t_j \leq 1 + \frac{\epsilon}{||\psi||_{op}}.$$

Now, $||\psi||_{op} = \psi(gd)$ for every $g \in G$ by point b). On the one hand,

$$\psi\left(\sum_{j=1}^{n} t_j g_j d - v\right) = ||\psi||_{op} \sum_{j=1}^{n} t_j - \psi(v) > ||\psi||_{op}.$$

On the other hand,

$$\psi\left(\sum_{j=1}^{n} t_j g_j d - v\right) \leq ||\psi||_{op} p_d\left(\sum_{j=1}^{n} t_j g_j d - v\right) \leq ||\psi||_{op} p_d\left(\sum_{j=1}^{n} t_j g_j d\right) \leq ||\psi||_{op} + \epsilon.$$

Therefore, $\psi\left(\sum_{j=1}^{n} t_j g_j d - v\right) \leq ||\psi||_{op}$, as ϵ was chosen arbitrarily. But this is the contradiction searched. Hence, ψ is positive. □

The inspiration to define p_d-norms comes from these last two propositions. In fact, such norms are the good ones to generate a topology for which positive functionals bounded on the majorizing vector subspace $\text{span}_{\mathbf{R}}\{gd : g \in G\}$ are continuous. We can think of these norms as a generalization of the order unit norm[1] for G-dominated ordered vector spaces.

The following example illustrates how these norms are essential to working with positive functionals defined on dominated spaces.

Example 4.1.3. Let $G = \mathbf{Z}$ be the additive group of the integers endowed with the discrete topology, and let $\mathrm{m}_{\mathbf{Z}}$ be a Haar measure for $\mathbf{Z}$, i.e., the counting measure. If the Riesz space $c_{00}(\mathbf{Z})$ is equipped with the p_{δ_0}-norm, where δ_0 is the Dirac mass at zero, then $\mathrm{m}_{\mathbf{Z}}$ is continuous thanks to point a) of Proposition 4.1.2. However, if $c_{00}(\mathbf{Z})$ is equipped with the supremum norm, then this is not true anymore. Indeed, let $(f_n)_n \subset c_{00}(\mathbf{Z})$ be the positive sequence given by $f_n = \frac{1}{n}\sum_{j=0}^{n^2} \delta_j$. Then

$$||f_n||_\infty = \frac{1}{n}\left|\left|\sum_{j=0}^{n^2} \delta_j\right|\right|_\infty = \frac{1}{n}$$

for every $n \in \mathbf{N}$. Hence, $(f_n)_n$ converges to zero in the $||\cdot||_\infty$-norm. Nevertheless,

$$\mathrm{m}_{\mathbf{Z}}(f_n) = \frac{1}{n}\mathrm{m}_{\mathbf{Z}}\left(\sum_{j=0}^{n^2} \delta_j\right) = \frac{1}{n}\sum_{j=0}^{n^2} \mathrm{m}_{\mathbf{Z}}(\delta_j) = \frac{1}{n}n^2 = n,$$

for every $n \in \mathbf{N}$.

[1] Let E be a normed Riesz space which admits an order unit $u \in E$. Then the order unit norm on E is the norm $||\cdot||_u$ defined by $||v||_u = \inf\{\alpha : v \leq \alpha u\}$.

4.1.B. Invariant normalized integrals. Let E be an ordered vector space and suppose that G has a representation π on E by positive linear automorphisms.

Definition 4.1.4. Fix a non-zero positive vector $d \in E$. Then a functional ψ defined on (E, d) is called:

- an **integral** if ψ is positive;
- a **normalized integral** if it is an integral and $\psi(d) = 1$;
- an **invariant normalized integral** if it is a normalized integral and $\pi^*(g)\psi = \psi$ for every $g \in G$, where π^* is the adjoint of π.

Remarks 4.1.5. 1) As before, if E is a normed Riesz space, then we ask that the representation of G on E is by positive linear isometries.

2) Let X be a set. Then a mean on an invariant subspace of $\ell^\infty(X)$, which contains the constant functions, is only a normalized integral. In fact, the definition of integral generalizes the one of mean to ordered vector spaces.

3) Suppose that E is Banach lattice. Then the definition above extends naturally to the vector subspaces of the form $(E, d)_\infty$ for every non-zero positive $d \in E$. In fact, we know that (E, d) is p_d-norm dense in $(E, d)_\infty$ (Proposition 3.3.4) and that $p_d^\infty = p_d$ on (E, d) (Theorem 3.3.5). Moreover, an integral is always continuous for the p_d-norm (Proposition 4.1.2). Therefore, we can extend uniquely every integral ψ on (E, d) to $(E, d)_\infty$. By continuity, the extension preserves the normalization and the invariance.

Given an ordered vector space E with a representation of a topological group G by positive linear automorphisms, we say that (E, d) **admits an invariant normalized integral** if there is an invariant normalized integral defined on it.

Example 4.1.6. The first and the easier examples of invariant integrals are invariant means. In fact,

1) every topological $\mathscr{U}$-amenable group G has an invariant normalized integral on the Banach lattice $\mathcal{C}_u^b(G, \mathscr{U}) = \big(\mathcal{C}_u^b(G, \mathscr{U}), \mathbf{1}_G\big)$. Here, $\mathscr{U}$ is a functionally invariant uniformity for G;

2) every topological group G admits an invariant normalized integral on the space $\mathcal{W}(G) = \mathcal{W}(G, \mathbf{1}_G)$ of weakly almost periodic functions, see [B70, Theorem 1.25];

3) more generally, every topological group G admits an invariant normalized integral on the space $\mathcal{A}sp(G) = \mathcal{A}sp(G, \mathbf{1}_G)$ of Asplund functions, see [GM12, Proposition 2.3].

A first property of invariant normalized integrals can be directly deduced from Proposition 4.1.1. In fact, the following holds:

Corollary 4.1.7. *Let E be an ordered vector space, and suppose that G acts on E by positive linear automorphisms. Let $d \in E$ be a non-zero positive vector.*

a) Suppose that the space (E, d) admits an invariant normalized integral I. *Then* I *is continuous with respect to the p_d-norm.*

b) Suppose that E is a Banach lattice and that the representation of G on E is by positive linear isometries. Suppose moreover that $(E, d)_\infty$ admits an invariant normalized integral I. *Then* I *is continuous with respect to the p_d^∞-norm, and it has operator norm equal to 1.*

Proof. For point a), I is continuous with respect to the p_d-norm by Proposition 4.1.1.

Let suppose we are in the situation of point b). Then I has operator norm equal to $||\mathrm{I}||_{op} = \mathrm{I}(d) = 1$ by point a) of Proposition 4.1.2. □

4.1.C. The invariant normalized integral property. Let E be an ordered vector space and let G be a topological group. Suppose that G has a representation on E by positive linear automorphisms.

Definition 4.1.8. We say that G, or π, **has the invariant normalized integral property for** E if the space (E, d) admits an invariant normalized integral for every non-zero positive vector $d \in E$.

Remarks 4.1.9. 1) As always, if E is a normed Riesz space, then we ask that the representation of G on E is by positive linear isometries.

2) Suppose that G has the invariant normalized integral property for a normed Riesz space E. Then it is not necessarily true that G has the invariant normalized integral property for the Banach lattice $\widehat{E}$, the topological completion of E.

The following proposition gives the first examples of group representations having the invariant normalized integral property.

Proposition 4.1.10. *Let G be a topological group with a representation on an ordered vector space E by positive linear automorphisms. Suppose that E admits a strictly positive invariant functional. Then G has the invariant normalized integral property for E.*

Proof. Let ψ be the strictly positive invariant functional on E and let $d \in E$ be a non-zero positive vector. Set $c = \psi(d) > 0$. Then an invariant normalized integral for (E, d) is given by $\mathrm{I} = \frac{1}{c}\psi$. □

Example 4.1.11. (Examples of the invariant normalized integral property)

1) Let G be a locally compact group and consider the left-translation representation of G on the Banach lattice $L^1(G)$. Then the formula $\mathrm{I}(f) = \int_G f dm_G$ defines a strictly positive invariant functional on $L^1(G)$. Therefore, every locally compact group G has the invariant normalized integral property for $L^1(G)$ by Proposition 4.1.10.

2) If, in addition, G is compact, then G has the invariant normalized integral property for $L^p(G)$ for $p \in [1, \infty]$ as $L^p(G) \subset L^1(G)$ for every $p \in [1, \infty]$, see [AB99, Corollary 13.3].

3) More generally, we claim that every representation of a topological group G on an AL-space by positive linear isometries has the invariant normalized integral property. Let G be a topological group and suppose that it has a representation π by positive linear isometries on an AL-space $(E, ||\cdot||)$. Then there is a measurable space (Ω, Σ, μ) such that $(E, ||\cdot||)$ is Riesz isometric to $\big(L^1(\Omega, \mu), ||\cdot||_1\big)$ by Theorem 2.4.21. Consequently, G has a representation by positive linear isometries on $L^1(\Omega, \mu)$, say π_Ω. By construction of (Ω, Σ, μ), the representation π_Ω comes from an action of G on Ω such that the measure μ is G-invariant. Therefore, there exists a strictly positive invariant functional I on $L^1(\Omega, \mu)$ given by

$$\mathrm{I}(f) = \int_\Omega f d\mu \quad \text{for every } f \in L^1(\Omega, \mu).$$

Thus, we can conclude that the representation π of G on E has the invariant normalized integral property.

4) Since the Haar measure can be interpreted as a strictly positive invariant functional on the Riesz space of compactly supported continuous functions $\mathcal{C}_{00}(G)$ ([Bou63, III §1 No.5 Théorème 1]), then every locally compact group G has the invariant integral property for $\mathcal{C}_{00}(G)$.

5) Let G be a topological group and consider the Banach lattice $\mathcal{AP}(G)$ of almost periodic functions on G, i.e., the spaces of all bounded continuous functions which have relatively compact orbit with respect to the $||\cdot||_\infty$-norm. Then $\mathcal{AP}(G)$ admits a strictly positive invariant functional, see [B70, Corollary 1.26]. Hence, every topological group G has the invariant normalized integral property for $\mathcal{AP}(G)$.

6) Let G a topological group and suppose that G has a unitary representation σ on a (complex) Hilbert space $\mathcal{H}$. Then the representation σ induces a representation Ad_σ by positive linear automorphisms on the ordered vector space $\mathcal{B}(\mathcal{H})$, of bounded linear operators from $\mathcal{H}$ to itself, given by

$$\mathrm{Ad}_\sigma(g)(T) = \sigma(g)T\sigma(g)^* \quad \text{for } g \in G \text{ and } T \in \mathcal{B}(\mathcal{H}).$$

Consider the ordered Ad_σ-invariant vector subspace $\mathrm{TC}(\mathcal{H})$ of the trace class operators, and define on it the map

$$\mathrm{I} : \mathrm{TC}(\mathcal{H}) \longrightarrow \mathbf{R}, \quad T \longmapsto \mathrm{I}(T) = \mathrm{tr}(T),$$

where tr denotes the trace function of $\mathcal{H}$. We claim that I is a strictly positive linear functional on $\mathrm{TC}(\mathcal{H})$. Indeed, the map I is linear thanks to [S18, Proposition 6.8] and strictly positive because for every positive T in $\mathrm{TC}(\mathcal{H})$ the lower bound

$\mathrm{tr}(T) \geq ||T||$ holds by [S18, Proposition 6.4]. Finally, I is Ad_σ-invariant because of [S18, Theorem 6.13 (b)]. In fact,

$$\mathrm{I}\left(\mathrm{Ad}_\sigma(g)T\right) = \mathrm{tr}\left(\sigma(g)T\sigma(g)^*\right) = \mathrm{tr}\left(T\right) = \mathrm{I}\left(T\right)$$

for every $g \in G$ and every $T \in \mathrm{TC}(\mathcal{H})$. We can conclude that G has the invariant normalized integral property for $\mathrm{TC}(\mathcal{H})$ by Proposition 4.1.10.

A peculiarity of these examples is that every invariant normalized integral was always the restriction of a strictly positive functional defined on a bigger space. These strictly positive invariant functionals are, almost always, unique up to constant and are defined on relatively small functions spaces. However, there are examples where different integrals appear. We will see that $L^\infty(G)$ has the invariant normalized integral property for every abelian non-compact locally compact group G. However, an invariant mean on $L^\infty(G)$ can not be strictly positive as explained in [P84, Proposition 21.2].

4.2 The translate property

The section is entirely dedicated to discussing Greenleaf's question.

4.2.A. First properties of the translate property. For instance, let G be a topological group acting on an ordered vector space E by positive linear automorphisms.

Definition 4.2.1. We say that a non-zero positive vector $d \in E$ has the **translate property** if

$$\sum_{j=1}^{n} t_j g_j d \geq 0 \quad \text{implies that} \quad \sum_{j=1}^{n} t_j \geq 0 \quad \text{for every } t_1, ..., t_n \in \mathbf{R} \text{ and } g_1, ..., g_n \in G.$$

Example 4.2.2. Suppose that G acts by positive linear automorphisms on an ordered vector space E. Then every G-fixed-point has the translate property because of the axioms of the vector ordering.

Before continuing, we present an example of a vector that does not have the translate property.

Example 4.2.3. Let $G = \mathbf{F}_2 = \langle a, b \rangle$ be the free group on 2 generators and consider the left-translation representation of $\mathbf{F}_2$ on the Banach lattice $\ell^\infty(\mathbf{F}_2)$. Let A be the set of all reduced words of $\mathbf{F}_2$ starting with the generator a and define the function

$$f = a^{-1}\mathbf{1}_A - b\mathbf{1}_A - \mathbf{1}_A.$$

We claim that f doesn't have the translate property. Indeed, if $w \in \mathbf{F}_2$ is a reduced word starting with a or b, then $f(w) = 0$. On the other hand, if $w \in \mathbf{F}_2$ is a reduced word starting with a^{-1}, b^{-1} or the identity element e, then $f(w) = 1$. This implies that $f \geq 0$ and that $f \neq 0$. However, the sum of its coefficients is equal to -1.

The following result shows that the translate property is somewhat encoded in the structure of a G-dominated space. Precisely, we can understand if a non-zero positive vector $d \in E$ has the translate property only by looking at the cone of positive functionals on (E, d) or at the norm p_d.

Proposition 4.2.4. *Let E be a normed Riesz space and suppose that G has a representation on E by positive linear isometries. Let $d \in E$ be a non-zero positive vector. Then the following assertions are equivalent:*

a) the vector d has the translate property;

b) there exists a normalized integral ψ on (E, d) invariant on $\operatorname{span}_{\mathbf{R}}\{gd : g \in G\}$;

c) we have that

$$p_d\left(\sum_{j=1}^n t_j g_j d\right) = \sum_{j=1}^n t_j \quad \text{for every } t_1, ..., t_n \in \mathbf{R}_+ \text{ and } g_1, ..., g_n \in G.$$

In other words, p_d is additive on $\operatorname{span}_{\mathbf{R}}\{gd : g \in G\}$.

Proof. We start showing that a) implies b). Let $d \in E$ be a non-zero positive vector with the translate property and define the linear map

$$\omega : \operatorname{span}_{\mathbf{R}}\{gd : g \in G\} \longrightarrow \mathbf{R}, \quad \sum_{j=1}^n t_j g_j d \longmapsto \omega\left(\sum_{j=1}^n t_j g_j d\right) = \sum_{j=1}^n t_j.$$

Note that ω is well-defined thanks to the translate property. Moreover, it is positive, G-invariant and $\omega(d) = 1$. As $\operatorname{span}_{\mathbf{R}}\{gd : g \in G\}$ is a majorizing vector subspace of (E, d), we can use Kantorovich Theorem (Theorem 2.2.7) to extend ω in a positive way to all (E, d). This extension is a normalized integral for (E, d) invariant on $\operatorname{span}_{\mathbf{R}}\{gd : g \in G\}$.

Now, we go for b) implies c). Let $t_1, ..., t_n \in \mathbf{R}_+$ and $g_1, ..., g_n \in G$. Then

$$p_d\left(\sum_{j=1}^n t_j g_j d\right) \le \sum_{j=1}^n t_j p_d(g_j d) \le \sum_{j=1}^n t_j$$

only because p_d is a monotone G-invariant norm. Let ψ be a normalized integral for (E, d) invariant on $\operatorname{span}_{\mathbf{R}}\{gd : g \in G\}$. By point a) of Proposition 4.1.2, ψ is continuous with operator norm equal to 1. This implies that

$$p_d\left(\sum_{j=1}^n t_j g_j d\right) \ge \psi\left(\sum_{j=1}^n t_j g_j d\right) = \sum_{j=1}^n t_j \psi(g_j d) = \sum_{j=1}^n t_j.$$

We can conclude that $p_d\left(\sum_{j=1}^n t_j g_j d\right) = \sum_{j=1}^n t_j$ as wished.

Finally, we show that c) implies a). Suppose that the translate property fails for the vector d. Therefore, there are $t_1, ..., t_n \in \mathbf{R}$ and $g_1, ..., g_n \in G$ such that

$$\sum_{j=1}^{n} t_j g_j d \geq 0 \quad \text{but} \quad \sum_{j=1}^{n} t_j < 0.$$

We can suppose that every t_j is in $\mathbf{Q}$. Indeed, if it is not the case, we can take $\epsilon > 0$ such that $\epsilon < \frac{-\sum_{j=1}^{n} t_j}{n}$, and we can chose $q_j \in \mathbf{Q}$ such that $q_j \geq t_j$ and $q_j - t_j < \epsilon$ for every j. Then

$$\sum_{j=1}^{n} q_j g_j d \geq \sum_{j=1}^{n} t_j g_j d \geq 0$$

and

$$\sum_{j=1}^{n} q_j = \sum_{j=1}^{n} (q_j - t_j) + \sum_{j=1}^{n} t_j \leq n\epsilon + \sum_{j=1}^{n} t_j < 0.$$

Now, there is $m \in \mathbf{N}$ st $mq_j = z_j \in \mathbf{Z}$ for every $j \in \{1, ..., n\}$. Then

$$0 \leq \sum_{j=1}^{n} mq_j g_j d = \sum_{j=1}^{n} z_j g_j d = \sum_{z_j \in I_+} z_j g_j d - \sum_{z_j \in I_-} |z_j| g_j d,$$

where $I_+ = \{z_j : z_j > 0\}$ and $I_- = \{z_j : z_j < 0\}$. Therefore,

$$\sum_{z_j \in I_-} |z_j| = p_d \left(\sum_{z_j \in I_-} |z_j| g_j d \right) \leq p_d \left(\sum_{z_j \in I_+} |z_j| g_j d \right) = \sum_{z_j \in I_+} z_j.$$

On the other side, we have that

$$0 > \sum_{j=1}^{n} mq_j = \sum_{j=1}^{n} z_j = \sum_{z_j \in I_+} z_j - \sum_{z_j \in I_-} |z_j|$$

which implies that $\sum_{z_j \in I_-} |z_j| > \sum_{z_j \in I_+} z_j$. But this is a contradiction. □

Remarks 4.2.5. 1) The equivalence between a) and b) is still true if E is supposed to be an ordered vector space and the representation of G on E is by positive linear automorphisms. Moreover, the theorem is still true if E is only an ordered normed vector space and G acts on it by positive linear isometries.

2) Point c) may be a little surprising at first glance but it is something which appears naturally when there is the absence of the translate property. For example, let f be the function of Example 4.2.3. Then

$$a^{-1}\mathbf{1}_A - b\mathbf{1}_A - \mathbf{1}_A \geq 0 \iff a^{-1}\mathbf{1}_A \geq b\mathbf{1}_A + \mathbf{1}_A.$$

Therefore,

$$p_{\mathbf{1}_A}(b\mathbf{1}_A + \mathbf{1}_A) \leq p_{\mathbf{1}_A}(a^{-1}\mathbf{1}_A) = 1.$$

Corollary 4.2.6. *Suppose that a group G has a representation by positive linear isometries on an AM-space $(E, ||\cdot||)$. Let $d \in E$ be a non-zero positive vector with the translate property and note $L = \mathrm{span}_{\mathbf{R}}\{gd : g \in G\}$. Then the $||\cdot||$-norm and the p_d-norm are equivalent on L if and only if L is finite-dimensional.*

Proof. Suppose that $||\cdot||$ and p_d are equivalent on L. Then the Banach lattices $(\overline{L}^{||\cdot||}, ||\cdot||)$ and $(\overline{L}^{p_d}, p_d)$ are Riesz homeomorphic. But the former is an AM-space while the latter an AL-space. Therefore, L has to be finite-dimensional by [AB99, Corollary 9.39].

The inverse is given by [K78, Theorem 2.4-5]. □

Definition 4.2.7. Let G be a topological group and let E be an ordered vector space. Suppose that G has a representation on E by positive linear automorphisms. Then we say that G **has the translate property for** E if every non-zero positive vector $d \in E$ has the translate property.

The following two corollaries of Proposition 4.2.4 give the first examples of group representations having the translate property.

Corollary 4.2.8. *Let E be an ordered vector space and suppose that G has a representation on E by positive linear automorphisms. If G has the invariant normalized integral property for E, then G has the translate property for E.*

Proof. The translate property for E is assured thanks to the fact that the point b) of Proposition 4.2.4 is satisfied. □

Example 4.2.9. (Examples of representations with the translate property)

1) Every locally compact group G has the translate property for $L^1(G)$ and for $\mathcal{C}_{00}(G)$.

2) In general, every representation of a topological group by positive linear isometries on an AL-space has the translate property.

3) Every compact group has the translate property for $L^p(G)$ for $p \in [1, \infty]$.

4) Every topological group G has the translate property for $\mathcal{AP}(G)$.

5) Suppose that a topological group G has a unitary representation σ on a Hilbert space $\mathcal{H}$. Then the representation Ad_σ of G on $\mathrm{TC}(\mathcal{H})$ has the translate property.

Corollary 4.2.10. *Let $(E.||\cdot||)$ be a normed Riesz space and suppose that a topological group G has a representation on E by positive linear isometries. Suppose that $||\cdot||$ is an L-norm. Then G has the translate property for E.*

Proof. Let $d \in E$ be a non-zero positive vector. Without loss of generality, we can suppose that $||d|| = 1$. Let $t_1, ..., t_n \in \mathbf{R}_+$ and $g_1, ..., g_n \in G$. Then

$$\sum_{j=1}^{n} t_j \geq p_d\left(\sum_{j=1}^{n} t_j g_j d\right) \geq ||\sum_{j=1}^{n} t_j g_j d|| = \sum_{j=1}^{n} t_j.$$

Therefore, condition c) of Proposition 4.2.4 is satisfied. Thus, d has the translate property. □

At this point, there are two questions we can ask about the translate property. The first question is: if every vector of a normed Riesz space E has the translate property, can we construct an invariant functional on E? Similarly, if we know that a non-zero positive vector d of a Banach lattice E has the translate property, can we construct an invariant integral on (E, d)? The remaining two subsections are dedicated to answering those questions. Note that the second question is the famous Greenleaf's question aforementioned.

4.2.B. Translate property and invariant means. Let E be a Banach lattice with order unit u and let G be a topological group. Suppose that G has a representation π on E by positive linear isometries. We recall that a mean on E is nothing but an integral m normalized on u. A mean m is said **localized** on a vector $d \in E$ if $\pi^*(g)\mathrm{m}(d) = \mathrm{m}(d)$ for every $g \in G$.

Theorem 4.2.11. *Let $(E, ||\cdot||)$ be a Banach lattice with an order unit u and suppose that G acts on E by positive linear isometries. Then for every non-zero positive vector $d \in E$ with the translate property, there is a mean on E localized on it.*

We follow the strategy used by Monod in [M17, Theorem 18].

Proof of Theorem 4.2.11. First of all, we can suppose that the norm $||\cdot||$ of E is the order unit norm $||\cdot||_u$ as explained in point 2) of Example 2.4.18.

Let $d \in E$ be a non-zero positive vector with the translate property, and consider the Banach subspace

$$D = \overline{\mathrm{span}_{\mathbf{R}}\{d - gd : g \in G\}}^{||\cdot||_u} \subset E.$$

Denote $Q = {}^{E}\!/_{D}$ the quotient space equipped with the quotient norm $||\cdot||_Q$. We claim that $||u_Q||_Q = 1$, where u_Q is the image of the order unit u of E in the quotient Q. Suppose that it is not the case, then there are $\epsilon > 0, t_1, ..., t_n \in \mathbf{R}$ and $g_1, ..., g_n \in G$ such that $v = \sum_{j=1}^{n} t_j(d - g_j d)$ satisfies $||u - v||_u \leq 1 - \epsilon$. This means that

$$u - v \leq (1 - \epsilon)u \iff \epsilon u \leq v.$$

Set $M = ||d||_u$ and note that

$$\frac{\epsilon}{M}d \leq \epsilon u \leq v \iff v - \frac{\epsilon}{M}d = \sum_{j=1}^{n} t_j(d - g_j d) - \frac{\epsilon}{M}d \geq 0.$$

However, this last inequality is a contradiction with the fact that d has the translate property. In fact, the sum of its coefficients is equal to $-\frac{\epsilon}{M} < 0$. Therefore, $||u_Q||_u = 1$. Using the Hahn-Banach Theorem, there is a continuous linear functional m_Q on Q such that $\mathrm{m}_Q(u_Q) = 1$. Now, define m as the lift of this functional. Then m is of norm one and positive. Moreover, $\mathrm{m}(d) = \mathrm{m}(gd)$ for every $g \in G$ as m vanishes on D □

Together with the following lemma, this last proposition gives us a partial answer to the first question we asked ourselves.

Suppose that G acts on a set X via the action γ and let E be a subspace of $\ell^\infty(X)$ which is π_γ-invariant and which contains the constant functions. Then E is said **introverted** if for every $v \in E$ and $\psi \in E'$, the real function $g \longmapsto \psi(gv)$ is in E. Examples of introverted spaces are given by $\mathcal{W}(G)$ for topological groups ([BJM89, Theorem 2.5]) and also by $\mathcal{C}(G)$ for compact groups ([BJM89, Example 2.5]).

Lemma 4.2.12. *Let G be a topological group with an action γ on a set X. Let E be a π_γ-invariant introverted subspace of $\ell^\infty(X)$ containing the constant functions. Then there is a π_γ-invariant mean on E if and only if there is a mean localized on f for every positive function $f \in E$.*

Proof. See [GR71, Lemma 1]. □

As the function space $\mathcal{C}^b_{ru}(G)$ is introverted by [N67, Lemma 1.2], we can conclude that:

Corollary 4.2.13. *If G has the translate property for $\mathcal{C}^b_{ru}(G)$, then G is amenable.*

Proof. By Proposition 4.2.11, the translate property for $\mathcal{C}^b_{ru}(G)$ implies that for every positive function $f \in \mathcal{C}^b_{ru}(G)$ there is a localized mean on f. Therefore, there is an invariant mean on $\mathcal{C}^b_{ru}(G)$ by Lemma 4.2.12. □

This last corollary can be seen as a topological version of [M17, Theorem 18].

Scholium 4.2.14. Monod showed Corollary 4.2.13 for discrete groups in his paper [M17]. If we read his proof carefully, we can see that he did not use every hypothesis at his disposal. In fact, he supposed that the group has the translate property for $\ell^\infty(G)$, but he only used it for the vectors of the form $\mathbf{1}_A$, where $A \subset G$. This was possible thanks to a result of Moore ([M13, Theorem 1.3]), which characterizes amenability of discrete groups using finitely additive measures localized on subsets $A \subset G$.

The reverse does not hold in general. Namely, there are amenable groups without the translate property.

Example 4.2.15. Let $G = Aff(\mathbf{R})$ be the group of affine transformations of the line endowed with the discrete topology. Then G is amenable as extension of amenable groups. However, it contains the free semigroup in two generators $\mathrm{T}_2 = \langle a, b\rangle$, where $a(r) = r + 1$ and $b(r) = 2r$. We claim that the vector $\mathbf{1}_{\mathrm{T}_2}$ doesn't have the translate property. Indeed, define the function

$$f = \mathbf{1}_{\mathrm{T}_2} - a\mathbf{1}_{\mathrm{T}_2} - b\mathbf{1}_{\mathrm{T}_2} = \mathbf{1}_{\mathrm{T}_2} - \mathbf{1}_{a\mathrm{T}_2} - \mathbf{1}_{b\mathrm{T}_2}$$

and note that f is non-zero and positive. It is positive because if $w \in \mathrm{T}_2$, then $w \in a\mathrm{T}_2$ or $w \in b\mathrm{T}_2$ but not both. Thus $f(w) \geq 0$. If $w \in a\mathrm{T}_2$, then $w \notin b\mathrm{T}_2$ and so $f(w) \geq 0$. Same for $w \in b\mathrm{T}_2$. Finally, f is non-zero as $f(e) = 1$, where $e \in \mathrm{T}_2$ is the identity element. But one can see that the sum of the coefficients of f is -1 which shows that f has not the translate property.

Scholium 4.2.16. Perhaps an attentive reader has noticed that, for the case of discrete groups, the translate property for vectors of the form $\mathbf{1}_A$, where $A \subset G$, is related to the concept of paradoxical decomposition. We refer to [K14] and to [R72] for an exposition and a discussion of this relationship.

4.2.C. Translate property and invariant normalized integrals. Let E be a normed Riesz space, and let G be a topological group with a representation on E by positive linear isometries.

Definition 4.2.17. Let $d \in E$ be a non-zero positive vector and consider (E, d) equipped with the p_d-norm. We define

$$\mathcal{I}_d(E) = \{\psi \in (E, d)'_+ : \psi(gd) = \psi(d) = 1 \text{ for all } g \in G\}$$

the **set of all normalized integrals on** (E, d).

Note that if E is an invariant subspace of $\ell^\infty(X)$, for some set X, which contains the constant functions, then $\mathcal{I}_{\mathbf{1}_G}(E) = \mathcal{M}(E)$.

Unlike the set of means, $\mathcal{I}_d(E)$ can be empty. Fortunately, we know precisely when it doesn't happen.

Proposition 4.2.18. *For every non-zero positive vector $d \in E$, the set $\mathcal{I}_d(E)$ is non-empty if and only if the vector d has the translate property.*

Proof. The proof is an application of Proposition 4.2.4. □

Using the Banach-Alaoglu Theorem ([M98, Theorem 2.6.18]) is straightforward to show that:

Proposition 4.2.19. *The set $\mathcal{I}_d(E)$ is convex and compact with respect to the weak-* topology for every non-zero positive vector $d \in E$.*

Proposition 4.2.20. *Let $\mathscr{U}$ be a functionally invariant uniformity for G for which G is $\mathscr{U}$-amenable and let $d \in E$ be a non-zero positive vector. Suppose that there is a non-zero $\psi_0 \in \mathcal{I}_d(E)$ such that the map $g \longmapsto g\psi_0$ is $\mathscr{U}$-uniformly continuous. Then the vector d has the translate property if and only if the space (E, d) admits an invariant normalized integral.*

Proof. The *if* part is straightforward. Let's look at the *only if* part. First of all, $\mathcal{I}_d(E)$ is non-empty thanks to Proposition 4.2.18. Now, the action of G on $\mathcal{I}_d(E)$ has a fixed-point I by Theorem 1.4.14 and because G is $\mathscr{U}$-amenable. The functional I is an invariant normalized integral for (E, d). □

Proposition 4.2.21. *Let G be an amenable group and let $d \in E$ be a non-zero positive vector. Suppose that the representation of G on (E, d) is continuous for the p_d-norm. Then the vector d has the translate property if and only if the space (E, d) admits an invariant normalized integral.*

Proof. If there is an invariant normalized integral on (E, d), then the vector d has the translate property by Corollary 4.2.8. Let's prove the converse. Suppose that d has the translate property. Therefore, the convex compact set $\mathcal{I}_d(E)$ is non-empty by Proposition 4.2.18. Now, the action of G on $\mathcal{I}_d(E)$ is orbitally continuous for the weak-* topology. This means that G fixes a point I in $\mathcal{I}_d(E)$ by theorem 1.4.17. This fixed-point is an invariant integral for (E, d). □

Those two last propositions imply the following corollaries.

Corollary 4.2.22. *Let G be an amenable group and suppose that E has an L-norm. For every non-zero positive vector $d \in E$ such that the representation of G on (E, d) is continuous for the p_d-norm, the space (E, d) admits an invariant normalized integral.*

Proof. By Corollary 4.2.10, the group G has the translate property for E. We can conclude using Proposition 4.2.21. □

Corollary 4.2.23 ([M17, Corollary 19])**.** *Let G be a discrete group. Then G has the invariant normalized integral property for $\ell^\infty(G)$ if and only if G has the translate property for $\ell^\infty(G)$.*

Proof. Note that G is amenable in both cases by Corollary 4.2.13. Therefore, we can conclude using Proposition 4.2.21. □

Corollary 4.2.24 (Rosenblatt)**.** *Let G be an amenable discrete group which acts on a set X. Then for every non-zero positive $f \in \ell^\infty(X)$, there is an invariant normalized integral on $\ell^\infty(X, f)$ if and only if the function f has the translate property.*

Proof. The proof is direct by Proposition 4.2.21. □

Chapter 5

The Fixed-Point Property for Cones

We generalise the fixed-point property for cones given by Monod in [M17] from the world of discrete groups to the universe of topological ones.

The chapter starts with a discussion by examples about groups acting on cones and fixed-points, which lead us to state the generalisation of the fixed-point property for cones. We use these examples to motivate and explain every assumption made in defining the fixed-point property for cones.

We continue giving different characterisations of the fixed-point property for cones using the invariant normalised integral property. The whole second section is dedicated to this purpose.

In the last part of the chapter, we abstract the characterisations developed previously. The interest of doing such abstraction work is to understand for which spaces a group with the fixed-point property for cones admits invariant normalized integrals. Similar characterisations were already developed for amenable groups and widely used by Zimmer to study rigidity of group actions, see [Z84]. Moreover, such an approach is helpful to get rid of some technical details, especially when working with locally compact groups.

5.1 A Fixed-point in mind

The geometric starting data is the following: we suppose to have a locally convex vector space E and a non-empty proper (convex) cone $C \subset E$.

The topological assumption on the cone is weakly completeness. Recall that weakly complete means that every net in C, which is Cauchy with respect to the canonical uniformity of the initial topology associated to the family of linear functionals E' converges. Considering only weakly complete cones is not a significant restriction. In fact, this condition includes most of the familiar cones we are used to working.

We refer to [C63] for examples of weakly complete cones and to [Bou81, II §6 No.8] for some properties of weakly complete cones in locally convex vector spaces.

We say that a group G has a representation on a non-empty proper convex cone C in a locally convex vector space E to designate that G has a representation on E, which leaves C invariant.

The pre-order defined by C on E is a vector ordering as C is supposed to be proper. Asking that the action of G preserves this order is equivalent to asking that the cone C is G-invariant. Indeed, suppose that the cone C is G-invariant and let $v, w \in E$ such that $v \leq_C w$. This means that $w - v \in C$. Therefore, $g(w - v) = gw - gv \in C$. We can conclude that $gv \leq_C gw$. Conversely, suppose that the action of G on E preserves the order induced by the cone C and let $c \in C$. Then $0 \leq_C c$ and so $0 \leq_C gc$ for every $g \in G$. This shows that the cone C is G-invariant.

In what follows, we illustrate various examples of representations of groups on cones without non-zero fixed points. This digression aims to motivate the choice of the definition of the fixed-point property for cones that we will state at the end of the section.

Starting from the roots: we defined a cone as a convex set. However, some authors do not ask for convexity. The choice of this assumption is mandatory to work with locally compact groups and, especially, with discrete ones.

Example A. *Every non-trivial locally compact group admits a non-zero fixed-point free representation on a non-empty closed non-convex cone.*

Proof. Let G be a non-trivial locally compact group. Then G is not extremely amenable by [V77, Theorem 2.2.1], i.e., there are no invariant multiplicative means on $\mathcal{C}^b_{ru}(G)$. Let $\mathcal{M}_m(\mathcal{C}^b_{ru}(G)) \subset \mathcal{C}^b_{ru}(G)'$ be the set of all multiplicative means on $\mathcal{C}^b_{ru}(G)$. Recall that a multiplicative mean on $\mathcal{C}^b_{ru}(G)$ is nothing but a mean m on $\mathcal{C}^b_{ru}(G)$ such that $\mathrm{m}(f_1 f_2) = \mathrm{m}(f_1)\mathrm{m}(f_2)$ for every $f_1, f_2 \in \mathcal{C}^b_{ru}(G)$. Then $\mathcal{M}_m(\mathcal{C}^b_{ru}(G))$ is a non-empty, invariant and compact set for the weak-* topology. But it is not convex. Consider now the weak-* closed non-convex cone given by

$$C = \left\{ \alpha \mathrm{m} : \alpha \in \mathbf{R} \text{ and } \mathrm{m} \in \mathcal{M}_m(\mathcal{C}^b_{ru}(G)) \right\} \subset \mathcal{C}^b_{ru}(G)'.$$

Then the adjoint representation of G on $\mathcal{C}^b_{ru}(G)'$ preserves C, but it fixes no non-zero points. □

Nonetheless, convexity is not enough to create attractive starting conditions.

Example B. *Every infinite topological group admits a non-zero fixed-point free representation on a non-empty proper cone.*

Proof. Let G be an infinite topological group. Consider the vector space $\mathbf{R}^G$ of all real functions from G to $\mathbf{R}$ and the left-translation representation π_L of G on it. Then $\mathbf{R}^G$ is a Hausdorff locally convex vector space when equipped with the pointwise topology. Fix a non-zero positive unbounded function $f \in \mathbf{R}^G$, and consider the π_L-invariant locally convex vector subspace given by

$$E = \operatorname{span}_{\mathbf{R}} \{gf : g \in G\} \subset \mathbf{R}^G.$$

Note that the pointwise closed cone C of positive functions of E is π_L-invariant. Nevertheless, there are no non-zero fixed-points in C as the constant function is not in E. □

The absence of non-zero fixed-points in the previous example is essentially due to the fact that every non-zero orbit of the representation of G on E is unbounded. Indeed, it suffices to show that the orbital action of G on f is unbounded. Let $V \subset E$ be the neighborhood of the origin given by $V = \{v \in E : |v(x)| < \alpha\}$, where $x \in G$ and $\alpha > 0$, and suppose that the set Gf is absorbed by V. This means that there is $\lambda > 0$ such that $Gf \subset \lambda V$. Now, we can chose $y \in G$ such that $f(y) > \lambda\alpha$ and $g \in G$ such that $g^{-1}x = y$. Therefore, $gf(x) = f(g^{-1}x) > \alpha\lambda$. But this is a contradiction with the fact that Gf is absorbed by V.

However, the unboundedness of the representation is not the only problem, since we can even find fixed-points free continuous positive linear isometric representations of groups on cones.

Example C. *Every non-compact locally compact group admits a non-zero fixed-point free continuous representation by positive linear isometries on a non-empty proper cone.*

Proof. Let G be a non-compact locally compact group and let $p \in [1, +\infty)$. Consider the left-translation representation of G on the Banach lattice $L^p(G)$. Then the action of G on $L^p(G)$ is continuous and by linear positive isometries and preserves the L^p-closed cone of positive functions $L^p(G)_+$. Nevertheless, G fixes no non-zero vectors in the positive cone of $L^p(G)$. □

Note that the action of G on $L^p(G)_+$ has a fixed-point if and only if G is compact.

Another example of this kind can help us understand why the representation is free of fixed points. Namely:

Example D. *Any infinite discrete group acts on some locally compact space without preserving any non-zero Radon measure.*

Proof. See [MR15, Proposition 4.3]. □

Despite that, we know that every cocompact action of a discrete supramenable group on a locally compact space fixes a non-zero Radon measure ([KMN13, Proposition 2.7]). On the one hand, the action considered in the proof of [MR15, Proposition 4.3] is not cocompact. On the other hand, the action of a group G on a locally compact space X is cocompact if and only if the Riesz space $C_{00}(X)$ admits a G-dominating element (Proposition 3.4.6). Moreover, the topological dual of $L^p(G)$ never allows a G-dominating element. Therefore, we can conclude that the topological dual of the locally convex space where the cone lies should not be *too big*.

After this short discussion, we are ready to generalize the fixed-point property for cones given by Monod.

For instance, we do not ask that every non-zero orbit is bounded. Instead, we ask that there is at least one non-zero point in the cone for which the boundedness condition is satisfied. As we want to define a fixed-point property for cones that takes care of the uniform structure of the group, we put a continuity condition on the action at this point.

Definition 5.1.1. Let G be a topological group and let $\mathscr{U}$ be a functionally invariant uniformity for G. We say that a representation of G on a non-empty proper cone C is **locally bounded** $(\mathscr{U}, \mathscr{U}_c)$**-uniformly continuous** if there is a non-zero $x_0 \in C$ such that the map

$$(G, \mathscr{U}) \longrightarrow (E, \mathscr{U}_c), \quad g \longmapsto gx_0$$

is bounded and uniformly continuous.

Remark 5.1.2. Some authors use the terminology *slightly* instead of *locally*, e.g. [P18] or [G17].

We state the following definition to overcome the problem contained in examples C and D.

Definition 5.1.3. Let G be a topological group that has a representation on a non-empty proper cone C. We say that the representation of G on C is of **cobounded type** if E' admits a G-dominating element for the adjoint representation of G on E' and for the order given by the polar cone of C.

We are finally ready to generalize the fixed-point property for cones.

Definition 5.1.4. Let G be a topological group and let $\mathscr{U}$ be a functionally invariant uniformity for G. We say that G has the $\mathscr{U}$**-fixed-point property for cones** if every representation of G on a non-empty weakly complete proper cone C in a (Hausdorff) locally convex vector space E which is locally bounded $(\mathscr{U}, \mathscr{U}_c)$-uniformly continuous and of cobounded type has a non-zero fixed-point.

Proposition 5.1.5. *Let G be a topological group and let $\mathscr{U}$ be a functionally invariant uniformity for G. If G has the $\mathscr{U}$-fixed-point property for cones, then G is $\mathscr{U}$-amenable.*

Proof. Suppose that G has the $\mathscr{U}$-fixed-point property for cones. Let $\mathcal{M}\left(\mathcal{C}_u^b(G, \mathscr{U})\right)$ be the set of means on $\mathcal{C}_u^b(G, \mathscr{U})$. Then $E = \mathcal{C}_u^b(G, \mathscr{U})'$ endowed with the weak-* topology is a locally convex vector space ([Bou81, I §6 No.2 Remarque 1)]). Consider the non-empty invariant proper cone given by

$$C = \left\{\alpha \mathrm{m} : \mathrm{m} \in \mathcal{M}\left(\mathcal{C}_u^b(G, \mathscr{U})\right) \text{ and } \alpha \in \mathbf{R}_+\right\} \subset \mathcal{C}_u^b(G, \mathscr{U})'.$$

Then C is weak-* closed in E, and hence weakly complete. The adjoint representation of G on C is locally bounded $(\mathscr{U}, \mathscr{U}_c^*)$-uniformly continuous.

This because the map $g \longmapsto gev_e$ is $(\mathscr{U}, \mathscr{U}_c^*)$-uniformly continuous and bounded by Lemma 1.4.13 with $(X, \mathscr{U}) = (G, \mathscr{U})$ and $\mathfrak{F}^b(X) = \mathcal{C}_u^b(G, \mathscr{U})$. Moreover, the adjoint representation of G on $\mathcal{C}_u^b(G, \mathscr{U})'$ is also of cobounded type. Indeed, the topological dual of $\mathcal{C}_u^b(G, \mathscr{U})'$ with respect to the weak-* topology is only $\mathcal{C}_u^b(G, \mathscr{U})$ ([Bou81, II §6 No.1 Remarque 1)]) and this last vector space is G-dominated by $\mathbf{1}_G$. Hence, we can use the $\mathscr{U}$-fixed-point property for cones to find a non-zero fixed-point in C, which is nothing but an invariant mean after normalization. □

We have a better result for discrete groups.

Proposition 5.1.6. *Let G be a discrete group and let $\mathscr{U}_d$ be the discrete uniformity of G. If G has the $\mathscr{U}_d$-fixed-point property for cones, then G is supramenable.*

Proof. It suffices to show that every cocompact action of G on a locally compact space fixes a non-zero Radon measure by [KMN13, Proposition 2.7]. Therefore, suppose that G acts cocompactly on a locally compact space X and let $C = \mathcal{M}(X)$ be the cone of Radon measures on X. Recall that it is possible identify C with $\mathcal{C}_{00}(X)_+^*$ the set of all positive functionals on $\mathcal{C}_{00}(X)$ ([Bou63, III §1 No.5 Théorème 1]). First of all, C is complete for the vague topology ([Bou63, III §1 No.9 Proposition 14]), and hence weakly complete since this topology is weak as it is the weak-* topology given by the duality with $\mathcal{C}_{00}(X)$. Moreover, the representation of G on C is locally bounded as witness by every evaluation map ev_x at a point $x \in X$. The proof of this last fact is given by Lemma 1.4.13 using $\mathfrak{F}^b(X) = \mathcal{C}_{00}(X)$. It is only left to check that the representation is of cobounded type, and then we can apply the $\mathscr{U}_d$-fixed-point property for cones. But, again by [Bou81, II §6 No.1 Remarque 1)], the topological dual of $\mathcal{C}_{00}(X)^*$ with respect to the weak-* topology is only $\mathcal{C}_{00}(X)$, and we know that this vector space is G-dominated by a positive function $\phi \in \mathcal{C}_{00}(X)$ because the action of G on X is cocompact and because of Proposition 3.4.6. Applying the $\mathscr{U}_d$-fixed-point property for cones, we have the existence of a non-zero Radon measure on X, which ensures the supramenability of G. □

Remark 5.1.7. Monod already proposed in [M17, Example 38] a possible generalization of his fixed-point property. However, the one he gave is not equal to ours in the non-locally compact case. We will motivate, and hopefully, we will persuade, the readers that our fixed-point property is the *right one* to generalize the work of Monod.

However, for locally compact groups, our generalization and the one of Monod coincide, see Subsection 6.3.C.

5.2 Characterizations of the $\mathscr{U}$-fixed-point property for cones

The section aims to characterize the $\mathscr{U}$-fixed-point property for cones in a similar way as done for $\mathscr{U}$-amenability (cf. Theorem 1.4.14).

Theorem 5.2.1. *Let G be a topological group and let $\mathscr{U}$ be a functionally invariant uniformity for G. Then the following assertions are equivalent:*

a) the group G has the $\mathscr{U}$-fixed-point property for cones;

b) the group G has the invariant normalized integral property for $\mathcal{C}_u^b(G, \mathscr{U})$;

c) for every action γ of G on a compact space K such that $\mathcal{C}(K)$ is π_γ-invariant and for every non-zero positive $\phi \in \mathcal{C}(K)$ for which there is $k_0 \in K$ such that $\phi(k_0) \neq 0$ and the map $g \longmapsto gk_0$ is $\mathscr{U}$-uniformly continuous, there is an invariant normalized integral on $\mathcal{C}(K, \phi)$;

d) for every action γ of G on a uniform space $(X, \mathscr{U}_X)$ such that $\mathcal{C}_u^b(X, \mathscr{U}_X)$ is π_γ-invariant and for every non-zero positive $f \in \mathcal{C}_u^b(X, \mathscr{U}_X)$ for which there is $x_0 \in X$ such that $f(x_0) \neq 0$ and the map $g \longmapsto gx_0$ is $(\mathscr{U}, \mathscr{U}_X)$-uniformly continuous, there is an invariant normalized integral on $\big(\mathcal{C}_u^b(X, \mathscr{U}_X), f\big)$.

Fixed-points in cones find their functional counterpart in invariant normalized integrals. This is not surprising as invariant normalized integrals are just a generalization of invariant means and every non-empty convex compact set generates a non-empty convex cone.

5.2.A. Proof of Theorem 5.2.1. The strategy to prove Theorem 5.2.1 is the following: firstly, we show the equivalence between a) and b), secondly, we prove the equivalence of b), c) and d).

We need the following lemma to prove that having the $\mathscr{U}$-fixed-point property for cones is equivalent to having the invariant normalized integral property for $\mathcal{C}_u^b(G, \mathscr{U})$. The lemma is also of independent interest as it displays an interesting property of functionally invariant uniformities for topological groups.

Lemma 5.2.2. *Let G be a topological group and let $\mathscr{U}$ be a functionally invariant uniformity for G. Then the maps*

$$\omega_L : (G, \mathscr{U}) \longrightarrow \Big(\mathcal{M}(\mathcal{C}_u^b(G, \mathscr{U})), \mathscr{U}_c^*\Big), \quad g \longmapsto \omega_L(g) = \pi_L(g)^* ev_x$$

and

$$\omega_R : (G, \mathscr{U}) \longrightarrow \Big(\mathcal{M}(\mathcal{C}_u^b(G, \mathscr{U})), \mathscr{U}_c^*\Big), \quad g \longmapsto \omega_R(g) = \pi_R(g)^* ev_x$$

are uniformly continuous for every $x \in G$.

Proof. We start showing that ω_L is $(\mathscr{U}, \mathscr{U}_c^*)$-uniformly continuous. Let $x \in G$ and take $A \in \mathscr{U}_c^*$. As before, we can suppose that A is of the form

$$A_V = \Big\{(\psi_1, \psi_2) \in \mathcal{M}(\mathcal{C}_u^b(G, \mathscr{U})) \times \mathcal{M}(\mathcal{C}_u^b(G, \mathscr{U})) : \psi_2 - \psi_1 \in V\Big\},$$

where $V = \{\psi \in \mathcal{C}_u^b(G,\mathscr{U})^* : |\psi(f_j)| < \epsilon\}$ for $f_1, ..., f_n \in \mathcal{C}_u^b(G,\mathscr{U})$ and $\epsilon > 0$. Because $\mathscr{U}$ is a functionally invariant uniformity, we have that the $\pi_R(x)f_j$'s are in $\mathcal{C}_u^b(G,\mathscr{U})$ for every $j = 1, ..., n$. Therefore, for every j, there is an $A_j \in \mathscr{U}$ such that if $(g_1, g_2) \in A_j$, then $(\pi_R(x)f_j(g_1), \pi_R(x)f_j(g_2)) \in A_\epsilon$. Here,

$$A_\epsilon = \{(r_1, r_2) \in \mathbf{R} \times \mathbf{R} : |r_2 - r_1| < \epsilon\} \in \mathscr{U}_c^{\mathbf{R}}.$$

Set $A_{\omega_L} = \bigcap_{j=1}^n A_j \in \mathscr{U}_c^{\mathbf{R}}$. Thus, if $(g_1, g_2) \in A_{\omega_L}$, then

$$\begin{aligned}
|\omega_L(g_1)(f_j) - \omega_L(g_2)(f_j)| &= |\pi_L(g_1)^* ev_x(f_j) - \pi_L(g_2)^* ev_x(f_j)| \\
&= |ev_{g_1 x}(f_j) - ev_{g_2 x}(f_j)| \\
&= |f_j(g_1 x) - f_j(g_2 x)| \\
&= |\pi_R(x)f_j(g_j) - \pi_R(x)f_j(g_2)| < \epsilon,
\end{aligned}$$

which implies that $\omega_L(g_2) - \omega_L(g_1) \in A_V$. We can conclude that ω_L is $(\mathscr{U}, \mathscr{U}_c^*)$-uniformly continuous.

For the map ω_R the proof is similar using the fact that

$$\pi_R(g) ev_x(f) = \pi_L(x^{-1}) f(g) \quad \text{for every } g, x \in G \text{ and } f \in \mathcal{C}_u^b(G,\mathscr{U}).$$

□

Proof of Theorem 5.2.1. We start showing that a) is equivalent to b). Thus, let $f \in \mathcal{C}_u^b(G,\mathscr{U})$ be a non-zero positive function and we want to prove that there is an invariant normalized integral I on $(\mathcal{C}_u^b(G,\mathscr{U}), f)$. Set $E = (\mathcal{C}_u^b(G,\mathscr{U}), f)^*$. Then E is a Hausdorff locally convex topological vector space when equipped with the weak-* topology, see [Bou81, I §6 No.2 Remarque 1)]. Consider now the cone $C = (\mathcal{C}_u^b(G,\mathscr{U}), f)_+^*$ of positive functionals on E. This cone is invariant and proper because spanned by positive elements. Recall that closed subspaces of complete spaces are complete ([Bou71, II §3 No.4 Proposition 8]) and that algebraic duals are weak-* complete ([Bou81, II §6 No.7]). Therefore, C is weakly complete as it is closed in the complete space E.

We claim that the representation of G on C is of locally bounded $(\mathscr{U}, \mathscr{U}_c^*)$-uniformly continuous type. Indeed, let $x \in G$ such that $f(x) \neq 0$. Then the map $g \longmapsto g ev_x$ is bounded $(\mathscr{U}, \mathscr{U}_c^*)$-uniformly continuous by Lemma 5.2.2.

The next step is to show that the action is of cobounded type. We have that $(E, \text{weak-*})' = (\mathcal{C}_u^b(G,\mathscr{U}), f)$ by [Bou81, II §6 No.1 Remarque 1)]. Therefore, the cobounded condition is verified.

Applying the $\mathscr{U}$-fixed-point property for cones to C, we have the existence of a non-zero G-invariant element $\mathrm{I} \in C$. It is clear that $\mathrm{I}(f) \neq 0$. Indeed, let $\phi \in E_+$ such that $\mathrm{I}(\phi) > 0$. Then $\phi \leq \sum_{j=1}^n g_j f$ for some $g_1, ..., g_n \in G$. Therefore,

$$0 < \mathrm{I}(\phi) \leq \mathrm{I}\left(\sum_{j=1}^n g_j f\right) = n\mathrm{I}(f).$$

After a normalization by $\mathrm{I}(f)$, I becomes an invariant normalized integral for E.

Let now prove that b) implies a). Take C a non-empty weakly complete invariant proper convex cone in a Hausdorff locally convex vector space E, and suppose that the action of G on E is of cobounded type and locally bounded $(\mathscr{U}, \mathscr{U}_c)$-uniformly continuous w.r.t. C. We have to show that there is a non-zero fixed-point in C. Let $x_0 \in C$ be the point which witness the locally bounded $(\mathscr{U}, \mathscr{U}_c)$-uniformly continuous condition and let $f \in E'$ be the G-dominating element given by the cobounded condition. For every $\lambda \in E'$, we define the map

$$\overline{\lambda} : G \longrightarrow \mathbf{R}, \quad g \longmapsto \overline{\lambda}(g) = \lambda(gx_0).$$

We claim that $\overline{\lambda} \in \mathcal{C}_u^b(G, \mathscr{U})$ for all $\lambda \in E'$. Indeed, we can write $\overline{\lambda}$ as

$$\overline{\lambda} : (G, \mathscr{U}) \longrightarrow (E, \mathscr{U}_c) \longrightarrow (\mathbf{R}, \mathscr{U}_c), \quad g \longmapsto gx_0 \longmapsto \lambda(gx_0).$$

Hence, $\overline{\lambda}$ is uniformly continuous as composition of uniformly continuous maps. Moreover, $\overline{\lambda}$ is bounded as the set $im(Gx_0)$ is bounded and continuous linear functionals map bounded sets to bounded sets ([Bou81, III §1 No.3 Corollaire 1]).

Thereby, we define the linear operator

$$\Lambda : E' \longrightarrow \mathcal{C}_u^b(G, \mathscr{U}), \quad \lambda \longmapsto \Lambda(\lambda) = \overline{\lambda}.$$

Note that Λ is equivariant because

$$\Lambda(a\lambda) = \overline{a\lambda} = a\overline{\lambda} = a\Lambda(\lambda) \quad \text{for every } a \in G \text{ and } \lambda \in E'.$$

Moreover, Λ is positive. In fact, if $\lambda \in E'_+$, then

$$\overline{\lambda}(g) = \lambda(gx_0) \geq 0 \quad \text{for every } g \in G.$$

Now, the image of Λ is contained in $\left(\mathcal{C}_u^b(G, \mathscr{U}), \overline{f}\right)$, where $\overline{f} = \Lambda(f)$. This is because if $\overline{\lambda} \in im(\Lambda)$, then there are $\lambda \in E'$ such that $\Lambda(\lambda) = \overline{\lambda}$ and $g_1, ..., g_n \in G$ such that $\pm\lambda \leq \sum_{j=1}^n g_j f$. This implies that

$$\pm\overline{\lambda} = \pm\Lambda(\lambda) \leq \Lambda\left(\sum_{j=1}^n g_j f\right) = \sum_{j=1}^n g_j \Lambda(f) = \sum_{j=1}^n g_j \overline{f}.$$

By hypothesis, there exists an invariant normalized integral I defined on $\left(\mathcal{C}_u^b(G, \mathscr{U}), \overline{f}\right)$ whereby one gets the linear map

$$\overline{\mathrm{I}} : E' \longrightarrow \mathbf{R}, \quad \lambda \longmapsto \overline{\mathrm{I}}(\lambda) = \mathrm{I}(\Lambda(\lambda)).$$

It easy to see that $\overline{\mathrm{I}}$ is positive, G-invariant and $\overline{\mathrm{I}}(f) = \mathrm{I}(\overline{f}) = 1$.

The functional $\bar{\mathrm{I}}$ is actually a fixed-point of the action of G on the algebraic dual of E'. If we can show that $\bar{\mathrm{I}}$ is in C, we are done. To prove this, embed C in its bidual $(E')^*$ via the canonical embedding, and, in order to find a contradiction, suppose that $\bar{\mathrm{I}} \notin C$. Because of the fact that C is weakly complete, C is closed in $(E')^*$ for the weak-* topology ([Bou71, II §3 No.3 Proposition 6] and [Bou71, II §3 No.4 Proposition 8]). So, we can apply the Hahn-Banach Separation Theorem ([R86, Thm.3.4]) and find $\lambda \in ((E')^*)'$ such that $\lambda \geq 0$ on C and $\lambda(\bar{\mathrm{I}}) < 0$. By [R86, 3.14], we have that $\lambda(\bar{\mathrm{I}}) = \bar{\mathrm{I}}(\lambda)$ and so

$$0 > \lambda(\bar{\mathrm{I}}) = \bar{\mathrm{I}}(\lambda) = \mathrm{I}(\overline{\lambda}) \geq 0.$$

However, this is a contradiction. Therefore, $\bar{\mathrm{I}} \in C$.

Now, we want to show that b) implies d). Let $(X, \mathscr{U}_X)$ as in the statement, and chose a non-zero positive function $f \in \mathcal{C}_u^b(X, \mathscr{U}_X)$. By hypothesis, there is $x_0 \in X$ such that $f(x_0) \neq 0$ and such that the map $g \longmapsto gx_0$ is $(\mathscr{U}, \mathscr{U}_X)$-uniformly continuous. We define the linear operator

$$T : \mathcal{C}_u^b(X, \mathscr{U}_X) \longrightarrow \mathcal{C}_u^b(G, \mathscr{U}), \quad \phi \longmapsto T(\phi)(g) = \phi(gx_0).$$

First of all, the operator T is well-defined, i.e., $T(\phi)$ is in $\mathcal{C}_u^b(G, \mathscr{U})$ for every $\phi \in \mathcal{C}_u^b(X, \mathscr{U}_X)$. This is because we can write $T(\phi)$ as the composition of uniformly continuous maps. In fact,

$$T(\phi) : (G, \mathscr{U}) \longrightarrow (X, \mathscr{U}_X) \longrightarrow \mathbf{R}, \quad g \longmapsto gx_0 \longmapsto \phi(gx_0)$$

for every $\phi \in \mathcal{C}_u^b(X, \mathscr{U}_X)$. A fast computation shows that T is also equivariant. Indeed,

$$T(a\phi)(g) = (a\phi)(gx_0) = \phi(a^{-1}gx_0) = T(\phi)(a^{-1}g) = aT(\phi)(g)$$

for every $a, g \in G$ and $\phi \in \mathcal{C}_u^b(X, \mathscr{U}_X)$. Finally, T is positive as $T(\phi)(g) = \phi(gx_0) \geq 0$ for every positive function $\phi \in \mathcal{C}_u^b(X, \mathscr{U}_X)$ and for every $g \in G$. Note that $T(f) > 0$ as $f(x_0) \neq 0$. Therefore, T maps $\left(\mathcal{C}_u^b(X, \mathscr{U}_X), f\right)$ into $\left(\mathcal{C}_u^b(G, \mathscr{U}), T(f)\right)$ and on this last space there is an invariant normalized integral I. The composition $\bar{\mathrm{I}} = \mathrm{I} \circ T$ is an invariant normalized integral for $\left(\mathcal{C}_u^b(X, \mathscr{U}_X), f\right)$.

Point d) implies c) directly. Indeed, consider the case where $(X, \mathscr{U}) = (K, \mathscr{U}_K)$.

Finally, suppose that point c) is true, and we want to show point b). Note that the Banach lattice $\mathcal{C}_u^b(G, \mathscr{U})$ is an AM-space with an order unit given by the constant function $\mathbf{1}_G$. Therefore, there is a compact space K such that $\mathcal{C}_u^b(G, \mathscr{U})$ is Riesz isometric to $\mathcal{C}(K)$ by Theorem 2.4.20. By the proof of this last cited theorem (see [AB99, Theorem 9.32] for details), the compact space K can be described explicitly by the set

$$K = \left\{\psi \in B_1^+ : \psi \text{ is an extreme point of } B_1^+ \text{ such that } \psi(\mathbf{1}_G) = 1\right\},$$

where $B_1^+ = \{\psi \in \mathcal{C}_u^b(G,\mathscr{U})'_+ : ||\psi||_{op} \leq 1\}$. Moreover, the Riesz isometry between $\mathcal{C}_u^b(G,\mathscr{U})$ and $\mathcal{C}(K)$ is given by the operator

$$T : \mathcal{C}_u^b(G,\mathscr{U}) \longrightarrow \mathcal{C}(K), \quad f \longmapsto T(f) = ev_f.$$

Note that $\mathcal{C}(K)$ is invariant for the representation induced by the action of G on K and the map T is equivariant. Let now $f \in \mathcal{C}_u^b(G,\mathscr{U})$ be a non-zero positive function. Then $T(f) \in \mathcal{C}(K)$ is a non-zero positive function. Set $k_0 = ev_{x_0}$. We can compute that

$$T(f)(k_0) = T(f)(ev_{x_0}) = ev_{x_0}(f) = f(x_0) \neq 0$$

and that the map $g \longmapsto gk_0$ is $\mathscr{U}$-uniformly continuous by Lemma 1.4.13. Therefore, there is an invariant normalized integral I on $(\mathcal{C}(K), T(f))$ by hypothesis. Consequently, the composition $\bar{\mathrm{I}} = \mathrm{I} \circ T$ defines an invariant normalized integral on $(\mathcal{C}_u^b(G,\mathscr{U}), f)$. □

Remarks 5.2.3. 1) We followed the strategy used by Monod in [M17, Theorem 7] for the proof of the equivalence of a) and b),

2) We could also have used a Gelfand-Naimark Representation Theorem type for commutative real Banach algebras (for example [A10, Theorem 1.1]) for the demonstration of d) implies c) because $\mathcal{C}_u^b(G,\mathscr{U})$ is a Banach algebra for every functionally invariant uniformity $\mathscr{U}$ for G as showed by Theorem 1.2.13.

As a direct consequences, we have that:

Corollary 5.2.4. *Let G be a compact group. Then G has the fixed-point property for cones.*

Note that we do not have to specify a functionally invariant uniform structure for compact groups as there is only one.

Proof of Corollary 5.2.4. Let G be a compact group. Then the normalized Haar measure of G is a strictly positive invariant functional on $\mathcal{C}(G)$. Therefore, G has the invariant normalized integral property for $\mathcal{C}(G)$ by Proposition 4.1.10. □

Corollary 5.2.5. *Let G be a topological group and let $\mathscr{U}$ be a functionally invariant uniformity for G. If G has the $\mathscr{U}$-fixed-point property for cones, then G has the translate property for $\mathcal{C}_u^b(G,\mathscr{U})$.*

Proof. By Theorem 5.2.1, the group G has the invariant normalized integral property for $\mathcal{C}_u^b(G,\mathscr{U})$. Therefore, G has the translate property for $\mathcal{C}_u^b(G,\mathscr{U})$ by Corollary 4.2.8. □

It is natural to ask when the converse also holds, i.e., the translate property implies the invariant normalized integral property. We know that this is true for discrete groups by Corollary 4.2.23. Unfortunately, this is no longer true outside the realm of discrete groups. However, there is a pleasant surprise for the case of locally compact groups, as we are going to see in Chapter 6.

Currently, the following proposition is the best that we can expect from general topological groups.

Proposition 5.2.6. *Let G be a topological group. Suppose that G is $\mathscr{R}$-amenable and let $f \in \mathcal{C}_u^b(G,\mathscr{R})$ be a non-zero positive support-dominating function. Then $\big(\mathcal{C}_u^b(G,\mathscr{R}), f\big)$ admits an invariant normalized integral if and only if f has the translate property.*

Proof. The *only if* part is given by Corollary 4.2.8. For the *if* part, note that the action of G on $\big(\mathcal{C}_u^b(G,\mathscr{R}), f\big)$ is continuous for the p_f-norm thanks to Lemma 3.4.4. Hence, we can conclude using Proposition 4.2.21. □

5.2.B. A look at the uniform structures $\mathscr{F}$, $\mathscr{R}$ and $\mathscr{L}$. The purpose of this subsection is to develop Theorem 5.2.1 in the particular cases of the uniform structures $\mathscr{F}$, $\mathscr{R}$ and $\mathscr{L}$.

Definition 5.2.7. Let G be a topological group that acts on a uniform space $(X, \mathscr{U}_X)$ by uniform isomorphisms.

a) We say that the action of G on $(X, \mathscr{U}_X)$ is **motion equicontinuous** if for every $A \in \mathscr{U}_X$ there is a neighborhood U of the identity of G such that $(x, gx) \in A$ for every $g \in U$ and $x \in X$.

b) We say that the action of G on $(X, \mathscr{U}_X)$ is **uniformly equicontinuous** if for every $A \in \mathscr{U}_X$ there is $B \in \mathscr{U}_X$ such that if $(x, y) \in B$, then $(gx, gy) \in A$ for all $g \in G$.

Some authors use the term bounded instead of motion equicontinuous, see [P06, Section 3.6] or [V93]. We decided to use this second terminology because every such action is, in particular, continuous and because we do not want to create confusion with the notion of bounded actions on vector spaces.

Lemma 5.2.8. *Let G be a topological group.*

a) The action of G on $(G, \mathscr{R})$ by left-translation is motion equicontinuous.

b) The action of G on $(G, \mathscr{L})$ by left-translation is uniformly equicontinuous.

Proof. Let $A \in \mathscr{R}$. We can suppose that A is of the form $A = \big\{(x, y) \in G \times G : xy^{-1} \in U\big\}$, where U is a neighborhood of the identity of G. Now, $x(gx)^{-1} \in U$ for every $x \in G$ and every $g \in U$. This implies that $(x, gx) \in A$ proving point a).

For point b), take $A \in \mathscr{L}$. As before, we can suppose that A is of the form $A = \big\{(x, y) \in G \times G : x^{-1}y \in U\big\}$, where U is a neighborhood of the identity of G. Set $B = A$ and note that for every $g \in G$, we have that $(gx, gy) \in A$ if and only if $(x, y) \in A$. This shows the uniform equicontinuity of the action. □

Other simple examples of uniformly equicontinuous actions are given by representations of topological groups by isometries on metric spaces. Indeed, let (X, d_X) be a metric space on which G acts by isometries and let $A \in \mathscr{U}_{d_X}$. We can suppose that A is of the form $A = \{(x, y) \in X \times X : d_X(x, y) < \epsilon\}$ for some $\epsilon > 0$. Therefore, $(x, y) \in A$ if and only if $(gx, gy) \in A$.

Lemma 5.2.9. *Let G be a topological group and let $(X, \mathscr{U}_X)$ be a uniform space. Suppose that G acts on $(X, \mathscr{U}_X)$ by uniform isomorphisms. Then for every non-zero positive function $f \in \mathcal{C}_u^b(X, \mathscr{U}_X)$ there is a positive equivariant linear operator $T_f : \mathcal{C}_u^b(X, \mathscr{U}_X) \longrightarrow \ell^\infty(G)$ such that $T_f(f) > 0$. Moreover,*

a) if the action of G on X is orbitally continuous, then the image of T_f is contained in $\mathcal{C}_u^b(G, \mathscr{F})$;

b) if the action of G on X is motion equicontinuous, then the image of T_f is contained in $\mathcal{C}_u^b(G, \mathscr{R})$;

c) if the action of G on X is uniformly equicontinuos, then the image of T_f is contained in $\mathcal{C}_u^b(G, \mathscr{L})$.

An action of a topological group on a uniform space is orbitally continuous if it is orbitally continuous w.r.t the topology induced by the uniform structure. This is equivalent to asking that the map

$$(G, \mathscr{F}) \longrightarrow (X, \mathscr{U}_X), \quad g \longmapsto gx$$

is uniformly continuous for every $x \in X$.

Proof of Lemma 5.2.9. Let $f \in \mathcal{C}_u^b(X, \mathscr{U}_X)$ be a non-zero positive function. Take $x_0 \in X$ such that $f(x_0) > 0$, and define

$$T_f : \mathcal{C}_u^b(X, \mathscr{U}_X) \longrightarrow \ell^\infty(G), \quad \phi \longmapsto T_f(\phi),$$

where $T_f(\phi)(g) = \phi(gx_0)$ for every $g \in G$. Clearly, T_f is well-defined, linear and positive. Moreover, $T_f(f) > 0$ because of the choice of x_0. Let now $a \in G$ and notice that

$$T_f(a\phi)(g) = \phi(a^{-1}gx_0) = \pi_L(a)\phi(gx_0) = \pi_L(a)T_f(\phi)(g)$$

for every $\phi \in \mathcal{C}_u^b(X, \mathscr{U}_X)$ and every $g \in G$. We can conclude that T_f is equivariant with respect to the induced representation of G on $\mathcal{C}_u^b(X, \mathscr{U}_X)$ and the left-translation representation of G on $\ell^\infty(G)$.

The proof of point a) is straightforward as the composition of continuous maps is continuous. For the proofs of point b) and c), we refer to [P06, Lemma 3.6.5]. □

Corollary 5.2.10. *If G has the $\mathscr{U}_d$-fixed-point property for cones, then for every action of G on a uniform space $(X, \mathscr{U}_X)$ by uniform isomorphisms, the induced representation of G on $\mathcal{C}_u^b(X, \mathscr{U}_X)$ has the invariant normalized integral property.*

Proof. By Lemma 5.2.9, for every non-zero positive function $f \in \mathcal{C}_u^b(X, \mathscr{U}_X)$ there is a positive equivariant operator T_f from $\mathcal{C}_u^b(X, \mathscr{U}_X)$ to $\ell^\infty(G)$ such that $T_f(f) > 0$. We can conclude using the fact that the $\mathscr{U}_d$-fixed-point property is equivalent to G having the invariant normalized integral property for $\ell^\infty(G)$ by Theorem 5.2.1. □

The following result is an elaboration of point d) of Theorem 5.2.1 for the uniform structures $\mathscr{F}$, $\mathscr{R}$ and $\mathscr{L}$.

Corollary 5.2.11. *Let G be a topological group.*

a) *The group G has the $\mathscr{F}$-fixed-point property for cones if and only if for every orbitally continuous action of G on a uniform space $(X, \mathscr{U}_X)$ by uniform isomorphisms, the induced representation of G on $\mathcal{C}_u^b(X, \mathscr{U}_X)$ has the invariant normalized integral property.*

b) *The group G has the $\mathscr{R}$-fixed-point property for cones if and only if for every motion equicontinuous action of G on a uniform space $(X, \mathscr{U}_X)$ by uniform isomorphisms, the induced representation of G on $\mathcal{C}_u^b(X, \mathscr{U}_X)$ has the invariant normalized integral property.*

c) *The group G has the $\mathscr{L}$-fixed-point property for cones if and only if for every uniformly equicontinuous action of G on a uniform space $(X, \mathscr{U}_X)$ by uniform isomorphisms, the induced representation of G on $\mathcal{C}_u^b(X, \mathscr{U}_X)$ has the invariant normalized integral property.*

Proof. We only give the proof of point a). The proofs of the points b) and c) are similar. Therefore, suppose that G has the $\mathscr{F}$-fixed-point property for cones and that it has an orbitally continuous action on a uniform space $(X, \mathscr{U}_X)$ by uniform isomorphisms. Using point a) of Proposition 5.2.11, the induced representation of G on $\mathcal{C}_u^b(X, \mathscr{U}_X)$ has the invariant normalized integral property. For the *if* part, we apply the hypothesis to the uniform space $(G, \mathscr{F})$, and then we conclude with Theorem 5.2.1. □

Corollary 5.2.12. *Let G be a topological group with the $\mathscr{L}$-fixed-point property for cones. For every orbitally continuous action of G on a discrete space X, the induced representation of G on $\ell^\infty(X)$ has the invariant normalized integral property.*

Proof. Since every continuous action of a topological group on a discrete space is uniformly equicontinuous, we can use point c) of Proposition 5.2.11 to show that $\ell^\infty(X)$ has the invariant normalized integral property. □

Remark 5.2.13. The previous two corollaries are still true if we change invariant normalized integral property with translate property.

Lemma 5.2.14. *Let G be a topological group and let $(E, ||\cdot||)$ be a Banach lattice. Suppose that G has a continuous representation on E by positive linear isometries. Then for every non-zero positive vector $v \in E$ there is a positive equivariant linear operator $T_v : E \longrightarrow \mathcal{C}_u^b(G, \mathscr{L})$ such that $T_v(v) > 0$.*

Be careful that here we mean equivariant with respect to the right-translation representation of G on $\mathcal{C}_u^b(G, \mathscr{L})$.

Proof of Lemma 5.2.9. Let $v \in E$ be a non-zero positive vector. By the Hahn-Banach Theorem there is a positive $\lambda \in E'$ such that $\lambda(v) > 0$. We define the operator

$$T_v : E \longrightarrow \mathcal{C}_u^b(G, \mathscr{L}), \quad w \longmapsto T_v(w),$$

where $T_v(w) = \lambda(gw)$ for every $g \in G$. Firstly, we should check that T_v is well-defined. Let $w \in E$ and note that

$$\sup_{g\in G} |\lambda(gw)| \leq ||\lambda||_{op}||w|| \quad \text{for every } g \in G.$$

Therefore, $T_v(w)$ is a bounded function. Then we have to check that $T_v(w)$ is left-uniformly continuous. To this end, let $\epsilon > 0$. As the representation of G on E is continuous, there is a neighborhood of the identity $A \subset G$ such that

$$||aw - w|| < \frac{\epsilon}{||\lambda||_{op}} \quad \text{for every } a \in A.$$

Hence,

$$\begin{aligned} |\pi_R(a)T_v(w)(g) - T_v(w)(g)| &= |\lambda(gaw) - \lambda(gw)| \\ &= |\lambda(gaw - gw)| \\ &\leq ||\lambda||_{op}||gaw - gw|| \\ &\leq ||\lambda||_{op}||aw - w|| < \epsilon \end{aligned}$$

for every $g \in G$ and $a \in A$. This shows that $T_v(w) \in \mathcal{C}_u^b(G, \mathscr{L})$ and so we can conclude that T_v is well-defined. Moreover, T_v is linear, positive and $T_v(v) > 0$. Therefore, we proceed to prove that T_v is equivariant. Let $a \in G$ and $w \in E$. Then

$$T_v(aw)(g) = \lambda(gaw) = T_v(w)(ga) = \pi_R(a)T_v(w)(g) \quad \text{for every } g \in G.$$

□

Compare the following result with the amenable case (Corollary 1.4.18).

Corollary 5.2.15. *Let G be a topological group. Then the following are equivalent:*

a) the group G has the $\mathscr{R}$-fixed-point property for cones;

b) the group G has the invariant normalized integral property for $\mathcal{C}_u^b(G, \mathscr{R})$;

c) for every jointly continuous action of G on a compact space K, the induced representation of G on $\mathcal{C}(K)$ has the invariant integral property;

d) for every orbitally continuous action of G on a topological space X, the induced representation of G on

$$\mathcal{C}^b(X)_c = \left\{ f \in \mathcal{C}^b(X) : g \longmapsto gf \text{ is } ||\cdot||_\infty\text{-continuous} \right\}$$

has the invariant normalized integral property.

Proof. Point a) and b) are equivalent by Theorem 5.2.1. We proceed to prove that points b) and c) are also equivalent.

For the first implication, let K be a compact set and suppose that G acts on it jointly continuous. Therefore, the induced representation of G on $\mathcal{C}(K)$ is continuous with respect to the $||\cdot||_\infty$-norm by Proposition 3.4.9. Fix a non-zero positive function $\phi \in \mathcal{C}(K)$. By Lemma 5.2.9, there is a positive equivariant linear operator T_ϕ from $\mathcal{C}(K)$ to $\mathcal{C}_u^b(G, \mathscr{L})$ such that $T_\phi(\phi) > 0$. Note that we consider the right-translation representation π_R of G on $\mathcal{C}_u^b(G, \mathscr{L})$. This representation has the invariant normalized integral property because the representation π_L of G on $\mathcal{C}_u^b(G, \mathscr{R})$ also has. Now the operator T_ϕ maps the space $\mathcal{C}(K, \phi)$ into $\left(\mathcal{C}_u^b(G, \mathscr{L}), T_\phi(\phi)\right)$. On this last one there is an invariant normalized integral I by hypothesis. Then the functional given by $\bar{\mathrm{I}} = \mathrm{I} \circ T_\phi$ is an invariant normalized integral for $\mathcal{C}(K, \phi)$.

We turn now to the inverse implication. By Theorem 2.4.20, $\mathcal{C}_u^b(G, \mathscr{R})$ is Riesz isometric to $\mathcal{C}(K)$, where K is the weak-* compact set defined as

$$K = \left\{\psi \in B_1^+ : \psi \text{ is an extreme point of } B_1^+ \text{ such that } \psi(\mathbf{1}_G) = 1\right\}.$$

Here, $B_1^+ = \left\{\psi \in \mathcal{C}_u^b(G, \mathscr{R})'_+ : ||\psi||_{op} \leq 1\right\}$. As the representation π_L of G on $\mathcal{C}_u^b(G, \mathscr{R})$ is $||\cdot||_\infty$-continuous, the action of G on K is orbitally continuous for the weak-* topology. Moreover, this action is jointly continuous for the weak-* topology as K is bounded. Hence, the representation of G on $\mathcal{C}(K)$ is $||\cdot||_\infty$-continuous by Proposition 3.4.9. Now, we can use the invariant normalized integral property of $\mathcal{C}(K)$ to conclude that G also has the invariant normalized integral property for $\mathcal{C}_u^b(G, \mathscr{R})$.

Finally, we conclude by showing that the points b) and d) are equivalent. Suppose that b) is true and suppose that G acts on a topological space X. Consider the vector subspace

$$\mathcal{C}^b(X)_c = \left\{f \in \mathcal{C}^b(X) : g \longmapsto gf \text{ is } ||\cdot||_\infty\text{-continuous}\right\} \subset \mathcal{C}^b(X).$$

Then $(\mathcal{C}^b(X)_c, ||\cdot||_\infty)$ is a Banach lattice on which G acts continuously, as explained in point 6) of Example 2.4.2. By Lemma 5.2.14, for every non-zero positive function $f \in \mathcal{C}^b(X)_c$ there is a positive equivariant linear operator from $\mathcal{C}^b(X)_c$ to $\mathcal{C}_u^b(G, \mathscr{L})$. Therefore, we can use the invariant normalized integral property of the representation π_R of G on $\mathcal{C}_u^b(G, \mathscr{L})$ to show that there is an invariant normalized integral on $(\mathcal{C}^b(X), f)$. The inverse implication is direct because $\mathcal{C}^b(G)_c = \mathcal{C}_{ru}^b(G) = \mathcal{C}_u^b(G, \mathscr{R})$. □

5.3 A functional perspective

We saw that the $\mathscr{U}$-fixed-point property for cones is characterized by the invariant normalized integral property for the Banach lattice of bounded $\mathscr{U}$-uniformly continuous functions. Here, we want to understand for which other ordered vector spaces the $\mathscr{U}$-fixed-point property for cones implies the invariant normalized integral property. Note that this kind of question was already studied and largely used in the case of amenability. See, for example, the work of Zimmer [Z84, Chapter 4].

In this section, every group representation on a normed Riesz space is by positive linear isometries.

Let G be a topological group and let $\mathscr{U}$ be a functionally invariant uniformity for G. Suppose that G has a representation π on a normed Riesz space E. Recall from Definition 5.1.1 that the adjoint representation π^* on E^* is locally bounded $(\mathscr{U}, \mathscr{U}_c^*)$-uniformly continuous if there is a non-zero positive functional $\lambda \in E^*$ such that the orbital map $g \longmapsto g\lambda$ is bounded and $(\mathscr{U}, \mathscr{U}_c^*)$-uniformly continuous. Here, $\mathscr{U}_c^*$ is the canonical uniformity of E^* with respect to the weak-* topology.

Theorem 5.3.1. *Let G be a topological group and let $\mathscr{U}$ be a functionally invariant uniformity for G. Then the following assertions are equivalent:*

a) the group G has the $\mathscr{U}$-fixed-point property for cones;

b) every representation π of G on a normed Riesz space E such that E is G-dominated and π^* is locally bounded $(\mathscr{U}, \mathscr{U}_c^*)$-uniformly continuous admits an invariant normalized integral;

c) every representation π of G on a Banach lattice E such that E is asymptotically G-dominated and π^* is locally bounded $(\mathscr{U}, \mathscr{U}_c^*)$-uniformly continuous admits an invariant normalized integral.

Proof. We start by showing that a) implies b). Let E and π be as in the hypothesis of b). Note that the positive polar cone E_+^* of E^* is convex and proper. If we equip the algebraic dual of E with the weak-* topology, then E_+^* is closed in E^* ([AT07, Theorem 2.13]). As closed subspaces of complete spaces are complete ([Bou71, II §3 No.4 Proposition 8]), we have that E_+^* is weak-* complete, and in particular weakly complete. Let's now look at the adjoint action of G on E^*. The adjoint representation is of cobounded type as the adjoint of the adjoint representation it is only the initial one, as the topological dual of $(E^*, \text{weak-}*)$ is E ([AB99, Theorem 5.93]). Moreover, π^* is locally bounded $(\mathscr{U}, \mathscr{U}_c^*)$-uniformly continuous by hypothesis. Thus, we can apply the $\mathscr{U}$-fixed-point property for cones to find an invariant normalized integral on E.

Now, we want to show that b) implies c). To this end, let E and π as in the hypothesis of c). By Remark 3.3.3, we can suppose that the norm of E is of the form p_d^∞, where d is the asymptotically G-dominating element of E. Thanks to Proposition 3.3.4, there is a G-dominated Riesz subspace of E, say D, which is p_d^∞-norm dense in E. Consider the G-equivariant continuous linear operator given by restriction

$$\text{res} : (E^*, \textit{weak-}*) \longrightarrow (D^*, \textit{weak-}*), \quad \psi \longmapsto \mathtt{res}(\psi) = \psi|_D.$$

Let $\lambda \in E_+^*$ be the non-zero positive functional which witnesses the locally bounded $(\mathscr{U}, \mathscr{U}_c^*)$-uniformly continuous condition. Then $\mathtt{res}(\lambda)$ is a non-zero positive functional defined on D, and the map $g \longmapsto g \cdot \mathtt{res}(\lambda)$ is bounded $(\mathscr{U}, \mathscr{U}_c^*)$-uniformly continuous. Thus, the adjoint representations of G on D^* is locally bounded $(\mathscr{U}, \mathscr{U}_c^*)$-uniformly continuous. This means that D and $\pi|_D$ respect the hypothesis of b). Accordingly, there is an invariant normalized integral I on D. Now, I is continuous with respect to the p_d^∞-norm by Corollary 4.1.7, and hence uniformly continuous. We can extend I to a linear functional $\bar{\mathrm{I}}$ defined on E because D is p_d^∞-dense in E. We claim that $\bar{\mathrm{I}}$ is an invariant normalized integral on E. It is clearly normalized on d. Thus, we only have to show that it is invariant. Let $v \in E$, then there is a sequence $(v_n)_n$ in D which converges to v in p_d^∞-norm. Therefore,

$$\begin{aligned}\bar{\mathrm{I}}(gv) &= \bar{\mathrm{I}}(g \lim_n v_n) = \bar{\mathrm{I}}(\lim_n g v_n) = \lim_n \bar{\mathrm{I}}(g v_n) = \lim_n \mathrm{I}(g v_n) \\ &= \lim_n \mathrm{I}(v_n) = \lim_n \bar{\mathrm{I}}(v_n) = \bar{\mathrm{I}}(\lim_n v_n) = \bar{\mathrm{I}}(v) \qquad \text{for every } g \in G.\end{aligned}$$

This shows that $\bar{\mathrm{I}}$ is an invariant normalized integral on E.

To conclude, we want to show that c) implies a). However, this is direct because c) implies that G has the invariant normalized integral property for $\mathcal{C}_u^b(G, \mathscr{U})$, and this implies that G has the $\mathscr{U}$-fixed-point property for cones by Theorem 5.2.1. Indeed, let $f \in \mathcal{C}_u^b(G, \mathscr{U})$ be a non-zero positive function. Then $(\mathcal{C}_u^b(G, \mathscr{U}), f)_\infty$ together with the p_f^∞-norm is a Banach lattice asymptotically G-dominated by f thanks to Theorem 3.2.14. Moreover, the adjoint representation π_L of the left-translation representation is locally bounded $(\mathscr{U}, \mathscr{U}_c^*)$-uniformly continuous by Lemma 1.4.13. Therefore, we can conclude that there exists an invariant normalized integral on $(\mathcal{C}_u^b(G, \mathscr{U}), f)_\infty$. □

We want to distil the essence of the translate property as done for invariant normalized integrals. Note that the translate property has a significant advantage: it does not depend on the ambient space. This will be useful in situations where we can not have control over the cobounded condition.

Definition 5.3.2. Let G be a topological group and let $\mathscr{U}$ be a functionally invariant uniformity for G. We say that G has the **abstract $\mathscr{U}$-translate property** if whenever G has a representation π on a Banach lattice E by positive linear isometries, then for every non-zero positive vector $v \in E$ for which there is a positive linear functional $\lambda \in E'$ with $\lambda(v) \neq 0$ and such that the orbital map $g \longmapsto g\lambda$ is bounded and $(\mathscr{U}, \mathscr{U}_c^*)$-uniformly continuous we have that

$$\sum_{j=1}^n t_j g_j v \geq 0 \quad \text{implies} \quad \sum_{j=1}^n t_j \geq 0 \quad \text{for every } t_1, ..., t_n \in \mathbf{R} \text{ and } g_1, ..., g_n \in G.$$

Theorem 5.3.3. *Let G be a topological group and let $\mathscr{U}$ be a functionally invariant uniformity for G. Then G has the abstract $\mathscr{U}$-translate property if and only if G has the translate property for $\mathcal{C}_u^b(G, \mathscr{U})$.*

Proof. We start by showing that the abstract $\mathscr{U}$-translate property implies the translate property for $\mathcal{C}_u^b(G, \mathscr{U})$. Let $f \in \mathcal{C}_u^b(G, \mathscr{U})$ be a non-zero positive function and let

$t_1, ..., t_n \in \mathbf{R}$ and $g_1, ..., g_n \in G$ such that $\sum_{j=1}^{n} t_j g_j f \geq 0$. Take $x \in G$ such that $f(x) \neq 0$ and consider the evaluation map ev_x at point $x \in G$. Then $ev_x(f) = f(x) \neq 0$ and $ev_x \geq 0$. Moreover, the orbital map $g \longmapsto g \cdot ev_x$ is bounded and $(\mathscr{U}, \mathscr{U}_c^*)$-uniformly continuous by Lemma 1.4.13. Therefore, we can apply the abstract $\mathscr{U}$-translate property to $\mathcal{C}_u^b(G, \mathscr{U})$ to conclude that $\sum_{j=1}^{n} t_j \geq 0$ as wished.

Let now prove the converse. Let E, v and λ as in the definition of the abstract $\mathscr{U}$-translate property for G and define the map

$$f : (G, \mathscr{U}) \longrightarrow (E', \mathscr{U}_c^*) \longrightarrow (\mathbf{R}, \mathscr{U}_c) \quad g \longmapsto g\lambda \longmapsto ev_v(g\lambda) = \lambda(gv),$$

where ev_v is the evaluation map at point $v \in E_+$. Then $f \in \mathcal{C}_u^b(G, \mathscr{U})$ as the orbital action on λ is bounded and $(\mathscr{U}, \mathscr{U}_c^*)$-uniformly continuous, and the evaluation map ev_v is $(\mathscr{U}, \mathscr{U}_c^*)$-uniformly continuous. Moreover, f is positive and non-zero as $f(e) = \lambda(v) \neq 0$. Let now $t_1, ..., t_n \in \mathbf{R}$ and $g_1, ..., g_n \in G$ such that $\sum_{j=1}^{n} t_j g_j v \geq 0$. Then

$$0 \leq g^{-1} \sum_{j=1}^{n} t_j g_j v = \sum_{j=1}^{n} t_j g^{-1} g_j v$$

and so

$$0 \leq \sum_{j=1}^{n} t_j \lambda\left(g^{-1} g_j v\right) = \sum_{j=1}^{n} t_j f(g_j^{-1} g) = \sum_{j=1}^{n} t_j g_j f(g) \quad \text{for every } g \in G.$$

We can employ the translate property of G for $\mathcal{C}_u^b(G, \mathscr{U})$ to conclude that $\sum_{j=1}^{n} t_j \geq 0$ as wished. □

In the following definition, we drop the locally uniformly continuous condition on the dual of the $\mathscr{U}$-abstract translate property, but we add continuity of the representation.

Definition 5.3.4. We say that a topological group G has the **abstract continuous translate property** if whenever G has a continuous representation on a Banach lattice E by positive linear isometries, then for every non-zero positive vector $v \in E$, we have that

$$\sum_{j=1}^{n} t_j g_j v \geq 0 \quad \text{implies} \quad \sum_{j=1}^{n} t_j \geq 0 \quad \text{for every } t_1, ..., t_n \in \mathbf{R} \text{ and } g_1, ..., g_n \in G.$$

The following theorem will be helpful later for studying hereditary properties of the translate property for $\mathcal{C}_u^b(G, \mathscr{R})$. In particular, it shows another time the strength of the uniform structure $\mathscr{R}$.

Theorem 5.3.5. *Let G be a topological group. Then G has the abstract continuous translate property if and only if G has the translate property for $\mathcal{C}_u^b(G, \mathscr{R})$.*

Proof. Suppose that G has the abstract continuous translate property. Then the left-translation representation of G on the Banach lattice $\mathcal{C}_u^b(G, \mathscr{R})$ is continuous. Therefore, G has the translate property for $\mathcal{C}_u^b(G, \mathscr{R})$.

Now, suppose that G has the translate property for $\mathcal{C}_u^b(G, \mathscr{R})$ and let π be a continuous representation of G on a Banach lattice E by positive linear isometries. Let $v \in E$ be a non-zero positive vector and let $t_1, ..., t_n \in \mathbf{R}$ and $g_1, ..., g_n \in G$ such that $\sum_{j=1}^n t_j \pi(g_j) v \geq 0$. By Lemma 5.2.14, there is a positive equivariant linear operator T_v from E to $\mathcal{C}_u^b(G, \mathscr{L})$ such that $T_v(v) > 0$. Therefore,

$$0 \leq T_v \left(\sum_{j=1}^n t_j \pi(g_j) v \right) = \sum_{j=1}^n t_j \pi_R(g_j) T_v(v).$$

Nevertheless, saying that the representation π_L of G on $\mathcal{C}_u^b(G, \mathscr{R})$ has the translate property is equivalently saying that the representation π_R of G on $\mathcal{C}_u^b(G, \mathscr{L})$ has the translate property. Thus, we can conclude that $\sum_{j=1}^n t_j \geq 0$ as wished. □

Chapter 6

The Locally Compact Case

The chapter handles locally compact groups. The central idea is to employ the theory of measure for locally compact spaces to develop new tools to investigate the fixed-point property for cones for locally compact groups.

The first section repeats some basics about Borel measures on locally compact groups and positive R-modules. Then we use positive modules of compactly supported Borel measures to adapt the definitions of dominated ordered vector space and invariant normalized integral to the measurable setting. Consequently, even dominated norms will have a measurable renovation. After that, we study the properties of positive functional on these new measurable dominated spaces. Finally, we apply the developed theory to the fixed-point property for cones.

Highlights of this Chapter are Theorem 6.3.4, which shows the equivalence of the invariant integral properties on the classical Banach lattices, Theorem 6.3.8, which gives a complete answer to the Greenleaf's question in the locally compact case and Theorem 6.3.10, which answers Greenleaf's question also for all the classical Banach lattices.

In this chapter, G is always a locally compact group, and m_G is a fixed left-invariant Haar measure for G.

6.1 Measures, modules and convolution

The section is devoted to recall some facts about measures on locally compact groups and define positive R-modules. Moreover, we explain how R-modules generalize group representations.

6.1.A. Measures and convolution. Let $\mathbb{B}(G)$ be the Borel σ-algebra of the group G, i.e., the σ-algebra generated by the open sets of the topology of G. Recall that a measure on G is said a **Borel measure** if it is defined on the σ-algebra $\mathbb{B}(G)$.

Definition 6.1.1. Let μ be a Borel measure on G. Then we say that:

- the measure μ is **regular**, if
  a) each compact subset K of G satisfies $\mu(K) < \infty$
  b) each set $A \in \mathbb{B}(G)$ satisfies
  $$\mu(A) = \inf\{\mu(U) : A \subset U \text{ and } U \subset G \text{ open}\}$$
  c) each open subset U of G satisfies
  $$\mu(U) = \sup\{\mu(K) : K \subset U \text{ and } K \subset G \text{ compact}\};$$
- the measure μ is **signed**, if it also takes negative values;
- the measure μ is **finite**, if it only takes finite values.

We write $\mathcal{M}(G)$ for the vector space of all **signed finite regular Borel measures on** G. Recall that $\mathcal{M}(G)$ equipped with the vector ordering defined by

$$\mu \leq \lambda \quad \Longleftrightarrow \quad \mu(A) \leq \lambda(A) \text{ for all } A \in \mathbb{B}(G)$$

is a Riesz space. In particular, this means that there is an absolute value operation on $\mathcal{M}(G)$ which can be explicit given by the formula

$$|\mu|(A) = \sup\left\{\sum_{j=1}^{n} |\mu(A_j)| : \{A_1, ..., A_n\} \text{ is a partition of } A\right\},$$

where $\mu \in \mathcal{M}(G)$ and $A \in \mathbb{B}(G)$. We refer to [AT07, p. 22] and to [AB99, Section 10.10] for details.

Definition 6.1.2. The **total variation** of a measure $\mu \in \mathcal{M}(G)$ is defined as the value $|\mu|(G)$. The **total variation norm** on $\mathcal{M}(G)$ is defined as $||\mu||_{\text{TV}} = |\mu|(G)$ for $\mu \in \mathcal{M}(G)$.

The Riesz space $\mathcal{M}(G)$ equipped with the total variation norm becomes a Banach space, see [C13, Proposition 4.1.8]. In particular, $\mathcal{M}(G)$ is a Banach lattice as the total variation norm is monotone by definition.

Definition 6.1.3. Let μ_1 and μ_2 two measures in $\mathcal{M}(G)$. We define their **convolution** as

$$(\mu_1 * \mu_2)(A) = \int_G \mu_2(x^{-1}A)d\mu_1(x) = \int_G \mu_1(Ay^{-1})d\mu_2(y),$$

where $A \in \mathbb{B}(G)$.

The preceding expression is well-defined and belongs to $\mathcal{M}(G)$, see [C13, Lemma 9.4.5]. Moreover,

$$||\mu_1 * \mu_2||_{\text{TV}} \leq ||\mu_1||_{\text{TV}}||\mu_2||_{\text{TV}} \quad \text{for every } \mu_1, \mu_2 \in \mathcal{M}(G).$$

It turns out that the Banach space $(\mathcal{M}(G), ||\cdot||_{\mathrm{TV}})$ considered with the convolution as multiplication is a unital Banach algebra ([C13, Proposition 9.4.6]).

We are not directly interested into $\mathcal{M}(G)$ but rather to its subspaces. But first to go through the structure of $\mathcal{M}(G)$, let's recall that a Borel measure μ_1 is **absolutely continuous** with respect to another Borel measure μ_2 if for every $A \in \mathbb{B}(G)$ such that $\mu_1(A) = 0$, then $\mu_2(A) = 0$.

Write $\mathcal{M}(G)_a$ for the set of all **signed finite regular Borel measures which are absolutely continuous with respect to m_G**. Then $\mathcal{M}(G)_a$ is an (algebraic) ideal of the Banach algebra $\mathcal{M}(G)$ ([C13, Proposition 9.4.7 (a)]).

Theorem 6.1.4. *The linear operator*

$$T : \left(L^1(G), ||\cdot||_1\right) \longrightarrow (\mathcal{M}(G)_a, ||\cdot||_{\mathrm{TV}}) \quad f \longmapsto T(f) = \mu_f,$$

where the measure μ_f is defined as $\mu_f(A) = \int_A f(g)d\mu(g)$, is an isometric isomorphism of Banach algebras.

Proof. See [R02, Theorem A.1.12]. □

Definition 6.1.5. The **support** $\mathrm{supp}(\mu)$ of a measure $\mu \in \mathcal{M}(G)$ is the complement of the largest open subset of G of μ-measure zero.

By definition, $\mathrm{supp}(\mu)$ is the smallest closed set whose complement has measure equal to zero under μ. Hence, the support of a measure is always a closed subset of G.

Write $\mathcal{M}_{00}(G)$ for the set of all **signed finite regular measures of G with compact support**.

Proposition 6.1.6. *The set $\mathcal{M}_{00}(G)$ is a normed subalgebra of $\mathcal{M}(G)$.*

Proof. First of all, note that the set inclusion $\mathcal{M}_{00}(G) \subset \mathcal{M}(G)$ holds by [Bou63, III §2 No.3 Proposition 11]. Next, we show that $\mathcal{M}_{00}(G)$ is a vector subspace of $\mathcal{M}(G)$. It is clear that $\mathcal{M}_{00}(G)$ is closed by multiplication by a scalar because

$$\mathrm{supp}(\alpha\mu) = \mathrm{supp}(\mu) \quad \text{for every non-zero } \alpha \in \mathbf{R} \text{ and } \mu \in \mathcal{M}_{00}(G).$$

Let now μ_1 and μ_2 be in $\mathcal{M}_{00}(G)$. By [Bou63, III §2 No.2 Proposition 4],

$$\mathrm{supp}(\mu_1 + \mu_2) \subset \mathrm{supp}(\mu_1) \cup \mathrm{supp}(\mu_2).$$

As the union of compact sets is compact and the support of a measure is closed by definition, we can conclude that the measure $\mu_1 + \mu_2$ has compact support. This proves that $\mathcal{M}_{00}(G)$ is a vector subspace of $\mathcal{M}(G)$. Finally, we check that $\mathcal{M}_{00}(G)$ is closed by convolution. Let μ_1 and μ_2 in $\mathcal{M}_{00}(G)$. Then

$$\mathrm{supp}(\mu_1 * \mu_2) \subset \overline{\mathrm{supp}(\mu_1)\mathrm{supp}(\mu_2)}$$

by [Bou59, VIII §1 No.4 Proposition 5 a)]. Now, the set $\mathrm{supp}(\mu_1)\mathrm{supp}(\mu_2)$ is compact by [HR63, Theorem (4.4)]. This implies that $\mathrm{supp}(\mu_1 * \mu_2)$ is compact, and hence that $\mathcal{M}_{00}(G)$ is a normed subalgebra of $\mathcal{M}(G)$. □

Write $\mathcal{M}_{00}(G)_a$ for the set of all **signed compactly supported regular measures of** G **which are absolutely continuous with respect to the Haar measure** m_G. In general, $\mathcal{M}_{00}(G)_a$ is not an (algebraic) ideal of $\mathcal{M}(G)$ but it is an (algebraic) ideal of $\mathcal{M}_{00}(G)$. This last assertion can be shown using the same proof of [C13, Proposition 9.4.7 (a)]. Moreover, the Riesz space $\mathcal{C}_{00}(G)$ can be identified to a subspace of $\mathcal{M}_{00}(G)_a$ using the operator T of Theorem 6.1.4.

6.1.B. About positive R-modules. Before defining the main object of study of this chapter, recall that a **R-module** $\mathcal{M}$ is the data of an abelian group $(\mathcal{M}, +)$, a ring R and an operation map $\mathrm{R} \times \mathcal{M} \longrightarrow \mathcal{M}$ such that:

(M1) $r \cdot (m_1 + m_2) = r \cdot m_1 + r \cdot m_2$ for every $r \in \mathrm{R}$ and $m_1, m_2 \in \mathcal{M}$;

(M2) $(r_1 + r_2) \cdot m = r_1 \cdot m + r_2 \cdot m$ for every $r_1, r_2 \in \mathrm{R}$ and $m \in \mathcal{M}$;

(M3) $(r_1 r_2) \cdot m = r_1 \cdot (r_2 \cdot m)$ for every $r_1, r_2 \in \mathrm{R}$ and $m \in \mathcal{M}$.

Definition 6.1.7. Suppose that R and $\mathcal{M}$ are ordered vector spaces. We say that $\mathcal{M}$ is a **positive R-module** if it is a R-module and

$$r \cdot m \geq 0 \quad \text{for every positive } r \in \mathrm{R} \text{ and every positive } m \in \mathcal{M}.$$

Example 6.1.8. (Examples of positive R-modules)

1) Let E be an ordered vector space and let $\mathcal{L}(E)$ be the set of all linear operators from E to E. Then E is a positive $\mathcal{L}(E)$-module with operation map given by

$$T \cdot v = T(v) \quad \text{for } T \in \mathcal{L}(E) \text{ and } v \in E.$$

2) Let $\mathcal{A}$ be a C^*-algebra equipped with its C^*-order. Then $\mathcal{A}$ is a positive $\mathcal{A}$-module with operation map given by

$$a \cdot b = aba^* \quad \text{for } a, b \in \mathcal{A}.$$

Indeed, the C^*-cone C_{sa} of $\mathcal{A}$ is invariant under conjugation, see [D77, 1.6.8].

We are only interested in positive $\mathcal{M}_{00}(G)$-module. Note that if E is a positive $\mathcal{M}_{00}(G)$-module, then there is a natural action by positive linear automorphisms of the group G on E given by $gv = \delta_g \cdot v$ for $g \in G$ and $v \in E$. If E is also a normed Riesz space, we always suppose that this action is by positive linear isometries.

Therefore, let's look at some specific example of positive $\mathcal{M}_{00}(G)$-modules.

Let $p \in [1, \infty]$. We define the **convolution between the functions** $f_1 \in L^1(G)$ and $f_2 \in L^p(G)$ as

$$(f_1 * f_2)(g) = \int_G f_1(x) f_2(g^{-1}x) dm_G(x) \quad \text{for } m_G\text{-almost every } g \in G.$$

This expression is well-defined by [HR63, Theorem (20.8)] and turns $L^p(G)$ into a positive $L^1(G)$-module ([HR70, Theorem (32.16)]).

Moreover, we define the **convolution between a measure and a function** by the formula

$$(\mu * f)(g) = \int_G f(x^{-1}g)d\mu(x) \quad \text{where } g \in G, \mu \in \mathcal{M}(G) \text{ and } f \in L^p(G).$$

This integral exists and is m_G-almost everywhere finite by [HR63, Theorem (20.12)]. If $\mu * f$ is defined to be zero when the integral is infinite, we have the following bound

$$||\mu * f||_p \leq ||\mu||_{\text{TV}}||f||_p \quad \text{for all } \mu \in \mathcal{M}(G) \text{ and } f \in L^p(G).$$

We can conclude that:

Proposition 6.1.9. *Let G be a locally compact group and let $p \in [1, \infty]$. Then the Banach lattice $L^p(G)$ is a positive $\mathcal{M}(G)$-module. In particular, $L^p(G)$ is a positive $\mathcal{M}_{00}(G)$-module.*

Proof. The proof is direct by the definition of convolution between a measure and a function. □

In the case $p = 1$, there is an intertwining between convolution between functions and convolution between measures since $L^1(G)$ can be identified with the ideal $\mathcal{M}(G)_a$ as explained in Theorem 6.1.4.

Theorem 6.1.10. *Let $\mu \in \mathcal{M}(G)$ and $f \in L^1(G)$. Then $\mu * f = \mu * \mu_f$, where for the left-hand side we consider convolution between measures and functions and for the right-hand side convolution between measures.*

Proof. See [HR63, Theorem (20.9)]. □

This last fact is helpful to give other examples of positive $\mathcal{M}_{00}(G)$-modules.

Proposition 6.1.11. *Let G be a locally compact group. Then each of the following Banach lattices*

$$L^\infty(G),\ \mathcal{C}^b(G),\ \mathcal{C}^b_{lu}(G),\ \mathcal{C}^b_{ru}(G),\ \mathcal{C}^b_u(G) \text{ and } \mathcal{C}_0(G),$$

is a positive $\mathcal{M}(G)$-module. In particular, they are all positive $\mathcal{M}_{00}(G)$-modules.

Proof. We only have to prove that every Banach lattice listed above is $\mathcal{M}(G)$-invariant.

The space $L^\infty(G)$ is a positive $\mathcal{M}(G)$-module by Proposition 6.1.9.

Let's consider the Banach lattice $\mathcal{C}^b(G)$ and let $f \in \mathcal{C}^b(G)$ and $\mu \in \mathcal{M}(G)$. Take a net $(g_\alpha)_\alpha$ in G such that $\lim_\alpha g_\alpha = e$, and compute that

$$\begin{aligned}\lim_\alpha (\mu * f)(g_\alpha) &= \lim_\alpha \int_G f(g^{-1}g_\alpha)d\mu(g) \\ &= \int_G \lim_\alpha f(g^{-1}g_\alpha)d\mu(g) \\ &= \int_G f(g^{-1})d\mu(g) = (\mu * f)(e).\end{aligned}$$

This shows that $f * \mu \in \mathcal{C}^b(G)$, and consequently, that $\mathcal{C}^b(G)$ is a positive $\mathcal{M}(G)$-module.

For the space $\mathcal{C}^b_{ru}(G)$, we use the fact that

$$L^1(G) * L^\infty(G) = \left\{\phi * f : \phi \in L^1(G) \text{ and } f \in L^\infty(G)\right\} = \mathcal{C}^b_{ru}(G).$$

This last assertion can be proved using to Cohen-Hewitt Factorization Theorem ([HR70, (32.22)] and [HR63, (32.45) (b)] for details). Let $\mu \in \mathcal{M}(G)$ and $f \in \mathcal{C}^b_{ru}(G)$. Then there are $\phi \in L^1(G)$ and $F \in L^\infty(G)$ such that $f = \phi * F$. We can compute that

$$\mu * f = \mu * (\phi * F) = \left(\mu * \mu_\phi\right) * F.$$

By Theorems 6.1.4 and 6.1.10, $\mu * \mu_\phi \in \mathcal{M}(G)_a$ and there is $v \in L^1(G)$ such that $\mu * \mu_\phi = \mu_v$. Therefore,

$$\mu * f = \mu_v * F = v * F \in \mathcal{C}^b_{ru}(G).$$

We can conclude that $\mathcal{C}^b_{ru}(G)$ is a positive $\mathcal{M}(G)$-module.

Let's move on the Banach lattice $\mathcal{C}^b_{lu}(G)$. By Cohen-Hewitt Factorization Theorem, we have that

$$L^\infty(G) * L^1(G) = \left\{f * \phi : f \in L^\infty(G) \text{ and } \phi \in L^1(G)\right\} = \mathcal{C}^b_{lu}(G),$$

see [HR63, (32.45) (d)]. Therefore, take $f \in \mathcal{C}^b_{lu}(G)$ and $\mu \in \mathcal{M}(G)$. Then there are $F \in L^\infty(G)$ and $\phi \in L^1(G)$ such that $f = F * \mu$. Thus,

$$\mu * f = \mu * (F * \phi) = (\mu * F) * \phi \in \mathcal{C}^b_{lu}(G)$$

as $\mu * F \in L^\infty(G)$.

Finally, the proof for the remaining two Banach lattices $\mathcal{C}^b_u(G)$ and $\mathcal{C}_0(G)$ is similar to the proof for $\mathcal{C}^b_{ru}(G)$. In fact,

$$L^1(G) * \mathcal{C}^b_{lu}(G) = \left\{\phi * f : \phi \in L^1(G) \text{ and } f \in \mathcal{C}^b_{lu}(G)\right\} = \mathcal{C}^b_u(G)$$

and

$$L^1(G) * \mathcal{C}_0(G) = \left\{\phi * f : \phi \in L^1(G) \text{ and } f \in \mathcal{C}_0(G)\right\} = \mathcal{C}_0(G)$$

by Cohen-Hewitt Factorization Theorem. We refer to [HR63, (32.45)] and to [HR63, (32.44) (f)], respectively, for details about the factorization of $\mathcal{C}^b_u(G)$ and $\mathcal{C}_0(G)$ above. □

Proposition 6.1.12. *Let G be a locally compact group. Then the normed Riesz space $\mathcal{C}_{00}(G)$ is a positive $\mathcal{M}_{00}(G)$-module.*

Proof. As before, we only have to prove that $\mathcal{C}_{00}(G)$ is $\mathcal{M}_{00}(G)$-invariant. Therefore, let $\mu \in \mathcal{M}(G)$ and $\phi \in \mathcal{C}_{00}(G)$. By Proposition 6.1.11, $\mu * \phi$ is in $\mathcal{C}_0(G)$. If we can show that $\mu * \phi$ has compact support, we are done. However, this is straightforward as

$$\text{supp}(\mu * \phi) = \text{supp}(\mu * \mu_\phi) \subset \text{supp}(\mu)\text{supp}(\mu_\phi) = \text{supp}(\mu)\text{supp}(\phi).$$

We can conclude that $\mathcal{C}_{00}(G)$ is a positive $\mathcal{M}_{00}(G)$-module. □

We introduce another example of a positive $\mathcal{M}(G)$-module that will be used in Appendix B. This example differs from the preceding ones as we are not looking at functions anymore but at linear operators between Hilbert spaces.

Let G be a locally compact group, and suppose that G has a continuous unitary representation σ on a (complex) Hilbert space $\mathcal{H}$. Recall that $\mathcal{B}(\mathcal{H})$, the vector spaces of all bounded linear operators of $\mathcal{H}$, is an ordered vector spaces when equipped with its C^*-order. As seen in point 6) of Example 4.1.11, the representation σ induces a representation Ad_σ on $\mathcal{B}(\mathcal{H})$ by positive linear automorphisms via the formula

$$\text{Ad}_\sigma(g)T = \sigma(g)T\sigma(g)^* \quad \text{for } g \in G \text{ and } T \in \mathcal{B}(\mathcal{H}).$$

In general, the representation Ad_σ is not continuous for the operator norm on $\mathcal{B}(\mathcal{H})$.

The representation Ad_σ leaves invariant many interesting subspaces of $\mathcal{B}(\mathcal{H})$. For instance, the subspace of trace-class operators $\text{TC}(\mathcal{H})$, the subspace of Hilbert-Schmidt operators $\text{HS}(\mathcal{H})$, the subspace of compact operators $\mathcal{B}_0(\mathcal{H})$ and the one of finite-rank operators $\mathcal{F}(\mathcal{H})$. All because of the fact that they are (algebraic) bi-ideals in $\mathcal{B}(\mathcal{H})$, see [S18, Theorem 6.6]. Moreover, it leaves invariant the subspace of self-adjoint operators $\mathcal{B}_{sa}(\mathcal{H})$ as

$$(\text{Ad}_\sigma(g)T)^* = (\sigma(g)T\sigma(g)^*)^* = \sigma(g)T^*\sigma(g)^* = \sigma(g)T\sigma(g)^* = \text{Ad}_\sigma(g)T$$

for every $g \in G$ and $T \in \mathcal{B}_{sa}(\mathcal{H})$.

Let's look at the vector subspace $\text{TC}(\mathcal{H})$ of all trace-class operators on $\mathcal{H}$ equipped with the **trace norm**

$$||T||_{\text{TC}} = \text{tr}\left(\sqrt{T^*T}\right) \quad \text{for } T \in \text{TC}(\mathcal{H}).$$

The pair $(\text{TC}(\mathcal{H}), ||\cdot||_{\text{TC}})$ is a Banach algebra ([S18, Theorem 6.15]) and its topological dual is equal to $\mathcal{B}(\mathcal{H})$ ([S18, Theorem 6.17]). The canonical duality map is given by

$$\mathcal{B}(\mathcal{H}) \times \text{TC}(\mathcal{H}) \longrightarrow \mathbf{R}, \quad (T, S) \longmapsto \langle T|S\rangle = \text{tr}(TS).$$

Lemma 6.1.13. *The representation Ad_σ of G on $TC(\mathcal{H})$ is continuous for the trace norm $||\cdot||_{TC}$.*

Before proving this lemma, recall that we have the following isometric isomorphism

$$(\mathrm{TC}(\mathcal{H}), ||\cdot||_{\mathrm{TC}}) \cong (\mathcal{H}' \otimes_\pi \mathcal{H}, ||\cdot||_\pi),$$

where $||\cdot||_\pi$ is the projective norm and $\otimes_\pi$ is the closure of the abstract tensor product with respect to the projective norm, see [Ry02, Corollary 4.8] for details. Moreover, the isomorphism is equivariant when we consider the Ad_σ representation of G on $\mathrm{TC}(\mathcal{H})$ and the extension of the tensor representation $\sigma^* \otimes \sigma$ of G on $\mathcal{H}' \otimes_\pi \mathcal{H}$.

Proof of Lemma 6.1.13. The the adjoint representation of G on $\mathcal{H}'$ is continuous for the operator norm as the unitary representation σ of G on $\mathcal{H}$ is continuous. This can be easily shown thanks to the Riesz Representation Theorem [C97, 3.8]. Therefore, the extension of the tensor representation $\sigma^* \otimes \sigma$ of G on $\mathcal{H}' \otimes \mathcal{H}$ is continuous for the projective norm. As continuity of a representation passes to the closure ([G17, Lemma 4.1.9]), the representation Ad_σ of G on $\mathrm{TC}(\mathcal{H})$ is continuous for the trace norm. □

Corollary 6.1.14. *The Ad_σ representation of G on $HS(\mathcal{H})$ and of $\mathcal{B}_0(\mathcal{H})$ is continuous for the Hilbert-Schmidt norm $||\cdot||_{HS}$ and the operator norm $||\cdot||_{op}$, respectively.*

Proof. This is due only to the facts that the finite-rank operators $\mathcal{F}(\mathcal{H})$ are dense in $\mathrm{HS}(\mathcal{H})$ and in $\mathcal{B}_0(\mathcal{H})$ for the Hilbert-Schmidt norm and the operator norm, respectively. Now the restriction of the representation Ad_σ to $\mathcal{F}(\mathcal{H})$ is continuous for both norms because

$$||T||_{op} \leq ||T||_{\mathrm{HS}} \leq ||T||_{\mathrm{TC}} \quad \text{for every } T \in \mathcal{F}(\mathcal{H})$$

by [S18, Proposition 6.4]. The corollary is proved using once again the fact that continuity of a representation passes to the closure ([G17, Lemma 4.1.9]). □

In particular, the adjoint representation $(\mathrm{Ad}_\sigma)^*$ of G on $\mathcal{B}(\mathcal{H})$ is continuous for the weak-* topology defined by the duality with $\mathrm{TC}(\mathcal{H})$.

We are finally ready to define an action of $\mathcal{M}(G)$ on $\mathcal{B}(\mathcal{H})$ using the theory of weak integration.

Let $\mu \in \mathcal{M}(G)$ and let $T \in \mathcal{B}(\mathcal{H})$. Then the map

$$F : (G, \mu) \longrightarrow \mathcal{B}(\mathcal{H}), \quad g \longmapsto F(g) = \mathrm{Ad}_\sigma(g)T$$

is weak-* μ-integrable as for every $S \in \mathrm{TC}(\mathcal{H})$ the function

$$\langle \cdot | S \rangle : (G, \mu) \longrightarrow \mathbf{R}, \quad g \longmapsto \langle F(g)|S\rangle = \mathrm{tr}(\mathrm{Ad}_\sigma(g)TS)$$

is continuous and its integral

$$\int_G \langle F(g)|S\rangle \, d\mu(g) = \int_G \mathrm{tr}\left(\mathrm{Ad}_\sigma(g)TS\right) d\mu(g) \leq ||\mu||_{\mathrm{TV}} ||T||_{op} ||S||_{\mathrm{TC}}$$

is finite. Therefore, the weak integral

$$\int_G F(g)d\mu(g) = \int_G \mathrm{Ad}_\sigma(g)Td\mu(g)$$

defined by the formula

$$\left\langle \int_G F(g)d\mu(g)|S \right\rangle = \int_G \langle F(g)|S\rangle \, d\mu(g) \quad \text{for every } S \in \mathrm{TC}(\mathcal{H})$$

is an element of $\mathcal{B}(\mathcal{H})$ by the Gelfand-Dunford Theorem ([Bou59, VI §1 No.4 Théorème 1]). Thus, the map

$$\mathcal{M}(G) \times \mathcal{B}(\mathcal{H}) \longrightarrow \mathcal{B}(\mathcal{H}), \quad (\mu, T) \longmapsto \mu \cdot T = \int_G \mathrm{Ad}_\sigma(g)Td\mu(g)$$

is well-defined. Moreover, by definition we have that

$$||\mu \cdot T||_{op} \leq ||\mu||_{\mathrm{TV}}||T||_{op} \quad \text{for every } \mu \in \mathcal{M}(G) \text{ and } T \in \mathcal{B}(\mathcal{H}).$$

Proposition 6.1.15. *Let G be a locally compact group. Then the ordered operator space $\mathcal{B}(\mathcal{H})$ is a positive $\mathcal{M}(G)$-module. In particular, $\mathcal{B}(\mathcal{H})$ is a positive $\mathcal{M}_{00}(G)$-module.*

Proof. We check that all the axioms which define a positive module are satisfied.

Let T_1 and T_2 be operators in $\mathcal{B}(\mathcal{H})$ and let $\mu \in \mathcal{M}(G)$. Then

$$\begin{aligned}\int_G \langle \mathrm{Ad}_\sigma(g)(T_1 + T_2)|S\rangle \, d\mu(g) &= \int_G \langle \mathrm{Ad}_\sigma(g)T_1 + \mathrm{Ad}_\sigma(g)T_2|S\rangle \, d\mu(g) \\ &= \int_G \langle \mathrm{Ad}_\sigma(g)T_1|S\rangle \, d\mu(g) + \int_G \langle \mathrm{Ad}_\sigma(g)T_2|S\rangle \, d\mu(g)\end{aligned}$$

for every $S \in \mathrm{TC}(\mathcal{H})$. Therefore, $\mu \cdot (T_1 + T_2) = \mu \cdot T_1 + \mu \cdot T_2$ and so **(M1)** is satisfied.

Let now μ_1 and μ_2 be measures in $\mathcal{M}(G)$ and $T \in \mathcal{B}(\mathcal{H})$. Then

$$\int_G \langle \mathrm{Ad}_\sigma(g)T|S\rangle \, d\left(\mu_1 + \mu_2\right)(g) = \int_G \langle \mathrm{Ad}_\sigma(g)T|S\rangle \, d\mu_1(g) + \int_G \langle \mathrm{Ad}_\sigma(g)T|S\rangle \, d\mu_2(g)$$

for every $S \in \mathrm{TC}(\mathcal{H})$. Consequently, $(\mu_1 + \mu_2) \cdot T = \mu_1 \cdot T + \mu_2 \cdot T$ and **(M2)** is true.

We continue showing **(M3)**. Let $\mu_1, \mu_2 \in \mathcal{M}(G)$ and $T \in \mathcal{B}(\mathcal{H})$. Then

$$\begin{aligned}\int_G \left\langle \mathrm{Ad}_\sigma(x) \left(\int_G \mathrm{Ad}_\sigma(g)Td\mu_2(g)\right) |S \right\rangle d\mu_1(x) &= \int_G \int_G \langle \mathrm{Ad}_\sigma(xg)T|S\rangle \, d\mu_2(g)d\mu_1(x) \\ &= \int_G \int_G \langle \sigma(xg)T\sigma(xg)^*|S\rangle \, d\mu_2(g)d\mu_1(x) \\ &= \int_G \langle \mathrm{Ad}_\sigma(xg)T|S\rangle \, d(\mu_1 * \mu_2)(xg)\end{aligned}$$

for every $S \in \mathrm{TC}(\mathcal{H})$. We can conclude that $\mu_1 \cdot (\mu_2 \cdot T) = (\mu_1 * \mu_2) \cdot T$ as wished.

The axiom **(M4)** is clear by the fact that

$$\delta_g \cdot T = \mathrm{Ad}_\sigma(g)T \quad \text{for every } g \in G \text{ and } T \in \mathcal{B}(\mathcal{H}).$$

It is only left to show that $\mathcal{B}(\mathcal{H})$ is a positive $\mathcal{M}(G)$-module, i.e., $\mu \cdot T$ is positive for every positive $\mu \in \mathcal{M}(G)$ and $T \in \mathcal{B}(\mathcal{H})$. Let $\mu \in \mathcal{M}(G)$ be a positive measure and $T \in \mathcal{B}(\mathcal{H})$ be a positive operator. We want to show that $\mu \cdot T$ is a positive operator. By [S18, Proposition 3.4], it suffices to show that the inequality $\langle (\mu \cdot T)(v), v\rangle \geq 0$ holds for every $v \in \mathcal{H}$ Using [M01, Lemma 3.2.1 (a)], we can directly compute that

$$\begin{aligned} \langle (\mu \cdot T)(v), v\rangle &= \left\langle \left(\int_G \mathrm{Ad}_\sigma(g) T d\mu(g) \right)(v), v \right\rangle \\ &= \left\langle \int_G \left(\mathrm{Ad}_\sigma(g)T\right)(v) d\mu(g), v \right\rangle \\ &= \int_G \langle \left(\mathrm{Ad}_\sigma(g)T\right)(v), v\rangle \, d\mu(g) \geq 0. \end{aligned}$$

□

Write $\mathcal{B}_c(\mathcal{H})$ for the set of the continuous vectors of the representation Ad_σ, i.e.,

$$\mathcal{B}_c(\mathcal{H}) = \{T \in \mathcal{B}(\mathcal{H}) : g \longmapsto \mathrm{Ad}_\sigma(g)T \text{ is } ||\cdot||\text{-continuous}\}.$$

Proposition 6.1.16. *Let G be a locally compact group. Then each of the following ordered operator spaces*

$$\mathcal{B}_{sa}(\mathcal{H}),\ \mathcal{B}_c(\mathcal{H}),\ \mathcal{B}_0(\mathcal{H}),\ \text{and}\ TC(\mathcal{H}),$$

is a positive $\mathcal{M}(G)$-module. In particular, they are all positive $\mathcal{M}_{00}(G)$-modules.

Proof. Thanks to the last proposition, we only need to prove that the above operator spaces are $\mathcal{M}(G)$-invariant.

For $\mathcal{B}_{sa}(\mathcal{H})$, we use the fact that taking the adjoint is a weak-* continuous linear operation together with [M01, Lemma 3.2.1].

For the operator space $\mathcal{B}_c(\mathcal{H})$, we use the same strategy used for $\mathcal{C}^b_{ru}(G)$ in Proposition 6.1.11. In fact, an application of Cohen-Hewitt Theorem gives that

$$L^1(G) \cdot \mathcal{B}(\mathcal{H}) = \left\{ \phi \cdot T : \phi \in L^1(G) \text{ and } T \in \mathcal{B}(\mathcal{H}) \right\} = \mathcal{B}_c(\mathcal{H}).$$

The same idea also applies to $\mathcal{B}_0(\mathcal{H})$ because

$$L^1(G) \cdot \mathcal{B}_0(\mathcal{H}) = \left\{ \phi \cdot T : \phi \in L^1(G) \text{ and } T \in \mathcal{B}_0(\mathcal{H}) \right\} = \mathcal{B}_0(\mathcal{H}).$$

Finally, the proof that $\mathrm{TC}(\mathcal{H})$ is $\mathcal{M}(G)$-invariant is straightforward always by [M01, Lemma 3.2.1]. □

Scholium 6.1.17. More examples of positive $\mathcal{M}(G)$-modules can be founded in [R20, Appendix F.1].

6.2 A measurable point-of-view

The section has the goal of employing measure theory to generalize the theory developed in Chapters 3 and 4 to locally compact groups. Note that when considering only discrete groups, everything done in this section coincides with what was done in Chapter 3. In fact, we want to develop specific tools to use for non-discrete locally compact groups.

For this section, let E be a positive $\mathcal{M}_{00}(G)$-module. If moreover, $(E, ||\cdot||)$ is a normed Riesz space, then we say that E is a normed Riesz positive $\mathcal{M}_{00}(G)$-module. In this case, we ask that the inequality

$$||\mu \cdot v|| \leq ||\mu||_{\mathrm{TV}}||v||$$

holds for every $\mu \in \mathcal{M}_{00}(G)$ and every $v \in E$.

6.2.A. Measurably dominated spaces. Let E be a positive $\mathcal{M}_{00}(G)$-module. A vector $v \in E$ is said **measurably dominated**, or $\mathcal{M}_{00}(G)_+$**-dominated**, by another vector $d \in E$ if there is a positive $\mu \in \mathcal{M}_{00}(G)$ such that $\pm v \leq \mu \cdot d$.

Definition 6.2.1. Let E be a positive $\mathcal{M}_{00}(G)$-module. For a non-zero positive vector $d \in E$, we define

$$(E, d)_{\mathcal{M}} = \{v \in E : \exists \mu \in \mathcal{M}_{00}(G)_+ \text{ s.t. } \pm v \leq \mu \cdot d\}.$$

In other words, the set $(E, d)_{\mathcal{M}}$ is the space of all vectors of E which are $\mathcal{M}_{00}(G)_+$-dominated by d. Note that (E, d) is a G-invariant subset of E as it is $\mathcal{M}_{00}(G)$-invariant by definition. Moreover, the following holds:

Proposition 6.2.2. *Let E be a positive $\mathcal{M}_{00}(G)$-module. Then the set $(E, d)_{\mathcal{M}}$ is a vector subspace of E for every non-zero positive vector $d \in E$. In particular, if E is a normed Riesz positive $\mathcal{M}_{00}(G)$-module, $(E, d)_{\mathcal{M}}$ is an ideal, and hence a Riesz subspace of E.*

Proof. Let $d \in E$ be a non-zero positive vector. We want to show that $(E, d)_{\mathcal{M}}$ is a vector subspace of E. Let $v_1, v_2 \in (E, d)_{\mathcal{M}}$. Then there are $\mu_1, \mu_2 \in \mathcal{M}_{00}(G)_+$ such that $\pm v_1 \leq \mu_1 \cdot d$ and $\pm v_2 \leq \mu_2 \cdot d$. Therefore,

$$\pm(v_1 + v_2) \leq \mu_1 \cdot d + \mu_2 \cdot d = (\mu_1 + \mu_2) \cdot d.$$

But now $\mu_1 + \mu_2 \in \mathcal{M}_{00}(G)_+$. Thus, $v_1 + v_2 \in (E, d)_{\mathcal{M}}$. Let now $\alpha \in \mathbf{R}$ and $v \in (E, d)_{\mathcal{M}}$. Then there is $\mu \in \mathcal{M}_{00}(G)_+$ such that $\pm v \leq \mu \cdot d$ and so

$$\pm(\alpha v) \leq |\alpha| v \leq |\alpha| (\mu \cdot d) = (|\alpha|\mu) \cdot d.$$

This shows that $\alpha v \in (E, d)_{\mathcal{M}}$ as $|\alpha|\mu$ is a positive element of $\mathcal{M}_{00}(G)$. We can conclude that $(E, d)_{\mathcal{M}}$ is an ideal of E.

Let's switch to the case where E is a normed Riesz positive $\mathcal{M}_{00}(G)$-module. It is clear that $(E, d)_{\mathcal{M}}$ is closed by taking the absolute value for every non-zero positive vector $d \in E$. Therefore, $(E, d)_{\mathcal{M}}$ is an ideal in E, and hence a Riesz subspace of E. □

If G is a discrete group, then $\mathcal{M}_{00}(G)_+ = c_{00}(G)_+$. Therefore, $(E,d)_{\mathcal{M}} = (E,d)$ for every non-zero $d \in E_+$. If G is a non-discrete locally compact group, we could not show, either disproved, this last equality. However, the relation between (E,d) and $(E,d)_{\mathcal{M}}$ is described in the following lemma.

Lemma 6.2.3. *Let E be a positive $\mathcal{M}_{00}(G)$-module, and let $d \in E$ be a non-zero positive vector. Then*

a) $(E,d) \subset (E,d)_{\mathcal{M}}$ and $(E,\mu_\phi \cdot d) \subset (E,d)_{\mathcal{M}}$ for every non-zero positive $\phi \in \mathcal{C}_{00}(G)$;

b) $(E,\mu_\phi \cdot d) = (E,\mu_{\tilde{\phi}} \cdot d)$ for every non-zero positive $\phi, \tilde{\phi} \in \mathcal{C}_{00}(G)$;

c) $(E,\mu_\phi \cdot d) = (E,\mu_\phi \cdot d)_{\mathcal{M}}$ for every non-zero positive $\phi \in \mathcal{C}_{00}(G)$;

d) $(E,\mu_\phi \cdot d)_{\mathcal{M}} = (E,\mu_{\tilde{\phi}} \cdot d)_{\mathcal{M}} \subset (E,d)_{\mathcal{M}}$ for every non-zero positive $\phi, \tilde{\phi} \in \mathcal{C}_{00}(G)$.

Proof. We start by proving point a). As every positive finite combination of Dirac masses is in $\mathcal{M}_{00}(G)_+$, the inclusion $(E,d) \subset (E,d)_{\mathcal{M}}$ holds. Similarly for every non-zero positive $\phi \in \mathcal{C}_{00}(G)$, the inclusion $(E,\mu_\phi \cdot d) \subset (E,d)_{\mathcal{M}}$ also holds, since $\mathcal{C}_{00}(G)$ can be seen as a subspace of $\mathcal{M}_{00}(G)$.

The proofs of points b) and c) are only a consequence of the fact that two non-zero positive compactly supported continuous functions always G-dominate each other, see Corollary 3.4.8. Indeed, fix two non-zero positive functions ϕ and $\tilde{\phi}$ in $\mathcal{C}_{00}(G)$. It suffices to show that there are elements $g_1,...,g_n,x_1,...,x_m \in G$ such that $\mu_\phi \cdot d \leq \sum_{j=1}^n g_j \left(\mu_{\tilde{\phi}} \cdot d\right)$ and $\mu_{\tilde{\phi}} \cdot d \leq \sum_{k=1}^m x_k \left(\mu_\phi \cdot d\right)$. As ϕ and $\tilde{\phi}$ are compactly supported continuous functions, there are $g_1,...,g_n \in G$ such that $\phi \leq \sum_{j=1}^n g_j \tilde{\phi}$ and $x_1,...,x_m \in G$ such that $\tilde{\phi} \leq \sum_{k=1}^m x_j \phi$. Therefore,

$$\mu_\phi \cdot d \leq \left(\sum_{j=1}^n g_j \mu_{\tilde{\phi}}\right) \cdot d = \sum_{j=1}^n g_j \left(\mu_{\tilde{\phi}} \cdot d\right)$$

and

$$\mu_{\tilde{\phi}} \cdot d \leq \left(\sum_{k=1}^m x_k \mu_\phi\right) \cdot d = \sum_{k=1}^m x_k \left(\mu_\phi \cdot d\right).$$

The proof to show that $(E,\mu_\phi \cdot d)_{\mathcal{M}} = (E,\mu_{\tilde{\phi}} \cdot d)_{\mathcal{M}}$ is similar.

Let's prove point c). Let $v \in (E,\mu_\phi \cdot d)$. Then $v \in (E,\mu_\phi \cdot d)_{\mathcal{M}}$ by point a). Therefore, $(E,\mu_\phi \cdot d) \subset (E,\mu_\phi \cdot d)_{\mathcal{M}}$. Conversely, let $v \in (E,\mu_\phi \cdot d)_{\mathcal{M}}$. Then there is a positive $\mu \in \mathcal{M}_{00}(G)$ such that $\pm v \leq \mu \cdot \left(\mu_\phi \cdot d\right) = \left(\mu * \mu_\phi\right) \cdot d$. But $\mu * \mu_\phi \in \mathcal{C}_{00}(G)$ as it can be seen as an (algebraic) ideal in $\mathcal{M}_{00}(G)$. So, $v \in \left(E, (\mu * \mu_\phi) \cdot d\right) = (E,\mu_\phi \cdot d)$ by point b). We can conclude that $(E,\mu_\phi \cdot d) = (E,\mu_\phi \cdot d)_{\mathcal{M}}$.

Let's look at point d). Let $\phi, \tilde{\phi} \in \mathcal{C}_{00}(G)$ be non-zero positive functions. Then

$$(E,\mu_\phi \cdot d)_{\mathcal{M}} = (E,\mu_\phi \cdot d) = (E,\mu_{\tilde{\phi}} \cdot d) = (E,\mu_{\tilde{\phi}} \cdot d)_{\mathcal{M}}$$

by points b) and c). Finally,

$$(E, \mu_\phi \cdot d)_{\mathcal{M}} = (E, \mu_\phi \cdot d) \subset (E, d)_{\mathcal{M}}$$

by point a). □

It is natural to extend the notion of dominated norm also for spaces of the form $(E, d)_{\mathcal{M}}$.

Definition 6.2.4. Let E be a positive $\mathcal{M}_{00}(G)$-module. For every non-zero positive vector $d \in E$, we define the possibly infinite value

$$\overline{p}_d(v) = \inf \{ ||\mu||_{\mathrm{TV}} : \pm v \leq \mu \cdot d \text{ for } \mu \in \mathcal{M}_{00}(G)_+ \} ,$$

where $v \in E$.

Note that $\overline{p}_d(v)$ is finite if and only if $v \in (E, d)_{\mathcal{M}}$. Therefore,

$$(E, d)_{\mathcal{M}} = \{ v \in E : \overline{p}_d(v) < \infty \} .$$

Similarly as in Chapter 3, we have the following proposition:

Proposition 6.2.5. *Let E be a positive $\mathcal{M}_{00}(G)$-module. Then the map*

$$\overline{p}_d : (E, d)_{\mathcal{M}} \longrightarrow \mathbf{R}, \quad v \longmapsto \overline{p}_d(v)$$

is absolutely homogeneous, sub-additive, G-invariant and monotone for every non-zero positive vector $d \in E$.

Proof. Let $d \in E$ be a non-zero positive vector. We start checking that $\overline{p}_d$ is absolutely homogeneous. Let $v \in (E, d)_{\mathcal{M}}$ and $\alpha \in \mathbf{R}$. Then for every $\epsilon > 0$ there is $\mu \in \mathcal{M}_{00}(G)_+$ such that

$$\pm v \leq \mu \cdot d \quad \text{ and } \quad \overline{p}_d(v) - \frac{\epsilon}{|\alpha|} \leq ||\mu||_{\mathrm{TV}} \leq \overline{p}_d(v) + \frac{\epsilon}{|\alpha|}.$$

Thus,

$$-((|\alpha|\mu) \cdot d) \leq \alpha v \leq (|\alpha|\mu) \cdot d \quad \text{ and } \quad |\alpha|\overline{p}_d(v) - \epsilon \leq |\alpha| \cdot ||\mu||_{\mathrm{TV}} \leq |\alpha|\overline{p}_d(v) + \epsilon.$$

As ϵ was arbitrary, we can conclude that $|\alpha|\overline{p}_d(v) = \overline{p}_d(\alpha v)$.

We continue proving that $\overline{p}_d$ is sub-additive. Let $v_1, v_2 \in (E, d)_{\mathcal{M}}$. Then for every $\epsilon > 0$ there are $\mu_1, \mu_2 \in \mathcal{M}_{00}(G)_+$ such that

$$\pm v_1 \leq \mu_1 \cdot d, \quad \pm v_2 \leq \mu_2 \cdot d, \quad ||\mu_1||_{\mathrm{TV}} \leq \overline{p}_d(v_1) + \frac{\epsilon}{2} \quad \text{ and } \quad ||\mu_2||_{\mathrm{TV}} \leq \overline{p}_d(v_2) + \frac{\epsilon}{2}.$$

Therefore,

$$\pm(v_1 + v_2) \leq (\mu_1 + \mu_2) \cdot d \quad \text{ and } \quad ||\mu_1 + \mu_2||_{\mathrm{TV}} \leq \overline{p}_d(v_1) + \overline{p}_d(v_2) + \epsilon.$$

As ϵ was arbitrary, we have that $\overline{p}_d$ is sub-additive.

We show the monotonicity of the map $\overline{p}_d$, i.e., if $v, w \in (E,d)_{\mathcal{M}}$ such that $v \leq w$, then $\overline{p}_d(v) \leq \overline{p}_d(w)$. Therefore, let $v, w \in (E,d)_{\mathcal{M}}$ such that $v \leq w$. Then there is $\mu \in \mathcal{M}_{00}(G)_+$ such that $\pm w \leq \mu \cdot d$. This implies that

$$-(\mu \cdot d) \leq -w \leq v \leq w \leq \mu \cdot d,$$

which suffices to ensure that $\overline{p}_d(v) \leq \overline{p}_d(w)$.

Finally, the G-invariance is a consequence of the fact that G acts on $\mathcal{M}_{00}(G)$ by positive linear isometries with respect to the total variation norm. □

In general, given a non-zero positive vector $d \in E$, the map $\overline{p}_d$ is not a norm on $(E,d)_{\mathcal{M}}$ (cf. example 3.1.9). However, it is a lattice seminorm when E is a normed Riesz positive $\mathcal{M}_{00}(G)$-module.

Corollary 6.2.6. *Let E be a normed Riesz positive $\mathcal{M}_{00}(G)$-module. Then the map $\overline{p}_d$ is a lattice seminorm for every non-zero positive vector $d \in E$. In particular, the pair $((E,d)_{\mathcal{M}}, \overline{p}_d)$ is a locally convex solid Riesz space.*

Proof. The map $\overline{p}_d$ is a lattice seminorm for every non-zero positive vector $d \in E$ by proposition 6.2.5. We can conclude using [AB99, Theorem 8.46]. □

Proposition 6.2.7. *Let E be a normed Riesz positive $\mathcal{M}_{00}(G)$-module. Fix a non-zero positive vector $d \in E$. Then the map $\overline{p}_d$ has the following properties:*

a) the inequality $||v|| \leq \overline{p}_d(v)||d||$ holds for every $v \in (E,d)_{\mathcal{M}}$;

b) let $h, v \in (E,d)_{\mathcal{M}}$ such that h is $\mathcal{M}_{00}(G)_+$-dominated by v and $v \geq 0$. Then

$$\overline{p}_d(h) \leq \overline{p}_v(h)\overline{p}_d(v).$$

Proof. Let $d \in E$ be a non-zero positive vector.

We start showing point a). Let $v \in (E,d)_{\mathcal{M}}$. Then for every $\epsilon > 0$ there is $\mu \in \mathcal{M}_{00}(G)_+$ such that

$$|v| \leq \mu \cdot d \quad \text{and} \quad ||\mu||_{\mathrm{TV}} \leq \overline{p}_d(v) + \frac{\epsilon}{||d||}.$$

It follows that

$$||v|| \leq ||\mu \cdot d|| \leq ||\mu||_{\mathrm{TV}} \cdot ||d|| \leq \overline{p}_d(v)||d|| + \epsilon.$$

As ϵ was arbitrary, we can conclude that $||v|| \leq \overline{p}_d(v)||d||$.

For point b) take an $\epsilon > 0$. Then there are $\mu_1, \mu_2 \in \mathcal{M}_{00}(G)_+$ such that

$$|h| \leq \mu_1 \cdot v, \quad v \leq \mu_2 \cdot d, \quad ||\mu_1||_{\mathrm{TV}} \leq \overline{p}_v(h) + \epsilon \quad \text{and} \quad ||\mu_2||_{\mathrm{TV}} \leq \overline{p}_d(v) + \epsilon.$$

We can compute that

$$|h| \leq \mu_1 \cdot v \leq \mu_1 \cdot (\mu_2 \cdot d) = (\mu_1 * \mu_2) \cdot d.$$

Using the fact that $\overline{p}_d$ is monotone, it follows

$$\begin{aligned}\overline{p}_d(h) &\leq ||\mu_1 * \mu_2||_{\mathrm{TV}} \\ &\leq ||\mu_1||_{\mathrm{TV}}||\mu_2||_{\mathrm{TV}} \\ &\leq (\overline{p}_v(h) + \epsilon)\,(\overline{p}_d(v) + \epsilon) \\ &= \overline{p}_v(h)\overline{p}_d(v) + \epsilon\overline{p}_v(h) + \epsilon\overline{p}_d(v) + \epsilon^2.\end{aligned}$$

As ϵ was arbitrary, we can conclude that $\overline{p}_d(h) \leq \overline{p}_v(h)\overline{p}_d(v)$ as wished. □

Corollary 6.2.8. *Let E be a normed Riesz positive $\mathcal{M}_{00}(G)$-module. Then, the pair $((E,d)_{\mathcal{M}}, \overline{p}_d)$ is a normed Riesz space for every non-zero positive vector $d \in E$. Moreover, the group G acts by positive linear isometries on it.*

Proof. The proof is direct by points a) and b) of Proposition 6.2.7. □

We continue by comparing the $\overline{p}_d$-norm and the p_d-norm.

Proposition 6.2.9. *Let E be a normed Riesz positive $\mathcal{M}_{00}(G)$-module, and let $d \in E$ be a non-zero positive vector. Let $(h_\alpha)_\alpha \subset (E,d)_{\mathcal{M}}$ be a net which converges to $h \in (E,d)_{\mathcal{M}}$ in p_v-norm, where v is a non-zero positive vector of $(E,d)_{\mathcal{M}}$. Then $(h_\alpha)_\alpha$ converges to h in $\overline{p}_d$-norm.*

Proof. As the net $(h_\alpha)_\alpha$ converges to h in p_v-norm, for every α there are a $n_\alpha \in \mathbf{N}$ and elements $g_1^\alpha, ..., g_{n_\alpha}^\alpha \in G$, $t_1^\alpha, ..., t_{n_\alpha}^\alpha \in \mathbf{R}_+$ such that

$$|h_\alpha - h| \leq \sum_{j=1}^{n_\alpha} t_j^\alpha g_j^\alpha v \quad \text{and} \quad \lim_\alpha \sum_{j=1}^{n_\alpha} t_j^\alpha = 0.$$

Now, there is $\mu \in \mathcal{M}_{00}(G)_+$ such that $|v| \leq \mu \cdot d$. This means that

$$|h_\alpha - h| \leq \sum_{j=1}^{n_\alpha} t_j^\alpha g_j^\alpha (\mu \cdot d) = \underbrace{\left(\left(\sum_{j=1}^{n_\alpha} t_j^\alpha \delta_{g_j^\alpha}\right) * \mu\right)}_{\in \mathcal{M}_{00}(G)_+} \cdot d$$

for every α, and thus

$$\begin{aligned}\lim_\alpha \left|\left| \left(\sum_{j=1}^{n_\alpha} t_j^\alpha \delta_{g_j^\alpha}\right) * \mu \right|\right|_{\mathrm{TV}} &\leq ||\mu||_{\mathrm{TV}} \lim_\alpha \left|\left| \sum_{j=1}^{n_\alpha} t_j^\alpha \delta_{g_j^\alpha} \right|\right|_{\mathrm{TV}} \\ &\leq ||\mu||_{\mathrm{TV}} \lim_\alpha \left(\sum_{j=1}^{n_\alpha} t_j^\alpha \delta_{g_j^\alpha}\right)(G) \\ &\leq \mu(G) \lim_\alpha \sum_{j=1}^{n_\alpha} t_j^\alpha = 0,\end{aligned}$$

which proves the convergence in $\overline{p}_d$-norm. □

We can conclude that if E is a normed Riesz positive $\mathcal{M}_{00}(G)$-module and $d \in E$ is a non-zero positive vector, then

$$\frac{||v||}{||d||} \leq \overline{p}_d(v) \leq p_d(v) \quad \text{for every } v \in (E, d).$$

Therefore, the $\overline{p}_d$-norm is weaker than the p_d-norm on the space (E, d). A situation where the two norms are actually equal is given in the following proposition.

Proposition 6.2.10. *Let E be one of the normed Riesz positive $\mathcal{M}_{00}(G)$-modules presented in Subsection 6.1.B. Fix a non-zero positive vector $d \in E$. Then*

$$\Big((E, \mu_\phi \cdot d), p_{\mu_\phi \cdot d}\Big) = \Big((E, \mu_\phi \cdot d)_{\mathcal{M}}, \overline{p}_{\mu_\phi \cdot d}\Big)$$

for every non-zero positive $\phi \in \mathcal{C}_{00}(G)$.

Proof. By Lemma 6.2.3, we can fix an arbitrary non-zero positive $\phi \in \mathcal{C}_{00}(G)$. We know that $(E, \mu_\phi \cdot d) = (E, \mu_\phi \cdot d)_{\mathcal{M}}$ and that $\overline{p}_{\mu_\phi \cdot d} \leq p_{\mu_\phi \cdot d}$ by point c) of Lemma 6.2.3 and Proposition 6.2.9, respectively. Therefore, it suffices to show that $p_{\mu_\phi \cdot d} \leq \overline{p}_{\mu_\phi \cdot d}$. To this aim, let $v \in (E, \mu_\phi \cdot d)_{\mathcal{M}}$. Then there is a positive net $(\mu_\alpha)_\alpha$ in $\mathcal{M}_{00}(G)$ such that $|v| \leq (\mu_\alpha * \mu_\phi) \cdot d$ for every α and $\lim_\alpha ||\mu_\alpha||_{\text{TV}} = \overline{p}_{\mu_\phi \cdot d}(v)$. Thus,

$$\begin{aligned}
p_{\mu_\phi \cdot d}(v) &\leq p_{\mu_\phi \cdot d}\left((\mu_\alpha * \mu_\phi) \cdot d\right) \\
&= p_{\mu_\phi \cdot d}\left(\mu_\alpha \cdot (\mu_\phi \cdot d)\right) \\
&= p_{\mu_\phi \cdot d}\left(\int_G x(\mu_\phi \cdot d) d\mu_\alpha(x)\right) \\
&\leq \int_G p_{\mu_\phi \cdot d}(x(\mu_\phi \cdot d)) d\mu_\alpha(x) = ||\mu_\alpha||_{\text{TV}}
\end{aligned}$$

for every α. Hence, $p_{\mu_\phi \cdot d}(v) \leq \overline{p}_{\mu_\phi \cdot d}(v)$. We can conclude that $\overline{p}_{\mu_\phi \cdot d} = p_{\mu_\phi \cdot d}$ as wished. □

For dominating norms of the form $p_{\mu_\phi * \cdot d}$ there is also an interesting convergence property illustrated in the following lemma.

Lemma 6.2.11. *Let E be a normed Riesz positive $\mathcal{M}_{00}(G)$-module, and let $d \in E$ and $\phi \in \mathcal{C}_{00}(G)$ be non-zero positive vectors. Then for every $\theta \in \mathcal{C}^1_{00}(G)$ and for every bounded approximate identity $(e_\alpha)_\alpha$ for $L^1(G)$ in $\mathcal{C}^1_{00}(G)$ with decreasing support, we have that*

$$\lim_\alpha (\mu_\theta * \mu_{e_\alpha}) \cdot v = \mu_\theta \cdot v \quad \text{and} \quad \lim_\alpha (\mu_{e_\alpha} * \mu_\theta) \cdot v = \mu_\theta \cdot v$$

in $p_{\mu_\phi \cdot d}$-norm for every $v \in (E, d)_{\mathcal{M}}$ and every $\theta \in \mathcal{C}_{00}(G)$.

The existence of bounded approximate identities with decreasing support is assured by Urysohn Lemma and the fact that G is locally compact.

Proof of Lemma 6.2.11. Let $v \in (E,d)_{\mathcal{M}}$, $\theta \in \mathcal{C}_{00}(G)$ and $(e_\alpha)_\alpha$ a bounded approximate identity for $L^1(G)$ in $\mathcal{C}^1_{00}(G)$ with decreasing support. We want to show that $\lim_\alpha(\mu_\theta * \mu_{e_\alpha}) \cdot v = \mu_\theta \cdot v$ in $p_{\mu_\phi \cdot d}$-norm. Without loss of generality, we can suppose that v is positive, since $(E,d)_{\mathcal{M}}$ is spanned by positive elements. Now the estimation

$$\begin{aligned} p_{\mu_\phi \cdot d}(\mu_\theta * \mu_{e_\alpha} \cdot v - \mu_\theta \cdot v) &= p_{\mu_\phi \cdot d}\left((\mu_\theta * \mu_{e_\alpha} - \mu_\theta) \cdot v\right) \\ &\leq p_{\mu_\phi \cdot v}\left((\mu_\theta * \mu_{e_\alpha} - \mu_\theta) \cdot v\right) p_{\mu_\phi \cdot d}(\mu_\phi \cdot v) \end{aligned}$$

holds for every α. The last inequality is possible thanks to point b) of Proposition 3.1.14. The second term is finite because $\mu_\phi \cdot v \in (E, \mu_\phi \cdot d)_{\mathcal{M}}$. So, let's focus on the first term. By Proposition 3.4.12, $\lim_\alpha \theta * e_\alpha = \theta$ in p_ϕ-norm. This means that $\lim_\alpha \mu_\theta * \mu_{e_\alpha} = \mu_\theta$ in p_{μ_ϕ}-norm. Therefore, for every α there is $n_\alpha \in \mathbf{N}$ and elements $g_1^\alpha, ..., g_{n_\alpha}^\alpha \in G$ and $t_1^\alpha, ..., t_{n_\alpha}^\alpha \in \mathbf{R}_+$ such that

$$|\mu_\theta * \mu_{e_\alpha} - \mu_\theta| \leq \sum_{j=1}^{n_\alpha} t_j^\alpha g_j^\alpha \mu_\phi \quad \text{and} \quad \lim_\alpha \sum_{j=1}^{n_\alpha} t_j^\alpha = 0.$$

This implies that

$$|\mu_\theta * \mu_{e_\alpha} - \mu_\theta| \cdot v \leq \left(\sum_{j=1}^{n_\alpha} t_j^\alpha g_j^\alpha \mu_\phi\right) \cdot v = \sum_{j=1}^{n_\alpha} t_j^\alpha g_j^\alpha (\mu_\phi \cdot v).$$

Hence,

$$\begin{aligned} p_{\mu_\phi \cdot v}\left(|\mu_\theta * \mu_{e_\alpha} - \mu_\theta| \cdot v\right) &\leq p_{\mu_\phi \cdot v}\left(\sum_{j=1}^{n_\alpha} t_j^\alpha g_j^\alpha (\mu_\phi \cdot v)\right) \\ &\leq \sum_{j=1}^{n_\alpha} t_j^\alpha p_{\mu_\phi \cdot v}(g_j^\alpha(\mu_\phi \cdot v)) = \sum_{j=1}^{n_\alpha} t_j^\alpha, \end{aligned}$$

for every α. Taking the limit with respect to α of this last inequality, it follows that $\lim_\alpha(\mu_\theta * \mu_{e_\alpha}) \cdot v = \mu_\theta \cdot v$ in $p_{\mu_\phi \cdot d}$-norm.

The proof to show that $\lim_\alpha(\mu_{e_\alpha} * \mu_\theta) \cdot v = \mu_\theta \cdot v$ in $p_{\mu_\phi \cdot d}$-norm is similar. □

Scholium 6.2.12. The theory developed in this section for positive $\mathcal{M}_{00}(G)$-modules could have also been developed for general positive R-modules for R an ordered normed vector space.

6.2.B. Continuous vectors for measurably dominating norms. Given a normed Riesz positive $\mathcal{M}_{00}(G)$-module E, we investigate which are the continuous vectors for the $\overline{p}_d$-norm. We don't know if the natural action of G on the normed Riesz space $((E,d)_{\mathcal{M}}, \overline{p}_d)$ is continuous. However, we can prove something sufficient for our future purposes.

Lemma 6.2.13. *Let E be a normed Riesz positive $\mathcal{M}_{00}(G)$-module, and let $d \in E$ be a non-zero positive vector. Define the $\mathcal{M}_{00}(G)$-invariant linear subspace*

$$D = \operatorname{span}_{\mathbf{R}} \left\{ \mu_\phi \cdot v : \phi \in \mathcal{C}_{00}(G) \text{ and } v \in (E,d)_{\mathcal{M}} \right\} \subset (E,d)_{\mathcal{M}}.$$

Then

a) the action of G on D is orbitally continuous with respect to the $p_{\mu_\phi \cdot d}$-norm for every non-zero positive $\phi \in \mathcal{C}_{00}(G)$;

b) the action of G on D is orbitally continuous with respect to the $\overline{p}_d$-norm.

Proof. First of all, we should show that D is genuinely a $\mathcal{M}_{00}(G)$-invariant linear subspace of $(E,d)_{\mathcal{M}}$. By definition, D naturally carries a structure of vector space. Therefore, we have only to check that D is a vector subspace of $(E,d)_{\mathcal{M}}$. Let $v \in D$. Then there are $\phi \in \mathcal{C}_{00}(G)$ and $w \in (E,f)_{\mathcal{M}}$ such that $v = \mu_\phi \cdot w$. As $w \in (E,f)_{\mathcal{M}}$, there is a $\mu \in \mathcal{M}_{00}(G)_+$ such that $|w| \le \mu \cdot d$. Thus,

$$|v| \le |\mu_\phi \cdot w| \le |\mu_\phi| \cdot |w| \le |\mu_\phi| \cdot (\mu \cdot d) = \underbrace{(|\mu_\phi| * \mu)}_{\in \mathcal{M}_{00}(G)_+} \cdot d.$$

This proves that D is a subspace of $(E,f)_{\mathcal{M}}$. The $\mathcal{M}_{00}(G)$-invariance is directed by the definition of D and because the measure μ_ϕ lies in the ideal $\mathcal{M}_{00}(G)_a$ for every $\phi \in \mathcal{C}_{00}(G)$.

We begin proving a). Fix $\mu_\psi \cdot v \in D$ and let $(g_\alpha)_\alpha \subset G$ be a net such that $\lim_\alpha g_\alpha = e$ the identity element of G. Without loss of generality, we can suppose that μ_ϕ is positive as each element in $\mathcal{M}_{00}(G)$ is the difference of two positive elements. Then the estimation

$$\begin{aligned} p_{\mu_\phi \cdot d}\left(g_\alpha\left(\mu_\psi \cdot v\right) - \mu_\psi \cdot v\right) &= p_{\mu_\phi \cdot d}\left(\left(g_\alpha \mu_\psi - \mu_\psi\right) \cdot v\right) \\ &\le p_{\mu_\phi \cdot d}\left(\left|g_\alpha \mu_\psi - \mu_\psi\right| \cdot |v|\right) \\ &\le p_{\mu_\psi \cdot |v|}\left(\left|g_\alpha \mu_\psi - \mu_\psi\right| \cdot |v|\right) p_{\mu_\phi \cdot d}\left(\mu_\psi \cdot |v|\right) \end{aligned}$$

holds for every α. Note that the last inequality is possible thanks to point b) of Proposition 3.1.14. Now $p_{\mu_\phi \cdot d}(\mu_\psi \cdot |v|)$ is finite, since $\mu_\psi \cdot |v| \in (E, \mu_\phi \cdot d)$. Therefore, we want to study the first term. To this aim, note that the action of G on $\mathcal{C}_{00}(G)$ is orbitally continuous with respect to the p_ψ-norm as showed in Lemma 3.4.10. In particular, as we can see $\mathcal{C}_{00}(G)$ as a subspace of $\mathcal{M}_{00}(G)_a$, for every α there are $n_\alpha \in \mathbf{N}$, $t_1^\alpha, ..., t_{n_\alpha}^\alpha \in \mathbf{R}_+$ and $g_1^\alpha, ..., g_{n_\alpha}^\alpha \in G$ such that

$$|g_\alpha \mu_\psi - \mu_\psi| \le \sum_{j=1}^{n_\alpha} t_j^\alpha g_j^\alpha \mu_\psi \quad \text{and} \quad \lim_\alpha \sum_{j=1}^{n_\alpha} t_j^\alpha = 0.$$

Multiplying both sides of the previous expression by $|v|$, we have that

$$|g_\alpha \mu_\psi - \mu_\psi| \cdot |v| \leq \left(\sum_{j=1}^{n_\alpha} t_j^\alpha g_j^\alpha \mu_\psi \right) \cdot |v| = \sum_{j=1}^{n_\alpha} t_j^\alpha g_j^\alpha \left(\mu_\psi \cdot |v| \right) \qquad \text{for every } \alpha.$$

Finally taking the $p_{\mu_\psi \cdot |v|}$-norm of this last inequality, we get the estimation

$$\begin{aligned} p_{\mu_\psi \cdot |v|} \left(|g_\alpha \mu_\psi - \mu_\psi| \cdot |v| \right) &\leq p_{\mu_\psi \cdot |v|} \left(\sum_{j=1}^{n_\alpha} t_j^\alpha g_j^\alpha \left(\mu_\psi \cdot |v| \right) \right) \\ &\leq \sum_{j=1}^{n_\alpha} t_j^\alpha p_{\mu_\psi \cdot |v|} \left(g_j^\alpha (\mu_\psi \cdot |v|) \right) = \sum_{j=1}^{n_\alpha} t_j^\alpha \qquad \text{for every } \alpha. \end{aligned}$$

This implies that $\lim_\alpha p_{\mu_\psi \cdot |v|}(|g_\alpha \mu_\psi - \mu_\psi| \cdot |v|) = 0$. We can conclude that the action is orbitally continuous.

The proof of point b) is straightforward using Proposition 6.2.9 and point a). □

6.2.C. A new kind of integral. The adaption of dominating spaces to the measurable setting gives rise to a new notion of normalized integral and consequently a new normalized integral property.

Define the set

$$\mathcal{M}_{00}^1(G) = \{\mu \in \mathcal{M}_{00}(G) : \mu \geq 0 \text{ and } ||\mu||_{\text{TV}} = \lambda(G) = 1\}.$$

Then $\mathcal{M}_{00}^1(G)$ is a unital monoid under convolution between measures with identity given by δ_e. Indeed, if μ_1 and μ_2 are in $\mathcal{M}_{00}^1(G)$, then

$$(\mu_1 * \mu_2)(G) = \int_G \mu_2(g^{-1}G) d\mu_1(g) = \int_G \mu_2(G) d\mu_1(g) = \mu_1(G)\mu_2(G) = 1.$$

Proposition 6.2.14. *Let E be a positive $\mathcal{M}_{00}(G)$-module. Take a non-zero positive vector $d \in E$ and a positive linear functional $\psi \in (E, d)^*_{\mathcal{M}}$. Suppose that ψ is uniformly bounded on the set $\{\mu \cdot d : \mu \in \mathcal{M}_{00}^1(G)\}$. Then ψ is continuous for the $\tau(\overline{p}_d)$-topology.*

Proof. Set $M = \sup\{\psi(\mu \cdot d) : \mu \in \mathcal{M}_{00}^1(G)\}$ and let $(v_\alpha)_\alpha$ be a net in $(E, d)_{\mathcal{M}}$ which converges to some $v \in (E, d)_{\mathcal{M}}$ for the $\tau(\overline{p}_d)$-topology. Then there is a positive net $(\epsilon_\alpha)_\alpha$ in $\mathbf{R}$ converging to zero and a net $(\mu_\alpha)_\alpha$ in $\mathcal{M}_{00}(G)$ such that

$$\pm(v_\alpha - v) \leq \mu_\alpha \cdot d \quad \text{and} \quad ||\mu_\alpha||_{\text{TV}} < \epsilon_\alpha \quad \text{for every } \alpha.$$

Therefore,

$$\begin{aligned} \pm\psi(v_\alpha - v) &\leq \psi\left(\mu_\alpha \cdot d\right) \\ &= ||\mu_\alpha||_{\text{TV}}\, \psi\left(\left(\frac{\mu_\alpha}{||\mu_\alpha||_{\text{TV}}} \right) \cdot d \right) \\ &\leq M ||\mu_\alpha||_{\text{TV}} \end{aligned}$$

for every α. The limit with respect of α of this last inequality gives

$$\lim_{\alpha} \pm\psi(v_\alpha - v) \leq M \lim_{\alpha} ||\mu_\alpha||_{\mathrm{TV}} \leq M \lim_{\alpha} \epsilon_\alpha = 0.$$

This implies that the net $(\psi(v_\alpha))_\alpha$ is converging to $\psi(v)$ for the $\tau(\overline{p}_d)$-topology showing the continuity of ψ. ☐

This last result can be improved for normed Riesz positive $\mathcal{M}_{00}(G)$-modules (cf. with Proposition 4.1.2).

Proposition 6.2.15. *Let E be a normed Riesz positive $\mathcal{M}_{00}(G)$-module. Fix a non-zero positive vector $d \in E$, and let ψ be a non-zero linear functional on $(E, d)_{\mathcal{M}}$. Then*

a) if the functional ψ is positive and uniformly bounded on the set $\{\mu \cdot d : \mu \in \mathcal{M}^1_{00}(G)\}$, then ψ is continuous for the $\overline{p}_d$-norm and

$$||\psi||_{op} \leq \sup \left\{ \psi(\mu \cdot d) : \mu \in \mathcal{M}^1_{00}(G) \right\}.$$

In particular, if ψ is constant on the set $\{\mu \cdot d : \mu \in \mathcal{M}^1_{00}(G)\}$, then $||\psi||_{op} = \psi(d)$;

b) if the functional ψ is continuous for the $\overline{p}$-norm, then

$$\sup \left\{ \psi(\mu \cdot d) : \mu \in \mathcal{M}^1_{00}(G) \right\} \leq ||\psi||_{op} \leq \sup \left\{ |\psi|(\mu \cdot d) : \mu \in \mathcal{M}^1_{00}(G) \right\}.$$

In particular, if ψ is positive, then $||\psi||_{op} = \sup \{|\psi|(\mu \cdot d) : \mu \in \mathcal{M}^1_{00}(G)\}$;

d) if the functional ψ is continuous for the $\overline{p}_d$-norm and constant on the set $\{\mu \cdot d : \mu \in \mathcal{M}^1_{00}(G)\}$, then ψ is positive.

Proof. We start by proving point a). We already know that ψ is continuous for the $\overline{p}_d$-norm. Hence, we want to show that ψ is bounded. Let

$$M = \sup\{\psi(\mu \cdot d) : \mu \in \mathcal{M}^1_{00}(G)\}$$

and let $v \in (E, d)_{\mathcal{M}}$ be such that $\overline{p}_d(v) = 1$. Then there is a positive net $(\mu_\alpha)_\alpha$ in $\mathcal{M}_{00}(G)$ such that $|v| \leq \mu_\alpha \cdot d$ for every α and such that $\lim_\alpha ||\mu_\alpha||_{\mathrm{TV}} = 1$. Therefore,

$$|\psi(v)| \leq \psi(|v|) \leq \psi(\mu_\alpha \cdot d) = ||\mu_\alpha||_{\mathrm{TV}}\, \psi\left(\left(\frac{\mu_\alpha}{||\mu_\alpha||_{\mathrm{TV}}}\right) \cdot d\right) \leq M||\mu_\alpha||_{\mathrm{TV}}$$

for every α. Hence, $|\psi(v)| \leq M$. This implies that $||\psi||_{op} \leq M$ as wished.

If ψ is constant on the set $\{\mu \cdot d : \mathcal{M}^1_{00}(G)\}$, then $||\psi||_{op} \geq M$, where $M > 0$ is the value of ψ on the set $\left\{\psi(\mu \cdot d) : \mu \in \mathcal{M}^1_{00}(G)\right\}$. This is because $\overline{p}_d(d) = 1$ and $\psi(d) = M$. Therefore, $||\psi||_{op} \geq M$ finishing the proof.

Before proving point b), recall that if ψ is a continuous functional on a Riesz space, then $|\psi|$ is also continuous by [AB99, Theorem 8.48] and $||\psi||_{op} = ||\,|\psi|\,||_{op}$. Now

$$0 \leq |\psi(\mu \cdot d)| \leq |\psi|(\mu \cdot d) \leq ||\psi||_{op}\overline{p}_d(\mu \cdot d) \leq ||\psi||_{op}||\mu||_{\mathrm{TV}} = ||\psi||_{op}$$

for every $\mu \in \mathcal{M}^1_{00}(G)$. Hence, we can conclude that

$$\sup\left\{\psi(\mu \cdot d) : \mu \in \mathcal{M}^1_{00}(G)\right\} \leq ||\psi||_{op}.$$

In order to prove the second inequality, set $M = \sup\{|\psi|(\mu \cdot d) : g \in G\}$, and note that this value is finite as $|\psi|$ is also continuous. Let $v \in (E,d)_{\mathcal{M}}$. Then for every $\epsilon > 0$, there is a positive $\mu \in \mathcal{M}_{00}(G)$ such that

$$|v| \leq \mu \cdot d \quad \text{and} \quad ||\mu||_{\mathrm{TV}} \leq \overline{p}_d(v) + \frac{\epsilon}{M}.$$

Thus,

$$|\psi|(v) \leq |\psi|(|v|) \leq |\psi|\,(\,\mu \cdot d) \leq M||\mu||_{\mathrm{TV}} \leq M\,\overline{p}_d(v) + \epsilon.$$

As ϵ and v were chosen arbitrarily, we can conclude that

$$||\psi||_{op} \leq \sup\left\{|\psi|(\mu \cdot d) : \mu \in \mathcal{M}^1_{00}(G)\right\}.$$

Finally, we show point d). In order to find a contradiction, suppose that ψ is not positive. Then there is a non-zero positive vector $v \in (E,d)_{\mathcal{M}}$ such that $\overline{p}_d(v) = 1$ and $\psi(v) < 0$. Consequently, for every $\epsilon > 0$ there is a positive $\mu \in \mathcal{M}_{00}(G)$ such that

$$v \leq \mu \cdot d \quad \text{and} \quad 1 \leq ||\mu||_{\mathrm{TV}} \leq 1 + \frac{\epsilon}{||\psi||_{op}}.$$

Now $||\psi||_{op} = |\psi(\mu \cdot d)|$ for every $\mu \in \mathcal{M}^1_{00}(G)$ by point c). This implies that

$$|\psi|\,(\mu \cdot d - v) = ||\psi||_{op} - \psi(v) > ||\psi||_{op}.$$

However,

$$\begin{aligned} |\psi|\,(\mu \cdot d - v) &\leq ||\psi||_{op}\overline{p}_d\,(\mu \cdot d - v) \\ &\leq ||\psi||_{op}\overline{p}_d\,(\mu \cdot d) \\ &\leq ||\psi||_{op}||\mu||_{\mathrm{TV}} \leq ||\psi||_{op} + \epsilon. \end{aligned}$$

Therefore, $|\psi|\,(\mu \cdot d - v) \leq ||\psi||_{op}$, since ϵ was chosen arbitrarily. But this is the contradiction we were searching. Hence, ψ is positive. □

This last result shows that positivity, continuity and be uniformly bounded on the majorizing subspace are linked proprieties of a linear functional on $(E,d)_{\mathcal{M}}$. The next proposition shows that also the $\mathcal{M}_{00}(G)$-module structure influences linear functionals.

Proposition 6.2.16. *Let E be a positive $\mathcal{M}_{00}(G)$-module. Fix a non-zero positive vector $d \in E$, and let ψ be a functional defined on $(E, d)_{\mathcal{M}}$. Suppose that there is $v \in (E, d)_{\mathcal{M}}$ such that ψ is constant on the set $\{\mu \cdot v : \mu \in \mathcal{M}^1_{00}(G)\}$. Then $\psi(\mu \cdot v) = \mu(G)\psi(v)$ for every $\mu \in \mathcal{M}_{00}(G)$.*

Proof. Let $\mu \in \mathcal{M}_{00}(G)$. Since the positive cone of a Riesz space is always generating, we can write

$$\mu = \mu_1 - \mu_2 \quad \text{where } \mu_1, \mu_2 \in \mathcal{M}_{00}(G)_+.$$

Therefore,

$$\begin{aligned} \psi(\mu \cdot v) &= \psi\left((\mu_1 - \mu_2) \cdot v\right) \\ &= \psi(\mu_1 \cdot v) - \psi(\mu_2 \cdot v) \\ &= \mu_1(G)\ \psi\left(\left(\frac{\mu_1}{\mu_1(G)}\right) \cdot v\right) - \mu_2(G)\ \psi\left(\left(\frac{\mu_2}{\mu_2(G)}\right) \cdot v\right) \\ &= \mu_1(G)\psi(v) - \mu_2(G)\psi(v) \\ &= \psi(v)(\mu_1 - \mu_2)(G) = \psi(v)\mu(G). \end{aligned}$$

□

We are now interested in specific linear functionals.

Definition 6.2.17. Let E by a positive $\mathcal{M}_{00}(G)$-module, and let $d \in E$ be a non-zero positive vector. Then a linear functional ψ on $(E, d)_{\mathcal{M}}$ is called:

- an **integral**, if ψ is positive;
- a **normalized integral**, if it is an integral and $\psi(d) = 1$;
- a **measurably invariant normalized integral**, if it is a normalized integral and $\psi(\mu \cdot v) = \psi(v)$ for every $\mu \in \mathcal{M}^1_{00}(G)$ and every $v \in (E, d)_{\mathcal{M}}$.

Remark 6.2.18. One might think that this last definition could create confusion with the Definition 4.1.4 given in Chapter 4. However, it is not the case because is possible to understand which type of integral we are considering only by looking at the space on which it is defined.

Example 6.2.19. Let G be an amenable locally compact group. Then by [G69, Theorem 2.2.1], there is a topological invariant mean m on $L^\infty(G)$, i.e., a mean m defined on $L^\infty(G)$ such that $\mathrm{m}(\phi * f) = \mathrm{m}(f)$ for every $f \in L^\infty(G)$ and every $\phi \in \mathrm{P}(G)$. Recall that

$$\mathrm{P}(G) = \left\{\phi \in L^1(G) : ||\phi||_1 = 1 \text{ and } \phi \geq 0\right\}.$$

We claim that m is also measurably invariant, and hence a measurably invariant normalized integral on $L^\infty(G, \mathbf{1}_G)_{\mathcal{M}} = L^\infty(G)$. Indeed, let $\mu \in \mathcal{M}^1_{00}(G)$ and take $\phi \in \mathrm{P}(G)$. We can directly compute that

$$\mathrm{m}(\mu * f) = \mathrm{m}\left(\phi * (\mu * f)\right) = \mathrm{m}(\underbrace{(\phi * \mu)}_{\in \mathrm{P}(G)} * f) = \mathrm{m}(f) \quad \text{for every } f \in L^\infty(G).$$

Conversely, every measurably invariant mean on $L^\infty(G)$ is topologically invariant. In fact, let m be a mean defined on $L^\infty(G)$ and let $\phi \in \mathrm{P}(G)$. Then there is a positive sequence $(\phi_n)_n$ in $\mathcal{C}_{00}(G)$ which converges to ϕ in L^1-norm. Therefore,

$$\begin{aligned} 0 &\leq |\mathrm{m}(\phi * f) - \mathrm{m}(f)| \\ &= |\mathrm{m}(\phi * f) - \mathrm{m}(\phi_n * f) + \mathrm{m}(\phi_n * f) - \mathrm{m}(f)| \\ &\leq |\mathrm{m}\left((\phi - \phi_n) * f\right)| + |\mathrm{m}(f) - \mathrm{m}(f)| \\ &\leq ||\mathrm{m}||_{op}||\left(\phi - \phi_n\right) * f||_\infty \\ &\leq ||\phi - \phi_n||_1 ||f||_\infty \end{aligned}$$

for every $n \in \mathbf{N}$ and every $f \in L^\infty(G)$. This implies that $\mathrm{m}(\phi * f) = \mathrm{m}(f)$ for every $f \in L^\infty(G)$. We can conclude that m is topologically invariant.

Let $d \in E$ be a non-zero positive vector. We say that $(E, d)_{\mathcal{M}}$ **admits a measurably invariant normalized integral** if there is a measurably invariant normalized integral I defined on $(E, d)_{\mathcal{M}}$.

Definition 6.2.20. Let E be a positive $\mathcal{M}_{00}(G)$-module. Then G **has the measurably invariant normalized integral property for** E if the space $(E, d)_{\mathcal{M}}$ admits a measurably invariant normalized integral for every non-zero positive vector $d \in E$.

Note that the definitions of measurably invariant normalized integral and invariant normalized integral coincide for discrete groups.

Proposition 6.2.21. *Let E be a positive $\mathcal{M}_{00}(G)$-module. If E admits a strictly positive measurably invariant functional, then G has the measurably invariant normalized integral property for E.*

Proof. Take a non-zero positive vector $d \in E$ and let $\psi \in E^*$ be the strictly positive measurably invariant functional of E. Set $c = \psi(d) > 0$. Then a measurably invariant normalized integral for $(E, d)_{\mathcal{M}}$ is given by $\mathrm{I} = \frac{1}{c}\psi$. Therefore, the group G has the measurably invariant integral property for E. □

Example 6.2.22. (Examples of the measurably invariant integral property)

1) Let G be a locally compact group and let $\mathcal{C}_{00}(G)$ be the Riesz space of all real compactly supported continuous functions on G. Consider the strictly positive functional I on $\mathcal{C}_{00}(G)$ given by

$$\mathrm{I}(f) = \int_G f(g) dm_G(g) \quad \text{for } f \in \mathcal{C}_{00}(G).$$

We claim that I is measurably invariant. Indeed, let $\mu \in \mathcal{M}^1_{00}(G)$. Thanks to Fubini Theorem ([C13, Proposition 7.6.4]), it is possible to compute that

$$\begin{aligned} \mathrm{I}(\mu * f) &= \int_G (\mu * f)(g) dm_G(g) \\ &= \int_G \int_G f(x^{-1}g) d\mu(x) dm_G(g) \\ &= \int_G \int_G f(x^{-1}g) dm_G(g) d\mu(x) \\ &= \int_G \int_G f(y) dm_G(y) d\mu(x) \\ &= \mu(G) \int_G f(y) dm_G(y) = \mathrm{I}(f) \quad \text{for every } f \in \mathcal{C}_{00}(G). \end{aligned}$$

By Proposition 6.2.21, we can conclude that every locally compact group has the measurably invariant integral property for $\mathcal{C}_{00}(G)$.

2) Let G be a locally compact group with a unitary representation σ on a Hilbert space $\mathcal{H}$. We claim that $\mathrm{TC}(\mathcal{H})$ admits a strictly positive measurably invariant functional. Indeed, the trace map $\mathrm{tr}(\cdot)$ is strictly positive as $\mathrm{tr}(T) \geq ||T||$ for every positive $T \in \mathrm{TC}(\mathcal{H})$ by [S18, Proposition 6.4]. Moreover, $\mathrm{tr}(\cdot)$ is measurably invariant. Indeed, let $T \in \mathrm{TC}(\mathcal{H})$ and $\mu \in \mathcal{M}^1_{00}(G)$. Then

$$\begin{aligned} \mathrm{tr}(\mu \cdot T) &= \mathrm{tr}\left(\int_G \mathrm{Ad}_\sigma(g) T d\mu(g)\right) \\ &= \int_G \mathrm{tr}\left(\sigma(g) T \sigma(g)^*\right) d\mu(g) \\ &= \int_G \mathrm{tr}\left(T\right) d\mu(g) \\ &= \mu(G)\mathrm{tr}(T) = \mathrm{tr}(T). \end{aligned}$$

Therefore, we can conclude that G has the measurably invariant normalized integral property for $\mathrm{TC}(\mathcal{H})$ by Proposition 6.2.21.

Proposition 6.2.23. *Let E be a positive $\mathcal{M}_{00}(G)$-module, and fix a non-zero positive vector $d \in E$. Let* I *be a measurably invariant normalized integral on $(E, d)_{\mathcal{M}}$. Then* I *is continuous with respect to the $\overline{p}_d$-norm. Moreover, if E is a normed Riesz positive $\mathcal{M}_{00}(G)$-module, then* I *has operator norm equal to one.*

Proof. The continuity of I for the $\overline{p}_d$-norm is given by Proposition 6.2.5. If E is a normed Riesz positive $\mathcal{M}_{00}(G)$-module, then $||\mathrm{I}||_{op} = \mathrm{I}(d) = 1$ by Proposition 6.2.15. ☐

Define

$$\mathcal{C}^1_{00}(G) = \{\phi \in \mathcal{C}_{00}(G) : \phi \geq 0 \text{ and } ||\phi||_1 = 1\}.$$

If E is a positive $\mathcal{M}_{00}(G)$-module, then a functional ψ defined on $(E,d)_{\mathcal{M}}$, for some non-zero positive vector $d \in E$, is said $\mathcal{C}^1_{00}(G)$-invariant if $\psi(\mu_\phi \cdot v) = \psi(v)$ for every $\phi \in \mathcal{C}^1_{00}(G)$ and every $v \in E$.

Proposition 6.2.24. *Let E be a positive $\mathcal{M}_{00}(G)$-module, and take a non-zero positive vector $d \in E$. Let* I *be a measurably invariant normalized integral defined on $(E,d)_{\mathcal{M}}$. Then*

a) the integral I *is G-invariant and $\mathcal{C}^1_{00}(G)$-invariant;*

b) $\mathrm{I}(\mu \cdot d) = \mu(G)$ for every $\mu \in \mathcal{M}_{00}(G)$.

Proof. Point a) is straightforward because G and $\mathcal{C}^1_{00}(G)$ can be represented as subsemigroups of $\mathcal{M}^1_{00}(G)$.

The proof of point b) is given by Proposition 6.2.16. □

Corollary 6.2.25. *Let E be a positive $\mathcal{M}_{00}(G)$-module. If G has the measurably invariant normalized integral property for E, then G has the invariant normalized integral property for E.*

Proof. Let $d \in E$ be a non-zero vector. Then $(E,d) \subset (E,d)_{\mathcal{M}}$ by point a) of Lemma 6.2.3. On the space $(E,d)_{\mathcal{M}}$ there is a measurably invariant normalized integral I by hypothesis. In particular, I is also invariant by point a) of Proposition 6.2.24. Therefore, the restriction of I on (E,d) is an invariant normalized integral. □

6.2.D. The measurably translate property . We can also generalise the translate property to the measurable setting, as done with the invariant normalized integral property.

Definition 6.2.26. Let E be a positive $\mathcal{M}_{00}(G)$-module. We say that a non-zero positive vector $d \in E$ has the **measurably translate property** if for every $\mu \in \mathcal{M}_{00}(G)$, we have that

$$\mu \cdot d \geq 0 \quad \text{implies} \quad \mu(G) \geq 0.$$

Example 6.2.27. Let $E = L^\infty(G)$ and consider the positive function $\mathbf{1}_G \in L^\infty(G)$. Take $\mu \in \mathcal{M}_{00}(G)$ such that $\mu * \mathbf{1}_G \geq 0$. Then

$$(\mu * \mathbf{1}_G)(g) = \int_G \mathbf{1}_G(x^{-1}g)d\mu(x) = \mu(G)\mathbf{1}_G(g) \quad \text{for every } g \in G.$$

This implies that $\mu(G) \geq 0$ showing that $\mathbf{1}_G$ has the measurably translate property.

A priori, the measurably translate property is stronger than the translate property.

Proposition 6.2.28. *Let E be a positive $\mathcal{M}_{00}(G)$-module and let $d \in E$ be a non-zero positive vector. If d has the measurably translate property, then d has the translate property.*

Proof. Let $d \in E$ be a non-zero positive vector and let $t_1, ..., t_n \in \mathbf{R}$ and $g_1, ..., g_n \in G$ such that $\sum_{j=1}^{n} t_j g_j d \geq 0$. Then

$$0 \leq \sum_{j=1}^{n} t_j g_j d = \underbrace{\left(\sum_{j=1}^{n} t_j \delta_{g_j} \right)}_{\in \mathcal{M}_{00}(G)} \cdot d.$$

Therefore, the measurably translate property implies that

$$0 \leq \Big(\sum_{j=1}^{n} t_j \delta_{g_j} \Big)(G) = \sum_{j=1}^{n} t_j.$$

□

Compare the following proposition with Proposition 4.2.4.

Proposition 6.2.29. *Let E be a normed Riesz positive $\mathcal{M}_{00}(G)$-module and let d be a non-zero positive vector of E. Then the following assertions are equivalent:*

a) the vector d has the measurably translate property;

b) there is a normalized integral ψ on $(E, d)_{\mathcal{M}}$ which is measurably invariant on the majorizing subspace $\{\mu \cdot d : \mu \in \mathcal{M}_{00}(G)\}$;

c) the equality $\overline{p}_d(\mu \cdot d) = \mu(G)$ holds for every positive $\mu \in \mathcal{M}_{00}(G)$.

Proof. We start showing that a) implies b). To this end, define the linear operator

$$\omega : \{\mu \cdot d : \mu \in \mathcal{M}_{00}(G)\} \longrightarrow \mathbf{R}, \quad \mu \cdot d \longmapsto \omega(\mu \cdot d) = \mu(G).$$

First, ω is well-defined. Indeed, if $\mu_1, \mu_2 \in \mathcal{M}_{00}(G)$ are such that $\mu_1 \cdot d = \mu_2 \cdot d$, then

$$(\mu_1 - \mu_2) \cdot d = 0 \quad \text{and} \quad (\mu_2 - \mu_1) \cdot d = 0.$$

Therefore, the measurably translate property implies that

$$(\mu_1 - \mu_2)\,(G) \geq 0 \quad \text{and} \quad (\mu_2 - \mu_1)\,(G) \geq 0.$$

From this, it is easy to deduce that $\mu_1(G) = \mu_2(G)$, which implies that $\omega(\mu_1 \cdot d) = \omega(\mu_2 \cdot d)$. Moreover, ω is positive. Now, the space $\{\mu \cdot d : \mu \in \mathcal{M}_{00}(G)\}$ is majorizing $(E, d)_{\mathcal{M}}$ by definition. Thus, we can apply Kantorovich Theorem (Theorem 2.2.7) to extend ω to a positive functional ψ defined on $(E, d)_{\mathcal{M}}$. We claim that ψ is a normalized integral which is measurably invariant on $\{\mu \cdot d : \mu \in \mathcal{M}_{00}(G)\}$. Indeed, let $\mu \in \mathcal{M}_{00}^{1}(G)$. Then

$$\psi(\mu \cdot d) = \omega(\mu \cdot d) = \mu(G) = 1$$

and

$$\psi(d) = \psi(\delta_e \cdot d) = \omega(\delta_e \cdot d) = \delta_e(G) = 1.$$

The proof that b) implies a) is straightforward. In fact, let $\mu \in \mathcal{M}_{00}(G)$ such that $\mu \cdot d \geq 0$ and let ψ be a normalized integral on $(E,d)_{\mathcal{M}}$ which is measurably invariant on $\{\mu \cdot d : \mu \in \mathcal{M}_{00}(G)\}$. Then

$$0 \leq \psi(\mu \cdot d) = \mu(G)$$

by Proposition 6.2.16. Thus, d has the measurably translate property.

Now we show that b) implies c). Let ψ be a normalized integral on $(E,d)_{\mathcal{M}}$ which is measurably invariant on $\{\mu \cdot d : \mu \in \mathcal{M}_{00}(G)\}$. Note that ψ is continuous with respect to the $\overline{p}_d$-norm, and it has operator norm equal to 1. Let $\mu \in \mathcal{M}_{00}(G)_+$. On the one hand,

$$\overline{p}_d(\mu \cdot d) \leq ||\mu||_{\text{TV}} = \mu(G).$$

On the other hand,

$$\overline{p}_d(\mu \cdot d) \geq \psi(\mu \cdot d) = \mu(G).$$

Therefore , $\overline{p}_d(\mu \cdot d) = \mu(G)$ as wished.

In conclusion, we give the proof that c) implies b). By Jordan Decomposition Theorem, every $\mu \cdot d$ in $\{\mu \cdot d : \mu \in \mathcal{M}_{00}(G)\}$ can be written as the difference of two vectors $\mu_1 \cdot d$ and $\mu_2 \cdot d$ in $\{\mu \cdot d : \mu \in \mathcal{M}_{00}(G)_+\}$. In fact,

$$\mu \cdot d = (\mu_1 - \mu_2) \cdot d = \mu_1 \cdot d - \mu_2 \cdot d.$$

This means that $\{\mu \cdot d : \mu \in \mathcal{M}_{00}(G)_+\}$ is the positive cone of the order vector space $\{\mu \cdot d : \mu \in \mathcal{M}_{00}(G)\}$. Let now define the positive homogeneous additive map

$$\omega_+ : \{\mu \cdot d : \mu \in \mathcal{M}_{00}(G)_+\} \longrightarrow \mathbf{R}, \quad \mu \cdot d \longmapsto \omega_+(\mu \cdot d) = \overline{p}_d(\mu \cdot d) = \mu(G).$$

By [AT07, Lemma 1.26 (1)], the map ω_+ extends to a positive linear functional

$$\omega : \{\mu \cdot d : \mu \in \mathcal{M}_{00}(G)\} \longrightarrow \mathbf{R}, \quad \mu \cdot d \longmapsto \omega(\mu \cdot d)$$

via the formula $\omega(\mu \cdot d) = \omega_+(\mu_1 \cdot d) - \omega_+(\mu_2 \cdot d)$, where $\mu_1, \mu_2 \in \mathcal{M}_{00}(G)_+$ are such that $\mu = \mu_1 - \mu_2$. Now we can utilise Kantorovich Theorem (Theorem 2.2.7) to extend ω to a positive functional ψ defined on $(E,d)_{\mathcal{M}}$. We claim that ψ is a normalized integral measurably invariant on $\{\mu \cdot d : \mu \in \mathcal{M}_{00}(G)\}$. Indeed,

$$\psi(\mu \cdot d) = \omega_+(\mu \cdot d) = \overline{p}_d(\mu \cdot d) = \mu(G) = 1$$

for every $\mu \in \mathcal{M}_{00}^1(G)$, and

$$\psi(d) = \psi(\delta_e \cdot d) = \omega_+(\delta_e \cdot d) = \overline{p}_d(\delta_e \cdot d) = \delta_e(G) = 1.$$

Therefore, point b) holds. □

Remark 6.2.30. The equivalence between a) and b) is still valid for positive $\mathcal{M}_{00}(G)$-module.

An application of the previous proposition gives:

Corollary 6.2.31. *Let E be a positive $\mathcal{M}_{00}(G)$-module, and let d be a non-zero positive vector in E. If $(E,d)_{\mathcal{M}}$ admits a measurably invariant normalized integral, then d has the measurably translate property.*

Proof. By hypothesis, $(E,d)_{\mathcal{M}}$ admits a normalized integral which is measurably invariant on $\{\mu \cdot d : \mu \in \mathcal{M}_{00}(G)\}$. Therefore, point b) of Proposition 6.2.29 is satisfied. This implies that d has the measurably translate property. □

Definition 6.2.32. Let E be a positive $\mathcal{M}_{00}(G)$-module. We say that G has the **measurably translate property for** E if every non-zero positive vector d of E has the measurably translate property.

Example 6.2.33. (Examples of the measurably translate property)

1) Every locally compact group G has the measurably translate property for $\mathcal{C}_{00}(G)$ by point 1) of Example 6.2.22 and Corollary 6.2.31.

2) Let G be a locally compact group and let $(\sigma, \mathcal{H})$ be a unitary representation of G on a Hilbert space $\mathcal{H}$. Then the induced representation Ad_σ of G on $\mathrm{TC}(\mathcal{H})$ has the measurably translate property by point 2) of Example 6.2.22 and Corollary 6.2.31.

Finally, it is worth mentioning that the measurably translate property for $\mathcal{C}^b_{ru}(G)$ implies amenability. In fact,

Corollary 6.2.34. *Suppose that G has the measurably translate property for $\mathcal{C}^b_{ru}(G)$, then G is amenable.*

Proof. Suppose that G has the measurably translate property for $\mathcal{C}^b_{ru}(G)$. Then G has the translate property for $\mathcal{C}^b_{ru}(G)$ by Proposition 6.2.28. Therefore, G is amenable by Corollary 4.2.13. □

Scholium 6.2.35. Paterson defined in [P88, (6.42)] a *generalized translate property* claiming that it could be a good definition for a topological version of the translate property for locally compact groups. The notion defined by Paterson is equivalent to our measurably translate property. The *hard* part of this equivalence can be shown directly using the notion of measurably invariant normalized integral.

Another topological translate property for locally compact groups was defined and studied by Jenkins under the name of *property* (P) in [J74]. Moreover, in a later publication, he claimed that this property implies the existence of invariant normalized integrals defined on particular linear subspaces of $L^\infty(G)$, see [J80, Proposition 5]. Sadly, the proof is incorrect because it erroneously claimed that G acts continuously on $L^\infty(G)$, which is not true in general. In any case, Jenkins's translate property is also equivalent to the measurably translate property.

6.3 Application to locally compact groups

In this section, we finally apply the positive $\mathcal{M}_{00}(G)$-module theory developed previously. In the first part, we use measurably invariant integrals to show the equivalence of the invariant normalized integral property on different Banach lattices. In the second, we study the translate property on different spaces. In particular, we give a complete answer to Greenleaf's question for locally compact groups. The third and last part aims to get rid of some technical details when working with the fixed-point property for cones for locally compact groups applying the results of the previous subsections.

6.3.A. The invariant normalized integral property for locally compact groups. The entire subsection is dedicated to the demonstration of Theorem 6.3.4. We start by stating and proving a small lemma that solve the technical core of the theorem. After that, two propositions prove intermediate results. The proof of the theorem can be founded at the end of the subsection.

But first recall that

$$\mathcal{C}_{00}^1(G) = \{\phi \in \mathcal{C}_{00}(G) : \phi \geq 0 \text{ and } ||\phi||_1 = 1\}$$

and note that $\mathcal{C}_{00}^1(G)$ is a semigroup under convolution between functions. Moreover, $\mathcal{C}_{00}^1(G)$ can be identified to a subsemigroup of $\mathcal{M}_{00}^1(G)_a = \mathcal{M}_{00}^1(G) \cap \mathcal{M}_{00}(G)_a$.

Lemma 6.3.1. *Let $f \in \mathcal{C}_{ru}^b(\mathrm{G})_+$ be a non-zero function and let $\phi \in \mathcal{C}_{00}^1(\mathrm{G})$. Consider the subspace*

$$D = \operatorname{span}_{\mathbf{R}} \left\{ \psi * h : \psi \in \mathcal{C}_{00}(G) \text{ and } h \in \mathcal{C}_{ru}^b(G, f)_{\mathcal{M}} \right\} \subset \mathcal{C}_{ru}^b(G, \phi * f).$$

Then every positive G-invariant $p_{\phi*f}$-continuous functional I on D is also measurably invariant, i.e., $\mathrm{I}(\mu * h) = \mathrm{I}(h)$ for every $\mu \in \mathcal{M}_{00}^1(G)$ and $h \in D$.

Proof. First of all, note that the action of G on the closure $\overline{D}^{p_{\phi*f}} \subset \mathcal{C}_{ru}^b(G, \phi * f)$ is orbitally continuous with respect to the $p_{\phi*f}$-norm because of point a) of Lemma 6.2.13 and [G17, Lemma 4.1.9]. Moreover, the functional I extends uniquely to a positive G-invariant linear functional on $\overline{D}^{p_{\phi*f}}$, since it is positive, G-invariant and $p_{\phi*f}$-continuous on D. Now take $h \in \overline{D}^{p_{\phi*f}}$ and $\mu \in \mathcal{M}_{00}^1(\mathrm{G})$ and consider the function

$$F : (G, \mu) \longrightarrow \overline{D}^{p_{\phi*f}}, \quad g \longmapsto F(g) = gh.$$

The map F is Bochner integrable, because it is continuous and the real integral

$$\int_G p_{\phi*f}(F(g)) d\mu(g) = \int_G p_{\phi*f}(gh) d\mu(g) \leq p_{\phi*f}(h)\mu(G)$$

is finite. Therefore, the Bochner integral $\int_G F(g)\mu(g) \in \overline{D}^{p_{\phi*f}}$, see [DE09, Appendix B]. The $p_{\phi*f}$-continuity of I on $\overline{D}^{p_{\phi*f}}$ gives the following:

$$\begin{aligned}\mathrm{I}(\mu * h) &= \mathrm{I}\left(\int_G gh\, d\mu(g)\right) = \int_G \mathrm{I}\,(gh)\, d\mu(g)\\ &= \int_G \mathrm{I}\,(h)\, d\mu(g) = \mathrm{I}(h)\mu(G) = \mathrm{I}(h).\end{aligned}$$

This proves that I is also measurably invariant on D.

□

We now prove the above-cited two propositions.

Proposition 6.3.2. *Suppose that G has the measurably invariant normalized integral property for $\mathcal{C}^b_{ru}(G)$, then G has the measurably invariant normalized integral property for $L^\infty(G)$.*

Proof. Let $f \in L^\infty(G)$ be a non-zero positive function and let $\phi \in \mathcal{C}^1_{00}(G)$. Define the linear operator

$$T : L^\infty(G, f)_{\mathcal{M}} \longrightarrow \mathcal{C}^b_{ru}(G, \phi * f)_{\mathcal{M}}, \quad h \longmapsto \phi * h.$$

First of all, we check that T is well-defined. It is clear that $\phi * h \in \mathcal{C}^b_{ru}(G)$ for every $h \in L^\infty(G)$ by Cohen-Hewitt Factorization Theorem. Suppose that $h \in L^\infty(G, f)_{\mathcal{M}}$, then there is $\lambda \in \mathcal{M}_{00}(G)_+$ such that $|h| \leq \lambda * f$. Consequently,

$$|\phi * h| \leq \phi * |h| \leq \underbrace{\phi * \lambda}_{\in \mathcal{C}_{00}(G)} * f,$$

and so $\phi * h \in \mathcal{C}^b_{ru}(G, (\phi * \lambda) * f)_{\mathcal{M}} = \mathcal{C}^b_{ru}(G, \phi * f)_{\mathcal{M}}$ by point d) of Lemma 6.2.3.

Let now I be a measurably invariant normalized integral on $\mathcal{C}^b_{ru}(G, \phi * f)_{\mathcal{M}}$ and define on $L^\infty(G, f)_{\mathcal{M}}$ the functional $\bar{\mathrm{I}} = \mathrm{I} \circ T$. We claim that $\bar{\mathrm{I}}$ is a measurably invariant normalized integral for $L^\infty(G, f)_{\mathcal{M}}$. Clearly, $\bar{\mathrm{I}}$ is linear as a composition of linear maps and it is normalized because $\bar{\mathrm{I}}(f) = \mathrm{I}(\phi * f) = 1$. Therefore, it is left to show that $\bar{\mathrm{I}}$ is $\mathcal{M}^1_{00}(G)$-invariant. To this end, let $\mu \in \mathcal{M}^1_{00}(G)$, $h \in L^\infty(G, f)_{\mathcal{M}}$ and take a bounded approximate identity $(e_\alpha)_\alpha$ for $L^1(G)$ in $\mathcal{C}^1_{00}(G)$ with decreasing support. Then

$$\begin{aligned}\bar{\mathrm{I}}(\mu * h) &= \mathrm{I}(\phi * \mu * h) = \mathrm{I}\left(\lim_\alpha (\phi * \mu) * e_\alpha * h\right) = \lim_\alpha \mathrm{I}\left((\phi * \mu) * e_\alpha * h\right)\\ &= \lim_\alpha \mathrm{I}\left(\phi * e_\alpha * h\right) = \mathrm{I}\left(\lim_\alpha \phi * e_\alpha * h\right) = \mathrm{I}\left(\phi * h\right) = \bar{\mathrm{I}}(h),\end{aligned}$$

since $\lim_\alpha (\phi * \mu) * e_\alpha * h = (\phi * \mu) * h$ and $\lim_\alpha \phi * e_\alpha * h = \phi * h$ in $p_{\phi*f}$-norm by Lemma 6.2.11. This shows that $\bar{\mathrm{I}}$ is measurably invariant and concludes the proof. □

Proposition 6.3.3. *Suppose that G has the invariant normalized integral property for $\mathcal{C}^b_{ru}(G)$, then G has the measurably invariant normalized integral property for $\mathcal{C}^b_{ru}(G)$.*

Proof. Let $f \in \mathcal{C}^b_{ru}(G)_+$ be a non-zero function and chose a $\phi \in \mathcal{C}^1_{00}(G)$.
Consider the linear operator

$$T : \mathcal{C}^b_{ru}(G, f)_{\mathcal{M}} \longrightarrow \mathcal{C}^b_{ru}(G, \phi * f),\ h \longmapsto \phi * h$$

and note that it is well-defined because if $h \in \mathcal{C}^b_{ru}(G, f)_{\mathcal{M}}$, then there is $\mu \in \mathcal{M}_{00}(G)_+$ such that $|h| \leq \mu * f$ which implies that

$$|\phi * h| \leq \phi * |h| \leq \underbrace{(\phi * \mu)}_{\in \mathcal{C}_{00}(G)} * f.$$

Hence, $\phi * h \in \mathcal{C}^b_{ru}(G, (\phi * \mu) * f) = \mathcal{C}^b_{ru}(G, \phi * f)$ by point b) of Lemma 6.2.3.
Moreover, note that T maps $\mathcal{C}^b_{ru}(G, f)_{\mathcal{M}}$ into

$$D = \operatorname{span}_{\mathbf{R}} \left\{ \psi * h : \psi \in \mathcal{C}_{00}(G) \text{ and } h \in \mathcal{C}^b_{ru}(G, f)_{\mathcal{M}} \right\} \subset \mathcal{C}^b_{ru}(G, \phi * f).$$

By hypothesis, there is an invariant normalized integral I on $\mathcal{C}^b_{ru}(G, \phi * f)$. We define the linear functional $\bar{\mathrm{I}}$ on $\mathcal{C}^b_{ru}(G, f)_{\mathcal{M}}$ as $\bar{\mathrm{I}} = \mathrm{I} \circ T$, and we claim that it is a measurably invariant normalized integral. Indeed, $\bar{\mathrm{I}}$ is normalized because $\bar{\mathrm{I}}(f) = \mathrm{I}(\phi * f) = 1$ and linear as composition of linear maps. Thus, it is left to show that $\bar{\mathrm{I}}$ is measurably invariant. To this end, let $\mu \in \mathcal{M}^1_{00}(G)$, $h \in \mathcal{C}^b_{ru}(G, f)_{\mathcal{M}}$ and take a bounded approximate identity $(e_\alpha)_\alpha$ for $L^1(G)$ in $\mathcal{C}^1_{00}(G)$ with decreasing support. Note that I is measurably invariant when restricted to D by Lemma 6.3.1. Then

$$\begin{aligned} \bar{\mathrm{I}}(\mu * h) &= \mathrm{I}(\phi * \mu * h) = \mathrm{I}\left(\lim_\alpha (\phi * \mu) * e_\alpha * h\right) = \lim_\alpha \mathrm{I}\left((\phi * \mu) * e_\alpha * h\right) \\ &= \lim_\alpha \mathrm{I}\left(\phi * e_\alpha * h\right) = \mathrm{I}\left(\lim_\alpha \phi * e_\alpha * h\right) = \mathrm{I}\left(\phi * h\right) = \bar{\mathrm{I}}(h), \end{aligned}$$

since $\lim_\alpha (\phi * \mu) * e_\alpha * h = (\phi * \mu) * h$ and $\lim_\alpha \phi * e_\alpha * h = \phi * h$ in $p_{\phi * f}$-norm by Lemma 6.2.11. This shows that $\bar{\mathrm{I}}$ is measurably invariant and concludes the proof. □

We are finally ready to give the proof of Theorem 6.3.4.

Theorem 6.3.4. *Let G be a locally compact group. If G has the invariant normalized integral property for one of the following Banach lattices*

$$L^\infty(G),\ \mathcal{C}^b(G),\ \mathcal{C}^b_{lu}(G),\ \mathcal{C}^b_{ru}(G) \text{ or } \mathcal{C}^b_u(G),$$

then G has the invariant normalized integral property for all the others.

Proof. Firstly, if G has the invariant normalized integral property for $L^\infty(G)$, then G has the invariant normalized integral property for all the other function spaces.

Secondly, the invariant normalized integral property for $\mathcal{C}_u^b(G)$ implies the invariant normalized integral property for $\mathcal{C}_{ru}^b(G)$. Indeed, let $f \in \mathcal{C}_{ru}^b(G)_+$ be a non-zero function. Fix a non-zero $\phi \in \mathcal{C}_{00}(G)$, and define the positive linear operator

$$T : \mathcal{C}_{ru}^b(G, f) \longrightarrow \mathcal{C}_u^b(G, f * \phi), \quad h \longmapsto T(h) = h * \phi.$$

The operator T is well-defined thanks to the Hewitt-Cohen Factorization Theorem. By hypothesis, there is an invariant normalized integral I on $\mathcal{C}_u^b(G, \phi * f)$. Then the functional $\bar{\mathrm{I}} = \mathrm{I} \circ T$ is an invariant normalized integral for $\mathcal{C}_{ru}^b(G, f)$.

Therefore, it is enough to show that the invariant normalized integral property for $\mathcal{C}_{ru}^b(G)$ implies the invariant normalized integral property for $L^\infty(G)$. Thus, suppose that G has the invariant normalized integral property for $\mathcal{C}_{ru}^b(G)$. Then G has the measurably invariant normalized integral property for $\mathcal{C}_{ru}^b(G)$ by Proposition 6.3.3. Therefore, G also has the measurably invariant normalized integral property for $L^\infty(G)$ by Proposition 6.3.2. We can finally conclude that G has the invariant normalized integral property for $L^\infty(G)$ thanks to Corollary 6.2.25. □

Remark 6.3.5. If we fix $f = \mathbf{1}_G$ in every proof of this subsection, then we obtain a proof of [G69, Theorem 2.2.1]. Hence, the case of amenability. This is because $\mathbf{1}_G$ is a fixed-point for the action by convolution of the semigroup $\mathcal{M}_{00}^1(G)$ on $L^\infty(G)$, and because the space $\mathcal{C}_{00}^1(G)$ is $||\cdot||_1$-dense in

$$\mathrm{P}(G) = \left\{ \phi \in L^1(G) : ||\phi||_1 = 1 \text{ and } \phi \geq 0 \right\}.$$

Let G be a locally compact group with the $\mathscr{U}$-fixed-point property for cones for $\mathscr{U} \in \{\mathscr{F}, \mathscr{R} \vee \mathscr{L}, \mathscr{R}, \mathscr{L}, \mathscr{R} \wedge \mathscr{L}\}$. Suppose that $\mathscr{U}'$ is another functionally invariant uniformity for G. Then we can observe that G also has the $\mathscr{U}'$-fixed-point property for cones. Indeed, G has the $\mathscr{F}$-fixed-point property for cones by this last theorem, and this suffices to ensure that G has the $\mathscr{U}'$-fixed-point property for cones, since $\mathscr{F}$ is the finest functionally invariant uniformity for G. Bearing this observation in mind, we state the following definition since we are primarily interested in the five standard uniform structures.

Definition 6.3.6. Let G be a locally compact group. Then we say that G has the **fixed-point property for cones** if G has the $\mathscr{U}$-fixed-point property for cones for one, and hence for all, of the following uniform structures:

$$\mathscr{F}, \ \mathscr{R} \vee \mathscr{L}, \ \mathscr{R}, \ \mathscr{L} \text{ or } \mathscr{R} \wedge \mathscr{L}.$$

The following corollary motivates the choice of the previous definition.

Corollary 6.3.7. *For every orbitally continuous action of a locally compact group G with the fixed-point property for cones on a topological set X, the induced representation of G on $\mathcal{C}^b(X)$ has the invariant normalized integral property.*

Proof. For every non-zero positive function $f \in \mathcal{C}^b(X)$, there is a positive equivariant linear operator T_f from $\mathcal{C}^b(X)$ to $\mathcal{C}^b(G)$ such that $T_f(f) > 0$ by point a) of Lemma 5.2.9. We can now use the invariant integral property of G for $\mathcal{C}^b(G)$ to conclude. □

6.3.B. The translate property. After seeing how the invariant normalized integral property for locally compact groups is so generous, two questions arise spontaneously about the translate property. The first one is: for which Banach lattices is the translate property equivalent to the invariant normalized integral property? While the second is: for which Banach lattices the translate property is equivalent?

Note that the first question is a generalized version of Greenleaf's question. We will be able to answer it only thanks to the positive answer to the original Greenleaf's question, which is presented in the following theorem.

Theorem 6.3.8. *Let G be a locally compact group. If G has the translate property for $\mathcal{C}^b_{ru}(G)$, then G has the invariant normalized integral property for $\mathcal{C}^b_{ru}(G)$.*

Proof. Let $f \in \mathcal{C}^b_{ru}(G)_+$ be a non-zero function and fix a $\phi \in \mathcal{C}^1_{00}(G)$. Consider the functional given by

$$\omega : \operatorname{span}_{\mathbf{R}} \{g(\phi * f) : g \in G\} \longrightarrow \mathbf{R}, \quad \sum_{j=1}^{n} t_j g_j(\phi * f) \longmapsto \omega\left(\sum_{j=1}^{n} t_j g_j(\phi * f)\right) = \sum_{j=1}^{n} t_j.$$

Thanks to the translate property, ω is well-defined. Moreover, ω is positive, G-invariant and $p_{\phi * f}$-continuous. Now we have the inclusion:

$$\operatorname{span}_{\mathbf{R}} \{g(\phi * f) : g \in G\} \subset \operatorname{span}_{\mathbf{R}} \left\{\psi * h : \psi \in \mathcal{C}_{00}(G) \text{ and } h \in \mathcal{C}^b_{ru}(G,f)_{\mathcal{M}}\right\} = D.$$

Note that the action of G on D is orbitally continuous with respect to the $p_{\phi*f}$-norm by point a) of Lemma 6.2.13. Because G is amenable (Corollary 4.2.13), we can apply [L90, Theorem 3.2 (j)] and extend ω to a positive linear G-invariant $p_{\phi*f}$-continuous functional $\overline{\omega}$ defined on all D. Note that the functional $\overline{\omega}$ is also measurably invariant by Lemma 6.3.1. Consider the positive linear operator

$$T : \mathcal{C}^b_{ru}(G,f) \longrightarrow D, \quad h \longmapsto \phi * h$$

and define the functional $\mathrm{I} = \overline{\omega} \circ T$ on $\mathcal{C}^b_{ru}(G,f)$. We claim that I is an invariant normalized integral. Indeed, I is positive as composition of positive operators and

$$\mathrm{I}(f) = \overline{\omega}(\phi * f) = \omega(\phi * f) = 1.$$

Therefore, it is left to show that I is invariant. Let $g \in G$, $h \in \mathcal{C}^b_{ru}(G,f)$ and take a bounded approximate identity $(e_\alpha)_\alpha$ for $L^1(G)$ in $\mathcal{C}^1_{00}(G)$ with decreasing support. Then

$$\begin{aligned}\mathrm{I}(gh) &= \overline{\omega}\left(\phi * (gh)\right) = \overline{\omega}\left(\phi_g * h\right) = \overline{\omega}\left(\lim_\alpha \phi_g * e_\alpha * h\right) \\ &= \lim_\alpha \overline{\omega}\left(\phi * e_\alpha * h\right) = \overline{\omega}\left(\lim_\alpha \phi * e_\alpha * h\right) = \overline{\omega}(\phi * h) = \mathrm{I}(h),\end{aligned}$$

since $\lim_\alpha \phi_g * e_\alpha * h = \phi_g * h$ and $\lim_\alpha \phi * e_\alpha * h = \phi * h$ in $p_{\phi*f}$-norm by Lemma 6.2.11. Therefore, I is invariant, and G has the invariant normalized integral property for $\mathcal{C}^b_{ru}(G)$.

□

Remark 6.3.9. Employing the same proof of Theorem 6.3.8, it is also possible to show that the invariant normalized integral property for $\mathcal{C}_0(G)$ is equivalent to the translate property for $\mathcal{C}_0(G)$ if G is an amenable locally compact group.

Thanks to this last result, it is possible to solve Greenleaf's question for all classical Banach spaces.

Theorem 6.3.10. *Let G be a locally compact group and let E be one of the following Banach lattices*

$$L^\infty(G),\ \mathcal{C}^b(G),\ \mathcal{C}^b_{ru}(G),\ \mathcal{C}^b_{lu}(G) \text{ or } \mathcal{C}^b_u(G).$$

Then G has the translate property for E if and only if G has the invariant normalized integral property for E.

Proof. If G has the invariant normalized property for E, then G has the translate property for E by Corollary 4.2.8. So, let's prove the other direction.

The case $E = \mathcal{C}^b_{ru}(G)$ has already been solved in Theorem 6.3.8.

Suppose that G has the translate property for $E \in \{L^\infty(G), \mathcal{C}^b(G)\}$. Then G has the translate property for $\mathcal{C}^b_{ru}(G)$. Therefore, G has the invariant normalized integral property for $\mathcal{C}^b_{ru}(G)$. This implies that G has the invariant integral property for E by Theorem 6.3.4.

Let's move onto the case $E = \mathcal{C}^b_{lu}(G)$. We claim that G also has the translate property for $L^\infty(G)$. Indeed, let $f \in L^\infty(G)$ be a non-zero function and let $t_1, ..., t_n \in \mathbf{R}$ and $g_1, ..., g_n \in G$ such that $\sum_{j=1}^n t_j g_j f \geq 0$. Chose a non-zero positive $\phi \in \mathcal{C}_{00}(G)$. Since taking the convolution with a positive function is a positive operation, the inequality

$$\left(\sum_{j=1}^n t_j g_j f\right) * \phi = \sum_{j=1}^n t_j g_j \underbrace{(f * \phi)}_{\in \mathcal{C}^b_{lu}(G)} \geq 0$$

holds. Therefore, $\sum_{j=1}^n t_j \geq 0$. We can hence conclude as in the previous case.

Finally, let $E = \mathcal{C}^b_u(G)$. Then G also has the translate property for $\mathcal{C}^b_{ru}(G)$ as $f * \phi \in \mathcal{C}^b_u(G)$ for every $\phi \in \mathcal{C}_{00}(G)$ and $f \in \mathcal{C}^b_{ru}(G)$. Thus, we can conclude as above. □

Finally, the answer to the second question we asked is contained in the following theorem:

Theorem 6.3.11. *Let G be a locally compact group. If G has the translate property for one of the following Banach lattices*

$$L^\infty(G),\ \mathcal{C}^b(G),\ \mathcal{C}^b_{lu}(G),\ \mathcal{C}^b_{ru}(G) \text{ or } \mathcal{C}^b_u(G),$$

then G has the translate property for all the others.

Proof. The proof is only a combination of Theorems 6.3.4 and 6.3.10. □

6.3.C. Another functional perspective. In Section 5.3, we distilled the essence of the invariant normalized integral property in a functional way using the theory of G-dominated normed Riesz spaces developed in Chapter 3. Here, we push this perspective even further to get rid of some technical details when working with the fixed-point property for cones for locally compact groups. The results here developed will be helpful later to study the class of locally compact groups which enjoy the fixed-point property for cones.

The following lemma shows that it is possible to suppose just continuity in the definition of the fixed-point property for cones for locally compact groups. In particular, the generalization of the fixed-point property for cones proposed by Monod in [M17, Example 38] and our generalization coincide for locally compact groups. In fact, point b) of the following lemma is exactly the generalization suggested by Monod.

A representation of a group on a locally convex vector space is said bounded if every orbit is bounded.

Lemma 6.3.12. *Let G be a locally compact group. The following assertions are equivalent:*

a) the group G has the fixed-point property for cones;

b) every bounded orbitally continuous representation of G on a non-empty weakly complete proper convex cone C in a locally convex vector space E which is of cobounded type has a non-zero fixed-point.

Proof. We start showing that a) implies b). Suppose that G has a bounded orbitally continuous representation on a non-empty weakly complete proper cone C in a locally convex space E which is of cobounded type. Then the representation of G on C is also locally bounded $(\mathscr{F}, \mathscr{U}_c)$-uniformly continuous. To guarantee a non-zero fixed-point in C, it suffices that G has the invariant normalized integral property for $\mathcal{C}^b(G)$ by Theorem 5.2.1 with $\mathscr{U} = \mathscr{F}$. Nevertheless, the fixed-point property for cones for locally compact groups is equivalent to the invariant normalized integral property by Theorem 6.3.4. Therefore, there is a non-zero fixed-point in C.

Let's prove the reverse implication. We show that G has the translate property for $\mathcal{C}^b_{ru}(G)$, which is equivalent to having the fixed-point property for cones for locally compact groups. Let $f \in \mathcal{C}^b_{ru}(G)_+$ be a non-zero function and let $E = \mathrm{span}_{\mathbf{R}}\{gf : g \in G\}$ together with the supremum norm. Consider the cone $C = (E^*)_+$ in the locally convex vector space $(E^*, \textit{weak-}*)$. Note that the adjoint action of G on E^* is orbitally continuous for the weak-* topology and $(E^*, \textit{weak-}*)' = E$. Therefore, there is a non-zero positive G-invariant functional ψ on E. By Proposition 4.2.4, this is sufficient to ensure that f has the translate property. □

We can improve Theorem 5.3.1 for locally compact groups.

Theorem 6.3.13. *Let G be a locally compact group. Then the following assertions are equivalent:*

a) the group G has the fixed-point property for cones;

b) *every continuous representation π of G on a normed Riesz space E by positive linear isometries has the invariant normalized integral property;*

c) *every continuous representation π of G on a Banach lattice E by positive linear isometries has the invariant normalized integral property.*

Proof. Let's look at the proof of a) implies b). Let π be a continuous representation of G by positive linear isometries on a normed Riesz space E and let $d \in E$ be a non-zero positive vector. Then the restriction of π on (E, d) is continuous. Therefore, the adjoint representation of G on the cone $(E, d)^*_+$ in the locally convex space $(E, d)^*$ is continuous. We can apply point b) of Lemma 6.3.12 to obtain an invariant normalized integral defined on (E, d). Thus, G has the invariant normalized integral property for E.

The proof that b) implies c) is straightforward as a Banach lattice is, in particular, a normed Riesz space.

It is left to show that c) implies a). Nonetheless, this is true because c) implies that G has the invariant normalized integral property for the Banach lattice $\mathcal{C}^b_{ru}(G)$, and this implies a) by Theorem 5.2.1. □

Example 6.3.14. 1) Let G be a locally compact group with the fixed-point property for cones. Then G has the invariant normalized integral property for $L^p(G)$ for every $p \in [1, \infty]$. This is thanks to point c) of Theorem 6.3.13. However, note that $L^p(G)$ admits an invariant functional if and only if G is compact as explained in Proposition 5.1.

2) Let G be a locally compact group with the fixed-point property for cones, and suppose that it has a unitary representation σ on a Hilbert lattice $\mathcal{H}$. Then the representation Ad_σ of G on the Banach lattice $\mathrm{HS}(\mathcal{H})$ has the invariant normalized integral property.

Corollary 6.3.15. *Let G be a locally compact group. Then G has the fixed-point property for cones if and only if it has the abstract continuous translate property.*

Proof. Let π be a continuous representation of G on a Banach lattice E by positive linear isometries and suppose that G has the fixed-point property for cones. Then G has the invariant normalized integral property for E by Theorem 6.3.13. Consequently, G has the translate property for E by Corollary 4.2.8. Therefore, G has the abstract continuous translate property.

Now, suppose that G has the abstract continuous translate property. Then G has the translate property for $\mathcal{C}^b_{ru}(G)$ by Theorem 5.3.5. However, for locally compact groups, this is equivalent to having the invariant normalized integral property for $\mathcal{C}^b_{ru}(G)$. Hence, G has the fixed-point property for cones. □

Remarks 6.3.16. 1) We mentioned in the introduction that one of our goals was also to unify the theories developed by Monod in [M17] and by Jenkins in [J76]. We can assert that this objective is achieved. Indeed, the fixed-point property studied by Jenkins under the name of *property F* is:

Suppose that a locally compact group G has a linear representation on a vector space E and that there is $v \in E$ and $\phi \in E^*$ such that the function $g \longmapsto \phi(gv)$ is a non-zero positive element of $L^\infty(G)$. Then there exists a net $(\phi_\alpha)_\alpha \subset E^*$ of finite positive G-translates of ϕ such that for every $g \in G$ the net $(\phi_\alpha(gv))_\alpha \subset \mathbf{R}$ converges to 1.

It is straightforward to see that *property F* implies the abstract continuous translate property, which is equivalent to the fixed-point property for cones by Corollary 6.3.15 (see also [J76, Remark p.348]). Now, suppose that G has the fixed-point property for cones. Therefore, G also has the invariant normalized integral property for $L^\infty(G)$ by Theorem 6.3.13. Take E, v and ϕ as in the hypothesis of the *property F* and consider the vector subspace $V = \mathrm{span}_{\mathbf{R}}\{gv : g \in G\} \subset E$. Then V^* endowed with the weak-* topology is a locally convex vector space such that the cone of positive linear functionals on V is closed. An argument similar to that of Theorem 5.2.1 implies the existence of a net as asked by the *property F*.

2) In Theorem 6.3.13, it is possible to check conditions b) and c) only for G-dominated normed Riesz spaces and asymptotically G-dominated Banach lattices, respectively. This is because they are the only spaces in which we are interested.

Chapter 7

Hereditary Properties

The chapter's primary goal is to understand and investigate the classes of topological groups with the fixed-point property for cones and with the translate property.

We saw in the previous chapter that these two families of groups coincide in the locally compact case. Although we do not have an example to guarantee it, they are a priori different for topological groups. For this reason, we study the two families of groups separately in the first part of the chapter dedicated to topological groups. After that, there will be no further need to treat the two notions independently in the second one when we deal with locally compact groups.

Finally, we study obstructions that prevent a group from having the fixed-point property for cones and the translate property.

7.1 The topological case

We start with one of the few general results that it is possible to state about functionally invariant uniformities.

Proposition 7.1.1. *Let G and H be two topological groups and let $\mathscr{U}_G$ and $\mathscr{U}_H$ be functionally invariant uniformities for G and H, respectively. Suppose that there is a uniformly continuous epimorphism of groups*

$$\phi : (G, \mathscr{U}_G) \longrightarrow (H, \mathscr{U}_H), \quad g \longmapsto \phi(g).$$

If G has the $\mathscr{U}_G$-fixed-point property for cones (resp. the translate property for $\mathcal{C}_u^b(G, \mathscr{U}_G)$), then H has the $\mathscr{U}_H$-fixed-point property for cones (resp. the translate property for $\mathcal{C}_u^b(H, \mathscr{U}_H)$).

We gives the proof only for the $\mathscr{U}$-fixed-point property for cones, since the proof for the translate property follows from it.

Proof of Proposition 7.1.1. We want to show that H has the invariant normalized integral property for $\mathcal{C}_u^b(H, \mathscr{U}_H)$. To this end, define the linear operator

$$T : \mathcal{C}_u^b(H, \mathscr{U}_H) \longrightarrow \mathcal{C}_u^b(G, \mathscr{U}_G), \quad f \longmapsto T(f) = f \circ \phi.$$

First, T is well-defined, as ϕ is uniformly continuous. Second, it is strictly positive. Indeed, if $f \in \mathcal{C}_u^b(H, \mathscr{U}_H)$ is a non-zero positive function, then there is $x \in H$ such that $f(x) > 0$. Thanks to the surjectivity of ϕ, there is $g \in G$ such that $\phi(g) = x$. Thus,

$$T(f)(g) = f(\phi(g)) = f(x) > 0.$$

Finally, ϕ is equivariant. To prove this last claim, let $x \in H$. Then there is $g \in G$ such that $\phi(g) = x$. Therefore,

$$\begin{aligned} T(xf)(a) &= T\left(\phi(g)f\right)(a) \\ &= (\phi(g)f)(\phi(a)) \\ &= f(\phi(g)^{-1}\phi(a)) \\ &= f(\phi(g^{-1}a)) = gT(f)(a), \end{aligned}$$

for every $f \in \mathcal{C}_u^b(H, \mathscr{U}_H)$ and every $a \in G$.

Now the function $T(f) \in \mathcal{C}_u^b(G, \mathscr{U}_G)$ is non-zero and positive for every non-zero positive function $f \in \mathcal{C}_u^b(H, \mathscr{U}_H)$. Therefore, the operator T maps $\left(\mathcal{C}_u^b(H, \mathscr{U}_H), f\right)$ into $\left(\mathcal{C}_u^b(G, \mathscr{U}_G), T(f)\right)$. On this last space, there is an invariant normalized integral I by hypothesis. Consequently, the functional $\bar{\mathrm{I}} = \mathrm{I} \circ T$ is an invariant normalized integral on $\left(\mathcal{C}_u^b(H, \mathscr{U}_H), f\right)$. This shows that H has the invariant normalized integral property for $\mathcal{C}_u^b(H, \mathscr{U}_H)$, and hence that H has the $\mathscr{U}_H$-fixed-point property for cones by Theorem 5.2.1. □

Recall that if G is a discrete group, then we write $\mathscr{U}_d$ for its discrete uniform structure, i.e., the uniformity on G definite by $\mathscr{U}_d = \{A \subset G \times G : \Delta_G \subset A\}$.

Corollary 7.1.2. *If G has the $\mathscr{U}_d$-fixed-point property for cones, then G has the $\mathscr{U}$-fixed-point property for cones for every group topology τ for G and every functionally invariant uniform structure $\mathscr{U}$ for (G, τ).*

Proof. Let τ be a group topology for G and let $\mathscr{U}$ be a functionally invariant uniformity for G with respect to τ. Then the identity map $\mathrm{Id} : (G, \mathscr{U}_d) \longrightarrow (G, \mathscr{U})$ is a uniformly continuous isomorphism of groups. Now G has the $\mathscr{U}_d$-fixed-point property for cones by hypothesis. Therefore, we can conclude that (G, τ) has the $\mathscr{U}$-fixed-point property for cones by Proposition 7.1.1. □

From now on, we only focus on the five standard uniform structures, and we separately investigate the fixed-point property for cones and the translate property as they are a priori two different notions.

7.1.A. Hereditary properties for the $\mathscr{U}$-fixed-point property for cones. First of all, it is possible to improve Proposition 7.1.1. In fact,

Corollary 7.1.3. *Let G and H be topological groups and let $\mathscr{U} \in \{\mathscr{F}, \mathscr{R} \vee \mathscr{L}, \mathscr{R}, \mathscr{L}, \mathscr{R} \wedge \mathscr{L}\}$. Suppose that there is a continuous epimorphism $\phi : G \longrightarrow H$. If G has the $\mathscr{U}$-fixed-point property for cones, then H also has the $\mathscr{U}$-fixed-point property for cones.*

Proof. If $\mathscr{U} \in \{\mathscr{F}, \mathscr{R} \vee \mathscr{L}, \mathscr{R}, \mathscr{L}, \mathscr{R} \wedge \mathscr{L}\}$, then the map $\phi : (G, \mathscr{U}) \longrightarrow (H, \mathscr{U})$ is uniformly continuous by [RD81, Proposition 2.25]. Therefore, H has the $\mathscr{U}$-fixed-point property for cones by Proposition 7.1.1. □

Corollary 7.1.4. *Suppose that G admits a topology τ_1 for which (G, τ_1) has the $\mathscr{U}$-fixed-point property for cones for $\mathscr{U} \in \{\mathscr{F}, \mathscr{R} \vee \mathscr{L}, \mathscr{R}, \mathscr{L}, \mathscr{R} \wedge \mathscr{L}\}$. Then the topological group (G, τ_2) has the $\mathscr{U}$-fixed-point property for cones for every stronger group topology τ_2 for G.*

Proof. Using the continuous isomorphism of groups given by the identity map $\mathrm{Id} : (G, \tau_1) \longrightarrow (G, \tau_2)$ and Corollary 7.1.3, (G, τ_2) has the $\mathscr{U}$-fixed-point property for cones. □

Before discussing the next result, we shall clarify the situation about uniform structures on quotient groups. Suppose that G is a topological group and Q a quotient of G. Then Q is a topological group when endowed with the quotient topology ([Bou71, III §2 No.6 Proposition 16]), and the quotient map $q : G \longrightarrow Q$ is continuous with respect to this topology ([HR63, Theorem (5.16)]). Let now $\mathscr{U}$ be a uniform structure on G. Then the **quotient uniform structure** $\mathscr{U}_Q$ on Q with respect to $\mathscr{U}$ is the finest uniformity on Q such that the map $q : (G, \mathscr{U}) \longrightarrow (Q, \mathscr{U}_Q)$ is uniformly continuous. An interesting fact to point out is that if $\mathscr{U}$ is one of the five standard uniformities $\{\mathscr{F}, \mathscr{R} \vee \mathscr{L}, \mathscr{R}, \mathscr{L}, \mathscr{R} \wedge \mathscr{L}\}$, then the quotient uniform structure $\mathscr{U}_Q$ of Q with respect to $\mathscr{U}$ is equal to the uniformity $\mathscr{U}$ of the topological group Q ([RD81, Proposition 5.30]).

Corollary 7.1.5. *Let G be a topological group and let $\mathscr{U} \in \{\mathscr{F}, \mathscr{R} \vee \mathscr{L}, \mathscr{R}, \mathscr{L}, \mathscr{R} \wedge \mathscr{L}\}$. Suppose that Q is a quotient group of G with the quotient topology. If G has the $\mathscr{U}$-fixed-point property for cones, then Q also has the $\mathscr{U}_Q$-fixed-point property for cones.*

Proof. The definition of quotient uniform structure and Corollary 7.1.3 are sufficient to ensure that the conclusion is true. □

We continue by studying open subgroups.

Proposition 7.1.6. *Let G be a topological group and let $\mathscr{U} \in \{\mathscr{F}, \mathscr{R} \vee \mathscr{L}, \mathscr{R}, \mathscr{L}, \mathscr{R} \wedge \mathscr{L}\}$. If G has the $\mathscr{U}$-fixed-point property for cones, then every open subgroups H of G has the $\mathscr{U}$-fixed-point property for cones.*

Proof. It suffices to show that H has the invariant normalized integral property for $\mathcal{C}_u^b(H, \mathscr{U})$ by Theorem 5.3.1. To this aim, let K be a set of representatives for the right H-cosets of G and define the operator

$$T : \mathcal{C}_u^b(H, \mathscr{U}) \longrightarrow \mathcal{C}_u^b(G, \mathscr{U}), \quad h \longmapsto T(h)(g) = h(x),$$

where $g = xk$ for a $k \in K$. Then T is well-defined and linear. Let now $h \in \mathcal{C}_u^b(H, \mathscr{U})$ be a non-zero positive function. Then for every $g \in G$ there is $x \in G$ and $k \in K$ such that $g = xk$. Therefore, $T(h)(g) = h(x) \geq 0$. Moreover, there is $g_0 \in H$ such that $h(g_0) > 0$, as h is a non-zero function. This implies that $T(h)(g_0) = h(g_0) > 0$ as g_0 lies in the coset of the identity element. Thus, we can conclude that T is strictly positive. We claim that T is also equivariant. Indeed, let $a \in H$ and $h \in \mathcal{C}_u^b(H, \mathscr{U})$. Then for every $g \in G$ there is $k \in K$ such that $g = xk$ for some $x \in H$. Consequently,

$$T(ah)(g) = (ah)(x) = h(a^{-1}x) = T(h)(a^{-1}g) = aT(h)(g).$$

Let $f \in \mathcal{C}_u^b(H, \mathscr{U})$ be a non-zero positive function and note that the image of the restriction of T on the space $\big(\mathcal{C}_u^b(H, \mathscr{U}), f\big)$ is a non-zero subspace of $\big(\mathcal{C}_u^b(G, \mathscr{U}), T(f)\big)$. By hypothesis, there is an invariant normalized integral I on $\big(\mathcal{C}_u^b(G, \mathscr{U}), T(f)\big)$. The composition $\bar{\mathrm{I}} = \mathrm{I} \circ T$ is an invariant normalized integral for $\big(\mathcal{C}_u^b(H, \mathscr{U}), f\big)$. □

Proposition 7.1.7. *Let G be a topological group and let $\mathscr{U} \in \{\mathscr{F}, \mathscr{R} \vee \mathscr{L}, \mathscr{R}, \mathscr{L}, \mathscr{R} \wedge \mathscr{L}\}$. Suppose that $F \trianglelefteq G$ is a finite normal subgroup of G such that the quotient group ${}^{G}\!/_{F}$ has the $\mathscr{U}$-fixed-point property for cones. Then G has the $\mathscr{U}$-fixed-point property for cones.*

Proof. Let E be a Banach lattice and suppose that G has a representation π on E by positive linear isometries. Moreover, suppose that π^* is locally bounded $(\mathscr{U}, \mathscr{U}_c^*)$-uniformly continuous and that E is asymptotically G-dominated. We want to show that there is an invariant normalized integral on E.

Let d be the asymptotically G-dominating element of E, and let $n = |F|$ be the order of the finite group F. Write E^F for the normed Riesz subspace of all vectors of E which are F-invariant, and consider the average operator

$$T : E \longrightarrow E^F, \quad v \longmapsto T(v) = \frac{1}{n}\sum_{j=1}^{n} x_j v \quad \text{where } x_1, ..., x_n \in F.$$

We can easily check that T is well-defined, continuous and linear. Moreover, T is strictly positive as F acts by (strictly) positive maps and the positive cone of a Banach lattice is always convex. Finally, T is equivariant because of the normality of F in G. In fact,

$$T(gv) = \frac{1}{n}\sum_{j=1}^{n} x_j g v = \frac{1}{n}\sum_{j=1}^{n} g x_j v = gT(v) \quad \text{for every } g \in G \text{ and } v \in E.$$

Now there is a natural representation π' of ${}^{G}\!/_{F}$ on E^F given by $\pi'(gF)w = \pi(g)w$. Note that π' is by positive linear isometries and $(\pi')^*$ is locally bounded $(\mathscr{U}, \mathscr{U}_c^*)$-uniformly continuous as $E' \subset (E^F)'$. We claim that the Banach lattice E^F is asymptotically ${}^{G}\!/_{F}$-dominated by the vector $d_F = T(d)$. Indeed, let $v_F \in E_F$. As T is surjective, there is a non-zero positive vector $v \in E$ such that $T(v) = |v_F|$, and there are sequences

$(g_j)_j$ in G and $(t_j)_j$ in $\mathbf{R}_+$ such that

$$|v| \leq \sum_{j=1}^{\infty} t_j g_j d \quad \text{and} \quad \sum_{j=1}^{\infty} t_j < \infty.$$

Therefore,

$$|v_F| = T(v) \leq T\left(\sum_{j=1}^{\infty} t_j g_j d\right) \leq \sum_{j=1}^{\infty} t_j g_j T(d) = \sum_{j=1}^{\infty} t_j g_j d_F.$$

As d_F is F-invariant, we can conclude that $|v_F| \leq \sum_{j=1}^{\infty} t_j q\left(g_j\right) d_F$, where q is the quotient map from G to ${}^{G}\!/_{F}$. This shows that E^F is asymptotically ${}^{G}\!/_{F}$-dominated.

Since ${}^{G}\!/_{F}$ has the $\mathscr{U}$-fixed-point property for cones, there is an ${}^{G}\!/_{F}$-invariant normalized integral on E^F, say I. Then the functional $\bar{\mathrm{I}} = \mathrm{I} \circ T$ provides a G-invariant normalized integral on E. The topological group G has the $\mathscr{U}$-fixed-point property for cones by Theorem 5.3.1. □

We have a slightly better result for the uniform structures $\mathscr{R}$ and $\mathscr{R} \wedge \mathscr{L}$.

Proposition 7.1.8. *Let G be a topological group and let $\mathscr{U} \in \{\mathscr{R}, \mathscr{R} \wedge \mathscr{L}\}$. Let $C \trianglelefteq G$ be a compact normal subgroup. If the quotient group ${}^{G}\!/_{C}$ has the $\mathscr{U}$-fixed-point property for cones, then G has the $\mathscr{U}$-fixed-point property for cones.*

Proof. Let $E = \mathcal{C}_u^b(G, \mathscr{U})$ for $\mathscr{U} \in \{\mathscr{R},\ \mathscr{R} \wedge \mathscr{L}\}$ and define the linear and positive operator given by the Bochner integral

$$T : E \longrightarrow E, \quad f \longmapsto T(f) = \int_C cf \, dm_C(c),$$

where m_C is the normalized Haar measure of C. The operator T is well-defined thanks to the fact that the action of C on E is continuous for the $||\cdot||_\infty$-norm. We claim that T is G-equivariant. Indeed, fix $g \in G$. Then, for every $c \in C$, the element $g^{-1}cg$ is in C as C is normal in G. Therefore, for every $c \in C$, there is $c' \in C$ such that $cg = gc'$. This leads to the conclusion

$$T(gf) = \int_C cgf \, dm_C(c) = \int_C gc'f \, dm_C(gcg^{-1}) = g \int_C c'f \, dm_C(c) = gT(f)$$

for every $f \in E$. Note that we used the fact that the group C is unimodular in the second-to-last equality. Moreover, $T(f) \in E^C$ for every $v \in E$. In fact, for a $x \in C$ fixed, the element $cx = y \in C$ for every $c \in C$. Therefore,

$$T(xf) = \int_C cxf \, dm_C(c) = \int_C yf \, dm_C(yx^{-1}) = \int_C yf \, dm_C(y) = T(f) \quad \text{for every } f \in E.$$

If $f \in E$ is a non-zero positive function, then the image of (E, f) under the operator T is contained in the normed Riesz space $(E^C, T(f))$. Now, the quotient group ${}^{G}\!/_{C}$ acts on E^C, and hence on $(E^C, T(f))$. Moreover, this last space is ${}^{G}\!/_{C}$-dominated and the adjoint action of ${}^{G}\!/_{C}$ on the dual of E^C is locally bounded $(\mathscr{U}, \mathscr{U}_c^*)$-uniformly continuous. Thus, there is a ${}^{G}\!/_{C}$-invariant normalized integral I on $(E^C, T(f))$ by Theorem 5.3.1. Taking the precomposition with T, we get a G-invariant normalized integral $\bar{\text{I}} = \text{I} \circ T$ on (E, f), which implies that G has the $\mathscr{U}$-fixed-point property for cones. □

Proposition 7.1.9. *Let G be a topological group and let $\mathscr{U} \in \{\mathscr{F}, \mathscr{R} \vee \mathscr{L}, \mathscr{R}, \mathscr{L}, \mathscr{R} \wedge \mathscr{L}\}$. Let $H < G$ be a topological subgroup of finite-index. If H has the $\mathscr{U}$-fixed-point property for cones, then G has the $\mathscr{U}$-fixed-point property for cones.*

Proof. Let E be a Banach lattice and suppose that G has a representation π on E by positive linear isometries. Suppose that π^* is locally bounded $(\mathscr{U}, \mathscr{U}_c^*)$-uniformly continuous and that E is asymptotically G-dominated. We want to show that there is an invariant normalized integral on E as done before.

Note that the restriction of π on the subgroup H defines a representation $\pi|_H$ of H by positive linear isometries on E such that π^* is locally bounded $(\mathscr{U}, \mathscr{U}_c^*)$-uniformly continuous. We claim that E is asymptotically H-dominated by the vector $d_H = \sum_{k=1}^n x_k d$, where d is the asymptotically G-dominating element of E and $\{x_1, ..., x_n\}$ is a set of representatives for the right H-cosets of G. Indeed, if $v \in E$, then there are sequences $(g_j)_j \subset G$ and $(t_j)_j \subset \mathbf{R}_+$ such that

$$|v| \leq \sum_{j=1}^{\infty} t_j g_j d \quad \text{and} \quad \sum_{j=1}^{\infty} t_j < \infty.$$

Now, for every $j \in \mathbf{N}$, we can write $g_j = h_j x_k$ for some $k \in \{1, ..., n\}$ and some $h_j \in H$. Thus,

$$|v| \leq \sum_{j=1}^{\infty} t_j g_j d = \sum_{j=1}^{\infty} t_j h_j x_k d \leq \sum_{j=1}^{\infty} t_j h_j \left(\sum_{k=1}^{n} x_k d \right) = \sum_{j=1}^{\infty} t_j h_j d_H.$$

As H has the $\mathscr{U}$-fixed-point property for cones, there is an H-invariant normalized integral I on E. Consequently, the expression $\bar{\text{I}} = \frac{1}{n} \sum_{k=1}^n x_k^{-1} \text{I}$ defines a G-invariant normalized integral on E. We can conclude that G has the $\mathscr{U}$-fixed-point property for cones using Theorem 5.3.1. □

Notice that the following proposition is not true for the fine uniform structure $\mathscr{F}$.

Proposition 7.1.10. *Let G be a topological group and let $\mathscr{U} \in \{\mathscr{R} \vee \mathscr{L}, \mathscr{R}, \mathscr{L}, \mathscr{R} \wedge \mathscr{L}\}$. Suppose that $D < G$ is a dense topological subgroup. If G has the $\mathscr{U}$-fixed-point property for cones, then D also has the $\mathscr{U}$-fixed-point property for cones.*

Proof. We can uniquely extend every function $f \in \mathcal{C}_u^b(D,\mathscr{U})$ to a function $\mathtt{ext}(f) \in \mathcal{C}_u^b(G,\mathscr{U})$ by [RD81, Proposition 3.24] and [Bou71, II §3 No.6 Théorème 2]. Therefore, consider the extension operator

$$\mathtt{ext} : \mathcal{C}_u^b(D,\mathscr{U}) \longrightarrow \mathcal{C}_u^b(G,\mathscr{U}), \quad f \longmapsto \mathtt{ext}(f)$$

which gives to every $f \in \mathcal{C}_u^b(D,\mathscr{U})$ is unique extension $\mathtt{ext}(f) \in G$. Then ext is a strictly positive, equivariant linear operator. Take now a non-zero positive function $f \in \mathcal{C}_u^b(D,\mathscr{U})$, and we want to show that there is an invariant normalized integral on $\left(\mathcal{C}_u^b(G,\mathscr{U}), f\right)$. As G has the $\mathscr{U}$-fixed-point property for cones, there is a G-invariant normalized integral on $\left(\mathcal{C}_u^b(G,\mathscr{U}), \mathtt{ext}(f)\right)$, say I. Then $\bar{\mathrm{I}} = \mathrm{I} \circ \mathtt{ext}$ is a D-invariant normalized integral on $\mathcal{C}_{ru}^b(D,f)$. We can conclude that D has the $\mathscr{U}$-fixed-point property for cones by Theorem 5.2.1. □

Example 7.1.11. We claim that precompact topological groups have the $\mathscr{U}$-fixed-point property for cones for $\mathscr{U} \in \{\mathscr{R} \vee \mathscr{L}, \mathscr{R}, \mathscr{L}, \mathscr{R} \wedge \mathscr{L}\}$. Indeed, recall that a topological group G is said **precompact** if it is isomorphic to a dense subgroup of a compact group K. Let ϕ be the continuous isomorphism from G to K and fix $\mathscr{U} \in \{\mathscr{R} \vee \mathscr{L}, \mathscr{R}, \mathscr{L}, \mathscr{R} \wedge \mathscr{L}\}$. Then $\phi(G)$ has the $\mathscr{U}$-fixed-point property for cones by Proposition 7.1.10. Considering the map $\phi^{-1} : \phi(G) \longrightarrow G$ and using Corollary 7.1.3, we can conclude that G also has the $\mathscr{U}$-fixed-point property for cones.

Proposition 7.1.12. *Let G be a topological group and let H be a closed subgroup of G. Assume that the space of right-cosets $G\backslash H$ is paracompact and that the fibration $p : G \longrightarrow G\backslash H$ is locally trivial. If G has the $\mathscr{F}$-fixed-point property for cones, then H also has the $\mathscr{F}$-fixed-point property for cones.*

We follow the strategy used by Rickert in [R67, Theorem 3.4].

Proof of Proposition 7.1.12. First, take an open cover $(U_j)_{j\in I}$ of $G\backslash H$ and a family of continuous sections $\left(\sigma_j : U_j \longrightarrow G\right)_{j\in I}$ such that $(p \circ \sigma_j)(g) = g$ for all $j \in I$ and $g \in U_j$. This is possible thanks to the fact that the fibration $p : G \longrightarrow G\backslash H$ is locally trivial. Since $G\backslash H$ is paracompact, it is possible to take a partition of the unit $(\varphi_j)_{j\in I}$ subordinates to $(U_j)_{j\in I}$ and define the linear operator

$$T : \mathcal{C}_u^b(H,\mathscr{F}) \longrightarrow \mathcal{C}_u^b(G,\mathscr{F}), \quad f \longmapsto T(f)(g) = \sum_{j\in I} \varphi_j(p(g)) f\left(g(\sigma_j \circ p)^{-1}(g)\right).$$

First of all, T is well-defined because, for every $f \in \mathcal{C}_u^b(H,\mathscr{F})$, the function $T(f)$ is continuous as the family of the supports of the partition of the unity $(\varphi_j)_{j\in I}$ is locally finite and $T(f)$ is bounded because $||T(f)||_\infty \leq ||f||_\infty$. Therefore, $T(f)$ is in $\mathcal{C}_u^b(G,\mathscr{F})$.

Moreover, T is equivariant. Indeed,

$$\begin{aligned}(aT(f))\,(g) &= T(f)(a^{-1}g)\\ &= \sum_{j\in I} \varphi_j(p(a^{-1}g))f\left(a^{-1}g(\sigma_j\circ p)^{-1}(a^{-1}g)\right)\\ &= \sum_{j\in I} \varphi_j(p(g))f\left(a^{-1}g(\sigma_j\circ p)^{-1}(a^{-1}g)\right) = T(af)(g)\end{aligned}$$

for every $a, g \in G$ and $f \in \mathcal{C}_u^b(H, \mathscr{F})$. Finally, T is strictly positive as the partition of the unit $(\varphi_j)_{j\in I}$ is positive.

Now for every non-zero positive function $f \in \mathcal{C}_u^b(H, \mathscr{F})$, there is an invariant normalized integral I defined on $\left(\mathcal{C}_u^b(G, \mathscr{F}), T(f)\right)$. Therefore, the composition $\bar{\mathrm{I}} = \mathrm{I}\circ T$ defines an invariant normalized integral on $\left(\mathcal{C}_u^b(H, \mathscr{F}), f\right)$. This shows that H has the invariant normalized integral property for $\mathcal{C}_u^b(H, \mathscr{F})$, and hence the $\mathscr{F}$-fixed-point property for cones by Theorem 5.3.1. □

Corollary 7.1.13. *Let G be a metrizable topological group. If G contains a discrete group without the fixed-point property for cones, then G does not have the $\mathscr{F}$-property for cones.*

Proof. All the hypothesis for applying Proposition 7.1.12 are satisfied as explained in [GH17, Corollary 4.6]. □

7.1.B. Hereditary properties for the translate property. We should point out that every proof used in the previous section is applicable for the translate property. Therefore, every result we stated remains true if the *$\mathscr{U}$-fixed-point property for cones* is replaced with the *translate property for $\mathcal{C}_u^b(G, \mathscr{U})$*.

However, we have some deeper result thanks to the fact that the translate property for $\mathcal{C}_u^b(G, \mathscr{U})$ is equivalent to the abstract $\mathscr{U}$-translate property, and that the translate property for $\mathcal{C}_u^b(G, \mathscr{R})$ is equivalent to the abstract continuous translate property.

Proposition 7.1.14. *Let G be a topological group and let $\mathscr{U} \in \{\mathscr{F}, \mathscr{R}\vee\mathscr{L}, \mathscr{R}, \mathscr{L}, \mathscr{R}\wedge\mathscr{L}\}$. Suppose that $G = \varinjlim H_\alpha$ is the direct limit of a family $(H_\alpha)_\alpha$ of topological groups with the inductive limit topology. If H_α has the translate property for $\mathcal{C}_u^b(H_\alpha, \mathscr{U})$ for every α, then G has the translate property for $\mathcal{C}_u^b(G, \mathscr{U})$.*

Proof. Let $f \in \mathcal{C}_u^b(G, \mathscr{U})$ be a non-zero positive function and let $t_1, \dots, t_n \in \mathbf{R}$ and $g_1, \dots, g_n \in G$ be such that $\sum_{j=1}^n t_j g_j f \geq 0$. We have to show that $\sum_{j=1}^n t_j \geq 0$. Take α such that $g_1, \dots, g_n \in H_\alpha$ and $f(\phi_\alpha(x)) \neq 0$ for some $x \in G$. Here, ϕ_α is the canonical homomorphism of group which sends each element of H_α to its equivalence class in the direct limit G. Then ϕ_α induces a positive, linear and H_α-equivariant operator

$$\phi_\alpha^* : \mathcal{C}_u^b(G, \mathscr{U}) \longrightarrow \mathcal{C}_u^b(H_\alpha, \mathscr{U}), \quad f \longmapsto \phi_\alpha^*(f) = f\circ\phi_\alpha.$$

Therefore,

$$0 \leq \phi_\alpha^* \left(\sum_{j=1}^n t_j g_j f \right) = \sum_{j=1}^n t_j g_j \phi_\alpha^*(f).$$

Applying the translate property for $\mathcal{C}_u^b(H_\alpha, \mathscr{U})$ of H_α, we can deduce that $\sum_{j=1}^n t_j \geq 0$ as wished. □

Example 7.1.15. Let $U(n)$ be the unitary group of degree $n \in \mathbf{N}$ endowed with its compact topology and let

$$\iota_n : U(n) \longrightarrow U(n+1), \quad u \longmapsto \iota_n(u) = \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix}$$

be the natural inclusion of $U(n)$ in $U(n+1)$. Define the topological group $U(\infty) = \varinjlim U(n)$ as the direct limit of the family $\{(U(n), \iota_n)\}_{n\in\mathbf{N}}$ with the inductive limit topology. Then $U(\infty)$ has the translate property for $\mathcal{C}_u^b(U(\infty), \mathscr{U})$ for every uniform structure $\mathscr{U} \in \{\mathscr{F}, \mathscr{R} \vee \mathscr{L}, \mathscr{R}, \mathscr{L}, \mathscr{R} \wedge \mathscr{L}\}$ by Proposition 7.1.14.

Proposition 7.1.16. *Let G be a topological group and let $D < G$ be a dense topological subgroup. Then G has the translate property for $\mathcal{C}_u^b(G, \mathscr{R})$ if and only if D has the translate property for $\mathcal{C}_u^b(D, \mathscr{R})$.*

Proof. The proof of the *only if* part is equal to the proof of Proposition 7.1.10. Let's prove the other direction. By Theorem 5.3.5, it suffices to show that G has the abstract continuous translate property. Therefore, suppose that G has a continuous representation on a Banach lattice E by positive linear isometries. Fix a non-zero positive vector $v \in E$ and let $t_1, ..., t_n \in \mathbf{R}$, $g_1, ..., g_n \in G$ such that $\sum_{j=1}^n t_j g_j v \geq 0$. Now there is a net $(h_\alpha^{(j)})_\alpha$ in D such that $\lim_\alpha h_\alpha^{(j)} = g_j$ for every $j = 1, ..., n$. Thus,

$$0 \leq \sum_{j=1}^n t_j g_j v = \sum_{j=1}^n t_j \lim_\alpha h_\alpha^{(j)} v = \lim_\alpha \sum_{j=1}^n t_j h_\alpha^{(j)} v.$$

We can conclude that $\sum_{j=1}^n t_j \geq 0$ thanks to the abstract continuous translate property of D. □

Example 7.1.17. 1) Every Lévy group G has the translate property for $\mathcal{C}_u^b(G, \mathscr{R})$ because of Propositions 7.1.14 and 7.1.16. Examples of Lévy groups can be found in Chapters 4 and 5 of [P06].[1]

[1]Recall that a topological group G is a Lévy group if there is a family $(G_\alpha)_\alpha$ of compact subgroups of G directed by inclusion having an everywhere dense union and such that for every α the normalized Haar measure on G_α concentrates with respect to the right uniform structure on G. We refer to [P06, Chapter 4] for more details about Lévy groups.

2) Let $q = p^k$, where p is a prime number and $k \in \mathbf{N}$, and let $\mathbf{SL}_n(q)$ be the special linear group of degree $n \in \mathbf{N}$ over the finite field of q elements. For every $n \in \mathbf{N}$, define the diagonal embedding

$$\phi_n : \mathbf{SL}_{2^n}(q) \longrightarrow \mathbf{SL}_{2^{n+1}}(q), \quad a \longmapsto \phi_n(a) = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$$

and write $A_0(q) = \varinjlim \mathbf{SL}_{2^n}(q)$ for the direct limit of the discrete family of groups $\{(\mathbf{SL}_{2^n}(q), \phi_n)\}_{n \in \mathbf{N}}$. Now on each group $\mathbf{SL}_{2^n}(q)$ consider the (normalized) rank distance d_{2^n}.[2] We can extend the rank-metric on the group $A_0(q)$, say d, and we define $A(q)$ as the metric completion of $A_0(q)$ with respect to d. With the topology given by d, the group $A(q)$ becomes a Polish non-locally compact topological group. We know that $A(q)$ has the translate property for $\mathcal{C}_u^b(A(q), \mathscr{R})$, since it is a Lévy group ([C18, Theorem 2.4]). Note that $A(q)$ is a SIN-group, as it is constructed as the completion of a group equipped with a bi-invariant metric. Therefore, we can directly deduce that $A(q)$ also has the translate property for $\mathcal{C}_u^b(A(q), \mathscr{L})$. However, if q is odd, then $A(q)$ contains $\mathbf{F}_2$ as a discrete subgroup ([C18, Theorem 4.2]), and hence it has not the translate property for $\mathcal{C}_u^b(A(q), \mathscr{F})$ by Corollary 7.1.13.

We recall that a locally compact group G is of **subexponential growth** if for every compact neighborhood of the identity $C \subset G$, we have that $\lim_n m_G(C^n)^{\frac{1}{n}} = 1$.

Theorem 7.1.18. *Let G be a topological group with the translate property for $\mathcal{C}_u^b(G, \mathscr{R})$ and let H be a compactly generated locally compact group of subexponential growth. Then the Cartesian product $G \times H$ has the translate property for $\mathcal{C}_u^b(G \times H, \mathscr{R})$.*

We follow the strategy used by Monod in [M17, Theorem 8 - (3)] for discrete groups. To this end, we need the following lemma due to Jenkins.

Lemma 7.1.19 (Jenkins)**.** *Let G be a locally compact group. Then the following assertions are equivalent:*

a) the group G is of subexponential growth;

b) for every compact subset K of G and every $\epsilon > 0$ there is a function $\phi \in L^1(G)$ such that

$$\phi(t) > 0 \quad \text{and} \quad \phi(ts) \leq (1+\epsilon)\phi(t) \quad \text{for every } t \in \langle K \rangle \text{ and } s \in K.$$

Here, $\langle K \rangle$ is the subgroup generated by K.

Proof. See [J76, Lemma 1]. □

[2] Write $\mathfrak{r}(a)$ for the rank of the matrix $a \in \mathrm{M}_n(\mathbf{F}_q)$. Then the (normalized) rank distance d_n on $\mathbf{SL}_n(q)$ is defined as $d_n(a,b) = \frac{1}{n}\mathfrak{r}(b-a) \in [0,1]$. Note that d_n is a bi-invariant metric.

Proof of Theorem 7.1.18. Let G be a topological group with the translate property for $\mathcal{C}^b_{ru}(G)$, and let H be a compactly generated locally compact group of subexponential growth. In order to find a contradiction, suppose that the Cartesian product $G \times H$ does not have the translate property for $\mathcal{C}^b_{ru}(G \times H)$. This means that there is a non-zero positive function $f \in \mathcal{C}^b_{ru}(G \times H)$ and elements $t_1, ..., t_n \in \mathbf{R}$, $g_1, ..., g_n \in G$, $h_1, ..., h_n \in H$ such that

$$\sum_{j=1}^{n} t_j g_j h_j f \geq 0 \quad \text{but} \quad \sum_{j=1}^{n} t_j < 0.$$

Chose an $0 < \epsilon < 1$ such that $\sum_{j=1}^{n} \overline{t_j} < 0$, where the coefficients $\overline{t_j}$ are given by

$$\overline{t_j} = \begin{cases} (1+\epsilon)t_j & \text{if } t_j > 0 \\ (1-\epsilon)t_j & \text{if } t_j \leq 0, \end{cases}$$

for every $j = 1, ..., n$. Then $\overline{t_j} \geq t_j$ for every $j = 1, ..., n$. By Jenkin's lemma there is a non-zero positive function $\phi \in L^1(H)$ such that $\pi_R(h_j)\phi(x) \leq (1+\epsilon)\phi(x)$ for every $x \in H$ and every $j = 1, ..., n$. Now it is important to notice that $\mathcal{C}^b_{ru}(G \times H)$ is a positive $L^1(H)$-module with operation map given by

$$L^1(H) \times \mathcal{C}^b_{ru}(G \times H) \longrightarrow \mathcal{C}^b_{ru}(G \times H), \quad (\psi, f) \longmapsto \psi * f = \int_H \psi(h)\pi_L(h) f \, dm_H(h).$$

The integral here is taken in the Bochner sense. We quickly check that the operation map is well-defined. First, for every $f \in \mathcal{C}^b_{ru}(G \times H)$ and every $\psi \in L^1(H)$, the map

$$(H, m_H) \longrightarrow \mathcal{C}^b_{ru}(G \times H), \quad h \longmapsto \psi(h)\pi_L(h) f$$

is m_H-measurable as ψ is an almost everywhere limit of continuous functions and the representation π_L of H on $\mathcal{C}^b_{ru}(G \times H)$ is continuous. Second,

$$\int_H ||\psi(h)\pi_L(h)f||_\infty \, dm_H(h) \leq \int_H |\psi(h)| \, ||\pi_L(h)f||_\infty \, dm_H(h) \leq ||\psi||_1 ||f||_\infty.$$

Therefore, the operation map is well-defined by [Bou63, IV §5 N.6 Théorème 5].

Two important properties of the operation map that we shall have in mind are that

$$(\pi_R(h)\psi) * f = \psi * (\pi_L(h)f) \quad \text{and} \quad \pi_L(g)(\psi * f) = \psi * (\pi_L(g)f)$$

for every $h \in H$, $g \in G$, $\psi \in L^1(H)$ and $f \in \mathcal{C}^b_{ru}(G \times H)$.

Returning to our problem. It is clear that

$$(1-\epsilon)\phi \leq \pi_R(h_j)\phi \leq (1+\epsilon)\phi \quad \text{for every } j = 1, ..., n.$$

Taking the convolution against f, we deduce that

$$(1-\epsilon)\phi * f \leq (\pi_R(h_j)\phi) * f \leq (1+\epsilon)\phi * f \quad \text{for every } j = 1, ..., n.$$

Multiplying for t_j this last expression, taking into account the sign of t_j, we have that

$$\overline{t_j}\phi * f \geq t_j(\pi_R(h_j)\phi) * f \geq t_j\phi * (\pi_L(h_j)f) \quad \text{for every } j = 1, ..., n.$$

We want to apply the abstract continuous translate property of G to the non-zero positive function $\phi * f \in \mathcal{C}^b_{ru}(G \times H)$. Thus,

$$\sum_{j=1}^{n} \overline{t_j}\pi_L(g_j)(\phi * f) \geq \sum_{j=1}^{n} t_j\pi_L(g_j)\left(\phi * (\pi_L(h_j)f)\right) \geq \phi * \left(\sum_{j=1}^{n} t_j g_j h_j f\right) \geq 0,$$

which implies that $\sum_{j=1}^{n} \overline{t_j} \geq 0$. But this is in contradiction with the fact that $\sum_{j=1}^{n} \overline{t_j} < 0$ by the choice of the $\overline{t_j}$'s. Therefore, $G \times H$ has the translate property for $\mathcal{C}^b_{ru}(G \times H)$. □

Example 7.1.20. Let $\mathcal{M}$ be an injective Von Neumann algebra with separable predual $\mathcal{M}_*$ and consider the group $U(\mathcal{M}) = \{u \in \mathcal{M} : uu^* = u^*u = 1\}$ of all unitary operators of $\mathcal{M}$. The group $U(\mathcal{M})$ is a Polish group, when endowed with the ultraweak topology, i.e., the weak-* topology given by the duality with $\mathcal{M}_*$. By a result of Giordano and Pestov ([GP07, Section 3]), $U(\mathcal{M})$ is the direct product of a compact group K and a Lévy group L. We know that L has the translate property for $\mathcal{C}^b_u(L, \mathscr{R})$ by Example 7.1.17 and that K is of subexponential growth. Therefore, we can apply Theorem 7.1.18 to deduce that $U(\mathcal{M})$ has the translate property for $\mathcal{C}^b_u(U(\mathcal{M}), \mathscr{R})$.

We want to conclude the discussion about topological groups with one last interesting example.

Example 7.1.21 (One example to rule them all)**.** Let $\Gamma = \mathrm{Sym}(\mathbf{N})$ be the group of all permutations of the natural numbers $\mathbf{N}$. How seen in Example 1.4.10, Γ has a unique Polish group topology which makes it $\mathscr{R}$-amenable but not $\mathscr{F}$-amenable.

Now, Γ is equal to the closure of the subgroup $\mathrm{Sym}_f(\mathbf{N})$ of all permutations of $\mathbf{N}$ with finite support. This last group is locally finite, and therefore has the $\mathscr{U}_d$-fixed-point property for cones by point a) of Theorem 7.2.2. In particular, $\mathrm{Sym}_f(\mathbf{N})$ has the translate property for $\mathcal{C}^b_u(\mathrm{Sym}_f(\mathbf{N}), \mathscr{U})$ for all the five standard uniformities $\mathscr{U}$. We can conclude that G has the translate property for $\mathcal{C}^b_u(\Gamma, \mathscr{R})$ by Proposition 7.1.16.

Moreover, Γ has $\mathbf{F}_2$ has a closed discrete subgroup as explained in [GH17, Proposition 5.5]. Therefore, Γ does not have the translate property for $\mathcal{C}^b_u(\Gamma, \mathscr{F})$ by Corollary 7.1.13. Γ also does not have the translate property for $\mathcal{C}^b_u(\Gamma, \mathscr{L})$ because of Corollary 5.2.12 and the fact that the action of Γ on $\mathbf{N}$ is continuous.

From these facts, we can draw the following conclusions for topological groups:

1) in general, a closed subgroup H of a topological group G with the translate property for $\mathcal{C}^b_u(G, \mathscr{R})$ needs not to have the translate property for $\mathcal{C}^b_u(H, \mathscr{R})$;

2) Proposition 7.1.16 is not true anymore for the uniformities $\mathscr{L}$ and $\mathscr{F}$;

3) suppose that a topological group G has the translate property for $\mathcal{C}_u^b(G, \mathscr{L})$. Then it does not imply that G is $\mathscr{L}$-amenable. Same for the uniform structure $\mathscr{F}$.

Finally, Γ allows us to apply another time Proposition 7.1.8. Indeed, let $p \in [1, \infty]$, $p \neq 2$, and consider the isometry group $\mathrm{Iso}(\ell^p(\mathbf{N}))$. By the work of Banach ([B78, XI §5 p. 178-179]), we can abstractly see $\mathrm{Iso}(\ell^p(\mathbf{N}))$ as the semidirect product $\{-1,1\}^{\mathbf{N}} \rtimes \Gamma$. Then the strong operator topology on $\mathrm{Iso}(\ell^p(\mathbf{N}))$ is equal to the product topology of the Polish topology of Γ and the compact one of $\{-1,1\}^{\mathbf{N}}$. Therefore, $\mathrm{Iso}(\ell^p(\mathbf{N}))$ has the translate property for $\mathcal{C}_u^b(\mathrm{Iso}(\ell^p(\mathbf{N})), \mathscr{R})$.

7.2 The locally compact case

We turn our attention to locally compact groups. We can go much deeper in the study of the class of locally compact groups with the fixed-point property for cones using the results of Chapter 6.

Recall that we only speak of the fixed-point property for cones without specifying any of the five standard uniformities when handling locally compact groups, since they are all equivalent (see Definition 6.3.6 and paragraph before). Furthermore, having the fixed-point property for cones is equivalent to having the translate property in the case of locally compact groups by Theorem 6.3.10.

We start showing that the fixed-point property for cones passes to closed subgroups. To prove this assertion, we borrow a tool from harmonic analysis on locally compact groups, namely Bruhat functions. To this end, recall that if G is a locally compact group and $H < G$ a closed subgroup, then a **Bruhat function** β for H is a positive continuous function on G such that $\mathrm{supp}(\beta|_{KH})$ is compact for every compact subset K of G and such that $\int_H \beta(gh) dm_H(h) = 1$ for every $g \in G$. Note that β depends on the choice of the Haar measure m_H of H. A proof that every closed subgroup of a locally compact group admits a Bruhat function β can be found in [R02, Proposition 1.2.6].

Theorem 7.2.1. *Suppose that G is a locally compact group with the fixed-point property for cones and let $H \leq G$ be a closed subgroup. Then H has the fixed-point property for cones.*

Proof. We want to show that H has the translate property for $L^\infty(H)$. Therefore, let β be a Bruhat function for H and define the linear operator

$$T : L^\infty(H) \longrightarrow L^\infty(G), \quad h \longmapsto T(h)(g) = \int_H h(x)\beta(g^{-1}x) dm_H(x).$$

Note that T is well-defined. Actually, $T(h) \in \mathcal{C}^b(G)$ for every $h \in L^\infty(H)$ by [P88, Proposition (1.12)] . Moreover, T is strictly positive because β is positive and the intersection

$\operatorname{supp}(\beta) \cap \operatorname{supp}(f)$ is non-empty. Finally, T is equivariant. Indeed,

$$\begin{aligned} T(ah)(g) &= \int_H (ah)(x)\beta(g^{-1}x)dm_H(x) = \int_H h(a^{-1}x)\beta(g^{-1}x)dm_H(x) \\ &= \int_H h(y)\beta(g^{-1}ay)dm_H(y) = \int_H h(y)\beta((a^{-1}g)^{-1}y)dm_H(y) \\ &= T(h)(a^{-1}g) = aT(h)(g) \end{aligned}$$

for every $a, g \in G$ and for every $h \in L^\infty(H)$. Now let $f \in L^\infty(H)$ be a non-zero positive function, and let $h_1, ..., h_n \in H$ and $t_1, ..., t_n \in \mathbf{R}$ be such that $\sum_{j=1}^n t_j h_j f \geq 0$. Therefore,

$$0 \leq T\Big(\sum_{j=1}^n t_j h_j f\Big) = \sum_{j=1}^n t_j h_j T(f).$$

As G has the fixed-point property for cones, then it also has the translate property for $\mathcal{C}^b(G)$ thanks to Theorem 6.3.11. Applying it to the non-zero positive function $T(f)$, we have that $\sum_{j=1}^n t_j \geq 0$ showing that H has the translate property for $L^\infty(H)$, and hence the fixed-point property for cones by Theorems 6.3.11. □

Theorem 7.2.2. *Let G be a locally compact group. Then:*

a) if G is the directed union of closed subgroups with the fixed-point property for cones, then G has the fixed-point property for cones;

b) if G has a dense subgroup D with the translate property for $\mathcal{C}_u^b(D, \mathscr{R})$, then G has the fixed-point property for cones.

We want to stress that the dense subgroup is not necessarily locally compact in point b). For this reason, we have to specify for which functionally invariant uniformity the dense subgroup has the translate property.

Proof of Theorem 7.2.2. The proof of point a) is direct by Proposition 5.3.3 and Proposition 7.1.14.

Let's look at the proof of point b). The *if* direction is given by Proposition 7.1.10. So let's prove the *only if* direction. Suppose that D is a dense subgroup of G with the translate property for $\mathcal{C}_u^b(D, \mathscr{R})$. Then G has the translate property for $\mathcal{C}_u^b(G, \mathscr{R})$ by Proposition 7.1.16. This implies that G has the fixed-point property for cones thanks to Theorem 6.3.10. □

As a consequence, for locally compact groups, the fixed-point property for cones is a local property, i.e., a locally compact group has the fixed-point property for cones if and only if every of its compactly generated subgroups has it.

Moreover, thanks to the fact that the fixed-point property for cones is equivalent to the translate property for locally compact groups, the following holds:

Corollary 7.2.3. *Locally compact groups of subexponential growth have the fixed-point property for cones.*

Proof. The proof is only an application of Theorem 7.1.18. □

Example 7.2.4. Let G be a discrete abelian group. Then G has polynomial growth ([P88, Corollary 6.19]), and hence it has subexponential growth. Therefore, G has the $\mathscr{U}_d$-fixed-point property for cones. More generally, let G be an abelian topological group and let $\mathscr{U}$ be a functionally invariant uniform structure for G. We can deduce that G has the $\mathscr{U}$-fixed-point property for cones by Corollary 7.1.2.

Whether the converse is true, i.e., a locally compact group with the fixed-point property is of subexponential growth, is still an open problem, even for discrete groups.

Corollary 7.2.5. *Virtually nilpotent locally compact groups have the fixed-point property for cones.*

Proof. Nilpotent locally compact groups have subexponential growth by [P88, (6.18)], and the fixed-point property for cones passes through finite-index subgroups by Proposition 7.1.9. □

We say that a locally compact group G is a **topologically finite conjugancy classes group**, or G is a $\overline{FC}$**-topologically group**, if the closure of each of his conjugacy classes is compact. These types of groups have subexponential growth, see [P78]. Therefore,

Corollary 7.2.6. *Let G be a locally compact $\overline{FC}$-topologically group. Then G has the fixed-point property for cones.*

Finally, we mention another type of extension for discrete groups that preserves the fixed-point property for cones. The proof is not obvious and quite technical. Accordingly, we decided not to repeat it here.

Recall that a group G is said to be a **central extension** of a group N if N is contained in the center of G.

Theorem 7.2.7 (Monod)**.** *Central extensions of discrete groups with the fixed-point property for cones have the fixed-point property for cones.*

Proof. See [M17, Section 7]. □

7.2.A. About groups of subexponential growth. There is a direct way to show that a locally compact group with subexponential growth has the fixed-point property for cones. The strategy was pointed out by Paterson in [P88, (6.42)(i)]. He claimed that a locally compact group G with subexponential growth has the measurably translate property for $L^\infty(G)$ and, consequently, the fixed-point property for cones.[3] Since Paterson only sketched the proof, we decided to reproduce here a formal and complete version of it.

[3] Actually, we took inspiration for the measurably translated property's definition and the measurably invariant normalized integral property by this idea of Paterson. To be corrected, it was Paterson the first to introduce the measurably translate property under the name of *generalized translate property*.

Theorem 7.2.8. *Let G be a locally compact group of subexponential growth. Then G has the measurably translate property for* $L^\infty(G)$*.*

Proof. Let $f \in L^\infty(G)$ be a non-zero positive function and let $\mu \in \mathcal{M}_{00}(G)$ be a regular Borel measure with compact support s.t. $\mu * f \geq 0$.

Take a symmetric compact neighborhood C of the identity such that $|\mu|(G \setminus K) = 0$ and $\mathbf{1}_C * f \neq 0$, and define the sequence $(a_n)_{n\in\mathbf{N}} \subset \mathbf{R}$ given by $a_n = \int_{C^n} f(x)dm_G(x)$. First of all, note that

$$(a_n)^{\frac{1}{n}} = \left(\int_{C^n} f(x)dm_G(x)\right)^{\frac{1}{n}} \leq m_G(C^n)^{\frac{1}{n}} \left(\sup_{x\in C^n} f(x)\right)^{\frac{1}{n}} \leq m_G(C^n)^{\frac{1}{n}} ||f||_\infty^{\frac{1}{n}}$$

for every $n \in \mathbf{N}$. Therefore, $0 < \lim_n (a_n)^{\frac{1}{n}} \leq 1$.

Further,

$$\frac{a_{n+1}}{a_n} = \frac{\int_{C^n} f(x)dm_G(x) + \int_{C^{n+1}\setminus C^n} f(x)dm_G(x)}{\int_{C^n} f(x)dm_G(x)} \geq 1$$

and

$$\liminf_n \frac{a_{n+1}}{a_n} \leq \liminf_n (a_n)^{\frac{1}{n}} \leq 1$$

for every $n \in \mathbf{N}$. This last affirmation is true because for a sequence of real positive numbers $(a_n)_n$, we always have that

$$\liminf_n \frac{a_{n+1}}{a_n} \leq \liminf_n (a_n)^{\frac{1}{n}}.$$

Therefore, we can take a subsequence $(a_{n_k})_k$ of $(a_n)_n$ such that $\lim_k \frac{a_{n_k+2}}{a_{n_k}} = 1$. Define the sequence of positive functions $(F_k)_k$ given by $F_k(x) = \frac{1}{a_{n_k}} \int_{C^{n_k+1}} f(xg)dm_G(g)$. We claim that $(F_k)_k$ converges uniformly on C to $\mathbf{1}_C$. Indeed, we have to show that $\lim_k F_k(s) = 1$ for every $s \in C$. Fix $s \in C$ and compute that

$$F_k(s) = \frac{1}{a_{n_k}} \int_{C^{n_k+1}} f(xg)dm_G(g) = \frac{1}{a_{n_k}} \int_{sC^{n_k+1}} f(y)dm_G(y),$$

where $sg = y$. Note that $C^{n_k} \subset sC^{n_k+1}$. In fact, if $x \in C^{n_k+1}$, then $x = c_1 \cdot ... \cdot c_{n_k}$ where $c_j \in C^j$ and $j \in \{i, ..., n_k\}$. So, $x = s \cdot e \cdot s^{-1} \cdot c_1 \cdot ... \cdot c_{n_k} \in sC^{n_k+1}$. We can conclude that $F_k(s) \geq 1$. Moreover,

$$F_k(s) = \frac{1}{a_{n_k}} \int_{C^{n_k+1}} f(xg)dm_G(g) \leq \frac{1}{a_{n_k}} \int_{C^{n_k+2}} f(y)dm_G(y) = \frac{a_{n_k+2}}{a_{n_k}}.$$

Putting together this two facts, we conclude that

$$1 \leq \lim_k F_k(s) \leq \lim_k \frac{a_{n_k+2}}{a_{n_k}} = 1$$

for every $s \in C$. This shows that $(F_k)_k$ converges uniformly on C to $\mathbf{1}_C$.

In conclusion,

$$\begin{aligned}
\mu(G) &= \int_G \mathbf{1}_G(s) d\mu(s) \\
&= \int_{\text{supp}(\mu)} \mathbf{1}_G(s) d\mu(s) \\
&= \int_C \mathbf{1}_G(s) d\mu(s) \\
&= \int_C \lim_k F_k(s) d\mu(s) \\
&= \lim_k \int_C F_k(s) d\mu(s) \\
&= \lim_k \frac{1}{a_{n_k}} \int_C \int_{C^{n_k+1}} f(sg) dm_G(g) d\mu(s) \\
&= \lim_k \frac{1}{a_{n_k}} \int_{C^{n_k+1}} \int_C f(sg) d\mu(s) dm_G(g) \\
&= \lim_k \frac{1}{a_{n_k}} \int_{C^{n_k+1}} (\mu * f)(g) dm_G(g) \geq 0
\end{aligned}$$

showing that G has the measurably translate property for $L^\infty(G)$. □

7.2.B. Obstructions to the fixed-point property for cones. The goal of this section is to find phenomena that prevents a group of having the fixed-point property for cones or the translate property.

We recall that a **uniformly discrete free subsemigroup in two generators** of a topological group G is a subsemigroup T_2 generated by two elements $a, b \in G$ such that there is a neighborhood W of the identity with the property that $sW \cap tW = \emptyset$ for every $s, t \in \mathrm{T}_2$, $s \neq t$.

Proposition 7.2.9. *Let G be a locally compact group that contains a uniformly discrete free subsemigroup in two generators T_2. Then G does not have the fixed-point property for cones.*

Proof. Suppose it is not the case. Then there is a neighborhood W of the identity such that $sW \cap tW = \emptyset$ for every $s, t \in \mathrm{T}_2$. Define the open subset $U = \mathrm{T}_2 \cdot W$. Consequently, $\mathbf{1}_U > 0$ since the set U has Haar measure strictly bigger than zero. By Theorem 6.3.4, there is an invariant normalized integral I on $L^\infty(G, \mathbf{1}_U)$. But $aU \cap bU = \emptyset$. This means that the function $\phi = \mathbf{1}_U - \mathbf{1}_{aU} - \mathbf{1}_{bU}$ is non-zero, and it belongs to $L^\infty(G, \mathbf{1}_U)_+$.

Therefore,

$$0 \geq \mathrm{I}(\phi) = \mathrm{I}(\mathbf{1}_U) - \mathrm{I}(\mathbf{1}_{aU}) - \mathrm{I}(\mathbf{1}_{bU}) = -1,$$

which is a contradiction. □

In particular, not every extension of groups with the fixed-point property for cones has the fixed-point property for cones. The easiest example is given by the discrete group of affine transformations of the line $\mathbf{R} \rtimes \mathbf{R}^*$. In fact, it contains the free subsemigroup T_2 generated by the elements $(0, 2)$ and $(1, 1)$.

However, thanks to this last result, it is possible show that there are particular cases where having the fixed-point property for cones is equivalent to being of subexponential growth.

Proposition 7.2.10. *Let G be a locally compact group.*

a) Suppose that G is connected. Then G has the fixed-point property for cones if and only if G is of subexponential growth.

b) Suppose that G is compactly generated and almost connected. Then G has the fixed-point property for cones if and only if G is of subexponential growth.

Proof. In the two cases, the *if* direction is true by Corollary 7.2.3.

For the *only if* direction of point a), suppose that G is not of subexponential growth. Then G contains a uniformly discrete subsemigroup T_2 in two generators by [P88, Proposition 6.39]. Therefore, G can not have the fixed-point property for cones by proposition 7.2.9.

For the *only if* direction of point b), suppose that G has the fixed-point property for cones and consider the exact sequence given by

$$\{e\} \longrightarrow G_e \longrightarrow G \longrightarrow {}^{G}\!/_{G_e} \longrightarrow \{e\},$$

where G_e is the connected component of the identity. Since G_e is closed in G and ${}^{G}\!/_{G_e}$ is compact, the groups G and G_e have the same growth by [G73, Theorem 4.1]. But G_e has the fixed-point property for cones by Theorem 7.2.1, and hence it is also of subexponential growth by point a). Therefore, we can conclude that G has subexponential growth. □

Similarly, we have a characterization of the fixed-point property for cones in terms of free subsemigroups for the class of elementary discrete groups. Recall that the class of **elementary discrete groups** is the smallest class of discrete groups containing the finite and abelian groups, and which is closed by taking subgroups, extensions and direct limit.

Proposition 7.2.11. *A discrete elementary group has the fixed-point property for cones if and only if it does not contains a free semigroup in two generators.*

Proof. If G is a group with the fixed-point property for cones, then it has not T_2 as a subsemigroup.

Suppose now that G does not have T_2 as a subsemigroup. Then G has polynomial growth by [W94, Theorem 12.18]. Thus, G has the fixed-point property for cones by Corollary 7.2.3. □

Chapter 8

An Application to Invariant Radon Measures

This short chapter wants to give a couple of easy but exciting applications of the fixed-point property for cones. In particular, the fixed-point property for cones can be efficiently applied in problems where a non-zero invariant Radon measure is required.

The first part shows that cocompact actions of groups with the fixed-point property for cones on locally compact uniform spaces always admit a non-zero invariant Radon measure. Then this result is applied to show the unimodularity of locally compact groups with the fixed-point property for cones and give another proof of a well-known theorem for finitely generated orderable groups.

8.1 Cocompact actions and invariant Radon measures

Let $(X, \mathscr{U}_X)$ be a uniform space. We say that $(X, \mathscr{U}_X)$ is a **locally compact uniform space** if the topology generated by the uniform structure $\mathscr{U}_X$ is locally compact. As we suppose that every locally compact space is Hausdorff, the uniformity $\mathscr{U}_X$ has to be separated.

Before starting, it is worth reminding that each Radon measure on a locally compact space X can be seen as a positive functional defined on the Riesz space $\mathcal{C}_{00}(X)$ ([Bou63, III §1 No.5 Théorème 1]).

Theorem 8.1.1. *Let G be a topological group and let $\mathscr{U}$ be a functionally invariant uniformity for G. Suppose that G has the $\mathscr{U}$-fixed-point property for cones. Then every cocompact action of G on a locally compact uniform space $(X, \mathscr{U}_X)$ by uniform isomorphisms for which there is a point $x_0 \in X$ such that the map $g \longmapsto gx_0$ is $(\mathscr{U}, \mathscr{U}_X)$-uniformly continuous admits a non-zero invariant Radon measure.*

Proof. First of all, recall that the space $\mathcal{C}_{00}(X)$ admits a G-dominating element ϕ as the action of G on X is cocompact (Proposition 3.4.6). In particular, $\mathcal{C}_{00}(X, \phi) = \mathcal{C}_{00}(X)$.

Without loss of generality, we can suppose that $\phi(x_0) \neq 0$. Therefore, we can apply point d) of Theorem 5.2.1 to obtain an invariant integral for $\mathcal{C}_{00}(X)$ normalized on ϕ which is nothing but a non-zero invariant Radon measure. □

Proposition 8.1.2. *Let G be a topological group and let $(X, \mathscr{U}_X)$ be a locally compact uniform space. Suppose that G acts cocompactly on $(X, \mathscr{U}_X)$ by uniform isomorphisms. Then*

a) if G has the $\mathscr{F}$-fixed-point property for cones and the action of G on X is orbitally continuous, then there is a non-zero invariant Radon measure on X;

b) if G has the $\mathscr{R}$-fixed-point property for cones and the action of G on X is motion equicontinuous, then there is a non-zero invariant Radon measure on X;

c) if G has the $\mathscr{L}$-fixed-point for cones and the action of G on X is uniformly equicontinuous, then there is a non-zero invariant Radon measure on X.

Proof. The proof is only a combination of Proposition 3.4.6 and Corollary 5.2.11. □

It is not surprising that the discussion becomes interesting when we look at the right uniform structure.

Theorem 8.1.3. *Let G be a topological group with the translate property for $\mathcal{C}_u^b(G, \mathscr{R})$. Then each jointly continuous cocompact action of G on any locally compact topological space X has a non-zero invariant Radon measure.*

Proof. As the action of G on X is jointly continuous, the induced representation of G on $\mathcal{C}_{00}(X)$ is continuous by proposition 3.4.10. Let now ϕ be the support-dominating and G-dominating element of $\mathcal{C}_{00}(X)$ given by proposition 3.4.6. Then the representation of G on $\mathcal{C}_{00}(X)$ is also continuous with respect to the p_ϕ-norm by Proposition 3.4.10. Using the fact that G also has the abstract continuous translate property (Theorem 5.3.5), we can find a p_ϕ-continuous positive invariant linear functional I on $\operatorname{span}_{\mathbf{R}} \{g\phi : g \in G\}$ normalized on ϕ. Finally, we can extend it in an invariant way to the whole space $\mathcal{C}_{00}(X)$ using Proposition 4.2.21 together with Corollary 4.2.13. □

8.2 Unimodularity

Let G be a locally compact group and write Δ_G for its modular function.

Corollary 8.2.1. *Let G be a locally compact group with the fixed-point property for cones. Then G is unimodular.*

Proof. Suppose it is not the case. This means that there is $g \in G$ such that $\Delta_G(g) = c \neq 1$. Let $H := \langle g \rangle$ be the group generated by g. We claim that $H \cong \mathbf{Z}$, and that H is closed and discrete as a subgroup of G. This is because of the following facts.

- *The element g has infinite order.* Suppose it is not the case. Then there is $n \in \mathbf{N}$ such that $g^n = e$. This implies that

$$\Delta_G(g)^n = \Delta_G(g^n) = \Delta_G(e) = 1 \quad \Longleftrightarrow \quad \Delta_G(g) = 1.$$

  But this is a contradiction.

- *H is closed.* Suppose it is not the case. Then there is a net $(g_\alpha)_\alpha \subset H$ such that $\lim_\alpha g_\alpha = x$ and $x \notin H$. Note that $(g_\alpha)_\alpha = (g^{n_\alpha})_\alpha$, where $(n_\alpha)_\alpha$ is a net in $\mathbf{N}$ converging to infinity. Then

$$\Delta_G(x) = \Delta_G(\lim_\alpha g^{n_\alpha}) = \lim_\alpha \Delta_G(g)^{n_\alpha} = \lim_\alpha c^{n_\alpha} = \begin{cases} 0 & \text{if } c < 1 \\ \infty & \text{if } c > 1. \end{cases}$$

  But this is a contradiction with the fact that $im(\Delta_G)$ is a subgroup of $\mathbf{R}^*$.

- *H is discrete.* Suppose it is not the case. Then there is a net $(g_\alpha)_\alpha \subset H$ such that $\lim_\alpha g_\alpha = e$. Employing the same strategy as before, we can find a contradiction.

Therefore, we can conclude that H is isomorphic to $\mathbf{Z}$. Now let G act on the locally compact space G/H, and note that this action is continuous and cocompact. By Theorem 8.1.3, there is a non-zero invariant Radon measure on G/H. But this is possible if and only if the restriction of Δ_G on the subgroup H is equal to Δ_H as explained in [Bou59, Chap. II §2 No.6 Corollaire 2]. As H is discrete, $\Delta_H = 1$. Thus,

$$1 \neq \Delta_G(g) = \Delta_H(g) = 1,$$

which is a contradiction. We can conclude that G is unimodular. □

Corollary 8.2.2. *Closed subgroups of groups with the fixed-point property for cones are unimodular.*

Proof. Closed subgroups of a group with the fixed-point property for cones have the fixed-point property for cones by Theorem 7.2.1. Thus, they are unimodular. □

Remark 8.2.3. In general, unimodularity does not pass to closed subgroups. An easy example is given by the unimodular locally compact Lie group $\mathbf{GL}_2(\mathbf{R})$ that contains the non-unimodular $(ax+b)$-group as a closed subgroup.

Corollary 8.2.4 ([G73, Lemme I.3.]). *Locally compact groups of subexponential growth are unimodular.*

Proof. We know that locally compact groups of subexponential growth have the fixed-point property for cones by Corollary 7.2.3. Therefore, they are unimodular. □

8.3 Fixing Radon measures on the line

We want to apply the close relationship between the fixed-point property for cones and non-zero invariant Radon measures to a dynamical problem. Namely, when the natural action of a subgroup of $\mathrm{Homeo}^+(\mathbf{R})$, the group of order-preserving homeomorphisms of the line, on $\mathbf{R}$ fixes a non-zero Radon measure. This problem was studied mainly by Plante in [P83] and [P75], who discovered that finitely generated virtually nilpotent subgroups of $\mathrm{Homeo}^+(\mathbf{R})$ always fix a non-zero Radon measure on the line (an exposition of those results can be found in [N11, Subsection 2.2.5]). This result is due to the fact that finitely generated virtually nilpotent groups are of subexponential growth, so it is not surprising that it si possible to generalise this theorem to the class of groups with the fixed-point property for cones. Interestingly, the proof for groups with the fixed-point property for cones is much more natural and less technical than the one for virtually nilpotent groups.

Recall that a subgroup G of $\mathrm{Homeo}^+(\mathbf{R})$ is said to be **boundedly generated** if there is a symmetric set of generators S of G and a point $x_0 \in \mathbf{R}$ such that the set $\{sx_0 : s \in S\}$ is a bounded subset of the line.

Theorem 8.3.1. *Let G be a boundedly generated subgroup of* $\mathrm{Homeo}^+(\mathbf{R})$ *with the fixed-point property for cones. Then there is a non-zero G-invariant Radon measure on* $\mathbf{R}$.

Proof. Let $S \subset G$ be a symmetric set that generates G boundedly. If the action of G on $\mathbf{R}$ has a global fixed point, then a Dirac mass on this point is a non-zero invariant Radon measure.

Thus, suppose that the action of G on $\mathbf{R}$ has no global fixed points. We claim that the action is of cocompact-type in this case. Indeed, let $x_0 \in \mathbf{R}$ be the point which witnesses the fact that S generates G boundedly and let $x_1 = \sup_{s\in S} s_j x_0$. Now, the set $I = [x_0, x_1]$ is compact because S generates G boundedly. If we can show that every orbit of G intersects the interval I, then we have that the action is cocompact. Let $x \in \mathbf{R}$ and note that the orbit Gx is unbounded in the two directions. Otherwise, its supremum and its infimum would be global fixed points. Thus, we can chose x'_0, x'_1 in Gx such that $x'_0 < x_0 < x_1 < x'_1$. Let $g = s_{j_n} \cdots s_{j_1} \in G$ be such that $gx'_0 = x'_1$ and let $m \in \{1, ..., n-1\}$ be the largest index for which the inequality $s_{j_m} \cdots s_{j_1} x'_0 < x_0$ holds. Then $s_{j_{m+1}} s_{j_m} \cdots s_{j_1} x'_0$ is in the orbit of x, it is greater than or equal to x_0, and it is smaller or equal to x_1 by definition. Therefore, we have that the orbit of x intersects I. Now that we know about the cocompactness of the action, we can conclude by applying Theorem 8.1.3. □

In particular, Theorem 8.3.1 generalizes [P75, Theorem (5.4)].

Remark 8.3.2. The assumption that G is boundedly generated cannot be dropped. In fact, there are even examples of abelian non-boundedly generated subgroups of $\mathrm{Homeo}^+(\mathbf{R})$ which fix no non-zero invariant Radon measure on the line (see [P83, Section 5]).

Scholium 8.3.3. Every result of this last subsection could also be proved using the slightly more general concept of supramenability instead of the fixed-point property for cones. This is possible thanks to [KMN13, Proposition 2.7].

Appendix A

Embeddings of Cones in Vector Spaces

This first appendix discusses embedding of abstract cones in vector spaces. It is a folkloric result that, given a convex compact set, it is always possible to embeds it into the continuous dual of a Banach space endowed with the weak-* topology ([G17, Proposition 5.1.7]). The question is: can we do the same with cones instead of compact sets? Namely, given an abstract cone, can we embed it in a suitable vector space? The answer to these questions is positive but more complicated than the one for compact sets.

The study of this problem was started by Rådström in his paper [R52]. He defined what is an abstract cone and explained how to construct a vector space out of it. Moreover, he showed that finding a linear embedding of the cone into its corresponding vector space was possible for a particular class of abstract cones. The construction of Rådström turned out to be very fruitful, and it was largely used to prove more sophisticated embedding theorems. See for examples [U76], [F85], [H55] and [S86]. The method of Rådström became so important and famous also for its applications in probability theory, mathematical economics, interval mathematics and related areas ([S86, Section 9]). We refer to [S85] for an excellent survey about results based on the Rådström construction.

However, there are other exciting embedding theorems for cones, such as the embedding theorem for locally compact cones of Edwards. He showed in [E64] that, given a generating locally compact cone in a locally convex vector space, it is possible to embed it continuously and linearly in the continuous dual of a Banach space endowed with the weak-* topology. This result generalizes the embedding theorem for convex compact sets quoted before. Note that, differently from the approach taken by Rådström, the locally convex vector space, where the locally compact cone lives, is part of the data for the construction of the embedding.

One could ask if it is also possible to state the fixed-point property for cones using abstract cones. We can already affirm that it is not possible. Indeed, in the fixed-point property for cones, the data of the locally convex vector space which contains the cone on which we are acting is essential. This is because of the group representation's cobounded condition, which also considers the dual of the vector space. Note that it is possible to state the definition of amenability using *abstract* convex compact sets. Furthermore, it is not possible to restrict the class of cones for which we have to check the fixed-point property for cones because of its generality.

In the first part of the appendix, the work of Rådström and some of its applications are discussed. In particular, it is explained how it is possible to extend an action of a group on an abstract cone to a linear representation on the vector space associated with it. After, the Edwards theorem for locally compact cones is presented and used to characterise amenability for topological groups.

A.1 Embedding abstract cones

We start by defining the concept of an abstract cone $\mathcal{C}$ and explain the Rådström construction introduced in [R52] of the vector space $E_{\mathcal{C}}$ and the embedding of $\mathcal{C}$ in $E_{\mathcal{C}}$. We are going to utilise it consistently in this first section. Afterwards, we look at abstract cones that admit a partial order and abstract cones that admit metrics.

A.1.A. The Rådström construction. Recall that a semigroup is a set together with a binary operation that satisfies the associative property.

Definition A.1.1. An **abstract cone** is a commutative semigroup $\mathcal{C}$ with a distinguished element $e \in \mathcal{C}$ such that $c + e = c$ for every $c \in \mathcal{C}$ and a map $\cdot : \mathbf{R}_+ \times \mathcal{C} \longrightarrow \mathcal{C}$, called scalar multiplication, satisfying:

(A1) $r \cdot (c_1 + c_2) = r \cdot c_1 + r \cdot c_2$ for all $r \in \mathbf{R}_+$ and $c_1, c_2 \in \mathcal{C}$;

(A2) $(r_1 + r_2) \cdot c = r_1 \cdot c + r_2 \cdot c$ for all $r_1, r_2 \in \mathbf{R}_+$ and $c \in \mathcal{C}$;

(A3) $(r_1 r_2) \cdot c = r_1(r_2 \cdot c)$ for all $r_1, r_2 \in \mathbf{R}_+$ and $c \in \mathcal{C}$;

(A4) $1 \cdot c = c$ and $0 \cdot c = e$ for all $c \in \mathcal{C}$.

The element e of an abstract cone $\mathcal{C}$ is also called the **zero element** of $\mathcal{C}$. Note that $r \cdot e = e$ for every $r \in \mathbf{R}_+$.

Example A.1.2. (Examples of abstract cones)

1) Let E be a vector space and let $C \subset E$ be a cone as in Definition 2.1.1. Then C is an abstract cone.

2) Let E be a vector space and let $\mathbb{P}(E)$ be the collection of all subsets of E. We equip it with the pointwise addition and the pointwise scalar multiplication of sets, i.e.,

$$A + B = \{a + b : a \in A \text{ and } b \in B\} \quad \text{and} \quad c \cdot A = \{ca : a \in A\}$$

for $A, B \in \mathbb{P}(E)$ and $c \in \mathbf{R}_+$. Then $\mathbb{P}(E)$ is an abstract cone with zero element given by the origin of E.

Definition A.1.3. An abstract cone $\mathcal{C}$ is said to have the **cancellation property** if for every pair $c_1, c_2 \in \mathcal{C}$ for which there is $z \in \mathcal{C}$ such that $c_1 + z = c_2 + z$, the equality $c_1 = c_2$ holds.

Note that every cone as in Definition 2.1.1 has the cancellation property because of the properties of an ordered vector space.

Example A.1.4. (Examples of abstract cones with the cancellation property)

1) Let E be a locally convex space and consider the collection $\mathbb{B}(E)$ of all closed bounded convex subsets of E. We define the closed sum of two elements A, B of $\mathbb{B}(E)$ as

$$A \overline{+} B = \overline{\{a + b : a \in A \text{ and } b \in B\}}.$$

Then $\mathbb{B}(E)$ equipped with this closed sum and the pointwise scalar multiplication becomes an abstract cone with zero element given by the origin of E.[1] Moreover, $\mathbb{B}(E)$ has the cancellation property as showed in [S86, Theorem 2.2].

2) Let E be a Banach space and let $d_{\mathcal{H}}$ be a Hausdorff metric. Define the collection

$$\mathbb{U}(E) = \{A \subset E : A \text{ is a non-empty closed convex set s.t. } d_{\mathcal{H}}(A, \{0\}) < \infty\}.$$

Then $\mathbb{U}(E)$ equipped with the closed sum and the pointwise scalar multiplication is an abstract cone with the cancellation property, see [B09, Proposition 2].

Suppose to have an abstract cone $\mathcal{C}$ with the cancellation property and define the equivalence relation $\sim$ on $\mathcal{C} \times \mathcal{C}$ by

$$(c_1, c_2) \sim (c_3, c_4) \iff c_1 + c_4 = c_2 + c_3 \quad \text{for every } (c_1, c_2) \text{ and } (c_3, c_4) \in \mathcal{C} \times \mathcal{C}.$$

Write $\langle c_1, c_2 \rangle$ for the equivalence class containing (c_1, c_2) and $E_{\mathcal{C}}$ for the **set of all equivalence classes of $\mathcal{C} \times \mathcal{C}$**.

On this last set, we define an addition by letting

$$\langle c_1, c_2 \rangle + \langle c_3, c_4 \rangle = \langle c_1 + c_3, c_2 + c_4 \rangle \quad \text{for every } \langle c_1, c_2 \rangle, \langle c_3, c_4 \rangle \in E_{\mathcal{C}}$$

and a scalar multiplication by

$$r \cdot \langle c_1, c_2 \rangle = \begin{cases} \langle r \cdot c_1, r \cdot c_2 \rangle & \text{if } r \in \mathbf{R}_+ \\ \langle (-r) \cdot c_2, (-r) \cdot c_1 \rangle & \text{otherwise.} \end{cases}$$

Furthermore, we define the map

$$j : \mathcal{C} \longrightarrow E_{\mathcal{C}}, \quad c \longmapsto j(c) = \langle c, e \rangle.$$

[1] Note that $\mathbb{B}(E)$ only equipped with the pointwise addition does not need to be a commutative semigroup as the sum of two closed sets may fail to be closed.

Theorem A.1.5 (Rådström's Basic Embedding Theorem)**.** *Let $\mathcal{C}$ be an abstract cone with the cancellation property. Then $E_{\mathcal{C}}$ is a real vector space satisfying $E_{\mathcal{C}} = j(\mathcal{C}) - j(\mathcal{C})$, and j is an additive positive homogeneous embedding. In particular, $\mathcal{C}$ can be seen as a generating cone in $E_{\mathcal{C}}$.*

Proof. See [R52, Theorem 1-A]. □

Let's now turn our attention to group actions on abstract cones.

Definition A.1.6. An action of G on an abstract cone $\mathcal{C}$ is nothing but a group action of G on the set $\mathcal{C}$ such that:

- $ge = e$ for every $g \in G$;
- $g(r \cdot c_1 + c_2) = r \cdot (gc_1) + gc_2$ for every $g \in G$, $r \in \mathbf{R}_+$ and $c_1, c_2 \in \mathcal{C}$.

Proposition A.1.7. *Suppose that a group G acts on an abstract cone $\mathcal{C}$ with the cancellation property. Then the action of G on $\mathcal{C}$ extends to a representation of G on $E_{\mathcal{C}}$ by linear automorphisms.*

Proof. As G acts on $\mathcal{C}$, then G also acts on $\mathcal{C} \times \mathcal{C}$ and on $E_{\mathcal{C}}$ vie the rule $g\langle c_1, c_2 \rangle = \langle gc_1, gc_2 \rangle$ for every $g \in E$ and $\langle c_1, c_2 \rangle \in E_{\mathcal{C}}$. Note that the action of G on $E_{\mathcal{C}}$ is linear. Now the map j is equivariant with respect to this last action of G on $E_{\mathcal{C}}$. In fact,

$$j(gc) = \langle gc, e \rangle = g\langle c, e \rangle = g\, j(c)$$

for every $g \in G$ and $c \in \mathcal{C}$. Therefore, the action of G on $\mathcal{C}$ extends to a representation of G on $E_{\mathcal{C}}$ by linear automorphisms. □

A.1.B. Abstract ordered cones.

Definition A.1.8. An **abstract ordered cone** is an abstract cone $\mathcal{C}$ together with and order relation $\leq$ such that the inequalities

$$c_1 + z \leq c_2 + z \quad \text{and} \quad r \cdot c_1 \leq r \cdot c_2$$

hold for every $c_1, c_2 \in \mathcal{C}$ such that $c_1 \leq c_2$ and for every $z \in \mathcal{C}$ and $r \in \mathbf{R}_+$.

The set $\mathcal{C}_+ = \{c \in \mathcal{C} : e \leq c\}$ is called the **positive abstract cone** of $\mathcal{C}$.

Definition A.1.9. Let $\mathcal{C}$ be an abstract ordered cone. Then:

- the cone $\mathcal{C}$ has the **order cancellation property** if $c_1 \leq c_2$ holds for every $c_1, c_2 \in \mathcal{C}$ satisfying $c_1 + z \leq c_2 + z$ for some $z \in \mathcal{C}$;
- the cone $\mathcal{C}$ is **Archimedean** if $c_1 \leq c_2$ holds for all $c_1, c_2 \in \mathcal{C}$ satisfying $nc_1 + z_1 \leq nc_2 + z_2$ for some $z_1, z_2 \in \mathcal{C}$ and all $n \in \mathbf{N}$;

- the cone $\mathcal{C}$ has the **Hukuhara property** if for all $c_1, c_2 \in \mathcal{C}$ satisfying $c_1 \leq c_2$ there exists some $z \in \mathcal{C}_+$ such that $c_1 + z = c_2$.

Note that the Archimedean property implies the order cancellation property, and that the cancellation property and the Hukuhara property together imply the order cancellation property.

Example A.1.10. (Examples of abstract ordered cone)

1) Let C be a proper cone in a vector space E. Then C and its induced order are an abstract ordered cone with the order cancellation property and the Hukuhara property.

2) Let E be a locally convex vector space. Then the abstract cone $\mathbb{B}(E)$ equipped with the set inclusion as order relation is an abstract ordered cone with the order cancellation property, see [S86, Theorem 3.1].

Let $\mathcal{C}$ be an abstract ordered cone, and define on $E_\mathcal{C}$ an order relation via the rule

$$\langle c_1, c_2 \rangle \leq \langle c_3, c_4 \rangle \iff c_1 + c_4 \leq c_2 + c_3.$$

Recall that a map f between two partially ordered sets $(X, \leq_X)$ and $(Y, \leq_Y)$ is said **order-preserving** if $f(x_1) \leq_Y f(x_2)$ whenever $x_1 \leq_X x_2$.

Theorem A.1.11. *Let $\mathcal{C}$ be an abstract ordered cone with the order cancellation property. Then $E_\mathcal{C}$ is an ordered vector space, and the maps j and j^{-1} are order-preserving. Moreover,*

a) the ordered vector space $E_\mathcal{C}$ is Archimedean if and only if $\mathcal{C}$ is Archimedean;

b) the abstract cone $\mathcal{C}$ has the Hukuhara property if and only if $(E_\mathcal{C})_+ = j(\mathcal{C}_+)$.

Proof. See [K77, Satzt 1]. □

In particular, when $\mathcal{C}$ has the Hukuhara property, it is possible to identify its positive cone with the positive cone of the ordered vector space $E_\mathcal{C}$.

Definition A.1.12. Suppose that a group G acts on an abstract ordered cone $\mathcal{C}$. We say that the action of G on $\mathcal{C}$ is **order-preserving** if for every $g \in G$ and $c_1, c_2 \in \mathcal{C}$ such that $c_1 \leq c_2$, then $gc_1 \leq gc_2$.

Proposition A.1.13. *Suppose that a group G acts on an abstract ordered cone $\mathcal{C}$ with the order cancellation property. If the action of G on $\mathcal{C}$ is order-preserving, then it extends to a representation of G on $E_\mathcal{C}$ by positive linear automorphisms.*

Proof. We already saw in Proposition A.1.7 how to extend the action on $\mathcal{C}$ to a representation on $E_\mathcal{C}$. We only have to show that the representation is by positive automorphisms. Therefore, suppose that the action of G on $\mathcal{C}$ is order-preserving and let $\langle c_1, c_2 \rangle, \langle c_3, c_4 \rangle \in \mathcal{C}$ such that $\langle c_1, c_2 \rangle \leq \langle c_3, c_4 \rangle$. This means that $c_1 + c_4 \leq c_2 + c_3$. Thus,

$$g(c_1 + c_4) = gc_1 + gc_4 \leq gc_2 + gc_3 = g(c_2 + c_3) \quad \text{for every } g \in G.$$

It is possible to conclude that

$$g\langle c_1, c_2\rangle = \langle gc_1, gc_2\rangle \le \langle gc_3, gc_4\rangle = g\langle c_3, c_4\rangle \quad \text{for every } g \in G.$$

This shows that the representation of G on $E_{\mathcal{C}}$ is by positive linear automorphisms. □

Definition A.1.14. Let $\mathcal{C}$ be an abstract ordered cone. We say that $\mathcal{C}$ is an **upper semilattice cone** if $c_1 \vee c_2 = \sup\{c_1, c_2\}$ exists for every two $c_1, c_2 \in \mathcal{C}$, and the identity

$$(c_1 + z) \vee (c_2 + z) = c_1 \vee c_2 + z \quad \text{holds for all } c_1, c_2, z \in \mathcal{C}.$$

Note that an upper semilattice cone with the cancellation property has the order cancellation property. Indeed, let $c_1, c_2 \in \mathcal{C}$ such that $c_1 + z \le c_2 + z$ for some $z \in \mathcal{C}$, and suppose that $c_1 > c_2$. Thus,

$$c_1 + z = c_1 \vee c_2 + z = (c_1 + z) \vee (c_2 + z) = c_2 + z$$

and $c_1 = c_2$ by the cancellation property. But this is a contradiction. Therefore, $c_1 \le c_2$.

Clearly, every lattice cone in a vector space is an upper semilattice cone with the order cancellation property. Another example of an upper semilattice cone is given by $\mathbb{B}(E)$ when E is a locally convex vector space ([S86, Theorem 4.1]).

Theorem A.1.15. *Let $\mathcal{C}$ be an upper semilattice cone with the cancellation property. Then $E_{\mathcal{C}}$ is a Riesz space.*

Proof. See [K77, Satzt 2]. □

In particular, the identities

$$\langle c_1, c_2\rangle \vee \langle c_3, c_4\rangle = \langle (c_1 + c_4) \vee (c_2 + c_3), c_2 + c_4\rangle$$

and

$$\langle c_1, c_2\rangle \wedge \langle c_3, c_4\rangle = \langle c_1 + c_3, (c_1 + c_4) \vee (c_2 + c_3)\rangle$$

hold for every $\langle c_1, c_2\rangle, \langle c_3, c_4\rangle \in E_{\mathcal{C}}$. We can deduce that the absolute value of an element $\langle c_1, c_2\rangle \in E_{\mathcal{C}}$ is given by

$$|\langle c_1, c_2\rangle| = \langle c_1, c_2\rangle \vee \langle c_2, c_1\rangle = \langle (c_1 + c_1) \vee (c_2 + c_2), c_1 + c_2\rangle.$$

Definition A.1.16. Suppose that a group G acts on an upper semilattice cone $\mathcal{C}$. We say that the action of G on $\mathcal{C}$ is **sup-preserving** if $g(c_1 \vee c_2) = gc_1 \vee gc_2$ for every $g \in G$ and $c_1, c_2 \in \mathcal{C}$.

A sup-preserving action of a group G on an upper semilattice $\mathcal{C}$ is automatically order-preserving. Indeed, if $c_1, c_2 \in \mathcal{C}$ such that $c_1 \le c_2$, then $c_2 - c_1 \ge 0$. Therefore, $0 \vee (c_2 - c_1) = c_2 - c_1$. Take $g \in G$ and note that

$$0 \vee (gc_2 - gc_1) = g0 \vee g(c_2 - c_1) = g(0 \vee (c_2 - c_1)) = gc_2 - gc_1.$$

This implies that $gc_2 - gc_1 \ge 0$, and hence that $gc_1 \le gc_2$ showing that the action is also order-preserving.

Proposition A.1.17. *Suppose that a group G acts on an upper semilattice cone $\mathcal{C}$ with the cancellation property. If the action of G on $\mathcal{C}$ is sup-preserving, then it extends to a representation of G on $E_{\mathcal{C}}$ by Riesz automorphisms.*

Proof. By Proposition A.1.13, the induced representation of G on $\mathcal{C}$ is by positive linear automorphisms. We have to prove that it preserves the supremum of vectors. To this end, let $\langle c_1, c_2\rangle$ and $\langle c_3, c_4\rangle$ be two vectors in $E_{\mathcal{C}}$, and compute that

$$\begin{aligned} g\left(\langle c_1, c_2\rangle \vee \langle c_3, c_4\rangle\right) &= g\langle (c_1+c_4)\vee(c_2+c_3), c_2+c_4\rangle \\ &= \langle g(c_1+c_4)\vee g(c_2+c_3), g(c_2+c_4)\rangle \\ &= \langle (gc_1+gc_4)\vee(gc_2+gc_3), gc_2+gc_4\rangle \\ &= \langle gc_1, gc_2\rangle \vee \langle gc_3, gc_4\rangle = g\langle c_1, c_2\rangle \vee g\langle c_3, c_4\rangle \end{aligned}$$

for every $g \in G$. We can conclude that the representation of G on $E_{\mathcal{C}}$ is by Riesz automorphisms. □

A.1.C. Abstract normed cones. We recall that a positively homogeneous and translation invariant metric $d_{\mathcal{C}}$ on a semigroup $\mathcal{C}$ is only a distance function on $\mathcal{C}$ such that

$$d_{\mathcal{C}}(rc_1, rc_2) = r\, d_{\mathcal{C}}(c_1, c_2) \quad \text{and} \quad d_{\mathcal{C}}(c_1+c, c_2+c) = d_{\mathcal{C}}(c_1, c_2)$$

for every $c, c_1, c_2 \in \mathcal{C}$ and every $r \in \mathbf{R}_+$.

Definition A.1.18. Suppose that $\mathcal{C}$ is an abstract cone and that $d_{\mathcal{C}}$ is a positively homogeneous and translation invariant metric defined on $\mathcal{C}$. Then the pair $(\mathcal{C}, d_{\mathcal{C}})$ is called an **abstract metric cone**.

Every abstract metric cone $(\mathcal{C}, d_{\mathcal{C}})$ has the cancellation property. In fact, let $c_1, c_2 \in \mathcal{C}$ and $z \in \mathcal{C}$ such that $c_1 + z = c_2 + z$. Then

$$d_{\mathcal{C}}(c_1, c_2) = d_{\mathcal{C}}(c_1+z, c_2+z) = 0$$

which implies $c_1 = c_2$ as wished.

Example A.1.19. (Examples of abstract metric cones)

1) Let C be a cone in a normed vector space $(E, ||\cdot||)$. Define the metric

$$d_{||\cdot||}(c, w) = ||c - w|| \quad \text{for } c, w \in C.$$

Then the pair $(C, d_{||\cdot||})$ is an abstract metric cone.

2) The collection $\mathbb{K}(E)$ of all compact convex subsets of a normed vector space E equipped with the Hausdorff distance $d_{\mathcal{H}}$ is an abstract metric cone, see [R52, Theorem 2].

3) Let E be a Banach space. Then the abstract cone $\mathbb{U}(E)$ equipped with the Hausdorff distance is an abstract metric cone, see [B09, Propositions 3-5].

Take now an abstract metric cone $(\mathcal{C}, d_{\mathcal{C}})$, and consider the vector space $E_{\mathcal{C}}$. We define on it the positive map

$$||\cdot||_{\mathcal{C}} : E_{\mathcal{C}} \longrightarrow \mathbf{R}, \quad \langle c_1, c_2\rangle \longmapsto ||\langle c_1, c_2\rangle||_{\mathcal{C}} = d_{\mathcal{C}}(c_1, c_2).$$

Theorem A.1.20. *Let $(\mathcal{C}, d_{\mathcal{C}})$ be an abstract metric cone. Then $(E_{\mathcal{C}}, ||\cdot||_{\mathcal{C}})$ is a normed vector space, and the embedding j is an isometry.*

Proof. See [R52, Theorem 1-B]. □

Proposition A.1.21. *Suppose that a group G acts isometrically on an abstract metric cone $(\mathcal{C}, d_{\mathcal{C}})$. Then the action of G on $\mathcal{C}$ extends to a representation of G on $E_{\mathcal{C}}$ by linear isometries.*

Proof. The proof is straightforward, since

$$||g\langle c_1, c_2\rangle||_{\mathcal{C}} = ||\langle gc_1, gc_2\rangle||_{\mathcal{C}} = d_{\mathcal{C}}(gc_1, gc_2) = d_{\mathcal{C}}(c_1, c_2) = ||\langle c_1, c_2\rangle||_{\mathcal{C}}$$

for every $\langle c_1, c_2\rangle \in \mathcal{C}$ and $g \in G$. □

Definition A.1.22. Let $\mathcal{C}$ be an upper semilattice cone and let $d_{\mathcal{C}}$ be a positively homogeneous translation invariant metric on $\mathcal{C}$ such that $d_{\mathcal{C}}(c_1, c_2) \leq d_{\mathcal{C}}(c_3, c_4)$ for every $c_1, c_2, c_3, c_4 \in \mathcal{C}$ such that $c_2 \leq c_1$, $c_4 \leq c_3$ and $c_1 + c_4 \leq c_2 + c_3$. Then the pair $(\mathcal{C}, d_{\mathcal{C}})$ is called an **upper semilattice metric cone**.

Every upper semilattice metric cone $(\mathcal{C}, d_{\mathcal{C}})$ has the Archimedean property. In fact, let $c_1, c_2 \in \mathcal{C}$ such that $nc_1 + z_1 \leq nc_2 + z_2$ for some $z_1, z_2 \in \mathcal{C}$ and every $n \in \mathbf{N}$. In order to find a contradiction, suppose that $c_1 > c_2$. This implies that the inequality $nc_1 + z_1 > nc_2 + z_1$ holds for every $n \in \mathbf{N}$. Therefore,

$$n^2 d_{\mathcal{C}}(c_2, c_1) = d_{\mathcal{C}}(nc_2 + z_1, nc_1 + z_1) \leq d_{\mathcal{C}}(nc_2 + z_2, nc_2 + z_1) = d_{\mathcal{C}}(z_1, z_2),$$

and so

$$d_{\mathcal{C}}(c_2, c_1) \leq \frac{1}{n^2} d_{\mathcal{C}}(z_1, z_2) \quad \text{for every } n \in \mathbf{N}.$$

Thus, $d_{\mathcal{C}}(c_2, c_1) = 0$. But this is a contradiction. Hence, $c_1 \leq c_2$. In particular, every upper semilattice cone has the order cancellation property.

Example A.1.23. (Examples of upper semilattice metric cones)

1) Let $(E, ||\cdot||)$ be a normed Riesz space and let C be the positive cone of E. Then C equipped with the metric $d_{||\cdot||}$ defined by $d_{||\cdot||}(c, w) = ||c - w||$, for $c, w \in C$, is an upper semilattice metric cone. Indeed, let c_1, c_2, c_3 and c_4 in C such that $c_1 \leq c_2$, $c_3 \leq c_4$ and $c_1 + c_4 \leq c_2 + c_3$. This implies that $0 \leq c_1 - c_2 \leq c_3 - c_4$. Therefore,

$$d_{||\cdot||}(c_1, c_2) = ||c_1 - c_2|| \leq ||c_3 - c_4|| = d_{||\cdot||}(c_3, c_4)$$

using the monotonicity of the norm $||\cdot||$.

2) Let E be a normed space. Then the upper semilattice cone $\mathbb{B}(E)$ equipped with the Hausdorff metric is an upper semilattice metric cone, see [S86, Theorem 6.1].

Theorem A.1.24. *Let $(\mathcal{C}, d_{\mathcal{C}})$ be an upper semilattice metric cone. Then $(E_{\mathcal{C}}, ||\cdot||_{\mathcal{C}})$ is a normed Riesz space, and the embedding j is an order-preserving isometry.*

Proof. We have that $(E_{\mathcal{C}}, ||\cdot||_{\mathcal{C}})$ is a normed vector space by Theorem A.1.20 and that $E_{\mathcal{C}}$ is a Riesz space by Theorem A.1.15. Moreover, the norm $||\cdot||_{\mathcal{C}}$ is monotone by definition of upper semilattice metric cone. Therefore, we can conclude that $(E_{\mathcal{C}}, ||\cdot||_{\mathcal{C}})$ is a normed Riesz space. □

Proposition A.1.25. *Suppose that a group G acts isometrically and order-preserving on an upper semilattice metric cone $(\mathcal{C}, d_{\mathcal{C}})$. Then the action of G on $\mathcal{C}$ extends to a representation of G on $E_{\mathcal{C}}$ by positive linear isometries.*

Proof. The proof is only a combination of Propositions A.1.13 and A.1.21. □

A.2 Embedding of locally compact cones

We present a theorem of Edwards for locally compact cones in locally convex vector spaces in this second section. First, we recall some results on locally compact cones, and then we discuss Edwards's theorem. Finally, we use it to characterise the amenability of topological groups via representations on locally compact cones.

A.2.A. Bases and topologies. Let C be a cone in a locally convex vector space E. A **base** for C is a non-empty convex set $B \subset C$ such that every non-zero $c \in C$ has a unique representation of the form $c = \alpha b$ for $b \in B$ and $\alpha \in \mathbf{R}_+$.

The following theorem is a famous and well-known result of Klee about cones that admit a compact basis.

Theorem A.2.1 (Klee)**.** *Let C be a cone in a locally convex vector space E. Then C is locally compact if and only if it admits a compact base.*

Proof. See [K55, (2.4)]. □

Example A.2.2. (Examples of locally compact cones)

1) Let C be a closed cone in a finite-dimensional vector space E. Then E is a locally compact vector space by [Bou81, I §2 No.4 Théorème 3]. As C is a closed subset of E, then it is also locally compact, and hence it admits a compact base. More generally, if C is a proper generating cone in the dual of a Banach space E, then C is locally compact with respect to the operator norm topology if and only if E is finite-dimensional, see [Y78].

2) Let E be a Banach lattice with an order unit u. Then the set E'_+ of continuous positive functionals on E is a locally compact cone in E'. A base for E'_+ is given by $\mathcal{M}(E)$. More generally, let G be a group with a representation by positive linear isometries on a Banach lattice E. Let $d \in E$ be a non-zero positive vector with the translate property. Then $(E,d)'_+$ is a locally compact cone in $(E,d)'$. A base for $(E,d)'_+$ is given by $\mathcal{I}_d(E)$.

A.2.B. Edwards theorem. Recall that a cone C in a vector space E is generating if every element in E can be written as the difference of two elements in C.

Theorem A.2.3 (Edwards). *Let C be a generating locally compact cone in a locally convex vector space E. Then C can be embedded in the continuous dual of a Banach space when endowed with the weak-* topology.*

We give a sketch of the construction made by Edwards. We refer to the original paper ([E64, Theorem 4]) for the details.

Let C be a generating locally compact cone in a locally convex vector space E, and let τ denotes the locally convex topology of E. By Klee's theorem, C admits a compact basis B. Consider the set

$$S = \left\{ \sum_{j=1}^{n} c_j b_j : \sum_{j=1}^{n} c_j = 1,\ c_j \in \mathbf{R}_+ \text{ and } b_j \in B \cup (-B) \text{ for every } j = 1, ..., n \right\}$$

of all finite convex combinations of points in $B \cup (-B)$. Then S is a convex absorbing neighbourhood of the origin of E. Note that S is absorbing, since C is a generating cone. Now let p_S be the Minkowski functional of S, i.e., $p_S(v) = \inf\{c : c > 0 \text{ and } v \in cS\}$ for $v \in E$. Then p_S is a norm on E, which turns the pair (E, p_S) into a Banach space. Moreover:

- the pair (E, p_S) is a Banach space and the topology generated by p_S is stronger than the original locally convex topology τ of E;

- the norm p_S is additive on C, i.e., $p_S(c_1 + c_2) = p_S(c_1) + p_S(c_2)$ for every $c_1, c_2 \in C$;[2]

- we have that $B = \{v \in C : p_S(v) = 1\}$ and $S = \{v \in E : p_S(v) \leq 1\}$;

- the norm p_S is independent of the choice of B, i.e., if B' is another basis for C, then the Minkowski functional $p_{S'}$ is a norm which is equivalent to the norm p_S. Here, S' is constructed in the same way as S but using B' instead of B.

[2]If, in addition, we assume that B is a Choquet simplex, then the vector space E equipped with the vector ordering given by C and the norm p_S is an AL-space ([E64, Corollary 3]).

Consider now the vector space given by

$$E_* = \{\psi : E \longrightarrow \mathbf{R} : \psi \text{ is a linear } \tau\text{-continuous on } p_S\text{-bounded sets functional}\} .$$

If we equip E_* with the operator norm, then E_* becomes a Banach space with continuous dual given by E. In other words, the pair $(E_*, ||\cdot||_{op})$ is a predual for E. We can consider on E the weak-* topology given by the duality between E_* and E. Then the locally convex topology τ and the weak-* topology on E induce identical relative topologies on p_S-bounded subsets of E and also on C. Therefore, C is locally compact for the weak-* topology. Moreover, the vector space E_* is generated by the cone K of all positive functionals belonging to E_*. This cone is Archimedean, and E_* admits an order unit when equipped with the vector ordering given by K. In conclusion, the identity map embeds C continuously in E when this last is endowed with the weak-* topology.

Scholium A.2.4. A similar construction to Edward's for *universally well-capped* cones was done in [AS68].

A.2.C. A conical characterization of amenability. Let A be a subset of a locally convex vector space E, and suppose that a group G has a representation by linear automorphisms on E. Then we say that G acts **uniformly bounded with respect to A** if there is $c_A > 0$ such that for every neighbourhood of the origin $U \subset E$ and every $v \in A$, the inclusion $Gv \subset c_A U$ holds.

Lemma A.2.5. *Suppose that G has a representation by linear automorphisms on a generating locally compact cone with basis B in a locally convex vector space E. If G acts uniformly bounded w.r.t. B, then $\sup_{g \in G} p_S(gv) < \infty$ for every $v \in E$. Here, $S = co\{B \cup (-B)\}$.*

Proof. Firstly, we claim that if G acts uniformly bounded w.r.t. B, then G also acts uniformly bounded w.r.t. S. Indeed, let $c_B > 0$ be the constant which witness the fact that G acts uniformly bounded w.r.t. B, $U \subset E$ be a neighborhood of the identity and $v \in S$. We want to show that $Gv \subset c_B U$. We can suppose that U is convex as E is a locally convex vector space. Now there are $c_1, ..., c_n \in \mathbf{R}_+$, $b_1, ..., b_n \in B \cup (-B)$ such that $\sum_{j=1}^n c_j = 1$ and $v = \sum_{j=1}^n c_j b_j$. Moreover, for every $g \in G$ and every $j = 1, ..., n$ there is $u_j \in U$ such that $gb_j = c_B u_j$. Therefore,

$$gv = \sum_{j=1}^n c_j g b_j = \sum_{j=1}^n c_j c_B u_j = c_B \underbrace{\sum_{j=1}^n c_j u_j}_{\in U} .$$

Hence, $gv \in c_B U$. The conclusion is that G acts uniformly bounded w.r.t. S, as g was chosen arbitrarily.

Fix now $v \in E$ and let $g \in G$. Suppose that $p_S(v) = c$. Then there is a net $(c_\alpha)_\alpha$ of positive real numbers such that $\lim_\alpha c_\alpha = c$ and $v \in c_\alpha S$. But now $gv \in c_\alpha g S$ and $gS \subset c_B S$. So, $gv \in c_\alpha c_B S$ for every α. We can conclude that $p_S(gv) \leq c_\alpha c_B$ for every α.

This implies that $p_S(gv) \leq c_B p_S(v)$. Therefore, $\sup_{g\in G} p_S(gv) \leq c_B p_S(v)$, since g was chosen arbitrarily. $\square$

Theorem A.2.6. *Let G be a topological group. Then the following assertions are equivalent:*

a) the group G is amenable;

b) every orbitally continuous representation of G by linear automorphisms on a locally compact cone C with basis B in a locally convex vector space E which is uniformly bounded w.r.t. B admits a non-zero fixed-point.

Proof. Let G be an amenable topological group that acts linearly, orbitally continuously, and with bounded orbits on a locally compact cone C in a locally convex vector space E. We can assume that C is generating because if it is not the case, we can consider the locally convex vector space $V = C - C$. Therefore, we can suppose by Edwards Theorem A.2.3 that C is a locally compact convex cone in the continuous dual E of a Banach space E_* with order unit, and the locally convex topology of E is the weak-* topology. Note that the action of G on E is still orbitally continuous with respect to the weak-* topology as the two topologies agree on p_S-bounded sets. We define the map

$$p_{is}(v) = \sup_{g\in G} p_S(gv) \quad \text{for every } v \in E.$$

Similarly to Remark 3.1.13, p_{is} is a norm on E which is equivalent to p_S-norm and for which the action of G on E is by linear isometries. Therefore, the operator norm $||\cdot||_{op}^{p_S}$ with respect to p_S on E_* is equivalent to the operator norm $||\cdot||_{op}^{p_{is}}$ with respect to p_{is}. For this last one the pre-adjoint action is by linear isometries. This implies that the set of mean $\mathcal{M}(E_*)$ of E_* is G-invariant. Moreover, G acts orbitally continuous on $\mathcal{M}(E_*)$. Therefore, it fixes a non-zero point which is also in C.

Suppose that G has the fixed-point property for locally compact cones described in point b). Then it suffices to apply it to the cone

$$C = \left\{cm : c \in \mathbf{R}_+ \text{ and } m \in \mathcal{M}(\mathcal{C}^b_{ru}(G))\right\} \subset \mathcal{C}^b_{ru}(G)',$$

which is locally compact when $\mathcal{C}^b_{ru}(G)'$ is endowed with the weak-* topology. $\square$

Appendix B

Looking at C^*-algebras

We mentioned earlier that it would have been possible to get all of the above results using the theory of C^*-algebra instead of the one of ordered vector spaces. It only depends on whether the cone we are considering lies in a real or complex vector space.

The first to use this approach has been Rørdam in [R19]. He considered actions of discrete groups on C^*-algebras, and he understood that invariant normalizing integrals could be seen as invariant traces.

The theory developed by Rørdam can be easily generalized to topological groups and uniform structures, as we are going to do in the first part of the chapter. In the second one, we look at only one particular but universal, C^*-algebra: the C^*-algebra of bounded linear operators of a Hilbert space.

In this chapter, $\mathcal{A}$ is always a complex C^*-algebra and $\mathcal{A}_{sa}$ the subspace of self-adjoint elements of $\mathcal{A}$, i.e., all the elements a of $\mathcal{A}$ such that $a^* = a$.

B.1 Ideals and traces

We recall that the C^*-order of a C^*-algebra $\mathcal{A}$ is the vector ordering induced by the C^*-cone

$$C_{sa} = \{a \in \mathcal{A}_{sa} : a = b^*b \text{ for some } b \in \mathcal{A}\} .$$

If we equip $\mathcal{A}$ with its C^*-order and with its C^*-norm, then $\mathcal{A}$ is an ordered Banach space but, in general, not a Banach lattice ([D77, 1.6.9]). We have that $\mathcal{A}_{sa}$ is a Banach lattice if and only if $\mathcal{A}$ is commutative ([S51, Theorems 1 & 2]).

In what follows, be careful that the terminology *ideal* is used in the algebraic sense and not in the order one. This means that $\mathcal{I}$ is a left ideal of a C^*-algebra $\mathcal{A}$ if and only if $\mathcal{I}$ is a C^*-subalgebra of $\mathcal{A}$ such that $ab \in \mathcal{I}$ for every $a \in \mathcal{A}$ and every $b \in \mathcal{I}$. The notions of right ideal and two-sided ideal are defined similarly. An ideal is said self-adjoint if it is closed by taking the adjoint.

Definition B.1.1. Let $\mathcal{I}$ be a subset of a C^*-algebra $\mathcal{A}$. We say that:

- the subset $\mathcal{I}$ is a **hereditary ideal** if it is a two-sided self-adjoint ideal such that if $0 \leq a \leq b$ for $a \in \mathcal{A}$ and $b \in \mathcal{I}$, then $a \in \mathcal{I}$;
- the subset $\mathcal{I}$ is **symmetric** if $a^*a \in \mathcal{I}$ whenever $aa^* \in \mathcal{I}$ for all $a \in \mathcal{A}$.

Example B.1.2. 1) Let $\mathcal{A}$ be a commutative C^*-algebra. Then $\mathcal{A}_{sa}$ equipped with the C^*-norm and the C^*-order is a Banach lattice by [S51, Theorems 1 & 2]. Therefore, an ideal $\mathcal{I}$ of $\mathcal{A}_{sa}$ is hereditary if and only if it is a Riesz subspace of $\mathcal{A}$. Actually, hereditary ideals are a generalization of Riesz subspaces to order vector spaces in the context of C^*-algebras.

2) Let $\mathcal{H}$ be a (complex) Hilbert space and consider the C^*-algebra $\mathcal{B}(\mathcal{H})$. Then the trace class operators $\mathrm{TC}(\mathcal{H})$ form a self-adjoint two-sided ideal in $\mathcal{B}(\mathcal{H})$, see [P89, Proposition 3.8]. We claim that $\mathrm{TC}(\mathcal{H})$ is a hereditary symmetric ideal. In fact, if $T_1, T_2 \in \mathrm{TC}(\mathcal{H})$ are such that $0 \leq T_1 \leq T_2$, then $\mathrm{tr}(T_1) \leq \mathrm{tr}(T_2)$ by [S18, p. 71]. Thus, $\mathrm{TC}(\mathcal{H})$ is hereditary. Moreover, if $T \in \mathcal{B}(\mathcal{H})$ such that $T^*T \in \mathrm{TC}(\mathcal{H})$, then $\mathrm{tr}(T^*T) < \infty$. But now $\mathrm{tr}(T^*T) = \mathrm{tr}(TT^*)$. Therefore, $TT^* \in \mathrm{TC}(\mathcal{H})$. We can conclude that $\mathrm{TC}(\mathcal{H})$ is also symmetric.

Note that every C^*-algebra $\mathcal{A}$ admits a unique minimal dense hereditary and symmetric ideal [P66, Theorem 1.3]. This ideal is called the **Pedersen ideal** of $\mathcal{A}$ and it is noted $\mathrm{Ped}(\mathcal{A})$.[1]

Example B.1.3. (Examples of Pedersen ideals)

1) Let $\mathcal{A}$ be a unital commutative C^*-algebra. Then by the Gelfand-Naimark Theorem ([K09, Theorem 2.2.7]), $\mathcal{A}$ is isometrically $*$-isomorphic to the C^*-algebra $\mathcal{C}(X)$ for some compact topological space X. A subset S of $\mathcal{C}(X)$ is dense if and only if it is a separating set and if $\mathbf{1}_X \in S$ by the Stone-Weierstrass Theorem ([K09, Theorem A.1.3]). If S is a hereditary ideal, then $S = \mathcal{C}(X)$, as $\mathbf{1}_X$ is also an order unit. Therefore, $\mathcal{A} = \mathrm{Ped}(\mathcal{A})$ for every commutative unital C^*-algebra $\mathcal{A}$.

2) Let $\mathcal{A}$ be a non-unital commutative C^*-algebra. Then $\mathcal{A}$ is isometrically $*$-isomorphic to $\mathcal{C}_0(X)$ for some locally compact space X. Since continuous compactly supported functions are uniformly dense in $\mathcal{C}_0(X)$, then $\mathrm{Ped}(\mathcal{C}_0(X)) \subset \mathcal{C}_{00}(X)$. However, $\mathrm{Ped}(\mathcal{C}_0(X))$ contains $\mathcal{C}_{00}(X)$ because it separates points. Therefore, $\mathrm{Ped}(\mathcal{C}_0(X)) = \mathcal{C}_{00}(X)$. In particular, for a non-unital commutative C^*-algebra, we always have that $\mathrm{Ped}(\mathcal{A}) \subset \mathcal{A}$ because the Pederson ideal of a C^*-algebra is preserved under $*$-homomorphism ([P66, Theorem 1.4]).

3) Let $\mathcal{H}$ be a (complex) Hilbert space and consider the C^*-algebra $\mathcal{B}_0(\mathcal{H})$ of all compact operators on $\mathcal{H}$. Then $\mathrm{Ped}(\mathcal{B}_0(\mathcal{H})) = \mathcal{F}(\mathcal{H})$ the finite-rank operators.

[1] The Pedersen ideal takes its name in honour of the homonymous Danish mathematician Gert Kjærgård Pedersen (1940-2004), which introduced it in his famous paper [P66].

4) Let $\mathcal{A}$ be a separable C^*-algebra. Then $\mathrm{Ped}(\mathcal{A}) = \mathcal{A}$ if and only if $\mathcal{A}$ is unital ([P66, Theorem 1.5]).

5) There are non-unital C^*-algebras $\mathcal{A}$ with $\mathrm{Ped}(\mathcal{A}) = \mathcal{A}$. An example is given by Pedersen in [P66, p. 136].

We built the theory of groups having the fixed-point property for cones around the notion of positive functionals. Precisely, invariant normalized integrals. The notion of trace gives the translation for the complex world of C^*-algebras. There are two different ways to define a trace on a C^*-algebra. Nevertheless, the two definitions are equivalent.

Definition B.1.4. Let $\mathcal{A}$ be a C^*-algebra.

- A **trace defined on the positive cone of** $\mathcal{A}$ is an additive homogeneous map $\mathrm{tr} : \mathcal{A}_+ \longrightarrow [0, \infty]$ satisfying the trace condition $\mathrm{tr}(a^*a) = \mathrm{tr}(aa^*)$ for all $a \in \mathcal{A}$.

- A **linear trace** is a positive linear map $\mathrm{tr} : \mathcal{I} \longrightarrow \mathbf{C}$ defined on a hereditary symmetric ideal $\mathcal{I}$ of $\mathcal{A}$ which satisfies the trace condition $\mathrm{tr}(a^*a) = \mathrm{tr}(aa^*)$ for all $a \in \mathcal{A}$ such that a^*a, and also aa^*, belongs to $\mathcal{I}$.

For a trace tr defined on the positive cone of a C^*-algebra $\mathcal{A}$, we define its **domain** as $\mathcal{D} = \{a \in \mathcal{A}_+ : \mathrm{tr}(a) < \infty\}$ and we say that tr is a **densely defined trace on** $\mathcal{A}$ if $\mathcal{D}$ is dense in $\mathcal{A}$. Clearly, if tr is a linear trace, then $\mathcal{D} = \mathcal{I}$ and tr is said densely defined if $\mathcal{I}$ is dense in $\mathcal{A}$.

Proposition B.1.5. *Let $\mathcal{A}$ be a C^*-algebra. Then there is a bijection between traces defined on the positive cone of $\mathcal{A}$ and linear traces. Precisely, for a trace tr_1 defined on the positive cone of $\mathcal{A}$, let $M = \{a \in \mathcal{A} : tr_1(a) < \infty\}$ and consider $\mathcal{I} = \mathrm{span}_{\mathbf{C}}\{M\}$. Then $\mathcal{I}$ is a hereditary symmetric ideal in $\mathcal{A}$, and there is only one linear trace tr_2 with domain $\mathcal{I}$ which agrees with tr_1 on M. Conversely, if tr_2 is a linear trace with domain $\mathcal{I}$, then we can recover tr_1 via the formula*

$$tr_1(a) = \begin{cases} tr_2(a) & \text{if } a \in \mathcal{I}, \\ \infty & \text{otherwise}, \end{cases}$$

for every $a \in \mathcal{A}_+$.

Proof. See [R19, Proposition 2.8]. □

Bearing this last proposition in mind , we only speak about traces without specifying if they are linear traces or traces defined on the positive cone of $\mathcal{A}$.

We privilege the point of view of linear traces as it provides a natural setting to work with proper cones. However, it is helpful to have in mind both definitions to understand better what is happening.

Definition B.1.6. Let $\mathcal{A}$ be a C^*-algebra. Suppose that tr is a trace on $\mathcal{A}$ with domain $\mathcal{D}$. Then tr is said **lower semicontinuous** if and only if whenever $(a_n)_n \subset \mathcal{D}$ is an increasing sequence of positive elements converging in norm to an element $a \in \mathcal{D}$, then $\mathrm{tr}(a) = \lim_n \mathrm{tr}(a_n)$.

Theorem B.1.7 (Pedersen Theorem). *The restriction of any densely defined trace of a C^*-algebra on its Pedersen ideal is lower semicontinuous.*

Proof. See [P66, Corollary 3.2]. □

Let tr be a trace on a C^*-algebra $\mathcal{A}$ with domain $\mathcal{I}$. Write $\overline{\mathcal{I}}$ for the closure of $\mathcal{I}$ in $\mathcal{A}$. Then $\mathrm{Ped}(\overline{\mathcal{I}}) \subset \mathcal{I} \subset \overline{\mathcal{I}}$. The restriction of tr to $\mathrm{Ped}(\overline{\mathcal{I}})$ is lower semicontinuous by Pedersen Theorem. If the restriction is equal to zero, then we say that the trace tr is **singular**. Therefore, every trace tr on a C^*-algebra $\mathcal{A}$ with domain $\mathcal{I}$ may be uniquely written as the sum $\mathrm{tr} = \mathrm{tr}_1 + \mathrm{tr}_2$ of a lower semicontinuous trace tr_1 and a singular one tr_2 both with domain $\mathcal{I}$.

Definition B.1.8. Let $\mathcal{A}$ be a C^*-algebra and $\mathcal{I}$ a hereditary symmetric ideal. Then we write $\mathcal{T}(\mathcal{A}, \mathcal{I})$ for the **set of traces of $\mathcal{A}$ with domain $\mathcal{I}$**.

The set $\mathcal{T}(\mathcal{A}, \mathcal{I})$ can be linearly and continuously embedded in the complex locally convex vector space $\mathcal{L}(\mathcal{I}, \mathbf{C})$ of all linear operators from $\mathcal{I}$ to $\mathbf{C}$. The mentioned locally convex topology on $\mathcal{L}(\mathcal{I}, \mathbf{C})$ is the one induced by $\mathcal{I}$, i.e., a net $(\phi_\alpha)_\alpha$ in $\mathcal{L}(\mathcal{I}, \mathbf{C})$ converges to an element $\phi \in \mathcal{L}(\mathcal{I}, \mathbf{C})$ if and only if for every $a \in \mathcal{I}$ the net $(\phi_\alpha(a))_\alpha$ converges to $\phi(a)$. Therefore, $\mathcal{T}(\mathcal{A}, \mathcal{I})$ can be seen as a proper convex cone in the locally convex vector space $\mathcal{L}(\mathcal{I}, \mathbf{C})$.

Be careful that the cone $\mathcal{T}(\mathcal{A}, \mathcal{I})$ can be degenerate, see for example [R19, Proposition 2.11]. Examples where $\mathcal{T}(\mathcal{A}, \mathcal{I})$ is different from zero are given by commutative C^*-algebras, since a non-zero trace on a commutative C^*-algebra $\mathcal{A}$ with domain $\mathcal{I}$ is nothing but a positive element of $\mathcal{L}(\mathcal{I}, \mathbf{C})$. Others (non-commutative) examples are given in [R19, Theorem 2.10].

Proposition B.1.9. *Let $\mathcal{A}$ be a C^*-algebra. Then for every hereditary symmetric ideal $\mathcal{I}$ of $\mathcal{A}$, the cone $\mathcal{T}(\mathcal{A}, \mathcal{I})$ is weakly complete in $\mathcal{L}(\mathcal{I}, \mathbf{C})$.*

Proof. See [R19, Proposition 3.1]. □

Scholium B.1.10. The set $\mathcal{T}(\mathcal{A}, \mathrm{Ped}(\mathcal{A}))$ of lower semicontinuous traces on $\mathcal{A}$ was deeply studied in [ERS01] but as an abstract cone. A particularity of $\mathcal{T}(\mathcal{A}, \mathrm{Ped}(\mathcal{A}))$ is that it does not have the cancellation property. Thus, it can not be embedded in a real vector space. Anyway, it was possible to put a *lattice structure* on it ([ERS01, Theorem 3.3]). An interesting feature of this point-of-view is that is possible to put a topology on $\mathcal{T}(\mathcal{A}, \mathrm{Ped}(\mathcal{A}))$ ([ERS01, Section 3.2]) for which it becomes Hausdorff compact topological space ([ERS01, Theorem 3.7]).

B.2 Traces and the $\mathscr{U}$-fixed-point property for cones

We are interested in giving characterizations of the $\mathscr{U}$-fixed-point property for cones in terms of (possibly non-commutative) C^*-algebras. Precisely, we want to link the $\mathscr{U}$-fixed-point property for cones with the existence of invariant traces on particular hereditary ideals.

Definition B.2.1. Let M be a subset of a C^*-algebra $\mathcal{A}$. Write $\mathcal{I}_{\mathcal{A}}(M)$ for the smallest hereditary ideal in $\mathcal{A}$ containing M. If a group G acts on $\mathcal{A}$, then write $\mathcal{I}_{\mathcal{A}}^G(M)$ for the smallest G-invariant hereditary ideal in $\mathcal{A}$ containing M.

Theorem B.2.2. *Let G be a topological group, and let $\mathscr{U}$ be a functionally invariant uniformity for G. Then G has the $\mathscr{U}$-fixed-point property for cones if and only if for every representation of G on a C^*-algebra $\mathcal{A}$ by positive linear isometries and for every hereditary symmetric ideal $\mathcal{I}$ of $\mathcal{A}$ such that $\mathcal{T}(\mathcal{A},\mathcal{I}) \neq 0$, and the induced action of G on $\mathcal{T}(\mathcal{A},\mathcal{I})$ is locally bounded $(\mathscr{U},\mathscr{U}_c)$-uniformly continuous and of cobounded type, there is a non-zero invariant trace on $\mathcal{A}$ with domain $\mathcal{I}$.*

Proof. Suppose that G has the $\mathscr{U}$-fixed-point property for cones for a functionally invariant uniform structure $\mathscr{U}$. Then we can apply it to the cone $\mathcal{T}(\mathcal{A},\mathcal{I})$, as it is weakly complete by Proposition B.1.9. Therefore, a non-zero point in $\mathcal{T}(\mathcal{A},\mathcal{I})$ is fixed by the adjoint representation of G on it. This fixed-point is a non-zero invariant trace on $\mathcal{A}$ with domain $\mathcal{I}$.

For the converse, it suffices to show that G has the invariant normalized integral property for the commutative C^*-algebra

$$\mathcal{C}_u^b((G,\mathscr{U}),\mathbf{C}) = \{f : (G,\mathscr{U}) \longrightarrow (\mathbf{C},\mathscr{U}_c) : f \text{ is uniformly continuous}\}.$$

Note that $\mathcal{C}_u^b((G,\mathscr{U}),\mathbf{C})$ is a (complex) C^*-algebra by Theorem 1.2.13. Thus, write $\mathcal{A} = \mathcal{C}_u^b((G,\mathscr{U}),\mathbf{C})$, and let $f \in \mathcal{A}$ be a non-zero positive function. Consider the hereditary and symmetric ideal $\mathcal{I} = (\mathcal{A}, f) = \mathcal{I}_{\mathcal{A}}^G(f)$. Then $\mathcal{T}(\mathcal{A},\mathcal{I}) \neq 0$, since every positive linear functional on $\mathcal{I}$ is a trace (the algebra being commutative). Moreover, the adjoint representation of G on $\mathcal{T}(\mathcal{A},\mathcal{I})$ is of cobounded type as the continuous dual of $\mathcal{L}(\mathcal{I},\mathbf{C})$ is $\mathcal{I}$. Finally, it is also locally bounded $(\mathscr{U},\mathscr{U}_c)$-uniformly continuous. This can be shown as in Lemma 5.2.2. Therefore, there is a non-zero invariant trace on $\mathcal{A}$ with domain $\mathcal{I}$, which is nothing but an invariant integral. After normalization, we become an invariant normalized integral for $\mathcal{I}$. We can conclude that G has the $\mathscr{U}$-fixed-point property for cones. □

Various conditions to assure that the adjoint representation of G on $\mathcal{T}(\mathcal{A},\mathcal{I})$ is of cobounded type have been given by Rørdam, see [R19, Section 3].

Theorem B.2.3. *Let G be a topological group, and let $\mathscr{U}$ be a functionally invariant uniformity for G. The following assertions are equivalent:*

a) the group G has the $\mathscr{U}$-fixed-point property for cones;

b) for every representation of G on a C^*-algebra $\mathcal{A}$ by positive linear isometries and for every non-zero positive $d \in \mathcal{A}$ such that $\mathcal{T}(\mathcal{A}, \mathcal{I}_{\mathcal{A}}^G(d)) \neq 0$ and the adjoint representation of G on $\mathcal{T}(\mathcal{A}, \mathcal{I}_{\mathcal{A}}^G(d))$ is locally bounded $(\mathscr{U}, \mathscr{U}_c)$-uniformly continuous, there is an invariant trace with domain $\mathcal{I}_{\mathcal{A}}^G(d)$ normalized on d;

c) for every representation of G on a commutative C^*-algebra $\mathcal{A}$ by positive linear isometries and for every non-zero positive $d \in \mathcal{A}$ such that the adjoint representation of G on $\mathcal{T}(\mathcal{A}, \mathcal{I}_{\mathcal{A}}^G(d))$ is locally bounded $(\mathscr{U}, \mathscr{U}_c)$-uniformly continuous, there is an invariant trace with domain $\mathcal{I}_{\mathcal{A}}^G(d)$ normalized on d.

Proof. We have that a) implies b) thanks to Theorem B.2.2. Moreover, the implication b) to c) is direct as the latter is a particular case of the former. Finally, c) implies a) only by applying the hypothesis to the commutative C^*-algebra $\mathcal{C}_u^b((G, \mathscr{U}), \mathbf{C})$. □

This theorem can be simplified considering only locally compact groups.

Theorem B.2.4. *Let G be a locally compact group. The following assertions are equivalent:*

a) the group G has the fixed-point property for cones;

b) for every continuous representation of G on a C^*-algebra $\mathcal{A}$ by positive linear isometries and for every non-zero positive $d \in \mathcal{A}$ such that $\mathcal{T}(\mathcal{A}, \mathcal{I}_{\mathcal{A}}^G(d)) \neq 0$, there is an invariant trace with domain $\mathcal{I}_{\mathcal{A}}^G(d)$ normalized on d;

c) for every continuous representation of G on a commutative C^*-algebra $\mathcal{A}$ by positive linear isometries and for every non-zero positive $d \in \mathcal{A}$, there is an invariant trace with domain $\mathcal{I}_{\mathcal{A}}^G(d)$ normalized on d.

Proof. The proof is similar to that of the previous theorem. □

B.3 An application to the C^*-algebra $\mathcal{B}(\mathcal{H})$

We study the case of the C^*-algebra of bounded linear operators on a Hilbert space. Our interest in this particular C^*-algebra comes mainly from Bekka's work on amenable representations, see [B90].

There are at least two ways to investigate this algebra. The first uses the theory developed in the previous section, while the second uses the peculiarities of $\mathcal{B}(\mathcal{H})$. We decided to use the second option as the proofs are constructive.

Recall that if we have a topological group G and a unitary representation $(\sigma, \mathcal{H})$ of G on $\mathcal{H}$, then there is an induced representation Ad_σ of G on $\mathcal{B}(\mathcal{H})$ given by

$$\mathrm{Ad}_\sigma(g)T = \sigma(g)T\sigma(g)^* \quad \text{for every } g \in G \text{ and } T \in \mathcal{B}(\mathcal{H}).$$

This representation preserves the C^*-cone of $\mathcal{B}(\mathcal{H})$, as it is stable under conjugation of unitary operators ([S18, Corollary 3.6]).

From now on, every unitary representation is considered continuous.

B.3.A. Invariant normalized integrals and the translate property.

Definition B.3.1. Let G be a topological group and let $(\sigma, \mathcal{H})$ be a unitary representation of G. Then

- the unitary representation $(\sigma, \mathcal{H})$ has the **invariant normalized integral property** if the representation Ad_σ of G on $\mathcal{B}(\mathcal{H})$ has the invariant normalized integral property;
- the unitary representation $(\sigma, \mathcal{H})$ has the **translate property** if the representation Ad_σ of G on $\mathcal{B}(\mathcal{H})$ has the translate property.

Clearly, if a unitary representation $(\mathcal{H}, \sigma)$ has the invariant normalized integral property, then it has the translate property.

We recall that two unitary representations $(\sigma_1, \mathcal{H}_1)$ and $(\sigma_2, \mathcal{H}_2)$ of a topological groups G are said **unitarily equivalent** if there is a unitary operator $U : \mathcal{H}_1 \longrightarrow \mathcal{H}_2$ such that $\sigma_1(g) = U^*\sigma_2(g)U$ for every $g \in G$. The unitary operator U is said the **intertwining operator** for the representations $(\sigma_1, \mathcal{H}_1)$ and $(\sigma_2, \mathcal{H}_2)$.

Proposition B.3.2. *The invariant normalized integral property (resp. the translate property) is preserved under unitary equivalence.*

We discuss only the proof for the invariant normalized integral property. The one for the translate property is similar.

Proof of Proposition B.3.2. Let $(\sigma_1, \mathcal{H}_1)$ and $(\sigma_2, \mathcal{H}_2)$ be two unitarily equivalent unitary representations of a topological group G. Suppose that $(\sigma_1, \mathcal{H}_1)$ has the invariant normalized integral property. We want to show that $(\sigma_2, \mathcal{H}_2)$ also has the invariant normalized integral property. Let U be the intertwining operator for the representations $(\sigma_1, \mathcal{H}_1)$ and $(\sigma_2, \mathcal{H}_2)$. Define the linear operator

$$\mathrm{Ad}_U : \mathcal{B}(\mathcal{H}_2) \longrightarrow \mathcal{B}(\mathcal{H}_1), \quad T \longmapsto \mathrm{Ad}_U(T) = U^*TU.$$

First of all, note that Ad_U is equivariant. In fact,

$$\begin{aligned} \mathrm{Ad}_U\left(\mathrm{Ad}_{\sigma_2}(g)T\right) &= \mathrm{Ad}_U\left(\sigma_2(g)T\sigma_2(g)^*\right) \\ &= U^*\sigma_2(g)T\sigma_2(g)^*U \\ &= \sigma_1(g)U^*TU\sigma_1(g) = \mathrm{Ad}_{\sigma_1}(g)\left(\mathrm{Ad}_U(T)\right) \end{aligned}$$

for every $g \in G$ and every $T \in \mathcal{B}(\mathcal{H}_2)$. Moreover, Ad_U is positive as the C^*-cone of a C^*-algebra is preserved under conjugation of unitary operators. In particular, for every positive operator $T \in \mathcal{B}(\mathcal{H}_2)$, the image of the restriction of Ad_U on $\mathcal{B}(\mathcal{H}_2, T)$ is contained in $\mathcal{B}(\mathcal{H}_1, \mathrm{Ad}_U(T))$. Let now $T \in \mathcal{B}(\mathcal{H}_2)$ be a non-zero positive operator and take an invariant normalized integral I on $\mathcal{B}(\mathcal{H}_1, U^*TU)$. Then the composition $\bar{\mathrm{I}} = \mathrm{I} \circ \mathrm{Ad}_U$ defines an invariant normalized integral on $\mathcal{B}(\mathcal{H}_2, T)$. We can conclude that the unitary representation $(\sigma_2, \mathcal{H}_2)$ has the invariant normalized integral property. □

Remark B.3.3. Let $(\sigma, \mathcal{H})$ be a unitary representation of a topological group G. Then it is possible to show, similarly as done in Proposition B.3.2, that $(\sigma, \mathcal{H})$ has the invariant normalized integral property (resp. the translate property) if and only if the complex conjugate representation $(\overline{\sigma}, \overline{\mathcal{H}})$ has the invariant normalized integral property (resp. the translate property).

Let $\mathcal{H}$ be a Hilbert space. Then a **faithful unitary invariant state** for $\mathcal{B}(\mathcal{H})$ is nothing but a positive functional $\mathrm{M} \in \mathcal{B}(\mathcal{H})'$ such that $\mathrm{M}(T) > 0$ for every $T \in \mathcal{B}(\mathcal{H})_+$ and $\mathrm{M}(U^*TU) = \mathrm{M}(T)$ for every $U \in U(\mathcal{H})$ and every $T \in \mathcal{B}(\mathcal{H})$.

Proposition B.3.4. *Let $(\sigma, \mathcal{H})$ be a unitary representation of a topological group G. If $\mathcal{B}(\mathcal{H})$ admits a faithful unitary invariant state, then $(\sigma, \mathcal{H})$ has the invariant normalized integral property (resp. the translate property).*

Proof. Let M be a faithful unitary invariant state for $\mathcal{H}$, and let $T \in \mathcal{B}(\mathcal{H})$ be a non-zero positive operator. Then the restriction of M to the vector space $\mathcal{B}(\mathcal{H}, T)$ defines an invariant normalized integral. Therefore, $(\sigma, \mathcal{H})$ has the invariant normalized integral property. □

Corollary B.3.5. *Every finite-dimensional unitary representation of a topological group G has the invariant normalized integral property (resp. the translate property).*

Proof. We can apply Proposition B.3.4 because the matricial trace is a faithful unitary invariant state for every finite-dimensional Hilbert space. □

Let $(\sigma, \mathcal{H})$ be a unitary representation of a topological group G. Recall that

$$\mathcal{B}_c(\mathcal{H}) = \{T \in \mathcal{B}(\mathcal{H}) : g \longmapsto \mathrm{Ad}_\sigma(g)T \text{ is } ||\cdot||\text{-continuous}\}\,.$$

We say that $(\sigma, \mathcal{H})$ has the **continuous invariant normalized integral property** if the representation Ad_σ of G on $\mathcal{B}_c(\mathcal{H})$ has the invariant normalized integral property. Similarly, we define the **continuous translate property for** $(\sigma, \mathcal{H})$.

Proposition B.3.6. *Let G be a topological group.*

a) If G has the invariant normalized integral property for $C^b_{lu}(G)$ (resp. the translate property for $C^b_{lu}(G)$), then every unitary representation $(\sigma, \mathcal{H})$ of G has the invariant normalized integral property (resp. the translate property).

b) If G has the invariant normalized integral property for $C^b_u(G)$ (resp. the translate property for $C^b_u(G)$), then every unitary representation $(\sigma, \mathcal{H})$ of G has the continuous invariant normalized integral property (resp. the continuous translate property).

Proof. First of all, take a positive operator $S \in \mathrm{TC}(\mathcal{H})$ such that $||S||_{\mathrm{TC}} = 1$ and consider the operator

$$\Lambda : \mathcal{B}(\mathcal{H}) \longrightarrow \mathcal{C}^b_{lu}(G), \quad T \longmapsto \Lambda(T) = f_T,$$

where the function f_T is defined pointwise by $f_T(g) = \mathrm{tr}(T\mathrm{Ad}_\sigma(g)S)$ for $g \in G$. Note that Λ is well-defined. Indeed, let $T \in \mathcal{B}(\mathcal{H})$, $\epsilon > 0$ and let $(g_\alpha)_\alpha$ be a net in G which converges to the identity element of G. By Lemma 6.1.13, there is α_0 such that

$$||\mathrm{Ad}_\sigma(g_\alpha)S - S||_{\mathrm{TC}} < \frac{\epsilon}{||T||} \quad \text{for every } \alpha \succ \alpha_0.$$

Therefore,

$$\begin{aligned} ||\pi_R(g_\alpha)f_T - f_T||_\infty &= \sup_{g\in G} |\pi_R(g_\alpha)f_T(g) - f_T(g)| \\ &= \sup_{g\in G} |f_T(gg_\alpha) - f_T(g)| \\ &= \sup_{g\in G} |\mathrm{tr}\left(T\mathrm{Ad}_\sigma(gg_\alpha)S - T\mathrm{Ad}_\sigma(g)S\right)| \\ &= \sup_{g\in G} |\mathrm{tr}\left(T\mathrm{Ad}_\sigma(g)\left(\mathrm{Ad}_\sigma(g_\alpha)S - S\right)\right)| \\ &\leq ||T||\; ||\mathrm{Ad}_\sigma(g_\alpha)S - S||_{\mathrm{TC}} < \epsilon \end{aligned}$$

for every $\alpha \succ \alpha_0$. Moreover, it is easy to see that f_T is bounded for every $T \in \mathcal{B}(\mathcal{H})$ as

$$f_T(g) = \mathrm{tr}\left(T\mathrm{Ad}_\sigma(g)S\right) \leq ||T||\; ||S||_{\mathrm{TC}} \quad \text{for every } g \in G.$$

This shows that $f_T \in \mathcal{C}^b_{lu}(G)$. The fact that Λ is a linear operator comes from the fact that the trace map is linear. Moreover, Λ is strictly positive and equivariant. Indeed, let's start showing that Λ is strictly positive. To this aim, let $T \in \mathcal{B}(\mathcal{H})$ be a non-zero positive operator. As S was chosen positive, there is a self-adjoint element Y in $\mathcal{B}(\mathcal{H})$ such that $S = Y^2$ ([S18, Proposition 3.4]). Thus,

$$\begin{aligned} f_T(g) &= \mathrm{tr}(T\mathrm{Ad}_\sigma(g)S) = \mathrm{tr}(T\sigma(g)Y^2\sigma(g)^*) \\ &= \mathrm{tr}((\sigma(g)Y)T(\sigma(g)Y)) \geq ||(\sigma(g)Y)T(\sigma(g)Y)|| > 0 \end{aligned}$$

for every $g \in G$. The second-to-last inequality was possible thanks to [S18, Proposition 6.4] and the fact that the C^*-cone is conjugation invariant. To prove the equivariance of the operator Λ, take $x \in G$ and $T \in \mathcal{B}(\mathcal{H})$. Then

$$\begin{aligned} \pi_L(x)f_T(g) &= f_T(x^{-1}g) \\ &= \mathrm{tr}(T\mathrm{Ad}_\sigma(x^{-1}g)S) \\ &= \mathrm{tr}(T\sigma(x^{-1}g)S\sigma(x^{-1}g)^*) \\ &= \mathrm{tr}(T\sigma(x^{-1})\sigma(g)S\sigma(g)^*\sigma(x)) \\ &= \mathrm{tr}(\sigma(x)T\sigma(x^{-1})\mathrm{Ad}_\sigma(g)S) \\ &= \mathrm{tr}(\mathrm{Ad}_\sigma(x)T\mathrm{Ad}_\sigma(g)S) = f_{\mathrm{Ad}_\sigma(x)T}(g) \end{aligned}$$

for every $g \in G$. We can conclude that

$$\Lambda(\mathrm{Ad}_\sigma(x)T) = \pi_L(x)\Lambda(T) \quad \text{for every } x \in G \text{ and } T \in \mathcal{B}(\mathcal{H}).$$

Finally, we can state the proof of point a). Let $(\sigma, \mathcal{H})$ be a unitary representation of G and suppose that G has the invariant normalized integral property for $\mathcal{C}^b_{lu}(G)$. Let $T \in \mathcal{B}(\mathcal{H})$ be a non-zero positive operator. Then Λ maps $\mathcal{B}(\mathcal{H}, T)$ into $\mathcal{C}^b_{lu}(G, f_T)$. Take an invariant normalized integral I on this last space. Then the composition $\bar{\mathrm{I}} = \mathrm{I} \circ \Lambda$ provides an invariant normalized integral on $\mathcal{B}(\mathcal{H}, T)$. This shows that $(\sigma, \mathcal{H})$ has the invariant normalized integral property.

For point b), we shall point out that if $T \in \mathcal{B}_c(\mathcal{H})$, then f_T is in $\mathcal{C}^b_u(G)$. Indeed, we know already that $f_T \in \mathcal{C}^b_{lu}(G)$. Therefore, it suffices to show that $f_T \in \mathcal{C}^b_{ru}(G)$. Let $\epsilon > 0$ and let $(g_\alpha)_\alpha$ be a net in G which converges to the identity element. As $T \in \mathcal{B}_c(\mathcal{H})$, there is α_0 such that

$$||\mathrm{Ad}_\sigma(g_\alpha)T - T|| < \frac{\epsilon}{||S||_{\mathrm{TC}}} \quad \text{for every } \alpha \succ \alpha_0.$$

Thus,

$$\begin{aligned}
||\pi_L(g_\alpha)f_T - f_T||_\infty &= \sup_{g\in G} |\pi_L(g_\alpha)f_T(g) - f_T(g)| \\
&= \sup_{g\in G} |f_T(g_\alpha^{-1}g) - f_T(g)| \\
&= \sup_{g\in G} |\mathrm{tr}(T\mathrm{Ad}_\sigma(g_\alpha^{-1}g)S) - \mathrm{tr}\,(T\mathrm{Ad}_\sigma(g)S)\,| \\
&= \sup_{g\in G} |\mathrm{tr}(T\sigma(g_\alpha^{-1}g)S\sigma(g_\alpha^{-1}g)^*) - \mathrm{tr}\,(T\mathrm{Ad}_\sigma(g)S)\,| \\
&= \sup_{g\in G} |\mathrm{tr}(\sigma(g_\alpha)T\sigma(g_\alpha^{-1})\sigma(g)S\sigma(g)^*) - \mathrm{tr}\,(T\mathrm{Ad}_\sigma(g)S)\,| \\
&= \sup_{g\in G} |\mathrm{tr}\,((\mathrm{Ad}_\sigma(g_\alpha)T - T)\mathrm{Ad}_\sigma(g)S)\,| \\
&\leq ||\mathrm{Ad}_\sigma(g_\alpha)T - T||\ ||S||_{\mathrm{TC}} < \epsilon
\end{aligned}$$

for every $\alpha \succ \alpha_0$. We can conclude that f_T belongs to $\mathcal{C}^b_{ru}(G)$, and hence to $\mathcal{C}^b_u(G)$. Therefore, we can use the same strategy of the proof of point a) to ensure that the representation Ad_σ of G on $\mathcal{B}_c(\mathcal{H})$ has the invariant normalized integral property. □

B.3.B. The locally compact case. Let G be a locally compact group and let

$$L^2_{\mathbf{C}}(G) = \left\{ f : G \longrightarrow \mathbf{C} : f \text{ is complex-measurable and } \int_{\mathbf{C}} |f|^2 dm_G < \infty \right\}.$$

The **left-regular representation** of G on $L^2_{\mathbf{C}}(G)$ is the unitary representation

$$\lambda : G \longrightarrow U(L^2_{\mathbf{C}}(G)), \quad g \longmapsto \lambda(g)$$

defined by $\lambda(g)f(x) = f(g^{-1}x)$ for every $g, x \in G$ and $f \in L^2_{\mathbf{C}}(G)$.

Theorem B.3.7. *Let G be a locally compact group. The following assertions are equivalent:*

a) the group G has the fixed-point property for cones;

b) every unitary representation $(\sigma, \mathcal{H})$ of G has the invariant normalized integral property (resp. the translate property);

c) the left-regular representation $(\lambda, L^2_{\mathbf{C}}(G))$ of G has the invariant normalized integral property (resp. the translate property).

In particular, for a locally compact group, it suffices to check point b) of Theorem B.2.4 only for the C^*-algebra $\mathcal{B}(L^2_{\mathbf{C}}(G))$.

Proof of Theorem B.3.7. Suppose that G has the fixed-point property for cones. In particular, G has the invariant normalized integral property for $\mathcal{C}^b_{lu}(G)$ by Theorem 6.3.4. Consequently, every unitary representation $(\sigma, \mathcal{H})$ of G has the invariant normalized integral property by point a) of Proposition B.3.6. We can conclude that a) implies b). Moreover, b) implies c) directly. Thus, we only need to prove that c) implies a). Actually, we show that if the left-regular representation of G has the invariant normalized integral property, then G has the invariant normalized integral property for $L^\infty(G)$. First of all, we define the linear operator

$$T : L^\infty(G) \longrightarrow \mathcal{B}\left(L^2_{\mathbf{C}}(G)\right), \quad f \longmapsto T(f) = T_f,$$

where T_f is defined by $T_f(\phi) = f \cdot \phi$ for every $\phi \in L^2_{\mathbf{C}}(G)$. Clearly, T is well-defined as $||T_f(\phi)||_2 \leq ||f||_\infty ||\phi||_2$ for every $\phi \in L^2_{\mathbf{C}}(G)$. Moreover, T is strictly positive as we have that $T_f = T_h^2$, where $h = \sqrt{f}$. Finally, T is equivariant. In fact,

$$\begin{aligned} \mathrm{Ad}_\lambda(x) T_f(\phi)(g) &= \lambda(x) T_f \lambda(x)^*(\phi)(g) \\ &= \lambda(x) T_f(\phi)(xg) \\ &= \lambda(x)\left((f)(g) \cdot (\phi)(xg)\right) \\ &= T_{\pi_L(x)f}(\phi)(g) \end{aligned}$$

for every $x, g \in G$, $f \in L^\infty(G)$ and $\phi \in L^2_{\mathbf{C}}(G)$. This implies that

$$\mathrm{Ad}_\lambda(x) T(f) = T(\pi_L(x) f) \quad \text{for every } x \in G \text{ and } f \in L^\infty(G).$$

Now, for every non-zero positive $f \in L^\infty(G)$, the operator T maps the space $L^\infty(G, f)$ into the space $\mathcal{B}(L^2_{\mathbf{C}}(G), T_f)$. Take an invariant normalized integral I on this last space. Then the composition $\bar{\mathrm{I}} = \mathrm{I} \circ T$ defines an invariant normalized integral on $L^\infty(G, f)$. We can conclude that G has the invariant integral property for $L^\infty(G)$, and consequently, the fixed-point property for cones by Theorem 5.2.1.

The proof for the translate property is similar. □

Remark B.3.8. Theorem B.3.7 can also be stated using the continuous invariant normalized integral property (resp. the continuous translate property) for a unitary representation.

Corollary B.3.9. *Let G be a locally compact group. Then the left-regular representation $(\lambda, L^2_{\mathbf{C}}(G))$ has the invariant normalized integral property if and only if it has the translate property.*

Proof. By Theorem B.3.7, $(L^2_{\mathbf{C}}(G), \lambda)$ has the invariant normalized integral property if and only if G has the fixed-point property for cones if and only if $(L^2_{\mathbf{C}}(G), \lambda)$ has the translate property. □

Recall that, for every (continuous) unitary representation $(\sigma, \mathcal{H})$ of a locally compact group G, the ordered vector space $\mathcal{B}(\mathcal{H})$ is a positive $\mathcal{M}(G)$-module by Proposition 6.1.16.

Definition B.3.10. Let $(\sigma, \mathcal{H})$ be a unitary representation of a locally compact group G. Then we say that:

- the unitary representation $(\sigma, \mathcal{H})$ has the **measurably invariant normalized integral property** if the representation Ad_σ of G on $\mathcal{B}(\mathcal{H})$ has the measurably invariant normalized integral property;

- the unitary representation $(\sigma, \mathcal{H})$ has the **measurably translate property** if the representation Ad_σ of G on $\mathcal{B}(\mathcal{H})$ has the measurably translate property.

Theorem B.3.11. *Let G be a locally compact group. The following assertions are equivalent:*

a) the group G has the fixed-point property for cones;

b) every unitary representation $(\sigma, \mathcal{H})$ of G has the measurably invariant normalized integral property (resp. the measurably translate property);

c) the left-regular representation $(\lambda, L^2_{\mathbf{C}}(G))$ of G has the measurably invariant normalized integral property (resp. the measurably translate property).

As before, we only present the proof for the measurably invariant normalized integral property.

Proof of Theorem B.3.11. For the proof that a) implies b), we only have to check that the operator Λ defined in Proposition B.3.6 is $\mathcal{M}(G)$-equivariant. To this end, let $T \in \mathcal{B}(\mathcal{H})$

and $\mu \in \mathcal{M}(G)$. Then

$$\begin{aligned}
\mu * \Lambda(T)(g) &= (\mu * f_T)(g) \\
&= \int_G f_T(x^{-1}g)d\mu(x) \\
&= \int_G f_{\mathrm{Ad}_\sigma(x)T}(g)d\mu(x) \\
&= \int_G \mathrm{tr}(\mathrm{Ad}_\sigma(x)T\mathrm{Ad}_\sigma(g)S)d\mu(x) \\
&= \mathrm{tr}\left(\int_G \mathrm{Ad}_\sigma(x)Td\mu(x)\mathrm{Ad}_\sigma(g)S\right) \\
&= \mathrm{tr}\left((\mu \cdot T)\mathrm{Ad}_\sigma(g)S\right) \\
&= \Lambda(\mu \cdot T)(g) \qquad \text{for every } g \in G.
\end{aligned}$$

Similarly, to show that b) implies c), it suffices to show that the linear operator T of Theorem B.3.7 is $\mathcal{M}(G)$-equivariant. Therefore, let $T \in \mathcal{B}(\mathcal{H})$ and $\mu \in \mathcal{M}(G)$. Then

$$\begin{aligned}
T(\mu * f)(\phi)(g) &= (\mu * f)(g)\phi(g) \\
&= \int_G f(x^{-1}g)d\mu(x)\phi(g) \\
&= \int_G f(x^{-1}g)\phi(g)d\mu(x) \\
&= (\mu \cdot T_f)(\phi)(g) \\
&= \mu \cdot T(f)(\phi)(g) \qquad \text{for every } g \in G \text{ and } \phi \in L^\infty(G).
\end{aligned}$$

□

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Symbols

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2021 — **Invariant Integrals on Topological Groups**, *arXiv:2011.03211*, submitted.

Talks

12.09.19 — **Geometry graduate colloquium**, *ETHZ*, Zürich, Switzerland.

09.11.19 — **Alumni 50th Homecoming**, *EPFL*, Lausanne, Switzerland.

10.06.21 — **Seminar EGG**, *EPFL*, Lausanne, Switzerland.

Organisational Activities

Fall 2019 — **Representation theory of locally compact groups**, *EPFL*, Lausanne, Switzerland.
Co-organizer of a working group on representation theory of locally compact groups.

Spring 2021 — **EGG Seminar**, *EPFL*, Lausanne, Switzerland.
Co-organizer of the *Ergodic Geometric Group theory* research group seminar.

Languages

Italian	Native	
French (C1)	Advanced	*very good understanding – very good speaking – good writing*
German (C1)	Advanced	*very good understanding – good speaking – good writing* *From March 2012 to May 2012 I attended German classes at the Goethe-institute in Berlin*
English (B2)	Intermediate	*good understanding – good speaking – good writing*

Comptuer Skills

Basic	Phyton, C++
Intermediate	XML, HTML, MATLAB, Microsoft Windows
Advanced	Java, Wolfram Alpha, LaTeX, Linux, Open Office

About me

Military service in Switzerland	In 2011, I started my military service in the Wind Orchestra of the military music playing the trumpet. Since Summer 2015, I was a cryptanalyst in the cryptology detachment of the army. In August 2019, I accomplished all my military duties.
SAC - Bern	I am an active member of the Swiss Alpine Club (SAC) section Bern, where I serve in the Environment committee. We contribute to developing environmentally friendly mountaineering with concrete and pragmatic steps.
Licenses	Driving License – Life-saver License and RCP – *Jeunesse et Sport* trekking licence
Interests	Playing the trumpet – Climbing – Trail running – Biking – Scout – Cooking

October 15, 2021
Vasco Schiavo