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Résumé

Dans cette thèse, nous étudions l'équation de la chaleur stochastic (SHE) sur des domaines bornés ou sur l'espace euclidien $\mathbb{R}^d$. Nous confirmons l'intuition que pour une suite de domaine croissant vers $\mathbb{R}^d$, la suite de solutions converge vers la solution sur $\mathbb{R}^d$. Nous considérons à la fois les conditions aux bords homogènes de Dirichlet et de Neumann.

Tout d'abord, nous étudions une version non linéaire de SHE en toute dimension spatiale, avec un bruit corrélé et une condition initiale bornée. Nous prouvons que les solutions de SHE sur une suite croissante de domaines convergent exponentiellement vite vers la solution sur $\mathbb{R}^d$. La convergence de tous les p-moments est uniforme sur les ensembles compacts. Les bruits admissibles sont les mêmes que ceux garantissant l'existence d'une solution ponctuelle. Nous itérons une inégalité de Gronwall et utilisons des bornes uniformes satisfaites par les solutions, qui seront étonnement valides pour toute la suite croissante de domaines.

Puis, nous étudions SHE en dimension spatiale $d \geq 2$, avec un bruit blanc additif et une condition initiale bornée. Bien que les solutions doivent être considérées au sens des distributions, nous montrons que leur difference est régulière. L'ordre de régularité dépend uniquement de la régularité des bords de la suite croissante de domaines. Nous prouvons que la transformée de Fourier, au sens des distributions, de la solution de SHE sur $\mathbb{R}^d$ n'admet aucune representation localement de carré intégrable. De ce fait, la convergence est étudiée dans des versions locales des espaces de Sobolev. A nouveau, une convergence exponentielle est obtenue.

Finalement, nous étudions le model de Anderson pour SHE avec un bruit corrélé et une condition initiale donnée par une mesure. Nous obtenons une expression spéciale pour le deuxième moment de la différence de la solution sur $\mathbb{R}^d$ avec celle sur un domaine borné. La contribution de la donnée initiale est rendue explicite. Par exemple, la convergence exponentielle sur les ensembles compacts est obtenue pour toute condition initiale de croissance polynomiale. Plus intéressant encore, à partir d'une convergence désirée, nous pouvons décider si la condition initiale est admissible.

Mots clés: equation de la chaleur stochastique non linéaire, erreurs de localisation, convergence exponentielle, bruits corrélés, fonction de Green.

Abstract

In this thesis, we study the stochastic heat equation (SHE) on bounded domains and on the whole Euclidean space $\mathbb{R}^d$. We confirm the intuition that as the bounded domain increases to the whole space, both solutions become arbitrarily close to one another. Both vanishing Dirichlet and Neumann boundary conditions are considered.

We first study the nonlinear SHE in any space dimension with multiplicative correlated noise and bounded initial data. We prove that the solutions to SHE on an increasing sequence of domains converge exponentially fast to the solution to SHE on $\mathbb{R}^d$. Uniform convergence on compact set is obtained for all p-moments. The conditions that need to be imposed on the noise are the same as those required to ensure existence of a random field solution. A Gronwall-type iteration argument is used together with uniform bounds on the solutions, which are surprisingly valid for the entire sequence of increasing domains.

We then study SHE in space dimension $d \geq 2$ with additive white noise and bounded initial data. Even though both solutions need to be considered as distributions, their difference is proved to be smooth. If fact, the order of smoothness depends only on the regularity of the boundary of the increasing sequence of domains. We prove that the Fourier transform, in the sense of distributions, of the solution to SHE on $\mathbb{R}^d$ do not have any locally mean-square integrable representative. Therefore, convergence is studied in local versions of Sobolev spaces. Again, exponential rate is obtained.

Finally, we study the Anderson model for SHE with correlated noise and initial data given by a measure. We obtain a special expression for the second moment of the difference of the solution on $\mathbb{R}^d$ with that on a bounded domain. The contribution of the initial condition is made explicit. For example, exponentially fast convergence on compact sets is obtained for any initial condition with polynomial growth. More interestingly, from a given convergence rate, we can decide whether some initial data is admissible.

Keywords: nonlinear stochastic heat equation, localization errors, exponential rate of convergence, correlated noises, Green function, Anderson model.

Contents

Chapter 1

Introduction

> Les équations différentielles de la propagation de la chaleur expriment les conditions les plus générales, et ramènent les questions physiques à des problèmes d'analyse pure.
>
> *Joseph Fourier*, 1822

Two centuries ago, the French mathematician/physicist Joseph Fourier introduced the heat diffusion equation (1.1) in his book "Théorie analytique de la chaleur" (The Analytic Theory of Heat). That equation is a mathematical model of heat diffusion in any medium (solid, liquid or gas). It enables to predict the temperature at any point of the medium at any future time. To derive this equation, it requires to identify and balance all the thermal energy transfers happening in the medium and apply a conservation principle, i.e. Fourier's law and conservation of energy. An explicit derivation is presented in [1, Chapter 2]. Some of the tools used to solve that equation are nowadays part of a branch of mathematics called Fourier analysis. Among many applications, it is essential to process digital music and medical imaging.

Equation (1.1) has been derived later in many other fields of physics. In molecular diffusion, from Fick's law and conservation of mass, it enables to predict the concentration of salt in water. In electricity, from Ohm's law and conservation of charge, it enables to predict the electric potential in a conductor. In hydrogeology, from Darcy's law and conservation of mass, it enables to predict the hydraulic head when studying the flow of water in porous material. For a thorough treatment of influences and connections with Fourier's work, see [42].

In the 20th century, major contributions were made on the formalization of random behaviours in the macroscopic scale. For example, Brownian motion can model the position of a small particle (e.g. pollen) moving randomly in an (a priori static) fluid. The movement of the particle is due to unpredictable shocks with the ambiant molecules of the fluid. The first mathematical descriptions of Brownian motion were independently made by Louis Bachelier in 1900 and Albert Einstein in 1905.

In 1944, the Japanese mathematician Kyoshi Itô introduced the theory of stochastic integration, with respect to Brownian motion. It is a tool of choice to model dynamical systems that are subject to random perturbations. For example, it is used in finance to model the evolution in time of stock prices. For a short survey describing the main achievements in probability in the 20th century, in relation with the work of Itô, see [47]. Stochastic partial differential equations (SPDE) are among them.

Combining these ideas of diffusion and random perturbations naturally leads to the study of the so-called stochastic heat equation:

$$\frac{\partial u}{\partial t}(t,x) = \nu \frac{\partial^2 u}{\partial x^2}(t,x) + \dot{W}(t,x). \tag{1.1}$$

The function $u(t,x)$ represents temperature at any time $t \geq 0$ and at any position $x \in \mathbb{R}$. The above equation links together the following four things:

- The variation in time of the temperature through its first derivative, i.e. the rate of change of the temperature, $\frac{\partial u}{\partial t}(t,x)$;
- The variation in space of the temperature through its second derivative, i.e. the curvature of the temperature profile, $\frac{\partial^2 u}{\partial x^2}(t,x)$;
- The thermal diffusivity coefficient ν of the medium. The larger the coefficient is, the faster heat propagates inside the medium.
- The (random) amount of thermal energy added to the system, i.e. the internal heat source. More precisely, $\dot{W}(t,x)$ represents the (random) rate per volume at which energy is generated in the medium, also known as the power density.

Randomness is expressed through a space-time white noise $\dot{W}(t,x)$. Loosely speaking, it has the following three properties: 1) On average, the thermal energy added to the system is zero. 2) The thermal energy added to the system has a gaussian law. 3) The thermal energy added within two disjoint regions are independent. More precisely, the random thermal energy added within any time-space region $E =]t_1, t_2] \times]x_1, x_2]$ follows a gaussian law with mean zero and variance $(t_2 - t_1)(x_2 - x_1)$, and it is independent of the thermal energy added to any other time-space region $F =]s_1, s_2] \times]y_1, y_2]$ if their intersection is empty, i.e. $E \cap F = \emptyset$.

We shall relax some of the above assumptions. If the thermal energy added to system does not have zero mean, then we could replace $\dot{W}(t,x)$ by $b(t,x) + \dot{W}(t,x)$, for some function $b(t,x)$ that would represent the average power density at time t and position x. In fact, the power density added to the system could well depend on the temperature of the system. For example, when the system is a conductor with electrical current flowing through it, the power density released is given by

$$b(t, x, u) = r_e(t, x, u)i_e(t, x)^2,$$

where i_e is the electrical flux and r_e is the specific electrical resistance of the material. For small variation of temperature, r_e and hence b vary linearly with respect to u. Assuming that the mean of power density added to the system behave linearly with respect to the increase of temperature, it seems natural to assume that the standard deviation of the noise may also vary with temperature. For this reason, $\dot{W}(t, x)$ could be replaced by $b(t, x, u) + \sigma(t, x, u)\dot{W}(t, x)$, where b and σ satisfy a linearity condition, see (2.6).

We shall consider a more general class of gaussian noises, those that are white in time and correlated in space. Loosely speaking, for a fixed position the behaviour of the noise at two different times are independent, whereas for a fixed time the behaviour of the noise at two different positions may be correlated. The letter $\dot{\mathrm{M}}$ shall represent such a noise.

Finally, we shall consider a medium that could well be two, three, or d-dimensional. In that case the second order space derivative inside (1.1) should be replaced by the Laplace operator

$$\Delta u(t, x) := \sum_{i=1}^{d} \frac{\partial^2 u}{\partial x_i^2}(t, x),$$

where $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$.

The above considerations in mind, we shall study the following stochastic heat equation:

$$\frac{\partial u}{\partial t}(t, x) = \nu\Delta u(t, x) + b(t, x, u(t, x)) + \sigma(t, x, u(t, x))\dot{\mathrm{M}}(t, x). \tag{1.2}$$

In what follows, we shall define rigorously what a correlated noises is and what it means to be a solution of (1.2).

1.1 Correlated noises and types of solutions

We start by recalling the solution to Equation (1.2), on the whole space $\mathbb{R}^d$ with $b \equiv 0$ and $\sigma \equiv 1$, in the particular case where the power density is given by a non-random bounded continuous function $\Phi(t, x)$. To analyse the temperature of the medium at future time, we need to know at least the initial (t=0) profile of temperature. Suppose that it is given by a bounded continuous function f, i.e. $u(0, x) = f(x)$. Then, it can be verified, see Section A.1, that

$$u(t, x) = \int_0^t \int_{\mathbb{R}^d} \Gamma_\nu(t - s, x - y)\Phi(s, y)\, dsdy + \int_{\mathbb{R}^d} \Gamma_\nu(t, x - y)f(y)\, dy \tag{1.3}$$

solves indeed problem (1.2), where Γ_ν is called the fundamental solution (to the heat equation) and is given by (A.8).

For a bounded regular domain D, we also need to impose boundary conditions, i.e. we need to specify how the medium interacts with the environment. Vanishing Dirichlet boundary conditions correspond to the case in which the temperature at the boundary is fixed to zero degree. Vanishing Neumann boundary conditions mean that the medium is perfectly insulated. In both cases, it can be verified, see Section A.2, that the solution is given by

$$u_D(t,x) = \int_0^t \int_D G_D(t-s,x,y)\Phi(s,y)\,dsdy + \int_D G_D(t,x,y)f(y)\,dy.$$

The function G_D, called the Green function, is different for Dirichlet or Neumann boundary conditions. When the geometry of the medium is simple, we can find explicitly the expression for the Green functions, see Section A.4.

In the present case of a random power density $\dot{\mathrm{M}}$, do these expressions make sense? What happens when the initial condition is also assumed to be random? These questions are the starting point of the theory of stochastic partial differential equations.

1.1.1 Correlated noise

We shall define the class of correlated noises in a somewhat abstract way. Examples of such noises will be given and in particular, we shall confirm the intuitive properties of space-time white noise given after equation (1.1).

We consider a Gaussian family[1] of mean zero random variables

$$M = \left\{ M(g),\ g \in C_c^\infty(\mathbb{R}^{d+1}) \right\},$$

indexed by $C_c^\infty(\mathbb{R}^{d+1})$ the space of infinitely differentiable functions with compact support, with covariance

$$\begin{aligned}\mathbb{E}\left[M(g)\,M(h)\right] &= \int_{\mathbb{R}} dt \int_{\mathbb{R}^d} \Lambda(dx)\left[g(t) * \tilde{h}(t)\right](x)\\ &= \int_{\mathbb{R}} dt \int_{\mathbb{R}^d} \mu(d\xi)\mathcal{F}g(t)(\xi)\overline{\mathcal{F}h(t)(\xi)}, \qquad (1.4)\end{aligned}$$

for all $g, h \in C_c^\infty(\mathbb{R}^{d+1})$, where "$*$" denotes convolution in the space variable, $\tilde{h}(t,x) := h(t,-x)$, with $\bar{z}$ the complex conjugate of $z \in \mathbb{C}$, and

$$\mathcal{F}f(\xi) := \int_{\mathbb{R}^d} f(x)e^{-2\pi i x\cdot\xi}\,dx$$

the Fourier transform of any (measurable) function $f : \mathbb{R}^d \to \mathbb{R}$. We need to impose the following natural conditions on the positive measure Λ :

[1] It means that every random vector $(M(g_1),\ldots,M(g_n))$ is Gaussian, for all $n \in \mathbb{N}$ and $g_1,\ldots,g_n \in C_c^\infty(\mathbb{R}^{d+1})$.

- It is symmetric, i.e. $\Lambda(B) = \Lambda(-B)$, for any Borel set $B \subseteq \mathbb{R}^d$. This guarantees that $\mathbb{E}\left[M(g)\, M(h)\right] = \mathbb{E}\left[M(h)\, M(g)\right]$.
- It is positive definite, i.e. $\int_{\mathbb{R}^d} \Lambda(dx)\, (\phi * \tilde{\phi})(x) \geq 0$, for all $\phi \in C_c^\infty(\mathbb{R}^d)$. This guarantees that $\mathbb{E}\left[M(g)\, M(g)\right] \geq 0$.
- It is a tempered measure[2], i.e. it satisfies the following integrability condition
$$\int_{\mathbb{R}^d} \frac{\Lambda(dx)}{(1+|x|)^p} < \infty,$$
for some $p \in \mathbb{R}$.

Being positive definite and tempered, it is in fact the Fourier transform, in the sense of distributions, of some (tempered) positive measure μ, i.e.

$$\int_{\mathbb{R}^d} \psi(x)\, \Lambda(dx) = \int_{\mathbb{R}^d} \mathcal{F}\psi(\xi)\, \mu(d\xi),$$

for all test functions $\psi \in \mathcal{S}(\mathbb{R}^d)$. We say that μ is the spectral measure of Λ. More informations about distributions are given in Section 6.1.

Two important properties should be mentioned here. First, it can be shown that M is linear, i.e. for all $\alpha, \beta \in \mathbb{R}$ and $g, h \in C_c^\infty(\mathbb{R}^{d+1})$, we have

$$M(\alpha g + \beta h) = \alpha M(g) + \beta M(h) \quad a.s.,$$

where equality is understood as random variables. This shall imply further down that stochastic integration is indeed linear. Another property is translation invariance. For example, the random vector $(M(g_y), M(h_y))$ has the same law as that of $(M(g), M(h))$, where $g_y(x) := g(x+y)$.

We now would like to extend the definition of M to a broader class of functions. First, we consider the space variable. Let U be the completion of the Schwartz space $\mathcal{S}(\mathbb{R}^d)$ with respect to the semi-inner product

$$\langle \phi, \psi \rangle_U := \int_{\mathbb{R}^d} \Lambda(dx)\, (\phi * \tilde{\psi})(x) = \int_{\mathbb{R}^d} \mu(d\xi) \mathcal{F}\phi(\xi) \overline{\mathcal{F}\psi(\xi)}.$$

Whereas the first expression would restrict our extension to a subclass of functions, the second equality enables to reach even Schwartz distributions. For example, the Dirac delta functional belong to U when μ is a finite measure. Another important example is the fundamental solution of the wave equation in any dimension, see [11].

For any fixed time horizon $T > 0$, we define the norm $\|\cdot\|_{U_T}$ as

$$\|g\|_{U_T}^2 := \int_0^T \langle g(s), g(s) \rangle_U \, ds,$$

[2] This is equivalent to assuming that Λ defines a distribution on the space of Schwartz test functions $\mathcal{S}(\mathbb{R}^d)$. It is the space of infinitely differentiable functions that, together with all their partial derivatives, decrease faster than any polynomial.

for any $g \in C_c^\infty([0,T] \times \mathbb{R}^d)$. In fact, the set of such test functions are dense in $U_T := L^2([0,T];U)$, the set of square integrable functions on $[0,T]$ with values in the separable Hilbert space U. See [17, Lemma 2.4] for the latter result.

Finally, we observe that the map $g \mapsto M(g)$ is an isometry between

$$\left(C_c^\infty([0,T] \times \mathbb{R}^d), \|\cdot\|_{U_T}\right) \to L^2(\Omega, \mathcal{F}, \mathbb{P}).$$

Following the standard method to extend an isometry, $M(g)$ can be defined for any $g \in U_T$, and the collection remains a Gaussian family of random variable with mean zero and covariance given by (1.4). We pause for a second and give two important examples of correlated noises.

Example 1.1 (Space-time white noise)**.** When the positive measure Λ is the Dirac delta measure, we say that M is a space-time white noise, and it is denoted by W. Its spectral measure is the Lebesgue measure $\mu(d\xi) = d\xi$. In particular, it can be extended to the space of square integrable functions $U_T = L^2(\mathbb{R} \times \mathbb{R}^d)$, with covariance given by

$$\mathbb{E}\left[W(g)\, W(h)\right] = \iint_{\mathbb{R}\times\mathbb{R}^d} g(t,x)h(t,x)\, dtdx.$$

The random thermal energy added to the system within the time-space region $E =]t_1,t_2] \times]x_1,x_2]$ is given by the mean zero Gaussian random variable $W(\mathbf{1}_E)$, whose variance is

$$\mathbb{E}\left[W(\mathbf{1}_E)\, W(\mathbf{1}_E)\right] = \iint_{\mathbb{R}\times\mathbb{R}^d} \mathbf{1}_E(t,x)\, \mathbf{1}_E(t,x)\, dtdx = (t_2 - t_1)(x_2 - x_1).$$

Compared to another time-space region $F =]s_1,s_2] \times]y_1,y_2]$, their covariance is given by

$$\mathbb{E}\left[W(\mathbf{1}_E)\, W(\mathbf{1}_F)\right] = \iint_{\mathbb{R}\times\mathbb{R}^d} \mathbf{1}_E(t,x)\, \mathbf{1}_F(t,x)\, dtdx = |E \cap F|\,,$$

the area of their intersection. If $E \cap F = \emptyset$, their covariance is zero and both random variables are independent, since they are Gaussian.

Example 1.2 (Riesz kernels)**.** It is the case when the positive measure Λ is given by $\Lambda(dx) = f(x)dx$ for $f(x) = |x|^{-d+\alpha}$, with $\alpha \in (0,d)$. Its spectral measure is given by $\mu(d\xi) = g(\xi)d\xi$ for $g(\xi) = c_\alpha |\xi|^{-\alpha}$ and some constant $c_\alpha > 0$. In particular, the covariance (1.4) is given by

$$\mathbb{E}\left[M(g)\, M(h)\right] = \iiint_{\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^d} g(t,x)f(x-y)h(t,y)\, dtdxdy.$$

Other examples of correlated noises are Poisson kernels, Cauchy kernels, Ornstein-Uhlenbeck-type kernels, and Bessel kernels. For some of their properties, see [24].

We have introduced, on purpose, two different notations for the space-time white noise. The first one, $\dot{W}(t,x)$, comes from a physical interpretation, and represents the random power density. Implicitly, this (wrongly) assumes that white noise can be evaluated at each point. Yet, once integrated on some time-space region E, it represents the random total energy produced within E. The second one, $W(\mathbf{1}_E)$, also represents the total energy produced within E. The following formal chain of equalities may link both notation styles.

$$\iint_E \dot{W}(t,x)\,dtdx = \iint_E W(dt,dx) = W(E) = W(\mathbf{1}_E).$$

If W were a measure with density given by the function $\dot{W}(t,x)$, then the above chain of equalities would perfectly make sense. In the general case of correlated noise, we shall also write $M(g)$ as the following integral

$$\iint_{\mathbb{R}\times\mathbb{R}^d} g(t,x)\,\mathrm{M}(dt,dx). \tag{1.5}$$

1.1.2 Different types of solutions

At the beginning the present section, we asked whether the first integral on the right hand side of (1.3) would make sense if the function Φ is replaced by $\dot{W}$ or $\dot{\mathrm{M}}$. The integral notation of the noise given in (1.5) is precisely what we need. In view of the previous subsection, the following integral

$$\int_0^t \int_{\mathbb{R}^d} \Gamma_\nu(t-s,x-y)\,W(dsdy)$$

would make sense if the function $(s,y)\mapsto\Gamma_\nu(t-s,x-y)$ belongs to $L^2(\mathbb{R}\times\mathbb{R}^d)$. In fact, it does if and only if $d=1$, and the same condition holds for the expression involving the Green function G_D on bounded domains. The former follows from (A.12), and the latter, in the particular case when the domain is a two dimensional cube with Neumann boundary conditions, is considered in the last paragraph of [55, Chapter 3].

Correlated noises are tailor-made to solve such problems. Indeed, an extra integrability assumption on the positive measure Λ enables to make sense of

$$\int_0^t \int_{\mathbb{R}^d} \Gamma_\nu(t-s,x-y)\,\mathrm{M}(ds,dy)$$

in any dimension $d\ge 1$. Such an assumption is given by (4.4) or equivalently by (4.3), and is called Dalang's condition. This integrability condition is a trade-off between the roughness of the noise and the regularity of the integrant. This condition was first formulated in the study of the stochastic wave equation, in [12], and then applied to a larger class of stochastic parabolic equations, in the seminal paper [11]. In the case of Riesz kernels, the latter condition becomes $d < 2 + \alpha$.

If we set $b \equiv 0$, $\sigma \equiv 1$, the initial temperature profile $f \equiv 0$, and we assume that the correlated noise satisfies Dalang's condition, then the solution to (1.2) should resemble formula (1.3), and an educated guess would be to set

$$u(t,x) = \int_0^t \int_{\mathbb{R}^d} \Gamma_\nu(t-s, x-y) \, \mathrm{M}(ds, dy).$$

The regularity of this Gaussian field depends on the regularity of the noise. Sharp Hölder continuity results can be found in [36] and [48]. In the particular case of Riesz kernel,

$$u \in C_{\frac{1}{2} - \frac{d-\alpha}{4} -, 1 - \frac{d-\alpha}{2} -}(\mathbb{R}_+^* \times \mathbb{R}^d),$$

and it is not differentiable! Thus, it cannot be a classical solution to (1.2).

We need to transform the general form of (1.2) into an integral equation. There are many possible ways to proceed, and each of them lead to a different definition of a solution to (1.2). A standard method is to first define a weak formulation by multiplying the equation on each side by a test function, and formally apply an integration by part to remove the unavailable derivatives of u. From the weak formulation, it is possible to reach another type of solution, which we shall shortly define as a random field solution. This procedure was carefully imagined and explained by Walsh in [55].

Definition 1.3. A real-valued, jointly measurable and adapted process

$$\big(u(t,x,\omega), t \in [0,T], x \in D, \, \omega \in \Omega\big)$$

is a *random field solution* of (1.2), if for all $(t,x) \in [0,T] \times D$,

$$\begin{aligned} u(t,x) = I_0(t,x) &+ \int_0^t \int_D G(t-s,x,y) b(s,y,u(s,y)) \, dyds \\ &+ \int_0^t \int_D G(t-s,x,y) \sigma(s,y,u(s,y)) \, \mathrm{M}(dy,ds) \quad a.s., \end{aligned} \tag{1.6}$$

where I_0 is the contribution of the initial condition $u(0,x) = u_0(x)$, and G is the Green function associated to the heat equation on some domain $D \subseteq \mathbb{R}^d$, under some possible boundary conditions.

The last term of (1.6) needs some comments. In Section 1.1.1, our definition of a noise was a Gaussian family of random variable indexed by the space $U_T := L^2([0,T];U)$. Now, we need to perform integration of random elements such as $(s, y, \omega) \to G(t-s, x, y)\sigma(s, y, u(s, y, \omega))$. Once more, we need to extend the definition of stochastic integration to such processes. Integration with respect to a (worthy) martingale measure is described in [55], and is called Walsh stochastic integral. An extension was presented in [11], which enables to use correlated noise as a (worthy) martingale measure, and therefore make sense of (1.6).

In the case of white noise in one space dimension, it is a standard result from [55] that if both functions b and σ are Lipschitz, and if the initial condition is bounded, then equation (1.2) admits a unique random field solution. Moreover, the solution has continuous trajectories. In the case of correlated noise in any space dimension, it is a standard result from [11] that under the same hypotheses, together with Dalang's condition (4.4), equation (1.2) admits a unique random field solution.

When the noise does not satisfy Dalang's condition (4.4), then the last term of (1.6) is not well defined and we cannot talk about random field solution. In particular, white noise in higher dimension $d \geq 2$, do not have any random field solution. Nevertheless, it can be defined as random linear functional in the space of Schwartz distributions, see [55, Chapter 5].

Final extension of the noise: For some specific class of random processes $P : \Omega \times (0,T) \times \mathbb{R}^d \to \mathbb{R}$, those that are adapted, jointly measurable, and satisfy $\mathbb{E}[|I_P(T)|^2] < \infty$, it is possible to define the stochastic integral, for $t \in [0,T]$,

$$I_P(t) := \iint_{(0,t)\times\mathbb{R}^d} P(\omega, s, y)\, \mathrm{M}(ds, dy).$$

The process $t \mapsto I_P(t)$ is a continuous square integrable martingale, whose quadratic variation is given by

$$\int_0^t \|P(\omega, s)\|_U^2\, ds.$$

We will mostly be interested in the case $P(\omega, s, y) = G(s, y)Z(\omega, s, y)$, where G will be either the heat kernel or the Green function associated to Dirichlet boundary conditions, and Z will typically be $Z(s, y) = \sigma(u(s, y))$.

1.1.3 Further concepts

Many generalisations of equation (1.2) are possible. For example, we could study more general type of equations, consider non-Lipschitz functions b and σ, or assign a broader class of noises.

In [11], the wave equation, the damped wave equation, and parabolic equations are considered. Dalang's condition (4.4) is still a necessary and sufficient condition to study these equations.

For the heat equation with white noise (in one dimension), the solution gets infinite energy almost surely in finite time, when $\sigma \equiv c \neq 0$, and $b \geq 0$ is convex and has super-linear growth. A precise result is given in [14], together with the existence of a continuous random field solution that doesn't blow-up when $|b(z)| = O\left(|z| \log |z|\right)$ and $|\sigma(z)| = o\left(|z| \left(\log |z|\right)^{1/4}\right)$, as $z \to \infty$.

Broader classes of noises exist. For example, Gaussian noises with covariance in both time and space variables. They have many applications, such as in biology, physics, or finance. In the case of the heat equation, see [54] for the additive case, and [2] for the Anderson model. For an operator given by a generator of a Lévy process, see [36]. Regularity results are given in each. A study of Levy white noise is given in [15]. The path regularity of the solution to the heat equation with such multiplicative noise is studied in [10]. The relation between the random field solution and the generalized solution is given in [16]. As one might expect, thanks to a newly derived version Fubini theorem.

Remark. A clear introduction to Walsh theory of SPDE is given [13]. Another important theory of stochastic integration is the one with respect to Hilbert-space-valued processes, presented in [46]. Both theories can be applied to get complementary informations about the solution of some stochastic partial differential equation. It is the primery objective of [17] to compare both theories.

1.2 Goals of the thesis

The driving scenario of this thesis is to compare the solution to the stochastic heat equation on the whole space $\mathbb{R}^d$, with the one on some (sequence of) bounded domains $D \subseteq \mathbb{R}^d$, to which boundary conditions are imposed, i.e.

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) = \Delta u(t,x) + b(u(t,x)) + \sigma(u(t,x))\,\dot{\mathrm{M}}, & t > 0,\, x \in \mathbb{R}^d, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^d, \end{cases}$$

and

$$\begin{cases} \dfrac{\partial u_D}{\partial t}(t,x) = \Delta u_D(t,x) + b(u_D(t,x)) + \sigma(u_D(t,x))\,\dot{\mathrm{M}}, & t > 0,\, x \in D, \\ u_D(t,x) = 0 \quad \text{or} \quad \dfrac{\partial u_D}{\partial \nu}(t,x) = 0, & t > 0,\, x \in \partial D, \\ u_D(0,x) = u_0(x), & x \in D, \end{cases}$$

where b, σ and u_0 satisfy the d-dimensional versions of (2.6) and (2.7), the noise $\dot{\mathrm{M}}$ on $\mathbb{R}_+ \times \mathbb{R}^d$ is white in time and correlated in space, and ν is the unit outward normal vector at the point $x \in \partial D$.

We study the regularity of their difference, as well as bounds of its p-moments. To the best of our knowledge, the former has not been solved yet, and the second has application in numerical approximations of SPDEs.

1.2.1 Regularity analysis

In one space dimension with white noise, when both b and σ are globally Lipschitz functions, it is well known that the random field solutions admit continuous modifications. In fact, they are Hölder continuous, with $u \in C_{1/4-,1/2-}(\mathbb{R}_+^* \times \mathbb{R})$, and $u_D \in C_{1/4-,1/2-}(\mathbb{R}_+^* \times D)$, but both are nowhere differentiable. The case of vanishing Neumann boundary conditions was already considered in [55], when the initial condition is bounded in expectation. To understand our future strategy, we give the main ideas of that proof, when $D = (-L, L)$, $b \equiv 0$, and $u_0 \equiv 0$. The random field solution admits the representation formula

$$u_D(t,x) = \int_0^t \int_{-L}^{L} G_D(t-s,x,y)\sigma(u_D(s,y))\, W(dsdy),$$

which can also be written as

$$u_D(t,x) = \int_0^t \int_{-L}^{L} \left[G_D - \Gamma\right](t-s,x,y)\sigma(u_D(s,y))\, W(dsdy), \\ + \int_0^t \int_{-L}^{L} \Gamma(t-s,x-y)\sigma(u_D(s,y))\, W(dsdy). \quad (1.7)$$

The trick is to observe that the difference $G_D - \Gamma$ is infinitely smooth, and each of its partial derivatives is bounded on $\mathbb{R}_+^* \times (-L, L) \times (-L, L)$, see Section 2.3 for an explicit formula. From that observation, it can be proved that the first integral of (1.7) admits a version that is in fact infinitely smooth. For an example of such a derivation, applying Kolmogorov continuity theorem to show that the guessed partial derivative is continuous and using Fubini theorem to show that it is indeed the derivative in the sense of distributions, see [40, Section 3.3]. The limiting regularity of u_D is therefore that of the last integral of (1.7), which involves the fundamental solution Γ.

A similar argument can be applied to the solution on $\mathbb{R}$. It admits the representation

$$u(t,x) = \int_0^t \int_{\mathbb{R}} \Gamma(t-s,x-y)\sigma(u(s,y))\, W(dsdy),$$

which can also be written as

$$u(t,x) = \int_0^t \int_{\mathbb{R}\setminus(-L,L)} \Gamma(t-s,x-y)\sigma(u(s,y))\, W(dsdy) \\ + \int_0^t \int_{-L}^{L} \Gamma(t-s,x-y)\sigma(u(s,y))\, W(dsdy). \quad (1.8)$$

For $x \in [-L+\varepsilon, L-\varepsilon]$ and $y \in \mathbb{R} \setminus (-L, L)$, the function $\Gamma(s, x-y)$ is infinitely smooth with bounded (exponentially fast decreasing) partial derivatives. Therefore, the first integral of (1.8) also admits a version that is infinitely smooth. The second integral of (1.8), which very much looks like the second integral of (1.7), is the one limiting the regularity of u.

In fact, for additive withe noise $\sigma \equiv 1$, the above argument becomes, for $x \in (-L, L)$,

$$u(t,x) - u_D(t,x) = \int_0^t \int_{\mathbb{R}\setminus(-L,L)} \Gamma(t-s, x-y)\, W(dsdy) + \int_0^t \int_{-L}^{L} [\Gamma - G_D]\,(t-s,x,y)\, W(dsdy),$$

which proves that the difference of interest $u - u_D$ is smooth. A similar procedure also applies for additive correlated noise in any dimension, if the boundary of the domain is smooth. Useful inequalities are given by (3.10) and (A.22).

In the case of additive white noise, in dimension $d \geq 2$, such formulas do not apply, for no random field solution exists. Yet, the regularity of the these distributions can be studied in Sobolev spaces. For a regular domain $D \subseteq \mathbb{R}^d$, it is known that u_D is a continuous process in time with values in H_{-n} for $n > d/2 - 1$, for some special versions of Sobolev spaces H_s. This result applies in fact to more general classes of equations, noises, and boundary conditions, see [55, Theorems 5.1 and 5.2]. Similar results holds for u on $\mathbb{R}^d$, in yet other versions of Sobolev spaces.

In Chapter 3, we prove that the difference of interest $u(t)-u_D(t)$, for any fixed time $t > 0$, has in fact very smooth trajectories in space. The precise statement is given in Theorem 3.7. In fact, the more regular the boundary of the domain is, the higher the regularity of the latter difference is. To reach this conclusion, we first need to find some versions of Sobolev spaces that can analyse both $u(t)$ and $u_D(t)$. In a failed first attempt, we did consider the ones given by

$$H^s = \left\{ f \in \mathcal{S}'(\mathbb{R}^d) : \int_{\mathbb{R}^d} \left(1 + |\xi|^2\right)^s |\mathcal{F}(f)(\xi)|^2 \, d\xi < \infty \right\}.$$

In fact, we proved in Proposition 3.5 that the Fourier transform of $u(t)$, in the sense of distributions, do not have any function valued representative. Yet, any product $\phi u(t)$, for $\phi \in C_c^\infty(\mathbb{R}^d)$, does. This translates the fact that $u(t)$, having already irregular local properties, has also a lack of integrability at infinity. Theorem 3.7 concludes that the local regularity of $u(t)$ and $u_D(t)$ is the same, which was to be expected from the one dimensional case.

A possible next step could be to analyse the regularity, say in space, of the difference $u(t)-u_D(t)$, for an additive correlated noise that do not satisfy Dalang's condition. We challenge the reader to do so.

1.2.2 Moment bounds

In the framework of non-stochastic partial differential equations, it is a fair objective to find a way to numerically approximate a phenomenon that is mathematically described on an unbounded domain, or a domain so large that the actual computation techniques are not sufficient. One obvious way is to restrict the problem on some bounded domain and impose artificial boundary conditions, such as (vanishing) Dirichlet or Neumann boundary conditions. The choice of the new bounded domain D is important to guarantee that the error made when approximating $u(t,x)$ by $u_D(t,x)$ is not too large. In finance, the price of an asset is profoundly linked with the solution to the heat equation, see [41, Chapter 5]. Introducing artificial vanishing Dirichlet boundary conditions on the interval $D=(-L,L)$ gives an approximative price. It is discussed below in Example 2.3, and in particular, for any $\alpha \in (0,1)$, we have the uniform bound

$$\sup_{t\in[0,T]} \sup_{x\in[-\alpha L,\alpha L]} |u(t,x)-u_D(t,x)| \leq \frac{c\sqrt{2\nu^2 T}}{L(1-\alpha)} \exp\left(-\frac{L^2(1-\alpha)^2}{2\nu^2 T}\right).$$

The further apart from the artificial boundary we are, the smaller the localization error is. Moreover, the error decreases exponentially fast as $L\to\infty$. To find such a bound, at least two approaches are possible. In [41], they used an application of Feynmann-Kac's formula. Another way, which we shall use throughout this thesis, is to carefully compare $\Gamma - G_D$.

It is now possible to introduce a numerical method to approximate the function u_D. The total error in the approximation process of $u(t,x)$ is the sum of the error from the artificial boundary and that of the numerical method. We can tune the order of the method and the size of the domain to reach a desire threshold error.

In Chapter 2, we explore the stochastic heat equation on $\mathbb{R}$ with white noise, and conclude in Theorem 2.4, that any p-moment of the difference $u-u_D$ admits the same exponential rate of convergence as $L\to\infty$. The cases of vanishing Dirichlet, mixed, or Neumann boundary conditions are studied on the symmetric interval $(-L,L)$, with bounded (random) initial data. (An obvious application to the non stochastic heat equation shows that the idea of studying the difference $\Gamma - G_D$ works well for the basic case.)

Numerical methods for the heat equation, with multiplicative white noise and vanishing Dirichlet boundary conditions on a finite interval, are already available. See [56] and [18] for finite element and finite difference methods. An interesting fact about a lower bound for the rate of convergence is given in the latter. It says that if h is the space step and k the time step, then any scheme, implicit of explicit, will have an error at least $O(h^{\frac{1}{2}}+k^{\frac{1}{4}})$. This is strongly related to the fact that the solution is only Hölder continuous, with exponent 1/2 in space and 1/4 in time. In fact, the latter remark is only true if the numerical scheme uses equidistant evaluations of the noise, see [38].

In Chapter 4, we explore the stochastic heat equation on $\mathbb{R}^d$ with correlated noises, and conclude in Theorem 4.2, that any p-moment of the difference admits the same exponential rate of convergence as $L \to \infty$. The cases of vanishing Dirichlet, mixed, or Neumann boundary conditions are studied with bounded (random) initial data.

In a series of four papers, Gerencsér and Gyöngy introduced two methods to study the numerical approximation of the solution in the whole space. In [34], [29], [30], they constructed a finite difference scheme directly applied to the whole space. In [31, Theorem 5.1], they introduced vanishing Dirichlet boundary conditions, and apply a new version of Feynmann-Kac formula. They reached error estimates in supremum norms via Sobolev's embeddings. In that regard, their results are stronger than ours.

We compare their second method to ours. Their result only apply for a sequence of increasing balls, yet this might be generalised later to C^1, since their Feynmann-Kac formula is valid for such domains. Our method apply to slightly more general boundary conditions, regular boundaries in the sense of [20]. For example, boundaries satisfying the exterior cone condition. Their approach is valid for a different class of noises. Concerning the choice of possible functions b and σ, the two classes of functions are different. They must impose W_p^1 regularity for the initial condition, whereas we can ask for a polynomially increasing one, in the case of the Anderson model. It should be mentioned that their method holds for an enormous class of parabolic equations, even degenerate ones.

In Chapter 5, we study the Anderson model in $\mathbb{R}^d$, i.e. for the special forms of $b \equiv 0$ and $\sigma(u) = \lambda u$. One reason to study this simplified model is that it allows a broader class of initial conditions, from the delta Dirac measure at the origin to exponentially growing measures. Other reasons are explained in [24], together with a detailed list of physic literature. In Theorem 5.1, we derived a very convenient expression for two points correlations

$$\begin{aligned}\mathbb{E}&\left[\{u(t,x) - u_D(t,x)\}\{u(t,x') - u_D(t,x')\}\right] \\ &= \iint_{\mathbb{R}^{2d}} \mu_0(d\alpha)\mu_0(d\alpha')\,\left[\mathcal{K}_{1,1} - \mathcal{K}_{1,2} - \mathcal{K}_{2,1} + \mathcal{K}_{2,2}\right](t,x,x',\alpha,\alpha'),\end{aligned}$$

for any initial condition μ_0 satisfying (5.1). Its derivation was similar to that of u given in [9]. The functions $\mathcal{K}$ gather informations about the nature of the heat equation and that of the noise. The initial condition appears only at the very end through integration. Second moment bounds are deduced from a careful analysis of the functions $\mathcal{K}$. For example, exponentially fast convergence on compact sets is obtained for any initial condition with polynomial growth. More interestingly, from a given convergence rate, we can decide whether some initial data is admissible.

Chapter 2

One space dimension with white noise

In this chapter, we will study the heat equation in one space dimension, with white noise and bounded initial data. We will compare the behavior of the solution on the whole real line $\mathbb{R}$ with the solution on the bounded symmetric interval $(-L, L)$. The former satisfies

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) = \dfrac{\partial^2 u}{\partial x^2}(t,x) + b(t,x,u(t,x)) & \\ \qquad\qquad\qquad + \sigma(t,x,u(t,x))\dot{W}, & t > 0,\ x \in \mathbb{R}, \\ u(0,x) = u_0(x), & x \in \mathbb{R}. \end{cases} \tag{2.1}$$

The latter satisfies

$$\begin{cases} \dfrac{\partial u_L}{\partial t}(t,x) = \dfrac{\partial^2 u_L}{\partial x^2}(t,x) + b(t,x,u_L(t,x)) & \\ \qquad\qquad\qquad + \sigma(t,x,u_L(t,x))\dot{W}, & t > 0,\ x \in (-L,L), \\ u_L(0,x) = u_0(x), & x \in (-L,L), \end{cases} \tag{2.2}$$

subjected to either vanishing Dirichlet, Neumann, or mixed boundary conditions,

$$u_L(t,-L) = 0 = u_L(t,L), \qquad t > 0, \tag{2.3}$$

$$\frac{\partial u_L}{\partial x}(t,-L) = 0 = \frac{\partial u_L}{\partial x}(t,L), \qquad t > 0, \tag{2.4}$$

$$u_L(t,-L) = 0 = \frac{\partial u_L}{\partial x}(t,L), \qquad t > 0. \tag{2.5}$$

Both equations (2.1) and (2.2) have the same white noise $\dot{W}$ on $\mathbb{R}_+ \times \mathbb{R}$.

Throughout this chapter, we will assume the following particular conditions on b, σ, and u_0 :

- The functions $b, \sigma : [0,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous, globally Lipschitz in the last variable and have linear growth, i.e., uniformly for all $t \in [0,T]$ and $x \in \mathbb{R}$, there exists some constants Lip and K such that

$$\begin{aligned} |b(t,x,u) - b(t,x,v)| \wedge |\sigma(t,x,u) - \sigma(t,x,v)| &\leq \mathrm{Lip}\,|u-v|\,, \\ |b(t,x,u)| \wedge |\sigma(t,x,u)| &\leq K(1+|u|), \end{aligned} \tag{2.6}$$

  where $a \wedge b := \max(a,b)$.

- The random measurable function $u_0 : \Omega \times \mathbb{R} \to \mathbb{R}$ is independent of $\dot{W}$ and bounded in expectation, i.e., for some $p \geq 2$,

$$\|u_0\| := \sup_{x\in\mathbb{R}} \mathbb{E}\left[|u_0(x)|^p\right]^{1/p} < \infty. \tag{2.7}$$

Those assumptions guarantee existence, uniqueness, and even some Hölder regularity of the random field solutions u, and u_L, see [55, Theorem 3.2, Corollary 3.4, and Exercise 3.4]. They satisfy the representation

$$\begin{aligned} u(t,x) = &\int_{\mathbb{R}} \Gamma(t,x-y)\,u_0(y)\,dy \\ &+ \int_0^t \int_{\mathbb{R}} \Gamma(t-s,x-y)\,b(s,y,u(s,y))\,dyds \\ &+ \int_0^t \int_{\mathbb{R}} \Gamma(t-s,x-y)\,\sigma(s,y,u(s,y))\,W(dyds), \end{aligned} \tag{2.8}$$

and

$$\begin{aligned} u_L(t,x) = &\int_{-L}^{L} G_L(t,x,y)\,u_0(y)\,dy \\ &+ \int_0^t \int_{-L}^{L} G_L(t-s,x,y) b(s,y,u_L(s,y))\,dyds \\ &+ \int_0^t \int_{-L}^{L} G_L(t-s,x,y)\sigma(s,y,u_L(s,y))\,W(dyds). \end{aligned} \tag{2.9}$$

where Γ is the heat kernel given by (A.8), and G_L is the Green function associated with the corresponding boundary conditions, see (A.48), (A.62), and (A.77).

2.1 Warm up

Before we get to the comparisons in full generality, we first consider the simplest possible assumptions $b = 0$, $\sigma = 0$, and $u_0 \geq 0$ non-random. From the representation of the solutions, see (A.9) and (A.47), (A.61), and (A.76), we have

$$u(t,x) = \int_{\mathbb{R}} \Gamma(t, x-y) u_0(y)\, dy,$$
$$u_L(t,x) = \int_{-L}^{L} G_L(t,x,y) u_0(y)\, dy,$$

where

$$\Gamma(t,x) := \Gamma_1(t,x) = \frac{1}{(4\pi t)^{1/2}} e^{-\frac{x^2}{4t}}$$

is the heat kernel and G_L is the corresponding Green function, different for every boundary conditions, see (A.48), (A.62), and (A.77).

Example 2.1. Vanishing Dirichlet boundary conditions (2.3) and constant positive initial data $u_0 = c > 0$. In that case, we have

$$u(t,x) = c \int_{\mathbb{R}} \Gamma(t, x-y)\, dy = c,$$

and

$$u_L(t,x) = c \int_{-L}^{L} G_L(t,x,y)\, dy = c(t,x) \leq c.$$

Moreover, from the properties of the Green function G_L, see Proposition A.8, we know that

$$u_L(t, \pm L) = 0,$$

for all $t > 0$, and

$$u_L(t,x) \longrightarrow 0, \text{ as } t \to \infty,$$

for all $x \in [-L, L]$, with convergence rate $t^{-1/2}$.

Example 2.2. Vanishing Neumann boundary conditions (2.4) and Gaussian initial data $u_0(x) = \Gamma(s,x)$. In that case, we have

$$u(t,x) = \int_{\mathbb{R}} \Gamma(t, x-y)\Gamma(s,y)\, dy = \Gamma(t+s, x),$$

and

$$u_L(t,x) = \int_{-L}^{L} G_L(t,x,y)\Gamma(s,y)\, dy \geq \Gamma(s, L),$$

by Proposition A.10. We see that $u_L(t,x)$ always stays greater than some positive threshold, whereas $u(t,x)$ will eventually decrease to zero, with convergence rate $t^{-1/2}$. Indeed, if u_0 is merely an integrable function, then

$$|u(t,x)| = \int_{\mathbb{R}} \Gamma(t, x-y) \left| u_0(y) \right| \, dy \leq (4\pi t)^{-1/2} \left\| u_0 \right\|_{L^1(\mathbb{R})} \longrightarrow 0,$$

as $t \to \infty$, uniformly for $x \in \mathbb{R}$.

We now analyse a slightly higher level example, when $b(u) = \lambda u + h$ and $\sigma \equiv 0$.

Example 2.3. To have an idea of the expected rate of convergence, we can analyse the deterministic equation,

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) = \dfrac{\nu^2}{2}\dfrac{\partial^2 u}{\partial x^2}(t,x) + \lambda u(t,x) + h(t,x), & t > 0,\, x \in \mathbb{R}, \\ u(0,x) = f(x), & x \in \mathbb{R}. \end{cases}$$

We let u_L the solution to the same equation, but with vanishing Dirichlet boundary conditions on the domain $(-L, L)$. When f and h are bounded and continuous on $\mathbb{R}$, respectively $\mathbb{R}_+ \times \mathbb{R}$, Feynman-Kac's formula can be applied to evaluate the difference

$$u(t,x) - u_L(t,x) = \mathbb{E}\left[\mathbf{1}_{\{\exists s \le t:\ X_s \notin (-L,L)\}}\, e^{\lambda t} f(X_t)\right] + \mathbb{E}\left[\int_\tau^t e^{\lambda s} h(t-s, X_s)\, ds\right]$$

where $X_s = \nu B_s + x$ for some Brownian motion B_s, and the exit time $\tau = \inf\{s \ge 0 :\ X_s \notin (-L,L)\} \wedge t$. For such a probabilistic resolution of the heat equation, we refer the reader to Section A.3.1. Using a similar argument to [41, chapter 5.2.1], we get the following bound: for any $0 < \alpha < 1$,

$$\sup_{t\in[0,T]} \sup_{x \in [-\alpha L, \alpha L]} |u(t,x) - u_L(t,x)| \le \frac{c\sqrt{2\nu^2 T}}{L(1-\alpha)} \exp\left(-\frac{L^2(1-\alpha)^2}{2\nu^2 T}\right), \tag{2.10}$$

where $c = c(\|f\|_\infty, \|h\|_\infty, \lambda, T)$.

Another way of finding inequality (2.10) is to write explicitly the difference of the two solutions, and since f and h are assumed to be bounded, it is enough to find effective bounds on the difference of the associated Green functions. We shall use the latter procedure.

From Example 2.1, we cannot hope to have a good approximation at the boundary points. In case of constant initial condition, we see that the difference $u(t,x) - u_L(t,x)$ is directly related to the following difference

$$\int_{\mathbb{R}} \Gamma(t, x-y)\, dy - \int_{-L}^{L} G_D(t,x,y)\, dy = 1 - \int_{-L}^{L} G_D(t,x,y)\, dy.$$

For such bounds, see Lemmas 2.7 and 4.5 or Equation (A.34).

From all three examples, we expect the difference $u - u_L$ to become more and more different as time evolves. We shall restrict our attention to a finite time horizon $T > 0$.

2.2 Main results and general ideas

We will prove that if $x \in (-L, L)$ is sufficiently far away from the boundary points, then both solutions are very close to one another, within some finite time horizon $t \in [0, T]$. In the present case of one space dimension, a point is close to the boundary if either $|x - L|$ or $|x + L|$ is small. The final bound will in fact be a function of these differences. As we shall see, Neumann boundary condition has to be treated separately.

The random field solutions to equations (2.1) and (2.2), subjected to either (2.3) or (2.5) satisfy the following convergence rate.

Theorem 2.4. *Fix $T > 0$. For all $t \in [0, T]$, $L > 0$, and $x \in [-L, L]$,*

$$\mathbb{E}\left[|u(t,x) - u_L(t,x)|^p\right]^{1/p} \le \tilde{c}\left(\exp\left(-\frac{(L-x)^2}{8t}\right) + \exp\left(-\frac{(L+x)^2}{8t}\right)\right),$$

where $\tilde{c} = \tilde{c}(T, Lip, K, \|u_0\|, p)$ is independent of t, x, and L. This bound is valid for the same p as in (2.7).

The solutions to equations (2.1), and (2.2) subjected to (2.4) satisfy the following convergence rate.

Theorem 2.5. *Fix $T > 0$ and $l > 0$. For all $t \in [0, T]$, $L \ge l\sqrt{T}$, and $x \in [-L, L]$,*

$$\mathbb{E}\left[|u(t,x) - u_L(t,x)|^p\right]^{1/p} \le \tilde{c}\left(\exp\left(-\frac{(L-x)^2}{8t}\right) + \exp\left(-\frac{(L+x)^2}{8t}\right)\right),$$

where $\tilde{c} = \tilde{c}(T, l, Lip, K, \|u_0\|, p)$ is independent of t, x, and L. This bound is valid for the same p as in (2.7).

In order to reach both of these conclusions in Section 2.5, we will apply the fact that the moments of the solutions are uniformly bounded. That fact was already known for u and u_L. We will find a bound that is valid uniformly on $L > 0$, or on $L \ge l\sqrt{T}$ respectively.

To simplify notations, we denote u_L^D, u_L^N, or u_L^M the solutions to equation (2.2) with Dirichlet boundary conditions (2.3), Neumann boundary conditions (2.4), or mixed boundary conditions (2.5), respectively.

Proposition 2.6. *The solutions u, and u_L^D, u_L^M, and u_L^N to the heat equations* (2.1) *and* (2.2) *satisfy*

$$\sup_{t\in[0,T]} \sup_{x\in\mathbb{R}} \mathbb{E}\left[|u(t,x)|^p\right] < \infty, \tag{2.11}$$

$$\sup_{t\in[0,T]} \sup_{L>0} \sup_{x\in[-L,L]} \mathbb{E}\left[\left|u_L^D(t,x)\right|^p\right] < \infty, \tag{2.12}$$

$$\sup_{t\in[0,T]} \sup_{L>0} \sup_{x\in[-L,L]} \mathbb{E}\left[\left|u_L^M(t,x)\right|^p\right] < \infty, \tag{2.13}$$

$$\sup_{t\in[0,T]} \sup_{L>l\sqrt{T}} \sup_{x\in[-L,L]} \mathbb{E}\left[\left|u_L^N(t,x)\right|^p\right] < \infty. \tag{2.14}$$

The first uniform bound is already well known in the literature, see for example [55, Theorem 3.2] for white noise in one space dimension or [11, Theorem 13] for colored noise in higher space dimension. The proof is given in Section 2.4.

Once these uniform bounds are obtained, we can use the representation formulas that satisfy the random field solutions, see equations (2.8) and (2.9), to write the difference of interest as

$$u(t,x) - u_L(t,x) = \sum_{i=0}^{6} I_i(t,x), \tag{2.15}$$

where,

$$\begin{aligned}
I_0(t,x) &= \int_{\mathbb{R}} \Gamma(t, x-y)\, u_0(y)\, dy - \int_{-L}^{L} G_L(t,x,y)\, u_0(y)\, dy, \\
I_1(t,x) &= \int_0^t \int_{-L}^{L} [\Gamma(t-s, x-y) - G_L(t-s,x,y)] b(s,y,u_L(s,y))\, dyds, \\
I_2(t,x) &= \int_0^t \int_{-L}^{L} [\Gamma(t-s, x-y) - G_L(t-s,x,y)] \sigma(s,y,u_L(s,y))\, W(dyds), \\
I_3(t,x) &= \int_0^t \int_{-L}^{L} \Gamma(t-s, x-y)[b(s,y,u(s,y)) - b(s,y,u_L(s,y))]\, dyds, \\
I_4(t,x) &= \int_0^t \int_{-L}^{L} \Gamma(t-s, x-y)[\sigma(s,y,u(s,y)) - \sigma(s,y,u_L(s,y))]\, W(dyds), \\
I_5(t,x) &= \int_0^t \int_{\mathbb{R}\setminus[-L,L]} \Gamma(t-s, x-y) b(s,y,u(s,y))\, dyds, \\
I_6(t,x) &= \int_0^t \int_{\mathbb{R}\setminus[-L,L]} \Gamma(t-s, x-y) \sigma(s,y,u(s,y))\, W(dyds).
\end{aligned}$$

For the first three terms I_0, I_1, and I_2, we will need to analyse the difference

$$\Gamma(t, x-y) - G_L(t,x,y)$$

between the heat kernel and the Green function, and its integral properties. This is carried in Section 2.3. For the last two terms I_5 and I_6, we will need to find some integral bounds of the heat kernel on $\mathbb{R} \setminus [-L, L]$. The terms I_3 and I_4, together with the Lipschitz conditions on b and σ, will enable a recursive argument. If we set

$$f_L(t,x) := \mathbb{E}\left[|u(t,x) - u_L(t,x)|^p\right]^{1/p},$$

for $x \in (-L, L)$, then we shall reach a Gronwall-type inequality

$$f_L(t,x)^2 \leq cJ(t,x)^2 + k \int_0^t \int_{-L}^{L} \bar{H}(t-s, x-y) f_L(s,y)^2\, dyds,$$

valid for $t \in [0,T]$, where $J(t,x) = e^{-\frac{(L-x)^2}{4t}} + e^{-\frac{(L+x)^2}{4t}}$ has precisely the behavior we are trying to prove, and

$$\bar{H}(r,z) := \Gamma(r,z) + \Gamma(r,z)^2.$$

The final steps include iterations of the latter Gronwall-type inequality, which will be possible thanks to precise integral bounds on $\bar{H}$.

2.3 Some Prerequisites

2.3.1 Properties of Dirichlet Green function

This subsection contains bounds on the following difference

$$\begin{aligned} F(t,x,y) = F_L^D(t,x,y) &:= \Gamma(t,x-y) - G_L^D(t,x,y) \\ &= \sum_{k=-\infty}^{\infty} \Gamma(t,x+y+(4k+2)L) - \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \Gamma(t,x-y+4kL), \end{aligned}$$

where Γ is the heat kernel, given by (A.38), and G_L^D is the Green function associated to the Dirichlet boundary conditions on the symmetric interval $(-L,L)$, given by (A.48).

Lemma 2.7. *Fix any $t > 0$ and $x, y \in D = [-L,L]$. The difference F is non-negative and symmetric in x and y. Furthermore, it satisfies*

$$\begin{aligned} F(t,x,y) &\leq \Gamma(t,x+y+2L) + \Gamma(t,x+y-2L) \\ &\leq 4\Gamma\left(4t, |x-y| + \operatorname{dist}(x,\partial D) + \operatorname{dist}(y,\partial D)\right) \\ &\leq 4\exp\left(-\frac{\operatorname{dist}(x,\partial D)^2}{16t}\right)\exp\left(-\frac{\operatorname{dist}(y,\partial D)^2}{16t}\right)\Gamma(4t,x-y), \end{aligned} \tag{2.16}$$

where $\operatorname{dist}(x,\partial D) = \min(L+x, L-x)$. *In particular,*

$$\sup_{y\in[-L,L]} F(t,x,y) \leq \Gamma(t,x-L) + \Gamma(t,x+L).$$

Proof. Both facts, $F(t,x,y) \geq 0$ and $F(t,x,y) = F(t,y,x)$, come from Proposition A.8 and the fact that $\Gamma(t,x-y) = \Gamma(t,y-x)$. We can rewrite the function F as follows:

$$\begin{aligned} F(t,x,y) = &\Gamma(t,x+y+2L) + \Gamma(t,x+y-2L) \\ &- \sum_{k=1}^{\infty} \left[\Gamma(t,x-y+4kL) - \Gamma(t,x+y+(4k+2)L)\right] \\ &- \sum_{k=-\infty}^{-1} \left[\Gamma(t,x-y+4kL) - \Gamma(t,x+y+(4k-2)L)\right], \end{aligned}$$

with both sums being non-negative.

We now show that $2(x+y+2L) \geq |x-y| + \operatorname{dist}(x, \partial D) + \operatorname{dist}(y, \partial D)$, for all $x, y \in [-L, L]$. First, we suppose that $x \geq y$, then

$$2x+2y+4L = x-y+(L+x)+3(L+y) \geq |x-y| + \operatorname{dist}(x,\partial D) + 3\operatorname{dist}(y,\partial D),$$

since $L+x \geq \operatorname{dist}(x,\partial D)$, and $L+y \geq \operatorname{dist}(y,\partial D)$. Second, we suppose that $x \leq y$, then

$$2x+2y+4L = y-x+3(L+x)+(L+y) \geq |x-y| + 3\operatorname{dist}(x,\partial D) + \operatorname{dist}(y,\partial D).$$

We can also show that $2|x+y-2L| \geq |x-y| + \operatorname{dist}(x,\partial D) + \operatorname{dist}(y,\partial D)$. This is done by observing that

$$2|x+y-2L| = 2(-x-y+2L) \geq |x-y| + \operatorname{dist}(x,\partial D) + \operatorname{dist}(y,\partial D),$$

since $-x, -y \in [-L, L]$. This concludes the proof. □

We are now interested in finding integral bounds of the function F.

Lemma 2.8. *For all $t > 0$ and $x \in [-L, L]$, we have the following integral bounds,*

$$\begin{aligned} \int_{-L}^{L} F(t,x,y)\,dy &\leq \frac{1}{2}\left(e^{-\frac{(L-x)^2}{4t}} + e^{-\frac{(L+x)^2}{4t}}\right), \\ \int_0^t \int_{-L}^{L} F(s,x,y)\,dyds &\leq \frac{t}{2}\left(e^{-\frac{(L-x)^2}{4t}} + e^{-\frac{(L+x)^2}{4t}}\right). \end{aligned} \tag{2.17}$$

Proof. Recall that $F(t,x,y) \geq 0$. If we bound

$$F(t,x,y) \leq \sum_{k=-\infty}^{\infty} \Gamma(t, x+y+(4k+2)L) + \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \Gamma(t, x-y+4kL),$$

and integrate

$$\int_{-L}^{L} F(t,x,y)\,dy \leq \int_{-\infty}^{x-L} \Gamma(t,y)\,dy + \int_{L-x}^{\infty} \Gamma(t,y)\,dy.$$

Using equation (A.16), the first inequality of (2.17) is proved.

The second inequality (2.17) follows since $\exp(-z^2/4s) \leq \exp(-z^2/4t)$, when $0 < s \leq t$ and $z \in \mathbb{R}$. It remains to integrate, in time, the constant one function. □

Lemma 2.9. *For all $t > 0$ and $x \in [-L, L]$, we have the following integral bounds,*

$$\begin{aligned} \int_{-L}^{L} F(t,x,y)^2\,dy &\leq \frac{1}{2}\frac{1}{\sqrt{4\pi t}}\left(e^{-\frac{(L-x)^2}{4t}} + e^{-\frac{(L+x)^2}{4t}}\right)^2, \\ \int_0^t \int_{-L}^{L} F(s,x,y)^2\,dyds &\leq \frac{\sqrt{t}}{\sqrt{4\pi}}\left(e^{-\frac{(L-x)^2}{4t}} + e^{-\frac{(L+x)^2}{4t}}\right)^2. \end{aligned} \tag{2.18}$$

Proof. The first inequality (2.18) is a consequence of Lemma 2.8 and the observation

$$\int_{-L}^{L} F(t,x,y)^2\,dy \le \sup_{y\in[-L,L]} F(t,x,y) \int_{-L}^{L} F(t,x,y)\,dy,$$

with $\sup_{y\in[-L,L]} F(t,x,y) \le \Gamma(t,x-L)+\Gamma(t,x+L)$, by Lemma 2.7.

The second inequality (2.18) follows since $\exp(-z^2/4s) \le \exp(-z^2/4t)$, when $0 < s \le t$ and $z \in \mathbb{R}$. It remains to integrate, in time, the function $1/2\sqrt{s}$. □

2.3.2 Properties of Mixed Green function

This subsection contains bounds on the following difference

$$F(t,x,y) = F_L^M(t,x,y) := \Gamma(t,x-y) - G_L^M(t,x,y),$$

where Γ is the heat kernel, given by (A.38), and G_L^M is the Green function associated to the mixed boundary conditions on the symmetric interval $(-L,L)$, given by (A.77).

Lemma 2.10. *Fix any $t > 0$ and $x, y \in D = [-L,L]$. The difference F is symmetric in x and y. Furthermore, it satisfies*

$$\begin{aligned} &F(t,x,y) \ge -\Gamma(t,x+y-2L),\\ &F(t,x,y) \le \Gamma(t,x+y+2L) + \Gamma(t,x-y+4L) + \Gamma(t,x+y-6L). \end{aligned} \tag{2.19}$$

In particular,

$$\begin{aligned} |F(t,x,y)| &\le 12\Gamma\left(4t, |x-y| + \operatorname{dist}(x,\partial D) + \operatorname{dist}(y,\partial D)\right)\\ &\le 12\exp\left(-\frac{\operatorname{dist}(x,\partial D)^2}{16t}\right)\exp\left(-\frac{\operatorname{dist}(y,\partial D)^2}{16t}\right)\Gamma(4t,x-y), \end{aligned}$$

and

$$\sup_{y\in[-L,L]} |F(t,x,y)| \le 3\left(\Gamma(t,x-L) + \Gamma(t,x+L)\right).$$

Proof. The fact that $F(t,x,y) = F(t,y,x)$ comes from Proposition A.12 and the fact that $\Gamma(t,x-y) = \Gamma(t,y-x)$. We can rewrite G_L^M in two different ways. First, we observe that

$$
\begin{aligned}
G_L^M(t,x,y) &= \Gamma(t,x-y) + \Gamma(t,x+y-2L) \\
&\quad - \sum_{n=0}^{\infty} \left[\Gamma(t,x-y+(8n+4)L) - \Gamma(t,x-y+(8n+8)L)\right] \\
&\quad - \sum_{n=-\infty}^{-1} \left[\Gamma(t,x-y+(8n+4)L) - \Gamma(t,x-y+8nL)\right] \\
&\quad - \sum_{n=0}^{\infty} \left[\Gamma(t,x+y+(8n+2)L) - \Gamma(t,x+y+(8n+6)L)\right] \\
&\quad - \sum_{n=-\infty}^{-1} \left[\Gamma(t,x+y+(8n+2)L) - \Gamma(t,x+y+(8n-2)L)\right],
\end{aligned}
$$

where all four series are positive for $x, y \in [-L, L]$. Another way of writing is as follows:

$$
\begin{aligned}
G_L^M(t,x,y) &= \Gamma(t,x-y) + \Gamma(t,x+y-2L) - \Gamma(t,x-y-4L) \\
&\quad - \Gamma(t,x+y+2L) - \Gamma(t,x-y+4L) - \Gamma(t,x+y-6L) \\
&\quad + \sum_{n=1}^{\infty} \left[\Gamma(t,x-y+8nL) - \Gamma(t,x-y+(8n+4)L)\right] \\
&\quad + \sum_{n=-\infty}^{-1} \left[\Gamma(t,x-y+8nL) - \Gamma(t,x-y+(8n-4)L)\right] \\
&\quad + \sum_{n=1}^{\infty} \left[\Gamma(t,x+y+(8n-2)L) - \Gamma(t,x+y+(8n+2)L)\right] \\
&\quad + \sum_{n=-\infty}^{-2} \left[\Gamma(t,x+y+(8n+6)L) - \Gamma(t,x+y+(8n+2)L)\right],
\end{aligned}
$$

where $\Gamma(t,x+y-2L) - \Gamma(t,x-y-4L) \geq 0$ and the four series are again positive for $x, y \in [-L, L]$.

We can show, as in the proof of Lemma 2.7, that each of $2\,|x+y-2L|$, $2\,|x+y-2L|$, $2\,|x-y+4L|$, and $2\,|x+y-6L|$ is greater than $|x-y| + \mathrm{dist}(x,\partial D) + \mathrm{dist}(y,\partial D)$. □

We are now interested in finding integral bounds of the function F.

Lemma 2.11. *For all $t > 0$ and $x \in [-L, L]$, we have the following integral bounds,*

$$
\begin{aligned}
\int_{-L}^{L} |F(t,x,y)|\, dy &\leq \frac{1}{2}\left(e^{-\frac{(L-x)^2}{4t}} + e^{-\frac{(L+x)^2}{4t}}\right), \\
\int_0^t \int_{-L}^{L} |F(s,x,y)|\, dyds &\leq \frac{t}{2}\left(e^{-\frac{(L-x)^2}{4t}} + e^{-\frac{(L+x)^2}{4t}}\right).
\end{aligned}
\tag{2.20}
$$

Proof. We apply a similar argument as the proof of Lemma 2.8, in which we use the bounds of Lemma 2.10 instead. □

Lemma 2.12. *For all $t > 0$ and $x \in [-L, L]$, we have the following integral bounds,*

$$\begin{aligned} \int_{-L}^{L} F(t,x,y)^2\, dy &\le \frac{3}{2}\frac{1}{\sqrt{4\pi t}}\left(e^{-\frac{(L-x)^2}{4t}} + e^{-\frac{(L+x)^2}{4t}}\right)^2, \\ \int_0^t \int_{-L}^{L} F(s,x,y)^2\, dyds &\le \frac{3\sqrt{t}}{\sqrt{4\pi}}\left(e^{-\frac{(L-x)^2}{4t}} + e^{-\frac{(L+x)^2}{4t}}\right)^2. \end{aligned} \tag{2.21}$$

Proof. We apply a similar argument as the proof of Lemma 2.9, in which we use the bounds of Lemma 2.10 instead. □

2.3.3 Properties of Neumann Green function

This subsection contains bounds on the following difference

$$\begin{aligned} F(t,x,y) = F_L^N(t,x,y) &:= G_L^N(t,x,y) - \Gamma(t, x-y) \\ &= \sum_{k=-\infty}^{\infty} \Gamma(t, x+y+(4k+2)L) + \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \Gamma(t, x-y+4kL), \end{aligned} \tag{2.22}$$

where Γ is the heat kernel, given by (A.38), and G_L^N is the Green function associated to the Neumann boundary conditions on the symmetric interval $(-L, L)$, given by (A.62).

For the following results, we need to introduce the Theta function, defined as

$$\theta(a) := \sum_{k=0}^{\infty} e^{-ak^2}, \tag{2.23}$$

for any real number $a > 0$. In Appendix B.2, inequalities (B.7) and (B.8) give some estimations of the Theta function. In particular, $\theta(a) \approx a^{-1/2}$ as $a \to 0$.

Lemma 2.13. *Fix any $t > 0$ and $x, y \in D = [-L, L]$. The difference F is non-negative and symmetric in x and y. Furthermore, it satisfies*

$$\sup_{y\in[-L,L]} F_L(t,x,y) \le \theta(L^2/t)\left[\Gamma(t, x-L) + \Gamma(t, x+L)\right],$$

and

$$\begin{aligned} F_L(t,x,y) \le 16 \exp\left(-\frac{\operatorname{dist}(x,\partial D)^2}{16t}\right) \exp\left(-\frac{\operatorname{dist}(y,\partial D)^2}{16t}\right) \\ \times\, \theta(4L^2/t)\, \Gamma(4t, x-y). \end{aligned}$$

Proof. The fact that $F(t,x,y) \geq 0$ comes from Equation (2.22). The fact that $F(t,x,y) = F(t,y,x)$ comes from Proposition A.10 and the fact that $\Gamma(t,x-y) = \Gamma(t,y-x)$.

We can bound, for $n \geq 0$ and $x, y \in [-L, L]$,

$$x - y + 4(n+1)L \geq (x+L) + (L-y) + (4n+2)L \geq (x+L) + (4n+2)L,$$
$$x + y + (4n+2)L \geq (x+L) + (L+y) + 4nL \geq (x+L) + 4nL,$$

and both are non-negative. Using $|(x+L) + 2kL|^2 \geq (x+L)^2 + 4k^2L^2$, we get

$$\sum_{n=0}^{\infty} \Gamma(t, x+y+(4n+2)L) + \sum_{n=1}^{\infty} \Gamma(t, x-y+4nL)$$
$$\leq \frac{1}{\sqrt{4\pi t}} e^{-\frac{(L+x)^2}{4t}} \sum_{k=0}^{\infty} e^{-\frac{k^2L^2}{t}} = \theta(L^2/t)\, \Gamma(t, x+L),$$

uniformly in $y \in [-L, L]$. We proceed in a similar way to get

$$\sum_{n=-\infty}^{-1} \Gamma(t, x+y+(4n+2)L) + \sum_{n=-\infty}^{-1} \Gamma(t, x-y+4nL)$$
$$\leq \frac{1}{\sqrt{4\pi t}} e^{-\frac{(L-x)^2}{4t}} \sum_{k=0}^{\infty} e^{-\frac{k^2L^2}{t}} = \theta(L^2/t)\, \Gamma(t, x-L).$$

Indeed, for $n \leq -1$ and $x, y \in [-L, L]$, we have

$$x + y + (4n+2)L = (x-L) + (y-L) + 4(n+1)L \leq (x-L) + 4(n+1)L,$$
$$x - y + 4nL = (x-L) + (-y-L) + (4n+2)L \leq (x-L) + (4n+2)L,$$

and both are non-positive. Then, we use the observation that for any $k \geq 0$, $|(x-L) - 2kL|^2 \geq (L-x)^2 + 4k^2L^2$.

For $k \geq 0$, we can show, as in the proof of Lemma 2.7, that

$$2\,|x+y+(4k+2)L| \geq 2\,|x+y+2L| + 8kL$$
$$\geq |x-y| + \operatorname{dist}(x, \partial D) + \operatorname{dist}(y, \partial D) + 8kL,$$

and

$$2\,|x-y+4(k+1)L| \geq 2\,|x-y+4L| + 8kL$$
$$\geq |x-y| + \operatorname{dist}(x, \partial D) + \operatorname{dist}(y, \partial D) + 8kL.$$

Therefore,

$$\sum_{k=0}^{\infty} \Gamma(t, x+y+(4k+2)L) + \sum_{k=1}^{\infty} \Gamma(t, x-y+4kL)$$
$$\leq 4\Gamma(4t, x-y) \exp\left(-\frac{\operatorname{dist}(x,\partial D)^2}{16t}\right) \exp\left(-\frac{\operatorname{dist}(y,\partial D)^2}{16t}\right) 2\sum_{k=0}^{\infty} e^{-\frac{4k^2L^2}{t}}.$$

As similar computation holds for the remaining terms. □

We are now interested in finding integral bounds of the function F.

Lemma 2.14. *For all $t > 0$ and $x \in [-L, L]$, we have the following integral bounds,*

$$\int_{-L}^{L} F(t,x,y)\,dy \le \frac{1}{2}\left(e^{-\frac{(L-x)^2}{4t}} + e^{-\frac{(L+x)^2}{4t}}\right),$$
$$\int_0^t \int_{-L}^{L} F(s,x,y)\,dyds \le \frac{t}{2}\left(e^{-\frac{(L-x)^2}{4t}} + e^{-\frac{(L+x)^2}{4t}}\right). \tag{2.24}$$

Proof. We apply a similar argument as the proof of Lemma 2.8, in which we use the bounds of Lemma 2.13 instead. □

Lemma 2.15. *For all $t > 0$ and $x \in [-L, L]$, we have the following integral bounds,*

$$\int_{-L}^{L} F(t,x,y)^2\,dy \le \frac{\gamma}{2}\frac{1}{\sqrt{4\pi t}}\left(e^{-\frac{(L-x)^2}{4t}} + e^{-\frac{(L+x)^2}{4t}}\right)^2,$$
$$\int_0^t \int_{-L}^{L} F(s,x,y)^2\,dyds \le \frac{\gamma\sqrt{t}}{\sqrt{4\pi}}\left(e^{-\frac{(L-x)^2}{4t}} + e^{-\frac{(L+x)^2}{4t}}\right)^2, \tag{2.25}$$

where $\gamma = \theta(L^2/t)$.

Proof. We apply a similar argument as the proof of Lemma 2.9, in which we use the bounds of Lemma 2.13 instead. □

2.4 Proof of uniform bounds, Proposition 2.6

We will be able to prove all four bounds in very similar ways. In fact, we introduce some notations that will allow a single argument instead of four. The letter D will stand for some domain, either $D = \mathbb{R}$, or $D = (-L, L)$ with any boundary conditions (2.3), (2.4), or (2.5). The solution u_D will be either u, or u_L with the corresponding boundary conditions. Finally, $G_D(t,x,y)$ will stand for either the heat kernel $\Gamma(t, x-y)$, or the Green function $G_L(t,x,y)$ associated with the corresponding boundary conditions. We gather two important properties.

Lemma 2.16. *For any $t \in [0,T]$ and $x \in D$, we have*

$$\int_0^t \int_D G_D(s,x,y)\,dyds \le t,$$
$$\int_0^t \int_D G_D(s,x,y)^2\,dyds \le c\sqrt{t},$$

where the constant c denotes either $(2\pi)^{-1/2}$, $(2/\pi)^{1/2}$, or $c(l)$ when $G_D = \Gamma$, $G_D = G_L^M$, or $G_D = G_L^N$, respectively.

It is very surprising that none of the two inequalities depends on the quantity L, which corresponds to the (half) length of the rod.

Proof. The first property is a direct consequence of equations (A.10), (A.52), (A.66), and (A.82). Indeed,

$$\int_0^t \int_D G_D(s,x,y)\,dyds \leq \int_0^t 1\,ds = t.$$

The second is a consequence of the semi-group properties (A.6), (A.51), (A.65), and (A.80), and the uniform bounds (A.50), (A.64), and (A.79). Indeed,

$$\int_0^t \int_D G_D(s,x,y)^2\,dyds = \int_0^t G_D(2s,x,x)\,ds \leq \int_0^t \frac{c}{2\sqrt{s}}\,ds = c\sqrt{t}.$$

The case of Neumann boundary conditions need the quantity $L^2/T \geq l^2$. In that case $1/L \leq l/\sqrt{t}$ for all $t \in [0,T]$, and

$$G_L^N(2s,x,x) \leq \frac{1}{2L} + \frac{1}{\sqrt{2\pi t}} \leq \frac{1}{2\sqrt{t}}\left(\frac{1}{l} + \frac{\sqrt{2}}{\sqrt{\pi}}\right) = \frac{c(l)}{2\sqrt{t}},$$

where $c(l) = l^{-1} + (2/\pi)^{1/2}$. □

In order to deduce properties of the solution $u_D(t,x)$ it is very often necessary to go back to the Picard iteration scheme, from which the solution was constructed. We recall it now. The initial condition initiates the recursive definition:

$$u_D^0(t,x) := \int_D G_D(t,x,y)u_0(y)\,dy.$$

For $n \geq 0$, we define recursively

$$\begin{aligned} u_D^{n+1}(t,x) = u_D^0(t,x) &+ \int_0^t \int_D G_D(t,x,y)\,b(s,y,u_D^n(s,y))\,dyds \\ &+ \int_0^t \int_D G_D(t-s,x,y)\,\sigma(s,y,u_D^n(s,y))\,W(dyds). \end{aligned}$$

Recalling the fact that the initial condition is bounded in expectation (2.7), we get, from either (A.52), (A.66), or (A.82), that

$$\left\|u_D^0(t,x)\right\|_{L^p(\Omega)} \leq \int_D G_D(t,x,y)\ \|u_0(y)\|_{L^p(\Omega)}\,dy \leq \|u_0\|\,.$$

Therefore, we have a uniform bound for the initiation of the recursion

$$C_0 := \sup_{t\in[0,T]} \sup_{x\in D} \left\|u_D^0(t,x)\right\|_{L^p(\Omega)} \leq \|u_0\|\,.$$

We define in a similar way

$$C_n := \sup_{t\in[0,T]} \sup_{x\in D} \|u_D^n(t,x)\|_{L^p(\Omega)}.$$

We can prove by induction that each C_n is bounded. Indeed, from Minkowski's and Burkholder's inequalities, given by (B.1) and (B.3), together with linear growth of the functions b and σ, assumed in (2.6), we get

$$\begin{aligned}\left\|u_D^{n+1}(t,x)\right\|_{L^p(\Omega)} &\leq \left\|u_D^0(t,x)\right\|_{L^p(\Omega)} \\ &+ \int_0^t \int_D G_D(t-s,x,y) K \left(1 + \|u_D^n(s,y)\|_{L^p(\Omega)}\right) dyds \\ &+ k_p \left(\int_0^t \int_D G_D(t-s,x,y)^2 K^2 \left(1 + \|u_D^n(s,y)\|_{L^p(\Omega)}\right)^2 dyds\right)^{1/2}.\end{aligned}$$

Using the induction hypothesis and Lemma 2.16

$$\begin{aligned}\left\|u_D^{n+1}(t,x)\right\|_{L^p(\Omega)} &\leq C_0 + K(1+C_n) \int_0^t \int_D G_D(s,x,y)\, dyds \\ &+ k_p K(1+C_n) \left(\int_0^t \int_D G_D(s,x,y)^2\, dyds\right)^{1/2} \\ &\leq C_0 + K(1+C_n)t + k_p K(1+C_n) \left(c\sqrt{t}\right)^{1/2}.\end{aligned}$$

Therefore,

$$C_{n+1} \leq C_0 + K(1+C_n)T + k_p K(1+C^n)\sqrt{c}T^{1/4} < \infty.$$

We can now show a much better bound using the Lipschitz assumptions instead of the linear growth. For further use, let us write

$$D_n(t) := \sup_{x\in D} \left\|u_D^{n+1}(t,x) - u_D^n(t,x)\right\|_{L^p(\Omega)},$$

and notice that

$$\sup_{t\in[0,T]} D_0(t) \leq K(1+C_0)T + k_p K(1+C_0)\sqrt{c}T^{1/4} < \infty. \tag{2.26}$$

From Minkowski's and Burkholder's inequalities, the Lipschitz condition of the functions b and σ, and Lemma 2.16, we get

$$\begin{aligned}
&\left\|u_D^{n+1}(t,x) - u_D^n(t,x)\right\|_{L^p(\Omega)} \\
&\leq \mathrm{Lip} \int_0^t \int_D G_D(t-s,x,y) \left\|u_D^n(s,y) - u_D^{n-1}(s,y)\right\|_{L^p(\Omega)} dyds \\
&+ k_p \mathrm{Lip} \left(\int_0^t \int_D G_D(t-s,x,y)^2 \left\|u_D^n(s,y) - u_D^{n-1}(s,y)\right\|_{L^p(\Omega)}^2 dyds \right)^{1/2} \\
&\leq \mathrm{Lip} \int_0^t D_{n-1}(s) \int_D G_D(t-s,x,y)\, dyds \\
&+ k_p \mathrm{Lip} \left(\int_0^t |D_{n-1}(s)|^2 \int_D G_D(t-s,x,y)^2\, dyds \right)^{1/2} \\
&\leq \mathrm{Lip} \int_0^t D_{n-1}(s)\, ds + k_p \mathrm{Lip} \left(\int_0^t |D_{n-1}(s)|^2 \frac{c}{2(t-s)^{1/2}}\, ds \right)^{1/2},
\end{aligned}$$

Therefore,

$$D_n(t) \leq \mathrm{Lip} \int_0^t D_{n-1}(s)\, ds + \sqrt{c/2}\, k_p \mathrm{Lip} \left(\int_0^t |D_{n-1}(s)|^2 (t-s)^{-1/2}\, ds \right)^{1/2}.$$

Taking squares on both sides and using the inequality $(a+b)^2 \leq 2(a^2+b^2)$, as well as Hölder's inequality, we obtain

$$\begin{aligned}
|D_n(t)|^2 &\leq 2\mathrm{Lip}^2 \left(\int_0^t D_{n-1}(s)\, ds \right)^2 + c k_p^2 \mathrm{Lip}^2 \int_0^t |D_{n-1}(s)|^2 (t-s)^{-1/2}\, ds \\
&\leq 2\mathrm{Lip}^2 t \int_0^t |D_{n-1}(s)|^2\, ds + c k_p^2 \mathrm{Lip}^2 \int_0^t |D_{n-1}(s)|^2 (t-s)^{-1/2}\, ds.
\end{aligned}$$

Thus

$$|D_n(t)|^2 \leq k \int_0^t |D_{n-1}(s)|^2 g(t-s), \tag{2.27}$$

where $k = \max(2\mathrm{Lip}^2 T, c k_p^2 \mathrm{Lip}^2)$ and $g(r) = 1 + r^{-1/2}$. Again, we emphasize the fact that we have no dependence on L for this Gronwall type inequality (2.27).

The extension of Gronwall's lemma, presented in the paper of Dalang [11, Lemmas 15 and 17], enables to conclude, thanks to (2.26) and the facts that $\int_0^T g(r)\, dr < \infty$, that the following series converges uniformly on $[0,T]$,

$$\sum_{n=0}^{\infty} D_n(t) \leq K_g \sup_{t\in[0,T]} D_0(t) < \infty,$$

where K_g is a constant only depending on k and g. Thus,

$$\begin{aligned}\|u_D(t,x)\|_{L^p(\Omega)} &= \lim_{n\to\infty} \|u_D^n(t,x)\|_{L^p(\Omega)} \\ &\leq \left\|u_D^0(t,x)\right\|_{L^p(\Omega)} + \sum_{n=0}^{\infty} D_n(t) \leq C_0 + K_g \sup_{t\in[0,T]} D_0(t).\end{aligned}$$

Therefore, we can conclude

$$\begin{aligned}&\sup_{t\in[0,T]} \sup_{x\in D} \|u_D(t,x)\|_{L^p(\Omega)} \\ &\qquad \leq \|u_0\| + K_g \left(K(1+\|u_0\|)T + k_p K(1+\|u_0\|)\sqrt{c}T^{1/4}\right) < \infty,\end{aligned}$$

and that the latter bound doesn't depend of L. This complete the proof of Proposition 2.6.

Addendum. If we wet $f_n(t) = |D_n(t)|^2$, then we would have

$$f_n(t) \leq k \int_0^t f_{n-1}(s) g(t-s) = k(g * f_{n-1})(t).$$

We can iterate this convolution pattern directly. Moreover, for $t \in [0,T]$, we could bound

$$g(t) \leq 1 + \frac{1}{t^{1/2}} \leq (T^{1/2}+1)t^{-1/2},$$

and iterate $f_n(t) \leq k(T^{1/2}+1)(\tilde{g}*f_{n-1})(t)$ for $\tilde{g}(t) = 1/\sqrt{t}$, which can be done using Beta integrals. An exact computation is given in [6, Proposition 2.2].

2.5 Proof of convergence rate, Theorem 2.4

We will be able to prove both convergence rates in very similar ways. In fact, we introduce some notations that will allow a single argument instead of two. We will set the difference $F(t,x,y) = \Gamma(t,x-y) - G_L(t,x,y)$, where G_L will stand for either one of the three Green functions G_D^D, G_L^M, or G_L^N, associated with Dirichlet, mixed, or Neumann boundary conditions, respectively. We summarize some properties of that difference. We introduce the quantity

$$J(t,x) = e^{-\frac{(L-x)^2}{4t}} + e^{-\frac{(L+x)^2}{4t}},$$

and precise that in the case of Neumann boundary conditions, we need to introduce the quantity

$$L^2/T \geq l^2 > 0.$$

Lemma 2.17. *For all $t \in [0,T]$ and $x \in [-L,L]$, we have*

$$\int_0^t \int_{-L}^{L} |F(s,x,y)|\, dyds \le c_1 t J(t,x),$$
$$\int_0^t \int_{-L}^{L} F(s,x,y)^2\, dyds \le \gamma \sqrt{t} J(t,x)^2,$$

where $c_1 = 1/2$ and $\gamma = 1/\sqrt{4\pi}$, $\gamma = 3/\sqrt{4\pi}$, or $\gamma = \theta(l^2)/\sqrt{4\pi}$, in the case of Dirichlet, mixed, or Neumann boundary conditions, respectively.

Proof. These are precisely Lemmas 2.8, 2.9, 2.11, 2.12, 2.14, 2.15. □

We need similar bounds concerning the heat kernel.

Lemma 2.18. *For all $t > 0$ and $x \in [-L,L]$, we have*

$$\int_0^t \int_{\mathbb{R}\setminus[-L,L]} \Gamma(t, x-y)\, dy \le c_1 t J(t,x),$$
$$\int_0^t \int_{\mathbb{R}\setminus[-L,L]} \Gamma(t, x-y)^2\, dy \le c_2 \sqrt{t} J(t/2,x) \le c_2 \sqrt{t} J(t,x)^2,$$

where $c_1 = 1/2$ and $c_2 = 1/\sqrt{8\pi}$.

Proof. These are direct consequences of inequalites (A.16) and (A.17). □

We make use of the uniform bounds found in Proposition 2.6. We set

$$C := \sup_{t\in[0,T]} \sup_{x\in\mathbb{R}} \|u(t,x)\|_{L^p(\Omega)},$$
$$C_D := \sup_{t\in[0,T]} \sup_{L>0} \sup_{x\in[-L,L]} \|u_L(t,x)\|_{L^p(\Omega)},$$

in both cases of Dirichlet or mixed boundary conditions. In the case of Neumann boundary conditions, the second supremum is over $L \ge l\sqrt{T}$.

Recalling the representation formulas (2.8) and (2.9) satisfied by the solutions, we can write

$$u(t,x) - u_L(t,x) = \sum_{i=0}^{6} I_i(t,x),$$

where,

$$
\begin{aligned}
I_0(t,x) &= \int_{\mathbb{R}} \Gamma(t,x-y)\,u_0(y)\,dy - \int_{-L}^{L} G_L(t,x,y)\,u_0(y)\,dy,\\
I_1(t,x) &= \int_0^t \int_{-L}^{L} [\Gamma(t-s,x-y) - G_L(t-s,x,y)] b(s,y,u_L(s,y))\,dyds,\\
I_2(t,x) &= \int_0^t \int_{-L}^{L} [\Gamma(t-s,x-y) - G_L(t-s,x,y)] \sigma(s,y,u_L(s,y))\,W(dyds),\\
I_3(t,x) &= \int_0^t \int_{-L}^{L} \Gamma(t-s,x-y)[b(s,y,u(s,y)) - b(s,y,u_L(s,y))]\,dyds,\\
I_4(t,x) &= \int_0^t \int_{-L}^{L} \Gamma(t-s,x-y)[\sigma(s,y,u(s,y)) - \sigma(s,y,u_L(s,y))]\,W(dyds),\\
I_5(t,x) &= \int_0^t \int_{\mathbb{R}\setminus[-L,L]} \Gamma(t-s,x-y) b(s,y,u(s,y))\,dyds,\\
I_6(t,x) &= \int_0^t \int_{\mathbb{R}\setminus[-L,L]} \Gamma(t-s,x-y) \sigma(s,y,u(s,y))\,W(dyds).
\end{aligned}
$$

We now evaluate each I_i. Rewriting the first one, we get

$$
I_0(t,x) = \int_{-L}^{L} [\Gamma(t,x-y) - G_L(t,x,y)] u_0(y)\,dy + \int_{\mathbb{R}\setminus[-L,L]} \Gamma(t,x-y) u_0(y)\,dy.
$$

Thus, using Minkowski's inequality (B.1) and Lemmas 2.17,

$$
\begin{aligned}
\|I_0(t,x)\|_{L^p(\Omega)} &\le \int_{-L}^{L} |F(t,x,y)|\, \|u_0(y)\|_{L^p(\Omega)}\,dy\\
&\quad + \int_{\mathbb{R}\setminus[-L,L]} \Gamma(t,x-y)\, \|u_0(y)\|_{L^p(\Omega)}\,dy\\
&\le 2c_1 \|u_0\|\, J(t,x).
\end{aligned}
\tag{2.28}
$$

Using Minkowski's inequality, linear growth of the function b, assumed in (2.6), and the uniform bounds of Proposition 2.6, we get

$$
\begin{aligned}
\|I_1(t,x)\|_{L^p(\Omega)} &\le \int_0^t \int_{-L}^{L} |F(t-s,x,y)|\, K \left(1 + \|u_L(s,y)\|_{L^p(\Omega)}\right) dyds\\
&\le K(1+C_D) \int_0^t \int_{-L}^{L} F(s,x,y)\,dyds\\
&\le K(1+C_D) c_1 t J(t,x).
\end{aligned}
$$

Using Burkholder's inequality (B.3) and linear growth of the function σ, we get

$$\begin{aligned}\|I_2(t,x)\|^2_{L^p(\Omega)} &\le k_p^2 \int_0^t \int_{-L}^{L} F(t-s,x,y)^2 \|\sigma(s,y,u_L(s,y))\|^2_{L^p(\Omega)}\, dyds \\ &\le k_p^2 K^2 (1+C_D)^2 \int_0^t \int_{-L}^{L} F(s,x,y)^2\, dyds \\ &\le k_p^2 K^2 (1+C_D)^2 \gamma \sqrt{t} J(t,x)^2.\end{aligned}$$

Using the Lipschitz condition of the functions b and σ, we can evaluate

$$\|I_3(t,x)\|_{L^p(\Omega)} \le \mathrm{Lip} \int_0^t \int_{-L}^{L} \Gamma(t-s,x-y) \|u(s,y)-u_L(s,y)\|_{L^p(\Omega)}\, dyds,$$

and

$$\begin{aligned}&\|I_4(t,x)\|^2_{L^p(\Omega)} \\ &\qquad \le k_p^2 \mathrm{Lip}^2 \int_0^t \int_{-L}^{L} \Gamma(t-s,x-y)^2 \|u(s,y)-u_L(s,y)\|^2_{L^p(\Omega)}\, dyds.\end{aligned}$$

We evaluate the remaining integrals as in the case of I_1 and I_2.

$$\begin{aligned}\|I_5(t,x)\|_{L^p(\Omega)} &\le K(1+C) \int_0^t \int_{\mathbb{R}\setminus[-L,L]} \Gamma(t-s,x-y)\, dyds \\ &\le K(1+C) c_1 t J(t,x),\end{aligned}$$

and

$$\begin{aligned}\|I_6(t,x)\|^2_{L^p(\Omega)} &\le k_p^2 K^2 (1+C)^2 \int_0^t \int_{\mathbb{R}\setminus[-L,L]} \Gamma(t-s,x-y)^2\, dyds \\ &\le k_p^2 K^2 (1+C)^2 c_2 \sqrt{t} J(t,x)^2.\end{aligned}$$

Putting everything together and setting

$$f_L(t,x) = \|u(t,x)-u_L(t,x)\|_{L^p(\Omega)},$$

we have

$$\begin{aligned}f_L(t,x) \le cJ(t,x) &+ \mathrm{Lip} \int_0^t \int_{-L}^{L} \Gamma(t-s,x-y) f_L(s,y)\, dyds \\ &+ k_p \mathrm{Lip} \left(\int_0^t \int_{-L}^{L} \Gamma(t-s,x-y)^2 f_L(s,y)^2\, dyds \right)^{1/2},\end{aligned}$$

where

$$\begin{aligned}c = 2c_1 \|u_0\| &+ c_1 K(2+C_D+C)T \\ &+ k_p K(1+C_D)\sqrt{\gamma} T^{1/4} + k_p K(1+C)\sqrt{c_2} T^{1/4}.\end{aligned}$$

Squaring both sides and using $(a+b+c)^2 \leq 3(a^2+b^2+c^2)$ lead to

$$f_L(t,x)^2 \leq 3c^2 J(t,x)^2 + 3\mathrm{Lip}^2 \left(\int_0^t \int_{-L}^{L} \Gamma(t-s,x-y) f_L(s,y)\, dyds \right)^2 + 3k_p^2 \mathrm{Lip}^2 \int_0^t \int_{-L}^{L} \Gamma(t-s,x-y)^2 f_L(s,y)^2\, dyds.$$

We can bound the middle term using Cauchy-Schwarz inequality,

$$\begin{aligned} &\left(\int_0^t \int_{-L}^{L} \Gamma(t-s,x-y) f_L(s,y)\, dyds \right)^2 \\ &\quad \leq \int_0^t \int_{-L}^{L} \Gamma(t-s,x-y)\, dyds \cdot \int_0^t \int_{-L}^{L} \Gamma(t-s,x-y) f_L(s,y)^2\, dyds \\ &\quad \leq t \int_0^t \int_{-L}^{L} \Gamma(t-s,x-y) f_L(s,y)^2\, dyds. \end{aligned}$$

Therefore

$$f_L(t,x)^2 \leq \tilde{J}(t,x) + k \int_0^t \int_{-L}^{L} \bar{H}(t-s,x-y) f_L(s,y)^2\, dyds, \tag{2.29}$$

where $\tilde{J}(t,x) = 3c^2 J(t,x)^2$, $k = \max(3\mathrm{Lip}^2 T, 3k_p^2 \mathrm{Lip}^2)$, and $\bar{H}(r,z) := \Gamma(r,z) + \Gamma(r,z)^2$.

If we iterate inequality (2.29), we get

$$\begin{aligned} f_L(t,x)^2 &\leq \tilde{J}(t,x) + k \int_0^t \int_{-L}^{L} \bar{H}(t-s,x-y) \tilde{J}(s,y)\, dyds \\ &+ k^2 \int_0^t \int_{-L}^{L} \bar{H}(t-s,x-y) \int_0^s \int_{-L}^{L} \bar{H}(s-r,y-z) f_L(r,z)^2\, dzdr\, dyds. \end{aligned}$$

Using Fubini's theorem, the last multiple integral can be re-written as

$$\int_0^t \int_{-L}^{L} f_L(r,z)^2 \int_r^t \int_{-L}^{L} \bar{H}(t-s,x-y) \bar{H}(s-r,y-z)\, dyds\, dzdr.$$

Anticipating the iteration behaviour, we want to prove

Lemma 2.19. *For any* $\alpha \geq 0$, $x, z \in \mathbb{R}$ *and* $0 \leq r < t \leq T$,

$$\begin{aligned} &\int_r^t (t-s)^\alpha \int_{\mathbb{R}} \tilde{H}(t-s,x-y) \tilde{H}(s-r,y-z)\, dyds \\ &\qquad \leq d \frac{\Gamma(\alpha+1/2)}{\Gamma(\alpha+1)} (t-r)^{\alpha+1/2} \tilde{H}(t-r,x-z), \end{aligned} \tag{2.30}$$

where Γ *is the Gamma function,* $d = \sqrt{T} + 1$, *and*

$$\begin{aligned}\bar{H}(r,z) &= \Gamma(r,z) + \Gamma(r,z)^2 = \Gamma(r,z) + \frac{1}{\sqrt{8\pi r}}\Gamma(r/2,z)\\ &\leq \Gamma(r,z)\left(1 + \frac{1}{\sqrt{4\pi r}}\right) =: \tilde{H}(r,z).\end{aligned}$$

For a review on the Gamma function, see Appendix B.2. There, it is recalled the log-convexity of the Gamma function and the relationship between the Beta and Gamma functions. Both of these facts will be used.

Addendum. A much easier iteration scheme would be to consider

$$\begin{aligned}\bar{H}(r,z) &= \Gamma(r,z) + \Gamma(r,z)^2 = \Gamma(r,z) + \frac{1}{\sqrt{8\pi r}}\Gamma(r/2,z)\\ &\leq \Gamma(r,z)\left(1 + \frac{1}{\sqrt{4\pi r}}\right) \leq c\frac{\Gamma(r,z)}{\sqrt{t}} =: \tilde{H}(r,z),\end{aligned}$$

where $c = \frac{\sqrt{4\pi T}+1}{\sqrt{4\pi}}$. In fact, an exact formula can be found for each iteration of such $\tilde{H}$, see [6, Proposition 2.2].

Proof. Using the semi-group property of Γ,

$$\begin{aligned}&\int_{\mathbb{R}} \tilde{H}(t-s, x-y)\tilde{H}(s-r, y-z)\,dy\\ &\qquad = \left(1 + \frac{1}{\sqrt{4\pi(t-s)}}\right)\left(1 + \frac{1}{\sqrt{4\pi(s-r)}}\right)\Gamma(t-r, x-z).\end{aligned}$$

We can now evaluate four integrals as follows:

$$\begin{aligned}\int_r^t (t-s)^\alpha\,ds &= \frac{1}{\alpha+1}(t-r)^{\alpha+1} \leq (t-r)^{\alpha+1/2}\sqrt{T}\frac{\Gamma(\alpha+1)}{\Gamma(\alpha+2)}\\ &\leq (t-r)^{\alpha+1/2}\sqrt{T}\frac{\Gamma(\alpha+1)}{\Gamma(\alpha+3/2)} \leq (t-r)^{\alpha+1/2}\sqrt{T}\frac{\Gamma(\alpha+1/2)}{\Gamma(\alpha+1)}.\end{aligned}$$

We used the facts that for all $\alpha \geq 0$, we have $\Gamma(\alpha+3/2) \leq \Gamma(\alpha+2)$, and that $x \mapsto \Gamma(x+1/2)/\Gamma(x)$ is an increasing function on $\mathbb{R}_+^*$ (see 4. and 5. in Appendix B.2.1). Then we observe that

$$\begin{aligned}\frac{1}{\sqrt{4\pi}}\int_r^t (t-s)^{\alpha-1/2}\,ds &= \frac{(t-r)^{\alpha+1/2}}{2\sqrt{\pi}(\alpha+1/2)} = \frac{(t-r)^{\alpha+1/2}}{2\sqrt{\pi}}\frac{\Gamma(\alpha+1/2)}{\Gamma(\alpha+3/2)}\\ &\leq \frac{(t-r)^{\alpha+1/2}}{\pi}\frac{\Gamma(\alpha+1/2)}{\Gamma(\alpha+1)},\end{aligned}$$

since $\Gamma(\alpha+1) \leq (2/\sqrt{\pi})\Gamma(\alpha+3/2)$, for all $\alpha \geq 0$, by 4. in Appendix B.2.1. Then, using Beta integrals (Lemma B.1), and the increasing function $x \mapsto \Gamma(x+1/2)/\Gamma(x)$, we have

$$\begin{aligned}\frac{1}{\sqrt{4\pi}}\int_r^t (t-s)^\alpha (s-r)^{-1/2}\,ds &= \frac{(t-r)^{\alpha+1/2}}{2\sqrt{\pi}}\frac{\Gamma(\alpha+1)\Gamma(1/2)}{\Gamma(\alpha+3/2)} \\ &\leq \frac{(t-r)^{\alpha+1/2}}{2}\frac{\Gamma(\alpha+1/2)}{\Gamma(\alpha+1)}.\end{aligned}$$

Finally, using Beta integrals,

$$\begin{aligned}\frac{1}{4\pi}\int_r^t (t-s)^{\alpha-1/2}(s-r)^{-1/2}\,ds &= \frac{(t-r)^\alpha}{4\pi}\frac{\Gamma(\alpha+1/2)\Gamma(1/2)}{\Gamma(\alpha+1)} \\ &= \frac{(t-r)^{\alpha+1/2}}{2}\frac{1}{\sqrt{4\pi(t-r)}}\frac{\Gamma(\alpha+1/2)}{\Gamma(\alpha+1)},\end{aligned}$$

which ends the proof. □

If we keep iterating inequality (2.29), we will deduce the following:

Proposition 2.20. *For all $t \in [0,T]$ and $x \in [-L,L]$,*

$$f_L(t,x)^2 \leq \tilde{J}(t,x) + k\int_0^t \phi_T(t-s)\int_{-L}^L \tilde{H}(t-s,x-y)\tilde{J}(s,y)\,dyds, \quad (2.31)$$

where $\phi_T(r) := \sqrt{\pi}E_{1/2,1/2}\left(kd\sqrt{r}\right)$ is defined via the Mittag-Leffler functions

$$E_{\alpha,\beta}(z) := \sum_{k=0}^\infty \frac{z^k}{\Gamma(\alpha k+\beta)}, \quad \alpha,\beta > 0.$$

Even though we restrict to real arguments $z \in \mathbb{R}$, the Mittag-Leffler functions are entire functions in the complex plane. In the special case where $\alpha = 1$ and $\beta = 1$, it reduces to the exponential function. It turns out that $E_{1/2,1/2}$, hence ϕ_T, can be bounded explicitly by the exponential function as in equation (B.6) of the Appendix B.2.

Proof. Iterating equation (2.29), using Fubini's theorem and Lemma 2.19 with $\alpha = 0$,

$$\begin{aligned}f_L(t,x)^2 \leq \tilde{J}(t,x) &+ k\int_0^t\int_{-L}^L \tilde{H}(t-s,x-y)\tilde{J}(s,y)\,dyds \\ &+ k(kd)\frac{\Gamma(1/2)}{\Gamma(1)}\int_0^t (t-s)^{1/2}\int_{-L}^L \tilde{H}(t-s,x-y)f_L(s,y)^2\,dyds.\end{aligned}$$

Again, using equation (2.29), Fubini's theorem and Lemma 2.19 with $\alpha = 1/2$,

$$
\begin{aligned}
f_L(t,x)^2 &\leq \tilde{J}(t,x) \\
&+ k \int_0^t \left(1 + \frac{\Gamma(1/2)}{\Gamma(1)} kd(t-s)^{1/2}\right) \int_{-L}^{L} \tilde{H}(t-s,x-y)\tilde{J}(s,y)\, dyds \\
&\qquad + k(kd)^2 \frac{\Gamma(1/2)}{\Gamma(3/2)} \int_0^t (t-s) \int_{-L}^{L} \tilde{H}(t-s,x-y) f_L(s,y)^2\, dyds.
\end{aligned}
$$

Repeating the same arguments as before with $\alpha = 1$, then $\alpha = 3/2$, and so on, we get

$$
\begin{aligned}
f_L(t,x)^2 &\leq \tilde{J}(t,x) \\
&+ k \int_0^t ds \left(1 + \frac{\Gamma(1/2)}{\Gamma(1)} kd(t-s)^{1/2} + \cdots + \frac{\Gamma(1/2)}{\Gamma(n/2)} (kd)^{n-1} (t-s)^{(n-1)/2}\right) \\
&\qquad \times \int_{-L}^{L} dy\, \tilde{H}(t-s,x-y)\tilde{J}(s,y) \\
&+ k(kd)^n \frac{\Gamma(1/2)}{\Gamma((n+1)/2)} \int_0^t (t-s)^{n/2} \int_{-L}^{L} \tilde{H}(t-s,x-y) f_L(s,y)^2\, dyds.
\end{aligned}
$$

We easily observe that the expression in the parenthesis in the middle term above is the partial sum, from 0 to $n-1$, of $\sqrt{\pi} E_{1/2,1/2}\left(kd\sqrt{t-s}\right)$. Therefore, for all $n \geq 1$,

$$
\begin{aligned}
f_L(t,x)^2 \leq \tilde{J}(t,x) &+ k \int_0^t \phi_T(t-s) \int_{-L}^{L} \tilde{H}(t-s,x-y)\tilde{J}(s,y)\, dyds \\
&+ k(kd)^n \frac{\Gamma(1/2)}{\Gamma((n+1)/2)} \int_0^t ds\, (t-s)^{n/2} \\
&\times \int_{-L}^{L} dy\, \tilde{H}(t-s,x-y) f_L(s,y)^2.
\end{aligned}
$$

Thanks to the uniform bounds of Proposition 2.6, the function $f_L(t,x)$ is uniformly bounded by some constant Q, for all $t \in [0,T]$, $x \in [-L,L]$, and $L > 0$. In the case of Neumann boundary conditions, we recall that we must restrict to $L \geq l\sqrt{T}$. If we bound

$$
\begin{aligned}
\int_0^t (t-s)^{n/2} \int_{-L}^{L} \tilde{H}(t-s,x-y)\, dyds &\leq \int_0^t s^{n/2} \left(1 + \frac{1}{\sqrt{4\pi s}}\right) ds \\
&= \frac{t^{(n+2)/2}}{(n+2)/2} + \frac{1}{\sqrt{4\pi}} \frac{t^{(n+1)/2}}{(n+1)/2},
\end{aligned}
$$

and use the following property of the Gamma function:

$$
\frac{n+2}{2} \Gamma\left((n+1)/2\right) \geq \frac{n+1}{2} \Gamma\left((n+1)/2\right) = \Gamma\left((n+3)/2\right),
$$

we obtain

$$f_L(t,x)^2 \leq \tilde{J}(t,x) + k\int_0^t \phi_T(t-s)\int_{-L}^{L} \tilde{H}(t-s,x-y)\tilde{J}(s,y)\,dyds$$
$$+ k(kd)^n \frac{\Gamma(1/2)}{\Gamma((n+3)/2)}\left(t^{(n+2)/2} + \frac{t^{(n+1)/2}}{\sqrt{4\pi}}\right)Q^2.$$

To conclude, we let $n \to \infty$. □

Proof of Theorems 2.4 and 2.5. Thanks to inequality (2.31) and the fact that $\phi_T(t)$ is bounded in $[0,T]$, it is sufficient to evaluate

$$\int_0^t \int_{-L}^{L} \tilde{H}(t-s,x-y)\tilde{J}(s,y)\,dyds.$$

Recalling that

$$\tilde{J}(t,x) = 3c^2 J(t,x)^2 \leq 6c^2\sqrt{2\pi t}\left[\Gamma(t/2,L-x) + \Gamma(t/2,L+x)\right],$$

it is then sufficient to evaluate the following, which uses the semi-group property of Γ and the Beta integrals:

$$\int_0^t \sqrt{2\pi s}\int_{-L}^{L} \tilde{H}(t-s,x-y)\Gamma(s/2,L-y)\,dyds$$
$$\leq \int_0^t \sqrt{2\pi s}\left(1 + \frac{1}{\sqrt{4\pi(t-s)}}\right)\Gamma(t-s/2,L-x)\,ds$$
$$\leq \sqrt{2}\Gamma(t,L-x)\int_0^t \sqrt{2\pi s}\left(1 + \frac{1}{\sqrt{4\pi(t-s)}}\right)ds$$
$$= \frac{e^{-\frac{(L-x)^2}{4t}}}{\sqrt{t}}\int_0^t \sqrt{s}\left(1 + \frac{1}{\sqrt{4\pi(t-s)}}\right)ds$$
$$= \frac{e^{-\frac{(L-x)^2}{4t}}}{\sqrt{t}}\left(\frac{2t\sqrt{t}}{3} + \frac{t\sqrt{\pi}}{4}\right) = e^{-\frac{(L-x)^2}{4t}}\left(\frac{2t}{3} + \frac{\sqrt{\pi t}}{4}\right).$$

The second term is bounded similarly, therefore,

$$f_L(t,x)^2 \leq 3c^2\left(e^{-\frac{(L-x)^2}{4t}} + e^{-\frac{(L+x)^2}{4t}}\right)^2$$
$$+ 6c^2 k\,\|\phi_T\|_{L^\infty([0,T])}\left(\frac{2t}{3} + \frac{\sqrt{\pi t}}{4}\right)\left(e^{-\frac{(L-x)^2}{4t}} + e^{-\frac{(L+x)^2}{4t}}\right),$$

which concludes the proof. □

2.6 Remark on Neumann boundary conditions

In the case of Neumann boundary conditions (2.4), we had to restrict to

$$L^2/T \geq l^2 > 0,$$

for some $l > 0$. A reason for this assumption is explained in [19, Corollary 3.2.8 and Theorem 3.2.9]. There it is proved that there is a fundamental difference between the Green functions associated to Dirichlet and Neumann boundary conditions. In fact, that difference is valid for a broader class of parabolic equations and in higher space dimension. There, it is proved that the Green function associated to Dirichlet boundary conditions satisfies

$$0 \leq G^D(s,x,y) \leq \frac{c_1}{s^{d/2}} e^{-\frac{|x-y|^2}{c_2 s}}, \quad \forall s > 0, \quad \forall x,y \in D,$$

for some constants c_1, c_2 that may depend on the domain D. In the case of Neumann boundary conditions, we don't have the same behaviour as $s \to \infty$, and a possible bound is

$$0 \leq G^N(s,x,y) \leq c_1 \max(1, s^{-d/2}) e^{-\frac{|x-y|^2}{c_2 s}}, \quad \forall s > 0, \quad \forall x,y \in D.$$

In the case of the heat equation in one space dimension in which $D = (-L, L)$, we have the dilation property

$$G_L^D(t,x,y) = \frac{1}{L} G_1^D \left(\frac{t}{L^2}, \frac{x}{L}, \frac{y}{L} \right), \quad G_L^N(t,x,y) = \frac{1}{L} G_1^N \left(\frac{t}{L^2}, \frac{x}{L}, \frac{y}{L} \right).$$

In the case of Dirichlet, the dilation property implies that

$$G_L^D(t,x,y) = \frac{1}{L} G_1^D \left(\frac{t}{L^2}, \frac{x}{L}, \frac{y}{L} \right) \leq \frac{c_1}{\sqrt{t}} e^{-\frac{(x-y)^2}{c_2 t}},$$

for all $t > 0$ and $x, y \in [-L, L]$, where the constant c_1, c_2 are now independent of the length L. Compare with (A.50).

In the case of Neumann, the dilation property implies that

$$G_L^N(t,x,y) = \frac{1}{L} G_1^N \left(\frac{t}{L^2}, \frac{x}{L}, \frac{y}{L} \right) \leq c_1 \max(1/L, 1/\sqrt{t}) e^{-\frac{(x-y)^2}{c_2 t}},$$

for all $t > 0$ and $x, y \in [-L, L]$, where the constant c_1, c_2 are now independent of the length L. Compare with (A.64).

The only hope to reach an upper bound with the heat kernel, is to impose the condition

$$\frac{1}{L} = \frac{\sqrt{t}}{L} \frac{1}{\sqrt{t}} \leq l^{-1} \frac{1}{\sqrt{t}},$$

with $\sqrt{t}/L \leq l^{-1}$, which is equivalent to $L^2/t \geq l^2 > 0$. This holds for all $t \in [0, T]$, if $L^2/T \geq l^2 > 0$.

The same problem occurs for the heat equation in higher space dimension. Indeed, in the case of a rectangular domain, the Green function is the product of the one dimensional Green functions.

Chapter 3

Distributional solutions in higher space dimension

In this chapter, we will study the heat equation in higher space dimension, $d \geq 2$, with additive white noise, i.e. the case where the functions $b = 0$ and $\sigma = 1$. First, we will assume vanishing initial condition, and then generalize to bounded initial data.

We will compare the behavior of the solution on the whole space $\mathbb{R}^d$ with the solution on some bounded domain $D \subseteq \mathbb{R}^d$, with Dirichlet boundary conditions,

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) = \Delta u(t,x) + \dot{W}, & t > 0,\, x \in \mathbb{R}^d, \\ u(0,x) = 0, & x \in \mathbb{R}^d, \end{cases} \tag{3.1}$$

and

$$\begin{cases} \dfrac{\partial u_D}{\partial t}(t,x) = \Delta u_D(t,x) + \dot{W}, & t > 0,\, x \in D, \\ u_D(t,x) = 0, & t > 0,\, x \in \partial D \\ u_D(0,x) = 0, & x \in D, \end{cases} \tag{3.2}$$

where $\dot{W}$ is white noise on $\mathbb{R}_+ \times \mathbb{R}^d$. As we shall see, the regularity of the boundary ∂D will play an important role.

In the case of additive white noise in dimension $d \geq 2$, it is well known that no random field solution exits to these problems. It is a consequence of the fact that the squared heat kernel is not integrable on $[0,T] \times \mathbb{R}^d$, and the fact that the squared Green function, associated to Dirichlet or Neumann boundary conditions, is not integrable on $[0,T] \times D$.

It is possible though to interpret solutions in the sense of distributions. To find their respective weak formulation, we informally multiply both sides by some test function $\psi(x)$, integrate in time and in space, and apply integration by parts. The weak formulation associated to problem (3.1) is to find a random process $\{u(t) : t > 0\}$ such that for all $\psi \in \mathcal{S}(\mathbb{R}^d)$,

$$\langle u(t), \psi \rangle = \int_0^t \langle u(s), \Delta\psi \rangle \, ds + \int_0^t \int_{\mathbb{R}^d} \psi(y) \, W(dyds). \tag{3.3}$$

The weak formulation associated to problem (3.2) is to find a random process $\{u_D(t) : t > 0\}$ such that for all $\psi \in \mathcal{S}(D)$,

$$\langle u_D(t), \psi \rangle = \int_0^t \langle u_D(s), \Delta\psi \rangle \, ds + \int_0^t \int_D \psi(y) \, W(dyds), \tag{3.4}$$

where $\mathcal{S}(D) = \{\psi \in C^\infty(\bar{D}) : \psi = 0 \text{ on } \partial D\}$. Functions in $C^\infty(\bar{D})$ are functions in $C^\infty(D)$ with the property that each derivative admits a continuous extension to $\bar{D}$. The requirement $\psi = 0$ on ∂D for (3.4) plays the role of a Dirichlet boundary condition. The requirement $\psi \in \mathcal{S}(\mathbb{R}^d)$ for (3.3) assumes that $u(t)$ and W should not grow too quickly at infinity. In fact, it makes sense a fortiori. Indeed, we have the following existence and uniqueness results from [55, Theorems 5.1 and 5.2]:

Theorem 3.1. *There exists a unique process $\{u(t) : t > 0\}$ with values in $\mathcal{S}'(\mathbb{R}^d)$ which satisfies (3.3). It is given by*

$$\langle u(t), \psi \rangle := \int_0^t \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} \psi(x) \Gamma(t-s, x-y) \, dx \right) W(dyds), \tag{3.5}$$

where Γ is the heat kernel given by $\Gamma(t,x) = (4\pi t)^{-d/2} e^{-|x|^2/4t}$.

Theorem 3.2. *There exists a process $\{u_D(t) : t > 0\}$ with values in $\mathcal{S}'(\mathbb{R}^d)$, which satisfies (3.4) for any $\psi \in \mathcal{S}(D) \cap \mathcal{S}(\mathbb{R}^d)$. It can be extended to a stochastic process $\{u(t,\psi) : t > 0, \psi \in \mathcal{S}(D)\}$; this process is unique. It is given by*

$$\langle u_D(t), \psi \rangle := \int_0^t \int_D \left(\int_D \psi(x) G_D(t-s, x, y) \, dx \right) W(dyds), \tag{3.6}$$

where G_D is the Green function to the heat equation associated to Dirichlet boundary conditions. As a consequence, the support of each distribution $u_D(t)$ is contained in $\bar{D}$.

Remark. The existence and uniqueness results from [55] are in fact much more general. They guarantee existence and uniqueness when the Laplacian operator is replaced by a uniformly elliptic self-adjoint second order differential operator with bounded smooth coefficients. White noise can be replaced by some space derivatives of some worthy (continuous) martingale measure with tempered dominating measure. Neumann boundary conditions are also covered. In the present case of white noise, both processes are continuous as a function of t.

We treat the deterministic case in Appendix D. There, a proof of existence and uniqueness is given in the class of continuous process with values in $\mathcal{S}'(\mathbb{R}^d)$.

3.1 Main results and general ideas

As before, we would like to conclude that $u_D(t)$ converges to $u(t)$ as the domain D expands to the whole space. Should we also expect some exponential rate of convergence? In Theorem 2.4 of the previous chapter, the expressions $(L-x)^2$ and $(L+x)^2$ correspond to the distance of some point $x \in (-L, L)$ to the right and left boundary of the interval. In the present case of distributions, we cannot evaluate the solutions at any given point. Thus, a pointwise convergence is out of range. We will analyse how the distributional solutions u and u_D differ from each other when evaluated against some test function $\psi \in C_c^\infty(\mathbb{R}^d)$. The quantity of interest is the difference

$$\langle u(t), \psi \rangle - \langle u_D(t), \psi \rangle \, .$$

How should translate the distance of some point to the boundary? If the support of the test function is contained into the domain, $\operatorname{supp}(\psi) \subseteq D$, then we could consider the distance between its support and the boundary of the domain ∂D. For that matter, we introduced $\delta = \operatorname{dist}(\operatorname{supp}(\psi), \partial D)$. In fact, we have

Proposition 3.3. *For any test function $\psi \in C_c^\infty(\mathbb{R}^d)$, with $\operatorname{supp}(\psi) \subseteq D$, and $t \in [0, T]$,*

$$\| \langle u(t), \psi \rangle - \langle u_D(t), \psi \rangle \|_{L^2(\Omega)} \leq c \, \|\psi\|_{L^2(\mathbb{R}^d)} \exp\left(-\frac{\delta^2}{4ct} \right),$$

where c is a constant independent of t and ψ, but which depends on the domain D.

The latter is not sufficient to conclude exponential convergence. Indeed, the fact that the constant c in the exponential depends on the domain D prevents from doing so. To bypass this problem, we fix, once and for all, some open bounded domain D, containing the origin, and consider the dilations

$$LD := \{Lx \in \mathbb{R}^d : x \in D\},$$

for any $L > 0$. This was in fact the procedure of the previous chapter in one space dimension. There, we considered the symmetric intervals $(-L, L)$, which are the scaled versions of $(-1, 1)$.

The computations will be in the same spirit as in the previous chapter. In order to evaluate the difference (3.1), we will need to consider the positive difference

$$F_D(t, x, y) := \Gamma(t, x - y) - G_D(t, x, y)$$

between the heat kernel and the Green function, see (A.30) for positivity. What we should really consider is the difference $\Gamma(t, x-y) - G_{LD}(t, x, y)$, where G_{LD} is the Green function associated with the dilated domain LD. In the particular case of the heat equation, the latter difference can be expressed from the former. This is a consequence of the scaling property of both the heat kernel and the Green function, see (A.32).

$$\begin{aligned}F_{LD}(t,x,y) &= \Gamma(t,x-y) - G_{LD}(t,x,y) \\ &= \frac{1}{L^d}\left[\Gamma(t/L^2,(x-y)/L) - G_D(t/L^2,x/L,y/L)\right] \\ &= \frac{1}{L^d}F_D(t/L^2,x/L,y/L).\end{aligned}$$

The last component we shall need is an appropriate bound on the latter difference F_D. In the present case of Dirichlet boundary conditions, the bound (2.16) can be generalized to any open domain. The result is given in Lemma 4.5.

We introduce the concise notation $u_L(t)$ instead of $u_{LD}(t)$. We also set $\delta(L) = \operatorname{dist}(\operatorname{supp}(\psi), \partial(LD))$ for any test function $\psi \in C_c^\infty(\mathbb{R}^d)$, whose support is contained into LD for some L large enough. The following result will be proved in Section 3.3.

Theorem 3.4. *For any open domain D and any test function $\psi \in C_c^\infty(\mathbb{R}^d)$ such that* $\operatorname{supp}(\psi) \subseteq LD$, *we have*

$$\|\langle u(t),\psi\rangle - \langle u_L(t),\psi\rangle\|_{L^2(\Omega)} \le C\,\|\psi\|_{L^2(\mathbb{R}^d)}\sqrt{t}e^{-\frac{\delta(L)^2}{4ct}}, \tag{3.7}$$

for all $t > 0$. In fact, both constants C and c depend neither on D, $L > 0$, nor ψ. In particular, if D contains the origin, then for L large enough,

$$\|\langle u(t),\psi\rangle - \langle u_L(t),\psi\rangle\|_{L^2(\Omega)} \le C\,\|\psi\|_{L^2(\mathbb{R}^d)}\sqrt{t}e^{-\frac{L^2}{4ct}}.$$

It is a fair question to ask whether the previous convergence result can be generalized. As we shall see, it does in some local version of Sobolev spaces. First, recall the usual Sobolev spaces

$$H^s = \left\{ f \in \mathcal{S}'(\mathbb{R}^d) : \int_{\mathbb{R}^d}\left(1+|\xi|^2\right)^s |\mathcal{F}(f)(\xi)|^2\,d\xi < \infty \right\}. \tag{3.8}$$

The index $s \in \mathbb{R}$ accounts for the regularity of the distribution, see Appendix C for some explanations. In particular, we have the following chain of inclusions:

$$\mathcal{S} \subseteq H^s \subseteq H^r \subseteq L^2(\mathbb{R}^d) \subseteq H^{-r} \subseteq H^{-s} \subseteq \mathcal{S}',$$

for any two real numbers $0 \le r \le s$.

In order to show convergence in the usual version of Sobolev spaces H^s, we would need to consider the Fourier transform, in the sense of distributions, of the solution $u(t)$, and hope that it is given by a (random) function. Unfortunately, it is not. The following result will be proved in Section 3.4.

Proposition 3.5. *Fix any $t > 0$. There exists no jointly measurable locally mean-square integrable $\mathbb{C}$-valued process $X : (x, \omega) \to X(x, \omega)$ such that a.s., for all $\psi \in C_c^\infty(\mathbb{R}^d)$,*

$$\langle \mathcal{F}(u(t)), \psi \rangle = \int_{\mathbb{R}^d} X(x)\psi(x)\, dx.$$

In order for a (non-random) function to belong to some H^s space, it must be regular enough locally, and regular enough at infinity, i.e. satisfy some integrability condition. For example, even though the constant one function is infinitely smooth locally, it does not belong to any of the H^s spaces because of its lack of integrability. In fact, its Fourier transform is the Dirac delta measure.

An effective way to focus on the local regularity is to multiply by some function that vanishes at infinity. For example, the product $1 \cdot \psi$ of the constant one function with any $\psi \in C_c^\infty(\mathbb{R}^d)$ belongs to all H^s spaces. The solution $u(t)$ has the same lack of regularity at infinity.

Lemma 3.6. *Fix $t > 0$. For any $\phi \in C_c^\infty(\mathbb{R}^d)$, the Fourier transform of the product $\phi \cdot u(t)$ is given by the random function*

$$\mathcal{F}(\phi \cdot u(t))(\xi) = \int_0^t \int_{\mathbb{R}^d} \mathcal{F}\left[\phi \cdot \Gamma(t-s, \bullet - y)\right](\xi)\, W(dsdy).$$

Moreover, it is a Gaussian process, and with probability one $u(t) \in H^r_{loc}$, for all $r < -d/2 + 1$.

To put this result in perspective, it was already known that $u_D(t) \in H_r(D)$, for $r < -d/2 + 1$, where $H_r(D)$ is some version of Sobolev spaces in the bounded (smooth) domain D, see [55, Remarks after Proposition 5.3].

To circumvent the lack of integrability of the solution $u(t)$, we introduce the local version of Sobolev spaces

$$H^s_{loc} = \left\{ f \in \mathcal{S}'(\mathbb{R}^d) : \forall \phi \in C_c^\infty(\mathbb{R}^d),\ \phi \cdot f \in H^s \right\}.$$

In Lemma C.2, we show the expected inclusion $H^s \subseteq H^s_{loc}$, for all $s \in \mathbb{R}$.

In order to analyse the difference $\phi u(t) - \phi u_D(t) = \phi(u(t) - u_L(t))$, we shall again use appropriate bounds on the difference $\Gamma - G_D$. This time though, we shall require bounds on the difference of their derivatives. One way to guarantee that G_D is sufficiently differentiable is to assume conditions on the regularity of the boundary of the domain. The following result will be proved in Section 3.6.

Theorem 3.7. *Let $\alpha \in (0, 1)$ and $n \in \mathbb{N}$. Fix a bounded domain D containing the origin whose boundary belongs to $C^{2+n+\alpha}$. Fix $t > 0$. Then $u_L(t) \to u(t)$*

in H^r_{loc}*, for all* $r < n - d/2$*, as* $L \to \infty$*. Moreover, for any test function* $\phi \in C_c^\infty(\mathbb{R}^d)$*, and* L *large enough so that* $\operatorname{supp}(\phi) \subset LD$*, we have*

$$\mathbb{E}\left[\|\phi(u(t) - u_L(t))\|^2_{H^r(\mathbb{R}^d)}\right] \leq c\,\|\phi\|^2_{H^n}\, te^{-\delta(L)^2/ct}\nu(\delta(L)),$$

where $\nu(\delta(L)) = O(1)$*, as* $L \to \infty$*, and* $c = c(\alpha, d, T, D, r, n)$ *does not depend on* $L > 0$*.*

In particular, if ∂D *is* C^∞*, then with probability one the difference* $u(t) - u_L(t) \in C^\infty(K)$ *for any compact set* $K \subseteq LD$*.*

The last statement is a consequence of the following Sobolev embedding [35, Theorem 3.32 and Exercise 3.34]:

Theorem 3.8. *Let* $l \geq 0$*. If* $s > l + d/2$*, then any distribution in* H^s *can be represented by a* $C^l(\mathbb{R}^d)$ *function.*

The last Section 3.7 focuses on generalizing both convergence results to the case of non-vanishing initial condition. In fact, Theorem 3.7 remains valid for vanishing Neumann boundary conditions.

Final remark: If we are not interested in the convergence rate at all, then the second part of Theorem 3.7 is expected to remain valid for a broader class of equations. Indeed, for second order parabolic differential equations, the fundamental solution and the Green function satisfy bounds such as (3.9), and their difference verifies (3.10), see [21, Theorem 1.1] and [22, Chapter 6].

3.2 Some Prerequisites

The precise relation between the regularity of the boundary ∂D of some open bounded domain D and the regularity of the Green function G_D, and its difference with the heat kernel

$$F_D(t, x, y) = \Gamma(t, x - y) - G_D(t, x, y)$$

is given in the following result, see [21, Theorem 1.1]. The fact that the Green function is bounded by the heat kernel was already known, see (A.30). The fact that the difference F_D is bounded by the heat kernel and the distance to the boundary was already known, see Lemmas 2.7 and 4.5. The fact that the derivatives of the heat kernel can be bounded by itself times some polynomial in t was already known, see (A.22).

Theorem 3.9. *Suppose the boundary* ∂D *belongs to* $C^{2+n+\alpha}$*, for some integer* $n \geq 0$ *and* $\alpha \in (0, 1)$*. Then, the Green function* G_D *is continuous on* $(0, T] \times \bar{D} \times \bar{D}$*. It satisfies* $G(t, x, y) = 0$ *if* $x \in \partial D$*, and*

$$\left|\partial_t^k \partial_x^l G_D(t, x, y)\right| \leq C t^{-k-|l|/2}\Gamma_c(t, x - y); \tag{3.9}$$

$$\left|\partial_t^k \partial_x^l F_D(t, x, y)\right| \leq C t^{-k-|l|/2}\Gamma_c\left(t, |x - y| + \operatorname{dist}(y, \partial D)\right), \tag{3.10}$$

for all $t \in (0,T]$ and $x, y \in D$, for as long as the total order of derivation $2k + |l| \le 2 + n$. Furthermore,

$$\left|\partial_t^k \partial_x^l G_D(t,x,y) - \partial_t^k \partial_x^l G_D(t,x_0,y)\right| \le C\left(\frac{|x-x_0|}{\sqrt{t}}\right)^{\alpha} t^{-n/2}\Gamma_c(t,\bar{x}-y), \quad (3.11)$$

$$\left|\partial_t^k \partial_x^l F_D(t,x,y) - \partial_t^k \partial_x^l F_D(t,x_0,y)\right| \le C\left(\frac{|x-x_0|}{\sqrt{t}}\right)^{\alpha} t^{-n/2}\Gamma_c(t,|\bar{x}-y| + \operatorname{dist}(y,\partial D)), \quad (3.12)$$

for all $t \in (0,T]$ and $x, x_0, y \in D$, only when $2k + |l| = 2 + n$, with $|\bar{x} - y| = \min(|x-y|, |x_0 - y|)$. Both constants c and C depend on d, T, n, α and the domain D.

The bounds on the difference F_D are deduced from the bounds satisfied by the Green function G_D, in which we replace the expressions of $|x-y|$ and $|\bar{x}-y|$ by $|x-y| + \operatorname{dist}(y,\partial D)$ and $|\bar{x}-y| + \operatorname{dist}(y,\partial D)$, respectively.

The fact that the constants c and C in equation (3.10) depend on the domain is very inconvenient. As already mentioned, we fix some open bounded domain D containing the origin and consider its dilations $LD = \{Lx \in \mathbb{R}^d : x \in D\}$, for $L > 0$. Using the scaling properties, we have

$$F_{LD}(t,x,y) = \frac{1}{L^d} F_D(t/L^2, x/L, y/L),$$

and the following result is easily deduced.

Corollary 3.10. *If the boundary ∂D belongs to $C^{2+n+\alpha}$, then*

$$\left|\partial_t^k \partial_x^l F_{LD}(t,x,y)\right| \le C t^{-k-|l|/2}\Gamma_c\left(t, |x-y| + \operatorname{dist}(y,\partial LD)\right), \quad (3.13)$$

for all $t \in (0, TL^2]$ and $x, y \in LD$, as long as $2k + |l| \le 2 + n$. The constant c and C depend only on n, α, d, T, and the domain D, but not on the constant $L > 0$.

Proof. We give the intuition by proving only the case $k = 1$ and $l = 0$. By the chain rule, inequality (3.10) and the scaling property of the heat kernel

$$\begin{aligned}
\frac{\partial F_{LD}}{\partial t}(t,x,y) &= \frac{1}{L^d}\frac{1}{L^2}\frac{\partial F_D}{\partial t}(t/L^2, x/L, y/L) \\
&\le \frac{1}{L^d}\frac{1}{L^2} C (t/L^2)^{-1}\Gamma_c\left(t/L^2, |x/L - y/L| + \operatorname{dist}(y/L,\partial D)\right) \\
&= \frac{1}{L^d} C t^{-1}\Gamma_c\left(t/L^2, \frac{|x-y| + \operatorname{dist}(y,\partial LD)}{L}\right) \\
&= C t^{-1}\Gamma_c\left(t, |x-y| + \operatorname{dist}(y,\partial LD)\right).
\end{aligned}$$

The equalities are true for all $t > 0$ and $x, y \in \mathbb{R}^d$. The inequality is valid only for $t/L^2 \in (0, T]$ and $x/L, y/L \in D$, thus $t \in (0, TL^2]$ and $x, y \in LD$. □

In the present case of the heat equation, the difference F_D is symmetric in the x and y variable, $F_D(t, x, y) = F_D(t, y, x)$ from equation (A.30). Therefore, the expression $\operatorname{dist}(y, \partial D)$ can be replaced by $\operatorname{dist}(x, \partial D)$ in the bound (3.10) at least when $k = 0$ and $l = 0$. If F_D were not symmetric, we could use the following trick: The bound $\operatorname{dist}(x, \partial D) \leq |x - y| + \operatorname{dist}(y, \partial D)$ implies easily

$$
\begin{aligned}
|x - y| + \operatorname{dist}(y, \partial D) &\geq \frac{1}{2}|x - y| + \frac{1}{2}\operatorname{dist}(y, \partial D) + \frac{1}{2}\operatorname{dist}(x, \partial D), \\
(|x - y| + \operatorname{dist}(y, \partial D))^2 &\geq \frac{1}{4}\left(|x - y|^2 + \operatorname{dist}(y, \partial D)^2 + \operatorname{dist}(x, \partial D)^2\right).
\end{aligned}
$$

Therefore, upon replacing c by $4c$, inequality (3.10) remains valid if the expression $\operatorname{dist}(y, \partial D)$ is replaced by $\operatorname{dist}(y, \partial D) + \operatorname{dist}(x, \partial D)$. A similar reasoning holds for the bound (3.13). We introduce the notation

$$
J_c(t, x; D) = \exp\left\{-\frac{\operatorname{dist}(x, \partial D)^2}{ct}\right\}.
$$

Corollary 3.11. *If the boundary ∂D belongs to $C^{2+n+\alpha}$, then*

$$
\begin{aligned}
\left|\partial_t^k \partial_x^l G_{LD}(t, x, y)\right| &\leq C t^{-k-|l|/2} \Gamma_c(t, x - y), \\
\left|\partial_t^k \partial_x^l F_{LD}(t, x, y)\right| &\leq C t^{-k-|l|/2} \Gamma_c(t, x - y) J_c(t, x; LD) J_c(t, y; LD),
\end{aligned}
\tag{3.14}
$$

for all $t \in (0, TL^2]$ and $x, y \in LD$, as long as $2k + |l| \leq 2 + n$. The constant c and C depend only on n, α, d, T, and the domain D, but not on the constant $L > 0$.

Remark. A similar result is valid for Neumann boundary conditions. Indeed, both inequalities (3.9) and (3.10) are valid in the case of Neumann boundary conditions by [21, Theorem 1.1] and the scaling property of the Green function is also satisfied.

3.3 Convergence in distribution

In order to prove Theorem 3.4, we explicitely evaluate the difference

$$
\langle u(t), \psi \rangle - \langle u_D(t), \psi \rangle = \sum_{i=1}^{3} I_i(t),
$$

for $\psi \in C_c^\infty(\mathbb{R}^d)$, where

$$I_1(t) = \int_0^t \int_D \left(\int_D \psi(x) \left[\Gamma(t-s, x-y) - G_D(t-s, x, y) \right] dx \right) W(dyds),$$
$$I_2(t) = \int_0^t \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d \setminus D} \psi(x) \Gamma(t-s, x-y)\, dx \right) W(dyds),$$
$$I_3(t) = \int_0^t \int_{\mathbb{R}^d \setminus D} \left(\int_D \psi(x) \Gamma(t-s, x-y)\, dx \right) W(dyds).$$

By assumption, $\text{supp}(\psi) \subseteq D$, and therefore $I_2(t)$ is identically zero. To simplify notations, we set $D' := \text{supp}(\psi)$, and $\delta = \text{dist}(D', \partial D)$.

We now analyse the $L^2(\Omega)$-norm of $I_3(t)$.

$$\begin{aligned} \|I_3(t)\|_{L^2(\Omega)} &= \left[\int_0^t ds \int_{\mathbb{R}^d \setminus D} dy \left(\int_D \psi(x) \Gamma(t-s, x-y)\, dx \right)^2 \right]^{1/2} \\ &= \left[\int_0^t J(t-s)^2\, ds \right]^{1/2} = \left[\int_0^t J(s)^2\, ds \right]^{1/2}, \end{aligned}$$

where

$$\begin{aligned} J(s) &= \left[\int_{\mathbb{R}^d \setminus D} dy \left(\int_{D'} \psi(x) \Gamma(s, x-y)\, dx \right)^2 \right]^{1/2} \\ &= \left[\int_{\mathbb{R}^d} dy\, \mathbf{1}_{\mathbb{R}^d \setminus D}(y) \left(\int_{\mathbb{R}^d} \mathbf{1}_{D'}(x) \psi(x) \Gamma(s, x-y)\, dx \right)^2 \right]^{1/2} \\ &= \left[\int_{\mathbb{R}^d} dy \left(\int_{\mathbb{R}^d} \mathbf{1}_{D'}(x+y)\, \mathbf{1}_{\mathbb{R}^d \setminus D}(y) \psi(x+y) \Gamma(s, x)\, dx \right)^2 \right]^{1/2} \\ &\leq \int_{\mathbb{R}^d} \Gamma(s, x) K(x)^{1/2}\, dx, \end{aligned}$$

thanks to Minkowski's inequality for integrals (B.1), where

$$\begin{aligned} K(x) &= \int_{\mathbb{R}^d} \mathbf{1}_{D'}(x+y)\, \mathbf{1}_{\mathbb{R}^d \setminus D}(y) \psi^2(x+y)\, dy \\ &= \int_{\mathbb{R}^d} \mathbf{1}_{D'}(y)\, \mathbf{1}_{\mathbb{R}^d \setminus D}(y-x) \psi^2(y)\, dy \\ &= \int_{\mathbb{R}^d} \mathbf{1}_{D'}(y)\, \mathbf{1}_{\mathbb{R}^d \setminus D}(y-x) \psi^2(y)\, dy. \end{aligned}$$

We can bound

$$\mathbf{1}_{D'}(y)\, \mathbf{1}_{\mathbb{R}^d \setminus D}(y-x) \leq \mathbf{1}_{D'}(y)\, \mathbf{1}_{\mathbb{R}^d \setminus B(0,\delta)}(x)$$

Indeed, for $y \in D'$ and $y - x \in \mathbb{R}^d \setminus D$, we have $\|x\| = \|y - (y-x)\| \geq \operatorname{dist}(D', \partial D) = \delta$. Finally, we get

$$\|I_3(t)\|_{L^2(\Omega)} \leq \|\psi\|_{L^2(\mathbb{R}^d)} \left[\int_0^t ds \left(\int_{\mathbb{R}^d} \Gamma(s,x)\, \mathbf{1}_{\mathbb{R}^d \setminus B(0,\delta)}(x)\, dx \right)^2 \right]^{1/2}. \tag{3.15}$$

We can estimate the inner integral as follows:

Lemma 3.12. *Independently of* $\delta > 0$, *there are* $c, C \geq 1$ *such that*

$$\int_{\|x\| \geq \delta} \Gamma(s,x)\, dx \leq C e^{-\frac{\delta^2}{4cs}}. \tag{3.16}$$

Proof. Applying spherical coordinates, setting $\phi(z) = e^{-z^2/2}/\sqrt{2\pi}$, and using inequalites (A.18) or (A.19), we get

$$\begin{aligned}
\int_{\|x\| \geq \delta} \Gamma(s,x)\, dx &= \frac{w_{d-1}}{(4\pi s)^{d/2}} \int_\delta^\infty z^{d-1} e^{-\frac{z^2}{4s}}\, dz \\
&= \frac{w_{d-1}}{(2\pi)^{(d-1)/2}} \int_{\delta/\sqrt{2s}}^\infty z^{d-1} \phi(z)\, dz \\
&\leq C P_{d-2}\left(\frac{\delta}{\sqrt{2s}} \right) e^{-\frac{\delta^2}{4s}}
\end{aligned}$$

where w_{d-1} is the area of the sphere in $\mathbb{R}^d$, and P_{d-2} a polynomial of degree $d-2$. We conclude with inequality (A.21). □

This leads directly to

$$\|I_3(t)\|_{L^2(\Omega)} \leq C\, \|\psi\|_{L^2(\mathbb{R}^d)} \sqrt{t} e^{-\frac{\delta^2}{4ct}}.$$

It remains to estimate $I_1(t)$:

$$\|I_1(t)\|^2_{L^2(\Omega)} = \int_0^t \int_D \left(\int_D \psi(x) F_D(t-s,x,y)\, dx \right)^2 dy\, ds.$$

Recall again that $\operatorname{supp}(\psi) = D' \subseteq D$, and $\delta = \operatorname{dist}(\operatorname{supp}(\psi), \partial D)$. Using Lemma 4.5, we have

$$\begin{aligned}
&\int_D \left(\int_D \psi(x) F_D(s,x,y)\, dx \right)^2 dy \\
&\leq C^2 \int_D \left(\int_{D'} |\psi(x)|\, \Gamma_c(s, x-y) J_c(s,x)\, dx \right)^2 dy \\
&\leq C^2 \exp\left(-\frac{\delta^2}{4cs} \right) \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |\psi(x)|\, \Gamma_c(s, y-x)\, dx \right)^2 dy \\
&= C^2 \exp\left(-\frac{\delta^2}{4cs} \right) \||\psi| * \Gamma_c(s)\|^2_{L^2(\mathbb{R}^d)} \\
&\leq C^2 \exp\left(-\frac{\delta^2}{4cs} \right) \|\psi\|^2_{L^2(\mathbb{R}^d)},
\end{aligned}$$

where the last inequality follows from the convolution inequality (B.2) applied to $f = |\psi|$ and $g(x) = \Gamma_c(s,x)$. Finally, we get

$$\|I_1(t)\|_{L^2(\Omega)} \le C\, \|\psi\|_{L^2(\mathbb{R}^d)} \sqrt{t} e^{-\frac{\delta^2}{4ct}}.$$

This conclude the proof of Theorem 3.4. Indeed, replacing D by LD everywhere, and setting $\delta(L) = \mathrm{dist}(D', LD)$, we observe that each bound for $I_i(t)$ remains valid thanks to Corollary 3.11.

Addendum. We give a simplification of the derivation involving I_3, that require Lemma 4.8. It applies to bound

$$\Gamma(s,x-y)\,\mathbf{1}_{\mathbb{R}^d\setminus D}(y)\,\mathbf{1}_{D'}(x) \le c^{d/2}\Gamma_c(s,x-y)J_c(s,x) \le c^{d/2}\Gamma_c(s,x-y)e^{-\frac{\delta^2}{4ct}},$$

for any $s \le t$. Therefore

$$\begin{aligned} J(s)^2 = \int_{\mathbb{R}^d} dy \left(\int_{\mathbb{R}^d} \mathbf{1}_{\mathbb{R}^d\setminus D}(y)\,\mathbf{1}_{D'}(x)\psi(x)\Gamma(s,x-y)\,dx \right)^2 \\ \le c^d e^{-\frac{\delta^2}{2ct}} \left\| |\psi| * \Gamma_c \right\|^2_{L^2(\mathbb{R}^d)} \le c^d e^{-\frac{\delta^2}{2ct}} \|\psi\|^2_{L^2(\mathbb{R}^d)}. \end{aligned}$$

Then

$$\|I_3(t)\|_{L^2(\Omega)} = \left[\int_0^t J(s)^2\,ds \right]^{1/2} \le c^{d/2} e^{-\frac{\delta^2}{4ct}} \|\psi\|_{L^2(\mathbb{R}^d)} \sqrt{t}.$$

which conclude the argument.

3.4 Regularity of the distributional solution $u(t)$

We start this section by some informal computations. Those should somehow reveal the lack of integrability of the distributional solution $u(t)$.

In one space dimension, we know that the (random field) solution $u(t,x)$ is continuous in both variables t, x. In the case of additive white noise, i.e. $b \equiv 0$ and $\sigma \equiv 1$, say with vanishing initial condition $u_0 \equiv 0$, it is given by

$$u(t,x) = \int_0^t \int_{\mathbb{R}} \Gamma(t-s,x-y)\,W(dyds),$$

see equation (2.8). Even in this very simple case, its Fourier transform in space is not well defined as a function. Indeed, an application of Fubini's theorem, whose hypotheses are not satisfied, see (B.4), would give

$$\begin{aligned} \mathcal{F}(u(t))(\xi) &= \int_{\mathbb{R}} e^{-2\pi i\xi x} \left(\int_0^t \int_{\mathbb{R}} \Gamma(t-s,x-y)\,W(dsdy) \right) dx \\ &\approx \int_0^t \int_{\mathbb{R}} \left(\int_{\mathbb{R}} e^{-2\pi i\xi x}\Gamma(t-s,x-y)\,dx \right) W(dsdy), \end{aligned} \tag{3.17}$$

from which we can conclude that it is not a well-defined process in $L^2(\Omega)$:

$$\begin{aligned}\mathbb{E}\left[|\mathcal{F}(u(t))(\xi)|^2\right] &= \int_0^t \int_{\mathbb{R}} \left| \int_{\mathbb{R}} e^{-2\pi i \xi x} \Gamma(t-s, x-y)\, dx \right|^2 dsdy \\ &= \int_0^t \int_{\mathbb{R}} \left| e^{-2\pi i y} e^{-4\pi^2 (t-s)\xi^2} \right|^2 dsdy = \infty,\end{aligned}$$

since $\mathcal{F}\Gamma(t)(\xi) = e^{-4\pi^2 t|\xi|^2}$.

A rigorous way to obtain the Fourier transform, as a distribution, in the general setting of $\mathbb{R}^d$, makes use of equation (3.5). For any $\psi \in \mathcal{S}(\mathbb{R}^d)$, we have

$$\begin{aligned}\langle \mathcal{F}(u(t)), \psi \rangle &:= \langle u(t), \mathcal{F}(\psi) \rangle \\ &= \int_0^t \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} \mathcal{F}(\psi)(\xi) \Gamma(t-s, \xi - y)\, d\xi \right) W(dyds) \\ &= \int_0^t \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} \psi(\xi) \left(\int_{\mathbb{R}^d} e^{-2\pi i \xi x} \Gamma(t-s, x-y)\, dx \right) d\xi \right) W(dsdy).\end{aligned} \tag{3.18}$$

where the first equality is obtained using Plancherel identity. Again, an informal computation using Fubini's theorem, whose hypotheses are not satisfied, would lead to

$$\langle \mathcal{F}(u(t)), \psi \rangle \approx \int_{\mathbb{R}^d} d\xi\, \psi(\xi) \int_0^t \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} e^{-2\pi i \xi x} \Gamma(t-s, x-y)\, dx \right) W(dsdy).$$

Therefore, if $\mathcal{F}(u(t))$ were to be a function, then it should resemble equation (3.17).

We now give the proof of Propositon 3.5. It was inspired by the proof of [11, Theorem 11]. In order to avoid back and forth, we recall that proposition here.

Proposition. *Fix $t > 0$. There exists no jointly measurable locally mean-square integrable $\mathbb{C}$-valued process $X : (x, \omega) \to X(x, \omega)$ such that a.s., for all $\psi \in C_c^\infty(\mathbb{R}^d)$,*

$$\langle \mathcal{F}(u(t)), \psi \rangle = \int_{\mathbb{R}^d} X(x) \psi(x)\, dx. \tag{3.19}$$

Proof. By contradiction, we suppose that such a process exists. We define the following approximation to the identity: let $\phi \in C_c^\infty(\mathbb{R}^d)$ be a non-negative bump function[1] with compact support and unit mass, and set

[1] In the present case, we choose ϕ and $\delta > 0$ so that $\phi \equiv 1$ in $B(0, \delta/2)$, $0 \le \phi \le c$, $\phi \equiv 0$ in $\mathbb{R}^d \setminus B(0, \delta)$, and $\int_{\mathbb{R}^d} \phi(x)\, dx = 1$.

$\phi_n(x) = n^d\phi(nx)$. We will estimate, in two different ways, the quantity

$$\mathbb{E}\left[|\langle \mathcal{F}(u(t)), \psi_n\rangle|^2\right],$$

for $\psi_n(x) = \phi_n(x_0 - x)$, and let $n \to \infty$. Using assumption (3.19), we shall show that the limit is finite. Using its definition (3.18), we shall show that the limit is infinite.

From (3.19) and Cauchy-Schwarz inequality, we get

$$\begin{aligned}\mathbb{E}\left[|\langle \mathcal{F}(u(t)), \psi_n\rangle|^2\right] &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \phi_n(x_0 - x)\phi_n(x_0 - y)\mathbb{E}\left[X(x)\overline{X(y)}\right] dydx \\ &\le \int_{\mathbb{R}^d} \phi_n(x_0 - x)\mathbb{E}\left[|X(x)|^2\right]^{1/2} dx \\ &\qquad \times \int_{\mathbb{R}^d} \phi_n(x_0 - y)\mathbb{E}\left[|X(y)|^2\right]^{1/2} dy.\end{aligned}$$

This is a convolution in $\mathbb{R}^d$ of the locally integrable function $\mathbb{E}\left[|X(x)|^2\right]^{1/2}$ with an approximation to the identity ϕ_n of compact support. Thus, using [52, Theorem 2.1 of Chapter 3], we get that for almost every $x_0 \in \mathbb{R}^d$,

$$\int_{\mathbb{R}^d} \phi_n(x_0 - x)\mathbb{E}\left[|X(x)|^2\right]^{1/2} dx \longrightarrow \mathbb{E}\left[|X(x_0)|^2\right]^{1/2}, \quad \text{as } n \to \infty,$$

and therefore,

$$\limsup_{n\to\infty} \mathbb{E}\left[|\langle \mathcal{F}(u(t)), \psi_n\rangle|^2\right] \le \mathbb{E}\left[|X(x_0)|^2\right] < \infty.$$

From (3.18) and Parseval's identity, we get

$$\begin{aligned}\mathbb{E}\left[|\langle \mathcal{F}(u(t)), \psi_n\rangle|^2\right] &= \int_0^t \int_{\mathbb{R}^d} |(\mathcal{F}(\psi_n) * \Gamma(t-s))(y)|^2 \, dyds \\ &= \int_0^t \int_{\mathbb{R}^d} |\psi_n(\xi)\mathcal{F}\Gamma(t-s)(\xi)|^2 \, d\xi ds \\ &= \int_0^t \int_{\mathbb{R}^d} |\phi_n(x_0 - \xi)|^2 \, \mathcal{F}\Gamma(2(t-s))(\xi)\, d\xi ds,\end{aligned}$$

the last equality follows from the fact that $\mathcal{F}\Gamma(t)(\xi) = e^{-4\pi^2 t|\xi|^2}$. The fact that ϕ_n is raised to the second power in the previous integral will imply that the limit is infinite. With our specific definition of ϕ, we have that $\phi_n(x_0 - \xi) = n^d$ for $\xi \in B(x_0, \delta/(2n))$, and we can define the following approximation to the identity: $\tilde{\phi}_n(x) = n^d\tilde{\phi}(nx)$, where $\tilde{\phi} := \tilde{c}\phi\,\mathbf{1}_{B(0,\delta/2)}$, where $\tilde{c}$ is such that $\int_{\mathbb{R}^d} \tilde{\phi}(x)\, dx = 1$, i.e. $\tilde{c} = \left(v_d(\delta/2)^d\right)^{-1}$, with v_d the volume of the unit ball in $\mathbb{R}^d$. We can estimate

$$\begin{aligned}\int_0^t \int_{\mathbb{R}^d} &|\phi_n(x_0-\xi)|^2 \, \mathcal{F}\Gamma(2(t-s))(\xi)\, d\xi ds \\ &\geq \int_0^t \int_{B(x_0,\delta/(2n))} |\phi_n(x_0-\xi)|^2 \, \mathcal{F}\Gamma(2(t-s))(\xi)\, d\xi ds \\ &= \int_0^t \frac{n^d}{\tilde{c}} \int_{B(x_0,\delta/(2n))} \tilde{\phi}_n(x_0-\xi)\mathcal{F}\Gamma(2(t-s))(\xi)\, d\xi ds \\ &\qquad = n^d v_d(\delta/2)^d \int_0^t \int_{\mathbb{R}^d} \tilde{\phi}_n(x_0-\xi)\mathcal{F}\Gamma(2(t-s))(\xi)\, d\xi ds\end{aligned}$$

By the same result [52, Theorem 2.1 in Chapter 3], we can ensure that for every $x_0 \in \mathbb{R}^d$,

$$\int_{\mathbb{R}^d} \tilde{\phi}_n(x_0-\xi)\mathcal{F}\Gamma(2(t-s))(\xi)\, d\xi \longrightarrow \mathcal{F}\Gamma(2(t-s))(x_0), \quad \text{as } n \to \infty,$$

since $\xi \mapsto \mathcal{F}\Gamma(2(t-s))(\xi)$ is continuous. Therefore, using Fatou's Lemma, we can conclude that

$$\begin{aligned}\liminf_{n\to\infty} \mathbb{E}&\left[|\langle \mathcal{F}(u(t)), \psi_n\rangle|^2\right] \\ &\geq v_d(\delta/2)^d \liminf_{n\to\infty} \left[n^d \int_0^t \int_{\mathbb{R}^d} \tilde{\phi}_n(x_0-\xi)\mathcal{F}\Gamma(2(t-s))(\xi)\, d\xi ds\right] \\ &\geq v_d(\delta/2)^d \lim_{n\to\infty} n^d \times \int_0^t \lim_{n\to\infty} \int_{\mathbb{R}^d} \tilde{\phi}_n(x_0-\xi)\mathcal{F}\Gamma(2(t-s))(\xi)\, d\xi ds \\ &= \infty,\end{aligned}$$

since

$$\begin{aligned}\int_0^t \lim_{n\to\infty} \int_{\mathbb{R}^d} &\tilde{\phi}_n(x_0-\xi)\mathcal{F}\Gamma(2(t-s))(\xi)\, d\xi ds \\ &= \int_0^t \mathcal{F}\Gamma(2(t-s))(x_0)\, ds > 0.\end{aligned}$$

This contradicts the fact that X was assumed to be locally mean-square integrable. □

As already mentioned, an effective way to remove the lack of integrability is to multiply by some regular test function with compact support. That product will reveal the local regularity.

Lemma. *Fix $t > 0$. For any $\phi \in C_c^\infty(\mathbb{R}^d)$, the Fourier transform of $\phi \cdot u(t)$ is given by*

$$\mathcal{F}(\phi \cdot u(t))(\xi) = \int_0^t \int_{\mathbb{R}^d} \mathcal{F}\left(\phi \cdot \Gamma(t-s, \bullet - y)\right)(\xi)\, W(dsdy). \tag{3.20}$$

Moreover, it is a Gaussian process, and with probability one $u(t) \in H^r_{loc}$, for all $r < -d/2 + 1$.

Proof. Fix $\phi \in C_c^\infty(\mathbb{R}^d)$ and $\psi \in \mathcal{S}(\mathbb{R}^d)$. By the definition of the Fourier transform and equation (3.5),

$$
\begin{aligned}
\langle \mathcal{F}(\phi \cdot u(t)), \psi \rangle &= \langle \phi \cdot u(t), \mathcal{F}(\psi) \rangle = \langle u(t), \phi \cdot \mathcal{F}(\psi) \rangle \\
&= \int_0^t \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} \phi(x) \mathcal{F}(\psi)(x) \, \Gamma(t-s, x-y) \, dx \right) W(dsdy) \\
&= \int_0^t \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} \psi(x) \mathcal{F}\left(\phi \cdot \Gamma(t-s, \bullet - y)\right)(x) \, dx \right) W(dsdy) \\
&= \int_{\mathbb{R}^d} \psi(x) \int_0^t \int_{\mathbb{R}^d} \mathcal{F}\left(\phi \cdot \Gamma(t-s, \bullet - y)\right)(x) \, W(dsdy),
\end{aligned}
$$

thanks to the Plancherel and Fubini theorems, and therefore, equation (3.20) follows. We verify Fubini's hypothesis. We consider the finite measure space $(\mathbb{R}^d, |\psi(x)| \, dx)$, and we need to show that the following integral is finite:

$$
\begin{aligned}
&\int_0^t ds \int_{\mathbb{R}^d} dy \int_{\mathbb{R}^d} |\psi(x)| \, dx \, |\mathcal{F}(\phi \cdot \Gamma(t-s, \bullet - y))(x)|^2 \\
&\qquad \le \|\psi\|_{L^1(\mathbb{R}^d)} \int_0^t ds \int_{\mathbb{R}^d} dy \sup_{x \in \mathbb{R}^d} |\mathcal{F}(\phi \cdot \Gamma(t-s, \bullet - y))(x)|^2 \\
&\qquad \le \|\psi\|_{L^1(\mathbb{R}^d)} \int_0^t ds \int_{\mathbb{R}^d} dy \left| \int_{\mathbb{R}^d} |\phi(x)| \, \Gamma(t-s, x-y) \, dx \right|^2 \\
&\qquad = \|\psi\|_{L^1(\mathbb{R}^d)} \int_0^t \left\| |\phi| * \Gamma(t-s) \right\|^2_{L^2(\mathbb{R}^d)} \, ds \\
&\qquad \le \|\psi\|_{L^1(\mathbb{R}^d)} \int_0^t \|\phi\|^2_{L^2(\mathbb{R}^d)} \, \|\Gamma(t-s)\|^2_{L^1(\mathbb{R}^d)} \, ds \\
&\qquad = t \, \|\psi\|_{L^1(\mathbb{R}^d)} \, \|\phi\|^2_{L^2(\mathbb{R}^d)} < \infty.
\end{aligned}
$$

The last inequality is the convolution inequality (B.2). The next careful computations will yield the proposed local Sobolev exponent $r < -d/2 + 1$.

$$
\begin{aligned}
\mathbb{E}\left[\left| \mathcal{F}\left(\phi \cdot u(t)\right)(\xi) \right|^2 \right] &= \int_0^t \int_{\mathbb{R}^d} \left| \mathcal{F}\left(\phi \cdot \Gamma(t-s, \bullet - y)\right)(\xi) \right|^2 \, dsdy \\
&= \int_0^t ds \int_{\mathbb{R}^d} dy \int_{\mathbb{R}^d} dx \, e^{2\pi i \xi \cdot x} \phi(x) \Gamma(t-s, x-y) \\
&\qquad \times \int_{\mathbb{R}^d} dz \, e^{-2\pi i \xi \cdot z} \phi(z) \Gamma(t-s, z-y) \\
&= \int_0^t ds \int_{\mathbb{R}^d} dx \, e^{2\pi i \xi \cdot x} \phi(x) \int_{\mathbb{R}^d} dz \, e^{-2\pi i \xi \cdot z} \phi(z) \\
&\qquad \times \int_{\mathbb{R}^d} dy \, \Gamma(t-s, x-y) \Gamma(t-s, z-y)
\end{aligned}
$$

$$= \int_0^t ds \int_{\mathbb{R}^d} dx\, e^{2\pi i\xi\cdot x}\phi(x) \int_{\mathbb{R}^d} dz\, e^{-2\pi i\xi\cdot z}\phi(z)\Gamma(2(t-s), z-x),$$

where the last equality comes from the semi-group property of the fundamental solution, equation (A.6). Using the fact the Fourier transform of a product is the convolution of the Fourier transforms, applied to the functions $\phi(z)$ and $z \mapsto \Gamma(2(t-s), z-x)$, we get

$$\begin{aligned}
&\mathbb{E}\left[\left|\mathcal{F}\left(\phi\cdot u(t)\right)(\xi)\right|^2\right] \\
&\quad= \int_0^t ds \int_{\mathbb{R}^d} dx\, e^{2\pi i\xi\cdot x}\phi(x) \int_{\mathbb{R}^d} dz\, \mathcal{F}\phi(z) e^{-2\pi i(\xi-z)\cdot x} e^{-8\pi^2(t-s)|\xi-z|^2} \\
&\qquad= \int_0^t ds \int_{\mathbb{R}^d} dz\, \mathcal{F}\phi(z) e^{-8\pi^2(t-s)|\xi-z|^2} \int_{\mathbb{R}^d} dx\, e^{2\pi i z\cdot x}\phi(x) \\
&\qquad\qquad= \int_0^t ds \int_{\mathbb{R}^d} dz\, \left|\mathcal{F}\phi(z)\right|^2 e^{-8\pi^2(t-s)|\xi-z|^2}.
\end{aligned}$$

To evaluate the last double integral, we will use the following easy facts. $(1-e^{-x})/x \le 2/(1+x)$, for all $x \ge 0$, and thus

$$\int_0^t e^{-\lambda s} ds = \frac{1-e^{-\lambda t}}{\lambda} = t\frac{1-e^{-\lambda t}}{\lambda t} \le t\frac{2}{1+\lambda t} = \frac{2}{1/t+\lambda}. \tag{3.21}$$

Therefore,

$$\int_0^t e^{-8\pi^2(t-s)|\xi-z|^2}\, ds \le \frac{2}{1/t + 8\pi^2\left|\xi-z\right|^2} \le \frac{2\max(t, 1/8\pi^2)}{1+\left|\xi-z\right|^2} \le c\frac{1+|z|^2}{1+|\xi|^2},$$

by equation (C.1) in Appendix C, with $c = 4\max(t, 1/8\pi^2)$. Hence

$$\begin{aligned}
\mathbb{E}\left[\left|\mathcal{F}\left(\phi\cdot u(t)\right)(\xi)\right|^2\right] &\le \frac{c}{1+|\xi|^2} \int_{\mathbb{R}^d} \left|\mathcal{F}\phi(z)\right|^2 (1+|z|^2)\, dz \\
&= \frac{c}{1+|\xi|^2} \left\|\phi\right\|_{H^1}^2 < \infty,
\end{aligned}$$

since $\phi \in C_0^\infty$. In fact, we can conclude that

$$\begin{aligned}
\mathbb{E}\left[\left\|\phi\cdot u(t)\right\|_{H^r}^2\right] &= \int_{\mathbb{R}^d} d\xi\, (1+|\xi|^2)^r \mathbb{E}\left[\left|\mathcal{F}\left(\phi\cdot u(t)\right)(\xi)\right|^2\right] \\
&\le c\left\|\phi\right\|_{H^1}^2 \int_{\mathbb{R}^d} d\xi\, (1+|\xi|^2)^{r-1}.
\end{aligned}$$

The last integral is finite if and only if $2(r-1) < -d$ which is equivalent to $r < -d/2+1$. □

3.5 Regularity of the distributional solution $u_D(t)$

We have the following result.

Lemma 3.13. *In the sense of distributions, the Fourier transform of $u_D(t)$ is a random function given by*

$$\mathcal{F}(u_D(t))(\xi) = \int_0^t \int_D \left(\int_D e^{-2\pi i \xi \cdot x} G_D(t-s,x,y)\, dx \right) W(dsdy). \tag{3.22}$$

Moreover, it is a mean-square bounded Gaussian process, and with probability one $u_D(t) \in H^r$, for all $r < -d/2$.

Proof. Recall that the definition of $u_D(t)$ is given in equation (3.6). By Plancherel identity, we get

$$\begin{aligned}
\langle \mathcal{F}(u_D(t)), \psi \rangle &:= \langle u_D(t), \mathcal{F}\psi \rangle \\
&= \int_0^t \int_D \left(\int_D \mathcal{F}(\psi)(x) G_D(t-s,x,y)\, dx \right) W(dyds) \\
&= \int_0^t \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} dx\, \psi(x) \int_{\mathbb{R}^d} d\xi\, e^{-2\pi i x \cdot \xi}\, \mathbf{1}_D(\xi)\, \mathbf{1}_D(y) G_D(t-s,\xi,y) \right) W(dyds) \\
&= \int_0^t \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} dx\, \psi(x) g(s,x,y) \right) W(dyds),
\end{aligned}$$

for $g(x,s,y) = \int_{\mathbb{R}^d} e^{-2\pi i x\cdot\xi}\, \mathbf{1}_D(\xi)\, \mathbf{1}_D(y) G_D(t-s,\xi,y)\, d\xi$. In order to use Fubini's theorem, applied to the finite measure space $(\mathbb{R}^d, |\psi(x)|\, dx)$, we need to check that the following integral is finite:

$$\begin{aligned}
\int_0^t ds \int_{\mathbb{R}^d} dy \int_{\mathbb{R}^d} dx\ |\psi(x)|\, |g(s,x,y)|^2 &\le \int_0^t ds \int_D dy \int_{\mathbb{R}^d} |\psi(x)|\ dx \\
&= t\, |D|\, \|\psi\|_{L^1(\mathbb{R}^d)} < \infty,
\end{aligned}$$

since $|g(s,x,y)| \le \mathbf{1}_D(y) \int_D G_D(t-s,\xi,y)\, d\xi \le \mathbf{1}_D(y)$, by inequality (A.31). Therefore,

$$\begin{aligned}
\langle \mathcal{F}(u_D(t)), \psi \rangle &= \int_{\mathbb{R}^d} dx\, \psi(x) \left(\int_0^t \int_{\mathbb{R}^d} g(s,x,y)\, W(dyds) \right) \\
&= \int_{\mathbb{R}^d} dx\, \psi(x) \int_0^t \int_D \left(\int_D e^{-2\pi i x\cdot\xi} G_D(t-s,\xi,y)\, d\xi \right) W(dyds),
\end{aligned}$$

which proves equation (3.22) and the fact that it is a well-defined stochastic integral. Moreover,

$$\begin{aligned}\mathbb{E}\left[|\mathcal{F}(u_D(t))(\xi)|^2\right] &= \int_0^t \int_D \left|\int_D e^{-2\pi i\xi\cdot x} G_D(t-s,x,y)\,dx\right|^2 dsdy \\ &\leq \int_0^t \int_D \left|\int_D G_D(t-s,x,y)\,dx\right|^2 dsdy \\ &\leq \int_0^t ds \int_D dy = t\,|D| < \infty,\end{aligned}$$

which proves that the process $\xi \mapsto \mathcal{F}(u_D(t))(\xi)$ is a mean-square bounded Gaussian process. To conclude the proof, we observe that

$$\begin{aligned}\mathbb{E}\left[\|u_D(t)\|_{H^r}^2\right] &= \int_{\mathbb{R}^d} d\xi\,(1+|\xi|^2)^r \mathbb{E}\left[|\mathcal{F}\left(u_D(t)\right)(\xi)|^2\right] \\ &\leq t\,|D| \int_{\mathbb{R}^d} d\xi\,(1+|\xi|^2)^r,\end{aligned}$$

which is finite if $r < -d/2$. □

In a similar manner, an analogous formula is also valid for the Fourier transform of $\phi \cdot u_D(t)$:

Lemma 3.14. *Fix $t > 0$. For any $\phi \in C_c^\infty(\mathbb{R}^d)$, we have*

$$\mathcal{F}(\phi \cdot u_D(t))(\xi) = \int_0^t \int_D \mathcal{F}\left(\phi\, \mathbf{1}_D\, G_D(t-s,\bullet,y)\right)(\xi)\, W(dsdy). \tag{3.23}$$

Moreover,

$$\mathbb{E}\left[|\mathcal{F}\left(\phi \cdot u_D(t)\right)(\xi)|^2\right] \leq t \cdot \|\phi\|_{L^2(\mathbb{R}^d)}^2 < \infty.$$

Proof. We use the same derivation as in the proof of Lemma 3.6, except we replace every occurence of $\Gamma(t-s,x-y)$ by $G_D(t-s,x,y)$ and $\mathbb{R}^d$ by D. In order to verify Fubini's hypothesis, we make use of inequality (A.31).

$$\begin{aligned}\mathbb{E}\left[|\mathcal{F}\left(\phi \cdot u_D(t)\right)(\xi)|^2\right] &= \int_0^t ds \int_D dy\, |\mathcal{F}\left(\phi\,\mathbf{1}_D\, G_D(t-s,\bullet,y)\right)(\xi)|^2 \\ &= \int_0^t ds \int_D dy \left|\int_D e^{-2\pi i x\cdot\xi} \phi(x) G_D(t-s,x,y)\,dx\right|^2 \\ &\leq \int_0^t ds \int_D dy \left|\int_D |\phi(x)|\, G_D(t-s,x,y)\,dx\right|^2 \\ &\leq \|\phi\|_{L^2(D)}^2 \int_0^t ds = t\,\|\phi\|_{L^2(D)}^2 .\end{aligned}$$

where the last inequality is derived by Cauchy-Schwarz inequatliy,

$$\left|\int_D \phi(x) G_D(s,x,y)\,dx\right|^2 \le \int_D |\phi(x)|^2\, G_D(s,x,y)\,dx \cdot \int_D G_D(s,x,y)\,dx$$
$$\le \int_D |\phi(x)|^2\, G_D(s,x,y)\,dx.$$

This concludes the proof. □

3.6 Convergence in local Sobolev spaces

We know that both distributional solutions $u(t)$ and $u_D(t)$ belong to some H^s_{loc} space. We shall see that their difference $u(t) - u_D(t)$ can be more regular that each of them, provided the boundary of the domain is sufficiently regular. We now prove Theorem 3.7

We will proceed as in the proof of Theorem 3.4. First we will consider the difference $u(t) - u_D(t)$ and evaluate it for some test function $\phi \in C_c^\infty(\mathbb{R}^d)$, whose support $D' := \operatorname{supp}(\psi) \subseteq D$. We set $\delta = \operatorname{dist}(D', D)$. We should need some constant $\nu = \nu(\delta)$.

All computations will remain valid if each occurence of D is replaced by the dilation LD. Each occurence of δ and ν should also be replaced by $\delta(L) = \operatorname{dist}(D', LD)$ and $\nu(L)$. This is possible thanks to Corollary 3.11.

We are interested in the values of $r \in \mathbb{R}$, for which the integral

$$\int_{\mathbb{R}^d} \left(1 + |\xi|^2\right)^r |\mathcal{F}(\phi\,[u(t) - u_D(t)])(\xi)|^2\; d\xi$$

is finite. Its expectation is given by

$$\int_{\mathbb{R}^d} d\xi \left(1 + |\xi|^2\right)^r \mathbb{E}\left[|\mathcal{F}(\phi\,[u(t) - u_D(t)])(\xi)|^2\right]$$
$$= \int_{\mathbb{R}^d} d\xi \left(1 + |\xi|^2\right)^r \mathbb{E}\left[I^2 + J^2\right],$$

where

$$I = \int_0^t \int_D \mathcal{F}\left(\phi \cdot F_D(t-s, \bullet, y)\right)(\xi)\, W(dsdy),$$
$$J = \int_0^t \int_{\mathbb{R}^d \setminus D} \mathcal{F}\left(\phi \cdot \Gamma(t-s, \bullet - y)\right)(\xi)\, W(dsdy).$$

Indeed, we can write the difference $\mathcal{F}(\phi \cdot u(t))(\xi) - \mathcal{F}(\phi \cdot u_D(t))(\xi)$ as

$$\int_0^t \int_{\mathbb{R}^d} \mathcal{F}\left(\phi \cdot \Gamma(t-s, \bullet - y)\right)(\xi)\, W(dsdy)$$
$$- \int_0^t \int_D \mathcal{F}\left(\phi\, \mathbf{1}_D\, G_D(t-s, \bullet, y)\right)(\xi)\, W(dsdy),$$

by (3.20) and (3.23). Thus,

$$\begin{aligned}
&\mathcal{F}(\phi \cdot u(t))(\xi) - \mathcal{F}(\phi \cdot u_D(t))(\xi) \\
&\qquad = \int_0^t \int_D \mathcal{F}\left(\phi \cdot F_D(t-s, \bullet, y)\right)(\xi)\, W(dsdy) \\
&\qquad - \int_0^t \int_{\mathbb{R}^d \setminus D} \mathcal{F}\left(\phi \cdot \Gamma(t-s, \bullet - y)\right)(\xi)\, W(dsdy) = I - J.
\end{aligned}$$

Both stochastic integrals have zero mean and are independent, indeed the integrals are over the disjoint sets $[0,t] \times D$ and $[0,t] \times \mathbb{R}^d \setminus D$. Thus, we have $\mathbb{E}[|I-J|^2] = \mathbb{E}[I^2] + \mathbb{E}[J^2]$.

We start with the computation of $\mathbb{E}[I^2]$. Fix $\xi \in \mathbb{R}^d$ and $\beta \in \mathbb{N}^d$ a multi-index with $|\beta| \leq n$. Later, β will depend on ξ. We will prove

$$\mathbb{E}[I^2] \leq \frac{C}{|\xi^\beta|^2} \, \|\phi\|_{H^n}^2 \, t e^{-\delta^2/ct} \nu. \tag{3.24}$$

Using integration by parts $|\beta|$ times,

$$\begin{aligned}
\mathbb{E}[I^2] &= \int_0^t \int_D \left|\mathcal{F}\left(\phi \cdot F_D(t-s, \bullet, y)\right)(\xi)\right|^2 \, dsdy \\
&= \int_0^t \int_D \left| \int_{\mathbb{R}^d} e^{-2\pi i \xi \cdot x} \phi(x) F_D(s,x,y)\, dx \right|^2 \, dsdy \\
&= \int_0^t \int_D \left| \int_{\mathrm{supp}(\phi)} \frac{e^{-2\pi i \xi \cdot x}}{(2\pi i)^{|\beta|} \xi^\beta} \frac{\partial^{|\beta|} \left(\phi \cdot F_D(s, \bullet, y)\right)}{\partial x^\beta}(x)\, dx \right|^2 \, dsdy.
\end{aligned}$$

We estimate now the inner integral. By the Leibniz derivation formula, and inequality (3.14)

$$\begin{aligned}
&\left| \int_{\mathrm{supp}(\phi)} \frac{e^{-2\pi i \xi \cdot x}}{(2\pi i)^{|\beta|} \xi^\beta} \frac{\partial^{|\beta|} \left(\phi \cdot F_D(s, \bullet, y)\right)}{\partial x^\beta}(x)\, dx \right| \\
&\qquad \leq \frac{C}{|\xi^\beta|} \int_{\mathrm{supp}(\phi)} \sum_{\gamma \leq \beta} \binom{\beta}{\gamma} \left| \frac{\partial^{|\beta - \gamma|} \phi}{\partial x^{\beta - \gamma}}(x) \right| \left| \frac{\partial^{|\gamma|} F_D}{\partial x^\gamma}(s,x,y) \right| dx \\
&\qquad \leq \frac{C}{|\xi^\beta|} \sum_{\gamma \leq \beta} \frac{e^{-\delta^2/cs}}{s^{|\gamma|/2}} \int_{\mathrm{supp}(\phi)} \left| \frac{\partial^{|\beta - \gamma|} \phi}{\partial x^{\beta - \gamma}}(x) \right| \Gamma_c(s, x-y)\, dx \\
&\qquad\qquad \leq \frac{C}{|\xi^\beta|} \sum_{\gamma \leq \beta} \frac{e^{-\delta^2/cs}}{s^{|\gamma|/2}} \left(\left| \frac{\partial^{|\beta - \gamma|} \phi}{\partial x^{\beta - \gamma}} \right| * \Gamma_c(s) \right)(y).
\end{aligned}$$

Thus, using the convolution inequality (B.2),

$$\mathbb{E}[I^2] \le \frac{C}{|\xi^\beta|^2} \int_0^t \left(\left\| \sum_{\gamma\le\beta} \frac{e^{-\delta^2/cs}}{s^{|\gamma|/2}} \left(\left| \frac{\partial^{|\beta-\gamma|}\phi}{\partial x^{\beta-\gamma}} \right| * \Gamma_c(s) \right) \right\|_{L^2(\mathbb{R}^d)} \right)^2 ds$$
$$\le \frac{C}{|\xi^\beta|^2} \int_0^t \left(\sum_{\gamma\le\beta} \frac{e^{-\delta^2/cs}}{s^{|\gamma|/2}} \left\| \frac{\partial^{|\beta-\gamma|}\phi}{\partial x^{\beta-\gamma}} \right\|_{L^2(\mathbb{R}^d)} \cdot \|\Gamma_c(s)\|_{L^1(\mathbb{R}^d)} \right)^2 ds.$$

Using Cauchy-Schwarz inequality, $\left(\sum_i a_i b_i\right)^2 \le \left(\sum_i a_i^2\right)\left(\sum_i b_i^2\right)$,

$$\mathbb{E}[I^2] \le \frac{C}{|\xi^\beta|^2} \|\phi\|^2_{H^n} \sum_{\gamma\le\beta} \int_0^t \frac{e^{-2\delta^2/cs}}{s^{|\gamma|}} ds$$

In the latter, we used the equivalent norm on H^n, given by the sum of all L^2 norms of the partial derivatives of order up to $|\beta| = n$, see Appendix C. Rewriting $1/s^{|\gamma|} = (\delta^{2|\gamma|}/s^{|\gamma|})/\delta^{2|\gamma|}$, and recalling (A.21), we can conclude

$$\mathbb{E}[I^2] \le \frac{C}{|\xi^\beta|^2} \|\phi\|^2_{H^n}\, t e^{-\delta^2/ct} \sum_{\gamma\le\beta} \frac{1}{\delta^{2|\gamma|}},$$

which proves (3.24) for $\nu = \sum_{\gamma\le\beta} \frac{1}{\delta^{2|\gamma|}}$.

We will show a similar bound

$$\mathbb{E}[J^2] \le \frac{C}{|\xi^\beta|^2} \|\phi\|^2_{H^n}\, t e^{-\delta^2/ct} \nu. \tag{3.25}$$

In a similar way, using integration by parts and Leibniz derivation formula, we get

$$\mathbb{E}[J^2] \le \frac{C}{|\xi^\beta|^2} \int_0^t ds \int_{\mathbb{R}^d\setminus D} dy \left| \sum_{\gamma\le\beta} \int_{\mathbb{R}^d} \left| \frac{\partial^{|\beta-\gamma|}\phi}{\partial x^{\beta-\gamma}}(x) \right| \left| \frac{\partial^{|\gamma|}\Gamma}{\partial x^\gamma}(s, x-y) \right| dx \right|^2.$$

An application of the usual Minkowski inequality in $L^2((0,t)\times(\mathbb{R}^d \setminus D))$ enables to take the sum out

$$\frac{C}{|\xi^\beta|^2} \left[\sum_{\gamma\le\beta} \left(\int_0^t ds \int_{\mathbb{R}^d\setminus D} dy \left| \int_{\mathbb{R}^d} \left| \frac{\partial^{|\beta-\gamma|}\phi}{\partial x^{\beta-\gamma}}(x) \right| \left| \frac{\partial^{|\gamma|}\Gamma}{\partial x^\gamma}(s, x-y) \right| dx \right|^2 \right)^{1/2} \right]^2.$$

Each partial derivative of the heat kernel is bounded as in (A.22). If we follow a similar reasoning as the one leading to (3.15), we get

$$\mathbb{E}[J^2] \leq \frac{C}{|\xi^\beta|^2} \left[\sum_{\gamma \leq \beta} \left\| \frac{\partial^{|\beta-\gamma|}\phi}{\partial x^{\beta-\gamma}} \right\|_{L^2(\mathbb{R}^d)} \left(\int_0^t \frac{ds}{s^{|\gamma|}} \left| \int_{\|x\|\geq\delta} \Gamma_c(s,x)\,dx \right|^2 \right)^{1/2} \right]^2$$

$$\leq \frac{C}{|\xi^\beta|^2} \|\phi\|_{H^n}^2 \sum_{\gamma\leq\beta} \int_0^t \frac{ds}{s^{|\gamma|}} \left(\int_{\|x\|\geq\delta} \Gamma_c(s,x)\,dx \right)^2$$

The constant $c > 1$ will increase from line to line. Using bounds (3.16) and (A.21),

$$\begin{aligned} \mathbb{E}[J^2] &\leq \frac{C}{|\xi^\beta|^2} \|\phi\|_{H^n}^2 \sum_{\gamma\leq\beta} \int_0^t \frac{ds}{s^{|\gamma|}} e^{-\frac{\delta^2}{4cs}} \\ &= \frac{C}{|\xi^\beta|^2} \|\phi\|_{H^n}^2 \sum_{\gamma\leq\beta} \frac{1}{\delta^{2|\gamma|}} \int_0^t \frac{\delta^{2|\gamma|}}{s^{|\gamma|}} e^{-\frac{\delta^2}{4cs}}\,ds \\ &\leq \frac{C}{|\xi^\beta|^2} \|\phi\|_{H^n}^2\, te^{-\frac{\delta^2}{4ct}} \sum_{\gamma\leq\beta} \frac{1}{\delta^{2|\gamma|}}, \end{aligned}$$

which proves (3.25).

In conclusion, we have proved that

$$\mathbb{E}\left[\int_{\mathbb{R}^d} d\xi \left(1+|\xi|^2\right)^r |\mathcal{F}(\phi(u(t)-u_D(t)))(\xi)|^2 \right] \leq C\|\phi\|_{H^n}^2\, te^{-\delta^2/ct}\nu \int_{\mathbb{R}^d} d\xi \left(1+|\xi|^2\right)^r \frac{1}{|\xi^\beta|^2} \quad (3.26)$$

We will now make use of the fact that β was any multi-index satisfying $0 \leq |\beta| \leq n$. For each $\xi \in \mathbb{R}^d$, we can choose β that maximizes $|\xi^\beta|$. For $\xi \in \mathbb{R}^d$ in the unit ball, we can choose $\beta = 0$, so that $|\xi^\beta| = 1$. For $\xi \in \mathbb{R}^d$ outside of the unit ball, at least one component satisfies $|\xi_i| \geq |\xi|/\sqrt{d}$, and setting $\beta_i = n$ yields $|\xi^\beta| \geq d^{-n/2}|\xi|^n$. This makes the integral in the right hand side of (3.26) converge for any value $r < n - d/2$.

This concludes the proof Theorem 3.7.

Addendum. Using Lemma 4.8, we can again simplify the derivation of $\mathbb{E}[J^2]$.

Final remark A similar reasoning can be applied to the case of vanishing Neumann boundary conditions. Every bound involving the Green function should be replaced by either Lemma 4.7, Corollary 3.11 or the remark following it.

3.7 Non-vanishing initial data

In this section, we generalize both convergence results, Theorem 3.4 and Theorem 3.7, to the case of a non-vanishing (random) initial data $u_0 : \Omega \times \mathbb{R}^d \to \mathbb{R}$ independent of $\dot{W}$ and bounded in expectation, i.e., for some $p \geq 2$,

$$\|u_0\| := \sup_{x \in \mathbb{R}^d} \mathbb{E}\left[|u_0(x)|^p\right]^{1/p} < \infty. \tag{3.27}$$

Recall that D is assumed to be an open bounded domain containing the origin and with boundary of class $C^{2+n+\alpha}$, for some integer $n \geq 0$ and $\alpha \in (0,1)$. We consider the weak formulations associated to the following initial value problems:

$$\begin{cases} \frac{\partial u}{\partial t}(t,x) = \Delta u(t,x) + \dot{W}, & t > 0,\, x \in \mathbb{R}^d, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^d, \end{cases} \tag{3.28}$$

and

$$\begin{cases} \frac{\partial u_D}{\partial t}(t,x) = \Delta u_D(t,x) + \dot{W}, & t > 0,\, x \in D, \\ u_D(t,x) = 0, & t > 0,\, x \in \partial D \\ u_D(0,x) = u_0(x), & x \in D, \end{cases} \tag{3.29}$$

where $\dot{W}$ is white noise on $\mathbb{R}_+ \times \mathbb{R}^d$. To find their respective weak formulation, we informally multiply by some test function $\psi(x)$, integrate in time and in space, and apply integration by parts.

The weak formulation to problem (3.28) is to find a process $\{u(t) : t > 0\}$ such that for all $\psi \in \mathcal{S}(\mathbb{R}^d)$,

$$\langle u(t), \psi \rangle = \int_{\mathbb{R}^d} u_0(y)\psi(y)\, dy + \int_0^t \langle u(s), \Delta\psi \rangle\, ds + \int_0^t \int_{\mathbb{R}^d} \psi(y)\, W(dyds). \tag{3.30}$$

The weak formulation to problem (3.29) is to find a process $\{u_D(t) : t > 0\}$ such that for all $\psi \in \mathcal{S}(D)$,

$$\langle u_D(t), \psi \rangle = \int_D u_0(y)\psi(y)\, dy + \int_0^t \langle u_D(s), \Delta\psi \rangle\, ds + \int_0^t \int_D \psi(y)\, W(dyds), \tag{3.31}$$

where $\mathcal{S}(D) = \{\psi \in C^\infty(\bar{D}) : \psi = 0 \text{ on } \partial D\}$.

Theorem 3.15. *There exists a unique process $\{u(t) : t > 0\}$ with values in $\mathcal{S}'(\mathbb{R}^d)$ which satisfies* (3.30). *It is given by*

$$\langle u(t), \psi \rangle := I_0^t(\psi) + \int_0^t \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} \psi(x)\Gamma(t-s, x-y)\, dx \right) W(dyds), \tag{3.32}$$

where Γ is the heat kernel in d-space dimensions given by (A.8)*, and*

$$I_0^t(\psi) := \int_{\mathbb{R}^d} dx\, \psi(x) \int_{\mathbb{R}^d} dy\, \Gamma(t, x-y) u_0(y)$$

is the contribution of the initial condition.

Proof. Using Theorem 3.1, it is sufficient to verify that

$$I_0^t(\psi) = \int_{\mathbb{R}^d} u_0(y)\psi(y)\,dy + \int_0^t I_0^s(\Delta\psi)\,ds.$$

Using Fubini's theorem, we get

$$\int_0^t I_0^s(\Delta\psi)\,ds = \int_{\mathbb{R}^d} dy\,u_0(y) \int_0^t ds \int_{\mathbb{R}^d} dx\,\Delta\psi(x)\Gamma(s, x-y).$$

Using twice integration by parts in the x variable, and recalling both properties (A.5) satisfied by Γ, we get

$$\begin{aligned}\int_0^t I_0^s(\Delta\psi)\,ds &= \int_{\mathbb{R}^d} dy\,u_0(y) \int_{\mathbb{R}^d} dx\,\psi(x) \int_0^t ds\,\frac{\partial\Gamma}{\partial t}(s, x-y)\\ &= \int_{\mathbb{R}^d} dy\,u_0(y) \int_{\mathbb{R}^d} dx\,\psi(x)\Gamma(t, x-y) - \int_{\mathbb{R}^d} u_0(y)\psi(y)\,dy,\end{aligned}$$

since $\Gamma(0, x-y) = \delta(x-y)$. □

Theorem 3.16. *There exists a process $\{u_D(t) : t > 0\}$ with values in $\mathcal{S}'(\mathbb{R}^d)$ which satisfies (3.31), for any $\psi \in \mathcal{S}(D) \cap \mathcal{S}(\mathbb{R}^d)$. It can be extended to a stochastic process $\{u(t,\psi) : t > 0, \psi \in \mathcal{S}(D)\}$; this process is unique. It is given by*

$$\langle u_D(t), \psi\rangle := J_0^t(\psi) + \int_0^t \int_D \left(\int_D \psi(x) G_D(t-s, x, y)\,dx\right) W(dyds) \quad (3.33)$$

where G_D is the Green function for the heat equation in D with Dirichlet boundary conditions, and

$$J_0^t(\psi) := \int_D dx\,\psi(x) \int_D dy\,G_D(t, x, y) u_0(y),$$

is the contribution of the initial condition. As a consequence, the support of each distribution $u_D(t)$ is contained in $\bar{D}$.

Proof. Using Theorem 3.2, it is sufficient to verify that

$$J_0^t(\psi) = \int_D u_0(y)\psi(y)\,dy + \int_0^t J_0^s(\Delta\psi)\,ds.$$

Fix $y \in D$, then two applications of integration by parts lead to

$$\int_D \Delta\psi(x) G_D(s, x, y)\,dx = \int_D \psi(x)\Delta_x G_D(s, x, y)\,dx,$$

since $\psi(x) = 0 = G(s,x,y)$ for any $x \in \partial D$. Then recalling the properties (A.26) satisfied by the Green function, we get

$$\begin{aligned}\int_0^t J_0^s(\Delta\psi)\,ds &= \int_D dy\, u_0(y) \int_0^t ds \int_D dx\, \Delta\psi(x) G_D(s,x,y) \\ &= \int_D dy\, u_0(y) \int_D dx\, \psi(x) \int_0^t ds\, \frac{\partial G_D}{\partial t}(s,x,y) \\ &= \int_D dy\, u_0(y) \int_D dx\, \psi(x) G_D(t,x,y) - \int_D u_0(y)\psi(y)\,dy,\end{aligned}$$

since $G_D(0,x,y) = \delta(x-y)$. □

We check that indeed the initial contributions I_0^t and J_0^t are Schwartz distributions.

Lemma 3.17. *If the initial condition u_0 satisfies* (3.27)*, then there exist versions of I_0^t and J_0^t with value in $\mathcal{S}'(\mathbb{R}^d)$.*

Proof. Let $(\psi_n)_{n=0}^{\infty} \subseteq \mathcal{S}(\mathbb{R}^d)$ be a sequence converging to 0 in $\mathcal{S}(\mathbb{R}^d)$. It is enough to show that both $I_0^t(\psi_n)$ and $J_0^t(\psi_n)$ converge to 0 in $L^2(\Omega)$.

$$\begin{aligned}\mathbb{E}\left[\left|I_0^t(\psi_n)\right|^2\right] &= \mathbb{E}\bigg[\int_{\mathbb{R}^d} dx\, \psi_n(x) \int_{\mathbb{R}^d} dy\, \Gamma(t,x-y) u_0(y) \\ &\qquad \times \int_{\mathbb{R}^d} dv\, \psi_n(v) \int_{\mathbb{R}^d} dw\, \Gamma(t,v-w) u_0(w)\bigg] \\ &= \int_{\mathbb{R}^d} dx\, \psi_n(x) \int_{\mathbb{R}^d} dv\, \psi_n(v) \int_{\mathbb{R}^d} dy\, \Gamma(t,x-y) \\ &\qquad \times \int_{\mathbb{R}^d} dw\, \Gamma(t,v-w) \mathbb{E}\left[u_0(y)u_0(w)\right] \\ &\le \|u_0\|^2 \int_{\mathbb{R}^d} dx\, |\psi_n(x)| \int_{\mathbb{R}^d} dv\, |\psi_n(v)| = \|u_0\|^2 \|\psi_n\|^2_{L^1(\mathbb{R}^d)},\end{aligned}$$

which converges to 0. We have used the fact that $\int_{\mathbb{R}^d} \Gamma(t,x-y)\,dy = 1$ for all $x \in \mathbb{R}^d$. The argument for J_0^t is similar, we instead use the fact that $\int_D G_D(t,x,y)\,dy \le 1$ for all $x \in D$. □

3.7.1 Convergence in distributions

The objectif of this section is to generalize Theorem 3.4 to the present case of random non-vanishing initial data.

Theorem 3.18. *For any open domain D and any test function $\psi \in C_c^\infty(\mathbb{R}^d)$ such that* $\mathrm{supp}(\psi) \subseteq LD$*, we have*

$$\|\langle u(t),\psi\rangle - \langle u_L(t),\psi\rangle\|_{L^2(\Omega)} \le C \max\left(\|\psi\|_{L^1(\mathbb{R}^d)}, \|\psi\|_{L^2(\mathbb{R}^d)}\right) \sqrt{t} e^{-\frac{\delta(L)^2}{4ct}},$$

for all $t > 0$. In fact, both constants C and c depend neither on D, $L > 0$, nor ψ. In particular, if D contains the origin, then for L large enough,

$$\|\langle u(t), \psi\rangle - \langle u_L(t), \psi\rangle\|_{L^2(\Omega)} \leq C \max\left(\|\psi\|_{L^1(\mathbb{R}^d)}, \|\psi\|_{L^2(\mathbb{R}^d)}\right) \sqrt{t} e^{-\frac{L^2}{4ct}}.$$

Proof. As in Section 3.3, we rewrite the difference

$$\langle u(t), \psi\rangle - \langle u_D(t), \psi\rangle = \sum_{i=1}^{6} I_i(t),$$

where

$$\begin{aligned}
I_1(t) &= \int_0^t \int_D \left(\int_D \psi(x) \left[\Gamma(t-s, x-y) - G_D(t-s, x, y)\right] dx\right) W(dyds),\\
I_2(t) &= \int_0^t \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d \setminus D} \psi(x)\Gamma(t-s, x-y)\, dx\right) W(dyds),\\
I_3(t) &= \int_0^t \int_{\mathbb{R}^d \setminus D} \left(\int_D \psi(x)\Gamma(t-s, x-y)\, dx\right) W(dyds),\\
I_4(t) &= \int_D dy\, u_0(y) \int_D dx\, \psi(x) \left[\Gamma(t, x-y) - G_D(t, x, y)\right],\\
I_5(t) &= \int_{\mathbb{R}^d} dy\, u_0(y) \int_{\mathbb{R}^d \setminus D} dx\, \psi(x)\Gamma(t, x-y),\\
I_6(t) &= \int_{\mathbb{R}^d \setminus D} dy\, u_0(y) \int_D dx\, \psi(x)\Gamma(t, x-y).
\end{aligned}$$

Observe that $I_i(t)$ and $I_{i+3}(t)$ play a similar role, for $i = 1, 2, 3$. To handle $I_{i+3}(t)$, we will use similar ideas as for $I_i(t)$.

By assumption, $\mathrm{supp}(\psi) \subseteq D$, and therefore $I_5(t)$ is identically zero. To simplify notations, we set $D' := \mathrm{supp}(\psi)$, and $\delta = \mathrm{dist}(D', \partial D)$. By Cauchy-Schwarz inequality, we have

$$\begin{aligned}
\mathbb{E}\left[|I_6(t)|^2\right] &= \mathbb{E}\Bigg[\int_{\mathbb{R}^d \setminus D} dy\, u_0(y) \int_{D'} dx\, \psi(x)\Gamma(t, x-y)\\
&\quad \times \int_{\mathbb{R}^d \setminus D} dw\, u_0(w) \int_{D'} dv\, \psi(v)\Gamma(t, v-w)\Bigg]\\
&= \int_{\mathbb{R}^d \setminus D} dy \int_{D'} dx\, \psi(x)\Gamma(t, x-y)\\
&\quad \times \int_{\mathbb{R}^d \setminus D} dw \int_{D'} dv\, \psi(v)\Gamma(t, v-w)\mathbb{E}\left[u_0(y)u_0(w)\right]\\
&\leq \|u_0\|^2 \left(\int_{\mathbb{R}^d \setminus D} dy \int_{D'} dx\, |\psi(x)|\, \Gamma(t, x-y)\right)^2.
\end{aligned}$$

By Lemma 4.8,

$$\int_{\mathbb{R}^d\setminus D} dy \int_{D'} dx \; |\psi(x)|\, \Gamma(t, x-y) \leq c^{d/2} e^{-\frac{\delta^2}{4ct}} \left\| |\psi| * \Gamma_c(t) \right\|_{L^1(\mathbb{R}^d)}$$
$$= c^{d/2} \left\| \psi \right\|_{L^1(\mathbb{R}^d)} e^{-\frac{\delta^2}{4ct}},$$

and thus

$$\left\| I_6(t) \right\|_{L^2(\Omega)} \leq c^{d/2} \left\| u_0 \right\| \left\| \psi \right\|_{L^1(\mathbb{R}^d)} e^{-\frac{\delta^2}{4ct}}.$$

For the remaining term $I_4(t)$, as in the computations of $I_6(t)$, we can bound

$$\mathbb{E}\left[|I_4(t)|^2 \right] \leq \left\| u_0 \right\|^2 \left(\int_D dy \int_{D'} dx \; |\psi(x)|\, F_D(t,x,y) \right)^2,$$

where $F_D = \Gamma - G_D$. As in the computations of $I_1(t)$, using Lemma 4.5, we can bound

$$\int_D dy \int_{D'} dx \; |\psi(x)|\, F_D(t,x,y)$$
$$\leq C \exp\left(-\frac{\delta^2}{4ct} \right) \int_{\mathbb{R}^d} dy \int_{\mathbb{R}^d} dx \; |\psi(x)|\, \Gamma_c(t, y-x)$$
$$= C \exp\left(-\frac{\delta^2}{4ct} \right) \left\| \psi \right\|_{L^1(\mathbb{R}^d)}.$$

Therefore,

$$\left\| I_4(t) \right\|_{L^2(\Omega)} \leq C \left\| u_0 \right\| \left\| \psi \right\|_{L^1(\mathbb{R}^d)} \exp\left(-\frac{\delta^2}{4ct} \right).$$

Replacing D by LD, and setting $\delta(L) = \operatorname{dist}(D', LD)$, we observe that each bound for $I_i(t)$ remains valid. This concludes the proof. □

3.7.2 Convergence in Sobolev spaces

The objectif of this section is to generalize Theorem 3.7 to the case of non-random and non-vanishing initial data.

Theorem 3.19. *Suppose u_0 is a deterministic bounded Borel function. Let $\alpha \in (0,1)$ and $n \in \mathbb{N}$. Fix a bounded domain D containing the origin whose boundary belongs to $C^{2+n+\alpha}$. Then $u_L(t) \to u(t)$ in H^r_{loc}, for all $r < n - d/2$, as $L \to \infty$, where $u(t)$ and $u_L(t)$ are defined in* (3.32) *and* (3.33). *The rate of convergence is the same as in Theorem 3.7.*

In particular, if ∂D is C^∞, then with probability one the difference $u(t) - u_L(t) \in C^\infty(K)$ for any compact set $K \subseteq LD$.

Remark. A similar reasoning can be applied to the case of vanishing Neumann boundary conditions. See the final remark of Section 3.6.

If u_0 is non-random, the contributions of the initial conditions are distributions given by the following two functions $I_0^t : \mathbb{R}^d \to \mathbb{R}$ and $J_0^t : \mathbb{R}^d \to \mathbb{R}$ defined by

$$I_0^t(x) = \int_{\mathbb{R}^d} u_0(y)\Gamma(t, x-y)\, dy,$$
$$J_0^t(x) = \mathbf{1}_D(x) \int_D u_0(y) G_D(t, x, y)\, dy.$$

We analyse their regularity now.

Lemma 3.20. *The function I_0^t is $C_b^\infty(\mathbb{R}^d) \subseteq \mathcal{S}'$ and the function J_0^t is $C^{2+n+\alpha}(D) \cap C_0(\mathbb{R}^d) \subseteq \mathcal{S}'$. Furthermore, let $\phi \in C_0^\infty(D)$. Then $\phi \cdot I_0^t \in \mathcal{S}(\mathbb{R}^d) \subset H^s(\mathbb{R}^d)$, for all $s \in \mathbb{R}$, and $\phi \cdot J_0^t \in H^s(\mathbb{R}^d)$, for all $s < 2 + n + \alpha$.*

Proof. Because u_0 is assumed to be bounded, and the derivatives of the heat kernel satisfy bound (A.22), dominated convergence theorem implies that

$$\frac{\partial^{|\beta|} I_0^t}{\partial x^\beta}(x) = \int_{\mathbb{R}^d} u_0(y) \frac{\partial^{|\beta|}\Gamma}{\partial x^\beta}(t, x-y)\, dy = \int_{\mathbb{R}^d} u_0(x-y) \frac{\partial^{|\beta|}\Gamma}{\partial x^\beta}(t, y)\, dy.$$

Thus every partial derivative of I_0^t is bounded, whence $I_0^t \in C_b^\infty(\mathbb{R}^d)$. For $\phi \in C_0^\infty(D)$, it is clear that $\phi \cdot I_0^t \in C_c^\infty(\mathbb{R}^d) \subset \mathcal{S}(\mathbb{R}^d)$.

Using bounds (3.9) and (3.11), we can deduce that $J_0^t \in C^{2+n+\alpha}(D)$, and thus $\phi \cdot J_0^t \in C_0^{2+n+\alpha}(\mathbb{R}^d) \subseteq H^s(\mathbb{R}^d)$, for any $s < 2+n+\alpha$ by Lemma C.3. □

We need to show that the distribution J_0^t converges to the distribution I_0^t in the local versions of the Sobolev spaces. We are interested in the functions $f : \mathbb{R}^d \to \mathbb{R}$ and $g : D \to \mathbb{R}$ defined by

$$f(x) := \int_{\mathbb{R}^d \setminus D} u_0(y)\Gamma(t, x-y)\, dy,$$
$$g(x) := \int_D u_0(y) \left[\Gamma(t, x-y) - G_D(t, x, y)\right] dy,$$

so that

$$\phi(x) \cdot (I_0^t(x) - J_0^t(x)) = \phi(x) \cdot (f(x) + g(x)),$$

for all $x \in D$. For $\phi \in C_0^\infty(D)$, we set $D' = \operatorname{supp}(\phi)$, and $\delta := \operatorname{dist}(D', \partial D)$.

Lemma 3.21. *The function f is $C^\infty(\mathbb{R}^d)$. Furthermore, for all $k \in \mathbb{N}$,*

$$\|\phi \cdot f\|_{H^k(\mathbb{R}^d)} \le C \|u_0\| \|\phi\|_{H^k(\mathbb{R}^d)} e^{-\frac{\delta^2 \lambda}{4t}} \nu(\delta), \tag{3.34}$$

where $C = C(k, \lambda, d)$, $\lambda \in (0, 1)$, and $\nu(\delta) = O(1)$ as $\delta \to \infty$.

We use the equivalent definitions of the Sobolev spaces and their related norms, see Appendix C.

Proof. We observe first that $f \in C^\infty(\mathbb{R}^d)$, with

$$\partial^\beta f(x) = \int_{\mathbb{R}^d \setminus D} u_0(y) \partial_x^\beta \Gamma(t, x-y)\, dy.$$

In order to show the desired inequality, it is sufficient to evaluate the $L^2(\mathbb{R}^d)$ norms of each partial derivative of the form $\partial^\alpha \phi\, \partial^\beta f$, for $|\alpha + \beta| \leq k$.

$$\begin{aligned}
&\int_{\mathbb{R}^d} \left| \partial^\alpha \phi(x)\, \partial^\beta f(x) \right|^2 dx \\
&\qquad = \int_{D'} dx\, |\partial^\alpha \phi(x)|^2 \left(\int_{\mathbb{R}^d \setminus D} dy\, u_0(y) \partial_x^\beta \Gamma(t, x-y) \right)^2 \\
&\qquad\qquad \leq \|u_0\|^2_{L^\infty(\mathbb{R}^d)} \int_{\mathbb{R}^d} dx\, |\partial^\alpha \phi(x)|^2 \\
&\qquad\qquad\qquad \times \left(\int_{\mathbb{R}^d} dy\, \mathbf{1}_{D'}(x)\, \mathbf{1}_{\mathbb{R}^d \setminus D}(y) \left| \partial_x^\beta \Gamma(t, x-y) \right| \right)^2,
\end{aligned}$$

Using Lemmas 4.8 and A.22, we can bound

$$\mathbf{1}_{D'}(x)\, \mathbf{1}_{\mathbb{R}^d \setminus D}(y) \left| \partial_x^\beta \Gamma(t, x-y) \right| \leq \frac{C}{t^{|\beta|}} e^{-\frac{\delta^2}{4ct}} \Gamma_c(t, x-y),$$

for some $C = C(d, c, \beta)$. Therefore,

$$\left\| \partial^\alpha \phi\, \partial^\beta f \right\|^2_{L^2(\mathbb{R}^d)} \leq \|u_0\|^2_{L^\infty(\mathbb{R}^d)} \|\partial^\alpha \phi\|^2_{L^2(\mathbb{R}^d)} \frac{C}{t^{|\beta|}} e^{-\frac{\delta^2}{2ct}}.$$

Rewriting $\frac{1}{t^{|\beta|}} = \frac{\delta^{2|\beta|}}{t^{|\beta|}} \frac{1}{\delta^{2|\beta|}}$ and setting

$$\nu(\delta) = \left(\sum_{|\beta| \leq k} \frac{1}{\delta^{2|\beta|}} \right)^{1/2},$$

we can conclude the proof using bound (A.21). □

Lemma 3.22. *The function g is $C^{2+n+\alpha}(D)$. Furthermore,*

$$\|\phi \cdot g\|_{H^{2+n}(\mathbb{R}^d)} \leq C \|u_0\| \|\phi\|_{H^{2+n}(\mathbb{R}^d)} e^{-\frac{\delta^2}{4ct}} \nu(\delta), \tag{3.35}$$

where both constants $c, C = c(d, \alpha, n, T, D)$, $\nu(\delta) = O(1)$ as $\delta \to \infty$.

Proof. Using bounds (3.10) and (3.12), we can deduce that $g \in C^{2+n+\alpha}(D)$ and thus $\phi \cdot g \in C_0^{2+n+\alpha}(\mathbb{R}^d) \subseteq H^s(\mathbb{R}^d)$, for any $s < 2 + n + \alpha$.

In order to show the desired inequality, it is sufficient to evaluate every partial derivative of the form $\partial^\alpha \phi \, \partial^\beta g$, for $|\alpha + \beta| \le 2 + n$. Using (3.14)

$$\begin{aligned}
\int_{\mathbb{R}^d} \left| \partial^\alpha \phi(x) \, \partial^\beta g(x) \right|^2 dx &= \int_{D'} dx \; |\partial^\alpha \phi(x)|^2 \left| \int_D dy \, u_0(y) \partial_x^\beta F_D(t, x, y) \right|^2 \\
&\le \|u_0\|^2_{L^\infty(\mathbb{R}^d)} \int_{D'} dx \; |\partial^\alpha \phi(x)|^2 \, \frac{C}{t^{|\beta|}} e^{-\frac{\operatorname{dist}(x, \partial D)^2}{2ct}} \\
&\le \|u_0\|^2_{L^\infty(\mathbb{R}^d)} \, \|\partial^\alpha \phi\|^2_{L^2(\mathbb{R}^d)} \, \frac{C}{t^{|\beta|}} e^{-\frac{\delta^2}{2ct}},
\end{aligned}$$

where both $C, c = c(d, T, D, n, \alpha)$. This concludes the proof. □

Recall that $\phi(x) \cdot (I_0^t(x) - J_0^t(x)) = \phi(x) \cdot (f(x) + g(x))$, for $x \in D$. Corollary 3.11 enables us to replace D by the scaled version LD in the preceding calculations. This concludes the proof of Theorem 3.19.

Chapter 4

Multiplicative correlated noise

Up to now, we have considered space-time white noise. In one space dimension with random field solutions and in higher dimension with distributional solutions. In this chapter, we will study the heat equation in any space dimension with other type of noises, those that are white in time and correlated in space. We shall fix some open bounded domain $D \subseteq \mathbb{R}^d$ containing the origin. In what follows, D will either be the d-dimensional square $(-1,1)^d$ or have a regular boundary. As in the previous chapter, we shall consider the dilation $LD = \{Lx \in \mathbb{R}^d : x \in D\}$, for any $L > 0$.

We will compare the behavior of the (random field) solution on the whole space with the (random field) solution on the dilated domain LD with vanishing Dirichlet boundary conditions, i.e.

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) = \Delta u(t,x) + \sigma(t,x,u(t,x))\,\dot{\mathrm{M}}, & t>0,\ x \in \mathbb{R}^d, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^d, \end{cases} \tag{4.1}$$

and

$$\begin{cases} \dfrac{\partial u_L}{\partial t}(t,x) = \Delta u_L(t,x) + \sigma(t,x,u_L(t,x))\,\dot{\mathrm{M}}, & t>0,\ x \in LD, \\ u_L(t,x) = 0, & t>0,\ x \in \partial(LD) \\ u_L(0,x) = u_0(x), & x \in LD, \end{cases} \tag{4.2}$$

where σ and u_0 satisfy the d-dimensional versions of (2.6) and (2.7), and $\dot{\mathrm{M}}$ is a noise on $\mathbb{R}_+ \times \mathbb{R}^d$ that is white in time and correlated in space. For an evident notational reason, we shall write $\sigma(u(t,x))$ instead of $\sigma(t,x,u(t,x))$.

As already mentioned, some trade off between the roughness of the noise and the regularity of the integrant must be imposed. Set

$$k(s,\nu) := \int_{\mathbb{R}^d} \Lambda(dz)\,\Gamma_\nu(s,z), \quad \lambda(t,\nu) := \int_0^t k(s,\nu)\,ds, \tag{4.3}$$

where Γ_ν is the heat kernel defined by (A.8), and Λ is the measure associated with the noise $\dot{\mathrm{M}}$. We ask the latter measure to satisfies the following properties.

Assumption 4.1. The positive measure Λ is tempered, symmetric, positive definite, and satisfies $\lambda(t,\nu) < \infty$ for all $t > 0$ (and for all $\nu > 0$).

Those assumptions guarantee that stochastic integration can be defined in the sense of Walsh as a (worthy) martingale measure. In the present case of the heat equation, we need to impose $\lambda(t,\nu) < \infty$ for all $t > 0$. If this assumption is satisfied by some $\nu > 0$, then it is satisfied for all $\nu > 0$. It is sometimes called Dalang's condition, and it may also be formulated with the help of the spectral measure as

$$\Upsilon(\beta) := \int_{\mathbb{R}^d} \frac{\mu(d\xi)}{\beta + |\xi|^2} < \infty, \quad \text{for some and hence for all } \beta > 0. \tag{4.4}$$

By computing (4.3), we get

$$\int_0^t k(s,\nu)\,ds = \int_0^t ds \int_{\mathbb{R}^d} \mu(d\xi)\, e^{-4\nu\pi^2 s|\xi|^2} = \int_{\mathbb{R}} \mu(d\xi)\, \frac{1 - e^{-4\nu\pi^2 t|\xi|^2}}{4\nu\pi^2\,|\xi|^2},$$

and the last fraction is equivalent to $(1/t + 4\nu\pi^2\,|\xi|^2)^{-1}$, see (3.21). With the help of the spectral measure, we observe that $k(t,\nu) = k(\nu t, 1)$ and that it is decreasing, i.e. $k(t,\beta) \leq k(t,\nu)$, for $\nu \leq \beta$.

Under Assumption 4.1, it is known that equations (4.1) and (4.2) admit unique random field solutions, that satisfy the representation formulas

$$u(t,x) = \int_{\mathbb{R}^d} \Gamma(t, x-y) u_0(y)\,dy + \int_0^t \int_{\mathbb{R}^d} \Gamma(t-s, x-y)\,\sigma(u(s,y))\,\mathrm{M}(ds,dy), \tag{4.5}$$

and

$$u_L(t,x) = \int_{LD} G_L(t,x,y) u_0(y)\,dy + \int_0^t \int_{LD} G_L(t-s,x,y)\sigma(u_L(s,y))\,\mathrm{M}(ds,dy), \tag{4.6}$$

where Γ is the heat kernel, given by (A.8), and G_L is the Green function of the heat equation on the bounded domain LD associated with Dirichlet boundary conditions, see Proposition A.6 for its existence and some basic properties. See [11] for existence and uniqueness of these random field solutions.

4.1 Main result and general ideas

We will prove that if $x \in LD$ is sufficiently far away from the boundary points, then both solutions u and u_L are very close to one another. In fact, we shall attain a similar convergence result to that of Chapter 2, given in Theorem 2.4. Therefore, the present results generalize those of Chapter 2.

The difference of the random field solutions (4.5) and (4.6) satisfies the following convergence rate, for any $p \geq 2$.

Theorem 4.2. *Uniformly for $t \in [0,T]$, $L > 0$, and $x \in LD$, we have*

$$\|u(t,x) - u_L(t,x)\|_{L^p(\Omega)} \leq \Theta(t) \exp\left(-\frac{\mathrm{dist}(x, \partial LD)^2}{4ct}\right),$$

for some increasing function $\Theta(t) = \Theta(t, T, Lip, K, \Lambda, u_0, D, p)$ and some positive constant $c = c(D)$.

Remark. It is important to emphasize that both the function Θ and the constant c do not depend on the dilation constant $L > 0$.

Remark. In the present case of Dirichlet boundary conditions, it is possible to find a positive constant c that works for all open set D bounded or not. See Lemma 4.5. In fact, we can also find $\Theta(t) = \Theta(t, T, Lip, K, \Lambda, u_0, p)$.

We shall use a similar approach as that of Chapter 2. Some steps will require only minor changes, whereas the crucial induction step will turn out to be very different.

First, we apply the fact that the p-moments of the solutions are uniformly bounded.

Proposition 4.3. *The solutions u and u_L to equations* (4.1) *and* (4.2) *satisfy*

$$\sup_{t \in [0,T]} \sup_{x \in \mathbb{R}} \mathbb{E}\left[|u(t,x)|^p\right] < \infty, \tag{4.7}$$

$$\sup_{t \in [0,T]} \sup_{L > 0} \sup_{x \in LD} \mathbb{E}\left[|u_L(t,x)|^p\right] < \infty. \tag{4.8}$$

Bound (4.7) is already well-known, see Dalang [11, Theorem 13]. The second bound (4.8) is obtained in a very similar way, in which we need to bound the Green function G_L by the heat kernel. The trick is to find a bound that is valid independently of the scaling factor $L > 0$, see (4.12). The procedure is as in Section 2.4, with some obvious changes.

Once the uniform bounds are obtained, we can use the representation formulas to write the difference

$$u(t,x) - u_L(t,x) = I_0(t,x) + I_2(t,x) + I_4(t,x) + I_6(t,x), \tag{4.9}$$

where,

$$I_0(t,x) = \int_{\mathbb{R}^d} \Gamma(t, x-y)\, u_0(y)\, dy - \int_{LD} G_L(t,x,y)\, u_0(y)\, dy,$$
$$I_2(t,x) = \int_0^t \int_{LD} [\Gamma(t-s, x-y) - G_L(t-s,x,y)]\sigma(u_L(s,y))\, \mathrm{M}(ds,dy),$$
$$I_4(t,x) = \int_0^t \int_{LD} \Gamma(t-s,x-y)[\sigma(u(s,y)) - \sigma(u_L(s,y))]\, \mathrm{M}(ds,dy),$$
$$I_6(t,x) = \int_0^t \int_{\mathbb{R}^d \setminus LD} \Gamma(t-s,x-y)\sigma(u(s,y))\, \mathrm{M}(ds,dy).$$

The first two terms I_0 and I_2 will require to bound the difference

$$F_L(t,x,y) := \Gamma(t,x-y) - G_L(t,x,y)$$

between the heat kernel and the Green function. Again, the trick is to find a bound that is valid independently of the scaling factor $L > 0$, see (4.12). The last term I_6 will require some integral bounds of the heat kernel on $\mathbb{R}^d \setminus LD$, see Lemma 4.8. The term I_4 will produce a recursive argument. More precisely, if we set

$$\begin{aligned} f_L(t,x) &:= \|u(t,x) - u_L(t,x)\|_{L^p(\Omega)}, \\ h(t,c) &:= \sup_{x \in LD} \frac{f_L(t,x)^2}{J_c(t,x)^2}, \end{aligned} \tag{4.10}$$

where

$$J_c(t,x) = \exp\left\{ -\frac{\operatorname{dist}(x, \partial LD)^2}{4ct} \right\},$$

then we shall find, see (4.24), that

$$h(t) \le c_1 + c_2(k * \mathbf{1})(t) + \frac{c_3}{2}(k \triangleright h)(t).$$

Both functions k and h are given by $k(s) = k(s,c)$ and $h(s) = h(s,c)$. The constants c, c_1, c_2, and c_3 shall not depend on the scaling factor $L > 0$. Convolution is represented by $*$, whereas $\triangleright$ will turn out to be a non-associative and non-commutative operator. In fact, the $\triangleright$ operator is sometimes comparable to convolution, see Lemma 4.27.

At this point, we do not know whether the function h is finite valued. Informal iterations and limits of the latter inequality yield the following educated guess

$$h(t) \le c_1 + [c_1 + c_2/c_3] \cdot (\mathcal{K} * \mathbf{1})(t), \tag{4.11}$$

where

$$\mathcal{K}(t) = \sum_{m=1}^{\infty} c_3^m k^{*m}(t).$$

Thus, we could set $\Theta(t) = \sqrt{c_1 + [c_1 + c_2/c_3] \cdot (\mathcal{K} * \mathbf{1})(t)}$. In order to confirm the validity of (4.11), we shall go back to the Picard iteration scheme defining both random field solutions u and u_L.

Finally, we shall extend both Theorem 4.2 and Proposition 4.3 in the following two directions. First, we shall consider more general equations, see (4.29). Second, we may admit more general vanishing boundary conditions. For square domains, we can require each side to have either Dirichlet or Neumann boundary conditions. For domains with regular boundary, we can have Neumann boundary conditions. Yet, Robin boundary conditions are not possible with the present method. This has to do with the scaling property of the Green function.

4.2 Some Prerequisites

The following inequalities describe the bounds, mentioned earlier, about the Green function and the difference of the Green function with the heat kernel. Under some assumptions on the domain D, and for some $l > 0$, we have

$$\begin{aligned} |G_L(t,x,y)| &\leq C\Gamma_c(t, x-y), \\ |\Gamma(t,x-y) - G_L(t,x,y)| &\leq C\Gamma_c(t,x-y)J_c(t,x)J_c(t,y), \end{aligned} \tag{4.12}$$

for all $t \in (0,T]$, $L \geq l$, and $x, y \in LD$. Both constants c and C depend only on T, l, and the domain D, but not on the scaling factor $L \geq l$.

In some specific cases, explicit values for both constants C and c are available, yet we do not, in general, pursue the optimal ones. We now give some examples of domains, for which inequalities (4.12) are satisfied:

- In the particular case of vanishing Dirichlet boundary conditions, the first inequality of (4.12) is satisfied with $C = c = 1$, see (A.30). In one space dimension, with $LD = (-L, L)$, the second inequality of (4.12) is satisfied with $C = c = 4$, see Lemma 2.7.

- In the particular case of vanishing mixed boundary conditions in one space dimension, with $LD = (-L, L)$, the first inequality of (4.12) is satisfied with $C = 2$ and $c = 1$, see (A.81). The second inequality of (4.12) is satisfied with $C = 12$ and $c = 4$, see Lemma 2.10.

- In the particular case of vanishing Neumann boundary conditions in one space dimension, with $LD = (-L, L)$, the second inequality of (4.12) is satisfied with $C = 16\theta(4L^2/t)$ and $c = 4$, see Lemma 2.13. The first inequality of (4.12) is in fact a consequence of the second one. Indeed,

  $$|G_L(t,x,y)| \leq |\Gamma(t,x-y) - G_L(t,x,y)| + \Gamma(t,x-y) \leq C\Gamma_c(t,x-y),$$

  for $C = 16\theta(4L^2/t) + 2^d$ and $c = 4$.

Observe that in the first two cases, the bounds are valid for all $L > 0$, and $t > 0$. In the third case, we have to restrict to $4L^2/T \geq l > 0$, because the theta function θ is decreasing and unbounded close to the origin. In that case, $\theta(4L^2/t) \leq \theta(l)$, for all $L \geq \sqrt{l/4T}$ and $t \in (0, T]$.

Square domains: The Green function associated to the square domain $D = (-L, L) \times \cdots \times (-L, L) \subseteq \mathbb{R}^d$ is given by the product of the one-dimensional Green functions associated to $(-L, L)$, see equation (A.27). For example, in dimension $d = 3$, if we are interested in the following vanishing boundary conditions:

$$u_L(t, -L, x_2, x_3) = 0, \quad u_L(t, x_1, -L, x_3) = 0, \quad \frac{\partial u_L}{\partial x_3}(t, x_1, x_2, -L) = 0,$$
$$u_L(t, L, x_2, x_3) = 0, \quad \frac{\partial u_L}{\partial x_2}(t, x_1, L, x_3) = 0, \quad \frac{\partial u_L}{\partial x_3}(t, x_1, x_2, L) = 0,$$

then its Green function is given by

$$G_L(t, x, y) = G_L^D(t, x_1, y_1) G_L^M(t, x_2, y_2) G_L^N(t, x_3, y_3), \tag{4.13}$$

where G_L^D, G_L^M, and G_L^N are the one dimensional Green functions associated with Dirichlet, mixed, and Neumann boundary conditions, respectively. As in Chapter 2, if only Dirichlet or mixed boundary conditions are imposed, then we shall find results uniformly for $L > 0$. If Neumann boundary condition is imposed, then we shall restrict to $L \geq \sqrt{l/4T}$, for any $l > 0$. For examples of this distinction, see Theorems 2.4 and 2.5 and Proposition 2.6.

Lemma 4.4. *For all $t \in (0, T]$, and $x, y \in LD = (-L, L)^d$, we have*

$$|G_L(t, x, y)| \leq C\Gamma_c(t, x - y),$$
$$|\Gamma(t, x - y) - G_L(t, x, y)| \leq C\Gamma_c(t, x - y) J_c(t, x) J_c(t, y),$$

where both C and c do not depend on the scaling variable L.

We need to restrict to finite time horizon only when Neumann boundary conditions are present.

Proof. The first estimate is a consequence of the examples given after (4.12). In the case of (4.13), we have

$$|G_L(t, x, y)| \leq 1 \cdot 2 \cdot (\theta(l) + 2^d) \cdot \Gamma_c(t, x - y),$$

for $c = 4$.

The second estimate is a consequence of Lemmas 2.7, 2.10, and 2.13, together with Lemma 5.16. For example, in the case of (4.13), we have

$$\begin{aligned}
\Gamma(t, x - y) - G_L(t, x, y) = \\
&\left[\Gamma(t, x_1 - y_1) - G_L^D(t, x_1, y_1)\right] \Gamma(t, x_2 - y_2) \Gamma(t, x_3 - y_3) \\
&+ G_L^D(t, x_1, y_1) \left[\Gamma(t, x_2 - y_2) - G_L^M(t, x_2, y_2)\right] \Gamma(t, x_3 - y_3) \\
&\quad + G_L^D(t, x_1, y_1) G_L^M(t, x_2, y_2) \left[\Gamma(t, x_3 - y_3) - G_L^N(t, x_3, y_3)\right],
\end{aligned}$$

and thus,

$$\begin{aligned}|\Gamma(t,x-y)-G_L(t,x,y)| &\le C\Gamma_c(t,x-y)J_c(t,x_1)J_c(t,y_1)\\ &+C\Gamma_c(t,x-y)J_c(t,x_2)J_c(t,y_2)+C\Gamma_c(t,x-y)J_c(t,x_3)J_c(t,y_3)\\ &\le C\Gamma_c(t,x-y)J_c(t,x)J_c(t,y),\end{aligned}$$

since

$$J_c(t,x)=\max_{i\in\{1,2,3\}} J_c(t,x_i)=\max_{i\in\{1,2,3\}} \exp\left\{-\frac{\min\left[|x_i-L|,|x_i+L|\right]^2}{4ct}\right\}.$$

The latter translates the fact that the distance of any point $x\in(-L,L)^d$ to the boundary is given by the smallest distance

$$\min\left[|x_1-L|,|x_1+L|,|x_2-L|,|x_2+L|,|x_3-L|,|x_3+L|\right].$$

This completes the proof. □

In the case of Dirichlet boundary conditions, we can show the following:

Lemma 4.5. *There exist two positive constants C and c, such that for any open set $D\subseteq\mathbb{R}^d$, we have*

$$0\le\Gamma(t,x-y)-G_D(t,x,y)\le C\Gamma_c(t,x-y)J_c(t,x)J_c(t,y), \tag{4.14}$$

for all $t>0$ and $x,y\in D$, where $J_c(t,x)=J_c(t,x;D)=e^{-\frac{\mathrm{dist}(x,\partial D)^2}{4ct}}$.

The proof relies on the fact that the present result is valid for cubes, see Lemma 4.4, which in turn was proven from the one dimensional case, see Lemma 2.7, which holds for arbitrary (large or small) interval.

Proof. The value of both constants C and c are not important. Without loss of generality, we can assume that $\mathrm{dist}(x,\partial D)\ge\mathrm{dist}(y,\partial D)$. Thus, $J_c(t,x)=J_{2c}(t,x)J_{2c}(t,x)\le J_{2c}(t,x)J_{2c}(t,y)$, and it is sufficient to show that $\Gamma(t,x-y)-G_D(t,x,y)\le C\Gamma_c(t,x-y)J_c(t,x)$. We consider two cases, whether or not $y\in B$, the open ball centered at x with radius $\mathrm{dist}(x,\partial D)/\sqrt{d}$.

First, we consider the case $y\in B$. Let Q be any open cube centered at x with side length $2\,\mathrm{dist}(x,\partial D)/\sqrt{d}$. Thus, $y\in B\subseteq Q\subseteq D$. Then, by (A.33),

$$\begin{aligned}\Gamma(t,x-y)-G_D(t,x,y)&\le\Gamma(t,x-y)-G_Q(t,x,y)\\ &\le\Gamma_c(t,x-y)J_c(t,x;Q)J_c(t,y;Q),\end{aligned}$$

for some C and c independent of the side length of the cube. Because $\mathrm{dist}(x,\partial D)=\sqrt{d}\,\mathrm{dist}(x,\partial Q)$, we have $J_c(t,x;Q)=J_{cd}(t,x;D)$, and hence

$$\Gamma(t,x-y)-G_D(t,x,y)\le\Gamma_c(t,x-y)J_{cd}(t,x;D),$$

since $J_c(t,y;Q) \leq 1$.

Second, we consider the case $y \notin B$, i.e. $|x-y|^2 \geq \mathrm{dist}(x,\partial D)^2/d$. We bound, for any $c \geq 2$,

$$\begin{aligned}\Gamma(t,x-y) - G_D(t,x-y) \leq \Gamma(t,x-y) &= c^{d/2}\Gamma_c(t,x-y)e^{-\frac{|x-y|^2}{4ct}}\\ &\leq c^{d/2}\Gamma_c(t,x-y)e^{-\frac{\mathrm{dist}(x,\partial D)^2}{4cdt}}\\ &= c^{d/2}\Gamma_c(t,x-y)J_{cd}(t,x;D),\end{aligned}$$

which completes the proof. □

Remark. Observe that $c = 4d$ is a possible value.

Corollary 4.6. *In the case of Dirichlet boundary conditions, both inequalities of* (4.12) *are satisfied for all* $t > 0$ *and* $L > 0$.

Regular domains: We consider some open bounded domain $D \subseteq \mathbb{R}^d$ containing the origin and whose boundary belongs to $C^{2+\alpha}$, for some $\alpha \in (0,1)$. In the case of the heat equation with vanishing Dirichlet and/or Neumann boundary conditions, we can observe that the Green function G_{LD} associated to the dilated domain LD is given by

$$G_{LD}(t,x,y) = \frac{1}{L^d}G_D(t/L^2, x/L, y/L), \tag{4.15}$$

for all $t > 0$ and $x, y \in LD$. This scaling formula is a consequence of the fact that if G_D satisfies (A.26), then G_{LD} satisfies the same system of equations in which D is replaced by LD, i.e. it is the Green function associated to the dilated domain LD. (The latter fact is not true if we consider Robin boundary conditions.) Thus,

$$\begin{aligned}F_{LD}(t,x,y) &= \Gamma(t,x-y) - G_{LD}(t,x,y)\\ &= \frac{1}{L^d}\left[\Gamma(t/L^2, x/L - y/L) - G_D(t/L^2, x/L, y/L)\right]\\ &= \frac{1}{L^d}F_D(t/L^2, x/L, y/L),\end{aligned} \tag{4.16}$$

for all $t > 0$ and $x, y \in LD$. For simplicity, we shall write G_L instead of G_{LD}, and F_L instead of F_{LD}.

For Neumann boundary conditions, the following result holds.

Lemma 4.7. *If the boundary* ∂D *belongs to* $C^{2+\alpha}$, *then*

$$\begin{aligned}|G_L(t,x,y)| &\leq C\Gamma_c(t,x-y),\\ |\Gamma(t,x-y) - G_L(t,x,y)| &\leq C\Gamma_c(t,x-y)J_c(t,x)J_c(t,y),\end{aligned}$$

for all $t \in (0, TL^2]$ *and* $x, y \in LD$. *The constant* c *and* C *depend only on* α, d, T, *and the domain* D, *but not on the constant* $L > 0$.

In particular, for $l = 1$, the latter inequalities are satisfied for all $t \in (0, T]$ and $L \geq l$.

Proof. It is a consequence of Corollary 3.11 and the remark following it. □

As we already mentioned, the first inequality of (4.12) is used to find the uniform bounds of Proposition 4.3 and the second inequality of (4.12) is used to bound the p-moment of I_0 and I_2. To bound the p-moment of I_0 and I_6, we shall need the following.

Lemma 4.8. *For any $x \in LD$, and $c \geq 2$,*

$$\Gamma(t, x-y)\, \mathbf{1}_{\mathbb{R}^d \setminus LD}(y) \leq c^{d/2} \Gamma_c(t, x-y) J_c(t, x).$$

Proof. We can rewrite $\Gamma(t, z) = 2^{d/2}\Gamma_2(t, z) \exp\left(-z^2/8t\right)$. We bound

$$\exp\left(-\frac{(x-y)^2}{8t}\right) \mathbf{1}_{\mathbb{R}^d \setminus LD}(y) \leq \exp\left(-\frac{\operatorname{dist}(x, \partial LD)^2}{8t}\right) \leq J_c(t, x),$$

and we use the fact that $\Gamma_{c_1}(t, z) \leq (c_2/c_1)^{d/2}\Gamma_{c_2}(t, z)$, for any $c_2/c_1 > 0$. □

4.3 Proof of uniform bounds, Proposition 4.3

We will prove both bounds (4.7) and (4.8) in very similar ways. If fact, we introduce some notations that will allow a single argument instead of two. The letter $\mathbb{D}$ will stand for either the whole space $\mathbb{R}^d$ or the dilated domain LD. The function v will stand for either u or u_L, the random field solutions to (4.1) or (4.2), respectively. Finally, $G(t, x, y)$ will stand for either the heat kernel $\Gamma(t, x-y)$ or the Green function $G_L(t, x, y)$ associated to the dilated domain LD with Dirichlet boundary conditions. Thanks to the first inequality of (4.12), we can use the fact that

$$|G(t, x, y)| \leq C\Gamma_c(t, x-y).$$

In order to deduce properties of the function v, it is useful to go back to the Picard iteration scheme, from which it was built. We recall it now. The initial condition initiates the recursive definition:

$$v^0(t, x) := \int_{\mathbb{D}} G(t, x, y) u_0(y)\, dy.$$

For $n \geq 0$, we define recursively

$$v^{n+1}(t, x) = v^0(t, x) + \int_0^t \int_{\mathbb{D}} G(t-s, x, y)\, \sigma(v^n(s, y))\, \mathrm{M}(ds, dy).$$

Recalling the fact that the initial condition is bounded in expectation (2.7), an application of Minkowski' inequality gives

$$\left\|v^0(t,x)\right\|_{L^p(\Omega)} \leq \int_{\mathbb{D}} |G(t,x,y)| \; \|u_0(y)\|_{L^p(\Omega)} \; dy \\ \leq C \|u_0\| \int_{\mathbb{R}^d} \Gamma_c(t,x-y)\, dy = C \|u_0\| .$$

Therefore, we have a uniform bound for the initiation of the recursion

$$C_0 := \sup_{t\in[0,T]} \sup_{x\in\mathbb{D}} \left\|v^0(t,x)\right\|_{L^p(\Omega)} \leq C \|u_0\| .$$

We define in a similar way

$$C_n := \sup_{t\in[0,T]} \sup_{x\in\mathbb{D}} \|v^n(t,x)\|_{L^p(\Omega)} .$$

We prove by induction that each C_n is bounded. First, we need to recall that

$$\tau \mapsto M_\tau := \int_0^\tau \int_{\mathbb{D}} G(t-s,x,y)\, \sigma(v^n(s,y))\, \mathrm{M}(ds,dy),$$

for $\tau \in [0,t]$, is a continuous square integrable martingale, whose quadratic variation is given by

$$\langle M\rangle_\tau = \int_0^\tau ds \int_{\mathbb{R}^d} \Lambda(dz) \int_{\mathbb{R}^d} dv \; \mathbf{1}_{\mathbb{D}}(v) G(t-s,x,v)\sigma(v^n(s,v)) \\ \times \mathbf{1}_{\mathbb{D}}(v-z) G(t-s,x,v-z)\sigma(v^n(s,v-z)).$$

By Minkowski's inequality,

$$\|\langle M\rangle_\tau\|_{L^{p/2}(\Omega)} \leq \int_0^\tau ds \int_{\mathbb{R}^d} \Lambda(dz) \int_{\mathbb{R}^d} dv \; \mathbf{1}_{\mathbb{D}}(v)\, |G(t-s,x,v)| \\ \times \mathbf{1}_{\mathbb{D}}(v-z)\, |G(t-s,x,v-z)| \\ \times \|\sigma(v^n(s,v))\sigma(v^n(s,v-z))\|_{L^{p/2}(\Omega)} .$$

By Cauchy-Schwarz inequality and linear growth of the function σ,

$$\|\sigma(v^n(s,v))\sigma(v^n(s,v-z))\|_{L^{p/2}(\Omega)} \\ \leq \|\sigma(v^n(s,v))\|_{L^p(\Omega)} \|\sigma(v^n(s,v-z))\|_{L^p(\Omega)} \leq K^2(1+C_n)^2.$$

Thus, by inequality (4.12) and the semi-group property of the heat kernel,

$$\|\langle M\rangle_\tau\|_{L^{p/2}(\Omega)} \leq C^2K^2(1+C_n)^2 \int_0^\tau ds \int_{\mathbb{R}^d} \Lambda(dz) \\ \times \int_{\mathbb{R}^d} dv\, \Gamma_c(t-s,x-v)\Gamma_c(t-s,x-(v-z)) \\ = C^2K^2(1+C_n)^2 \int_0^\tau ds \int_{\mathbb{R}^d} \Lambda(dz)\Gamma_{2c}(t-s,z) \\ = C^2K^2(1+C_n)^2 \int_0^\tau k(t-s,2c)\, ds.$$

By Burkholder's inequality,

$$\|M_t\|_{L^p(\Omega)} \leq k_p \|\langle M\rangle_t\|_{L^{p/2}(\Omega)}^{1/2},$$

and hence, by Minkowski's inequalitiy,

$$\left\|v^{n+1}(t,x)\right\|_{L^p(\Omega)} \leq \left\|v^0(t,x)\right\|_{L^p(\Omega)} + k_p C K(1+C_n)\left(\int_0^t k(s,2c)\,ds\right)^{1/2}.$$

Therefore,

$$C_{n+1} \leq C\|u_0\| + k_p CK(1+C_n)\lambda(T,2c)^{1/2} < \infty,$$

by Assumption 4.1.

We can now show a much better bound using the Lipschitz assumption instead of the linear growth. We set

$$D_n(t) := \sup_{x\in\mathbb{D}} \left\|v^{n+1}(t,x) - v^n(t,x)\right\|_{L^p(\Omega)}.$$

Notice that

$$v^1(t,x) - v^0(t,x) = \int_0^t \int_{\mathbb{D}} G(t-s,x,y)\,\sigma(v^0(s,y))\,\mathrm{M}(ds,dy)$$

satisfies, with a similar argument as before,

$$\left\|v^1(t,x) - v^0(t,x)\right\|_{L^p(\Omega)} \leq k_p CK(1+\|u_0\|)\lambda(t,2c)^{1/2},$$

and thus

$$\sup_{t\in[0,T]} D_0(t) \leq k_p CK(1+\|u_0\|)\lambda(T,2c)^{1/2} < \infty. \tag{4.17}$$

For any $n \geq 1$, we have

$$\begin{aligned} v^{n+1}(t,x) - v^n(t,x) = \int_0^t \int_{\mathbb{D}} & G(t-s,x,y) \\ & \times \left[\sigma(v^n(s,y)) - \sigma(v^{n-1}(s,y))\right]\,\mathrm{M}(ds,dy). \end{aligned}$$

From a similar argument as before, involving Burkholder's, Minkowski's, and Cauchy-Schwarz' inequalities, we have

$$\begin{aligned} \left\|v^{n+1}(t,x) - v^n(t,x)\right\|_{L^p(\Omega)}^2 &\leq k_p^2 \int_0^t ds \int_{\mathbb{R}^d} \Lambda(dz) \\ &\times \int_{\mathbb{R}^d} dv\; \mathbf{1}_{\mathbb{D}}(v)\,|G(t-s,x,v)|\,\mathbf{1}_{\mathbb{D}}(v-z)\,|G(t-s,x,v-z)| \\ &\times \left\|\sigma(v^n(s,v)) - \sigma(v^{n-1}(s,v))\right\|_{L^p(\Omega)} \\ &\times \left\|\sigma(v^n(s,v-z)) - \sigma(v^{n-1}(s,v-z))\right\|_{L^p(\Omega)}. \end{aligned}$$

From (4.12), the Lipschitz assumption of the function σ, and the semi-group property of the heat kernel, we have

$$
\begin{aligned}
\left\|v^{n+1}(t,x)-v^n(t,x)\right\|_{L^p(\Omega)}^2 &\le k_p^2 C^2 \mathrm{Lip}^2 \int_0^t ds \int_{\mathbb{R}^d} \Lambda(dz) \\
&\quad \times \int_{\mathbb{R}^d} dv\, \Gamma_c(t-s,x-v)\Gamma_c(t-s,x-(v-z)) \\
&\quad \times \left\|v^n(s,v)-v^{n-1}(s,v)\right\|_{L^p(\Omega)} \left\|v^n(s,v-z)-v^{n-1}(s,v-z)\right\|_{L^p(\Omega)} \\
&\le k_p^2 C^2 \mathrm{Lip}^2 \int_0^t D_{n-1}(s)^2 \int_{\mathbb{R}^d} \Lambda(dz)\, \Gamma_{2c}(t-s,z) \\
&= k_p^2 C^2 \mathrm{Lip}^2 \int_0^t D_{n-1}(s)^2 k(t-s,2c).
\end{aligned}
$$

Therefore,

$$
D_n(t)^2 \le k_p^2 C^2 \mathrm{Lip}^2 (k * D_{n-1}^2)(t), \tag{4.18}
$$

for all $t \in [0,T]$, $L \ge l$, and for $k(t) = k(t,2c)$.

Remark. In the present case of Dirichlet boundary conditions, the latter inequality (4.18) is in fact valid for all $t > 0$ and $L > 0$.

The extension of Gronwall's lemma, presented in the paper of Dalang [11, Lemmas 15 and 17], enables to conclude, thanks to (4.17) and the facts that $\int_0^T k(t)\,dt < \infty$, that the following series converges uniformly on $[0,T]$,

$$
\sum_{n=0}^{\infty} D_n(t) \le \bar{C} \sup_{t\in[0,T]} D_0(t) < \infty.
$$

where $\bar{C} = c(T,\Lambda,\mathrm{Lip},p)$. Thus,

$$
\begin{aligned}
\|v(t,x)\|_{L^p(\Omega)} &= \lim_{n\to\infty} \|v^n(t,x)\|_{L^p(\Omega)} \le \sup_{n\ge 0} \|v^n(t,x)\|_{L^p(\Omega)} \\
&\le \left\|v^0(t,x)\right\|_{L^p(\Omega)} + \sum_{n=0}^{\infty} D_n(t) \le C\|u_0\| + \bar{C} \sup_{t\in[0,T]} D_0(t).
\end{aligned}
$$

Therefore, we can conclude

$$
\sup_{t\in[0,T]} \sup_{x\in\mathbb{D}} \|v(t,x)\|_{L^p(\Omega)} \le C\|u_0\| + \bar{C} k_p C K (1+\|u_0\|)\lambda(T,2c)^{1/2}, \tag{4.19}
$$

which is finite by Assumption 4.1. Observe that the latter bound doesn't depend on the scaling variable L. This complete the proof of Proposition 4.3.

4.3.1 Another way to solve Gronwall type inequalities

Instead of using the extension of Gronwall's lemma presented in [11], we can estimate the series $\sum_{n=0}^{\infty} D_n(t)$ using Laplace transform. We will use a method presented in Lemma 3.8 of [2]. Definition of Laplace transform, for some positive function $f : \mathbb{R}_+ \to \mathbb{R}_+$, is

$$\mathcal{L}(f)(\gamma) := \int_0^\infty f(t)\, e^{-\gamma t}\, dt,$$

for $\gamma \in \mathbb{R}$. It acts on convolution as follows $\mathcal{L}(f * g)(\gamma) = \mathcal{L}(f)(\gamma) \cdot \mathcal{L}(g)(\gamma)$, and satisfies, by Hölder's inequality,

$$\begin{aligned}\mathcal{L}(f^{1/p})(\gamma) &= \int_0^\infty f(t)^{1/p} e^{-\gamma t}\, dt \\ &\leq \left(\int_0^\infty f(t) e^{-\gamma t}\, dt\right)^{1/p} \left(\int_0^\infty e^{-\gamma t}\, dt\right)^{1/q} = \mathcal{L}(f)(\gamma)^{1/p} \gamma^{-1/q},\end{aligned}$$

for any $p > 1$ and $1/p + 1/q = 1$.

We are mainly interested in computing $\sum_{n=0}^{\infty} D_n(t)$, for it led to the following upper bound

$$\|v(t,x)\|_{L^p(\Omega)} = \lim_{n\to\infty} \|v^n(t,x)\|_{L^p(\Omega)} \leq \left\|v^0(t,x)\right\|_{L^p(\Omega)} + \sum_{n=0}^{\infty} D_n(t).$$

In fact, we have a more explicit formula.

Proposition 4.9. *There exist two constants $\gamma, c_\gamma > 0$, both depending on (Lip, Λ, p) such that*

$$\sup_{x\in\mathbb{D}} \|v(t,x)\|_{L^p(\Omega)} \leq C \|u_0\| + c_\gamma k_p C K (1 + \|u_0\|) e^{\gamma t},$$

for all $t \in [0,T]$, and $L \geq l$.

At the end of the present section, we give a brief comparison between the present approach and the extension of Gronwall's lemma presented in [11].

Remark. In the present case of Dirichlet boundary conditions, the latter inequality of Proposition 4.9 is in fact valid for all $t > 0$ and $L > 0$.

We first iterate inequality (4.18). To simplify notations, we set $\bar{c} = k_p C\mathrm{Lip}$. For $n \geq 1$,

$$D_n(t)^2 \leq \bar{c}^{2n} (k^{*n} * D_0^2)(t),$$

for all $t \in [0,T]$, where

$$k^{*n} = \underbrace{k * k * \cdots * k}_{n \text{ times}}.$$

Recall the initial bound $D_0(t)^2 \leq k_p^2 C^2 K^2 (1 + \|u_0\|)^2 \lambda(t)$. Thus,

$$\begin{aligned} D_n(t)^2 &\leq k_p^2 C^2 K^2 (1 + \|u_0\|)^2 \bar{c}^{2n} (k^{*n} * \lambda)(t), \\ D_n(t) &\leq k_p C K (1 + \|u_0\|) \bar{c}^n \left[(k^{*n} * \lambda)(t)\right]^{1/2}, \end{aligned}$$

for any $n \geq 0$. Adding them together yields

$$\sum_{n=0}^{\infty} D_n(t) \leq k_p C K (1 + \|u_0\|) \sum_{n=0}^{\infty} \bar{c}^n \left[(k^{*n} * \lambda)(t)\right]^{1/2},$$

where the right hand side is an increasing functions. Indeed, it is a consequence of Lemma 4.10 below, and the following observation:

$$\lambda(t) = \int_0^t k(s) ds = (k * \mathbf{1})(t).$$

Since $\mathbf{1}$ is increasing, so is the convolution $k * \mathbf{1}$, and each convolution $k^{*n} * \lambda$.

Lemma 4.10. *If k and h are positive and if h is increasing, then $k * h$ is increasing.*

Proof. For $\varepsilon > 0$,

$$\begin{aligned} (k * h)(t + \varepsilon) &= \int_0^{t+\varepsilon} k(s) h(t + \varepsilon - s)\, ds \geq \int_0^t k(s) h(t + \varepsilon - s)\, ds \\ &\geq \int_0^t k(s) h(t - s)\, ds = (k * h)(t), \end{aligned}$$

which completes the proof. □

We now show that $\sum_{n=0}^{\infty} D_n(t)$ is finite-valued using Laplace transform. Using Hölder's inequality with $p = 2$, the Laplace transform of the following series is bounded by

$$\begin{aligned} \mathcal{L}\left(\sum_{n=0}^{\infty} \bar{c}^n (k^{*n} * \lambda)^{1/2}\right)(\gamma) &\leq \gamma^{-1/2} \sum_{n=0}^{\infty} \bar{c}^n \left[\mathcal{L}(k^{*n} * \lambda)(\gamma)\right]^{1/2} \\ &= \frac{1}{\gamma} \mathcal{L}(k)(\gamma)^{1/2} \sum_{n=0}^{\infty} \bar{c}^n \left[\mathcal{L}(k)(\gamma)\right]^{n/2}, \end{aligned}$$

We used the fact that $\mathcal{L}(\lambda)(\gamma) = \mathcal{L}(k * \mathbf{1})(\gamma) = \mathcal{L}(k)(\gamma)/\gamma$. Observe that the latter series is finite if and only if $\bar{c}^2 \mathcal{L}(k)(\gamma) < 1$, which is valid for γ large enough. Indeed,

$$\begin{aligned}\mathcal{L}(k)(\gamma) &= \int_0^\infty k(t,2c)\,e^{-\gamma t}\,dt = \int_0^\infty dt\,e^{-\gamma t}\int_{\mathbb{R}^d}\Lambda(dz)\,\Gamma_{2c}(t,z)\\ &= \int_0^\infty dt\,e^{-\gamma t}\int_{\mathbb{R}^d}\mu(d\xi)\,\mathcal{F}\Gamma_{2c}(t)(\xi)\\ &= \int_{\mathbb{R}^d}\mu(d\xi)\int_0^\infty dt\,\exp\left(-\gamma t - 8c\pi^2|\xi|^2 t\right)\\ &= \int_{\mathbb{R}^d}\frac{\mu(d\xi)}{\gamma + 8c\pi^2|\xi|^2},\end{aligned}$$

which is finite precisely by Dalang's condition (4.4). By dominated convergence, $\mathcal{L}(k)(\gamma) < 1/\bar{c}^2$ for γ sufficiently large.

By applying Lemma 4.11 below, we can conclude that

$$\sum_{n=0}^{\infty} D_n(t) \le k_p C K (1 + \|u_0\|) c_\gamma e^{\gamma t},$$

for some $\gamma > 0$ and $c_\gamma > 0$, both depending on $(\bar{c}, c, \Lambda)$. This complete the proof of Proposition 4.9.

The next result was found in [2, Lemma A.1]. The present proof is simpler.

Lemma 4.11. *For a positive increasing function $H : \mathbb{R}_+ \to \mathbb{R}_+$, if*

$$\gamma_0 := \inf\left\{\gamma \in \mathbb{R} : \mathcal{L}(H)(\gamma) = \int_0^\infty H(t)\,e^{-\gamma t}\,dt < \infty\right\} < \infty,$$

then

$$\limsup_{t\to\infty}\frac{1}{t}\log H(t) \le \gamma_0.$$

Furthermore, for every $\gamma > \gamma_0$, there exists $c_\gamma > 0$, such that

$$H(t) \le c_\gamma e^{\gamma t}, \quad \forall t \ge 0.$$

Proof. Fix $\varepsilon > 0$ and $\gamma > \gamma_0$. Because $\int_0^\infty H(s)\,e^{-\gamma s}\,ds < \infty$, there exists $T = T(\varepsilon,\gamma) > 0$ such that

$$\int_t^\infty H(s)\,e^{-\gamma s}\,ds < \varepsilon, \quad \forall t \ge T.$$

In particular,

$$\varepsilon > \int_t^{t+\varepsilon} H(s)\,e^{-\gamma s}\,ds \ge \varepsilon H(t)\,e^{-\gamma(t+\varepsilon)}, \quad \forall t \ge T,$$

and hence,

$$H(t) < e^{\gamma\varepsilon} e^{\gamma t}, \quad \forall t \ge T.$$

This proves both our conclusions. First, recalling that H is increasing yields

$$H(t) \le c_\gamma e^{\gamma t}, \quad \forall t \ge 0.$$

Second,

$$\frac{1}{t}\log H(t) \le \frac{\gamma\varepsilon}{t} + \gamma, \quad \forall t \ge T,$$

implies that

$$\limsup_{t\to\infty} \frac{1}{t}\log H(t) \le \gamma.$$

This concludes the proof since this last inequality is true for all $\gamma > \gamma_0$. □

Remark. We do expect an exponential growth in time of any p-moment. For $p = 2$, such lower and upper bounds can be found in [9]. The special case of Riesz kernels is treated in their Example 1.2. In one space dimension, the special case of white noise is treated in their Example 1.5.

In the special case of Anderson model, $\sigma(u) = \lambda u$, with white noise and constant one initial data, the second moment has been explicitly computed in [6, Corollary 2.5]. It is given by

$$\mathbb{E}\left[u(t,x)^2\right] = 2e^{\lambda^4 t/8}\Phi(\lambda^2\sqrt{t/4}),$$

where $\Phi(s) = \int_{-\infty}^{s}(2\pi)^{-1/2}e^{-y^2/2}\,dy$. The key step is the explicit computation of $\sum_{n=1}^{\infty} a^n k^{*n}(t)$, for all $a > 0$.

In the extension of Gronwall's lemma presented in [11, Lemma 17], it is proved that the latter sum is indeed finite using a large deviations argument. An application of Hölder's inequality implies that for any $p > 1$, we also have $\sum_{n=1}^{\infty} a^n k^{*n}(t)^{1/p} < \infty$. More precisely, observe that

$$\sum_{n=1}^{\infty} a^{n/2}\left[(k^{*n} * \lambda)(t)\right]^{1/2} \le \lambda(t)^{1/2}\sum_{n=1}^{\infty}\lambda(t)^{n/2}a^{n/2}\mathbb{P}(S_n \le t)^{1/2},$$

where $S_n = \sum_{i=1}^{n} Y_i$, for (Y_i) a sequence of i.i.d. random variables with law given by the density $f_Y(y) = k(y)/\int_0^t k(s)\,ds = k(y)/\lambda(t)$, for $y \in (0,t)$. This is a consequence of the fact that λ is an increasing function and that $\int_0^t k^{*n}(s)\,ds = \lambda(t)^n\mathbb{P}(S_n \le t)$. Large deviations enable to bound $\mathbb{P}(S_n \le t) \le b^{-n}$ for any $b \ge 1$, and n large enough.

The precise exponent, of the time exponential growth, can be found in the preceding references. In fact, the same growth in time can be obtained for any p-th moment, for $p \ge 1$, see [2].

4.4 Proof of convergence rate, Theorem 4.2

Recall both representation formulas (4.5) and (4.6), as well as the decomposition of the difference $u(t,x) - u_L(t,x)$, given in (4.9). To evaluate the

p-moment of each of I_0, I_2, I_4, and I_6, we make use of the uniform bounds of Proposition 4.3. We set

$$C_1 := \sup_{t\in[0,T]} \sup_{x\in\mathbb{R}^d} \|u(t,x)\|_{L^p(\Omega)}, \tag{4.20}$$

$$C_2 := \sup_{t\in[0,T]} \sup_{L>0} \sup_{x\in LD} \|u_L(t,x)\|_{L^p(\Omega)}. \tag{4.21}$$

As mentioned earlier, we first prove that the function h, see (4.10), satisfies a Gronwall type inequality.

Lemma 4.12. *For all $t \in [0,T]$, we have*

$$h(t) \le c_1 + c_2 \int_0^t k(s)\, ds + c_3 \int_0^t k(t-s)h(s)\, ds, \tag{4.22}$$

for $c_1 = c(\|u_0\|, D)$, $c_2 = c(T, K, Lip, \|u_0\|, \Lambda, D, p)$, $c_3 = c(Lip, p)$, and $k(s) = k(s, 2c)$, $h(t) = h(t,c)$.

The values of the constants are not important, yet they should not depend on the scaling variable L.

Proof. We start with the p-moment of I_0. By Minkowski's inequality (B.1) applied to the decomposition

$$I_0(t,x) = \int_{LD} [\Gamma(t, x-y) - G_L(t,x,y)]\, u_0(y)\, dy \\ + \int_{\mathbb{R}^d\setminus LD} \Gamma(t, x-y) u_0(y)\, dy,$$

we get,

$$\|I_0(t,x)\|_{L^p(\Omega)} \le \int_{LD} |\Gamma(t,x-y) - G_L(t,x,y)|\, \|u_0(y)\|_{L^p(\Omega)}\, dy \\ + \int_{\mathbb{R}^d\setminus LD} \Gamma(t,x-y)\, \|u_0(y)\|_{L^p(\Omega)}\, dy.$$

Using (4.12) and Lemma 4.8, we can bound

$$\begin{aligned} \|I_0(t,x)\|_{L^p(\Omega)} &\le (C + c^{d/2})\, \|u_0\|\, J_c(t,x) \int_{\mathbb{R}^d} \Gamma_c(t, x-y)\, dy \\ &= (C + c^{d/2})\, \|u_0\|\, J_c(t,x). \end{aligned} \tag{4.23}$$

In order to compute the p-moments of I_i, for $i \in \{2,4,6\}$, we shall use the following method. By Burkholder inequality,

$$\|I_2(t,x)\|^2_{L^p(\Omega)} = \|M_t\|^2_{L^p(\Omega)} \le k_p^2\, \|\langle M\rangle_t\|_{L^{p/2}(\Omega)},$$

where $\tau \mapsto M_\tau$ is the following continuous square integrable martingale

$$M_\tau = \int_0^\tau \int_{LD} [\Gamma(t-s, x-y) - G_L(t-s,x,y)]\sigma(u_L(s,y))\,\mathrm{M}(ds,dy),$$

for $\tau \in [0,t]$, whose quadratic variation is given by

$$\begin{aligned}\langle M\rangle_\tau = \int_0^\tau ds \int_{\mathbb{R}^d} \Lambda(dz) \int_{\mathbb{R}^d} dv\ \mathbf{1}_{LD}(v) F_L(t-s,x,v)\sigma(u_L(s,v))\\ \times \mathbf{1}_{LD}(v-z) F_L(t-s,x,v-z)\sigma(u_L(s,v-z)).\end{aligned}$$

Thus, by Minkowski's inequality,

$$\begin{aligned}\|\langle M\rangle_\tau\|_{L^{p/2}(\Omega)} \le \int_0^\tau ds \int_{\mathbb{R}^d} \Lambda(dz) \int_{\mathbb{R}^d} dv\ \mathbf{1}_{LD}(v)\,|F_L(t-s,x,v)|\\ \times \mathbf{1}_{LD}(v-z)\,|F_L(t-s,x,v-z)|\\ \times \|\sigma(u_L(s,v))\sigma(u_L(s,v-z))\|_{L^{p/2}(\Omega)}.\end{aligned}$$

By Cauchy-Schwarz inequality, under the following form,

$$\|XY\|_{L^{p/2}(\Omega)} \le \|X\|_{L^p(\Omega)}\,\|Y\|_{L^p(\Omega)},$$

together with linear growth of the function σ and (4.21), we get

$$\begin{aligned}\|\langle M\rangle_\tau\|_{L^{p/2}(\Omega)} \le K^2(1+C_2)^2 \int_0^\tau ds \int_{\mathbb{R}^d} \Lambda(dz)\\ \times \int_{\mathbb{R}^d} dv\ \mathbf{1}_{LD}(v)\,|F_L(t-s,x,v)|\,\mathbf{1}_{LD}(v-z)\,|F_L(t-s,x,v-z)|.\end{aligned}$$

Using inequality (4.12) and the semi-group property of the heat kernel, we get

$$\begin{aligned}\|\langle M\rangle_\tau\|_{L^{p/2}(\Omega)} &\le C^2K^2(1+C_2)^2 J_c(t,x)^2 \int_0^\tau ds \int_{\mathbb{R}^d} \Lambda(dz)\\ &\quad\times \int_{\mathbb{R}^d} dv\,\Gamma_c(t-s,x-v)\Gamma_c(t-s,x-(v-z))\\ &= C^2K^2(1+C_2)^2 J_c(t,x)^2 \int_0^\tau ds \int_{\mathbb{R}^d} \Lambda(dz)\Gamma_{2c}(t-s,z).\end{aligned}$$

Therefore,

$$\|I_2(t,x)\|^2_{L^p(\Omega)} \le k_p^2 C^2K^2(1+C_2)^2 J_c(t,x)^2 \int_0^t k(s,2c)\,ds.$$

To compute the p-moment of I_6, we use Burkholder's inequality

$$\|I_6(t,x)\|^2_{L^p(\Omega)} = \|M_t\|^2_{L^p(\Omega)} \le k_p^2\,\|\langle M\rangle_t\|_{L^{p/2}(\Omega)},$$

where $\tau \mapsto M_\tau$ is the following continuous square integrable martingale

$$M_\tau = \int_0^t \int_{\mathbb{R}^d \setminus LD} \Gamma(t-s, x-y)\sigma(u(s,y))\, \mathrm{M}(ds, dy),$$

for $\tau \in [0,t]$, whose quadratic variation is given by

$$\begin{aligned}\langle M\rangle_\tau = \int_0^\tau ds \int_{\mathbb{R}^d} \Lambda(dz) \int_{\mathbb{R}^d} dv\ &\mathbf{1}_{\mathbb{R}^d\setminus LD}(v)\Gamma(t-s, x-v)\sigma(u(s,v)) \\ &\times \mathbf{1}_{\mathbb{R}^d\setminus LD}(v-z)\Gamma(t-s, x-(v-z))\sigma(u(s,v-z)).\end{aligned}$$

Minkowski's and Cauchy-Schwarz' inequality, together with linear growth of the function σ and (4.20), yield

$$\begin{aligned}\|\langle M\rangle_\tau\|_{L^{p/2}(\Omega)} \leq K^2(1+C_1)^2 &\int_0^\tau ds \int_{\mathbb{R}^d} \Lambda(dz) \\ &\times \int_{\mathbb{R}^d} dv\ \mathbf{1}_{\mathbb{R}^d\setminus LD}(v)\Gamma(t-s, x-v) \\ &\qquad \times \mathbf{1}_{\mathbb{R}^d\setminus LD}(v-z)\Gamma(t-s, x-(v-z)).\end{aligned}$$

Two applications of inequality (4.8) and the semi-group property of the heat kernel yield

$$\begin{aligned}\|\langle M\rangle_\tau\|_{L^{p/2}(\Omega)} &\leq c^d K^2(1+C_1)^2 J_c(t,x)^2 \int_0^\tau ds \int_{\mathbb{R}^d} \Lambda(dz) \\ &\quad \times \int_{\mathbb{R}^d} dv\, \Gamma_c(t-s, x-v)\Gamma_c(t-s, x-(v-z)) \\ &= c^d K^2(1+C_1)^2 J_c(t,x)^2 \int_0^\tau ds \int_{\mathbb{R}^d} \Lambda(dz)\, \Gamma_{2c}(t-s, z).\end{aligned}$$

Therefore,

$$\|I_6(t,x)\|^2_{L^p(\Omega)} \leq k_p^2 c^d K^2(1+C_1)^2 J_c(t,x)^2 \int_0^t k(s, 2c)\, ds.$$

Finally, to compute the p-moment of I_4, we use Burkholder's inequality

$$\|I_4(t,x)\|^2_{L^p(\Omega)} = \|M_t\|^2_{L^p(\Omega)} \leq k_p^2\, \|\langle M\rangle_t\|_{L^{p/2}(\Omega)},$$

where $\tau \mapsto M_\tau$ is the following continuous square integrable martingale

$$M_\tau = \int_0^\tau \int_{LD} \Gamma(t-s, x-y)[\sigma(u(s,y)) - \sigma(u_L(s,y))]\, \mathrm{M}(ds, dy),$$

for $\tau \in [0,t]$, whose quadratic variation is given by

$$\begin{aligned}\langle M\rangle_\tau = \int_0^\tau ds \int_{\mathbb{R}^d} \Lambda(dz) \int_{\mathbb{R}^d} dv\ &\mathbf{1}_{LD}(v)\Gamma(t-s, x-v) \\ &\times \mathbf{1}_{LD}(v-z)\Gamma(t-s, x-(v-z)) \\ \times \{\sigma(u(s,v)) - \sigma(u_L(s,v))\}&\{\sigma(u(s,v-z)) - \sigma(u_L(s,v-z))\}.\end{aligned}$$

Using Minkowski's and Cauchy-Schwarz' inequality, together with the fact that σ is a Lipschitz function, we get

$$\begin{aligned}\|\langle M\rangle_\tau\|_{L^{p/2}(\Omega)} \le \mathrm{Lip}^2 \int_0^\tau ds \int_{\mathbb{R}^d} \Lambda(dz) \int_{\mathbb{R}^d} dv\, \mathbf{1}_{LD}(v)\Gamma(t-s,x-v) \\ \times \mathbf{1}_{LD}(v-z)\Gamma(t-s,x-(v-z)) \\ \times \|u(s,v)-u_L(s,v)\|_{L^p(\Omega)} \|u(s,v-z)-u_L(s,v-z)\|_{L^p(\Omega)}\end{aligned}$$

Recall the definitions of (4.10). We multiply and divide by $J_c(t,x)^2$ and $J_c(s,y)^2$ to get

$$\begin{aligned}\|\langle M\rangle_\tau\|_{L^{p/2}(\Omega)} &\le \mathrm{Lip}^2 J_c(t,x)^2 \int_0^\tau ds \int_{\mathbb{R}^d} \Lambda(dz) \int_{\mathbb{R}^d} dv \\ &\quad \times \mathbf{1}_{LD}(v)\Gamma(t-s,x-v)\frac{J_c(s,v)}{J_c(t,x)}\frac{f_L(s,v)}{J_c(s,v)} \\ &\quad \times \mathbf{1}_{LD}(v-z)\Gamma(t-s,x-(v-z))\frac{J_c(s,v-z)}{J_c(t,x)}\frac{f_L(s,v-z)}{J_c(s,v-z)} \\ &\le \mathrm{Lip}^2 J_c(t,x)^2 \int_0^\tau ds\, h(s,c) \int_{\mathbb{R}^d} \Lambda(dz) \int_{\mathbb{R}^d} dv\, q(s,v;t,x) q(s,v-z;t,x),\end{aligned}$$

where

$$q(s,v;t,x) := \mathbf{1}_{LD}(v)\Gamma(t-s,x-v)\frac{J_c(s,v)}{J_c(t,x)}.$$

By Lemma 4.13 below, we can conclude that

$$\|I_4(t,x)\|^2_{L^p(\Omega)} \le k_p^2 c^d \mathrm{Lip}^2 J_c(t,x)^2 \int_0^t h(s,c)k(t-s,2c).$$

If we put together each bound for I_0, I_2, I_6, and I_4, we reach the following

$$\begin{aligned}h(t)^{1/2} \le (C+c^{d/2})\|u_0\| + k_p C K(1+C_2)(k*\mathbf{1})(t)^{1/2} \\ + k_p c^{d/2} K(1+C_1)(k*\mathbf{1})(t)^{1/2} + k_p c^{d/2}\mathrm{Lip}(k*h)(t)^{1/2},\end{aligned}$$

where $k(s)=k(s,2c)$, and $h(s)=h(s,c)$. If we define $\bar{C}=C+c^{d/2}$, and $C_3=\max(C_1,C_2)$, we get

$$h(t)^{1/2} \le \bar{C}\|u_0\| + k_p\bar{C}K(1+C_3)(k*\mathbf{1})(t)^{1/2} + k_p c^{d/2}\mathrm{Lip}(k*h)(t)^{1/2}.$$

We now simplify the latter inequality. Observe that we have different powers of the function h on both sides. Squaring both side, we get

$$h(t) \le 3\bar{C}^2\|u_0\|^2 + 3k_p^2\bar{C}^2K^2(1+C_3)^2(k*\mathbf{1})(t) + 3k_p^2 c^d \mathrm{Lip}^2(k*h)(t).$$

The proof is complete if we set $c_1 = 3\bar{C}^2\|u_0\|^2$, $c_2 = 3k_p^2\bar{C}^2K^2(1+C_3)^2$, and $c_3 = 3k_p^2c^d\mathrm{Lip}^2$. □

Remark. If we used Lemmas 4.14 or 4.15 below instead of Lemma 4.13, then convolution would not appear in (4.22). Its last integral would become

$$\frac{c_3}{2}\int_0^t \frac{s}{t}k(2s(t-s)/t)h(s)\,ds.$$

We don't know which one is the most effective bound. We shall analyse the worst possible case, i.e.

$$h(t) \le c_1 + c_2\int_0^t k(s)\,ds + \frac{c_3}{2}\int_0^t k(2s(t-s)/t)h(s)\,ds, \tag{4.24}$$

for $k(s) = k(s,c)$, since $c \mapsto k(s,c)$ is decreasing, and $k(t-s,2c) = k(2(t-s),c) \le k(2s(t-s)/t,c)$, for all $0 \le s \le t$. In the next section, we shall iterate inequality (4.24).

Lemma 4.13. *Independently of $x \in LD$, and $c \ge 2$, we have*

$$\int_{\mathbb{R}^d}\Lambda(dz)\int_{\mathbb{R}^d} dv\, q(s,v;t,x)q(s,v-z;t,x) \le c^d k(t-s,2c).$$

Proof. Using the same trick as in Lemma 4.8, we can bound, for $c \ge 2$,

$$\Gamma(t-s,x-v) \le c^{d/2}\Gamma_c(t-s,x-v)e^{-\frac{|x-v|^2}{4c(t-s)}}. \tag{4.25}$$

Let w be (one of) the closest points of v in ∂LD, and y be (one of) the closest points of x in ∂LD, thus

$$q(s,v;t,x) \le c^{d/2}\Gamma_c(t-s,x-v)\frac{e^{-\frac{|x-v|^2}{4c(t-s)}}e^{-\frac{|v-w|^2}{4cs}}}{e^{-\frac{|x-y|^2}{4ct}}}.$$

Because $|x-y| \le |x-w| \le |x-v| + |v-w|$, a version of Cauchy-Schwarz inequality, Lemma 5.14, applies and

$$\frac{|x-y|^2}{t} \le \frac{(|x-v|+|v-w|)^2}{t} \le \frac{|x-v|^2}{t-s} + \frac{|v-w|^2}{s}.$$

Therefore, the quotient of exponentials is bounded by one and

$$q(s,v;t,x) \le c^{d/2}\Gamma_c(t-s,x-v).$$

The double integral becomes, using the semi-group property (A.6),

$$\begin{aligned}\int_{\mathbb{R}^d}\Lambda(dz)\int_{\mathbb{R}^d} dv\, q(s,v;t,x)q(s,v-z;t,x)&\\ \le c^d\int_{\mathbb{R}^d}\Lambda(dz)\int_{\mathbb{R}^d} dv\,\Gamma_c(t-s,x-v)\Gamma_c(t-s,x-(v-z))&\\ = c^d\int_{\mathbb{R}^d}\Lambda(dz)\,\Gamma_{2c}(t-s,z) = c^d k(t-s,2c),&\end{aligned}$$

which concludes the proof. □

In the particular case of the symmetric interval $D = (-1, 1)$, we can also find the following.

Lemma 4.14. *Independently of $x \in LD = (-L, L)$ and $c \geq 1$, we have*

$$\int_{\mathbb{R}} \Lambda(dz) \int_{\mathbb{R}} dv\, q(s, v; t, x) q(s, v - z; t, x) \leq 16c \frac{s}{t} k(2s(t-s)/t, c).$$

Proof. First, observe that the inner most integral is a convolution, evaluated at the point z, i.e. $(q * \tilde{q})(z)$. In the present case of one space dimension, we can bound

$$J_c(t, x) \leq \sqrt{4c\pi t}\, [\Gamma_c(t, x - L) + \Gamma_c(t, x + L)] \leq 2J_c(t, x),$$

for all $x \in (-L, L)$. Furthermore, for $c \geq 1$, $\Gamma(r, x) \leq \sqrt{c}\Gamma_c(r, x)$. Thus

$$\begin{aligned} q(s, v; t, x) &\leq 2\sqrt{c}\frac{\sqrt{s}}{\sqrt{t}} \Gamma_c(t - s, x - v) \frac{\Gamma_c(s, v - L) + \Gamma_c(s, v + L)}{\Gamma_c(t, x - L) + \Gamma_c(t, x + L)} \\ &\leq 2\sqrt{c}\frac{\sqrt{s}}{\sqrt{t}} \left[b(s, v; t, x, L) + b(s, v; t, x, -L)\right], \end{aligned}$$

where

$$\begin{aligned} b(s, v; t, x, L) &= \frac{\Gamma_c(t - s, x - v)\Gamma_c(s, v - L)}{\Gamma_c(t, x - L)} \\ &= \Gamma_c\left(\frac{s(t-s)}{t}, \frac{(t-s)(L-v) + s(x-v)}{t}\right) \\ &= \Gamma_c\left(\frac{s(t-s)}{t}, -v + \frac{s}{t}x + \frac{t-s}{t}L\right). \end{aligned}$$

The second equality is an application of the multiplication formula for the heat kernel, see (A.11). The function $v \mapsto b(s, v; t, x, L)$ is the density of a scaled brownian bridge starting at the point L at time $s = 0$, and finishing at the point x at time $s = t$. This observation is made rigorous in Lemma 6.17. Because it is a Schwartz function, we can compute its Fourier transform in the v variable,

$$\mathcal{F}b(s, t, x, L)(\xi) = \exp\left(-4c\pi^2 \frac{s(t-s)}{t} |\xi|^2\right) \exp\left(-2\pi i \xi \left(\frac{s}{t}x + \frac{t-s}{t}L\right)\right).$$

Thus,

$$\begin{aligned} &|\mathcal{F}b(s, t, x, L)(\xi) + \mathcal{F}b(s, t, x, -L)(\xi)|^2 \\ &\qquad = 4\exp\left(-4c\pi^2 \frac{2s(t-s)}{t} |\xi|^2\right) \cos^2\left(2\pi\xi \frac{t-s}{t} L\right). \end{aligned}$$

Putting every bounds together yields

$$
\begin{aligned}
\int_{\mathbb{R}} \Lambda(dz) \int_{\mathbb{R}} & dv\, q(s,v;t,x) q(s,v-z;t,x) \\
&\leq 4c\frac{s}{t}\int_{\mathbb{R}} \Lambda(dz) \int_{\mathbb{R}} dv\ [b(s,v;t,x,L) + b(s,v;t,x,-L)] \\
&\qquad \times [b(s,v-z;t,x,L) + b(s,v-z;t,x,-L)] \\
&= 4c\frac{s}{t}\int_{\mathbb{R}} \mu(d\xi)\, |\mathcal{F}\,[b(s,t,x,L) + b(s,t,x,-L)]\,(\xi)|^2 \\
&\leq 16c\frac{s}{t}\int_{\mathbb{R}} \mu(d\xi) \exp\left(-4c\pi^2 \frac{2s(t-s)}{t}\,|\xi|^2\right) \\
&= 16c\frac{s}{t}\int_{\mathbb{R}} \Lambda(dz)\Gamma_c\left(\frac{2s(t-s)}{t}, z\right) = 16c\frac{s}{t}k(2s(t-s)/t, c).
\end{aligned}
$$

This completes the proof. □

The generalization to the square $D = (-1,1)^d$ is given now.

Lemma 4.15. *Independently of $x \in LD = (-L,L)^d$, and $c \geq 1$, we have*

$$
\int_{\mathbb{R}^d} \Lambda(dz) \int_{\mathbb{R}^d} dv\, q(s,v;t,x) q(s,v-z;t,x) \leq (2d)^4 c^d \frac{s}{t} k(2s(t-s)/t, c).
$$

Proof. We use the special notation Γ^d for the heat kernel in d space dimensions and Γ for the heat kernel in one space dimension. We would like to bound

$$
q(s,v;t,x) = \mathbf{1}_D(v)\Gamma^d(t-s, x-v) J_c(s,v)/J_c(t,x).
$$

We first observe that

$$
\begin{aligned}
\frac{J_c(t,x)}{\sqrt{4c\pi t}} &\leq \Gamma_c(t, x_1 - L) + \Gamma_c(t, x_1 + L) + \cdots + \Gamma_c(t, x_d - L) + \Gamma_c(t, x_d + L) \\
&\leq 2d \frac{J_c(t,x)}{\sqrt{4c\pi t}},
\end{aligned}
$$

and thus,

$$
\begin{aligned}
\frac{J_c(s,v)}{J_c(t,x)} &\leq 2d\frac{\sqrt{s}}{\sqrt{t}} \\
&\times \frac{\Gamma_c(s, v_1 - L) + \Gamma_c(s, v_1 + L) + \cdots + \Gamma_c(s, v_d - L) + \Gamma_c(s, v_d + L)}{\Gamma_c(t, x_1 - L) + \Gamma_c(t, x_1 + L) + \cdots + \Gamma_c(t, x_d - L) + \Gamma_c(t, x_d + L)}.
\end{aligned}
$$

To simplify notations, we set $\mathbb{R}^{d-1} \ni \hat{x}^j = (x_1, \cdots, x_{j-1}, x_{j+1}, \ldots, x_d)$ the vector whose j-th component has been removed. We need to bound $2d$ terms, each of the form

$$\begin{aligned}\Gamma^d(t-s,x-v)&\frac{\Gamma_c(s,v_1-L)}{\Gamma_c(t,x_1-L)}\\&\le c^{d/2}\Gamma_c^{d-1}(t-s,\hat{x}^1-\hat{v}^1)\frac{\Gamma_c(t-s,x_1-v_1)\Gamma_c(s,v_1-L)}{\Gamma_c(t,x_1-L)}\\&=c^{d/2}\Gamma_c^{d-1}(t-s,\hat{x}^1-\hat{v}^1)b(s,v_1;t,x_1,L),\end{aligned}$$

where

$$b(s,v_1;t,x_1,L):=\frac{\Gamma_c(t-s,x_1-v_1)\Gamma_c(s,v_1-L)}{\Gamma_c(t,x_1-L)},$$

is the density of a scaled brownian bridge starting at the point L at time $s=0$, and finishing at the point x_1 at time $s=t$. This observation is made rigorous in Lemma 6.17. We apply the multiplication formula for the heat kernel, see (A.11), to get

$$b(s,v_1;t,x_1,L)=\Gamma_c\left(\frac{s(t-s)}{t},-v_1+\frac{s}{t}x_1+\frac{t-s}{t}L\right),$$

whose one dimensional Fourier transform in the v_1 variable is given by

$$\mathcal{F}b(s;t,x_1,L)(\xi_1)=\exp\left\{-4c\pi^2\frac{s(t-s)}{t}\left|\xi_1\right|^2\right\}e^{-2\pi i\xi_1\left(\frac{s}{t}x_1+\frac{t-s}{t}L\right)}.$$

Observe that the following d dimensional Fourier transform in the v variable can be split into a $d-1$ dimensional Fourier transform in the $\hat{v}^1$ variable and a one dimensional Fourier transform in the v_1 variable,

$$\begin{aligned}\mathcal{F}&\left\{\Gamma_c^{d-1}(t-s,\hat{x}^1-\hat{v}^1)b(s,v_1;t,x_1,L)\right\}(\xi)\\&=\exp\left\{-4c\pi^2(t-s)\left|\hat{\xi}^1\right|^2\right\}e^{-2\pi i\hat{\xi}^1\cdot\hat{x}^1}\\&\qquad\times\exp\left\{-4c\pi^2\frac{s(t-s)}{t}\left|\xi_1\right|^2\right\}e^{-2\pi i\xi_1\left(\frac{s}{t}x_1+\frac{t-s}{t}L\right)},\end{aligned}$$

with module given by

$$\begin{aligned}\exp&\left\{-4c\pi^2(t-s)\left|\hat{\xi}^1\right|^2\right\}\exp\left\{-4c\pi^2\frac{s(t-s)}{t}\left|\xi_1\right|^2\right\}\\&\le\exp\left\{-4c\pi^2\frac{s(t-s)}{t}\left|\xi\right|^2\right\}.\end{aligned}$$

Therefore,

$$\begin{aligned}\int_{\mathbb{R}^d}&\Lambda(dz)\int_{\mathbb{R}^d}dv\,q(s,v;t,x)q(s,v-z;t,x)\\&\le(2d)^4c^d\frac{s}{t}\int_{\mathbb{R}^d}\mu(d\xi)\exp\left\{-4c\pi^2\frac{2s(t-s)}{t}\left|\xi\right|^2\right\}\\&=(2d)^4c^d\frac{s}{t}k(2s(t-s)/t,c),\end{aligned}$$

which concludes the proof. □

4.4.1 Iteration of some Gronwall type inequality

It is time to iterate inequality (4.24). To do so, we define a triangle operator

$$(f \triangleright g)(t) := \int_0^t f\left(2\frac{s(t-s)}{t}\right) g(s)\, ds. \tag{4.26}$$

The iteration scheme could be written as

$$h(t) \leq c_1 + c_2(k * 1)(t) + \frac{c_3}{2}(k \triangleright h)(t),$$

for all $t \in [0,T]$. The latter is very similar to the one of [11, Lemma 15], with the difference that the above triangle operator is replaced by convolution. As we shall see, both operators can sometimes be compared to one another. We iterate the latter inequality to get

$$\begin{aligned} h(t) \leq c_1 + c_2(k * 1)(t) + c_1\frac{c_3}{2}(k \triangleright 1)(t) \\ + c_2\frac{c_3}{2}(k \triangleright k * 1)(t) + \left(\frac{c_3}{2}\right)^2 (k \triangleright k \triangleright h)(t). \end{aligned}$$

If we iterate it again, we get

$$\begin{aligned} h(t) \leq c_1 + c_2(k * 1)(t) + c_1\frac{c_3}{2}(k \triangleright 1)(t) \\ + c_2\frac{c_3}{2}(k \triangleright k * 1)(t) + c_1\left(\frac{c_3}{2}\right)^2 (k \triangleright k \triangleright 1)(t) \\ + c_2\left(\frac{c_3}{2}\right)^2 (k \triangleright k \triangleright k * 1)(t) + \left(\frac{c_3}{2}\right)^3 (k \triangleright k \triangleright k \triangleright h)(t). \end{aligned}$$

It will be checked that the present triangle operator is neither commutative nor associative, see Lemma 4.16 below. In fact, we wrote

$$f \triangleright g \triangleright h := f \triangleright (g \triangleright h) \neq (f \triangleright g) \triangleright h.$$

The fact that both convolution and the triangle operator appear in the Gronwall inequality makes the computations harder. In fact, computations involving the triangle operator are much harder, see Lemma 5.23.

Lemma 4.17 below enables to bound the triangle operator with convolution, which is associative and commutative. Using Lemma 4.10 together with Lemma 4.17, we can bound, for example,

$$\begin{aligned} (k \triangleright k \triangleright k * 1)(t) &:= (k \triangleright (k \triangleright (k * 1)))(t) \\ &\leq 2(k \triangleright (k * (k * 1)))(t) \\ &\leq 4(k * k * k * 1)(t). \end{aligned}$$

In fact, we can replace each occurence of $f \triangleright g$ by $2f * g$, if we can make sure that the hypotheses of Lemma 4.17 are verified. In particular, we have

$$k \triangleright 1 \leq 2k * 1, \quad k \triangleright k * 1 \leq 2k * k * 1, \quad \text{and} \quad k \triangleright k \triangleright 1 \leq 2^2 k * k * 1.$$

Thus,

$$\begin{aligned} h(t) \leq c_1 + c_2(k * 1)(t) + c_1 c_3 (k * 1)(t) \\ + c_2 c_3 (k * k * 1)(t) + c_1 c_3^2 (k * k * 1)(t) \\ + c_2 c_3^2 (k * k * k * 1)(t) + \left(\frac{c_3}{2}\right)^3 (k \triangleright k \triangleright k \triangleright h)(t). \end{aligned}$$

If we keep iterating, finitely many time, we get

$$\begin{aligned} h(t) \leq c_1 + \left(c_1 + \frac{c_2}{c_3}\right) \sum_{m=1}^{n-1} c_3^m (k^{*m} * 1)(t) \\ + \frac{c_2}{c_3} c_3^n (k^{*n} * 1)(t) + \left(\frac{c_3}{2}\right)^n (k^{\triangleright n} \triangleright h)(t). \end{aligned}$$

We would like to let $n \to \infty$, and somehow hope that

$$h(t) \leq c_1 + \left(c_1 + \frac{c_2}{c_3}\right) \sum_{m=1}^{\infty} c_3^m (k^{*m} * 1)(t).$$

We shall later see that such a bound is valid. To have such a conclusion, we should show that both remaining terms

$$c_3^n (k^{*n} * 1)(t) \longrightarrow 0 \quad \text{and} \quad \left(\frac{c_3}{2}\right)^n (k^{\triangleright n} \triangleright h)(t) \longrightarrow 0,$$

as $n \to \infty$. The fact that the latter series is convergent is a consequence of either [11, Lemma 17] or the Laplace transform argument in Proposition 4.9 above. Hence, the first remaining term has to converge to zero. If h were to be a bounded function, say $h(s) \leq M$ for all $s \leq t$, then we could argue as before and conclude that the second remaining term is bounded by

$$\left(\frac{c_3}{2}\right)^n (k^{\triangleright n} \triangleright h)(t) \leq M c_3^n (k^{*n} * 1)(t) \longrightarrow 0, \quad \text{as } n \to \infty.$$

Unfortunately, the boundedness (and more generally the finiteness) of the function h is precisely what we are trying to prove! To overcome the latter circular argument, we need to go back to the Picard iteration schemes defining both $u(t,x)$ and $u_L(t,x)$. This will be done in the next Section 4.4.2.

We now gather two properties, mentioned above, regarding the triangle operator.

Lemma 4.16. *The triangle operator, defined by* (4.26), *is neither commutative nor associative.*

Proof. Non-commutativity is obvious from the definition. To show non-associativity, we first compute

$$\begin{aligned}
((f \triangleright g) \triangleright h)(t) &= \int_0^t (f \triangleright g)\left(2s - \frac{2s^2}{t}\right) h(s)\, ds \\
&= \int_0^t ds \int_0^{2\frac{s(t-s)}{t}} dr\, f\left(2r - \frac{2r^2}{2s - \frac{2s^2}{t}}\right) g(r)h(s).
\end{aligned}$$

Then, we compute

$$\begin{aligned}
(f \triangleright (g \triangleright h))(t) &= \int_0^t f\left(2s - \frac{2s^2}{t}\right)(g \triangleright h)(s)\, ds \\
&= \int_0^t ds \int_0^s dr\, f\left(2s - \frac{2s^2}{t}\right) g\left(2r - \frac{2r^2}{s}\right) h(r) \\
&= \int_0^t dr \int_r^t ds\, f\left(2s - \frac{2s^2}{t}\right) g\left(2r - \frac{2r^2}{s}\right) h(r).
\end{aligned}$$

For r fixed, we can make the change of variable $x = 2r(s-r)/s$, or equivalently $s = 2r^2/(2r - x)$, which yields

$$\begin{aligned}
&(f \triangleright (g \triangleright h))(t) \\
&= \int_0^t dr \int_0^{2\frac{r(t-r)}{t}} dx\, \frac{2r^2}{(2r-x)^2} f\left(2\left(\frac{2r^2}{2r-x}\right) - \frac{2\left(\frac{2r^2}{2r-x}\right)^2}{t}\right) g(x)h(r).
\end{aligned}$$

This concludes the proof. □

Under the following special assumptions, the triangle operator is comparable to convolution.

Lemma 4.17. *Suppose $g \geq 0$ is increasing and $k \geq 0$ is decreasing, then*

$$\frac{1}{2}(k * g)(t) \leq (k \triangleright g)(t) \leq 2(k * g)(t). \tag{4.27}$$

The author discovered this result before encountering the second inequality in [2, Lemma 3.6].

Proof. First, we observe that $(t-s)/t \geq 1/2$ for $s \in (0, t/2)$, and $s/t \geq 1/2$ for $s \in (t/2, t)$. Using the facts that k is decreasing, g is increasing, and that both are positive, we get

$$\begin{aligned}
\int_0^t g(s)k\left(2\frac{s(t-s)}{t}\right) ds &\leq \int_0^{t/2} g(s)k(s)\, ds + \int_{t/2}^t g(s)k(t-s)\, ds \\
&= \int_0^{t/2} [g(s) + g(t-s)]\, k(s)\, ds \\
&\leq 2\int_0^{t/2} g(t-s)k(s)\, ds \leq 2\int_0^t g(t-s)k(s)\, ds.
\end{aligned}$$

Similarly, we observe that $(t-s)/t \leq 1$ for $s \in (0, t/2)$, and $s/t \leq 1$ for $s \in (t/2, t)$. Using the facts that k is decreasing, g is positive and increasing, we get

$$\begin{aligned}
\int_0^t g(s) k\left(2\frac{s(t-s)}{t}\right) ds &\geq \int_0^{t/2} g(s)k(2s)\, ds + \int_{t/2}^t g(s)k(2(t-s))\, ds \\
&= \int_0^{t/2} \left[g(s) + g(t-s)\right] k(2s)\, ds \\
&\geq \int_0^{t/2} g(t-s)k(2s)\, ds = \frac{1}{2}\int_0^t g(t-r/2)k(r)\, dr \\
&\geq \frac{1}{2}\int_0^t g(t-r)k(r)\, dr,
\end{aligned}$$

which concludes the proof. □

4.4.2 Last step of the proof

The main purpose of this section is to confirm the previous guess

$$h(t) \leq c_1 + [c_1 + c_2/c_3] \cdot (\mathcal{K} * \mathbf{1})(t),$$

for all $t \in [0, T]$, where

$$\mathcal{K}(t) = \sum_{m=1}^{\infty} c_3^m k^{*m}(t).$$

To do so, we need to go back to the Picard iteration schemes associated to equations (4.1) and (4.2), which are defined by

$$\begin{aligned}
u^0(t,x) &:= \int_{\mathbb{R}^d} \Gamma(t, x-y)\, u_0(y)\, dy, \\
u_L^0(t,x) &:= \int_{LD} G_L(t,x,y)\, u_0(y)\, dy, \\
u^{n+1}(t,x) &:= u^0(t,x) + \int_0^t \int_{\mathbb{R}^d} \Gamma(t-s, x-y)\, \sigma\left(u^n(s,y)\right) \mathrm{M}(ds, dy), \\
u_L^{n+1}(t,x) &:= u_L^0(t,x) + \int_0^t \int_{LD} G_L(t-s, x, y)\, \sigma\left(u_L^n(s,y)\right) \mathrm{M}(ds, dy).
\end{aligned}$$

We define the corresponding functions of (4.10),

$$\begin{aligned}
f_L^n(t,x) &:= \left\| u^n(t,x) - u_L^n(t,x) \right\|_{L^p(\Omega)}, \\
h_n(t,c) &:= \sup_{x \in LD} \frac{f_L^n(t,x)^2}{J_c(t,x)^2},
\end{aligned}$$

It is possible to bound the function h by the functions h_n in the following way:

Lemma 4.18. *For all $t > 0$,*

$$h(t) \leq \liminf_{n\to\infty} h_n(t). \tag{4.28}$$

Proof. Because $u^n(t,x)$ and $u_L^n(t,x)$ converge to respectively $u(t,x)$ and $u_L(t,x)$, in $L^p(\Omega)$, we get that $f_L^n(t,x)$ converges to $f_L(t,x)$. Now recall that for any function ϕ of two parameters, we have

$$\sup_y \inf_x \phi(x,y) \leq \inf_x \sup_y \phi(x,y).$$

Thus,

$$\begin{aligned} h(t) &= \sup_{x\in LD} \frac{f_L(t,x)^2}{J_c(t,x)^2} = \sup_{x\in LD} \liminf_{n\to\infty} \frac{f_L^n(t,x)^2}{J_c(t,x)^2} \\ &= \sup_{x\in LD} \sup_{N\geq 1} \inf_{n\geq N} \frac{f_L^n(t,x)^2}{J_c(t,x)^2} = \sup_{N\geq 1} \sup_{x\in LD} \inf_{n\geq N} \frac{f_L^n(t,x)^2}{J_c(t,x)^2} \\ &\leq \sup_{N\geq 1} \inf_{n\geq N} \sup_{x\in LD} \frac{f_L^n(t,x)^2}{J_c(t,x)^2} = \liminf_{n\to\infty} h_n(t), \end{aligned}$$

which concludes the proof. □

We are now interested in computing each function h_n. For $n = 0$, we observe that $u^0(t,x) - u_L^0(t,x) = I_0(t,x)$, and thus by (4.23),

$$f_L^0(t,x) = \|I_0(t,x)\|_{L^p(\Omega)} \leq (C + c^{d/2}) \|u_0\| J_c(t,x).$$

Hence $h_0(t) \leq (C+c^{d/2})^2 \|u_0\|^2$. The fact that the initial iteration is bounded by a constant function will play a significant role. We can argue similarly as in Lemma 4.12, and the remark that follows it, to get the following recursive formula

$$h_{n+1}(t) \leq c_1 + c_2(k * \mathbf{1})(t) + \frac{c_3}{2}(k \triangleright h_n)(t).$$

Without lost of generality, we can assume that $(C + c^{d/2})^2 \|u_0\|^2 \leq c_1$. (In fact, we had $c_1 = 3(C + c^{d/2})^2 \|u_0\|^2$.) In order to use the comparison between the triangle operator and convolution, see Lemma 4.17, we need to recursively define another sequence of functions $\{\bar{h}_n\}$, with $\bar{h}_0(t) = c_1$ and

$$\bar{h}_{n+1}(t) = c_1 + c_2(k * \mathbf{1})(t) + c_3(k * \bar{h}_n)(t).$$

That sequence satisfies the following two properties:

Lemma 4.19. *Each function $\bar{h}_n$ is increasing and dominates h_n, i.e. for any $n \in \mathbb{N}$ and $\varepsilon, t > 0$, we have $\bar{h}_n(t) \leq \bar{h}_n(t+\varepsilon)$ and $h_n(t) \leq \bar{h}_n(t)$.*

This result was inspired by [9, Lemma 2.6].

Proof. The fact that each function $\bar{h}_n$ is increasing is a consequence of Lemma 4.10 and the fact that the initial function $\bar{h}_0$ is constant. The fact that each $\bar{h}_n$ dominates h_n is a consequence of Lemma 4.17 and the facts k is decreasing and each $\bar{h}_n$ is increasing. The proof is done by induction. For $n = 0$, we have $h_0(t) \leq c_1 = \bar{h}_0(t)$. Moreover, for $n \geq 0$,

$$\frac{c_3}{2}(k \rhd h_n)(t) \leq \frac{c_3}{2}(k \rhd \bar{h}_n)(t) \leq c_3(k * \bar{h}_n)(t),$$

and thus $h_{n+1}(t) \leq \bar{h}_{n+1}(t)$. □

We now give the exact expression for the dominating sequence, i.e.

$$\begin{aligned}\bar{h}_1(t) &= c_1 + c_2(k * \mathbf{1})(t) + c_3(k * \bar{h}_0)(t) \\ &= c_1 + c_2(k * \mathbf{1})(t) + c_1 c_3(k * \mathbf{1})(t) \\ &= c_1 + [c_2 + c_1 c_3]\,(k * \mathbf{1})(t);\end{aligned}$$

$$\begin{aligned}\bar{h}_2(t) &= c_1 + c_2(k * \mathbf{1})(t) + c_3(k * \bar{h}_1)(t) \\ &= c_1 + c_2(k * \mathbf{1})(t) + c_1 c_3(k * \mathbf{1})(t) + c_3\,[c_2 + c_1 c_3]\,(k * k * \mathbf{1})(t) \\ &= c_1 + [c_2 + c_1 c_3]\,(k * \mathbf{1})(t) + c_3\,[c_2 + c_1 c_3]\,(k * k * \mathbf{1})(t);\end{aligned}$$

and by induction, for $n \geq 1$,

$$\begin{aligned}\bar{h}_n &= c_1 + [c_2 + c_1 c_3] \cdot \left(\sum_{m=1}^{n} c_3^{m-1} k^{*m} \right) * \mathbf{1} \\ &= c_1 + [c_1 + c_2/c_3] \cdot \left(\sum_{m=1}^{n} c_3^{m} k^{*m} \right) * \mathbf{1}\,.\end{aligned}$$

Therefore,

$$h(t) \leq \liminf_{n\to\infty} h_n(t) \leq \lim_{n\to\infty} \bar{h}_n(t) = c_1 + [c_1 + c_2/c_3] \cdot (\mathcal{K} * \mathbf{1})(t),$$

which completes the proof of (4.11).

We are left to show that the increasing function $\mathcal{K} * \mathbf{1}$ is finite-valued. As in the proof of Proposition 4.9, we compute its Laplace transform $\mathcal{L}(\mathcal{K} * \mathbf{1})(\gamma)$ and observe that it is finite if and only if $c_3 \mathcal{L}(k)(\gamma) < 1$. By dominated convergence, this is possible for γ sufficiently large since

$$c_3 \mathcal{L}(k)(\gamma) = c_3 \int_{\mathbb{R}^d} \frac{\mu(d\xi)}{\gamma + 4c\pi^2 \left|\xi\right|^2},$$

which is finite for all $\gamma > 0$, by Dalang's condition (4.4).

The proof of Theorem 4.2 is completed. Indeed, we have proved that for all $t \in [0, T]$,

$$h(t) = \sup_{x \in LD} \frac{\|u(t,x) - u_L(t,x)\|_{L^p(\Omega)}^2}{J_c(t,x)^2} \leq c_1 + [c_1 + c_2/c_3] \cdot (\mathcal{K} * \mathbf{1})(t),$$

where nothing on the right hand side depends on the scaling variable L. Therefore, uniformly for $t \in [0,T]$, $L > 0$, and $x \in LD$, we have

$$\|u(t,x) - u_L(t,x)\|_{L^p(\Omega)} \leq \Theta(t) \exp\left(-\frac{\mathrm{dist}(x, \partial LD)^2}{4ct}\right),$$

for $\Theta(t) = \sqrt{c_1 + [c_1 + c_2/c_3] \cdot (\mathcal{K} * \mathbf{1})(t)}$.

4.5 Possible generalizations

We describe here two generalizations of Theorem 4.2. First, we consider a more general type of equations, and second, vanishing Dirichlet boundary conditions are replaced by combinations of vanishing Dirichlet and Neumann boundary conditions. The open bounded domain $D \subseteq \mathbb{R}^d$ is as before. It is either the d-dimensional square $(-1,1)^d$ or has a regular boundary (and contains the origin). Its boundary is decomposed into $\partial D = S_1 \cup S_2$.

The method used in the previous sections applies for both generalizations mentioned above. More precisely, we compare the behavior of the (random field) solution on the whole space $\mathbb{R}^d$ with the (random field) solution on the dilated domain LD, with vanishing Dirichlet and/or Neumann boundary conditions, i.e.

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) = \Delta u(t,x) + b(u(t,x)) + \sigma(u(t,x))\,\dot{\mathrm{M}}, & t > 0,\ x \in \mathbb{R}^d, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^d, \end{cases} \tag{4.29}$$

and

$$\begin{cases} \dfrac{\partial u_L}{\partial t}(t,x) = \Delta u_L(t,x) + b(u_L(t,x)) + \sigma(u_L(t,x))\,\dot{\mathrm{M}}, & t > 0,\ x \in LD, \\ u_L(t,x) = 0, & t > 0,\ x \in LS_1, \\ \dfrac{\partial u_L}{\partial \nu}(t,x) = 0, & t > 0,\ x \in LS_2, \\ u_L(0,x) = u_0(x), & x \in LD, \end{cases} \tag{4.30}$$

where b, σ and u_0 satisfy the d-dimensional versions of (2.6) and (2.7), the noise $\dot{\mathrm{M}}$ on $\mathbb{R}_+ \times \mathbb{R}^d$ is white in time and correlated in space, i.e. it satisfies Assumption 4.1, and ν is the unit outward normal vector at the point $x \in \partial LD$.

Under those assumptions, equations (4.29) and (4.30) admit unique random field solutions $u(t,x)$ and $u_L(t,x)$, that satisfy the following representation formulas

$$u(t,x) = \int_{\mathbb{R}^d} \Gamma(t, x-y) u_0(y)\, dy \\ + \int_0^t ds \int_{\mathbb{R}^d} \Gamma(t-s, x-y)\, b(u(s,y))\, dy \\ + \int_0^t \int_{\mathbb{R}^d} \Gamma(t-s, x-y)\, \sigma(u(s,y))\, \mathrm{M}(ds, dy), \quad (4.31)$$

and

$$u_L(t,x) = \int_{LD} G_L(t,x,y) u_0(y)\, dy \\ + \int_0^t ds \int_{LD} G_L(t-s, x, y) b(u_L(s,y))\, dy \\ + \int_0^t \int_{LD} G_L(t-s, x, y) \sigma(u_L(s,y))\, \mathrm{M}(ds, dy), \quad (4.32)$$

where Γ is the heat kernel, given by (A.8), and G_L is the Green function of the heat equation on the bounded domain LD associated with the boundary conditions of (4.30). See [11] for existence and uniqueness of these random field solutions. See also [9] for unbounded (or measure valued) initial conditions.

Remark. The latter Green function G_L is associated to the heat equation $\frac{\partial u}{\partial t} = \Delta u$ but not to the operator $\frac{\partial u}{\partial t} = \Delta u + b(u)$. In the particular case $b(u) = b\cdot u$, for $b \in \mathbb{R}$, the operator $\frac{\partial u}{\partial t} = \Delta u + b\cdot u$ admits the following Green function $e^{bt} G_L(t,x,y)$. Similarly, the fundamental solution is $e^{bt}\Gamma(t, x-y)$. In that case, we would expect the solution to satisfy the following representation formulas

$$v(t,x) = \int_{\mathbb{R}^d} e^{bt} \Gamma(t, x-y)\, u_0(y)\, dy \\ + \int_0^t \int_{\mathbb{R}^d} e^{b(t-s)} \Gamma(t-s, x-y)\, \sigma(v(s,y))\, \mathrm{M}(ds, dy),$$

and

$$v_L(t,x) = \int_{LD} e^{bt} G_L(t,x,y)\, u_0(y)\, dy \\ + \int_0^t \int_{LD} e^{b(t-s)} G_L(t-s, x, y) \sigma(v_L(s,y))\, \mathrm{M}(ds, dy).$$

In the present case, we don't have the scaling properties (4.15) and (4.16). But we have similar upper bounds when $b < 0$. Indeed,

$$e^{bt} G_{LD}(t,x,y) = e^{bt/L^2} \frac{1}{L^d} G_D(t/L^2, x/L, y/L) e^{bt(1-1/L^2)},$$

for all $t > 0$ and $x, y \in LD$. Thus,

$$e^{bt} |G_{LD}(t,x,y)| \leq e^{bt/L^2} \frac{1}{L^d} \left|G_D(t/L^2, x/L, y/L)\right|,$$

whenever $b < 0$ and $L \geq 1$. A similar argument holds for the difference of the fundamental solution and the Green function. Therefore, both inequalities of (4.12) are verified for the Green function associated to the present operator $\frac{\partial u}{\partial t} = \Delta u + b \cdot u$. Whence, the convergence result, Theorem 4.2, also holds for the latter operator.

As in Chapter 2, the difference of interest is

$$u(t,x) - u_L(t,x) = \sum_{i=0}^{6} I_i(t,x),$$

where

$$\begin{aligned}
I_0(t,x) &= \int_{\mathbb{R}^d} \Gamma(t, x-y)\, u_0(y)\, dy - \int_{LD} G_L(t,x,y)\, u_0(y)\, dy,\\
I_1(t,x) &= \int_0^t ds \int_{LD} [\Gamma(t-s, x-y) - G_L(t-s,x,y)] b(u_L(s,y))\, dy,\\
I_2(t,x) &= \int_0^t \int_{LD} [\Gamma(t-s, x-y) - G_L(t-s,x,y)] \sigma(u_L(s,y))\, \mathrm{M}(ds,dy),\\
I_3(t,x) &= \int_0^t ds \int_{LD} \Gamma(t-s, x-y)[b(u(s,y)) - b(u_L(s,y))]\, dy,\\
I_4(t,x) &= \int_0^t \int_{LD} \Gamma(t-s, x-y)[\sigma(u(s,y)) - \sigma(u_L(s,y))]\, \mathrm{M}(ds,dy),\\
I_5(t,x) &= \int_0^t ds \int_{\mathbb{R}^d \setminus LD} \Gamma(t-s, x-y) b(u(s,y))\, dy,\\
I_6(t,x) &= \int_0^t \int_{\mathbb{R}^d \setminus LD} \Gamma(t-s, x-y) \sigma(u(s,y))\, \mathrm{M}(ds,dy).
\end{aligned}$$

The p-moments of I_0, I_2, I_4, and I_6 were already handled in Lemma 4.12. The computations of the p-moments of I_1, I_3, and I_5 are similar to those of Chapter 2. Recall the definitions of (4.10). To get a better understanding on what shall happen, we set

$$\begin{aligned}
k_\Lambda(t,c) &:= \int_{\mathbb{R}^d} \Lambda(dz)\, \Gamma_c(t,z),\\
k_\lambda(t,c) &:= \int_{\mathbb{R}^d} \Gamma_c(t,z)\, dz.
\end{aligned}$$

The second definition is a particular case of the first, in which $\Lambda(dz) = dz$ is the Lebesgue measure on $\mathbb{R}^d$. For some $l > 0$, we have the following uniform bounds,

$$C_1 := \sup_{t\in[0,T]} \sup_{x\in\mathbb{R}^d} \|u(t,x)\|_{L^p(\Omega)} < \infty, \tag{4.33}$$

$$C_2 := \sup_{t\in[0,T]} \sup_{L>l} \sup_{x\in LD} \|u_L(t,x)\|_{L^p(\Omega)} < \infty. \tag{4.34}$$

These uniform bounds are proved in a similar way as in Propositions 2.6 and 4.3, in which the recursive equation (4.18) should be replaced by

$$D_n(t)^2 \leq \bar{c}\,\mathrm{Lip}^2\left[(k_\lambda + k_\Lambda) * D_{n-1}^2\right](t),$$

for all $t \in [0,T]$, where

$$D_n(t) := \sup_{x\in\mathbb{D}} \left\|v^{n+1}(t,x) - v^n(t,x)\right\|_{L^p(\Omega)},$$

and $\bar{c} = c(T,p,D)$. Do deduce the latter recursive bound, the computations in the next Lemma may help.

We have the following Gronwall type inequality.

Lemma 4.20. *For all* $t \in [0,T]$, *we have*

$$h(t) \leq c_1 + c_2\left[(k_\lambda + k_\Lambda) * \mathbf{1}\right](t) + c_3\left[(k_\lambda + k_\Lambda) * h\right](t),$$

for some constants $c_1 = c(\|u_0\|, D)$, $c_2 = c(T, K, Lip, \|u_0\|, \Lambda, D, p)$, *and* $c_3 = c(T, Lip, p)$, *and* $k_\Lambda(s) = k_\Lambda(s, 2c)$, $h(t) = h(t,c)$.

Going back to the Picard iteration scheme, as in Section 4.4.2, we can show that equation (4.11) is still valid, i.e.

$$h(t) \leq c_1 + [c_1 + c_2/c_3] \cdot (\mathcal{K} * \mathbf{1})(t),$$

for all $t \in [0,T]$, where

$$\mathcal{K}(t) = \sum_{m=1}^{\infty} c_3^m (k_\lambda + k_\Lambda)^{*m}(t).$$

The convolution $\mathcal{K} * \mathbf{1}$ is again a finite valued increasing function with exponential growth. Indeed, its Laplace transform $\mathcal{L}(\mathcal{K} * \mathbf{1})(\gamma)$ is finite if and only if $c_3\mathcal{L}(k_\lambda + k_\Lambda)(\gamma) < 1$. This is the case for γ large enough since

$$\mathcal{L}(k_\lambda + k_\Lambda)(\gamma) = \frac{1}{\gamma} + \int_{\mathbb{R}^d} \frac{\mu(d\xi)}{\gamma + 8c\pi^2 |\xi|^2},$$

and the latter integral is finite for all $\gamma > 0$, by Assumption 4.1.

Proof. The computations of Lemma 4.12 can be written as follows:

$$\|I_0(t,x)\|_{L^p(\Omega)} \leq (C + c^{d/2})\,\|u_0\|\,J_c(t,x),$$
$$\|I_2(t,x)\|_{L^p(\Omega)} \leq k_p CK(1+C_2)J_c(t,x)(k_\Lambda * \mathbf{1})(t)^{1/2},$$
$$\|I_6(t,x)\|_{L^p(\Omega)} \leq k_p c^{d/2}K(1+C_1)J_c(t,x)(k_\Lambda * \mathbf{1})(t)^{1/2},$$
$$\|I_4(t,x)\|_{L^p(\Omega)} \leq k_p c^{d/2}\mathrm{Lip}J_c(t,x)(k_\Lambda * h)(t)^{1/2}.$$

We are left to compute I_1, I_3, and I_5. By Minkowski's inequality and linear growth of the function b, we get

$$\begin{aligned}\|I_1(t,x)\|_{L^p(\Omega)} &\leq \int_0^t ds \int_{LD} |F_L(t-s,x,y)|\,K\left(1+\|u_L(s,y)\|_{L^p(\Omega)}\right)\,dy \\ &\leq K(1+C_2)\int_0^t ds \int_{LD} |F_L(s,x,y)|\,dy.\end{aligned}$$

By (4.12) and the definition of k_λ,

$$\begin{aligned}\|I_1(t,x)\|_{L^p(\Omega)} &\leq CK(1+C_2)J_c(t,x)\int_0^t ds \int_{\mathbb{R}^d} \Gamma_c(s,x-y)\,dy. \\ &= CK(1+C_2)J_c(t,x)\int_0^t k_\lambda(s,c)\,ds.\end{aligned}$$

Similarly, by Minkowski's inequality, linear growth of b, and Lemma (4.8), we get

$$\begin{aligned}\|I_5(t,x)\|_{L^p(\Omega)} &\leq \int_0^t ds \int_{\mathbb{R}^d\setminus LD} \Gamma(t-s,x-y)K\left(1+\|u(s,y)\|_{L^p(\Omega)}\right)\,dy \\ &\leq c^{d/2}K(1+C_1)J_c(t,x)\int_0^t ds \int_{\mathbb{R}^d} \Gamma_c(s,x-y)\,dy \\ &= c^{d/2}K(1+C_1)J_c(t,x)\int_0^t k_\lambda(s,c)\,ds.\end{aligned}$$

By Minkowski's inequality, the fact that b is Lipschitz, and recalling definitions of (4.10), we get

$$\begin{aligned}\|I_3(t,x)\|_{L^p(\Omega)} &\leq \int_0^t ds \int_{LD} \Gamma(t-s,x-y)\,\|b(u(s,y)) - b(u_L(s,y))\|_{L^p(\Omega)}\,dy \\ &\leq \mathrm{Lip}\int_0^t ds \int_{LD} \Gamma(t-s,x-y)\,\|u(s,y)-u_L(s,y)\|_{L^p(\Omega)}\,dy \\ &= \mathrm{Lip}\,J_c(t,x)\int_0^t ds \int_{LD} \Gamma(t-s,x-y)\frac{J_c(s,y)}{J_c(t,x)}\frac{f_L(s,y)}{J_c(s,y)}\,dy \\ &\leq \mathrm{Lip}\,J_c(t,x)\int_0^t ds\, h(s,c)^{1/2}\int_{LD} \Gamma(t-s,x-y)\frac{J_c(s,y)}{J_c(t,x)}\,dy.\end{aligned}$$

The arguments in the proof of Lemma 4.13 yield

$$\Gamma(t-s,x-y)\frac{J_c(s,y)}{J_c(t,x)} \leq c^{d/2}\Gamma_c(t-s,x-y),$$

for all $x, y \in LD$, and $c \geq 2$. Thus,

$$\begin{aligned}\|I_3(t,x)\|_{L^p(\Omega)} &\leq c^{d/2}\mathrm{Lip}\, J_c(t,x)\int_0^t ds\, h(s,c)^{1/2}\int_{\mathbb{R}^d}\Gamma_c(t-s,x-y)\,dy \\ &= c^{d/2}\mathrm{Lip}\, J_c(t,x)\int_0^t h(s,c)^{1/2}k_\lambda(t-s,c)\,ds.\end{aligned}$$

If we put these previous seven bounds together, we reach the following

$$\begin{aligned}h(t)^{1/2} &\leq (C+c^{d/2})\,\|u_0\| \\ &\quad + CK(1+C_2)(k_\lambda * \mathbf{1})(t) + c^{d/2}K(1+C_1)(k_\lambda * \mathbf{1})(t) \\ &\quad + k_pCK(1+C_2)(k_\Lambda * \mathbf{1})(t)^{1/2} + k_pc^{d/2}K(1+C_1)(k_\Lambda * \mathbf{1})(t)^{1/2} \\ &\quad + c^{d/2}\mathrm{Lip}(k_\lambda * h^{1/2})(t) + k_p\mathrm{Lip}c^{d/2}(k_\Lambda * h)(t)^{1/2},\end{aligned}$$

where $k_\lambda(s) = k_\lambda(s,c)$, $k_\Lambda(s) = k_\Lambda(s,2c)$, and $h(s) = h(s,c)$. If we define $\bar{C} = C + c^{d/2}$, and $C_3 = \max(C_1, C_2)$, we get

$$\begin{aligned}h(t)^{1/2} &\leq \bar{C}\,\|u_0\| \\ &\quad + \bar{C}K(1+C_3)(k_\lambda * \mathbf{1})(t) + k_p\bar{C}K(1+C_3)(k_\Lambda * \mathbf{1})(t)^{1/2} \\ &\quad + c^{d/2}\mathrm{Lip}(k_\lambda * h^{1/2})(t) + k_pc^{d/2}\mathrm{Lip}(k_\Lambda * h)(t)^{1/2}.\end{aligned}$$

We now simplify the latter inequality. Observe that we have different powers of the function h on both sides. Squaring both side, we get

$$\begin{aligned}h(t) &\leq 5\bar{C}^2\,\|u_0\|^2 \\ &\quad + 5\bar{C}^2K^2(1+C_3)^2(k_\lambda * \mathbf{1})(t)^2 + 5k_p^2\bar{C}^2K^2(1+C_3)^2(k_\Lambda * \mathbf{1})(t) \\ &\quad + 5c^d\mathrm{Lip}^2(k_\lambda * h^{1/2})(t)^2 + 5k_p^2c^d\mathrm{Lip}^2(k_\Lambda * h)(t).\end{aligned}$$

Using Cauchy-Schwarz inequality, we can bound

$$(k_\lambda * h^{1/2})(t)^2 \leq (k_\lambda * \mathbf{1})(t)\cdot(k_\lambda * h)(t).$$

Furthermore, recall that $k_\lambda(t) = 1$, for all $t > 0$, and thus $(k_\lambda * \mathbf{1})(t) = t \leq T$. If we set $c_1 = 5\bar{C}^2\,\|u_0\|^2$, $c_2 = 5\bar{C}^2K^2(1+C_3)^2\max(T,k_p^2)$, and $c_3 = 5c^d\mathrm{Lip}^2\max(T,k_p^2)$, then

$$h(t) \leq c_1 + c_2(k_\lambda * \mathbf{1})(t) + c_2(k_\Lambda * \mathbf{1})(t) + c_3(k_\lambda * h)(t) + c_3(k_\Lambda * h)(t),$$

for all $t \in [0,T]$, which concludes the proof. □

We summarize our findings in the next three results. First, in the case of vanishing Dirichlet boundary conditions, for any open set $D \ni 0$ we have:

Theorem 4.21. *Uniformly for $t \in [0,T]$, $L > 0$, and $x \in LD$, we have*

$$\|u(t,x) - u_L(t,x)\|_{L^p(\Omega)} \leq \Theta(t) \exp\left(-\frac{\operatorname{dist}(x, \partial LD)^2}{4ct}\right),$$

for some increasing function $\Theta(t) = \Theta(t, T, Lip, K, \Lambda, u_0, p)$ and some positive constant c.

Remark. By Lemma 4.5, a possible value is $c = 4d$.

Second, for the square domain $D = (-1,1)^d$, which on each side have either Dirichlet or Neumann boundary conditions, we have:

Theorem 4.22. *Fix any $l > 0$. Uniformly for $t \in [0,T]$, $L > \sqrt{l/4T}$, and $x \in LD$, we have*

$$\|u(t,x) - u_L(t,x)\|_{L^p(\Omega)} \leq \Theta(t) \exp\left(-\frac{\operatorname{dist}(x, \partial LD)^2}{4ct}\right),$$

for some increasing function $\Theta(t) = \Theta(t, T, Lip, K, \Lambda, u_0, p, l)$ and some positive constant c.

Remark. If none of both opposite surfaces have Neumann boundary conditions, then the results holds for all $L > 0$. Furthermore, by Lemma 4.4, $c = 4$ is a possible value.

Third, in the case of Neumann boundary conditions, for any open regular domain $D \ni 0$ (i.e. its boundary belongs to $C^{2+\alpha}$ for some $\alpha \in (0,1)$), we have:

Theorem 4.23. *Uniformly for $t \in [0,T]$, $L \geq 1$, and $x \in LD$, we have*

$$\|u(t,x) - u_L(t,x)\|_{L^p(\Omega)} \leq \Theta(t) \exp\left(-\frac{\operatorname{dist}(x, \partial LD)^2}{4ct}\right),$$

for some increasing function $\Theta(t) = \Theta(t, T, Lip, K, \Lambda, u_0, p, D)$ and some positive constant $c = c(T, D)$.

Chapter 5

Anderson model with correlated noise

In this chapter, we will study the heat equation in any space dimension, $d \geq 1$, with a particular form of multiplicative noise. Anderson's model is the case where the functions $b \equiv 0$ and $\sigma(u) = \lambda u$, for some non zero constant $\lambda \in \mathbb{R}$. As in the previous chapter, correlated noises will be considered. Unlike any previous discussion, we will allow the initial condition to be a positive measure μ_0 that satisfies the growth condition:

$$\int_{\mathbb{R}^d} e^{-a|x|^2} \mu_0(dx) < \infty, \quad \text{for all a } > 0. \tag{5.1}$$

This condition allows the homogeneous heat equation to have the following solution

$$u^0(t,x) = \int_{\mathbb{R}^d} \Gamma(t, x-y)\, \mu_0(dy).$$

We denote $\mathcal{M}_H(\mathbb{R}^d)$ the set of positive measures for which (5.1) holds.

We will compare the behavior of the (random field) solution on the whole space $\mathbb{R}^d$ with the (random field) solution on some bounded open domain $D \subseteq \mathbb{R}^d$, with vanishing Dirichlet boundary conditions, i.e.

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) = \Delta u(t,x) + \lambda u\, \dot{\mathrm{M}}, & t > 0,\, x \in \mathbb{R}^d, \\ u(0,dx) = \mu_0(dx), & x \in \mathbb{R}^d, \end{cases} \tag{5.2}$$

and

$$\begin{cases} \dfrac{\partial u_D}{\partial t}(t,x) = \Delta u_D(t,x) + \lambda u_D\, \dot{\mathrm{M}}, & t > 0,\, x \in D, \\ u_D(t,x) = 0, & t > 0,\, x \in \partial D, \\ u_D(0,dx) = \mu_0(dx), & x \in D, \end{cases} \tag{5.3}$$

where $\dot{\mathrm{M}}$ is a noise on $\mathbb{R}_+ \times \mathbb{R}^d$ that is white in time and correlated in space. Recall Assumption 4.1 about the present correlated noise. The positive measure Λ is tempered, symmetric, and positive definite, and satisfies

$$\int_0^t \int_{\mathbb{R}^d} \Lambda(dz)\, \Gamma_\nu(s,z) < \infty,$$

for all $t > 0$ (and for all $\nu > 0$). The latter integral condition is equivalent to Dalang's condition

$$\Upsilon(\beta) := \int_{\mathbb{R}^d} \frac{\mu(d\xi)}{\beta + |\xi|^2} < \infty, \quad \text{for some and hence for all } \beta > 0. \tag{5.4}$$

The positive measure μ is also tempered and positive definite. It is called the spectral measure of Λ. In the sense of distributions, Λ is the Fourier transform of μ, i.e. $\mathcal{F}\mu = \Lambda$.

Under those assumptions, the Anderson models (5.2) and (5.3) admit unique random field solutions, that satisfy the representation formulas

$$\begin{aligned} u(t,x) = &\int_{\mathbb{R}} \Gamma(t, x-y)\, \mu_0(dy) \\ &+ \lambda \int_0^t \int_{\mathbb{R}} \Gamma(t-s, x-y) u(s,y)\, \mathrm{M}(ds,dy), \end{aligned} \tag{5.5}$$

and

$$\begin{aligned} u_D(t,x) = &\int_D G_D(t,x,y)\, \mu_0(dy) \\ &+ \lambda \int_0^t \int_D G_D(t-s,x,y) u_D(s,y)\, \mathrm{M}(ds,dy), \end{aligned} \tag{5.6}$$

where Γ is the heat kernel, given by (A.8), and G_D is the Green function associated with Dirichlet boundary conditions. See [9, Theorem 2.4] for existence and uniqueness of these random field solutions.

5.1 Main results and general ideas

As in the previous chapters, we will conclude that $u_D(t)$ converges to $u(t)$ as the domain D expands to the whole space. We will in fact be able to say much more about the rate of convergence. In some special cases, we shall not be restricted to any finite time horizon.

All these good news are consequences of an “explicit" formula for the two points correlation

$$\begin{aligned} &\mathbb{E}\left[\{u(t,x) - u_D(t,x)\}\{u(t,y) - u_D(t,x')\}\right] \\ &\quad = \mathbb{E}\left[u(t,x)u(t,x')\right] - \mathbb{E}\left[u(t,x)u_D(t,x')\right] \\ &\qquad - \mathbb{E}\left[u_D(t,x)u(t,x')\right] + \mathbb{E}\left[u_D(t,x)u_D(t,x')\right]. \end{aligned}$$

In the present case of Anderson's model, Le Chen and Kunwoo Kim found an explicit formula for the two points correlation

$$\mathbb{E}\left[u(t,x)u(t,x')\right] = \iint_{\mathbb{R}^{2d}} \mu_0(d\alpha)\mu_0(d\alpha')\,\mathcal{K}(t,x,x',\alpha,\alpha').$$

The latter formula is amazing! The mysterious function $\mathcal{K}$ will gather informations about the noise and the fundamental solution to the heat equation only. The initial condition μ_0 appears only when integrating that function. Thus, any information about the function $\mathcal{K}$ turns into a precious insight for the computation of the two points correlation. In this direction, Lemmas 5.21 and 5.23 could turn into precious allies. These computations were possible thanks to a clearer way of writing the function $\mathcal{K}$ and its components. In fact, Section 6 will be all about potential generalization to other classes of noises, to other classes of parabolic equations, and to distribution-valued initial condition, provided that the stochastic integral still makes sense. A derivation of this formula is given in Lemma 5.5.

We shall derive similar "explicit" expressions for $\mathbb{E}\left[u_D(t,x)u_D(t,y)\right]$ and $\mathbb{E}\left[u(t,x)u_D(t,y)\right]$, which will lead to

Theorem 5.1. *For all $\mu_0 \in \mathcal{M}_H(\mathbb{R}^d)$, and $t>0$, $x,x' \in \mathbb{R}^d$,*

$$\begin{aligned} 0 &\le \mathbb{E}\left[\left\{u(t,x)-u_D(t,x)\right\}\left\{u(t,x')-u_D(t,x')\right\}\right] \\ &= \iint_{\mathbb{R}^{2d}} \mu_0(d\alpha)\mu_0(d\alpha')\,\left[\mathcal{K}_{1,1}-\mathcal{K}_{1,2}-\mathcal{K}_{2,1}+\mathcal{K}_{2,2}\right](t,x,x',\alpha,\alpha'). \end{aligned}$$

The main idea is to extract as much information as possible from the difference $\mathcal{K}_{1,1}-\mathcal{K}_{1,2}-\mathcal{K}_{2,1}+\mathcal{K}_{2,2}$, and combine it with the behavior of the initial condition. For example,

Theorem 5.2. *If the initial measure $\mu_0(dx) = u_0(x)dx$ for some non-negative bounded function $0 \le u_0(x) \le M$, then there exists some constants c, γ, $c_\gamma > 0$ such that for all $t \in [0,T]$, $x,x' \in D$, we have*

$$\begin{aligned} &\mathbb{E}\left[\left\{u(t,x)-u_D(t,x)\right\}\left\{u(t,x')-u_D(t,x')\right\}\right] \\ &\qquad \le c_\gamma e^{\gamma t} M \exp\left\{-\frac{\operatorname{dist}(x,\partial D)^2}{ct}\right\}\exp\left\{-\frac{\operatorname{dist}(x',\partial D)^2}{ct}\right\}. \end{aligned} \tag{5.7}$$

As was done in the previous chapters, we shall apply the previous result to the dilations

$$LD := \{Lx \in \mathbb{R}^d : x \in D\}, \quad L>0,$$

of some bounded open domain D, containing the origin, and prove that the above constants c,γ,c_γ do not depend on the parameter L. That is a direct consequence of Corollary 3.11, when the domain has a regular boundary.

Corollary 5.3. *If the boundary ∂D is sufficiently regular, i.e. belongs to $C^{2+\alpha}$, then under the same hypotheses of Theorem 5.2, we have*

$$\mathbb{E}\left[\{u(t,x)-u_{LD}(t,x)\}\{u(t,x')-u_{LD}(t,x')\}\right] \le c_\gamma e^{\gamma t} M \exp\left\{-\frac{\operatorname{dist}(x,\partial LD)^2}{ct}\right\}\exp\left\{-\frac{\operatorname{dist}(x',\partial LD)^2}{ct}\right\}, \quad (5.8)$$

for all $t\in[0,L^2T]$ and $x,x'\in LD$. The constants c,γ,c_γ depend on α, d, T, λ, Λ, and on the domain D, but not on the constant L.

In some particular cases, such as rectangle domains, inequalities (3.9) and (3.10) are in fact valid for all $t>0$ instead of $t\in[0,T]$. An application of Corollary 3.11 translates into the validity of bound (5.8) for $t>0$ and $x,x'\in LD$. We had such results in one space dimension in the case of Dirichlet or mixed boundary conditions, see (2.16) and (2.19). An application of (A.27) concludes the arguments for rectangular domains.

If we restrict ourselves to points x,x' within some compact set, then we can allow the initial condition to have polynomial growth.

Here is how we shall extract informations about the difference $\mathcal{K}_{1,1}-\mathcal{K}_{1,2}-\mathcal{K}_{2,1}+\mathcal{K}_{2,2}$. First, each of the $\mathcal{K}_{i,j}$'s functions will be expressed as an infinite series of the form

$$\mathcal{K}_{i,j}=\sum_{k=1}^{\infty}\mathcal{L}_{i,j}^{\triangleright k},$$

where $\mathcal{L}_{i,j}^{\triangleright k}$ is the k-th iteration of some associative triangle operator,

$$\mathcal{L}_{i,j}^{\triangleright k}=\underbrace{\mathcal{L}_{i,j}\triangleright\mathcal{L}_{i,j}\triangleright\cdots\triangleright\mathcal{L}_{i,j}}_{k\text{-times}}.$$

In fact, the function $\mathcal{L}_{i,j}$ will be expressed solely in terms of the fundamental solution or the Green function, and the triangle operator will solely contain information about the noise. The difference of interest can be decomposed into

$$\mathcal{K}_{1,1}-\mathcal{K}_{1,2}-\mathcal{K}_{2,1}+\mathcal{K}_{2,2}=\sum_{k=1}^{\infty}\left[\mathcal{L}_{1,1}^{\triangleright k}-\mathcal{L}_{1,2}^{\triangleright k}-\mathcal{L}_{2,1}^{\triangleright k}+\mathcal{L}_{2,2}^{\triangleright k}\right].$$

The plan is to find an appropriate bound for each component. The intuition is easily carried for the first iteration. Indeed, we shall see that

$$0\le\left[\mathcal{L}_{1,1}-\mathcal{L}_{1,2}-\mathcal{L}_{2,1}+\mathcal{L}_{2,2}\right](t,x,x',\alpha,\alpha') \le \Gamma_c(t,x-\alpha)\Gamma_c(t,x'-\alpha')f_c(t,x,\alpha)f_c(t,x',\alpha'),$$

where

$$f_c(t,x,\alpha)=\mathbf{1}_{D\times D}(x,\alpha)e^{-\frac{\operatorname{dist}(x,\partial D)^2}{ct}}e^{-\frac{\operatorname{dist}(\alpha,\partial D)^2}{ct}}+\mathbf{1}_{(\mathbb{R}^d\times\mathbb{R}^d)\setminus(D\times D)}(x,\alpha).$$

The intuition is as follows. First, we will only consider points $x, x' \in D$. Second, recall that the α and α' variables are to be integrated with respect to the initial condition. Third, observe that the bound splits into the set of variables (t, x, α) and (t, x', α'). Thus, the double integral could become a product of two integrals. Finally, we consider one of the two integrals, say the one with the (t, x, α) variables. For $\alpha \in D$, the expected rate of convergence, $\exp\{-\operatorname{dist}(x, \partial D)^2/(ct)\}$, appears in the function $f_c(t, x, \alpha)$, and can be taken outside of integration. For $\alpha \in \mathbb{R}^d \setminus D$, we expect the heat kernel $\Gamma_c(t, x-\alpha)$ to give it to us. This will be the case when bounded initial data is assumed.

That procedure is exactly the one used in Chapters 2 and 4. See how we decomposed the difference $u(t,x) - u_L(t,x)$ in Sections 2.5 and 4.4.

The objective is to get a similar bound for each difference $\mathcal{L}_{1,1}^{\triangleright k} - \mathcal{L}_{1,2}^{\triangleright k} - \mathcal{L}_{2,1}^{\triangleright k} + \mathcal{L}_{2,2}^{\triangleright k}$, and eventually for the whole series.

Theorem 5.4. *For any $x, x', \alpha, \alpha' \in \mathbb{R}^d$, we have*

$$\begin{aligned} 0 &\leq \left[\mathcal{K}_{1,1} - \mathcal{K}_{1,2} - \mathcal{K}_{2,1} + \mathcal{K}_{2,2}\right](t, x, x', \alpha, \alpha') \\ &\leq Q(t, x - x', \alpha - \alpha')\Gamma_c(t, x-\alpha)\Gamma_c(t, x'-\alpha') f_c(t, x, \alpha) f_c(t, x', \alpha'), \end{aligned}$$

where Q is expressed as an infinite series

$$Q(t, x, y) = \sum_{k=1}^{\infty} c^k \, q^{\triangleleft k}(t, x, y),$$

for some inverse triangle $\triangleleft$ operator.

We shall study the relation between these two triangle operators. In particular, the density of a Brownian bridge will make an appearance, from which associativity of the inverse triangle operator will be deduced.

The final ingredient to prove Theorem 5.2 is a uniform bound for the infinite series $Q(t, x, y) \leq Q(t)$. The latter was obtained in [9, Lemma 2.7]. It is an increasing function with exponential growth.

Remark. In the case of vanishing Neumann boundary conditions, similar computations are possible thanks to Lemma 4.7. In the case of vanishing Dirichlet boundary conditions, assumptions regarding the regularity of the boundary are no longer necessary thanks to Lemma 4.5.

5.2 Two points correlation, proof of Theorem 5.1

The two point correlation formulas obtained in this section are strongly inspired by [9, Section 2.2]. The present derivations look somehow more natural and allow, once more, the use of the Picard iteration scheme. We set

$$u^0(t,x) = \int_{\mathbb{R}^d} \Gamma(t, x-y)\, \mu_0(dy),$$
$$u^{n+1}(t,x) = u^0(t,x) + \lambda \iint_{[0,t)\times\mathbb{R}^d} \Gamma(t-s, x-y) u^n(s,y)\, \mathrm{M}(ds, dy).$$

The fact that the initial condition μ_0 is non-random and positive, together with positivity of the fundamental solution $\Gamma(t,x)$, imply

$$\begin{aligned}\mathcal{J}(t,x,x') &:= \mathbb{E}\left[u^0(t,x)u^0(t,x')\right] = u^0(t,x)u^0(t,x') \\ &= \iint_{\mathbb{R}^{2d}} \mu_0(dy)\mu_0(dy')\, \Gamma(t,x-y)\Gamma(t,x'-y') \geq 0.\end{aligned}$$

The fact that the stochastic integral has zero mean implies that

$$\begin{aligned}&\mathbb{E}\left[u^{n+1}(t,x)u^{n+1}(t,x')\right] = \\ &\mathcal{J}(t,x,x') + \lambda^2 \int_0^t ds \int_{\mathbb{R}^d} \Lambda(dz) \int_{\mathbb{R}^d} dy\, \Gamma(t-s, x-y)\Gamma(t-s, x'-(y-z)) \\ &\qquad \times \mathbb{E}\left[u^n(s,y)u^n(s,y-z)\right].\end{aligned}$$

To simplify computations, we introduce the triangle operator. For $f, g : \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$, we set

$$\begin{aligned}(f \triangleright g)(t,x,x',\alpha,\alpha') := \lambda^2 \int_0^t ds \int_{\mathbb{R}^d} \Lambda(dz) \int_{\mathbb{R}^d} dy\, f(t-s, x, x', y, y-z) \\ \times g(s, y, y-z, \alpha, \alpha'). \quad (5.9)\end{aligned}$$

If we set $g^n(t,x,x') := \mathbb{E}\left[u^n(t,x)u^n(t,x')\right]$ and

$$\mathcal{L}(t,x,x',\alpha,\alpha') := \Gamma(t, x-\alpha)\Gamma(t, x'-\alpha'),$$

then we can write in a concise form

$$g^{n+1}(t,x,x') = \mathcal{J}(t,x,x') + (\mathcal{L} \triangleright g_n)(t,x,x',0,0). \quad (5.10)$$

For functions of only two space variables $g : \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$, we will use the following extension to four space variables

$$g(t,x,x',\alpha,\alpha') := g(t,x,x').$$

Corollary 5.8 below ensures associativity of the triangle $\triangleright$ operation. It is thus non-ambiguous to define the k-th iteration

$$\mathcal{L}^{\triangleright k} = \underbrace{\mathcal{L} \triangleright \mathcal{L} \triangleright \cdots \triangleright \mathcal{L}}_{k\text{-times}}.$$

Applying these definitions to the Picard iteration scheme gives

$$\begin{aligned}
g^0(t,x,x') &= \mathcal{J}(t,x,x'),\\
g^1(t,x,x') &= \mathcal{J}(t,x,x') + (\mathcal{L} \triangleright \mathcal{J})(t,x,x',0,0),\\
g^2(t,x,x') &= \mathcal{J}(t,x,x') + \left(\mathcal{L} \triangleright g^1\right)(t,x,x',0,0)\\
&= \mathcal{J}(t,x,x') + (\mathcal{L} \triangleright \mathcal{J})(t,x,x',0,0) + \left(\mathcal{L}^{\triangleright 2} \triangleright \mathcal{J}\right)(t,x,x',0,0),\\
&\ \vdots\\
g^n(t,x,x') &= \mathcal{J}(t,x,x') + \left[\left(\sum_{k=1}^{n} \mathcal{L}^{\triangleright k}\right) \triangleright \mathcal{J}\right](t,x,x',0,0).
\end{aligned}$$

Observe that each iteration is positive. Indeed, the initial step $\mathcal{J}$ is positive and each step is obtained from it by applications of the triangle operator, which contains the positive measure Λ. By monotone convergence, the limit $g(t,x,x') := \lim_{n\to\infty} g^n(t,x,x')$ exists and equals

$$g(t,x,x') = \mathcal{J}(t,x,x') + [\mathcal{K} \triangleright \mathcal{J}](t,x,x',0,0), \tag{5.11}$$

where

$$\mathcal{K}(t,x,x',\alpha,\alpha') = \left(\sum_{k=1}^{\infty} \mathcal{L}^{\triangleright k}\right)(t,x,x',\alpha,\alpha').$$

By (5.10) and monotone convergence, we can see that the limit g satisfies

$$g = \mathcal{J} + (\mathcal{L} \triangleright g)\,.$$

From the fact that $u^n(t,x) \to u(t,x)$ in $L^2(\Omega,\mathbb{P})$, we get that the product $u^n(t,x)u^n(t,x') \to u(t,x)u(t,x')$ in $L^1(\Omega,\mathbb{P})$ and

$$\begin{aligned}
\mathbb{E}\left[u(t,x)u(t,x')\right] &= \lim_{n\to\infty} \mathbb{E}\left[u^n(t,x)u^n(t,x')\right]\\
&= \lim_{n\to\infty} g^n(t,x,x') = g(t,x,x').
\end{aligned}$$

Formula (5.11) gives the expectation of the solution as a combination of the initial condition, through $\mathcal{J}$, and the kernel $\mathcal{K}$. The next result gives a more convenient expression. It is an adaptation of [9, Lemma 2.8].

Lemma 5.5. *For all $\mu_0 \in \mathcal{M}_H(\mathbb{R}^d)$, and $t>0$, $x,x' \in \mathbb{R}^d$,*

$$\begin{aligned}
0 \le \mathbb{E}\left[u(t,x)u(t,x')\right] &= \mathcal{J}(t,x,x') + [\mathcal{K} \triangleright \mathcal{J}](t,x,x',0,0)\\
&= \iint_{\mathbb{R}^{2d}} \mu_0(d\alpha)\mu_0(d\alpha')\,\mathcal{K}(t,x,x',\alpha,\alpha').
\end{aligned}$$

Before we give the proof, we need to observe the following property about $\mathcal{J}$ and the kernel $\mathcal{K}$:

$$\mathcal{J}(t,x,x') = \iint_{\mathbb{R}^{2d}} \mu_0(d\alpha)\mu_0(d\alpha')\,\mathcal{L}(t,x,x',\alpha,\alpha'),$$

$$\mathcal{K} \triangleright \mathcal{L} = \sum_{k=1}^{\infty} \mathcal{L}^{\triangleright(k+1)} = \sum_{k=2}^{\infty} \mathcal{L}^{\triangleright k} = \mathcal{K} - \mathcal{L}.$$

Proof. This property comes from the link between the definition of $\mathcal{J}$, $\mathcal{L}$, and $\mathcal{K}$. Indeed,

$$\begin{aligned}
&\mathcal{K} \rhd \mathcal{J}(t,x,x',0,0) \\
&= \lambda^2 \int_0^t ds \int_{\mathbb{R}^d} \Lambda(dz) \int_{\mathbb{R}^d} dy\, \mathcal{K}(t-s,x,x',y,y-z)\mathcal{J}(s,y,y-z) \\
&\quad = \iint_{\mathbb{R}^{2d}} \mu_0(d\alpha)\mu_0(d\alpha')\, \lambda^2 \int_0^t ds \int_{\mathbb{R}^d} \Lambda(dz) \int_{\mathbb{R}^d} dy \\
&\quad \times \mathcal{K}(t-s,x,x',y,y-z)\mathcal{L}(s,y,y-z,\alpha,\alpha') \\
&\quad = \iint_{\mathbb{R}^{2d}} \mu_0(d\alpha)\mu_0(d\alpha')(\mathcal{K} \rhd \mathcal{L})(t,x,x',\alpha,\alpha') \\
&\qquad = \iint_{\mathbb{R}^{2d}} \mu_0(d\alpha)\mu_0(d\alpha')\, \mathcal{K}(t,x,x',\alpha,\alpha') - \mathcal{J}(t,x,x'),
\end{aligned}$$

which concludes the proof. □

Remark. In equation (5.2), the term $\lambda u \dot{\mathrm{M}}$ could be understood either as $(\lambda u)\dot{\mathrm{M}}$ or as $u(\lambda\dot{\mathrm{M}})$. The second may be more appropriate. It corresponds to a scaling of the correlated noise. Indeed, when taking covariance, the expression λ^2 appears every time the measure Λ appears. For example, see (5.9). Therefore, upon replacing Λ by $\lambda^2\Lambda$, we may assume that $\lambda = 1$. Why not $\lambda = -1$? Well, in the present case of Gaussian noise, both would have the same laws. Moreover, the stochastic integral has mean zero by definition and we are solely interested in second moment computations.

Remark. We can now confirm the prediction that the function $\mathcal{L}$ is expressed solely in terms of the fundamental solution to the heat equation, and the triangle $\rhd$ operator solely contains information about the noise. They interact through the function $\mathcal{K}$, which is computed without any information about the initial condition μ_0.

Similar expressions for $\mathbb{E}\left[u_D(t,x)u_D(t,y)\right]$ and $\mathbb{E}\left[u(t,x)u_D(t,y)\right]$ are now derived. To simplify notations, we set

$$\begin{aligned}
G_1(t,x,y) &:= \Gamma(t,x-y) \quad \text{and} \quad G_2(t,x,y) := G_D(t,x,y); \\
\mathcal{L}_{i,j}(t,x,x',y,y') &:= G_i(t,x,y)G_j(t,x',y'), \quad \text{for any } i,j \in \{1,2\}. \qquad (5.12)
\end{aligned}$$

In fact, the exact same procedure can be applied. The triangle $\rhd$ operator remains unchanged, and is given by (5.9). The k-th iteration are

$$\mathcal{L}_{i,j}^{\rhd k} = \underbrace{\mathcal{L}_{i,j} \rhd \mathcal{L}_{i,j} \rhd \cdots \rhd \mathcal{L}_{i,j}}_{k\text{-times}}.$$

In order for integrals to make sense, we extend the Green function G_D : $\mathbb{R}_+ \times D \times D$ to $\mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d$, as a vanishing function on the complement $\mathbb{R}_+ \times \left((\mathbb{R}^d \times \mathbb{R}^d) \setminus (D \times D)\right)$. If we denote $u_1(t,x) = u(t,x)$ and $u_2(t,x) = u_D(t,x)$, then we have

Lemma 5.6. *For all $\mu_0 \in \mathcal{M}_H(\mathbb{R}^d)$, and $t > 0$, $x, x' \in \mathbb{R}^d$,*

$$0 \leq \mathbb{E}\left[u_i(t,x)u_j(t,x')\right] = \iint_{\mathbb{R}^{2d}} \mu_0(d\alpha)\mu_0(d\alpha')\,\mathcal{K}_{i,j}(t,x,x',\alpha,\alpha'),$$

where $\mathcal{K}_{i,j} = \sum_{k=1}^{\infty} \mathcal{L}_{i,j}^{\triangleright k}$.

Proof. Positivity of the two points correlation is a consequence of positivity of the Green function, see (A.30). □

The present notations $\mathcal{L}_{1,1}$ and $\mathcal{K}_{1,1}$ correspond to $\mathcal{L}$ and $\mathcal{K}$, previously defined.

We now show that the triangle $\triangleright$ operator is indeed associative. In fact, much more can be proved.

Lemma 5.7. *For any (positive) measure χ on $\mathbb{R}^n$ and any two (positive) functions $f, g : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+$, the following operation is associative*

$$(f \triangleright g)(u,w) = \int_{\mathbb{R}^n} f(u,v)g(v,w)\,\chi(dv).$$

See Lemma 6.13 for an extension to the case where χ is a distribution.

Proof. Thanks to Fubini's theorem, we have

$$\begin{aligned}
(f \triangleright (g \triangleright h))(u,w) &= \int_{\mathbb{R}^n} f(u,x)(g \triangleright h)(x,w)\,\chi(dx) \\
&= \int_{\mathbb{R}^n} \chi(dx) \int_{\mathbb{R}^n} \chi(dy)\, f(u,x)g(x,y)h(y,w) \\
&= \int_{\mathbb{R}^n} (f \triangleright g)(u,y)h(y,w)\,\chi(dy) = ((f \triangleright g) \triangleright h))(u,w),
\end{aligned}$$

which proves associativity. □

Corollary 5.8. *The triangle $\triangleright$ operator defined in* (5.9) *is associative.*

Proof. It can easily be shown explicitly. We will argue differently. Recall the notation for the reflection of some translated meaure

$$\int f(z)\,\widetilde{T_y\Lambda}(dz) := \int f(y-z)\Lambda(dz).$$

Taking $n = 2d$, $u = (x,x')$, $v = (y,z)$, and $w = (\alpha,\alpha')$ in the previous Lemma 5.7, as well as $\chi(dydz) = dy\,\widetilde{T_y\Lambda}(dz)$, we have

$$\begin{aligned}
\int_{\mathbb{R}^n} f(u,v)g(v,w)\,\chi(dv) &= \iint_{\mathbb{R}^{2d}} f(x,x',y,z)g(y,z,\alpha,\alpha')\,\chi(dydz) \\
&= \int_{\mathbb{R}^d} dy \int_{\mathbb{R}^d} \Lambda(dz) f(x,x',y,y-z)g(y,y-z,\alpha,\alpha') \\
&= \int_{\mathbb{R}^d} \Lambda(dz) \int_{\mathbb{R}^d} dy\, f(x,x',y,y-z)g(y,y-z,\alpha,\alpha'),
\end{aligned}$$

which is the part of (5.9) containing space integration. The time integration part of (5.9) corresponds to convolution, which is associative. □

To complete the proof of Theorem 5.1, it remains to show that the difference $\mathcal{K}_{1,1} - \mathcal{K}_{1,2} - \mathcal{K}_{2,1} + \mathcal{K}_{2,2}$ is positive. In fact, we will show that each difference $\mathcal{L}_{1,1}^{\triangleright n} - \mathcal{L}_{1,2}^{\triangleright n} - \mathcal{L}_{2,1}^{\triangleright n} + \mathcal{L}_{2,2}^{\triangleright n}$ is positive. It is basically a consequence of the fact that the Green function is dominated by the heat kernel, and that the latter difference can be written is a special way. First, we observe that

$$\begin{aligned}
&[\mathcal{L}_{1,1} - \mathcal{L}_{1,2} - \mathcal{L}_{2,1} + \mathcal{L}_{2,2}]\,(t,x,x',\alpha,\alpha') \\
&\qquad = G_1(t,x,\alpha)G_1(t,x',\alpha') - G_1(t,x,\alpha)G_2(t,x',\alpha') \\
&\qquad\qquad - G_2(t,x,\alpha)G_1(t,x',\alpha') + G_2(t,x,\alpha)G_2(t,x',\alpha'),
\end{aligned}$$

which can be factorized as

$$\begin{aligned}
&[\mathcal{L}_{1,1} - \mathcal{L}_{1,2} - \mathcal{L}_{2,1} + \mathcal{L}_{2,2}]\,(t,x,x',\alpha,\alpha') \\
&\qquad = [G_1(t,x,\alpha) - G_2(t,x,\alpha)]\left[G_1(t,x',\alpha') - G_2(t,x',\alpha')\right] \geq 0. \quad (5.13)
\end{aligned}$$

To compute the difference of the higher terms, we need to write explicitly each $\mathcal{L}_{i,j}^{\triangleright n}$. The notation is simpler when considering the index $n+1$ instead of n.

Lemma 5.9. *For each $n \geq 0$, we have*

$$\begin{aligned}
&\mathcal{L}_{i,j}^{\triangleright(n+1)}(t,x,x',\alpha,\alpha') \\
&\quad = \int_0^{s_0} ds_1 \cdots \int_0^{s_{n-1}} ds_n \int_{\mathbb{R}^d} \Lambda(dz_1) \cdots \int_{\mathbb{R}^d} \Lambda(dz_n) \int_{\mathbb{R}^d} dy_1 \cdots \int_{\mathbb{R}^d} dy_n \\
&\qquad\quad \prod_{k=0}^{n} G_i(s_k - s_{k+1}, y_k, y_{k+1}) G_j(s_k - s_{k+1}, y_k - z_k, y_{k+1} - z_{k+1}),
\end{aligned}$$

where we set $s_0 = t, y_0 = x, z_0 = x - x', s_{n+1} = 0, y_{n+1} = \alpha, z_{n+1} = \alpha - \alpha'$.

We interpret the case $n = 0$ in the obvious sense of (5.12).

Proof. For the case $n = 1$, by definition we have

$$\begin{aligned}
&\mathcal{L}_{i,j}^{\triangleright 2}(t,x,x',\alpha,\alpha') \\
&\qquad = \int_0^t ds \int_{\mathbb{R}^d} \Lambda(dz) \int_{\mathbb{R}^d} dy\, G_i(t-s,x,y) G_j(t-s,x',y-z) \\
&\qquad\qquad\qquad \times G_i(s,y,\alpha) G_j(s,y-z,\alpha'),
\end{aligned}$$

which has the desired form. For the induction step, we want to compute

$$\begin{aligned}
\mathcal{L}_{i,j}^{\triangleright(n+2)}(t,x,x',\alpha,\alpha') &= \left(\mathcal{L}_{i,j} \triangleright \mathcal{L}_{i,j}^{\triangleright(n+1)}\right)(t,x,x',\alpha,\alpha') \\
&= \int_0^t ds_1 \int_{\mathbb{R}^d} \Lambda(dz_1) \int_{\mathbb{R}^d} dy_1\, G_i(t-s_1,x,y_1) G_j(t-s_1,x',y_1-z_1) \\
&\quad \times \mathcal{L}_{i,j}^{\triangleright(n+1)}(s_1,y_1,y_1-z_1,\alpha,\alpha')
\end{aligned}$$

If we use the induction hypothesis, we can write

$$\begin{aligned}
&\mathcal{L}_{i,j}^{\triangleright(n+1)}(s_1, y_1, y_1 - z_1, \alpha, \alpha') \\
&= \int_0^{s_1} ds_2 \cdots \int_0^{s_n} ds_{n+1} \int_{\mathbb{R}^d} \Lambda(dz_2) \cdots \int_{\mathbb{R}^d} \Lambda(dz_{n+1}) \int_{\mathbb{R}^d} dy_2 \cdots \int_{\mathbb{R}^d} dy_{n+1} \\
&\qquad \prod_{k=1}^{n+1} G_i(s_k - s_{k+1}, y_k, y_{k+1}) G_j(s_k - s_{k+1}, y_k - z_k, y_{k+1} - z_{k+1}),
\end{aligned}$$

where we set $s_{n+2} = 0, y_{n+2} = \alpha, z_{n+2} = \alpha - \alpha'$. Putting everything back together, and applying Fubini's theorem, concludes the proof. □

In Lemma 5.21 below, we shall give another representation of the n-th iteration $\mathcal{L}_{1,1}^{\triangleright n} = \mathcal{L}^{\triangleright n}$ in terms of the inverse triangle $\triangleleft$ operator.

The following result concludes the proof of Theorem 5.1.

Corollary 5.10. *For each $n \geq 0$, we have*

$$\begin{aligned}
&\left[\mathcal{L}_{1,1}^{\triangleright(n+1)} - \mathcal{L}_{1,2}^{\triangleright(n+1)} - \mathcal{L}_{2,1}^{\triangleright(n+1)} + \mathcal{L}_{2,2}^{\triangleright(n+1)}\right](t, x, x', \alpha, \alpha') \\
&\quad = \int_0^{s_0} ds_1 \cdots \int_0^{s_{n-1}} ds_n \int_{\mathbb{R}^d} \Lambda(dz_1) \cdots \int_{\mathbb{R}^d} \Lambda(dz_n) \int_{\mathbb{R}^d} dy_1 \cdots \int_{\mathbb{R}^d} dy_n \\
&\qquad \left[\prod_{k=0}^{n} G_1(s_k - s_{k+1}, y_k, y_{k+1}) - \prod_{k=0}^{n} G_2(s_k - s_{k+1}, y_k, y_{k+1})\right] \\
&\qquad\qquad \times \left[\prod_{k=0}^{n} G_1(s_k - s_{k+1}, y_k - z_k, y_{k+1} - z_{k+1}) \right. \\
&\qquad\qquad\qquad \left. - \prod_{k=0}^{n} G_2(s_k - s_{k+1}, y_k - z_k, y_{k+1} - z_{k+1})\right], \qquad (5.14)
\end{aligned}$$

where we set $s_0 = t, y_0 = x, z_0 = x - x', s_{n+1} = 0, y_{n+1} = \alpha, z_{n+1} = \alpha - \alpha'$. In particular,

$$\left[\mathcal{L}_{1,1}^{\triangleright(n+1)} - \mathcal{L}_{1,2}^{\triangleright(n+1)} - \mathcal{L}_{2,1}^{\triangleright(n+1)} + \mathcal{L}_{2,2}^{\triangleright(n+1)}\right](t, x, x', \alpha, \alpha') \geq 0.$$

We interpret the case $n = 0$ in the obvious sense of (5.13).

Proof. This is a direct consequence of Lemma 5.9, equation (A.30), and the following factorization

$$\begin{aligned}
&\prod_{k=0}^{n} a_k b_k - \prod_{k=0}^{n} a_k d_k - \prod_{k=0}^{n} c_k b_k + \prod_{k=0}^{n} c_k d_k \\
&\qquad = \left[\prod_{k=0}^{n} a_k - \prod_{k=0}^{n} c_k\right] \times \left[\prod_{k=0}^{n} b_k - \prod_{k=0}^{n} d_k\right],
\end{aligned}$$

for all $a_k, b_k, c_k, d_k \in \mathbb{R}$. □

For the fun of it, we express now symmetries satisfied by each iteration. They are consequences of the symmetry of the heat kernel and the Green function, and that of the measure Λ.

Lemma 5.11. *For all $t > 0$, $x, x'\alpha, \alpha' \in \mathbb{R}^d$, and $i, j \in \{1, 2\}$, we have*

$$\begin{aligned}\mathcal{L}_{i,j}^{\triangleright n}(t, x, x', \alpha, \alpha') &= \mathcal{L}_{i,j}^{\triangleright n}(t, \alpha, \alpha', x, x'),\\ &= \mathcal{L}_{j,i}^{\triangleright n}(t, x', x, \alpha', \alpha),\end{aligned}$$

for all $n \in \mathbb{N}$.

Proof. From the definition of $\mathcal{L}_{i,j}(t, x, x', \alpha, \alpha') = G_i(t, x, \alpha)G_j(t, x', \alpha')$, we easily get

$$\mathcal{L}_{i,j}(t, x, x', \alpha, \alpha') = \mathcal{L}_{j,i}(t, x', x, \alpha', \alpha).$$

The symmetry property of both the heat kernel and the Green function translates into $G_i(t, x, \alpha) = G_i(t, \alpha, x)$ for any $i \in \{1, 2\}$, see (A.30). Thus,

$$\begin{aligned}\mathcal{L}_{i,j}(t, x, x', \alpha, \alpha') &= \mathcal{L}_{i,j}(t, \alpha, x', x, \alpha') = \mathcal{L}_{i,j}(t, x, \alpha', \alpha, x')\\ &= \mathcal{L}_{i,j}(t, \alpha, \alpha', x, x').\end{aligned}$$

We prove now the first symmetry. By induction and the change of variable $\bar{s} = t - s$, we have

$$\begin{aligned}&\mathcal{L}_{i,j}^{\triangleright(n+1)}(t, x, x', \alpha, \alpha') = \left(\mathcal{L}_{i,j}^{\triangleright n} \triangleright \mathcal{L}_{i,j}\right)(t, x, x', \alpha, \alpha')\\ &= \int_0^t ds \int_{\mathbb{R}^d} \Lambda(dz) \int_{\mathbb{R}^d} dy\, \mathcal{L}_{i,j}^{\triangleright n}(t - s, x, x', y, y - z)\mathcal{L}_{i,j}(s, y, y - z, \alpha, \alpha')\\ &= \int_0^t d\bar{s} \int_{\mathbb{R}^d} \Lambda(dz) \int_{\mathbb{R}^d} dy\, \mathcal{L}_{i,j}^{\triangleright n}(\bar{s}, y, y - z, x, x')\mathcal{L}_{i,j}(t - \bar{s}, \alpha, \alpha', y, y - z)\\ &\qquad = \left(\mathcal{L}_{i,j} \triangleright \mathcal{L}_{i,j}^{\triangleright n}\right)(t, \alpha, \alpha', x, x') = \mathcal{L}_{i,j}^{\triangleright(n+1)}(t, \alpha, \alpha', x, x'),\end{aligned}$$

For the second, we make use of the symmetry of Λ, i.e. for any Borel set B, we have the equality $\Lambda(B) = \Lambda(-B)$. By induction and the changes of variables $\bar{y} = y - z$, and $\bar{z} = -z$, we have

$$\begin{aligned}&\mathcal{L}_{j,i}^{\triangleright(n+1)}(t, x', x, \alpha', \alpha)\\ &= \int_0^t ds \int_{\mathbb{R}^d} \Lambda(dz) \int_{\mathbb{R}^d} dy\, \mathcal{L}_{j,i}^{\triangleright n}(t - s, x', x, y, y - z)\mathcal{L}_{j,i}(s, y, y - z, \alpha', \alpha)\\ &= \int_0^t ds \int_{\mathbb{R}^d} \Lambda(dz) \int_{\mathbb{R}^d} dy\, \mathcal{L}_{i,j}^{\triangleright n}(t - s, x, x', y - z, y)\mathcal{L}_{i,j}(s, y - z, y, \alpha, \alpha')\\ &= \int_0^t ds \int_{\mathbb{R}^d} \Lambda(d\bar{z}) \int_{\mathbb{R}^d} d\bar{y}\, \mathcal{L}_{i,j}^{\triangleright n}(t - s, x, x', \bar{y}, \bar{y} - \bar{z})\mathcal{L}_{i,j}(s, \bar{y}, \bar{y} - \bar{z}, \alpha, \alpha')\\ &\qquad = \mathcal{L}_{i,j}^{\triangleright(n+1)}(t, x, x', \alpha, \alpha').\end{aligned}$$

The proof is completed. □

In fact, another property is satisfied by the n-th iteration $\mathcal{L}_{1,1}^{\triangleright n} = \mathcal{L}^{\triangleright n}$. As we shall see in Lemma 5.21, its dependence on the four variables x, x', α, α' is only an illusion. It depends only through their differences $x - x', \alpha - \alpha', x - \alpha, x' - \alpha'$.

5.3 Estimations for each iteration difference

The purpose of this section is to prove the following estimation about each iteration $\mathcal{L}_{1,1}^{\triangleright n} - \mathcal{L}_{1,2}^{\triangleright n} - \mathcal{L}_{2,1}^{\triangleright n} + \mathcal{L}_{2,2}^{\triangleright n}$, which will be used later to estimate the difference $\mathcal{K}_{1,1} - \mathcal{K}_{1,2} - \mathcal{K}_{2,1} + \mathcal{K}_{2,2}$, in Theorem 5.4.

Proposition 5.12. *For each $n \in \mathbb{N}$, there exists some function h_{n+1} such that for all $t \in [0,T]$, $x, x', \alpha, \alpha' \in \mathbb{R}^d$, we have*

$$\begin{aligned}\Big[\mathcal{L}_{1,1}^{\triangleright(n+1)} - \mathcal{L}_{1,2}^{\triangleright(n+1)} - \mathcal{L}_{2,1}^{\triangleright(n+1)} + \mathcal{L}_{2,2}^{\triangleright(n+1)}\Big]&(t,x,x',\alpha,\alpha') \\ &\leq c_{n+1}^2 \mathcal{L}_c^{\triangleright(n+1)}(t,x,x',\alpha,\alpha') f_c(t,x,\alpha) f_c(t,x',\alpha'),\end{aligned}$$

The constants are not important. Yet they can be expressed as follows: $c = 2\max(c_1, 4)$, $c_{n+1} = C^{n+1}(n+2)2^{(n+1)d/2}$, and

$$C = \max\left(C_1, \max(c_1,4)/\min(c_1,4)\right).$$

In order to understand the notation, we treat the case $n = 0$. Recall factorization (5.13),

$$\begin{aligned}\left[\mathcal{L}_{1,1} - \mathcal{L}_{1,2} - \mathcal{L}_{2,1} + \mathcal{L}_{2,2}\right]&(t,x,x',\alpha,\alpha') \\ &= \left[G_1(t,x,\alpha) - G_2(t,x,\alpha)\right]\left[G_1(t,x',\alpha') - G_2(t,x',\alpha')\right],\end{aligned}$$

and the extended Green function $G_2 = G_D : \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d$, which vanishes on $\mathbb{R}_+ \times \big((\mathbb{R}^d \times \mathbb{R}^d) \setminus (D \times D)\big)$. In case both $x, \alpha \in D$, we can use Corollary 3.11 to bound

$$G_1(t,x,\alpha) - G_2(t,x,\alpha) \leq C_1 \, \Gamma_{c_1}(t, x-\alpha) e^{-\frac{\mathrm{dist}(x,\partial D)^2}{c_1 t}} e^{-\frac{\mathrm{dist}(\alpha,\partial D)^2}{c_1 t}}.$$

In case x or α belongs to $\mathbb{R}^d \setminus D$, we have

$$G_1(t,x,\alpha) - G_2(t,x,\alpha) = G_1(t,x,\alpha) = \Gamma(t, x-\alpha).$$

We can gather theses two observations in one,

$$G_1(t,x,\alpha) - G_2(t,x,\alpha) \leq C \, \Gamma_c(t, x-\alpha) f_c(t,x,\alpha),$$

where

$$f_c(t,x,\alpha) = \mathbf{1}_{D\times D}(x,\alpha) e^{-\frac{\mathrm{dist}(x,\partial D)^2}{ct}} e^{-\frac{\mathrm{dist}(\alpha,\partial D)^2}{ct}} + \mathbf{1}_{(\mathbb{R}^d\times\mathbb{R}^d)\setminus(D\times D)}(x,\alpha),$$

and $c = \max(c_1, 4)$, $C = \max\left(C_1, \max(c_1, 4)/\min(c_1, 4)\right)$. Those constants are not important. In fact, they will change in the general case, but will be fixed for $n \geq 1$. Therefore,

$$\begin{aligned}
&\left[\mathcal{L}_{1,1} - \mathcal{L}_{1,2} - \mathcal{L}_{2,1} + \mathcal{L}_{2,2}\right](t, x, x', \alpha, \alpha') \\
&\qquad \leq C^2\, \Gamma_c(t, x-\alpha)\Gamma_c(t, x'-\alpha') f_c(t, x, \alpha) f_c(t, x', \alpha') \\
&\qquad\qquad = C^2\, \mathcal{L}_c(t, x, x', \alpha, \alpha') f_c(t, x, \alpha) f_c(t, x', \alpha'), \quad (5.15)
\end{aligned}$$

where $\mathcal{L}_c(t, x, x', \alpha, \alpha') := \Gamma_c(t, x-\alpha)\Gamma_c(t, x'-\alpha')$.

For the general case $n \geq 1$, we use equation (5.14), in which we make explicit the following difference:

$$\begin{aligned}
&\prod_{k=0}^{n} G_1(s_k - s_{k+1}, y_k, y_{k+1}) - \prod_{k=0}^{n} G_2(s_k - s_{k+1}, y_k, y_{k+1}) \\
&\qquad = \mathbf{1}_{D\times\cdots\times D}(x, y_1, \ldots, y_n, \alpha) \\
&\qquad \times \left[\prod_{k=0}^{n} G_1(s_k - s_{k+1}, y_k, y_{k+1}) - \prod_{k=0}^{n} G_2(s_k - s_{k+1}, y_k, y_{k+1})\right] \\
&\qquad + \mathbf{1}_{(D\times\cdots\times D)^c}(x, y_1, \ldots, y_n, \alpha) \prod_{k=0}^{n} G_1(s_k - s_{k+1}, y_k, y_{k+1}).
\end{aligned}$$

We will use two different strategies, depending on the value of the indicator function

$$\mathbf{1}_{D\times\cdots\times D}(x, y_1, \ldots, y_n, \alpha).$$

When its value is zero we will apply Lemma 5.13, and when it is one we will apply Lemma 5.15.

Lemma 5.13. *If $y_k \in \mathbb{R}^d \setminus D$ for some $k \in \{1, \ldots, n\}$, then*

$$\begin{aligned}
\prod_{k=0}^{n} G_1(s_k - s_{k+1}, y_k, y_{k+1}) &\leq 2^{(n+1)d/2} \prod_{k=0}^{n} G_1(2(s_k - s_{k+1}), y_k, y_{k+1}) \\
&\quad \times \exp\left\{-\frac{\operatorname{dist}(x, \partial D)^2}{8t}\right\} \exp\left\{-\frac{\operatorname{dist}(\alpha, \partial D)^2}{8t}\right\}.
\end{aligned}$$

Proof. First, we use the fact that

$$G_1(s, y, z) = 2^{d/2} G_1(2s, y, z) \exp\left\{-\frac{|y-z|^2}{8s}\right\}.$$

We fix $j \in \{1, \ldots, n\}$ such that $y_j \in \mathbb{R}^d \setminus D$. Recall that $y_0 = x$ and $y_{n+1} = \alpha$.

It remains to show that both

$$\prod_{k=0}^{j-1} \exp\left\{-\frac{|y_k - y_{k+1}|^2}{8(s_k - s_{k+1})}\right\} \leq \exp\left\{-\frac{|x-y_j|^2}{8(t-s_j)}\right\} \leq \exp\left\{-\frac{\mathrm{dist}(x,\partial D)^2}{8t}\right\},$$

$$\prod_{k=j}^{n} \exp\left\{-\frac{|y_k - y_{k+1}|^2}{8(s_k - s_{k+1})}\right\} \leq \exp\left\{-\frac{|y_j-\alpha|^2}{8s_j}\right\} \leq \exp\left\{-\frac{\mathrm{dist}(\alpha,\partial D)^2}{8t}\right\}.$$

We prove only the first one since the second one is analogous. We can rewrite

$$\prod_{k=0}^{j-1} \exp\left\{-\frac{|y_k - y_{k+1}|^2}{8(s_k - s_{k+1})}\right\} = \exp\left\{-\frac{1}{8(t-s_j)}\sum_{k=0}^{j-1} |y_k - y_{k+1}|^2 \lambda_k^{-1}\right\},$$

where $\lambda_k = \frac{s_k - s_{k+1}}{t-s_j}$. We can observe that $\sum_{k=0}^{j-1} \lambda_k = 1$. If we set $\theta_k = |y_k - y_{k+1}|$, then the triangle inequality and an application of Lemma 5.14 lead to

$$|x-y_j|^2 \leq \left(\sum_{k=0}^{j-1} |y_k - y_{k+1}|\right)^2 \leq \sum_{k=0}^{j-1} |y_k - y_{k+1}|^2 \lambda_k^{-1}.$$

Therefore,

$$\exp\left\{-\frac{1}{8(t-s_j)}\sum_{k=0}^{j-1} |y_k - y_{k+1}|^2 \lambda_k^{-1}\right\} \leq \exp\left\{-\frac{|x-y_j|^2}{8(t-s_j)}\right\},$$

which concludes the proof. □

Lemma 5.14. *For* $\theta_i > 0$, *and* $\lambda_i \in (0,1)$ *such that* $\sum_{i=1}^k \lambda_i = 1$, *we have*

$$\left(\sum_{i=1}^k \theta_i\right)^2 \leq \sum_{i=1}^k \theta_i^2 \lambda_i^{-1}.$$

Proof. It is an application of Cauchy-Schwarz's inequality applied to the measure λ on the discrete space $\{1,\dots,k\}$. Indeed,

$$\sum_{i=1}^k \theta_i = \sum_{i=1}^k \left(\frac{\theta_i}{\lambda_i}\right)\lambda_i \leq \left[\sum_{i=1}^k \left(\frac{\theta_i}{\lambda_i}\right)^2 \lambda_i\right]^{1/2} \left[\sum_{i=1}^k \lambda_i\right]^{1/2} = \left[\sum_{i=1}^k \theta_i^2 \lambda_i^{-1}\right]^{1/2},$$

which concludes the proof. □

Lemma 5.15. *If each* $y_k \in D$, *for* $k \in \{1,\dots,n\}$, *then*

$$\left[\prod_{k=0}^n G_1(s_k - s_{k+1}, y_k, y_{k+1}) - \prod_{k=0}^n G_2(s_k - s_{k+1}, y_k, y_{k+1})\right]$$
$$\leq C^{n+1}\left[\prod_{k=0}^n \Gamma_c(s_k - s_{k+1}, y_k, y_{k+1})\right]\left[\sum_{j=0}^n f_c(s_j - s_{j+1}, y_j, y_{j+1})\right],$$

where $c = \max(c_1, 4)$, *and* $C = \max\left(C_1, \max(c_1,4)/\min(c_1,4)\right)$.

Proof. To bound the difference of products, we use Lemma 5.16 below, with the values $a_k = G_1(s_k - s_{k+1}, y_k, y_{k+1})$ and $c_k = G_2(s_k - s_{k+1}, y_k, y_{k+1})$. Inequality (A.30) says that $a_k \geq c_k \geq 0$, and Corollary 3.11 is used to bound each difference $a_k - c_k$. □

Lemma 5.16. *Suppose each $a_k \geq c_k \geq 0$, then*

$$\prod_{k=0}^{n} a_k - \prod_{k=0}^{n} c_k \leq \sum_{k=0}^{n} (a_k - c_k) \left[\prod_{l \neq k}^{n} a_l \right].$$

Proof. We can rewrite

$$\prod_{k=0}^{n} a_k - \prod_{k=0}^{n} c_k = \sum_{k=0}^{n} (a_k - c_k) \prod_{l<k} c_l \prod_{l>k} a_l.$$

Using the fact that $0 \leq c_l \leq a_l$, for all l, concludes the proof. □

Lemma 5.17. *For any $j \in \{0, \ldots, n\}$, we can bound*

$$f_c(s_j - s_{j+1}, y_j, y_{j+1}) \prod_{k=0}^{n} \Gamma_c(s_k - s_{k+1}, y_k, y_{k+1})$$
$$\leq 2^{(n+1)d/2} f_{2c}(t, y_0) f_{2c}(t, y_{n+1}) \prod_{k=0}^{n} \Gamma_{2c}(s_k - s_{k+1}, y_k, y_{k+1}).$$

Proof. This is a repetition of the argument of Lemma 5.13. □

We can now conclude the proof of Proposition 5.12. We did consider the integrant of equation (5.14). We were able to bound it using the strategy in which the indicator function was either one or zero. We can put together those bounds to reach

$$\prod_{k=0}^{n} G_1(s_k - s_{k+1}, y_k, y_{k+1}) - \prod_{k=0}^{n} G_2(s_k - s_{k+1}, y_k, y_{k+1})$$
$$\leq c_{n+1} f_c(t, y_0) f_c(t, y_{n+1}) \prod_{k=0}^{n} \Gamma_c(s_k - s_{k+1}, y_k, y_{k+1}),$$

and a similar formula for

$$\prod_{k=0}^{n} G_1(s_k - s_{k+1}, y_k - z_k, y_{k+1} - z_{k+1})$$
$$- \prod_{k=0}^{n} G_2(s_k - s_{k+1}, y_k - z_k, y_{k+1} - z_{k+1})$$
$$\leq c_{n+1} f_c(t, y_0 - z_0) f_c(t, y_{n+1} - z_{n+1})$$
$$\times \prod_{k=0}^{n} \Gamma_c(s_k - s_{k+1}, y_k - z_k, y_{k+1} - z_{k+1}).$$

It remains to integrate and use the factorization formula of Lemma 5.9. Recall that $y_0 = x$, $y_0 - z_0 = x'$, $y_{n+1} = \alpha$, and $y_{n+1} - z_{n+1} = \alpha'$.

5.3.1 Proof of Theorem 5.4

For a fresh start, we recall it here

Theorem. *For any $x, x', \alpha, \alpha' \in \mathbb{R}^d$, we have*

$$[\mathcal{K}_{1,1} - \mathcal{K}_{1,2} - \mathcal{K}_{2,1} + \mathcal{K}_{2,2}]\,(t, x, x', \alpha, \alpha') \\ \leq Q(t, x - x', \alpha - \alpha')\Gamma_c(t, x, \alpha)\Gamma_c(t, x', \alpha')f_c(t, x, \alpha)f_c(t, x', \alpha'),$$

where

$$f_c(t, x, \alpha) = \boldsymbol{1}_{D\times D}(x, \alpha)e^{-\frac{\operatorname{dist}(x,\partial D)^2}{ct}}e^{-\frac{\operatorname{dist}(\alpha,\partial D)^2}{ct}} + \boldsymbol{1}_{(\mathbb{R}^d\times\mathbb{R}^d)\setminus(D\times D)}(x, \alpha).$$

Proof. From Proposition 5.12, we can bound

$$[\mathcal{K}_{1,1} - \mathcal{K}_{1,2} - \mathcal{K}_{2,1} + \mathcal{K}_{2,2}]\,(t, x, x', \alpha, \alpha') \\ \leq f_c(t, x, \alpha)f_c(t, x', \alpha')\sum_{n=1}^{\infty}\bar{c}^n\mathcal{L}_c^{\triangleright n}(t, x, x', \alpha, \alpha'),$$

for some constants c and $\bar{c}$. We now use Lemma 5.21 to write each iteration term as

$$\mathcal{L}_c^{\triangleright n}(t, x, x', \alpha, \alpha') = \Gamma_c(t, x, \alpha)\Gamma_c(t, x', \alpha')q_c^{\triangleleft n}(t, x - x', \alpha - \alpha').$$

By setting

$$Q(t, x, y) = \sum_{n=1}^{\infty}\bar{c}^n q_c^{\triangleleft n}(t, x, y),$$

the proof is completed. □

We hoped that explicit formulations such as the one found in Lemma 5.23 could lead to a fine description of the function Q. Yet, time forces us to abort that project. Instead we use the following, found in [9, Lemma 2.7].

Lemma 5.18. *There exists an increasing function $Q(t)$ such that for all $x, y \in \mathbb{R}^d$,*

$$Q(t, x, y) \leq Q(t).$$

Furthermore, there are $\gamma > 0$ and c_γ such that

$$Q(t) \leq c_\gamma e^{\gamma t}.$$

5.4 Convergence rate, Proofs of Theorems 5.2

Thanks to Theorems 5.1 and 5.4, together with Lemma 5.18 we can conclude that

$$\begin{aligned}\mathbb{E}\left[\left(u(t,x)-u_D(t,x)\right)\left(u(t,x')-u_D(t,x')\right)\right] \\ \le Q(t)\int_{\mathbb{R}^d}\mu_0(d\alpha)\,\Gamma_c(t,x,\alpha)f_c(t,x,\alpha) \\ \times\int_{\mathbb{R}^d}\mu_0(d\alpha')\,\Gamma_c(t,x',\alpha')f_c(t,x',\alpha').\end{aligned}$$

In particular, when the initial condition μ_0 is given by a non-negative bounded function $0 \le u_0(x) \le M$, we can further bound each integral as

$$\begin{aligned}&\int_{\mathbb{R}^d}\mu_0(d\alpha)\,\Gamma_c(t,x,\alpha)f_c(t,x,\alpha) \\ &\quad\le f_c(t,x)\int_D\mu_0(d\alpha)\,\Gamma_c(t,x,\alpha)+\int_{\mathbb{R}^d\setminus D}\mu_0(d\alpha)\,\Gamma_c(t,x,\alpha) \\ &\quad\le e^{-\frac{\mathrm{dist}(x,\partial D)^2}{ct}}M\int_{\mathbb{R}^d}\Gamma_c(t,x,\alpha)d\alpha+M\int_{|\alpha|>\mathrm{dist}(x,\partial D)}\Gamma_c(t,\alpha)\,d\alpha.\end{aligned}$$

The first integral is one and the second can be bounded as in (3.16). This concludes the proof of Theorem 5.2.

A similar proof can be applied for any initial condition with polynomial growth. Indeed, for $u_0(x) = 1 + |x|^n$, we would get, in the first integral, the n-th moment of the a Gaussian. In the second integral, we apply the following generalisation of (3.16)

Lemma 5.19. *For $\beta > 0$, $\lambda \in (0,1)$, there exists $c = c(d,\beta,\lambda)$ such that for all $t \in (0,T)$,*

$$\int_{|x|>\delta}\Gamma(s,x)\,|x|^\beta\,dx \le cT^{\beta/2}e^{-\frac{\delta^2\lambda}{4s}}.$$

Proof. With spherical coordinates, the left hand side is equal to

$$w^{d-1}(4\pi s)^{-d/2}\int_\delta^\infty e^{-\frac{r^2}{4s}}r^{d-1+\beta}.$$

With the change of variable $z = r/\sqrt{2s}$, it is in turn equal to

$$w^{d-1}(2s)^{\beta/2}(2\pi)^{-d/2}\int_{\delta/\sqrt{2s}}^\infty z^{\beta+d-1}e^{-\frac{z^2}{2}}\,dz.$$

The latter integral is bounded, thanks to (A.18), (A.19) by

$$P_{d+\beta-2}\left(\frac{\delta}{\sqrt{2s}}\right)e^{-\frac{\delta^2}{4s}} \le ce^{-\frac{\delta^2\lambda}{4s}},$$

which concludes the proof. □

5.5 Relation between triangle operators $\triangleright$ and $\triangleleft$

We can prove another representation of $\mathcal{L}^{\triangleright n}$. The objective is given in Lemma 5.21. In the next computation, we took $\lambda = 1$. Recall that is possible if we replace Λ by $\lambda^2\Lambda$.

Recall that the triangle $\triangleright$ operator contained only properties of the noise. And the definition of $\mathcal{L}$ contained only properties of the heat kernel. The present inverse triangle $\triangleleft$ will contain information about both. We define it as follows. For $f, g : \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}_+$, we set

$$(f \triangleleft g)(t,x,y) = \int_0^t ds \int_{\mathbb{R}^d} \Lambda(dz)\, f(t-s,x,z) g(s,z,y) \times \Gamma\left(\frac{2s(t-s)}{t}, -z + \frac{s}{t}x + \frac{t-s}{t}y\right). \quad (5.16)$$

By a change of variable $s' = t - s$, we get

$$(f \triangleleft g)(t,x,y) = \int_0^t ds \int_{\mathbb{R}^d} \Lambda(dz)\, f(s,x,z) g(t-s,z,y) \times \Gamma\left(\frac{2s(t-s)}{t}, -z + \frac{t-s}{t}x + \frac{s}{t}y\right). \quad (5.17)$$

This new operation is obviously not commutative.

Lemma 5.20. *The operation $\triangleleft$ is associative.*

Proof. For $f, g, h : \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}_+$, we need to verify that

$$[f \triangleleft (g \triangleleft h)](t,x,y) = [(f \triangleleft g) \triangleleft h](t,x,y).$$

For the former, we first use equation (5.17), then equation (5.16), to get

$$[f \triangleleft (g \triangleleft h)](t,x,y) = \int_0^t ds \int_{\mathbb{R}^d} \Lambda(dz_1) f(s,x,z_1)\, (g \triangleleft h)(t-s,z_1,y) \times \Gamma\left(\frac{2s(t-s)}{t}, -z_1 + \frac{t-s}{t}x + \frac{s}{t}y\right),$$

and

$$(g \triangleleft h)(t-s,z_1,y) = \int_0^{t-s} dr \int_{\mathbb{R}^d} \Lambda(dz_2) g(t-s-r,z_1,z_2) h(r,z_2,y) \times \Gamma\left(\frac{2r(t-s-r)}{t-s}, -z_2 + \frac{r}{t-s}z_1 + \frac{t-s-r}{t-s}y\right),$$

so that

$$\begin{aligned}[f \lhd (g \lhd h)](t,x,y) = \int_0^t ds \int_0^{t-s} dr \int_{\mathbb{R}^d} \Lambda(dz_1) \int_{\mathbb{R}^d} \Lambda(dz_2) \\ \times f(s,x,z_1) g(t-s-r,z_1,z_2) h(r,z_2,y) \\ \times \Gamma\left(\frac{2s(t-s)}{t}, -z_1 + \frac{t-s}{t}x + \frac{s}{t}y\right) \\ \times \Gamma\left(\frac{2r(t-s-r)}{t-s}, -z_2 + \frac{r}{t-s}z_1 + \frac{t-s-r}{t-s}y\right).\end{aligned}$$

For the latter, we first use equation (5.16), then equation (5.17), to get

$$\begin{aligned}[(f \lhd g) \lhd h](t,x,y) = \int_0^t dr \int_{\mathbb{R}^d} \Lambda(dz_2)\ (f \lhd g)(t-r,x,z_2) h(r,z_2,y) \\ \times \Gamma\left(\frac{2r(t-r)}{t}, -z_2 + \frac{r}{t}x + \frac{t-r}{t}y\right),\end{aligned}$$

and

$$\begin{aligned}(f \lhd g)(t-r,x,z_2) = \int_0^{t-r} ds \int_{\mathbb{R}^d} \Lambda(dz_1) f(s,x,z_1) g(t-r-s,z_1,z_2) \\ \times \Gamma\left(\frac{2s(t-r-s)}{t-r}, -z_1 + \frac{t-r-s}{t-r}x + \frac{s}{t-r}z_2\right),\end{aligned}$$

so that

$$\begin{aligned}[(f \lhd g) \lhd h](t,x,y) = \int_0^t dr \int_0^{t-r} ds \int_{\mathbb{R}^d} \Lambda(dz_1) \int_{\mathbb{R}^d} \Lambda(dz_2) \\ \times f(s,x,z_1) g(t-r-s,z_1,z_2) h(r,z_2,y) \\ \times \Gamma\left(\frac{2r(t-r)}{t}, -z_2 + \frac{r}{t}x + \frac{t-r}{t}y\right) \\ \times \Gamma\left(\frac{2s(t-r-s)}{t-r}, -z_1 + \frac{t-r-s}{t-r}x + \frac{s}{t-r}z_2\right).\end{aligned}$$

We permute the dr and the ds integral in the last formula. For any function $q(r,s)$, we have

$$\int_0^t dr \int_0^{t-r} ds\, q(r,s) = \int_0^t ds \int_0^{t-s} dr\, q(r,s).$$

Therefore,

$$\begin{aligned}[f \lhd (g \lhd h)](t,x,y) = \int_0^t ds \int_0^{t-s} dr \int_{\mathbb{R}^d} \Lambda(dz_1) \int_{\mathbb{R}^d} \Lambda(dz_2) \\ \times f(s,x,z_1) g(t-s-r,z_1,z_2) h(r,z_2,y) F(r,s,t,x,z_1,z_2,y),\end{aligned}$$

and

$$[(f \triangleleft g) \triangleleft h]\,(t,x,y) = \int_0^t ds \int_0^{t-s} dr \int_{\mathbb{R}^d} \Lambda(dz_1) \int_{\mathbb{R}^d} \Lambda(dz_2) \\ \times f(s,x,z_1) g(t-r-s,z_1,z_2) h(r,z_2,y) G(r,s,t,x,z_1,z_2,y),$$

where

$$F(r,s,t,x,z_1,z_2,y) = \Gamma\left(\frac{2s(t-s)}{t}, -z_1 + \frac{t-s}{t}x + \frac{s}{t}y\right) \\ \times \Gamma\left(\frac{2r(t-s-r)}{t-s}, -z_2 + \frac{r}{t-s}z_1 + \frac{t-s-r}{t-s}y\right),$$

and

$$G(r,s,t,x,z_1,z_2,y) = \Gamma\left(\frac{2r(t-r)}{t}, -z_2 + \frac{r}{t}x + \frac{t-r}{t}y\right) \\ \times \Gamma\left(\frac{2s(t-r-s)}{t-r}, -z_1 + \frac{t-r-s}{t-r}x + \frac{s}{t-r}z_2\right).$$

To conclude the proof, we can verify that $F = G$. □

We can relate both operations ▷ and ◁ as follows.

Lemma 5.21.

$$\mathcal{L}^{\triangleright n}(t,x,x',\alpha,\alpha') = \mathcal{L}(t,x,x',\alpha,\alpha') q_n(t,x,x',\alpha,\alpha'), \tag{5.18}$$

where

$$q_n(t,x,x',\alpha,\alpha') = q^{\triangleleft n}(t,x-x',\alpha-\alpha'),$$

with the initial function $q = 1$.

Proof. For $n = 1$, by definition we have $q_1 = 1$, thus we can take $q = 1$. By induction, we have

$$\mathcal{L}^{\triangleright(n+1)}(t,x,x',\alpha,\alpha') = (\mathcal{L} \triangleright \mathcal{L}^{\triangleright n})(t,x,x',\alpha,\alpha') \\ = \int_0^t ds \int_{\mathbb{R}^d} \Lambda(dz) \int_{\mathbb{R}^d} dy\, \mathcal{L}(t-s,x,x',y,y-z) \mathcal{L}^{\triangleright n}(s,y,y-z,\alpha,\alpha') \\ = \int_0^t ds \int_{\mathbb{R}^d} \Lambda(dz) \int_{\mathbb{R}^d} dy\, \mathcal{L}(t-s,x,x',y,y-z) \mathcal{L}(s,y,y-z,\alpha,\alpha') q_n(s,y,y-z,\alpha,\alpha') \\ = \int_0^t ds \int_{\mathbb{R}^d} \Lambda(dz)\, q^{\triangleleft n}(s,z,\alpha-\alpha') \\ \times \int_{\mathbb{R}^d} dy\, \mathcal{L}(t-s,x,x',y,y-z) \mathcal{L}(s,y,y-z,\alpha,\alpha').$$

We will show, in Lemma 5.22, that the last integral can be simplified as follows

$$\int_{\mathbb{R}^d} dy\, \mathcal{L}(t-s,x,x',y,y-z)\mathcal{L}(s,y,y-z,\alpha,\alpha') \\ = \mathcal{L}(t,x,x',\alpha,\alpha')\Gamma\left(\frac{2s(t-s)}{t}, -z+\frac{s}{t}(x-x')+\frac{t-s}{t}(\alpha-\alpha')\right).$$

Thus,

$$\mathcal{L}^{\triangleright(n+1)}(t,x,x',\alpha,\alpha') = \mathcal{L}(t,x,x',\alpha,\alpha')\int_0^t ds \int_{\mathbb{R}^d} \Lambda(dz)\, q^{\triangleleft n}(s,z,\alpha-\alpha') \\ \times \Gamma\left(\frac{2s(t-s)}{t}, -z+\frac{s}{t}(x-x')+\frac{t-s}{t}(\alpha-\alpha')\right).$$

Using the fact that $q(t-s,x-x',z)=1$, the last double integral can be rewritten as

$$\int_0^t ds \int_{\mathbb{R}^d} \Lambda(dz)\, q(t-s,x-x',z)q^{\triangleleft n}(s,z,\alpha-\alpha') \\ \times \Gamma\left(\frac{2s(t-s)}{t}, -z+\frac{s}{t}(x-x')+\frac{t-s}{t}(\alpha-\alpha')\right),$$

which is by definition $(q \triangleleft q^{\triangleleft n})(t,x-x',\alpha-\alpha') = q^{\triangleleft(n+1)}(t,x-x',\alpha-\alpha')$. This concludes the proof. □

Lemma 5.22. *We have the following identity*

$$\int_{\mathbb{R}^d} \mathcal{L}(t-s,x,x',y,y-z)\mathcal{L}(s,y,y-z,\alpha,\alpha')\, dy \\ = \mathcal{L}(t,x,x',\alpha,\alpha')\Gamma\left(\frac{2s(t-s)}{t}, -z+\frac{s}{t}(x-x')+\frac{t-s}{t}(\alpha-\alpha')\right).$$

Proof. Recall that $\mathcal{L}(t,x,x',\alpha,\alpha') = G(t,x,\alpha)G(t,x',\alpha')$. Thus

$$\mathcal{L}(t-s,x,x',y,y-z)\mathcal{L}(s,y,y-z,\alpha,\alpha') \\ = G(t-s,x,y)G(t-s,x',y-z)G(s,y,\alpha)G(s,y-z,\alpha')$$

We can rewrite the following product as

$$G(t-s,x,y)G(s,y,\alpha) = G(t,x,\alpha)\frac{G(t-s,x,y)G(s,y,\alpha)}{G(t,x,\alpha)}.$$

The Levy bridge starting at the point α at time $s=0$ and finishing at the point x at time $s=t$, has density function $B(y;t,s,x,\alpha)$ at any other time $s \in (0,t)$, where

$$B(y;t,s,x,\alpha) := \frac{G(t-s,x,y)G(s,y,\alpha)}{G(t,x,\alpha)}.$$

Thus, we can rewrite

$$\mathcal{L}(t-s,x,x',y,y-z)\mathcal{L}(s,y,y-z,\alpha,\alpha') \\ = \mathcal{L}(t,x,x',\alpha,\alpha')B(y;t,s,x,\alpha)B(y-z;t,s,x',\alpha'), \quad (5.19)$$

and

$$\int_{\mathbb{R}^d} \mathcal{L}(t-s,x,x',y,y-z)\mathcal{L}(s,y,y-z,\alpha,\alpha')\,dy \\ = \mathcal{L}(t,x,x',\alpha,\alpha') \int_{\mathbb{R}^d} B(y;t,s,x,\alpha)B(y-z;t,s,x',\alpha')\,dy.$$

Now, recall that for any two independent random vectors Z_1 and Z_2 with density function f_{Z_1} and f_{Z_2}, the density function of the difference $Z_1 - Z_2$ is given by

$$f_{Z_1-Z_2}(z) = (f_{Z_1} * \tilde{f}_{Z_2})(z) = \int_{\mathbb{R}^d} f_{Z_1}(y) f_{Z_2}(y-z)\,dy.$$

We can therefore conclude that

$$\int_{\mathbb{R}^d} B(y;t,s,x,\alpha)B(y-z;t,s,x',\alpha')\,dy = f_{Z_1-Z_2}(z)$$

is the density function at time s, evaluated at the point z, of the difference $Z_1 - Z_2$ of two independent Levy bridge, where Z_1 is starting at the point α at time $s = 0$ and finishing at the point x at time $s = t$, and Z_2 is staring at the point α' at time $s = 0$ and finishing at the point x' at time $s = t$. See Lemma 6.17 for more details. □

The purpose of the next result is to give an expression of $q^{\triangleleft n}(t,x,y)$ in term of the spectral measure. Recall that by definition,

$$q^{\triangleleft(n+1)}(t,x,y) = (q \triangleleft q^{\triangleleft n})(t,x,y),$$

where $q = q^{\triangleleft 1} \equiv 1$, and

$$(f \triangleleft g)(t,x,y) = \int_0^t ds \int_{\mathbb{R}^d} \Lambda(dz)\, f(t-s,x,z) g(s,z,y) \\ \times \Gamma\left(\frac{2s(t-s)}{t}, -z + \frac{s}{t}x + \frac{t-s}{t}y\right).$$

Lemma 5.23. *For $t > 0$, and $x, y \in \mathbb{R}^d$, we have*

$$
\begin{aligned}
&q^{\lhd(n+1)}(t,x,y)\\
&= \int_0^t ds_1 \int_0^{s_1} ds_2 \cdots \int_0^{s_{n-1}} ds_n \int_{\mathbb{R}^d} \mu(d\xi_n)\, \mathcal{F}\Gamma\left(2\frac{s_n(s_{n-1}-s_n)}{s_{n-1}}\right)(\xi_n)\\
&\quad \times \int_{\mathbb{R}^d} \mu(d\xi_{n-1})\, \mathcal{F}\Gamma\left(2\frac{s_{n-1}(s_{n-2}-s_{n-1})}{s_{n-2}}\right)\left(\xi_{n-1}+\frac{s_n}{s_{n-1}}\xi_n\right)\\
&\quad \times \cdots\\
&\quad \times \int_{\mathbb{R}^d} \mu(d\xi_1)\, \mathcal{F}\Gamma\left(2\frac{s_1(t-s_1)}{t}\right)\left(\xi_1+\frac{s_2}{s_1}\xi_2+\cdots+\frac{s_n}{s_1}\xi_n\right)\\
&\quad \times \exp\left\{-2\pi i x\left(\frac{s_1\xi_1+\cdots+s_n\xi_n}{t}\right)\right\}\\
&\quad \times \exp\left\{-2\pi i y\left(\frac{(t-s_1)\xi_1+\cdots+(t-s_n)\xi_n}{t}\right)\right\}.
\end{aligned}
$$

After writing this proof, we found that a related result was given in [2, Lemma 3.4] The present derivation is somewhat simpler.

Proof. This is easy induction procedure. Indeed, by definition and induction hypothesis, we get

$$
\begin{aligned}
&q^{\lhd(n+1)}(t,x,y)\\
&= \int_0^t ds_1 \int_{\mathbb{R}^d} \Lambda(dz)\, \Gamma\left(\frac{2s_1(t-s_1)}{t}, -z+\frac{s_1}{t}x+\frac{t-s_1}{t}y\right) q^{\lhd n}(s_1,z,y)\\
&\quad = \int_0^t ds_1 \int_{\mathbb{R}^d} \Lambda(dz)\, \Gamma\left(\frac{2s_1(t-s_1)}{t}, -z+\frac{s_1}{t}x+\frac{t-s_1}{t}y\right)\\
&\quad \times \int_0^{s_1} ds_2 \cdots \int_0^{s_{n-1}} ds_n \int_{\mathbb{R}^d} \mu(d\xi_n)\, \mathcal{F}\Gamma\left(2\frac{s_n(s_{n-1}-s_n)}{s_{n-1}}\right)(\xi_n)\\
&\quad \times \int_{\mathbb{R}^d} \mu(d\xi_{n-1})\, \mathcal{F}\Gamma\left(2\frac{s_{n-1}(s_{n-2}-s_{n-1})}{s_{n-2}}\right)\left(\xi_{n-1}+\frac{s_n}{s_{n-1}}\xi_n\right)\\
&\quad \times \cdots\\
&\quad \times \int_{\mathbb{R}^d} \mu(d\xi_2)\, \mathcal{F}\Gamma\left(2\frac{s_2(s_1-s_2)}{s_1}\right)\left(\xi_2+\frac{s_3}{s_2}\xi_3+\cdots+\frac{s_n}{s_2}\xi_n\right)\\
&\quad \times \exp\left\{-2\pi i z\left(\frac{s_2\xi_2+\cdots+s_n\xi_n}{s_1}\right)\right\}\\
&\quad \times \exp\left\{-2\pi i y\left(\frac{(s_1-s_2)\xi_2+\cdots+(s_1-s_n)\xi_n}{s_1}\right)\right\}.
\end{aligned}
$$

Using Fubini's theorem, we need to compute

$$\int_{\mathbb{R}^d} \Lambda(dz)\, \Gamma\left(\frac{2s_1(t-s_1)}{t}, -z + \frac{s_1}{t}x + \frac{t-s_1}{t}y\right) \times \exp\left\{-2\pi i z\left(\frac{s_2\xi_2 + \cdots + s_n\xi_n}{s_1}\right)\right\}.$$

Recall that for $g(z) = e^{-2\pi i z\mu} f(-z+c)$, we have

$$\mathcal{F}g(\xi) = e^{-2\pi i c(\xi+\mu)} \mathcal{F}f(-\xi-\mu).$$

We apply the latter with $\mu = \left(\frac{s_2\xi_2+\cdots+s_n\xi_n}{s_1}\right)$, and $c = \frac{s_1}{t}x + \frac{t-s_1}{t}y$. With Parseval's identity, the latter integral is

$$\int_{\mathbb{R}^d} \mu(d\xi_1) \exp\left\{-2\pi i\left(\frac{s_1}{t}x + \frac{t-s_1}{t}y\right)\left(\xi_1 + \frac{s_2\xi_2 + \cdots + s_n\xi_n}{s_1}\right)\right\} \times \mathcal{F}\Gamma\left(2\frac{s_1(t-s_1)}{t}\right)\left(-\xi_1 - \left(\frac{s_2\xi_2 + \cdots + s_n\xi_n}{s_1}\right)\right)$$

Using the fact that $\mathcal{F}\Gamma(t,\xi)$ is even in the ξ variable, we can replace the expression $-\xi_1 - \left(\frac{s_2\xi_2+\cdots+s_n\xi_n}{s_1}\right)$ by $\xi_1 + \left(\frac{s_2\xi_2+\cdots+s_n\xi_n}{s_1}\right)$. We are left to simplify the remaining exponential terms in x and y. For the former, we have

$$\exp\left\{-2\pi i\frac{s_1}{t}x\left(\xi_1 + \frac{s_2\xi_2 + \cdots + s_n\xi_n}{s_1}\right)\right\} = \exp\left\{-2\pi i x\left(\frac{s_1\xi_1 + s_2\xi_2 + \cdots + s_n\xi_n}{t}\right)\right\},$$

and for the latter, we have

$$\exp\left\{-2\pi i y\left(\frac{(s_1-s_2)\xi_2 + \cdots + (s_1-s_n)\xi_n}{s_1}\right)\right\} \times \exp\left\{-2\pi i\frac{t-s_1}{t}y\left(\xi_1 + \frac{s_2\xi_2 + \cdots + s_n\xi_n}{s_1}\right)\right\} = \exp\left\{-2\pi i y\left(\frac{(t-s_1)\xi_1 + \cdots + (t-s_n)\xi_n}{t}\right)\right\}.$$

Indeed, we use the fact that for $k \in \{2, \ldots, n\}$

$$\frac{s_1 - s_k}{s_1} + \frac{t-s_1}{t}\frac{s_k}{s_1} = 1 - \frac{s_k}{s_1} + \frac{s_k}{s_1} - \frac{s_k}{t} = \frac{t-s_k}{t}.$$

This concludes the proof. □

In the previous chapter, we had the following definition (4.26):

$$(f \star g)(t) = \int_0^t f\left(2\frac{s(t-s)}{t}\right) g(s)\, ds,$$

which was neither commutative, nor associative. That operation appeared in a similar context. In fact, from a more careful observation of Lemma 2.7 in [9], the following can be shown. If we let $p_1 \equiv 1$ and $p_{n+1} = k \star p_n$, where

$$k(t) = \int_{\mathbb{R}} \Lambda(dz)\, \Gamma(t,z) = \int_{\mathbb{R}} \mu(d\xi)\, \mathcal{F}\Gamma(t)(\xi),$$

then

$$q^{\triangleleft n} \leq p_n.$$

Using Lemma 4.17 we can get the bound

$$q^{\triangleleft n} \leq p_n \leq 2^{n-1} g_n$$

where $g_1 \equiv 1$, and $g_{n+1} = k * g_n$.

5.6 Time dilation

In this section, we give properties of both triangle operators, when the time is scaled. The following observation motivates the next result. For any $\nu \geq 1$, we have

$$\Gamma(\nu t, x) = \nu^{-d/2}\Gamma(t, x/\sqrt{\nu}) \geq \nu^{-d/2}\Gamma(t,x),$$

and thus

$$\Gamma(t,x) \leq \nu^{d/2}\Gamma(\nu t, x).$$

We will get a similar time scaled bound for the iterations $q^{\triangleleft n}$.

Lemma 5.24. *For any $\nu \geq 1$, we have*

$$q^{\triangleleft n}(t,x,y) \leq c^{n-1} q^{\triangleleft n}(\nu t, x, y),$$

where $c = \nu^{d/2-1}$.

Proof. The case $n = 1$ is clear since $q^{\triangleleft 1}$ is the constant one function. We proceed by induction.

$$\begin{aligned} q^{\triangleleft (n+1)}(t,x,y) &= (q^{\triangleleft 1} \triangleleft q^{\triangleleft n})(t,x,y) \\ &= \int_0^t ds \int_{\mathbb{R}^d} \Lambda(dz)\, q^{\triangleleft 1}(t-s,x,z) q^{\triangleleft n}(s,z,y) \\ &\qquad \times \Gamma\left(\frac{2s(t-s)}{t}, -z + \frac{s}{t}x + \frac{t-s}{t}y\right) \end{aligned}$$

By a linear change of variable $r = \nu s$, we observe that

$$\Gamma\left(\frac{2s(t-s)}{t}, -z + \frac{s}{t}x + \frac{t-s}{t}y\right) = \Gamma\left(\frac{2\frac{r}{\nu}(t-\frac{r}{\nu})}{t}, -z + \frac{\frac{r}{\nu}}{t}x + \frac{t-\frac{r}{\nu}}{t}y\right)$$
$$= \Gamma\left(\frac{1}{\nu}\frac{2r(\nu t - r)}{\nu t}, -z + \frac{r}{\nu t}x + \frac{\nu t - r}{\nu t}y\right),$$

and

$$q^{\triangleleft(n+1)}(t,x,y) = \frac{1}{\nu}\int_0^{\nu t} dr \int_{\mathbb{R}^d} \Lambda(dz)\, q^{\triangleleft n}(r/\nu, z, y)$$
$$\times \Gamma\left(\frac{1}{\nu}\frac{2r(\nu t - r)}{\nu t}, -z + \frac{r}{\nu t}x + \frac{\nu t - r}{\nu t}y\right).$$

Using the induction hypothesis and the bound for Γ, we get

$$q^{\triangleleft(n+1)}(t,x,y) \le \frac{1}{\nu}c^{n-1}\nu^{d/2}\int_0^{\nu t} dr \int_{\mathbb{R}^d} \Lambda(dz)\, q^{\triangleleft n}(r, z, y)$$
$$\times \Gamma\left(\frac{2r(\nu t - r)}{\nu t}, -z + \frac{r}{\nu t}x + \frac{\nu t - r}{\nu t}y\right)$$
$$= \nu^{d/2-1}c^{n-1}q^{\triangleleft(n+1)}(\nu t, x, y),$$

which concludes the proof. □

We define the time-scaled version

$$\mathcal{L}_\nu(t,x,x',\alpha,\alpha') := \mathcal{L}(\nu t, x, x', \alpha, \alpha'),$$

and write $\mathcal{L}_\nu^{\triangleright n} := (\mathcal{L}_\nu)^{\triangleright n}$. For $f, g : \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}_+$, we define a similar operation

$$(f \triangleleft_\nu g)(t,x,y) = \int_0^t ds \int_{\mathbb{R}^d} \Lambda(dz)\, f(t-s,x,z)g(s,z,y)$$
$$\times \Gamma\left(\nu\frac{2s(t-s)}{t}, -z + \frac{s}{t}x + \frac{t-s}{t}y\right). \quad (5.20)$$

The latter operator can be rewritten in term of that of $\triangleleft$. We

Lemma 5.25. *For any f, g, we have*

$$(f \triangleleft_\nu g)(t,x,y) = \frac{1}{\nu}(f_{1/\nu} \triangleleft g_{1/\nu})(\nu t, x, y),$$

where $f_\alpha(t,x,y) := f(\alpha t, x, y)$ is the dilation in time.

Proof. First, we observe that

$$\begin{aligned} f(t-s,x,z)g(s,z,y)\Gamma\left(\nu\frac{2s(t-s)}{t}, -z+\frac{s}{t}x+\frac{t-s}{t}y\right) \\ = f_{1/\nu}(\nu t-\nu s,x,z)g_{1/\nu}(\nu s,z,y) \\ \times\Gamma\left(\frac{2\nu s(\nu t-\nu s)}{\nu t}, -z+\frac{\nu s}{\nu t}x+\frac{\nu t-\nu s}{\nu t}y\right). \end{aligned}$$

Using a change of variable $r=\nu s$, we get

$$\begin{aligned} (f\lhd_\nu g)(t,x,y) = \frac{1}{\nu}\int_0^{\nu t} dr \int_{\mathbb{R}^d} \Lambda(dz)\, f_{1/\nu}(\nu t-r,x,z)g_{1/\nu}(r,z,y) \\ \times\Gamma\left(\frac{2r(\nu t-r)}{\nu t}, -z+\frac{r}{\nu t}x+\frac{\nu t-r}{\nu t}y\right), \end{aligned}$$

which concludes the proof. □

We can give a time dilation result for the iterations $\mathcal{L}_\nu^{\rhd n}$.

Lemma 5.26. *For any $\nu>0$, we have*

$$\mathcal{L}_\nu^{\rhd n}(t,x,x',\alpha,\alpha') = \frac{1}{\nu^{n-1}}\mathcal{L}^{\rhd n}(\nu t,x,x',\alpha,\alpha').$$

Furthermore,

$$q^{\lhd_\nu n}(t,x,y) = \frac{1}{\nu^{n-1}}q^{\lhd n}(\nu t,x,y),$$

which can be written as $(q^{\lhd_\nu n})_{1/\nu} = \frac{1}{\nu^{n-1}}q^{\lhd n}$.

Proof. We proceed by induction. By the linear change of variable $r=\nu s$.

$$\begin{aligned} &\mathcal{L}_\nu^{\rhd(n+1)}(t,x,x',\alpha,\alpha') = (\mathcal{L}_\nu \rhd \mathcal{L}_\nu^{\rhd n})(t,x,x',\alpha,\alpha') \\ &= \int_0^t ds \int_{\mathbb{R}^s}\Lambda(dz)\int_{\mathbb{R}^d} dv\, \mathcal{L}_\nu(t-s,x,x',v,v-z)\mathcal{L}_\nu^{\rhd n}(s,v,v-z,\alpha,\alpha') \\ &= \int_0^t ds \int_{\mathbb{R}^s}\Lambda(dz)\int_{\mathbb{R}^d} dv\, \mathcal{L}(\nu t-\nu s,x,x',v,v-z)\frac{1}{\nu^{n-1}}\mathcal{L}^{\rhd n}(\nu s,v,v-z,\alpha,\alpha') \\ &= \frac{1}{\nu^{n-1}}\frac{1}{\nu}\int_0^{\nu t} dr \int_{\mathbb{R}^s}\Lambda(dz)\int_{\mathbb{R}^d} dv\, \mathcal{L}(\nu t-r,x,x',v,v-z)\mathcal{L}^{\rhd n}(r,v,v-z,\alpha,\alpha') \\ &= \frac{1}{\nu^n}\mathcal{L}^{\rhd(n+1)}(\nu t,x,x',\alpha,\alpha'). \end{aligned}$$

Applying the preceding Lemma 5.25 to the constant function $q\equiv 1$, for which $q_{1/\nu}=q$, we get by induction

$$\begin{aligned} q^{\lhd_\nu(n+1)}(t,x,y) &= (q\lhd_\nu q^{\lhd_\nu n})(t,x,y) \\ &= \frac{1}{\nu}(q_{1/\nu}\lhd(q^{\lhd_\nu n})_{1/\nu})(\nu t,x,y) \\ &= \frac{1}{\nu^n}(q\lhd q^{\lhd n})(\nu t,x,y), \end{aligned}$$

which concludes the proof. □

We could have also deduced one property from the other. Indeed, we have the following relation

Corollary 5.27.

$$\mathcal{L}_\nu^{\triangleright n}(t,x,x',\alpha,\alpha') = \mathcal{L}_\nu(t,x,x',\alpha,\alpha') q^{\triangleleft_\nu n}(t,x-x',\alpha-\alpha').$$

Proof. Applying the previous result and Lemma 5.21, we get

$$\begin{aligned}\mathcal{L}_\nu^{\triangleright n}(t,x,x',\alpha,\alpha') &= \frac{1}{\nu^{n-1}}\mathcal{L}^{\triangleright n}(\nu t,x,x',\alpha,\alpha')\\ &= \frac{1}{\nu^{n-1}}\mathcal{L}(\nu t,x,x',\alpha,\alpha')q^{\triangleleft n}(\nu t,x-x',\alpha-\alpha')\\ &= \mathcal{L}_\nu(t,x,x',\alpha,\alpha')q^{\triangleleft_\nu n}(t,x-x',\alpha-\alpha'),\end{aligned}$$

which concludes the proof. □

The latter result could have been observed directly from the definition and Lemma 5.22. Indeed, by induction, we have

$$\begin{aligned}&\mathcal{L}_\nu^{\triangleright(n+1)}(t,x,x',\alpha,\alpha') = (\mathcal{L}_\nu \triangleright \mathcal{L}_\nu^{\triangleright n})(t,x,x',\alpha,\alpha')\\ &= \int_0^t ds \int_{\mathbb{R}^s} \Lambda(dz) \int_{\mathbb{R}^d} dv\, \mathcal{L}_\nu(t-s,x,x',v,v-z)\mathcal{L}_\nu^{\triangleright n}(s,v,v-z,\alpha,\alpha')\\ &\qquad = \int_0^t ds \int_{\mathbb{R}^s} \Lambda(dz) \int_{\mathbb{R}^d} dv\, \mathcal{L}_\nu(t-s,x,x',v,v-z)\\ &\qquad\quad \times \mathcal{L}_\nu(s,v,v-z,\alpha,\alpha')q^{\triangleleft_\nu n}(s,z,\alpha-\alpha')\\ &\qquad\qquad = \int_0^t ds \int_{\mathbb{R}^s} \Lambda(dz)\, q^{\triangleleft_\nu n}(s,z,\alpha-\alpha')\\ &\quad \times \int_{\mathbb{R}^d} dv\, \mathcal{L}(\nu t-\nu s,x,x',v,v-z)\mathcal{L}(\nu s,v,v-z,\alpha,\alpha')\\ &\quad = \mathcal{L}(\nu t,x,x',\alpha,\alpha') \int_0^t ds \int_{\mathbb{R}^s} \Lambda(dz)\, q^{\triangleleft_\nu n}(s,z,\alpha-\alpha')\\ &\quad \times \Gamma\left(\nu\frac{2s(t-s)}{t}, -z+\frac{s}{t}(x-x')+\frac{t-s}{t}(\alpha-\alpha')\right)\\ &\qquad\qquad = \mathcal{L}(\nu t,x,x',\alpha,\alpha')(q \triangleleft_\nu q^{\triangleleft_\nu n})(t,x-x',\alpha-\alpha').\end{aligned}$$

Chapter 6

Parabolic Anderson Dream

We will first define a very general version of the noise $\dot{\mathrm{M}}$, then apply it to the resolution of the Anderson model, for parabolic or elliptic equations that are compatible with the noise.

6.1 Important results on distributions

The next result is a corner stone in the theory of distributions. It can be found in Chapter 1, Section 1 of [28], under Theorem 3, Theorem 5, and Theorem 6.

Theorem 6.1 (The Kernel Theorem)**.** *Every bilinear functional $B(\phi, \psi)$ on the space $C_c^\infty(\mathbb{R}^d)$, respectivement on $\mathcal{S}(\mathbb{R}^d)$, which is continuous in each of the arguments ϕ and ψ has the form*

$$B(\phi, \psi) = \langle T, \phi \otimes \psi \rangle = \langle T_{x,y}, \phi(x)\psi(y) \rangle \,, \tag{6.1}$$

where T is a continuous linear functional on the space $C_c^\infty(\mathbb{R}^{2d})$, respectivement on $\mathcal{S}(\mathbb{R}^{2d})$. Furthermore, there exists some constant $C > 0$ and some norms $\|\cdot\|_n$ and $\|\cdot\|_m$ on $C_c^\infty(\mathbb{R}^d)$, respectively on $\mathcal{S}(\mathbb{R}^d)$, such that

$$|B(\phi, \psi)| \leq C \|\phi\|_n \|\psi\|_m \,. \tag{6.2}$$

In regard of the Kernel Theorem, we introduce the notion of positive-definiteness of a distribution.

Definition 6.2. A distribution T on $C_c^\infty(\mathbb{R}^{2d})$ is said to be positive-definite, if

$$\langle T, \phi \otimes \phi \rangle = \langle T_{x,y}, \phi(x)\phi(y) \rangle \geq 0, \tag{6.3}$$

for all $\phi \in C_c^\infty(\mathbb{R}^d)$.

Another notion of positve-definiteness is also possible, yet on lower space dimensional distribution.

Definition 6.3. A distribution T on $C_c^\infty(\mathbb{R}^d)$ is said to be positive-definite, if

$$\langle T, \phi * \phi^* \rangle \geq 0, \tag{6.4}$$

for all $\phi \in C_c^\infty(\mathbb{R}^d)$, where $\phi^*(x) = \phi(-x)$.

Both notions will be used further down. The former for general type of noises, and the latter with the restricted class of translation invariant noises. For the latter, we have the following useful result. It can be found in Chapter 2, section 3 of [28], under Theorem 1, and Theorem 3.

Theorem 6.4 (Bochner-Schwartz)**.** *Every positive-definite distribution T on $C_c^\infty(\mathbb{R}^d)$, or on $\mathcal{S}(\mathbb{R}^d)$, is the Fourier transform of a positive tempered measure μ, i.e. it can be written as*

$$\langle T, \phi \rangle = \int_{\mathbb{R}^d} \mathcal{F}\phi(\xi)\, \mu(d\xi). \tag{6.5}$$

If B is translation invariant bilinear functional continuous in each argument on $C_c^\infty(\mathbb{R}^d)$, or on $\mathcal{S}(\mathbb{R}^d)$, it can be written as

$$B(\phi, \psi) = \langle T, \phi \otimes \psi \rangle = \langle Q, \phi * \psi^* \rangle,$$

where Q is a distribution on $C_c^\infty(\mathbb{R}^d)$, or on $\mathcal{S}(\mathbb{R}^d)$ respectively. See [28, page 169].

Suppose that B is translation invariant. If B, or equivalently T, is positive definite, then so is Q. In that case, using the Bochner-Schwartz theorem, we conclude that

$$B(\phi, \psi) = \int \mathcal{F}\phi(\xi)\overline{\mathcal{F}\psi(\xi)}\, \mu(d\xi).$$

The last ingredient we will need about distributions is the existence of tensor product. For two distributions S on $C_c^\infty(\mathbb{R}^m)$ and T on $C_c^\infty(\mathbb{R}^n)$, there exists a unique distribution on $C_c^\infty(\mathbb{R}^{m+n})$, denoted by $S \otimes T$, that satisfies

$$\langle S \otimes T, \phi \otimes \psi \rangle = \langle S_x \otimes T_y, \phi(x)\psi(y) \rangle = \langle S, \phi \rangle \cdot \langle T, \psi \rangle, \tag{6.6}$$

for any pair of test functions $\phi \in C_c^\infty(\mathbb{R}^m)$ and $\psi \in C_c^\infty(\mathbb{R}^n)$. Uniqueness and existence can be found in Chapter IV of [50], under Theorem III and Theorem IV. Uniqueness of this distribution is a consequence of the following fact.

Theorem 6.5 (Uniqueness of tensor product)**.** *Linear combinations of functions of the form $\phi(x)\psi(y)$ are dense in $C_c^\infty(\mathbb{R}^{m+n})$.*

Existence is guaranteed by setting either

$$\langle S_x \otimes T_y, \varphi(x,y) \rangle := \langle S_x, \langle T_y, \varphi(x,y) \rangle \rangle,$$

or

$$\langle S_x \otimes T_y, \varphi(x,y)\rangle := \langle T_y, \langle S_x, \varphi(x,y)\rangle\rangle\,.$$

Any of the two indeed defines a distribution on $C_c^\infty(\mathbb{R}^{m+n})$, and both coincide for functions of the form $\phi(x)\psi(y)$. By uniqueness, they are the same distribution. In particular, it is possible to perform evaluation successively and in any order. This is a general version of Fubini's Theorem for distributions.

The tensor product extends to any finite number of distributions and that product is associative. For example, if R, S, and T are three distributions on $C_c^\infty(\mathbb{R}^l)$, $C_c^\infty(\mathbb{R}^m)$, and $C_c^\infty(\mathbb{R}^n)$ respectively, then

$$(R_x \otimes S_y) \otimes T_z = R_x \otimes (S_y \otimes T_z)\,.$$

In fact, we can directely define the tensor product

$$\langle R_x \otimes S_y \otimes T_z, u(x)v(y)w(z)\rangle := \langle R, u\rangle \cdot \langle S, v\rangle \cdot \langle T, w\rangle\,,$$

for any functions $u \in C_c^\infty(\mathbb{R}^l)$, $v \in C_c^\infty(\mathbb{R}^m)$, and $w \in C_c^\infty(\mathbb{R}^n)$. Evaluation can also be performed successively and in any order. For example,

$$\begin{aligned}\langle R_x \otimes S_y \otimes T_z, \varphi(x,y,z)\rangle &= \langle R_x, \langle S_y, \langle T_z, \varphi(x,y,z)\rangle\rangle\rangle \\ &= \langle T_z, \langle R_x, \langle S_y, \varphi(x,y,z)\rangle\rangle\rangle\,,\end{aligned}$$

for any $\varphi \in C_c^\infty(\mathbb{R}^{l+m+n})$.

Analog results hold for tempered distributions on $\mathcal{S}(\mathbb{R}^d)$.

6.2 Decomposition of distributions

Suppose $T = R \otimes S$.

Lemma 6.6. *T is the zero distribution iff R or S is the zero distribution.*

Proof. If R or S is the zero distribution, then so is $R \otimes S$ by the density result, Theorem 6.5. If both R and S are not the zero distribution, then there exist ϕ and ψ such that $\langle R, \phi\rangle \neq 0 \neq \langle S, \psi\rangle$. But this contradicts the fact that

$$\langle R, \phi\rangle \cdot \langle S, \psi\rangle = \langle R_x \otimes S_y, \phi(x)\psi(y)\rangle = 0.$$

This concludes the proof. □

To avoid tautology, we assume that both R and S are not the zero distributions.

We now consider distributions $R_{u,v}$ on $C_c^\infty(\mathbb{R}^{2m})$, and $S_{x,y}$ on $C_c^\infty(\mathbb{R}^{2n})$, whose tensor product is denoted by $T_{u,x,v,y}$. We could have chosen the other notation $T_{u,v,x,y}$, which look simpler. The reason is that we want to consider

T as a distribution on $\mathbb{R}^{m+n} \times \mathbb{R}^{m+n}$, rather that on $\mathbb{R}^{2m} \times \mathbb{R}^{2n}$. To make this distinction more obvious, we could have written $T_{(u,x),(v,y)}$.

A distribution $S_{x,y}$ is said to be symmetric if

$$\langle S_{x,y}, \varphi(x,y)\rangle = \langle S_{x,y}, \varphi(y,x)\rangle ,$$

for all $\varphi \in C_c^\infty(\mathbb{R}^{2n})$. This can be written as $S_{x,y} = S_{y,x}$.

A distribution S is said to by antisymmetric if

$$\langle S_{x,y}, \varphi(x,y)\rangle = - \langle S_{x,y}, \varphi(y,x)\rangle ,$$

for all $\varphi \in C_c^\infty(\mathbb{R}^{2n})$. This can be written as $S_{x,y} = -S_{y,x}$. This is in fact equivalent to $\langle S_{x,y}, \phi(x)\phi(y)\rangle = 0$ for all $\phi \in C_c^\infty(\mathbb{R}^n)$. Indeed, this is a direct consequence of Theorem 6.5 and the following observation

$$\begin{aligned} 0 = \langle S_{x,y}, (\phi(x) + \psi(x))(\phi(y) + \psi(y))\rangle \\ = \langle S_{x,y}, \phi(x)\phi(y)\rangle + \langle S_{x,y}, \phi(x)\psi(y)\rangle \\ + \langle S_{x,y}, \psi(x)\phi(y)\rangle + \langle S_{x,y}, \psi(x)\psi(y)\rangle \\ = \langle S_{x,y}, \phi(x)\psi(y)\rangle + \langle S_{x,y}, \psi(x)\phi(y)\rangle . \end{aligned}$$

It is clear that a distribution that is both symmetric and antisymmetric is the zero distribution. As we mentioned earlier, we will avoid those cases.

The tensor product has the same symmetries as that of its components.

Lemma 6.7. *$T_{u,x,v,y}$ and $R_{u,v}$ have the same symmetry property in the pair of indices (u, v).*

Similarily, $T_{u,x,v,y}$ and $S_{x,y}$ have the same symmetry property in the pair of indices (x, y).

Proof. This is a consequence of the density result, Theorem 6.5, and the following observation

$$\begin{aligned} \langle T_{u,x,v,y}, \phi_1(u)\psi_1(x)\phi_2(v)\psi_2(y)\rangle = \langle R_{u,v}, \phi_1(u)\phi_2(v)\rangle \cdot \langle S_{x,y}, \psi_1(x)\psi_2(y)\rangle \\ = \pm \langle R_{u,v}, \phi_1(v)\phi_2(u)\rangle \cdot \langle S_{x,y}, \psi_1(x)\psi_2(y)\rangle \\ = \pm \langle T_{u,x,v,y}, \phi_1(v)\psi_1(x)\phi_2(u)\psi_2(y)\rangle , \end{aligned}$$

which holds for all $\phi_1, \phi_2 \in C_c^\infty(\mathbb{R}^m)$ and $\psi_1, \psi_2 \in C_c^\infty(\mathbb{R}^n)$. □

This does not say that if $S_{x,y}$ is symmetric, then S is the tensor product of two distributions.

For T to be symmetric as a distribution on $\mathbb{R}^{m+n} \times \mathbb{R}^{m+n}$, we mean that $T_{(u,x),(v,y)} = T_{(v,y),(u,x)}$.

Lemma 6.8. *T is symmetric if and only if either R and S are both symmetric, or R and S are both antisymmetric.*

Proof. Assume that T is symmetric. That means

$$\langle T_{u,x,v,y}, \chi(u,x,v,y)\rangle = \langle T_{u,x,v,y}, \chi(v,y,u,x)\rangle\,,$$

for all $\chi \in C_c^\infty(\mathbb{R}^{2(m+n)})$. If we take the special case

$$\chi(u,x,v,y) = \phi_1(u)\psi_1(x)\phi_2(v)\psi_2(y),$$

for $\phi_1, \phi_2 \in C_c^\infty(\mathbb{R}^m)$ and $\psi_1, \psi_2 \in C_c^\infty(\mathbb{R}^n)$, the left hand side becomes

$$\langle R_{u,v}, \phi_1(u)\phi_2(v)\rangle \cdot \langle S_{x,y}, \psi_1(x)\psi_2(y)\rangle\,,$$

whereas the right hand side becomes

$$\langle R_{u,v}, \phi_1(v)\phi_2(u)\rangle \cdot \langle S_{x,y}, \psi_1(y)\psi_2(x)\rangle\,.$$

We now use the fact that R is not the zero distribution. We have two cases to handle. First, if we can find some $\bar\phi \in C_c^\infty(\mathbb{R}^m)$ such that

$$\left\langle R_{u,v}, \bar\phi(u)\bar\phi(v)\right\rangle \neq 0,$$

then we can conclude, using $\chi(u,x,v,y) = \bar\phi(u)\psi_1(x)\bar\phi(v)\psi_2(y)$, that

$$\langle S_{x,y}, \psi_1(x)\psi_2(y)\rangle = \langle S_{x,y}, \psi_1(y)\psi_2(x)\rangle\,,$$

for all $\psi_1, \psi_2 \in C_c^\infty(\mathbb{R}^n)$. Thus S is symmetric. In turn, this implies that R is symmetric.

Second, if we cannot find such $\bar\phi \in C_c^\infty(\mathbb{R}^m)$, this means that R is antisymmetric. In turn, this implies that S is antisymmetric. □

The previous proof can be slightly modified to yield the following result.

Lemma 6.9. *T is antisymmetric if and only if either R is symmetric and S is antisymmetric, or R is antisymmetric and S is symmetric.*

We would like to show an analog relation between R, S, and T concerning positive definiteness. As the next expression shows, caution is rigor. If T is positive definite, then for $\Phi(u,x) = \phi(u)\psi(x)$, we must have

$$0 \leq \langle T_{u,x,v,y}, \Phi(u,x)\Phi(v,y)\rangle = \langle R_{u,v}, \phi(u)\phi(v)\rangle \cdot \langle S_{x,y}, \psi(x)\psi(y)\rangle\,.$$

This could mean that both R and S are positive definite, or that both R and S are negative definite, or that any of R and S is antisymmetric. Fortunately, we can conclude something when symmetry is also assumed.

Lemma 6.10. *T is symmetric and positive definite implies that R and S are both symmetric.*

In comparison with Lemma 6.8, the extra assumption that T is positive definite excludes the possibility that R and S are both antisymmetric.

Proof. We assume that T is symmetric and positive definite, and we will show that if R and S are both antisymmetric, then T must also be antisymmetric. A contradiction with the fact that T is not the zero distribution. We must show that

$$\langle T_{u,x,v,y}, \Phi(u,x)\Phi(v,y)\rangle = 0,$$

for all $\Phi \in C_c^\infty(\mathbb{R}^{m+n})$. By density result, Theorem 6.5, it is sufficient to show it for $\Phi(u,x) = \sum_{k=1}^{l} \alpha_k \phi_k(u)\psi_k(x)$. We will show this by induction on the number of term in the linear combination. The starting point, $l = 1$, is precisely the hypothesis that R and S are antisymmetric. Indeed, for $\Phi(u,x) = \alpha\phi(u)\psi(x)$, we have

$$\langle T_{u,x,v,y}, \Phi(u,x)\Phi(v,y)\rangle = \alpha^2 \langle R_{u,v}, \phi(u)\phi(v)\rangle \cdot \langle S_{x,y}, \psi(x)\psi(y)\rangle = 0.$$

Assuming that the result holds for $\chi(u,x) = \sum_{k=1}^{l} \alpha_k \phi_k(u)\psi_k(x)$, for any $\alpha_k \in \mathbb{R}$, $\phi_k \in C_c^\infty(\mathbb{R}^m)$, and $\psi_k \in C_c^\infty(\mathbb{R}^n)$, we will show that it still holds for $\Phi(u,x) = \chi(u,x) + \alpha\phi(u)\psi(x)$. Because T is positive definite,

$$\begin{aligned} 0 \le &\langle T_{u,x,v,y}, \Phi(u,x)\Phi(v,y)\rangle \\ &= \langle T_{u,x,v,y}, \chi(u,x)\chi(v,y)\rangle + \alpha \langle T_{u,x,v,y}, \chi(u,x)\phi(v)\psi(y)\rangle \\ &+ \alpha \langle T_{u,x,v,y}, \phi(u)\psi(x)\chi(v,y)\rangle + \alpha^2 \langle T_{u,x,v,y}, \phi(u)\psi(x)\phi(v)\psi(y)\rangle . \end{aligned}$$

By induction hypothesis, the first term $\langle T_{u,x,v,y}, \chi(u,x)\chi(v,y)\rangle = 0$. The last term is zero as before since R and S are antisymmetric. Using the fact that T is symmetric, we conclude that

$$0 \le \langle T_{u,x,v,y}, \Phi(u,x)\Phi(v,y)\rangle = 2\alpha \langle T_{u,x,v,y}, \chi(u,x)\phi(v)\psi(y)\rangle ,$$

for all $\alpha \in \mathbb{R}$. By taking $\tilde{\alpha} = -\alpha$, we must have that

$$\langle T_{u,x,v,y}, \chi(u,x)\phi(v)\psi(y)\rangle = 0.$$

In turn, this shows that $\langle T_{u,x,v,y}, \Phi(u,x)\Phi(v,y)\rangle = 0$. This proves the induction step, and concludes the proof. □

We are now ready for the following result, which will be useful for the general noises.

Proposition 6.11. *T is symmetric and positive definite only if R and S are both symmetric, and either R and S are both positive definite or R and S are both negative definite.*

If R and S are negative definite, then $-R$ and $-S$ would be positive definite, and $R \otimes S = (-R) \otimes (-S)$.

We will need the following definitions. For $\phi_1, \phi_2 \in C_c^\infty(\mathbb{R}^m)$, we set

$$\langle \phi_1, \phi_2\rangle_R := \langle R_{u,v}, \phi_1(u)\phi_2(v)\rangle .$$

We shall see that $\langle\cdot,\cdot\rangle_R$ acts as an inner product when R is a symmetric and positive definite distribution. In that case, usual arguments show that $\langle\cdot,\cdot\rangle_R$ satisfies both Cauchy-Schwarz and Minkowski's inequalities, i.e.

$$|\langle\phi_1,\phi_2\rangle_R|^2 \le \langle\phi_1,\phi_1\rangle_R \cdot \langle\phi_2,\phi_2\rangle_R\,,$$
$$\langle\phi_1+\phi_2,\phi_1+\phi_2\rangle_R^{1/2} \le \langle\phi_1,\phi_1\rangle_R^{1/2} + \langle\phi_2,\phi_2\rangle_R^{1/2}\,,$$

for all $\phi_1,\phi_2 \in C_c^\infty(\mathbb{R}^m)$.

Proof. Suppose that T is symmetric and positive definite. By Lemma 6.10, this implies that both R and S are symmetric. We want to show that either $(\phi,\phi)_R \ge 0$ for all $\phi \in C_c^\infty(\mathbb{R}^m)$, or $(\phi,\phi)_R \le 0$ for all $\phi \in C_c^\infty(\mathbb{R}^m)$. We cannot have $(\phi,\phi)_R = 0$ for all $\phi \in C_c^\infty(\mathbb{R}^m)$, for R would be both symmetric and antisymmetric, and that would contradict the fact that $R \ne 0$. Suppose, by contradiction, that there are $\phi_1,\phi_2 \in C_c^\infty(\mathbb{R}^m)$ such that

$$\langle\phi_1,\phi_1\rangle_R > 0, \quad \text{and} \quad \langle\phi_2,\phi_2\rangle_R < 0.$$

Since S is not the zero distribution and is symmetric, there is $\psi \in C_c^\infty(\mathbb{R}^n)$ such that $\langle\psi,\psi\rangle_S \ne 0$. Using the fact that T is positive definite, we have

$$\begin{aligned} 0 &\le \langle T_{u,x,v,y}, (\alpha\phi_1(u)+\beta\phi_2(u))\psi(x)(\alpha\phi_1(v)+\beta\phi_2(v))\psi(y)\rangle \\ &= \left(\alpha^2\langle\phi_1,\phi_1\rangle_R + 2\alpha\beta\langle\phi_1,\phi_2\rangle_R + \beta^2\langle\phi_2,\phi_2\rangle_R\right)\langle\psi,\psi\rangle_S\,, \end{aligned}$$

for all $\alpha,\beta \in \mathbb{R}$. We have two cases to cover. If $\langle\psi,\psi\rangle_S > 0$, we can take $\alpha = 0$ and $\beta = 1$ to conclude that

$$0 \le \langle T_{u,x,v,y}, \phi_2(u)\psi(x)\phi_2(v)\psi(y)\rangle = \langle\phi_2,\phi_2\rangle_R\langle\psi,\psi\rangle_S < 0,$$

which is a contradiction. If $\langle\psi,\psi\rangle_S < 0$, we could take $\alpha = 1$ and $\beta = 0$ to conclude that

$$0 \le \langle T_{u,x,v,y}, \phi_1(u)\psi(x)\phi_1(v)\psi(y)\rangle = \langle\phi_1,\phi_1\rangle_R\langle\psi,\psi\rangle_S < 0,$$

which is again a contradiction. Therefore, we showed that either $(\phi,\phi)_R \ge 0$ for all $\phi \in C_c^\infty(\mathbb{R}^m)$, or $(\phi,\phi)_R \le 0$ for all $\phi \in C_c^\infty(\mathbb{R}^m)$. The former says that R is positive definite. In turn, that implies that S is also positive definite. The latter says that R is negative definite. In turn, that implies that S is also negative definite. □

The converse is trickier. Here are some ideas but we couldn't conclude. Second, we show that if both R and S are symmetric and positive definite, then T is symmetric and positive definite. The symmetry of T from that of R and S was already proved in Lemma 6.8. We are left to show that T is indeed positive definite. We need to show that

$$\langle T_{u,x,v,y}, \Phi(u,x)\Phi(v,y)\rangle \ge 0,$$

for all $\Phi \in C_c^\infty(\mathbb{R}^{m+n})$. By the density result, Theorem 6.5, it sufficient to show it for any linear combinations $\Phi(u,x) = \sum_{k=1}^{l} \alpha_k \phi_k(u)\psi_k(x)$. We will show this by induction of the number of term in the linear combination. The starting point, $l = 1$, is a precisely the hypothesis that R and S are positive definite. Indeed, for $\Phi(u,x) = \alpha\phi(u)\psi(x)$, we have

$$\langle T_{u,x,v,y}, \Phi(u,x)\Phi(v,y)\rangle = \alpha^2 \langle \phi, \phi\rangle_R \cdot \langle \psi, \psi\rangle_S \geq 0.$$

Assuming that the result holds for $\chi(u,x) = \sum_{k=1}^{l} \alpha_k \phi_k(u)\psi_k(x)$, for any $\alpha_k \in \mathbb{R}$, $\phi_k \in C_c^\infty(\mathbb{R}^m)$, and $\psi_k \in C_c^\infty(\mathbb{R}^n)$, we want to show that it still holds for $\Phi(u,x) = \chi(u,x) + \alpha\phi(u)\psi(x)$.

This part seems hard. Maybe use the fact that the (de Gram) matrix, $a_{i,j} = \langle e_i, e_j\rangle$ for an inner product $\langle \cdot, \cdot \rangle$, is positive definite. It also looks equivalent as showing that the Cauchy-Schwarz inequality is still valid on the finite $l+1$ dimensional space, from that of the finite l dimensional space.

6.3 General type of noise

This section is strongly inspired by the generalized random field introduced on page 242 of [28]

We want to assign to each $\phi \in C_c^\infty(\mathbb{R}^d)$ a random variable $M(\phi)$. If all the finite dimensional laws $(M(\phi_1), \cdots, M(\phi_n))$ are compatible, we call M a random functional.

The random functional M is called linear if for any $\phi, \psi \in C_c^\infty(\mathbb{R}^d)$, and $\alpha \in \mathbb{R}$, we have

$$M(\phi + \alpha\psi) = M(\phi) + \alpha M(\psi) \quad \mathbb{P}\text{-almost surely.}$$

In order to enlarge the domain of definition of the noise, we need to impose some continuity assumption. We could ask the following. If a sequence of test functions converges to another test function, then we would expect the corresponding random variables to converges weakly. In fact, we need to extend that property to random vectors. If each $\phi_{m,k}$ converges to ϕ_m in the space $C_c^\infty(\mathbb{R}^d)$, for $m \in \{1, \ldots, n\}$, then the random vectors $(M(\phi_{1,k}), \ldots, M(\phi_{n,k}))$ converge weakly to $(M(\phi_1), \ldots, M(\phi_n))$, i.e. for any bounded continuous function $f : \mathbb{R}^n \to \mathbb{R}$, we have

$$\mathbb{E}\left[f\left(M(\phi_{1,k}), \ldots, M(\phi_{n,k})\right)\right] \longrightarrow \mathbb{E}\left[f\left(M(\phi_1), \ldots, M(\phi_n)\right)\right].$$

A continuous linear random functional will be called a generalized random function. Such noise is also called a generalized random process (when $d = 1$) or a generalized random field (when $d \geq 2$).

For our purposes we need to restrict to random functional that admits mean and correlation. In fact, we shall restrict to the one with zero mean.

Assumption 6.12. For our purpose, it is sufficient to consider noises with zero mean, i.e.

$$\mathbb{E}[M(\phi)] = 0,$$

for all $\phi \in C_c^\infty(\mathbb{R}^d)$.

If $\mathbb{E}[M(\phi)M(\psi)]$ exists for all $\phi, \psi \in C_c^\infty(\mathbb{R}^d)$, and is continuous in each of the arguments ϕ and ψ, we define the correlation functional

$$B(\phi, \psi) := \mathbb{E}[M(\phi)M(\psi)].$$

By the Kernel Theorem, Theorem 6.1, there exists a distribution on $C_c^\infty(\mathbb{R}^{2d})$ such that

$$B(\phi, \psi) = \langle T_{x,y}, \phi(x)\psi(y) \rangle .$$

For example, we could define a map $M : C_c^\infty(\mathbb{R}^d) \to L^2(\Omega, \mathbb{P})$, whose range is very convenient since it is the largest space in which we can consider covariance. The noise could be a set of mean zero random variables $\{M(\phi), \phi \in C_c^\infty(\mathbb{R}^d)\}$ that are jointly gaussian. If the covariance satisfies an isometry property, e.g. $\mathbb{E}[M(\phi)M(\psi)] = \int_{\mathbb{R}^d} \phi(x)\psi(x)\mu(dx)$, for some positive sigma finite measure μ, then the domain of definition of the noise can be extended. It is in fact a mean zero gaussian noise $M : L^2(\mathbb{R}^d, \mu) \to L^2(\Omega, \mathbb{P})$. That process is usually called the isonormal process. For this example, we can deduce that $T_{x,y}$ is a measure σ on $\mathbb{R}^{2d}$ supported on the diagonal $\Delta = \{(x, x) \in \mathbb{R}^{2d} : x \in \mathbb{R}^d\}$, whose measure is given by $\sigma(B) = \mu(B^\Delta)$, where B is any Borel set $B \subseteq \mathbb{R}^{2d}$, and $B^\Delta = \{x \in \mathbb{R}^d : (x, x) \in B\}$.

What happen when it is supported on the other diagonal? It would lead to

$$B(\phi, \psi) = \int \phi(x)\psi(-x)\, \mu(dx).$$

The latter cannot define a noise since is not positive definite. Indeed for any odd function ϕ, the quantity $B(\phi, \phi) \leq 0$.

As in the previous example, the structure of the noise is strongly related to the properties satisfied by the covariance, when it exists. It should satisfy the following three basic properties of covariance, i.e. linearity, symmetry, and positive-definiteness:

$$\begin{aligned} B(\phi, \psi + \alpha\theta) &= B(\phi, \psi) + \alpha B(\phi, \theta) \\ B(\phi, \psi) &= B(\psi, \phi); \\ B(\phi, \phi) &\geq 0 \end{aligned}$$

for all $\alpha \in \mathbb{R}$.

6.4 Anderson model

From here, every argument is purely formal. We assume existence of some mean zero stochastic integral. We also assume that correlations can be computed in the usual way, see (6.8). The purpose of this section is to generalize the triangle $\triangleright$ operator of Chapter 5, and see that it remains associative. We can also generalize the two points correlation formula.

We would also like to convince the reader that it is sometime more convenient to write

$$\mathbb{E}\left[M(\phi)M(\psi)\right] = \langle T_{t,x,s,y}, \phi(t,x)\psi(s,y)\rangle$$

rather than any of the very informative

$$\begin{aligned}\mathbb{E}\left[M(\phi)M(\psi)\right] &= \int_{\mathbb{R}\times\mathbb{R}} dtds \int_{\mathbb{R}^d\times\mathbb{R}^d} dxdy\, \gamma(t-s)\lambda(x-y)\phi(t,x)\psi(s,y) \\ &= \int_{\mathbb{R}} \gamma(dt) \int_{\mathbb{R}^d} \lambda(dx)\, (\phi * \tilde{\psi})(t,x),\end{aligned}$$

in the example of correlated noise in time and space. The latter are very convenient to see that the bilinear form $B(\phi,\psi) = \mathbb{E}\left[M(\phi)M(\psi)\right]$ is bilinear, symmetric, and positive definite, when γ and λ are symmetric and positive-definite measures. The former will show great effectiveness to show associativity of the triangle $\triangleright$ operator in the following Anderson model.

We are interested in the general parabolic equation,

$$\begin{cases} \frac{\partial u}{\partial t}(t,x) = \mathcal{L}u(t,x) + \lambda u\, \dot{\mathrm{M}}, & t>0,\, x\in\mathbb{R}^d, \\ u(s,\cdot) = S^0, & x\in\mathbb{R}^d, \end{cases} \tag{6.7}$$

where $\mathcal{L}$ is a parabolic operator having a Green function $G(t,x;s,y)$, i.e. the solution to the homogeneous equations is given by

$$u^0(t,x) = \left\langle S^0_y, G(t,x;s,y)\right\rangle,$$

for some (non-random) initial condition at time s given by a distribution S^0.

The Picard iteration scheme would be given by

$$\begin{aligned} u^0(t,x) &= \langle S_y, G(t,x;s,y)\rangle, \\ u^{n+1}(t,x) &= u^0(t,x) + \lambda \iint_{\mathbb{R}\times\mathbb{R}^d} G(t,x;r,y)u^n(r,y)\,\mathrm{M}(dr,dy). \end{aligned}$$

To simplify notation we set $G(t,x;r,y) = \mathbf{1}_{[s,t]}(r)G(t,x;r,y)$, so that instead of having $\int_s^t dr$ we can have $\int_{\mathbb{R}} dr$.

Observe that

$$\mathbb{E}[u^0(t,x)u^0(t',x')] = u^0(t,x)u^0(t',x')$$

since the initial condition is non-random, and

$$\begin{aligned}\mathbb{E}[u^{n+1}(t,x)u^{n+1}(t',x')] &= u^0(t,x)u^0(t',x') \\ &\quad + \lambda^2\mathbb{E}\Big[\iint_{\mathbb{R}\times\mathbb{R}^d} G(t,x;r,y)u^n(r,y)\,\mathrm{M}(dr,dy) \\ &\qquad \times \iint_{\mathbb{R}\times\mathbb{R}^d} G(t',x';r',y')u^n(r',y')\,\mathrm{M}(dr',dy')\Big].\end{aligned}$$

To simplify notations, we introduce the variable $z = (t, x)$, $z' = (t', x')$, $v = (r, y)$, $w = (r', y')$, and $G(z, v) = G(t, x; r, y)$, so that we can rewrite

$$\begin{aligned}\mathbb{E}[u^{n+1}(z)u^{n+1}(z')] &= u^0(z)u^0(z') \\ &\quad + \lambda^2\mathbb{E}\left[\iint_{\mathbb{R}\times\mathbb{R}^d} G(z,v)u^n(v)\,\mathrm{M}(dv)\iint_{\mathbb{R}\times\mathbb{R}^d} G(z',w)u^n(w)\,\mathrm{M}(dw)\right].\end{aligned}$$

We suppose that the last expectation can be computed as

$$\begin{aligned}\mathbb{E}&\left[\iint_{\mathbb{R}\times\mathbb{R}^d} G(z,v)u^n(v)\,\mathrm{M}(dv)\iint_{\mathbb{R}\times\mathbb{R}^d} G(z',w)u^n(w)\,\mathrm{M}(dw)\right] \\ &= \left\langle T_{v,w}, G(z,v)G(z',w)\mathbb{E}\left[u^n(v)u^n(w)\right]\right\rangle. \qquad (6.8)\end{aligned}$$

If we set

$$g^n(z,z') = \mathbb{E}[u^n(z)u^n(z')],$$

then we have the following relation, for $n \geq 0$,

$$g^{n+1}(z,z') = g^0(z,z') + \lambda^2\left\langle T_{v,w}, G(z,v)G(z',w)g^n(v,w)\right\rangle.$$

We define the general operation $\triangleright$ as follows. For $f, g : \mathbb{R}^{4(d+1)} \to \mathbb{R}$, we set

$$(f \triangleright g)(z,z',y,y') = \lambda^2\left\langle T_{v,w}, f(z,z',v,w)f(v,w,y,y')\right\rangle.$$

As was already observed, we could assume $\lambda = 1$, upon replacing each occurence of the distribution T by $\lambda^2 T$. Furthermore, we set

$$\mathcal{L}(z,z',y,y') := G(z,y)G(z',y')$$

and

$$g^n(z,z',y,y') := g^n(z,z').$$

In that case, we have

$$g^{n+1}(z,z') = g^0(z,z') + \left(\mathcal{L} \triangleright g^n\right)(z,z',0,0).$$

We show that the triangle $\triangleright$ operator is associative. This is a consequence of associativity of tensor product for distributions.

Lemma 6.13. *The operation $\triangleright$ is associative.*

Proof. We need to show that $[(f \triangleright g) \triangleright h] = [f \triangleright (g \triangleright h)]$, for any $f, g, h : \mathbb{R}^{4(d+1)} \to \mathbb{R}$. First, observe that

$$\begin{aligned}
&[(f \triangleright g) \triangleright h]\,(z, z', y, y') = \left\langle T_{v_2,w_2}, (f \triangleright g)(z, z', v_2, w_2)\, h(v_2, w_2, y, y') \right\rangle \\
&\quad = \left\langle T_{v_2,w_2}, \left\langle T_{v_1,w_1}, f(z, z', v_1, w_1) g(v_1, w_1, v_2, w_2) \right\rangle h(v_2, w_2, y, y') \right\rangle \\
&\qquad = \left\langle T_{v_1,w_1} \otimes T_{v_2,w_2}, f(z, z', v_1, w_1) g(v_1, w_1, v_2, w_2) h(v_2, w_2, y, y') \right\rangle .
\end{aligned}$$

We also have

$$\begin{aligned}
&[f \triangleright (g \triangleright h)]\,(z, z', y, y') = \left\langle T_{v_1,w_1}, f(z, z', v_1, w_1)(g \triangleright h)(v_1, w_1, y, y') \right\rangle \\
&\quad = \left\langle T_{v_1,w_1}, f(z, z', v_1, w_1) \left\langle T_{v_2,w_2}, g(v_1, w_1, v_2, w_2) h(v_2, w_2, y, y') \right\rangle \right\rangle \\
&\qquad = \left\langle T_{v_1,w_1} \otimes T_{v_2,w_2}, f(z, z', v_1, w_1) g(v_1, w_1, v_2, w_2) h(v_2, w_2, y, y') \right\rangle .
\end{aligned}$$

This concludes the proof. □

Using the associativity of the operation $\triangleright$, we can define

$$\mathcal{L}^{\triangleright n} = \underbrace{\mathcal{L} \triangleright \mathcal{L} \triangleright \cdots \triangleright \mathcal{L}}_{n\text{-times}},$$

and deduce that

$$\begin{aligned}
g^1(z, z') &= g^0(z, z') + \left(\mathcal{L} \triangleright g^0\right)(z, z', 0, 0), \\
g^2(z, z') &= g^0(z, z') + \left(\mathcal{L} \triangleright g^1\right)(z, z', 0, 0) \\
&= g^0(z, z') + \left(\mathcal{L} \triangleright g^0\right)(z, z', 0, 0) + \left(\mathcal{L}^{\triangleright 2} \triangleright g^0\right)(z, z', 0, 0), \\
&\;\vdots \\
g^n(z, z') &= g^0(z, z') + \left[\left(\sum_{k=1}^{n} \mathcal{L}^{\triangleright k}\right) \triangleright g^0\right](z, z', 0, 0).
\end{aligned}$$

From that induction procedure, we see that the fonction

$$\mathcal{K}(z, z', y, y') = \left(\sum_{k=1}^{\infty} \mathcal{L}^{\triangleright k}\right)(z, z', y, y')$$

plays an important role. If the sequence converges, the limit $g = \lim_{n\to\infty} g^n$ would satisfy

$$g(z, z') = g^0(z, z') + \left[\mathcal{K} \triangleright g^0\right](z, z', 0, 0).$$

The formula gives the expectation of the solution as a combination of the initial condition, through g^0, and the kernel $\mathcal{K}$. The next result gives a more convenient expression. It is an adaptation of [9, Lemma 2.8].

Theorem 6.14. *For any initial distribution* S^0,

$$\mathbb{E}\left[u(z)u(z')\right] = g^0(z,z') + \left[\mathcal{K} \rhd g^0\right](z,z',0,0)$$
$$= \left\langle S^0_\alpha \otimes S^0_{\alpha'}, \mathcal{K}(z,z',(s,\alpha),(s,\alpha'))\right\rangle .$$

Before we give the proof, we need to observe the following property about g^0 and the kernel $\mathcal{K}$:

$$g^0(z,z') = u^0(z)u^0(z') = \left\langle S^0_\alpha, G(z,(s,\alpha))\right\rangle \left\langle S^0_{\alpha'}, G(z',(s,\alpha'))\right\rangle$$
$$= \left\langle S^0_\alpha \otimes S^0_{\alpha'}, G(z,(s,\alpha))G(z',(s,\alpha'))\right\rangle$$
$$= \left\langle S^0_\alpha \otimes S^0_{\alpha'}, \mathcal{L}(z,z',(s,\alpha),(s,\alpha'))\right\rangle ,$$

and

$$\mathcal{K} \rhd \mathcal{L} = \sum_{k=1}^{\infty} \mathcal{L}^{\rhd(k+1)} = \sum_{k=2}^{\infty} \mathcal{L}^{\rhd k} = \mathcal{K} - \mathcal{L}.$$

Proof. This property comes from the link between the definition of g^0, $\mathcal{L}$, and $\mathcal{K}$. Indeed,

$$(\mathcal{K} \rhd g^0)(z,z',0,0) = \left\langle T_{v,w}, \mathcal{K}(z,z',v,w)g^0(v,w)\right\rangle$$
$$= \left\langle T_{v,w}, \mathcal{K}(z,z',v,w)\left\langle S^0_\alpha \otimes S^0_{\alpha'}, \mathcal{L}(v,w,(s,\alpha),(s,\alpha'))\right\rangle\right\rangle$$
$$= \left\langle T_{v,w} \otimes S^0_\alpha \otimes S^0_{\alpha'}, \mathcal{K}(z,z',v,w)\mathcal{L}(v,w,(s,\alpha),(s,\alpha'))\right\rangle$$
$$= \left\langle S^0_\alpha \otimes S^0_{\alpha'}, \left\langle T_{v,w}, \mathcal{K}(z,z',v,w)\mathcal{L}(v,w,(s,\alpha),(s,\alpha'))\right\rangle\right\rangle$$
$$= \left\langle S^0_\alpha \otimes S^0_{\alpha'}, (\mathcal{K} \rhd \mathcal{L})(z,z',(s,\alpha),(s,\alpha'))\right\rangle$$
$$= \left\langle S^0_\alpha \otimes S^0_{\alpha'}, \mathcal{K}(z,z',(s,\alpha),(s,\alpha'))\right\rangle$$
$$- \left\langle S^0_\alpha \otimes S^0_{\alpha'}, \mathcal{L}(z,z',(s,\alpha),(s,\alpha'))\right\rangle ,$$

which concludes the proof. □

6.4.1 Inverse triangle operator

One effective way to get a bound on the kernel $\mathcal{K}$, is to get a bound for each of the $\mathcal{L}^{\rhd k}$.

Instead of using the operation $\rhd$, we can directly give an expression for $\mathbb{E}\left[u(z)u(z')\right]$ from g^0, or $S^0 \otimes S^0$, and the tensor products

$$T^{\otimes n}_{v,w} := \underbrace{T_{v_1,w_1} \otimes T_{v_2,w_2} \otimes \cdots \otimes T_{v_n,w_n}}_{n\text{-times}} .$$

For any $n \geq 1$, we have

$$\mathbb{E}[u^n(z)u^n(z')] = u^0(z)u^0(z')$$
$$+ \sum_{k=1}^{n} \Bigg\langle T^{\otimes k}_{v,w}\, ,\, G(z,v_k)G(v_k,v_{k-1}) \cdots G(v_2,v_1)u^0(v_1)$$
$$\times G(z',w_k)G(w_k,w_{k-1}) \cdots G(w_2,w_1)u^0(w_1)\Bigg\rangle .$$

$$\mathbb{E}[u^n(z)u^n(z')] = u^0(z)u^0(z') + \sum_{k=1}^{n} \left\langle T_{v,w}^{\otimes k}, \left(\prod_{j=1}^{k} G(v_{j+1}, v_j) G(w_{j+1}, w_j) \right) u^0(v_1) u^0(w_1) \right\rangle$$

where for each k, we have set $v_{k+1} = z$, $w_{k+1} = z'$.

Lemma 6.15. *If we set $v_0 = (s, \alpha)$, and $w_0 = (s, \alpha')$, we also have*

$$\mathbb{E}[u^n(z)u^n(z')] = \sum_{k=0}^{n} \left\langle T_{v,w}^{\otimes k} \otimes S_\alpha^0 \otimes S_{\alpha'}^0, \prod_{j=0}^{k} G(v_{j+1}, v_j) G(w_{j+1}, w_j) \right\rangle.$$

From here we would like to know if the product

$$\prod_{j=0}^{k} G(v_{j+1}, v_j) G(w_{j+1}, w_j),$$

or if the expression involving that product,

$$\left\langle T_{v,w}^{\otimes k}, \prod_{j=0}^{k} G(v_{j+1}, v_j) G(w_{j+1}, w_j) \right\rangle,$$

could be simplified. For example, could we write

$$\prod_{j=0}^{k} G(v_{j+1}, v_j) G(w_{j+1}, w_j) = G(z, v_0) G(z', w_0) X$$

where X is a function of all variables v_j, w_j, for $j \in \{0, \ldots, k+1\}$?

If the noise is correlated in space, then we could find the corresponding inverse triangle $\triangleleft$ operator. From the fact that G satisfies the semi-group property,

$$\int_{\mathbb{R}^d} G(t, x; r_1, y_1) G(r_1, y_1; r_2, y_2)\, dy_1 = G(t, x; r_2, y_2),$$

we could rewrite

$$G(z, v_1) G(v_1, v_2) = G(z, v_2) \frac{G(z, v_1) G(v_1, v_2)}{G(z, v_2)} = G(z, v_2) X(z, v_1, v_2),$$

where we define

$$X(z, v_1, v_2) := \frac{G(z, v_1) G(v_1, v_2)}{G(z, v_2)}.$$

If the Green function $G(z, v_2)$ vanishes for some z or v_2, then so should $X(z, v_1, v_2)$, so it is set to be zero for all v_1.

From the semi-group property, we know that $X(z, v_1, v_2)$ is a probability distribution, in the space variable y_1 of $v_1 = (r_1, y_1)$. We would like to know if it has the same law as a Lévy bridge? At least in the following case it does.

Example 6.16. For the heat equation on $\mathbb{R}^d$, the Green function is given by the fundamental solution, i.e.

$$G(t, x; r, y) = G(t - r, x - y),$$

where $G(t, x) = \frac{1}{\sqrt{4\pi t}} e^{\frac{-x^2}{4t}}$. Let $B(t)$ be a brownian motion with variance $2t$. We will show that the quotient

$$X_s^{t,z,r,x}(y) = \frac{G(t, z; s, y)G(s, y; r, x)}{G(t, z; r, x)}, \tag{6.9}$$

is the density law at time s and point y of a brownian bridge that starts at time r and point x, and finishes at time t and point z. Of course, we assume that $r \leq s \leq t$. We let

$$\begin{aligned} Y_s = Y_s^{t,z,r,x} = B(s) - \frac{t-s}{t-r}B(r) - \frac{s-r}{t-r}B(t) + \frac{t-s}{t-r}x + \frac{s-r}{t-r}z, \\ Z_s = Z_s^{t,z,r,x} = B(s)|(B(r) = x, B(t) = z), \end{aligned}$$

and will show that both random variable Y_s and Z_s have density $X_s^{t,z,r,x}(y)$. Z_s should be read as $B(s)$ knowing $B(r) = x$ and $B(t) = z$.

Lemma 6.17. *The density of both random variables Y_s and Z_s is given by the quotient* (6.9)*.*

Proof. By definition of the conditional density, we have

$$f_{Z_s}(y) = \frac{f_{B(t),B(s),B(r)}(z, y, x)}{f_{B(t),B(r)}(z, x)}.$$

By a change of variable, we can simplify as

$$f_{Z_s}(y) = \frac{f_{B(t)-B(s),B(s)-B(r),B(r)}(z - y, y - x, x)}{f_{B(t)-B(r),B(r)}(z - x, x)}.$$

Using the independence of the increments of Brownian motion, we can simplify further

$$\begin{aligned} f_{Z_s}(y) &= \frac{f_{B(t)-B(s)}(z - y)f_{B(s)-B(r)}(y - x)}{f_{B(t)-B(r)}(z - x)} \\ &= \frac{G(t - s, z - y)G(s - r, y - x)}{G(t - r, z - x)}. \end{aligned}$$

This is what needed to be proved for Z_s. For Y_s, we can rewrite it as

$$Y_s = -\frac{s-r}{t-r}(B(t) - B(s)) + \frac{t-s}{t-r}(B(s) - B(r)) + \frac{(s-r)z + (t-s)x}{t-r}.$$

By independence of increments, the fact that the increments have normal distributions with variance $2(t-s)$ and $2(s-r)$, we conclude that Y_s follows a normal distribution with mean μ_s and variance σ_s^2, given by

$$\mu_s = \frac{(s-r)z + (t-s)x}{t-r}$$

and

$$\sigma_s^2 = \left(\frac{s-r}{t-r}\right)^2 2(t-s) + \left(\frac{t-s}{t-r}\right)^2 2(s-r) = 2\frac{(t-s)(s-r)}{t-r}.$$

To conclude the proof, we will give the exact form of the quotient, and find that it is indeed the law of a normal distribution with mean μ_s and variance σ_s^2 as above. We apply the product identity (A.11) of the heat kernel to get

$$\begin{aligned} &G(t-s, z-y)G(s-r, y-x) \\ &= G(t-r, z-x)G\left(\frac{(t-s)(s-r)}{t-r}, \frac{(s-r)(z-y) - (t-s)(y-x)}{t-r}\right) \\ &\qquad = G(t-r, z-x)G\left(\frac{(t-s)(s-r)}{t-r}, \frac{(s-r)z + (t-s)x}{t-r} - y\right), \end{aligned}$$

which concludes the proof. □

Observe that for the derivation of the law of Z_s, we “solely" needed the fact that the increments of brownian motion are independent.

Appendix A

Parabolic equations

We restrict our discussion to second order parabolic type equations. We will distinguish the Cauchy problem and the boundary value problem. The former is an equation on the whole space $\mathbb{R}^d$, whereas the latter is on some (simply connected) domain $D \subseteq \mathbb{R}^d$. The domain D, bounded or unbounded, is a proper subset of $\mathbb{R}^d$. The concepts of fundamental solutions, respectively Green functions, are key tools to find the solutions to the Cauchy problem, respectively the boundary value problem. Most ideas in that chapter can be found in [44].

A.1 Cauchy problem

The Cauchy problem can be expressed as

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) - (Au)(t,x) = \Phi(t,x), & t > 0,\ x \in \mathbb{R}^d, \\ u(0,x) = f(x), & x \in \mathbb{R}^d, \end{cases} \tag{A.1}$$

where A is a second order differential operator

$$(Au)(t,x) = \sum_{i,j=1}^{d} a_{i,j}(t,x) \frac{\partial^2 u}{\partial x_i \partial x_j}(t,x) + \sum_{i=1}^{d} a_i(t,x) \frac{\partial u}{\partial x_i}(t,x) + a(t,x)u(t,x), \tag{A.2}$$

with the property of uniform ellipticity, i.e.

$$\sum_{i,j=1}^{d} a_{i,j}(t,x)\xi_i\xi_j \geq \nu \sum_{i=1}^{d} \xi_i^2, \quad \nu > 0. \tag{A.3}$$

Equations (A.1) is said to be homogeneous if $\Phi \equiv 0$, and have vanishing initial condition if $f \equiv 0$.

Very strong results are known for the class of such parabolic equations. Under some precise assumptions on the regularity of the operators A, the solution can be expressed as

$$u(t,x) = \int_0^t \int_{\mathbb{R}^d} \Gamma(t,x;r,y)\Phi(s,y)\,dydr + \int_{\mathbb{R}^d} \Gamma(t,x;0,y)f(y)\,dy, \quad \text{(A.4)}$$

where Γ is the fundamental solution of the Cauchy problem (A.1). As we shall see, the fundamental solution doesn't depend of the functions Φ and f. In fact, it satisfies the following homogeneous equation

$$\begin{cases} \dfrac{\partial \Gamma}{\partial t}(t,x;r,y) - (A_x\Gamma)(t,x;r,y) = 0, & t > 0,\ x, \in \mathbb{R}^d, \\ \Gamma(t,x;t,y) = \delta(x-y), & t \geq 0,\ x \in \mathbb{R}^d. \end{cases} \quad \text{(A.5)}$$

In (A.5), the quantities r and y are free parameters, with $t > r \geq 0$ and $y \in \mathbb{R}^d$, and δ is the Dirac delta functional.

Remark. If the coefficients of the operator A are independent of the time variable t, then the fundamental solution depends only on three arguments, $\Gamma(t,x;r,y) = \Gamma(t-r,x,y)$. Furthermore, if the coefficients of the operator A are constant, then the fundamental solution depends only on two arguments, $\Gamma(t,x;r,y) = \Gamma(t-r,x-y)$.

This section was strongly inspired by [44, Section 16.2.2].

The fundamental solution satisfies the following semi-group property.

Lemma A.1. *For all $t > s > r \geq 0$ and $x, z \in \mathbb{R}^d$, we have*

$$\int_{\mathbb{R}^d} \Gamma(t,z;s,y)\Gamma(s,y;r,x)\,dy = \Gamma(t,z;r,x). \quad \text{(A.6)}$$

This result is given in [22, Theorem VI.I]. For many bounds satisfied by the fundamental solution, see the whole Chapter 6 of [22]. In the particular assumptions of [22, Theorem VI.6], the fundamental solution is positive and have both lower and upper bounds given by the heat kernel. Therefore, Dalang's condition (4.4) is satisfied exactly for the same class of correlated noises.

A.1.1 Heat equation

The heat equation is the particular case of (A.1) where the operator A has constant coefficients of the form

$$(Au)(t,x) = \nu\,\Delta u(t,x) := \nu \sum_{i=1}^d \frac{\partial^2 u}{\partial x_i^2}(t,x), \quad \text{(A.7)}$$

where $\nu > 0$ represents the heat diffusivity of the medium. Φ represents the volume thermal source. Its fundamental solution can be expressed as $\Gamma(t,x;r,y) = \Gamma_\nu(t-r,x-y)$, with

$$\Gamma_\nu(t,x) = \mathbf{1}_{\mathbb{R}^+}(t) \cdot \frac{1}{(4\nu t\pi)^{d/2}} \exp\left\{-\frac{|x|^2}{4\nu t}\right\}, \quad \text{(A.8)}$$

where $\mathbf{1}_{\mathbb{R}^+}$ is the Heaviside unit step function, i.e. $\mathbf{1}_{\mathbb{R}^+}(t) = 0$ for $t \leq 0$, and $\mathbf{1}_{\mathbb{R}^+}(t) = 1$ for $t > 0$. The function Γ_ν will also be called the heat kernel. The notion of (good) kernel, or approximation to the identity, is linked to the fact that indeed $\Gamma_\nu(t, \cdot)$ converges to the Dirac delta functional as $t \to 0$. That fact can be found in [51, Theorem 1.6 of Chapter 5].

Therefore, the solution to the homogeneous heat equation, with initial condition $u_0(x)$, is given by

$$u(t,x) = \int_{\mathbb{R}^d} \frac{1}{(4\nu t\pi)^{d/2}} e^{-\frac{|x-y|^2}{4\nu t}} u_0(y)\, dy. \tag{A.9}$$

In addition to the semi-group property (A.6), the fundamental solution to the heat equation has the following two properties.

Lemma A.2. *For any $x, z \in \mathbb{R}^d$, and $r, t > 0$, we have*

$$\int_{\mathbb{R}^d} \Gamma_\nu(t,y)\, dy = 1, \tag{A.10}$$

$$\Gamma_\nu(r,x)\Gamma_\nu(t,z) = \Gamma_\nu\left(\frac{rt}{r+t}, \frac{rz+tx}{r+t}\right)\Gamma_\nu(r+t, x-z). \tag{A.11}$$

These can be verified directly. We omit the proof.

A.1.2 Properties of the heat kernel

We derive here some properties of the heat kernel. By the semi group property,

$$\int_{\mathbb{R}^d} \Gamma_\nu(t,y)^2\, dy = \Gamma_\nu(2t, 0) = (8\nu t\pi)^{-d/2}. \tag{A.12}$$

We deduce in any dimension

$$\int_0^t \int_{\mathbb{R}^d} \Gamma_\nu(s,y)\, dyds = t. \tag{A.13}$$

And in one space dimension $d = 1$,

$$\int_0^t \int_{\mathbb{R}} \Gamma_\nu(s,y)^2\, dyds = \int_0^t \Gamma_\nu(2s,0)\, ds = \frac{\sqrt{t}}{\sqrt{2\nu\pi}}. \tag{A.14}$$

For all $x, \gamma > 0$, the following bound

$$\begin{aligned} \int_x^\infty e^{-\gamma y^2}\, dy &= \int_0^\infty e^{-\gamma(z+x)^2}\, dz = e^{-\gamma x^2} \int_0^\infty e^{-2\gamma xz} e^{-\gamma z^2}\, dz \\ &\leq e^{-\gamma x^2} \int_0^\infty e^{-\gamma z^2}\, dz = \frac{\sqrt{\pi}}{2\sqrt{\gamma}} e^{-\gamma x^2}, \end{aligned} \tag{A.15}$$

can be used to estimate integrals of the heat kernel:

$$\int_x^\infty \Gamma_\nu(s,y)\,dy = \frac{1}{\sqrt{4\nu s\pi}}\int_x^\infty e^{\frac{-y^2}{4\nu s}}\,dy \le \frac{1}{2}e^{-\frac{x^2}{4\nu s}}, \tag{A.16}$$

$$\int_x^\infty \Gamma_\nu(s,y)^2\,dy = \frac{1}{4\nu s\pi}\int_x^\infty e^{\frac{-y^2}{2\nu s}}\,dy \le \frac{1}{4\sqrt{2\nu s\pi}}e^{-\frac{x^2}{2\nu s}}. \tag{A.17}$$

In [43, Table 1], we can found more general integral bounds for the heat kernel. For example, if we consider the function

$$\phi(y) = \Gamma_{1/2}(1,y) = \frac{1}{\sqrt{2\pi}}e^{-\frac{y^2}{2}},$$

then for any nonnegative integer k we have

$$\int_x^\infty y^{2k+1}\phi(y)\,dy = \phi(x)(2k)!!\sum_{j=0}^{k}\frac{x^{2j}}{(2j)!!}, \tag{A.18}$$

$$\int_x^\infty y^{2k+2}\phi(y)\,dy = \phi(x)(2k+1)!!\sum_{j=0}^{k}\frac{x^{2j+1}}{(2j+1)!!} + (2k+1)!!\Phi(x), \tag{A.19}$$

where the double factorial of a nonnegative integer is defined as follows: $n!! = n\cdot(n-2)\cdots 4\cdot 2$ if n is even, $n!! = n\cdot(n-2)\cdots 5\cdot 3\cdot 1$ if n is odd, and $0!! = 1 = 1!!$. The function $\Phi(x)$ is defined by $\int_x^\infty \phi(y)\,dy$ and can be estimated with (A.16).

Another interesting property concerns the derivatives of $\Gamma_\nu(t,x,y) := \Gamma_\nu(t,x-y)$ in one space dimension.

Lemma A.3. *For any $k,n\in\mathbb{N}$ and $\lambda>1$, there exists $c=c(k,n,\lambda)$ such that*

$$\left|\frac{\partial^k\partial^n\Gamma_\nu}{\partial t^k\partial x^n}(t,x,y)\right| \le \frac{c}{\nu^{n/2}t^{k+n/2}}\Gamma_{\nu\lambda}(t,x,y). \tag{A.20}$$

Proof. Recalling that the heat kernel satisfies the heat equation, we observe that

$$\frac{\partial\Gamma_\nu}{\partial t}(t,x,y) = \frac{\partial\Gamma_\nu}{\partial t}(t,x-y) = \nu\frac{\partial^2\Gamma_\nu}{\partial x^2}(t,x-y),$$

and thus, we only need to get estimates for the space derivatives. By induction, we can show that

$$\frac{\partial^n\Gamma_\nu}{\partial x^n}(t,x) = (\nu t)^{-n/2}P_n\left(\frac{x}{\sqrt{4\nu t}}\right)\Gamma_\nu(t,x),$$

where P_n is a polynomial of degree n. Indeed, the cases $n=1$ and $n=2$ follow from direct computations

$$\frac{\partial\Gamma_\nu}{\partial x}(t,x) = \frac{-2x}{4\nu t}\Gamma_\nu(t,x) = -\frac{1}{\sqrt{\nu t}}\frac{x}{\sqrt{4\nu t}}\Gamma_\nu(t,x),$$

and

$$\frac{\partial^2\Gamma_\nu}{\partial x^2}(t,x) = \left(\frac{-1}{2\nu t} + \left(\frac{-2x}{4\nu t}\right)^2\right)\Gamma_\nu(t,x)$$
$$= \frac{1}{\nu t}\left(-\frac{1}{2} + \left(\frac{x}{\sqrt{4\nu t}}\right)^2\right)\Gamma_\nu(t,x).$$

The general case is deduced by the Leibniz derivation formula. Thus

$$\frac{\partial^k\partial^n\Gamma_\nu}{\partial t^k\partial x^n}(t,x,y) = \nu^k\frac{\partial^{2k+n}\Gamma_\nu}{\partial x^{2k+n}}(t,x,y)$$
$$= \nu^{-n/2}t^{-k-n/2}P_{2k+n}\left(\frac{x-y}{\sqrt{4\nu t}}\right)\Gamma_\nu(t,x-y).$$

Finally, we observe that for any $r \geq 0$ and $1/\lambda = \theta \in (0,1)$, we have

$$z^r e^{-z} \leq \frac{(r+1)!}{(1-\theta)^{r+1}}e^{-\theta z} = c(r,\lambda)e^{-\theta z}. \tag{A.21}$$

Therefore,

$$\left|P_{2k+n}\left(\frac{x-y}{\sqrt{4\nu t}}\right)\right|\Gamma_\nu(t,x-y) \leq c(k,n,\lambda)\Gamma_{\nu\lambda}(t,x-y),$$

which concludes the proof. □

In any space dimension, we get

Lemma A.4. *For any $k \in \mathbb{N}$, $\gamma \in \mathbb{N}^d$, and $\lambda > 1$, there exists a constant $C = C(d,k,\gamma,\lambda)$ such that*

$$\left|\frac{\partial^k\partial^{|\gamma|}\Gamma_\nu}{\partial t^k\partial x^\gamma}(t,x,y)\right| \leq \frac{C}{\nu^{|\gamma|/2}t^{k+|\gamma|/2}}\Gamma_{\nu\lambda}(t,x,y). \tag{A.22}$$

Proof. It is a direct consequence of the one dimensional result (A.20) and the fact that

$$\Gamma_\nu(t,x,y) = \Gamma_\nu^1(t,x_1,y_1)\cdots\Gamma_\nu^1(t,x_d,y_d),$$

where Γ_ν^1 is the heat kernel in one space dimension. □

A.2 Boundary value problems

To express the boundary value problem, we need further information on the behavior that is expected to happen at the boundary $S = \partial D$.

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) - (Au)(t,x) = \Phi(t,x), & t>0,\ x\in D,\\ (Bu)(t,x) = g(t,x), & t>0,\ x\in S,\\ u(0,x) = f(x), & x \in D, \end{cases} \tag{A.23}$$

where A is given by (A.2) and satisfies (A.3), and B is a first order differential operator

$$(Bu)(t,x) = \sum_{i=1}^{d} b_i(t,x) \frac{\partial u}{\partial x_i}(t,x) + b(t,x)u(t,x). \tag{A.24}$$

Equation (A.23) is said to be homogeneous if $\Phi \equiv 0$, have vanishing boundary condition if $g \equiv 0$, and have vanishing initial condition if $f \equiv 0$.

We say that (A.23) is the "first" boundary value problem, when the boundary condition reduces to $u(t,x) = g(t,x)$ for $t > 0$, and $x \in S$, i.e. each $b_i \equiv 0$ and $b \equiv 1$ in (A.24). It is also called the Dirichlet boundary condition. "Second" and "third" boundary value problems can also be defined, with special forms of the boundary operator B. In the case of the heat equation, the "second" boundary value problem is the case in which the boundary condition reduces to $\sum_{i=1}^{d} \nu_i(x) \frac{\partial u}{\partial x_i}(t,x) = g(t,x)$, where $\nu(x)$ is the unit outward normal to the surface S, at the point $x \in S$. It is also called the Neumann boundary condition.

Very strong results are known for the class of such parabolic equations. Under some precise assumptions on the regularity of the operators A and B, and on the smoothness of the boundary S, the solution can be expressed as

$$\begin{aligned} u(t,x) = \int_0^t \int_D G(t,x;r,y)\Phi(r,y)\,dydr + \int_F G(t,x;0,y)f(y)\,dy \\ + \int_0^t \int_S \mathcal{G}(t,x;r,y)g(r,y)\,S(dy)dr, \end{aligned} \tag{A.25}$$

where G is the Green function of the boundary value problem (A.23). As we shall see, the Green function doesn't depend of the functions Φ, f, and g. In fact, it satisfies the following homogeneous equation

$$\begin{cases} \dfrac{\partial G}{\partial t}(t,x;r,y) - (A_x G)(t,x;r,y) = 0, & t > 0,\ x \in D, \\ (B_x G)(t,x;r,y) = 0 & t > 0,\ x \in S, \\ G(t,x;t,y) = \delta(x-y), & t \geq 0,\ x \in D. \end{cases} \tag{A.26}$$

In (A.26), the quantities r and y are free parameters, with $t > r \geq 0$ and $y \in D$, and δ is the Dirac delta functional.

The function $\mathcal{G}$ appearing in the representation formula (A.25) can in fact be expressed via the Green function G. Its expression may depend on both operators A and B.

Remark. If the coefficients of the operator A and B are independent of the time variable t, then the Green function depends only on three arguments, $G(t,x;r,y) = G(t-r,x,y)$. Furthermore, if the coefficients of the operator A and B are constant, then the Green function depends only on two arguments, $G(t,x;r,y) = G(t-r,x-y)$.

This section was strongly inspired by [44, Section 17.4.2]. Further details about the expression of the function $\mathcal{G}$ can be found there.

Remark. For general type equations, the construction of the Green function requires solving an associated eigenvalue problem, see [44, Section 17.5.1]. This method is also applicable for other classes of equations, such as elliptic and hyperbolic.

A.2.1 Heat equation

As we shall see in Section A.4, the method of separation of variables can be applied to find the Green function to the heat equation in one space dimension. To solve the heat equation in higher space dimension, at least for rectangular domains, the following general construction applies.

Consider the rectangular domain $D = [\alpha_1, \beta_1] \times \cdots \times [\alpha_d, \beta_d]$, together with the parabolic equation

$$\frac{\partial u}{\partial t} - A_1 u \cdots - A_d u = \Phi,$$

where each term A_i is a second-order linear differential operator in only one space variable x_i, such that each coefficient depends only on t and x_i, i.e.

$$(A_i u)(t,x) = a_{i,i}(t,x_i)\frac{\partial^2 u}{\partial x_i^2}(t,x) + a_i(t,x_i)\frac{\partial u}{\partial x_i}(t,x) + \bar{a}_i(t,x_i)u(t,x),$$

for $i = 1,\ldots,d$. In fact, this relates to general operator A given by (A.2), in which the matrix $a_{i,j}$ is diagonal, with diagonal coefficient $a_{i,i}(t,x) = a_{i,i}(t,x_i)$. The coefficients $a_i(t,x) = a_i(t,x_i)$ and $a(t,x) = \sum_{i=1}^d \bar{a}_i(t,x_i)$.

Suppose that the boundary conditions at each faces of the domain are prescribed as in a one dimensional problem, then the Green function has the product form

$$G(t,x;r,y) = \prod_{i=1}^{d} G_i(t,x_i,r,y_i), \tag{A.27}$$

where each G_i is the Green function of the one dimensional equation, i.e. it verifies

$$\begin{cases} \dfrac{\partial G_i}{\partial t}(t,x_i;r,y_i) - (A_1 G)(t,x_i;r,y_i) = 0, & t > 0,\ x_i \in (\alpha_i,\beta_i), \\ (B_i G)(t,x_i;r,y_i) = 0 & t > 0,\ x \in \{\alpha_i,\beta_i\}, \\ G(t,x_i;t,y_i) = \delta(x_i - y_i), & t \geq 0,\ x \in (\alpha_i,\beta_i). \end{cases} \tag{A.28}$$

In (A.28), the quantities r and y_i are free parameters, with $t > r \geq 0$ and $y_i \in (\alpha_i,\beta_i)$, and δ is the Dirac delta functional.

This section was strongly inspired by [44, Section 17.5.2].

A.3 Probabilistic methods

Some powerful probabilistic methods can be applied to solve partial differential equations. To solve the Laplace equation, see [53, Theorem 6.1 of Chapter 6]. To solve the heat equation, see [20, Chapter 4]. To solve more general parabolic equations, see [23, Chapter 6.5].

A.3.1 Heat equation

The fundamental solution $\Gamma_\nu(t,x)$, given by (A.8), to the heat equation can be seen as the probability density of a Gaussian random variable with zero mean and variance $2\nu t$. If we let $t \mapsto B_t$ be a standard Brownian motion, then the solution to the heat equation (A.9) can be rewritten as

$$\begin{aligned} u(t,x) &= \int_{\mathbb{R}^d} \frac{1}{(4\nu t\pi)^{d/2}} e^{-\frac{|x-y|^2}{4\nu t}} u_0(y)\, dy \\ &= \mathbb{E}\left[u_0(x+\sqrt{2\nu}B_t)\right] = \mathbb{E}\left[u_0(X(t))\right]. \quad \text{(A.29)} \end{aligned}$$

For latter use, we let $X(t) := x + \sqrt{2\nu}B_t$ be a Brownian motion that starts at x at time $t=0$, whose variance is $2\nu t$.

It is also possible to solve the homogeneous heat equation with Dirichlet boundary condition via a Brownian motion argument. Fix $D \subseteq \mathbb{R}^d$ any open set and $x \in D$. Let $X(t)$ be a Brownian motion starting at x at time $t=0$, with variance $2\nu t$. Let $T = \inf\{t > 0 : X(t) \notin D\}$ be the first time the process X leaves the domain D, and $T_t = \min(t,T)$.

Interpretation of the process $t \mapsto X(T_t)$: The Brownian particule $X(t)$ moves freely until, if ever, it touches the boundary ∂D. When it does, at time T, it stops and forever remains at that boundary point $X(T)$. We say that $X(T_t)$ is “killed" at the boundary.

Theorem A.5. *The solution to the homogeneous heat equation, with Dirichlet boundary condition, is given by*

$$u(t,x) = \mathbb{E}\left[u(t-T_t, X(T_t))\right],$$

when both the initial condition and the boundary condition are continuous and bounded (up to time t).

Proof. This is [20, Theorem 4 of Chapter 4]. □

This expectation is computed solely from the initial condition $u(0,x)$, for $x \in D$, the boundary condition $u(t,x)$, for $t \geq 0$ and $x \in \partial D$, and of course the law of the random vector $(T_t, X(T_t))$. Indeed, if $t < T$, that means the process X has remained within D until time t. In that case $u(t-T_t, X(T_t)) = u(0, X(t))$ depends only on the initial condition. If $T \leq t$, that means the process X has left the domain D at time T through some boundary point $X(T) \in \partial D$. In that case $u(t-T_t, X(T_t)) = u(t-T, X(T))$ depends only on the boundary condition.

The random variable $X(T_t)$ induces a probability measure on the closure $\bar{D}$ of D by

$$\mu_{t,x}(A) = \mathbb{P}\left[X(T_t) \in A\right],$$

for any Borel set $A \subseteq \bar{D}$. It can be shown that the restriction of $\mu_{t,x}$ to the open domain D is precisely the associated Green function, i.e.

$$\mathbb{P}\left[X(T_t) \in A\right] = \int_A G_D(t,x,y)\, dy,$$

for any Borel set $A \subseteq D$.

Proposition A.6. *Let $D \subseteq \mathbb{R}^d$ be any open set and $t > 0$. The Green function $G_D : \mathbb{R}^+ \times D \times D$ associated to the heat equation with Dirichlet boundary conditions is continuous. Furthermore, for any $x, y \in D$,*

$$0 \le G_D(t,x,y) = G_D(t,y,x) \le \Gamma_\nu(t, x-y), \tag{A.30}$$

$$\int_D G_D(t,x,y)\, dy \le 1, \tag{A.31}$$

$$G_{\delta D}(\delta^2 t, \delta x, \delta y) = \frac{1}{\delta^d} G_D(t,x,y), \quad \forall \delta > 0, \tag{A.32}$$

where Γ_ν is the fundamental solution given in (A.8).

Proof. Observe that the event $\{X(T_t) \in A\} \subseteq \{X(t) \in A\}$, for any Borel set $A \subseteq D$. Therefore,

$$\int_A G_D(t,x,y)\, dy = \mathbb{P}\left[X(T_t) \in A\right] \le \mathbb{P}\left[X(t) \in A\right] = \int_A \Gamma_\nu(t, x-y)\, dy.$$

The inequalities of (A.30) follows by continuity, and (A.31) is a consequence of (A.10). The scaling property (A.32) follows from the same scaling property of Brownian motion, or it can be shown that $G_{\delta D}(t,x,y) := G_D(t/\delta^2, x/\delta, y/\delta)/\delta^d$ satisfies the conditions (A.26) of a Green function on δD if G_D does on D. □

Details can be found in [20, Theorems 6–9 of Chapter 4]. Moreover, if $D_1 \subseteq D_2$ are two open sets, then

$$G_{D_1}(t,x,y) \le G_{D_2}(t,x,y), \tag{A.33}$$

for all $t > 0$ and $x, y \in D_1$. It follows directly from the interpretation of the Green function in term of the killed Brownian motion, or from [20, Theorem 7].

In fact, more can be said when we impose a special geometry of the domain D. When it is connected, we have

$$G_D(t,x,y) > 0.$$

When $\mathbb{R}^d \setminus D$ has positive measure,

$$\int_D G_D(t,x,y)\,dy < 1.$$

When D is bounded and convex, an application of the maximum principle gives

$$\int_D G_D(t,x,y)\,dy \geq 1 - 2\int_{\mathbb{R}^d\setminus D} \Gamma_\nu(t,x-y)\,dy, \tag{A.34}$$

for any $t > 0$, and $x \in D$. See [20, Theorem 14 of Chapter 4].

A.4 How to find the Green function in one space dimension

The method of separation of variable is very useful to solve the homogeneous heat equation and find the associated Green function, on some particular domain and with some prescribed boundary conditions. We will give a complete derivation of the Green functions for the vanishing Dirichlet boundary conditions on both domains $[0, L]$ and $[-L, L]$. In fact, the latter will be derived easily from the former. Similar derivations, with less details, will also be done for Neumann and mixed boundary conditions.

The method of separation of variable is described in [44, Section 15.1.2]. Green functions for many other domains and boundary conditions can be found in Sections 3.1.1 and 3.1.2 of [44].

For many boundary conditions, it is known that the Green function is related to the density of some process that is linked to a Brownian motion. As a rule of thumb, Dirichlet boundary conditions will be associated with a Brownian motion that is killed at the boundary points, whereas Neumann boundary condition will be associated with a Brownian motion that is reflected at the boundary points.

A.4.1 Dirichlet boundary conditions

The method of separation of variables is very useful to solve the homogeneous heat equation with vanishing Dirichlet boundary condition,

$$\begin{cases} \frac{\partial u}{\partial t}(t,x) = \frac{\partial^2 u}{\partial x^2}(t,x), & t > 0, x \in (0,L), \\ u(t,0) = u(t,L) = 0, & t > 0, \\ u(0,x) = u_0(x), & x \in (0,L). \end{cases} \tag{A.35}$$

It leads to solutions of the form

$$u(t,x) = \sum_{k=1}^{\infty} a_k \sin\left(\tfrac{\pi}{L}kx\right) e^{\frac{-\pi^2 k^2}{L^2}t}. \tag{A.36}$$

When the initial condition u_0 is an integrable function, we impose the Fourier coefficients

$$a_k = \frac{2}{L}\int_0^L u_0(x)\sin\left(\tfrac{\pi}{L}kx\right)\,dx.$$

Recall that if these coefficients are uniformly bounded, $a_k \le M$, $\forall k \in \mathbb{N}$, then formula (A.36) gives a C^∞ solution on $0 \le x \le L$, $t > 0$. Furthermore, if $\sum_{k=1}^{\infty}|a_k| < \infty$, then the solution is continuous up to the boundary. To get the right initial temperature distribution

$$u_0(x) = u(0,x) = \sum_{k=1}^{\infty} a_k \sin\left(\frac{\pi}{L}kx\right),$$

sufficient conditions are as follow: u_0 is continuous and $\sum_{k=1}^{\infty}|a_k| < \infty$. With such conditions, and for $t > 0$, the solution given by (A.36) can be rewritten as

$$\begin{aligned} u(t,x) &= \sum_{k=1}^{\infty} \sin\left(\tfrac{\pi}{L}kx\right) e^{\frac{-\pi^2 k^2}{L^2}t}\,\frac{2}{L}\int_0^L u_0(y)\sin\left(\tfrac{\pi}{L}ky\right)\,dy \\ &= \int_0^L u_0(y)\left(\frac{2}{L}\sum_{k=1}^{\infty}\sin\left(\tfrac{\pi}{L}kx\right)\sin\left(\tfrac{\pi}{L}ky\right) e^{\frac{-\pi^2 k^2}{L^2}t}\right)\,dy \\ &= \int_0^L u_0(y)\,\bar{G}(t,x,y)\,dy, \end{aligned}$$

where

$$\bar{G}(t,x,y) := \frac{2}{L}\sum_{k=1}^{\infty}\sin\left(\frac{\pi}{L}kx\right)\sin\left(\frac{\pi}{L}ky\right) e^{\frac{-\pi^2 k^2}{L^2}t} \tag{A.37}$$

is the Green function associated to the problem (A.35). That is,

$$\begin{cases} \frac{\partial\bar{G}}{\partial t}(t,x,y) = \frac{\partial^2\bar{G}}{\partial x^2}(t,x,y), & t>0,\ x,y\in(0,L), \\ \bar{G}(t,0,y) = \bar{G}(t,L,y) = 0, & t>0,\ y\in(0,L), \\ \bar{G}(0,x,y) = \delta(x-y), & x,y\in(0,L). \end{cases}$$

The first two equalities are easily verified. The last equality should be understood in the sense of distributions, i.e.

$$\int_0^L u_0(y)\bar{G}(t,x,y)\,dy \xrightarrow{t\to 0} u_0(x),$$

for any test function $u_0 \in C_c^\infty((0,L))$. Results in Fourier analysis handle the convergence for more general class of functions. In particular, the concepts of Poisson kernel and Abel summability, introduced in [51, Section 5.4 of Chapter 2], enable to conclude that the limit exists for the general class on integrable function, and the limit is indeed $u_0(x)$ for all continuity point $x \in (0,L)$.

We now describe some properties of the Green function $\bar{G}$, which for later use will also be denoted as $\bar{G}_L$. Formula (A.37) defines in fact a continuous function on $\mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}$. We can relate that Green function to the heat kernel (A.8) in one space dimension

$$H(t,x) := \mathbf{1}_{\mathbb{R}^+}(t) \frac{1}{\sqrt{4\pi t}} e^{\frac{-x^2}{4t}}. \tag{A.38}$$

Proposition A.7. *Fix any $s,t > 0$ and $x, y \in \mathbb{R}$. The Green function $\bar{G}_L$ is symmetric in x and y, $(2L,0)$- and (L,L)-periodic, odd, and antisymmetric around L:*

$$\begin{aligned}\bar{G}_L(t,y,x) = \bar{G}_L(t,x,y) = \bar{G}_L(t,x+2L,y) = \bar{G}_L(t,x+L,y+L)\\ = -\bar{G}_L(t,-x,y) = -\bar{G}_L(t,2L-x,y);\end{aligned} \tag{A.39}$$

is uniformly bounded in space

$$\left|\bar{G}_L(t,x,y)\right| \le \frac{1}{\sqrt{\pi t}} \exp\left(\frac{-\pi^2 t}{L^2}\right)\left(1 + \frac{2\sqrt{\pi t}}{L}\right); \tag{A.40}$$

is uniformly bounded in the variables x, y and $L > 0$:

$$\left|\bar{G}_L(t,x,y)\right| \le H(t,x-y) \le \frac{1}{\sqrt{4\pi t}}; \tag{A.41}$$

satisfies the semi-group property:

$$\int_0^L \bar{G}_L(s,x,z)\bar{G}_L(t,z,y)\,dz = \bar{G}_L(s+t,x,y); \tag{A.42}$$

can be represented with the heat kernel:

$$\bar{G}_L(t,x,y) = \sum_{n=-\infty}^{\infty} \left[H(t,x-y+2nL) - H(t,x+y+2nL)\right]; \tag{A.43}$$

and is integrable

$$\int_0^L \left|\bar{G}_L(t,x,y)\right| dy \le \int_{-\infty}^{\infty} H(t,y)\,dy = 1. \tag{A.44}$$

Furthermore, if $x, y \in [0,L]$ then $\bar{G}_L(t,x,y) \ge 0$.

Equation (A.43) will be a consequence of the Poisson summation formula, which we now recall: if $f \in \mathcal{S}(\mathbb{R})$ is a function in the Schwartz space, and $\hat{f}(\xi) = \int_{\mathbb{R}} f(y)e^{-2\pi i \xi y}\,dy$ denotes its Fourier transform, then

$$\sum_{n=-\infty}^{\infty} f(z+2nL) = \frac{1}{2L}\sum_{n=-\infty}^{\infty} \hat{f}\left(\tfrac{n}{2L}\right) e^{\frac{\pi}{L}inz}. \tag{A.45}$$

A clear proof of this fact can be read in [51, Theorem 3.1 of Chapter 5].

The fact that the Green function, for particular domains, can be represented with the fundamental solution is also true in higher dimension, see [44, Page 1230].

Proof of Proposition A.7. Equalities in (A.39) follow easily from (A.37) and the properties of the sine function. Uniformly in $x, y \in \mathbb{R}$, we have

$$\left|\bar{G}_L(t,x,y)\right| \le \frac{2}{L}\sum_{k=1}^{\infty} e^{\frac{-\pi^2k^2}{L^2}t} = \frac{2}{L}e^{\frac{-\pi^2}{L^2}t}\left(1+\sum_{k=2}^{\infty} e^{\frac{-\pi^2(k^2-1)}{L^2}t}\right).$$

We can bound the series by the following integral,

$$\sum_{k=2}^{\infty} e^{\frac{-\pi^2(k^2-1)}{L^2}t} \le \int_1^{\infty} e^{\frac{-\pi^2(x^2-1)}{L^2}t}\,dx \le \int_0^{\infty} e^{\frac{-\pi^2x^2}{L^2}t}\,dx = \frac{\sqrt{\pi}}{2}\sqrt{\frac{L^2}{\pi^2 t}}.$$

By setting $\lambda = \sqrt{\pi^2 t/L^2}$, we get

$$\begin{aligned}\left|\bar{G}_L(t,x,y)\right| &\le \frac{2}{L}e^{-\lambda^2}\left(1+\frac{\sqrt{\pi}}{2}\frac{1}{\lambda}\right) = \frac{2\lambda}{\sqrt{\pi^2 t}}e^{-\lambda^2}\left(1+\frac{\sqrt{\pi}}{2}\frac{1}{\lambda}\right)\\ &= \frac{1}{\sqrt{\pi t}}e^{-\lambda^2}\left(\frac{2\lambda}{\sqrt{\pi}}+1\right),\end{aligned}$$

which is inequality (A.40). We observe that $e^{-\lambda^2}\left(\frac{2\lambda}{\sqrt{\pi}}+1\right)$ is a bounded function in the argument λ. Uniformly in $x, y \in \mathbb{R}$, and $L > 0$,

$$\left|\bar{G}_L(t,x,y)\right| \le \frac{2}{L}\sum_{k=1}^{\infty} e^{\frac{-\pi^2k^2}{L^2}t} \le \frac{2}{L}\int_0^{\infty} e^{\frac{-\pi^2x^2}{L^2}t}\,dx = \frac{1}{\sqrt{\pi t}}.$$

The semi-group property comes from the facts that

$$\frac{2}{L}\int_0^L \sin\left(\frac{\pi}{L}mz\right)\sin\left(\frac{\pi}{L}nz\right)\,dz = \delta(n-m),$$

and that integration can be interchanged with double summation.

For the representation with heat kernel, we rewrite the products of sines as sums of exponentials:

$$\begin{aligned}\sin\left(\tfrac{\pi}{L}kx\right)\sin\left(\tfrac{\pi}{L}ky\right) &= \frac{e^{\frac{\pi}{L}ikx}-e^{-\frac{\pi}{L}ikx}}{2i}\cdot\frac{e^{\frac{\pi}{L}iky}-e^{-\frac{\pi}{L}iky}}{2i}\\ &= \frac{1}{4}\left(e^{\frac{\pi}{L}ik(x-y)}+e^{-\frac{\pi}{L}ik(x-y)}-e^{\frac{\pi}{L}ik(x+y)}-e^{-\frac{\pi}{L}ik(x+y)}\right).\end{aligned}$$

Therefore,

$$\begin{aligned}\bar{G}_L(t,x,y) &= \frac{2}{L}\sum_{k=1}^{\infty}\frac{1}{4}\left(e^{\frac{\pi}{L}ik(x-y)} + e^{-\frac{\pi}{L}ik(x-y)}\right)e^{-\frac{\pi^2k^2}{L^2}t} \\ &\quad - \frac{2}{L}\sum_{k=1}^{\infty}\frac{1}{4}\left(e^{\frac{\pi}{L}ik(x+y)} + e^{-\frac{\pi}{L}ik(x+y)}\right)e^{-\frac{\pi^2k^2}{L^2}t} \\ &= \frac{1}{2L}\sum_{k=-\infty}^{\infty} e^{\frac{\pi}{L}ik(x-y)}e^{-\frac{4\pi^2k^2}{4L^2}t} - \frac{1}{2L}\sum_{k=-\infty}^{\infty} e^{\frac{\pi}{L}ik(x+y)}e^{-\frac{4\pi^2k^2}{4L^2}t} \\ &= \sum_{n=-\infty}^{\infty}\frac{1}{\sqrt{4\pi t}}e^{-\frac{(x-y+2nL)^2}{4t}} - \sum_{n=-\infty}^{\infty}\frac{1}{\sqrt{4\pi t}}e^{-\frac{(x+y+2nL)^2}{4t}},\end{aligned}$$

where the last equality is obtained by applying the Poisson summation formula with $f(z) = \frac{1}{\sqrt{4\pi t}}e^{-\frac{z^2}{4t}}$, and $\hat{f}(\xi) = e^{-4\pi^2\xi^2 t}$.

Finally, the integral bound is obtained from the representation with the heat kernel as follows:

$$\begin{aligned}\int_0^L \left|\bar{G}_L(t,x,y)\right| dy &\leq \sum_{k=-\infty}^{\infty}\int_0^L \left[H(t,x-y+2kL) + H(t,x+y+2kL)\right] dy \\ &= \sum_{k=-\infty}^{\infty}\left[\int_{x-L+2kL}^{x+2kL} H(t,y)\,dy + \int_{x+2kL}^{x+L+2kL} H(t,y)\,dy\right] \\ &= \int_{-\infty}^{\infty} H(t,y)\,dy = 1.\end{aligned}$$

Non-negativity for $x, y \in [0, L]$ can be deduced from the maximum principle. Indeed, for any non-negative continuous function $u_0 : [0, L] \to \mathbb{R}$, with $u_0(0) = u_0(L) = 0$,

$$u(t,x) = \int_0^L u_0(y)\,\bar{G}_L(t,x,y)\,dy$$

is continuous in $[0, T] \times [0, L]$ and satisfy the heat equation in $(0, T) \times (0, L)$. The maximum principle, see [32, Theorem 3.1], guarantees in the present case of vanishing Dirichlet boundary conditions that $u(t, x)$ is non-negative in $[0, T] \times [0, L]$. Therefore, $\bar{G}_L(t, x, y)$ is non-negative.

Non-negativity for $x, y \in [0, L]$ can also be deduced thanks to the link between Green function and Brownian motion, introduced in subsection A.3.1. More precisely, it is (A.30). If we let $t \mapsto X(t)$ be a Brownian motion starting at x, with variance $2t$, and define $t \mapsto X(T_t)$ to be the "killed" process at the boundary points 0 and L, then

$$0 \leq \mathbb{P}\left[X(T_t) \in I\right] = \int_I \bar{G}_L(t,x,y)\,dy,$$

for any open sub-interval of $I \subseteq (0, L)$.

A probabilistic argument involving killed Brownian motion can also lead to (A.43). For example [20, Theorem 2 of Chapter 4] or [39, Proposition 8.10 of Section 2.8]. □

Shifted rod

We are now interested in considering a rod of length $2L$, centered at the origin, with the corresponding vanishing Dirichlet boundary conditions, i.e.

$$\begin{cases} \frac{\partial u}{\partial t}(t,x) = \frac{\partial^2 u}{\partial x^2}(t,x) + f(t,x), & t > 0,\ x \in (-L, L), \\ u(t,-L) = u(t,L) = 0, & t > 0, \\ u(0,x) = u_0(x), & x \in (-L, L). \end{cases} \tag{A.46}$$

In that case, using (A.25), the solution becomes

$$u(t,x) = \int_{-L}^{L} G_L(t,x,y)\, u_0(y)\, dy + \int_0^t \int_{-L}^{L} G_L(t-s,x,y)\, f(s,y)\, dyds, \tag{A.47}$$

where

$$\begin{aligned} G_L(t,x,y) &= \bar{G}_{2L}(t, x+L, y+L) \\ &= \frac{1}{L}\sum_{k=1}^{\infty} \sin\left(\frac{\pi}{2L}k(x+L)\right) \sin\left(\frac{\pi}{2L}k(y+L)\right) e^{\frac{-\pi^2 k^2}{4L^2}t} \\ &= \sum_{k=-\infty}^{\infty} \left[H(t, x-y+4kL) - H(t, x+y+(4k+2)L)\right]. \end{aligned} \tag{A.48}$$

To get the preceding formulas (A.48), we first considered the boundary value problem on $[0, 2L]$, and then shifted it to $[-L, L]$. These induced the following changes: every occurence of L is replaced by $2L$, and every occurence of x or y is replaced by $x + L$ or $y + L$, respectively.

In similar ways, as previously done for $\bar{G}_L$, we can show that

Proposition A.8. *Fix any $s, t > 0$ and $x, y \in \mathbb{R}$. The Green function G_L is symmetric in x and y, $(4L, 0)$- and $(2L, 2L)$-periodic, and antisymmetric around $-L$ and L :*

$$\begin{aligned} G_L(t,y,x) = G_L(t,x,y) &= G_L(t, x+4L, y) = G_L(t, x+2L, y+2L) \\ &= -G_L(t, -2L-x, y) = -G_L(t, 2L-x, y); \end{aligned} \tag{A.49}$$

is uniformly bounded in the variable $L > 0$:

$$|G_L(t,x,y)| \le H(t, x-y) \le \frac{1}{\sqrt{4\pi t}}; \tag{A.50}$$

satisfies the semi-group property:

$$\int_{-L}^{L} G_L(s,x,z)G_L(t,z,y)\,dz = G_L(s+t,x,y); \tag{A.51}$$

and is integrable:

$$\int_{-L}^{L} |G_L(t,x,y)|\;dy \leq 1. \tag{A.52}$$

Furthermore, if $x,y \in [-L,L]$ *then* $G_L(t,x,y) \geq 0$.

Remark. We can deduce the Green function for any interval $[a,b]$. It is given by $G(t,x,y) = \bar{G}_{b-a}(t,x-a,y-a)$.

A.4.2 Neumann boundary conditions

The method of separation of variables is very useful to solve the homogeneous heat equation with vanishing Neumann boundary conditions

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) = \dfrac{\partial^2 u}{\partial x^2}(t,x), & t>0,\ x\in(0,L) \\ \dfrac{\partial u}{\partial x}(t,0) = 0 = \dfrac{\partial u}{\partial x}(t,L), & t>0, \\ u(0,x) = u_0(x), & x\in(0,L), \end{cases} \tag{A.53}$$

It leads to solutions of the form

$$u(t,x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos\left(\tfrac{\pi}{L}kx\right) e^{-\frac{\pi^2k^2}{L^2}t}.$$

Identifying the Fourier coefficients a_k as in Section A.4.1, we get that

$$u(t,x) = \int_0^L u_0(y)\bar{G}(t,x,y)\,dy,$$

where

$$\bar{G}(t,x,y) = \frac{1}{L} + \frac{2}{L}\sum_{k=1}^{\infty} \cos\left(\frac{\pi}{L}kx\right)\cos\left(\frac{\pi}{L}ky\right)e^{-\frac{\pi^2k^2}{L^2}t} \tag{A.54}$$

is the Green function associated to the problem (A.53). That is,

$$\begin{cases} \frac{\partial \bar{G}}{\partial t}(t,x,y) = \frac{\partial^2 \bar{G}}{\partial x^2}(t,x,y), & t>0,\ x,y\in(0,L), \\ \frac{\partial \bar{G}}{\partial x}(t,0,y) = 0 = \frac{\partial \bar{G}}{\partial x}(t,L,y), & t>0,\ y\in(0,L) \\ \bar{G}(0,x,y) = \delta(x-y), & x,y\in(0,L). \end{cases}$$

The first two equalities are easily verified. The last should be understood in the sense of distributions, i.e.

$$\int_0^L u_0(y)\bar{G}(t,x,y)\,dy \xrightarrow{t\to 0} u_0(x),$$

for any test function $u_0 \in C_c^\infty((0,L))$. This can be verified for the more general class of continuous functions, see [51, Corollary 3.4 of Chapter 5].

We now describe some properties of the Green function $\bar{G}$, which for later use will also be denoted as $\bar{G}_L$. Formula (A.54) defines in fact a continuous function on $\mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}$. We can relate that Green function to the heat kernel (A.38).

Proposition A.9. *Fix any $s, t > 0$, and $x, y \in \mathbb{R}$. The Green function $\bar{G}_L$ is symmetric in x and y, $(2L, 0)$- and (L, L)- periodic, even, symmetric around L, and non-negative:*

$$\begin{aligned}\bar{G}_L(t,y,x) = \bar{G}_L(t,x,y) &= \bar{G}_L(t,x+2L,y) = \bar{G}_L(t,x+L,y+L) \\ &= \bar{G}_L(t,-x,y) = \bar{G}_L(t,2L-x,y) \geq 0;\end{aligned} \tag{A.55}$$

is uniformly bounded in space:

$$\left|\bar{G}_L(t,x,y) - \frac{1}{L}\right| \leq \frac{1}{\sqrt{\pi t}}; \tag{A.56}$$

satisfies the semi-group property:

$$\int_0^L \bar{G}_L(s,x,z)\bar{G}_L(t,z,y)\,dz = \bar{G}_L(s+t,x,y); \tag{A.57}$$

can be represented with the heat kernel:

$$\bar{G}_L(t,x,y) = \sum_{n=-\infty}^{\infty} \left[H(t,x-y+2nL) + H(t,x+y+2nL)\right]; \tag{A.58}$$

and is integrable:

$$\int_0^L \bar{G}_L(t,x,y)\,dy = \int_{-\infty}^{\infty} H(t,y)\,dy = 1. \tag{A.59}$$

Proof. The equalities in (A.55) follow easily from (A.54) and the properties of the cosine function. Uniformly in $x, y \in \mathbb{R}$, we have

$$\left|\bar{G}_L(t,x,y) - \frac{1}{L}\right| \leq \frac{2}{L}\sum_{k=1}^{\infty} e^{-\frac{\pi^2 k^2}{L^2}t} \leq \frac{2}{L}\int_0^\infty e^{-\frac{\pi^2 x^2}{L^2}t}\,dx = \frac{1}{\sqrt{\pi t}}.$$

Equality (A.57) is obtained by interchanging integration with double summation and by orthogonality of the cosine functions, i.e. for integers m and n,

$$\frac{2}{L}\int_0^L \cos\left(\frac{\pi}{L}mz\right)\cos\left(\frac{\pi}{L}nz\right)\,dz = \delta(n-m).$$

The Poisson summation formula (A.45) has been used to deduce (A.58). It is sufficient to write the product of cosines as a sum of exponentials and follow the same derivation as in the proof of Proposition A.7.

Non-negativity and (A.59) follow directly from (A.58). □

Let $X(t)$ be a Brownian motion starting at x at time $t = 0$, with variance $2t$. We can define the doubly reflected Brownian motion as $\phi(X(t))$, where $\phi : \mathbb{R} \to [0, L]$ satisfies

$$\phi(2nL) = 0, \quad \phi((2n+1)L) = L, \quad \text{for all } n \in \mathbb{Z},$$

and piecewise linear between those points. Then

$$\mathbb{P}\left[\phi(X(t)) \in I\right] = \int_I \bar{G}_L(t, x, y)\, dy,$$

for any open sub-interval I of $(0, L)$. This follows from (A.58) and [39, Exercise 8.9 of Chapter 2.8].

Inequality (A.56) can be compared to [3, Theorem 2.4], where it is proved that for the Neumann boundary condition on any Hölder domain D, its Green function satisfies

$$\left| G(t, x, y) - \frac{1}{|D|} \right| \leq C e^{-ct},$$

for some positive constants C, c and $t \geq t_0 > 0$. This means that the Green function approches the stationary distribution uniformly and exponentially. In the present case, this bound can be achieved using the following estimations. Uniformly in $x, y \in \mathbb{R}$, we have

$$\left| \bar{G}_L(t, x, y) - \frac{1}{L} \right| \leq \frac{2}{L} \sum_{k=1}^{\infty} e^{\frac{-\pi^2 k^2}{L^2} t} = \frac{2}{L} e^{\frac{-\pi^2}{L^2} t} \left(1 + \sum_{k=2}^{\infty} e^{\frac{-\pi^2 (k^2-1)}{L^2} t} \right).$$

We can bound the series by the following integral,

$$\sum_{k=2}^{\infty} e^{\frac{-\pi^2 (k^2-1)}{L^2} t} \leq \int_1^{\infty} e^{\frac{-\pi^2 (x^2-1)}{L^2} t}\, dx \leq \int_0^{\infty} e^{\frac{-\pi^2 x^2}{L^2} t}\, dx = \frac{\sqrt{\pi}}{2} \sqrt{\frac{L^2}{\pi^2 t}}.$$

Thus,

$$\left| \bar{G}_L(t, x, y) \right| \leq \frac{2}{L} e^{-\pi^2 t / L^2} \left(1 + \frac{L}{2\sqrt{\pi t}} \right) = e^{-\pi^2 t / L^2} \left(\frac{2}{L} + \frac{1}{\sqrt{\pi t}} \right).$$

Shifted rod

We are now interested in considering a rod of length $2L$, centered at the origin, with the corresponding vanishing Neumann boundary conditions, i.e.

$$\begin{cases} \dfrac{\partial u}{\partial t}(t, x) = \dfrac{\partial^2 u}{\partial x^2}(t, x) + f(t, x), & t > 0,\ x \in (-L, L), \\ \dfrac{\partial u}{\partial x}(t, 0) = 0 = \dfrac{\partial u}{\partial x}(t, L), & t > 0, \\ u(0, x) = u_0(x), & x \in (-L, L). \end{cases} \tag{A.60}$$

In that case, using (A.25), the solution becomes

$$u(t,x) = \int_{-L}^{L} G_L(t,x,y)\, u_0(y)\, dy + \int_0^t \int_{-L}^{L} G_L(t-s,x,y)\, f(s,y)\, dyds, \tag{A.61}$$

where

$$\begin{aligned} G_L(t,x,y) &= \bar{G}_{2L}(t,x+L,y+L) \\ &= \frac{1}{2L} + \frac{1}{L}\sum_{k=1}^{\infty} \cos\left(\frac{\pi}{2L}k(x+L)\right)\cos\left(\frac{\pi}{2L}k(y+L)\right) e^{\frac{-\pi^2 k^2}{4L^2}t} \\ &= \sum_{k=-\infty}^{\infty} \left[H(t,x-y+4kL) + H(t,x+y+(4k+2)L)\right]. \end{aligned} \tag{A.62}$$

In similar ways, as previously done for $\bar{G}_L$, we can show that

Proposition A.10. *Fix any $s,t>0$, and $x,y\in\mathbb{R}$. The Green function G_L is symmetric in x and y, $(4L,0)$- and $(2L,2L)$- periodic, symmetric around $-L$ and L, and non-negative:*

$$\begin{aligned} G_L(t,y,x) = G_L(t,x,y) &= G_L(t,x+4L,y) = G_L(t,x+2L,y+2L) \\ &= G_L(t,-2L-x,y) = G_L(t,2L-x,y) \geq 0; \end{aligned} \tag{A.63}$$

is uniformly bounded in space:

$$\left| G_L(t,x,y) - \frac{1}{2L} \right| \leq \frac{1}{\sqrt{\pi t}}; \tag{A.64}$$

satisfies the semi-group property:

$$\int_{-L}^{L} G_L(s,x,z) G_L(t,z,y)\, dz = G_L(s+t,x,y); \tag{A.65}$$

and is integrable:

$$\int_{-L}^{L} G_L(t,x,y)\, dy = \int_{-\infty}^{\infty} H(t,y)\, dy = 1. \tag{A.66}$$

Remark. We can deduce the Green function for any interval $[a,b]$. It is given by $G(t,x,y) = \bar{G}_{b-a}(t,x-a,y-a)$.

A.4.3 Mixed boundary conditions

The method of separation of variables is very useful to solve the homogeneous heat equation with vanishing mixed boundary conditions,

$$\begin{cases} \frac{\partial u}{\partial t}(t,x) = \frac{\partial^2 u}{\partial x^2}(t,x), & t>0, x\in(0,L) \\ u(t,0) = 0 = \frac{\partial u}{\partial x}(t,L), & t>0, \\ u(0,x) = u_0(x), & x\in(0,L). \end{cases} \tag{A.67}$$

It leads to solutions of the form

$$u(t,x) = \sum_{k=0}^{\infty} a_k \sin\left(\tfrac{\pi}{2L}(2k+1)x\right) e^{-\frac{\pi^2(2k+1)^2}{4L^2}t}.$$

Identifying the Fourier coefficients a_k as in Section A.4.1, we get that

$$u(t,x) = \int_0^L u_0(y)\bar{G}(t,x,y)\,dy,$$

where

$$\bar{G}(t,x,y) = \frac{2}{L}\sum_{k=0}^{\infty} \sin\left(\tfrac{\pi}{2L}(2k+1)x\right)\sin\left(\tfrac{\pi}{2L}(2k+1)y\right) e^{-\frac{\pi^2(2k+1)^2}{4L^2}t} \qquad \text{(A.68)}$$

is the Green function associated to the problem (A.67). That is,

$$\begin{cases} \frac{\partial\bar{G}}{\partial t}(t,x,y) = \frac{\partial^2\bar{G}}{\partial x^2}(t,x,y), & t>0,\ x,y\in(0,L),\\ \bar{G}(t,0,y) = 0 = \frac{\partial\bar{G}}{\partial x}(t,L,y), & t>0,\ y\in(0,L),\\ \bar{G}(0,x,y) = \delta(x-y), & x,y\in(0,L). \end{cases}$$

The first two equalities are easily verified. The last should be understood in the sense of distributions, i.e.

$$\int_0^L u_0(y)\bar{G}(t,x,y)\,dy \xrightarrow{t\to 0} u_0(x),$$

for any test function $u_0 \in C_c^\infty((0,L))$. This can be proven from the same property satisfied by the Green function $\bar{G}^D_{2L}(t,x,y)$ associated to the Dirichlet problem on the interval $[0,2L]$, see definition (A.37), and the following representation (A.73). Indeed, extend u_0 to the interval $(0,2L)$ by $u_0(x)=0$ for $x\in[L,2L)$. Then

$$\begin{aligned} &\int_0^L \bar{G}(t,x,y)u_0(y)\,dy \\ &\quad = \int_0^{2L} \bar{G}^D_{2L}(t,x,y)u_0(y)\,dy + \int_0^{2L} \bar{G}^D_{2L}(t,x,2L-y)u_0(y)\,dy \\ &\qquad = \int_0^{2L} \bar{G}^D_{2L}(t,x,y)u_0(y)\,dy + \int_0^{2L} \bar{G}^D_{2L}(t,x,y)u_0(2L-y)\,dy. \end{aligned}$$

The first integral on the right hand side converges to $u_0(x)$, and the second integral to $u_0(2L-x)=0$, for any $x\in(0,L)$ by definition of the extension.

We now describe some properties of the Green function $\bar{G}$, which for later use will also be denoted as $\bar{G}_L$. Formula (A.68) defines in fact a continuous function on $\mathbb{R}_+\times\mathbb{R}\times\mathbb{R}$. We can relate that Green function to the heat kernel (A.38).

Proposition A.11. *Fix any* $s, t > 0$, *and* $x, y \in \mathbb{R}$. *The Green function* $\bar{G}_L$ *is symmetric in* x *and* y, $(2L, 0)$*-antiperiodic, odd, and thus* $(4L, 0)$*- and* $(2L, 2L)$*-periodic and symmetric around* L :

$$\begin{aligned}\bar{G}_L(t, y, x) &= \bar{G}_L(t, x, y) = -\bar{G}_L(t, x + 2L, y) = -\bar{G}_L(t, -x, y) \\ &= \bar{G}_L(t, x + 4L, y) = \bar{G}_L(t, x + 2L, y + 2L) = \bar{G}_L(t, 2L - x, y); \end{aligned} \quad \text{(A.69)}$$

is uniformly bounded in space:

$$\left|\bar{G}_L(t, x, y)\right| \leq \frac{2}{\sqrt{4\pi t}}; \quad \text{(A.70)}$$

satisfies the semi-group property:

$$\int_0^L \bar{G}_L(s, x, z)\bar{G}_L(t, z, y)\, dz = \bar{G}_L(s + t, x, y); \quad \text{(A.71)}$$

can be represented with the heat kernel:

$$\begin{aligned}\bar{G}_L(t, x, y) &= \sum_{n=-\infty}^{\infty} (-1)^n \left[H(t, x - y + 2nL) - H(t, x + y + 2nL)\right] \quad \text{(A.72)} \\ &= \bar{G}^D_{2L}(t, x, y) + \bar{G}^D_{2L}(t, x, 2L - y); \quad \text{(A.73)}\end{aligned}$$

and is integrable:

$$\int_0^L \left|\bar{G}_L(t, x, y)\right|\, dy \leq \int_{-\infty}^{\infty} H(t, y)\, dy = 1. \quad \text{(A.74)}$$

Furthermore, if $x, y \in [0, L]$ *then* $\bar{G}_L(t, x, y) \geq 0$.

Proof. Equalities in (A.69) follow easily from (A.68) and the properties of the sine function. Uniformly in $x, y \in \mathbb{R}$, we have

$$\left|\bar{G}_L(t, x, y)\right| \leq \frac{2}{L}\sum_{k=1}^{\infty} e^{-\frac{\pi^2 k^2}{4L^2}t} \leq \frac{2}{L}\int_0^{\infty} e^{\frac{-\pi^2 x^2}{4L^2}t}\, dx = \frac{2}{\sqrt{\pi t}}.$$

Equality (A.71) is obtained by interchanging integration with double summation and by orthogonality of the sine functions, i.e. for integers m and n,

$$\frac{2}{L}\int_0^L \sin\left(\frac{\pi}{2L}(2m + 1)z\right)\sin\left(\frac{\pi}{2L}(2n + 1)z\right)\, dz = \delta(n - m).$$

For the representation with heat kernel (A.72), we apply the Poisson summation formula (A.45) to deduce that

$$\sum_{n=-\infty}^{\infty} H(t, x-y+4nL) - \sum_{n=-\infty}^{\infty} H(t, x+y+4nL)$$
$$- \sum_{n=-\infty}^{\infty} H(t, x-y+(4n+2)L) + \sum_{n=-\infty}^{\infty} H(t, x+y+(4n+2)L)$$
$$= \frac{1}{4L} \sum_{k=-\infty}^{\infty} e^{-\frac{\pi^2 k^2 t}{4L^2}} e^{i\frac{\pi}{2L}k(x-y)} - \frac{1}{4L} \sum_{k=-\infty}^{\infty} e^{-\frac{\pi^2 k^2 t}{4L^2}} e^{i\frac{\pi}{2L}k(x+y)}$$
$$- \frac{1}{4L} \sum_{k=-\infty}^{\infty} e^{-\frac{\pi^2 k^2 t}{4L^2}} e^{i\frac{\pi}{2L}k(x-y+2L)} + \frac{1}{4L} \sum_{k=-\infty}^{\infty} e^{-\frac{\pi^2 k^2 t}{4L^2}} e^{i\frac{\pi}{2L}k(x+y+2L)},$$

the four components with $k = 0$ cancel out, and we rewrite each summation from $k = -\infty$ to $k = +\infty$ as two summations from $k = 1$ to $k = +\infty$, hence this is equal to

$$\frac{1}{4L} \sum_{k=1}^{\infty} e^{-\frac{\pi^2 k^2 t}{4L^2}} \left[e^{i\frac{\pi}{2L}k(x-y)} + e^{-i\frac{\pi}{2L}k(x-y)} \right]$$
$$- \frac{1}{4L} \sum_{k=1}^{\infty} e^{-\frac{\pi^2 k^2 t}{4L^2}} \left[e^{i\frac{\pi}{2L}k(x+y)} + e^{-i\frac{\pi}{2L}k(x+y)} \right]$$
$$- \frac{1}{4L} \sum_{k=1}^{\infty} e^{-\frac{\pi^2 k^2 t}{4L^2}} \left[e^{i\frac{\pi}{2L}k(x-y+2L)} + e^{-i\frac{\pi}{2L}k(x-y+2L)} \right]$$
$$+ \frac{1}{4L} \sum_{k=1}^{\infty} e^{-\frac{\pi^2 k^2 t}{4L^2}} \left[e^{i\frac{\pi}{2L}k(x+y+2L)} + e^{-i\frac{\pi}{2L}k(x+y+2L)} \right]$$
$$= \frac{1}{4L} \sum_{k=1}^{\infty} e^{-\frac{\pi^2 k^2 t}{4L^2}} \left[e^{i\frac{\pi}{2L}k(x-y)} + e^{-i\frac{\pi}{2L}k(x-y)} - e^{i\frac{\pi}{2L}k(x+y)} - e^{-i\frac{\pi}{2L}k(x+y)} \right]$$
$$- \frac{1}{4L} \sum_{k=1}^{\infty} e^{-\frac{\pi^2 k^2 t}{4L^2}} \left[e^{i\frac{\pi}{2L}k(x-y+2L)} + e^{-i\frac{\pi}{2L}k(x-y+2L)} - e^{i\frac{\pi}{2L}k(x+y+2L)} - e^{-i\frac{\pi}{2L}k(x+y+2L)} \right],$$

recall that $e^{i\theta} + e^{-i\theta} = 2\cos(\theta)$, and that $\cos(b) - \cos(a) = 2\sin\left(\frac{a+b}{2}\right)\sin\left(\frac{a-b}{2}\right)$, hence this is equal to

$$\frac{1}{L} \sum_{k=1}^{\infty} e^{-\frac{\pi^2 k^2 t}{4L^2}} \sin\left(\tfrac{\pi}{2L}kx\right) \sin\left(\tfrac{\pi}{2L}ky\right)$$
$$- \frac{1}{L} \sum_{k=1}^{\infty} e^{-\frac{\pi^2 k^2 t}{4L^2}} \sin\left(\tfrac{\pi}{2L}k(x+2L)\right) \sin\left(\tfrac{\pi}{2L}ky\right)$$

$$= \frac{1}{L}\sum_{k=1}^{\infty} e^{-\frac{\pi^2 k^2 t}{4L^2}} \left[\sin\left(\tfrac{\pi}{2L}kx\right) - \sin\left(\tfrac{\pi}{2L}k(x+2L)\right)\right] \sin\left(\tfrac{\pi}{2L}ky\right)$$
$$= \frac{2}{L}\sum_{k=0}^{\infty} \sin\left(\tfrac{\pi}{2L}(2k+1)x\right) \sin\left(\tfrac{\pi}{2L}(2k+1)y\right) e^{-\frac{\pi^2(2k+1)^2}{4L^2}t},$$

since

$$\sin(\theta) - \sin(\theta + k\pi) = \begin{cases} 0, & \text{if } k \text{ is even,} \\ 2\sin(\theta), & \text{if } k \text{ is odd.} \end{cases}$$

Prove that $\bar{G}_L(t,x,y) = \bar{G}^D_{2L}(t,x,y) - \bar{G}^D_{2L}(t,x+2L,y)$ using the definition (A.37), and the fact that for $\theta = \frac{\pi k}{2L}$,

$$\sin(\theta) - \sin(\theta + k\pi) = \begin{cases} 0, & \text{if } k \text{ is even,} \\ 2\sin(\theta), & \text{if } k \text{ is odd.} \end{cases}$$

Then use the representation (A.43) to deduce the present representation (A.72).

The Inequality (A.74) follows from (A.72) exactly as (A.44) followed from (A.43),

$$\begin{aligned} \int_0^L \left|\bar{G}_L(t,x,y)\right| dy &\le \sum_{k=-\infty}^{\infty} \int_0^L \left[H(t,x-y+2kL) + H(t,x+y+2kL)\right] dy \\ &= \sum_{k=-\infty}^{\infty} \left[\int_{x-L+2kL}^{x+2kL} H(t,y)\,dy + \int_{x+2kL}^{x+L+2kL} H(t,y)\,dy\right] \\ &= \int_{-\infty}^{\infty} H(t,y)\,dy = 1. \end{aligned}$$

Non-negativity of $\bar{G}_L(t,x,y)$ for $x,y \in [0,L]$ follows from non-negativity of $\bar{G}^D_{2L}(t,u,v)$ for $u,v \in [0,2L]$, see Proposition A.7. Indeed,

$$\begin{aligned} \bar{G}_L(t,x,y) &= \sum_{n=-\infty}^{\infty} (-1)^n \left[H(t,x-y+2nL) - H(t,x+y+2nL)\right] \\ &= \sum_{n=-\infty}^{\infty} \left[H(t,x-y+4nL) - H(t,x+y+4nL)\right] \\ &\quad - \sum_{n=-\infty}^{\infty} \left[H(t,x-y+(4n+2)L) - H(t,x+y+(4n+2)L)\right] \\ &= \bar{G}^D_{2L}(t,x,y) - \bar{G}^D_{2L}(t,x+2L,y) \\ &= \bar{G}^D_{2L}(t,x,y) - \bar{G}^D_{2L}(t,x,y-2L) \\ &= \bar{G}^D_{2L}(t,x,y) + \bar{G}^D_{2L}(t,x,2L-y) \ge 0, \end{aligned}$$

since $x, y, 2L-y \in [0,2L]$. □

We will define a Brownian motion $\phi(X(T_t))$ that is killed at the origin (Dirichlet boundary condition) and reflected at the boundary point L (Neumann boundary condition), and show that its density law is given by $\bar{G}_L$ in the open interval $(0, L)$. First, let $X(t)$ be a Brownian motion starting at x, with variance $2t$. Then define the Brownian motion that is reflected at the boundary point L as $\phi(X(t))$, where $\phi : \mathbb{R} \to (-\infty, L]$ satisfies

$$\begin{cases} \phi(x) = x, & x \leq L, \\ \phi(x) = 2L - x, & x \geq L. \end{cases}$$

Finally, let

$$T = \inf\{t > 0 : \phi(X(t)) = 0\} = \inf\{t > 0 : X(t) \notin (0, 2L)\},$$

be the first time the reflected Brownian motion $\phi(X(t))$ hits 0, or equivalently the first time the Brownian motion $X(t)$ leaves the domain $(0, 2L)$, and $T_t = \min(t, T)$.

We now find the density law of $\phi(X(T_t))$ in the open interval $(0, L)$. Recall that the density law of $X(T_t)$ in the open interval $(0, 2L)$ was already found to be the Green function $\bar{G}_{2L}^D$ associated to the Dirichlet problem on $(0, 2L)$. Thus

$$\begin{aligned} \mathbb{P}\left[\phi(X(T_t)) \in (a, b)\right] &= \mathbb{P}\left[X(T_t) \in (a, b)\right] + \mathbb{P}\left[X(T_t) \in (2L - b, 2L - a)\right] \\ &= \int_a^b \bar{G}_{2L}^D(t, x, y)\, dy + \int_{2L-b}^{2L-a} \bar{G}_{2L}^D(t, x, y)\, dy \\ &= \int_a^b \bar{G}_{2L}^D(t, x, y) + \bar{G}_{2L}^D(t, x, 2L - y)\, dy \\ &= \int_a^b \bar{G}_L(t, x, y)\, dy, \end{aligned}$$

for any interval $(a, b) \subseteq (0, L)$.

Shifted rod

We are now interested in considering a rod of length $2L$, centered at the origin, with the corresponding vanishing mixed boundary conditions, i.e.

$$\begin{cases} \dfrac{\partial u}{\partial t}(t, x) = \dfrac{\partial^2 u}{\partial x^2}(t, x) + f(t, x), & t > 0,\ x \in (-L, L), \\ u(t, 0) = 0 = \dfrac{\partial u}{\partial x}(t, L), & t > 0, \\ u(0, x) = u_0(x), & x \in (-L, L). \end{cases} \tag{A.75}$$

In that case, using (A.25), the solution becomes

$$u(t, x) = \int_{-L}^{L} G_L(t, x, y)\, u_0(y)\, dy + \int_0^t \int_{-L}^{L} G_L(t - s, x, y)\, f(s, y)\, dyds, \tag{A.76}$$

where

$$\begin{aligned} G_L(t,x,y) &= \bar{G}_{2L}(t, x+L, y+L) \\ &= \frac{1}{L}\sum_{k=0}^{\infty} \sin\left(\tfrac{\pi}{2L}(2k+1)(x+L)\right) \sin\left(\tfrac{\pi}{2L}(2k+1)(y+L)\right) e^{-\frac{\pi^2(2k+1)^2}{16L^2}t} \\ &= \sum_{n=-\infty}^{\infty} (-1)^n \left[H(t, x-y+4nL) - H(t, x+y+(4n+2)L)\right]. \end{aligned} \tag{A.77}$$

In similar ways, as previously done for $\bar{G}_L$, we can show that

Proposition A.12. *Fix any $s, t > 0$, and $x, y \in \mathbb{R}$. The Green function G_L is symmetric in x and y, $(4L, 0)$-antiperiodic, $(8L, 0)$- and $(4L, 4L)$-periodic, antisymmetric around $-L$, and symmetric around L :*

$$\begin{aligned} & G_L(t,y,x) = G_L(t,x,y) = -G_L(t, x+4L, y) = G_L(t, x+8L, y) \\ &= G_L(t, x+4L, y+4L) = -G_L(t, -2L-x, y) = G_L(t, 2L-x, y); \end{aligned} \tag{A.78}$$

is uniformly bounded in space:

$$|G_L(t,x,y)| \le \frac{2}{\sqrt{4\pi t}}; \tag{A.79}$$

satisfies the semi-group property:

$$\int_0^L G_L(s,x,z) G_L(t,z,y)\, dz = G_L(s+t, x, y); \tag{A.80}$$

can be represented with the Dirichlet Green function:

$$G_L(t,x,y) = G^D_{2L}(t, x-L, y-L) + G^D_{2L}(t, x-L, L-y); \tag{A.81}$$

and is integrable:

$$\int_0^L |G_L(t,x,y)|\, dy \le \int_{-\infty}^{\infty} H(t,y)\, dy = 1. \tag{A.82}$$

Furthermore, if $x, y \in [-L, L]$ then $G_L(t,x,y) \ge 0$.

Remark. We can deduce the Green function for any interval $[a, b]$. It is given by $G(t,x,y) = \bar{G}_{b-a}(t, x-a, y-a)$.

Appendix B

Some general results

B.1 Minkowski, Burkholder, Fubini

The next two inequalities will be used to evaluate the $L^p(\Omega)$ norm of the stochastic heat solution.

Fact 1 (Minkowski's inequality for integrals). *Let (X_1, μ_1) and (X_2, μ_2) be two (σ-finite) measure spaces, and $1 \leq p \leq \infty$. If $f(x_1, x_2)$ is measurable on $X_1 \times X_2$ and non-negative, then*

$$\left\| \int_{X_2} f(x_1, x_2)\, d\mu_2 \right\|_{L^p(X_1)} \leq \int_{X_2} \|f(x_1, x_2)\|_{L^p(X_1)}\, d\mu_2. \tag{B.1}$$

Some indications on the proof can be found in [53, Exercise 15 of Chapter 1]. We will use this inequality in the special case where $(X_1, \mu_1) = (\Omega, \mathbb{P})$, and where X_2 is some (measurable) subset of the Euclidean space $\mathbb{R}$ or $\mathbb{R}^d$ and μ_2 is the Lebesgue measure. Another application is the following:

$$\begin{aligned} \left\| \int_{X_2} f(x_2) K(x_1, x_2)\, d\mu_2 \right\|_{L^p(X_1)} &\leq \int_{X_2} |f(x_2)|\, \|K(x_1, x_2)\|_{L^p(X_1)}\, d\mu_2 \\ &\leq \|f\|_{L^1(X_2)}\, \|K\|_{L^{p,\infty}(X_1, X_2)}, \end{aligned}$$

where

$$\|K\|_{L^{p,\infty}(X_1, X_2)} = \sup_{x_2 \in X_2} \left(\int_{X_1} |K(x_1, x_2)|^p\, d\mu_1 \right)^{1/p}.$$

In the particular case of convolution, $(X_1, \mu_1) = (X_2, \mu_2) = (\mathbb{R}^d, dx)$, we get

$$\|f * g\|_{L^p(\mathbb{R}^d)} \leq \|f\|_{L^p(\mathbb{R}^d)}\, \|g\|_{L^1(\mathbb{R}^d)}. \tag{B.2}$$

Fact 2 (Burkholder's inequality). *Let $\{M_t\}_{t \geq 0}$ be a continuous (local) martingale, with initial condition $M_0 = 0$ and with quadratic variation $\langle M \rangle_t$ at time t. There exist universal constants k_p, only depending on $p > 0$, such that*

$$\|M_t\|^2_{L^p(\Omega)} \leq k_p^2\, \|\langle M \rangle_t\|_{L^{p/2}(\Omega)}. \tag{B.3}$$

We will only need the cases $p \geq 2$. In those cases, Itô's formula can be applied and the proof is almost done. More details can be found in [13, Theorem 5.27].

Fact 3 (Fubini's theorem)**.** *Let $\mathcal{P}$ the set of predicable functions in $\Omega \times \mathbb{R}_+$, and let $(G, \mathcal{G}, \mu)$ be a finite measure space. If a function $f : \Omega \times \mathbb{R}_+ \times \mathbb{R}^d \times G$ is $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{G}$-measurable and satisfies*

$$\mathbb{E}\left[\int_{[0,T]} \int_{\mathbb{R}^d} \int_G |f(\omega, s, x, z)|^2 \, dsdx\mu(dz)\right] < \infty,$$

then almost surely,

$$\int_G \left(\int_0^t \int_{\mathbb{R}^d} f(\omega, s, x, z) W(dsdx)\right) \mu(dz) = \int_0^t \int_{\mathbb{R}^d} \left(\int_G f(\omega, s, x, z)\mu(dz)\right) W(dsdx). \quad \text{(B.4)}$$

This is [55, Theorem 2.6].

B.2 Special functions

B.2.1 Properties of Gamma function

Let us recall the definition of the Gamma function and its derivatives:

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} \, dt, \quad \Gamma^{(k)}(x) = \int_0^\infty e^{-t} t^{x-1} \log(t)^k \, dt.$$

Lemma B.1 (Beta integrals)**.** *For any $t > r \geq 0$, and $x, y > 0$,*

$$\int_r^t (t-s)^{x-1}(s-r)^{y-1} \, ds = (t-r)^{x+y-1} \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}. \quad \text{(B.5)}$$

In the special case where $r = 0$ and $t = 1$, the left hand side is denoted by $B(x, y)$ and is called the Beta distribution.

Proof. We only prove the case $r = 0$ and $t = 1$ since the general result then follows. Using the change of variable $s = pq$ and $t = p(1-q)$, we get

$$\begin{aligned}\Gamma(x)\Gamma(y) &= \int_0^\infty e^{-s} s^{x-1} \, ds \int_0^\infty e^{-t} t^{y-1} \, dt \\ &= \int_0^\infty \int_0^1 e^{-pq} p^{x-1} q^{x-1} e^{-p(1-q)} p^{y-1} (1-q)^{y-1} p \, dpdq \\ &= \int_0^\infty e^{-p} p^{x+y-1} \, dp \int_0^1 q^{x-1}(1-q)^{y-1} \, dq = \Gamma(x+y)B(x, y). \square\end{aligned}$$

We now recall some properties of the Gamma function:

1. It takes the special values $\Gamma(1) = 1$ and $\Gamma(1/2) = \sqrt{\pi}$. The first is trivial, and the second comes from the change of variable $s = \sqrt{t}$ and the Gauss integral.

2. For all $x > 0$, $\Gamma(x+1) = x\Gamma(x)$. Hence, $\Gamma(2) = 1$ and $\Gamma(3/2) = \frac{\sqrt{\pi}}{2}$. This comes from an integration by parts.

3. It is log-convex, i.e., for $x > 0$, $x \mapsto \log(\Gamma(x))$ is convex. Indeed, the second derivative of $\log(\Gamma(x))$ is non-negative as can be seen by applying the Cauchy-Schwarz inequality to the measure $e^{-t}t^{x-1}dt$:
$$[\log(\Gamma(x))]'' = \frac{\Gamma''(x)\Gamma(x) - \Gamma'(x)^2}{\Gamma(x)^2} \geq 0.$$

4. For any $\alpha > 0$, the function $x \mapsto \Gamma(x+\alpha)/\Gamma(x)$ in increasing on $\mathbb{R}_+^*$. This fact follows directly from log-convexity, but can also be proved using the Beta distribution. Indeed
$$\Gamma(x+\alpha)/\Gamma(x) = \Gamma(\alpha)/B(x,\alpha),$$
and $x \mapsto B(x,\alpha)$ is decreasing on $\mathbb{R}_+^*$. In particular for $\alpha = 1/2$ and $x = y+1$, we have $\Gamma(y+1) \leq (2/\sqrt{\pi})\Gamma(y+3/2)$ for all $y \geq 0$.

5. Since Γ is a strictly convex function on $\mathbb{R}_+^*$ (because $\Gamma''(x) > 0$ for all $x > 0$) and $\Gamma(1) = \Gamma(2)$, it admits a unique minimum at some point $\gamma \in (1,2)$. Furthermore, Gamma is strictly decreasing before γ and strictly increasing after γ. It can be shown that $\gamma \approx 1.4616$ and $\Gamma(\gamma) \approx 0.8856$, which is very close to $\Gamma(3/2) = \frac{\sqrt{\pi}}{2} \approx 0.8862$.

We may use the following result found in Gelfand, book 1, page 53.
$$\int_0^\infty t^x \left(e^{-at} - e^{-bt}\right) dt = \left(\frac{1}{a^{x+1}} - \frac{1}{b^{x+1}}\right)\Gamma(x+1).$$
That integral is defined for $\Re(x) > -2$.

B.2.2 Bounds on Mittag-Leffler functions

Recall the definition of the Mittag-Leffler functions: for $\alpha, \beta > 0$,
$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad \alpha, \beta > 0.$$
In the text, we were interested in the special case where $\alpha = \beta = 1/2$. We bound $E_{1/2,1/2}$ by polynomials and the exponential function:

Lemma B.2. *For $z \geq 0$,*

$$\frac{1}{\sqrt{\pi}} - 1 + (z+1)\exp\left(z^2\right) \leq E_{1/2,1/2}(z) \leq \frac{1}{\sqrt{\pi}} + \left(\frac{2}{\pi} - 1\right) z^2 + (z^2 + z)\exp\left(z^2\right). \tag{B.6}$$

In fact, it is shown in [45, equation (1.147)] that

$$E_{1/2,1/2}(z) \leq c_1 + c_2(z+1)\exp(z^2),$$

for some constants c_1 and c_2.

Proof. We separate the summation over the even and odd integers,

$$E_{1/2,1/2}(z) = \sum_{k=0}^{\infty} \frac{z^{2k}}{\Gamma(k+1/2)} + z\sum_{k=0}^{\infty} \frac{z^{2k}}{\Gamma(k+1)}.$$

The second series is equal to $\exp\left(z^2\right)$. Recalling that $\Gamma(1/2) = \sqrt{\pi}$, and $\Gamma(x) < \Gamma(y)$ if $3/2 \leq x \leq y$, the first series can be decomposed as

$$\begin{aligned}
\sum_{k=0}^{\infty} \frac{z^{2k}}{\Gamma(k+1/2)} &= \frac{1}{\sqrt{\pi}} + \sum_{k=1}^{\infty} \frac{z^{2k}}{\Gamma(k+1/2)} \\
&\geq \frac{1}{\sqrt{\pi}} + \sum_{k=1}^{\infty} \frac{z^{2k}}{\Gamma(k+1)} \\
&= \frac{1}{\sqrt{\pi}} - 1 + \sum_{k=0}^{\infty} \frac{z^{2k}}{k!} = \frac{1}{\sqrt{\pi}} - 1 + \exp\left(z^2\right),
\end{aligned}$$

or as

$$\begin{aligned}
\sum_{k=0}^{\infty} \frac{z^{2k}}{\Gamma(k+1/2)} &= \frac{1}{\sqrt{\pi}} + \frac{z^2}{\Gamma(3/2)} + \sum_{k=2}^{\infty} \frac{z^{2k}}{\Gamma(k+1/2)} \\
&\leq \frac{1}{\sqrt{\pi}} + \frac{2}{\sqrt{\pi}} z^2 + z^2 \sum_{k=1}^{\infty} \frac{z^{2k}}{\Gamma(k+1)} \\
&= \frac{1}{\sqrt{\pi}} + \frac{2}{\sqrt{\pi}} z^2 - z^2 + z^2 \sum_{k=0}^{\infty} \frac{z^{2k}}{k!} \\
&= \frac{1}{\sqrt{\pi}} + \left(\frac{2}{\sqrt{\pi}} - 1\right) z^2 + z^2 \exp\left(z^2\right).
\end{aligned}$$

We used the fact that $\Gamma(3/2) = \frac{\sqrt{\pi}}{2}$. To improve the inequality, we should use a generalization of [45, equation (1.60)]. □

B.2.3 Bounds on Theta function

Recall the definition of the theta function, for $a > 0$,

$$\theta(a) = \sum_{k=0}^{\infty} e^{-ak^2}.$$

We can bound it as follows:

$$\frac{\sqrt{\pi}}{2\sqrt{a}} = \int_0^{\infty} e^{-ax^2}\,dx \leq \theta(a) \leq 1 + \int_0^{\infty} e^{-ax^2}\,dx = 1 + \frac{\sqrt{\pi}}{2\sqrt{a}}, \tag{B.7}$$

$$1 \leq \theta(a) \leq \sum_{k=0}^{\infty} e^{-ak} = \frac{1}{1-e^{-a}}. \tag{B.8}$$

The behaviour of the theta function is easily deduced. When $a \to 0$, we see that the bound (B.7) goes as $1/\sqrt{a}$, whereas the bound (B.8) goes as $1/a$. When $a \to \infty$, the bound (B.8) is much better. Many definitions of the theta function exist in the literature, for example $\vartheta(s) = \sum_{n=-\infty}^{\infty} e^{-\pi n^2 s}$, for $s > 0$. Some important properties of ϑ can be found in [51, Section 3.1 of Chapter 5].

B.3 Generalization of Convolution

We define, for some function $\phi : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{C}$,

$$\begin{aligned}(f \star g)(t) &= \int_0^t f(t-s)\phi(t-s,s)\overline{g(s)}\,ds \\ &= \int_0^t f(s)\phi(s,t-s)\overline{g(t-s)}\,ds.\end{aligned}$$

Lemma B.3. *Applied to real-valued functions, the $\star$ operator is associative if, for all $r, s, t \in \mathbb{R}$,*

$$\phi(s,t-s)\overline{\phi(t-s-r,r)} = \phi(s,t-s-r)\phi(t-r,r). \tag{B.9}$$

This operator is hermitian if $\phi(r,s) = \overline{\phi(s,r)}$. In particular, it is both associative and hermitian when ϕ is some power of the harmonic mean, i.e.

$$\phi(r,s) = \left(\frac{rs}{r+s}\right)^{\beta} = \left(\frac{1}{\frac{1}{r}+\frac{1}{s}}\right)^{\beta},$$

for any $\beta \in \mathbb{R}$.

Equation (B.9) could be closely related to reversibility?

Proof. Suppose that $\phi(r,s) = \overline{\phi(s,r)}$, then

$$(g \star f)(t) = \int_0^t g(s)\phi(s,t-s)\overline{f(t-s)}\,ds = \int_0^t g(s)\overline{\phi(t-s,s)}\,\overline{f(t-s)}\,ds,$$

and thus $\overline{(g \star f)(t)} = (f \star g)(t)$. We proceed with associativity.

$$\begin{aligned}
[f \star (g \star h)](t) &= \int_0^t ds\, f(s)\phi(s,t-s)\overline{(g \star h)(t-s)} \\
&= \int_0^t ds\, f(s)\phi(s,t-s)\int_0^{t-s} dr\, \overline{g(t-s-r)}\,\overline{\phi(t-s-r,r)}h(r).
\end{aligned}$$

$$\begin{aligned}
[(f \star g) \star h](t) &= \int_0^t dr\, (f \star g)(t-r)\phi(t-r,r)\overline{h(r)} \\
&= \int_0^t dr \int_0^{t-r} ds\, f(s)\phi(s,t-r-s)\overline{g(t-r-s)}\phi(t-r,r)\overline{h(r)} \\
&= \int_0^t ds \int_0^{t-s} dr\, f(s)\phi(s,t-r-s)\overline{g(t-r-s)}\phi(t-r,r)\overline{h(r)},
\end{aligned}$$

which concludes the proof when h is real. □

Application to white noise when $\phi(r,s) = \left(\frac{rs}{r+s}\right)^{-1/2}$. And to Riesz kernel.

Corollary B.4. *For $q(t) \equiv 1$, and $\phi(r,s) = \left(\frac{r+s}{rs}\right)^{\alpha}$, for $\alpha < 1$, we define $q^{\star 1} = q$, and*

$$q^{\star(n+1)} = q \star q^{\star n}.$$

We can explicitly compute

$$q^{\star n}(t) = t^{(n-1)(1-\alpha)}\frac{\Gamma(1-\alpha)^n}{\Gamma\left(n(1-\alpha)\right)}.$$

Proof. We proceed by induction. By definition, induction hypothesis, and the change of variable $r = s/t$, we get

$$\begin{aligned}
q^{\star(n+1)}(t) &= \int_0^t q(t-s)\phi(t-s,s)q^{\star n}(s)\,ds \\
&= t^{\alpha}\int_0^t (t-s)^{-\alpha}s^{-\alpha}s^{(n-1)(1-\alpha)}\frac{\Gamma(1-\alpha)^n}{\Gamma(n(1-\alpha))}\,ds \\
&= t^{\alpha+1}\frac{\Gamma(1-\alpha)^n}{\Gamma(n(1-\alpha))}\int_0^1 (t-tr)^{-\alpha}(tr)^{-\alpha}(tr)^{(n-1)(1-\alpha)}\,dr \\
&= t^{n(1-\alpha)}\frac{\Gamma(1-\alpha)^n}{\Gamma(n(1-\alpha))}\int_0^1 (1-r)^{(1-\alpha)-1}r^{n(1-\alpha)-1}\,dr,
\end{aligned}$$

since $(\alpha+1)-\alpha-\alpha+(n-1)(\alpha-1) = n(\alpha-1)$. Using Beta integrals, Lemma B.1, we can rewrite the last integral as follows

$$\int_0^1 (1-r)^{(1-\alpha)-1} r^{n(1-\alpha)-1}\,dr = \frac{\Gamma(1-\alpha)\Gamma(n(1-\alpha))}{\Gamma((n+1)(1-\alpha))},$$

which conclude the proof. □

Appendix C

Sobolev spaces

Let us recall the Sobolev spaces H^s. For any $s \in \mathbb{R}$,

$$H^s = \left\{ f \in \mathcal{S}'(\mathbb{R}^d) : \int_{\mathbb{R}^d} \left(1 + |\xi|^2\right)^s |\mathcal{F}(f)(\xi)|^2 \, d\xi < \infty \right\}$$

is a Hilbert space with the following inner product: for any $f, g \in H^s$,

$$\langle f, g \rangle_{H^s} = \int_{\mathbb{R}^d} \left(1 + |\xi|^2\right)^s \mathcal{F}(f)(\xi) \overline{\mathcal{F}(g)(\xi)} \, d\xi.$$

Another definition of the Sobolev spaces is in term of the weak derivatives. In the case $s = n \in \mathbb{N}$, we have the equivalent definition [35, Remark 3.14]:

$$H^n = W_2^n := \left\{ f \in L^2(\mathbb{R}^d) : \forall \beta \in \mathbb{N}^d \text{ such that } |\beta| \le n, \; D^\beta f \in L^2(\mathbb{R}^d) \right\},$$

where $D^\beta f$ is a derivative in the sense of distributions. It has the equivalent norm

$$\|f\|_{W_2^n}^2 = \sum_{|\beta| \le n} \left\| D^\beta f \right\|_{L^2(\mathbb{R}^d)}^2 .$$

In the case $s = n + \sigma$, for $n \in \mathbb{N}$ and $0 < \sigma < 1$, we have the equivalent definition [35, Theorem 3.24]:

$$H^s = W_2^s := \left\{ f \in W_2^n : \|f\|_{W_2^s} < \infty \right\},$$

where

$$\|f\|_{W_2^s}^2 = \|f\|_{W_2^n}^2 + \sum_{|\alpha| = n} \int_{h \le 1} \frac{dh}{|h|^{d+2\sigma}} \int_{\mathbb{R}^d} dx \, |D^\alpha f(x+h) - D^\alpha f(x)|^2 .$$

Therefore, the index s in the definition of H^s accounts for the regularity of the distribution. In particular, we have the following chain of inclusions:

$$\mathcal{S} \subseteq H^s \subseteq H^r \subseteq L^2(\mathbb{R}^d) \subseteq H^{-r} \subseteq H^{-s} \subseteq \mathcal{S}',$$

for any two real numbers $0 \le r \le s$. It can be shown that H^{-s} is in fact the dual space of H^s.

C.1 Local versions of Sobolev spaces

We introduce the local version of Sobolev spaces. For any $s \in \mathbb{R}$,

$$H^s_{loc} = \left\{ f \in \mathcal{S}'(\mathbb{R}^d) : \forall \phi \in C_c^\infty(\mathbb{R}^d),\ \phi \cdot f \in H^s \right\}.$$

Lemma C.1. *For any $v, w \in \mathbb{R}^d$,*

$$\left(1 + |v+w|^2\right) \le 2\left(1+|v|^2\right)\left(1+|w|^2\right). \tag{C.1}$$

In particular, we get $\left(1+|v|^2\right)^{-1} \le 2\left(1+|v+w|^2\right)^{-1}\left(1+|w|^2\right)$.

Lemma C.2. *For any $s \in \mathbb{R}$, the Sobolev space $H^s(\mathbb{R}^d)$ is contained into its local version $H^s_{loc}(\mathbb{R}^d)$.*

Proof. Let us fix $f \in H^s(\mathbb{R}^d)$. We begin with the following claim: for any $\phi \in C_0^\infty$, the distribution $\mathcal{F}(\phi \cdot f)$ is given by the function $\mathcal{F}f * \mathcal{F}\phi$. By definition, for any $\psi \in \mathcal{S}$,

$$\begin{aligned}
\langle \mathcal{F}(\phi \cdot f), \psi \rangle &= \langle \phi \cdot f, \mathcal{F}\psi \rangle = \langle f, \phi \cdot \mathcal{F}\psi \rangle = \left\langle \mathcal{F}f, \mathcal{F}^{-1}(\phi \cdot \mathcal{F}\psi) \right\rangle \\
&= \left\langle \mathcal{F}f, (\mathcal{F}^{-1}\phi) * \psi \right\rangle = \int_{\mathbb{R}^d} \mathcal{F}f(\xi) \left((\mathcal{F}^{-1}\phi) * \psi\right)(\xi)\, d\xi \\
&= \int_{\mathbb{R}^d} d\xi\, \mathcal{F}f(\xi) \int_{\mathbb{R}^d} dy\, \mathcal{F}^{-1}\phi(\xi - y)\psi(y) \\
&= \int_{\mathbb{R}^d} dy\, \psi(y) \int_{\mathbb{R}^d} d\xi\, \mathcal{F}f(\xi)\, \underbrace{\mathcal{F}^{-1}\phi(\xi - y)}_{= \mathcal{F}\phi(y-\xi)} \\
&= \int_{\mathbb{R}^d} \psi(y)\, (\mathcal{F}f * \mathcal{F}\phi)(y)\, dy,
\end{aligned}$$

where Fubini's theorem has been used in the next-to-last equality. Before checking Fubini's hypothesis, we observe that the function $\mathcal{F}f * \mathcal{F}\phi$ is well-defined. Indeed,

$$\begin{aligned}
&\int_{\mathbb{R}^d} |\mathcal{F}f(\xi)|\, |\mathcal{F}\phi(y-\xi)|\, d\xi \\
&\quad = \int_{\mathbb{R}^d} |\mathcal{F}f(\xi)| \left(1+|\xi|^2\right)^{s/2} |\mathcal{F}\phi(y-\xi)| \left(1+|\xi|^2\right)^{-s/2} d\xi \\
&\qquad \le \|f\|_{H^s(\mathbb{R}^d)} \left(\int_{\mathbb{R}^d} |\mathcal{F}\phi(y-\xi)|^2 \left(1+|\xi|^2\right)^{-s} d\xi \right)^{1/2}.
\end{aligned}$$

The last integral is finite for all $s \in \mathbb{R}$ since $\mathcal{F}\phi \in \mathcal{S}(\mathbb{R}^d)$. In the case $s \le 0$, we can apply inequality (C.1) with $v = -y$ and $w = y - \xi$, so that

$$\left(1+|\xi|^2\right)^{-s} \le 2^{-s}\left(1+|y|^2\right)^{-s}\left(1+|y-\xi|^2\right)^{-s}.$$

In the case $s \geq 0$, we apply inequality (C.1) with $v = \xi$ and $w = y - \xi$, so that

$$\left(1+|\xi|^2\right)^{-s} \leq 2^s \left(1+|y|^2\right)^{-s} \left(1+|y-\xi|^2\right)^{s}.$$

In both cases, we get

$$\begin{aligned}\left(\int_{\mathbb{R}^d} |\mathcal{F}\phi(y-\xi)|^2 \left(1+|\xi|^2\right)^{-s} d\xi\right)^{1/2} & \\ \leq 2^{|s|/2} \left(1+|y|^2\right)^{-s/2} \left(\int_{\mathbb{R}^d} |\mathcal{F}\phi(y-\xi)|^2 \left(1+|y-\xi|^2\right)^{|s|} d\xi\right)^{1/2} & \\ = 2^{|s|/2} \|\phi\|_{H^{|s|}(\mathbb{R}^d)} \left(1+|y|^2\right)^{-s/2}, &\end{aligned}$$

and therefore,

$$\int_{\mathbb{R}^d} |\mathcal{F}f(\xi)|\,|\mathcal{F}\phi(y-\xi)|\, d\xi \leq 2^{|s|/2} \|f\|_{H^s(\mathbb{R}^d)} \|\phi\|_{H^{|s|}(\mathbb{R}^d)} \left(1+|y|^2\right)^{-s/2}. \tag{C.2}$$

Upon replacing $v = \xi$ and $w = -y$ in the second case $s \geq 0$, we can derive the similar bound:

$$\int_{\mathbb{R}^d} |\mathcal{F}f(\xi)|\,|\mathcal{F}\phi(y-\xi)|\, d\xi \leq 2^{|s|/2} \|f\|_{H^s(\mathbb{R}^d)} \|\phi\|_{H^{-s}(\mathbb{R}^d)} \left(1+|y|^2\right)^{|s|/2}. \tag{C.3}$$

We are now able to prove Fubini's hypothesis we needed above. Indeed,

$$\begin{aligned}\int_{\mathbb{R}^d} dy\; |\psi(y)| \int_{\mathbb{R}^d} d\xi\; |\mathcal{F}f(\xi)|\,|\mathcal{F}\phi(y-\xi)| & \\ \leq 2^{|s|/2} \|f\|_{H^s(\mathbb{R}^d)} \|\phi\|_{H^{-s}(\mathbb{R}^d)} \int_{\mathbb{R}^d} |\psi(y)| \left(1+|y|^2\right)^{|s|/2} dy < \infty, &\end{aligned}$$

since $\psi \in \mathcal{S}$.

We are left to show that the distribution $\phi \cdot f \in H^s(\mathbb{R}^d)$. We will apply Minkowski's inequality for integrals (B.1) to the function

$$(\xi, y) \mapsto \left(1+|\xi|^2\right)^{s/2} \mathcal{F}\phi(y)\mathcal{F}f(\xi-y),$$

as well as inequality (C.1) to the following two cases: if $s \geq 0$ we set $v = y$

and $w = \xi - y$, and if $s < 0$ we set $v = \xi$ and $w = -y$. Hence,

$$
\begin{aligned}
\|\phi \cdot f\|_{H^s(\mathbb{R}^d)} &= \left(\int_{\mathbb{R}^d} \left(1+|\xi|^2\right)^s |(\mathcal{F}f * \mathcal{F}\phi)(\xi)|^2 \, d\xi\right)^{1/2} \\
&= \left(\int_{\mathbb{R}^d} d\xi \, \left(1+|\xi|^2\right)^s \left|\int_{\mathbb{R}^d} dy \, \mathcal{F}\phi(y)\mathcal{F}f(\xi - y)\right|^2\right)^{1/2} \\
&\leq \int_{\mathbb{R}^d} dy \, |\mathcal{F}\phi(y)| \left(\int_{\mathbb{R}^d} d\xi \, \left(1+|\xi|^2\right)^s |\mathcal{F}f(\xi-y)|^2\right)^{1/2} \\
&\leq 2^{|s|/2} \int_{\mathbb{R}^d} dy \, \left(1+|y|^2\right)^{|s|/2} |\mathcal{F}\phi(y)| \\
&\quad \times \left(\int_{\mathbb{R}^d} d\xi \, \left(1+|\xi-y|^2\right)^s |\mathcal{F}f(\xi-y)|^2\right)^{1/2} \\
&= 2^{|s|/2} \|f\|_{H^s(\mathbb{R}^d)} \int_{\mathbb{R}^d} \left(1+|y|^2\right)^{|s|/2} |\mathcal{F}\phi(y)| \, dy < \infty,
\end{aligned}
$$

which concludes the proof. □

Notice that the same proof applies if we replace the hypothesis $\phi \in C_0^\infty$ by $\phi \in \mathcal{S}$.

We have the trivial inclusions for compactly supported functions.

Lemma C.3. *Let $n \in \mathbb{N}$ and $0 < \sigma < 1$. We have $C_0^n \subseteq H^n$, and $C_0^{n+\sigma} \subseteq H^{n+\sigma-\varepsilon}$, for all $\varepsilon > 0$.*

Proof. For a function $f \in C_0^n$, and $|\beta| \leq n$, the derivative $D^\beta f$ in the sense of distribution is given by its usual derivative, which is continuous and compactly supported, hence in $L^2(\mathbb{R}^d)$. Hence $C_0^n \subseteq W_2^n = H^n$.

For a function $f \in C_0^{n+\sigma}$, we evaluate its $W_2^{n+\sigma-\varepsilon}$ norm. We already know that $\|f\|_{W_2^n} < \infty$, by the previous argument, so we are left with the estimates of the form

$$
\begin{aligned}
&\int_{h\leq 1} \frac{dh}{|h|^{d+2\sigma-2\varepsilon}} \int_{\mathbb{R}^d} dx \, |D^\alpha f(x+h) - D^\alpha f(x)|^2 \\
&\qquad \leq \int_{h\leq 1} \frac{dh}{|h|^{d+2\sigma-2\varepsilon}} \int_{\mathrm{supp}(f)+B(0,1)} dx \, C^2 |h|^{2\sigma} \\
&\qquad\qquad \leq c \int_{h\leq 1} \frac{dh}{|h|^{d-2\varepsilon}} < \infty,
\end{aligned}
$$

for all multi-index $\alpha \in \mathbb{N}^d$ with $|\alpha| = n$. □

For more information about Sobolev spaces, and a proof of the Sobolev embedding, see [35, Section 3.2 and 3.3].

Appendix D

Uniqueness of solution to the heat equation

The weak formulation (3.3) to the heat equation admits a unique solution in the space of $C^0([0,\infty),\mathcal{S}'(\mathbb{R}^d))$. Indeed, if u, v are two such solutions, then the difference $w := u - v$ is a solution to

$$\langle w(t), \psi\rangle = \int_0^t \langle w(s), \Delta\psi\rangle \, ds. \tag{D.1}$$

Observe that this is the weak formulation to the following initial value heat equation:

$$\begin{cases} \frac{\partial w}{\partial t}(t,x) = \Delta w(t,x) & t > 0,\, x \in \mathbb{R}^d, \\ w(0,x) = 0, & x \in \mathbb{R}^d. \end{cases} \tag{D.2}$$

Theorem D.1. *The unique solution to* (D.1) *in the space* $C^0([0,\infty),\mathcal{S}'(\mathbb{R}^d))$ *is the zero distribution, i.e.* $w(t) = 0 \in \mathcal{S}'(\mathbb{R}^d)$, *for all* $t \geq 0$.

D.1 Calculus in $\mathcal{S}'(\mathbb{R}^d)$

Before proving this result, we need to clarify whether the weak or the strong topology is used in $\mathcal{S}'(\mathbb{R}^d)$. Let T_n be a sequence in $\mathcal{S}'(\mathbb{R}^d)$. In the weak topology, we say that this sequence converges to $T \in \mathcal{S}'(\mathbb{R}^d)$, if $\langle T_n, \phi\rangle \to \langle T, \phi\rangle$, for all $\phi \in \mathcal{S}(\mathbb{R}^d)$. In the strong topology, we further require that the convergence holds uniformly on any bounded set in $\mathcal{S}(\mathbb{R}^d)$. A useful result about weak and strong topologies is the following:

Lemma D.2. *Any continuous map* $w \in C^0([0,\infty),\mathcal{S}'(\mathbb{R}^d))$ *in the weak topology is also continuous in the strong topology.*

This is Theorem XIII of Chapter III together with Section 4 in Chapter VII of [50]. It is also a consequence of the fact that the class of bounded sets in the weak topology coincide with the class of bounded sets in the strong topology, see Section 5.5 in Chapter I of [27]. It is therefore sufficient to consider convergence in the weak topology.

Lemma D.3. *Suppose* $w \in C^0([0,\infty), \mathcal{S}'(\mathbb{R}^d))$ *and satisfies* (D.1)*, then* $w \in C^\infty([0,\infty), \mathcal{S}'(\mathbb{R}^d))$.

Proof. We first show that $t \mapsto w(t)$ is (weakly) differentiable. For $h > 0$,

$$\frac{\langle w(t+h), \psi\rangle - \langle w(t), \psi\rangle}{h} = \frac{1}{h}\int_t^{t+h} \langle w(s), \Delta\psi\rangle \, ds \longrightarrow \langle w(t), \Delta\psi\rangle ,$$

as $h \to 0$, for any $\psi \in \mathcal{S}(\mathbb{R}^d)$, since each map $s \mapsto \langle w(s), \Delta\psi\rangle$ is continuous by assumption. A similar argument can be applied for $h < 0$. Thus,

$$\left\langle w'(t), \psi \right\rangle = \langle w(t), \Delta\psi\rangle . \tag{D.3}$$

The fact that $t \mapsto w'(t)$ is continuous follows at once from continuity of $t \mapsto w(t)$. Proceeding inductively, we conclude that

$$\left\langle w^{(n)}(t), \psi \right\rangle = \langle w(t), \Delta^n \psi\rangle ,$$

and again, continuity of $w^{(n)}$ follows from that of w. □

Proof of Theorem D.1. From equation (D.1), it is clear that $w(0)$ is the zero distribution. Fix $T > 0$ and $\psi \in \mathcal{S}(\mathbb{R}^d)$. We want to show that $\langle w(T), \psi\rangle = 0$. We consider the dual problem

$$\begin{cases} \frac{\partial U}{\partial t}(t,x) = -\Delta U(t,x) & t \in (0,T),\ x \in \mathbb{R}^d, \\ U(T,x) = \psi(x), & x \in \mathbb{R}^d, \end{cases} \tag{D.4}$$

for which $U(t) := \Gamma(T-t) * \psi$ is a solution, and belongs to $\mathcal{S}(\mathbb{R}^d)$ for all $t \in [0,T]$. To conclude the proof, it is sufficient to show that the function $H : [0,T] \to \mathbb{R}$ defined by $H(t) = \langle w(t), U(t)\rangle$ is continuous on $[0,T]$ and differentiable on $(0,T)$ with $H'(t) = 0$. In that case,

$$\langle w(T), \psi\rangle = \langle w(T), U(T)\rangle = \langle w(0), U(0)\rangle = 0.$$

We are left to analyse the function H. Assuming that $t \mapsto U(t)$ is differentiable in $\mathcal{S}(\mathbb{R}^d)$, we could apply Lemma D.4 and deduce that

$$\begin{aligned} H'(t) &= \left\langle w'(t), U(t)\right\rangle + \left\langle w(t), U'(t)\right\rangle \\ &= \langle w(t), \Delta U(t)\rangle + \langle w(t), -\Delta U(t)\rangle = 0, \end{aligned}$$

thanks to equations (D.3) and (D.4).

In turn, we are left to analyse the regularity of the function $t \mapsto U(t) = \Gamma(T-t)*\psi$. In Proposition D.10, we show that $U \in C^\infty([0,T], \mathcal{S}(\mathbb{R}^d))$, which concludes the present proof. □

This proof was strongly inspired by Chapter II of [26].

Lemma D.4. *Suppose* $w : [0,T] \to \mathcal{S}'(\mathbb{R}^d)$ *is continuous and (weakly) differentiable on* $(0,T)$, *and* $U : [0,T] \to \mathcal{S}(\mathbb{R}^d)$ *is continuous and differentiable on* $(0,T)$. *Then, the function* $H : [0,T] \to \mathbb{R}$ *defined by* $H(t) = \langle w(t), U(t) \rangle$ *is continuous and differentiable on* $(0,T)$, *with*

$$H'(t) = \big\langle w'(t), U(t) \big\rangle + \big\langle w(t), U'(t) \big\rangle .$$

Proof. This is precisely Appendix 2 in Chapter II of [27]. □

D.2 Calculus in $\mathcal{S}(\mathbb{R}^d)$

In order to complete the proof of the uniqueness theorem, we need to introduce the class of $C^k([0,\infty), \mathcal{S}(\mathbb{R}^d))$ functions. They are k-times continuously differentiable functions taking values in the Schwartz space $\mathcal{S}(\mathbb{R}^d)$. The latter space is endowed with its usual topology generated by the family of norms $\{\|\cdot\|_N\}_{N\in\mathbb{N}}$: for $\psi \in \mathcal{S}(\mathbb{R}^d)$, and any two multi-indices $\alpha, \beta \in \mathbb{N}^d$,

$$\|\psi\|_{\alpha,\beta} = \sup_{x\in\mathbb{R}^d} \left| x^\alpha \frac{\partial^{|\beta|}\psi}{\partial x^\beta}(x) \right|, \quad \text{and} \quad \|\psi\|_N = \sup_{|\alpha|,|\beta|\le N} \|\psi\|_{\alpha,\beta} .$$

In order to prove any continuity (or differentiability) property of some function $f : (a,b) \to \mathcal{S}(\mathbb{R}^d)$, it will be sufficient (and it is in fact equivalent) to prove continuity (or differentiability) of

$$f : ((a,b), |\cdot|) \to \Big(\mathcal{S}(\mathbb{R}^d), \|\cdot\|_{\alpha,\beta} \Big) ,$$

as a map between normed vector spaces, for all $\alpha, \beta \in \mathbb{N}^d$. For example, to say that f is differentiable at some point $c \in (a,b)$, it means that there exists some Schwartz function $g \in \mathcal{S}(\mathbb{R}^d)$ such that for all $\alpha, \beta \in \mathbb{N}^d$

$$\left\| \frac{f(c+h) - f(c)}{h} - g \right\|_{\alpha,\beta} \longrightarrow 0, \quad \text{as } h \to 0.$$

As usual, if such a function g exists, it is written as $f'(c)$.

Using general differential calculus results in normed vector spaces (non-necessarily Banach spaces), we can deduce the following two lemmas.

Lemma D.5. *Suppose* $\phi : (a,b) \to (r,s)$ *is differentiable at some point* $c \in (a,b)$, *and* $f : (r,s) \to \mathcal{S}(\mathbb{R}^d)$ *is differentiable at the point* $\phi(c)$. *Then, their composition* $f \circ \phi$ *is differentiable at the point* c, *with*

$$(f \circ \phi)'(c) = f'(\phi(c)) \cdot \phi'(c).$$

Moreover, if both ϕ *and* f *are assumed to be of class* C^k, *then so is their composition.*

Proof. This is a copy of the arguments in Theorem 2.2.1 and in Theorem 5.4.2 of [4], applied to each norm $\|\cdot\|_{\alpha,\beta}$. ☐

Lemma D.6. *Suppose $f, g : (a, b) \to \mathcal{S}(\mathbb{R}^d)$ are differentiable at some point $c \in (a, b)$. Then, their product $f \cdot g$ is differentiable at the point c, with $(f \cdot g)'(c) = f'(c) \cdot g(c) + f(c) \cdot g'(c)$. Moreover, if both f and g are assumed to be of class C^k, then so is their product.*

Proof. Let us remark that $\mathcal{S}(\mathbb{R}^d)$ is an algebra with continuous multiplication. Indeed, for all $\alpha, \beta \in \mathbb{N}^d$,

$$\|f \cdot g\|_{\alpha,\beta} \leq c_{\alpha,\beta} \|f\|_{\alpha,\beta} \cdot \|g\|_{\alpha,\beta} .$$

The rest is a copy of the arguments in Proposition 2.5.2 of [4], and induction on the order of differentiability. ☐

We want to consider the function $t \mapsto \Gamma(t) * \psi$, for any $\psi \in \mathcal{S}(\mathbb{R}^d)$. Under the Fourier transformation, convolution becomes multiplication and we are left to analyse $\mathcal{F}\Gamma(t) \cdot \mathcal{F}\psi$.

Lemma D.7. *A function f is in $C^k([0,\infty), \mathcal{S}(\mathbb{R}^d))$ if and only if its Fourier transform (in the space variables) $\mathcal{F}f$ is in $\in C^k([0,\infty), \mathcal{S}(\mathbb{R}^d))$.*

Proof. This is an easy consequence of the inequality

$$\|\mathcal{F}f\|_N \leq c_N \|f\|_{N+d+1} ,$$

which can be found in Section 1.5 within Chapter 3 of [53]. ☐

The next two results will show that first $t \mapsto \mathcal{F}\Gamma(t)$ is $C^\infty((0,\infty), \mathcal{S}(\mathbb{R}^d))$, and second that $t \mapsto \mathcal{F}\Gamma(t) \cdot \mathcal{F}\psi$ extends to $C^\infty([0,\infty), \mathcal{S}(\mathbb{R}^d))$.

Let $f \in \mathcal{S}(\mathbb{R}^d)$, and define $g : (0,\infty) \times \mathbb{R}^d$ by $g(t,x) = t^{-d} f(x/t)$. To simplify notations, we set $F(y) := \mathcal{F}f(y)$ and $G(t,y) := (\mathcal{F}g(t))(y)$. Thus,

$$\begin{aligned} G(t,y) &= \int_{\mathbb{R}^d} e^{-2\pi i y \cdot x} g(t,x)\, dx = t^{-d} \int_{\mathbb{R}^d} e^{-2\pi i y \cdot x} f(x/t)\, dx \\ &= \int_{\mathbb{R}^d} e^{-2\pi i (ty) \cdot x} f(x)\, dx = F(ty). \end{aligned}$$

Observe that the function G can be continuously extended at $t = 0$ by $G(0,y) = F(0) = \int_{\mathbb{R}^d} f(x)\, dx$. Therefore, the function $G(0,\cdot)$ is a constant function, and hence may not be in $\mathcal{S}(\mathbb{R}^d)$.

Lemma D.8. *The map $t \mapsto G(t,\cdot)$ is $C^\infty((0,\infty), \mathcal{S}(\mathbb{R}^d))$.*

Proof. The fact that F is a Schwartz function implies at once that $G \in C^\infty([0,\infty) \times \mathbb{R}^d)$. We prove differentiability in $\mathcal{S}(\mathbb{R}^d)$ of any order. Fix $s > 0$ and h sufficiently small as well as any integer $k \in \mathbb{N}$ and multi-indices

$\alpha, \beta \in \mathbb{N}^d$. Then, by the mean value theorem, for all $y \in \mathbb{R}^d$ there exists $\theta = \theta(y)$ such that $|\theta| \le |h|$ and

$$y^\alpha \partial_y^\beta \left(\frac{\frac{\partial^k G}{\partial t^k}(s+h,y) - \frac{\partial^k G}{\partial t^k}(s,y)}{h} - \frac{\partial^{k+1} G}{\partial t^{k+1}}(s,y) \right) \\ = y^\alpha \left(\partial_t^{k+2} \partial_y^\beta G \right) (s+\theta(y), y) \cdot h.$$

We can explicit the partial derivative

$$\left(\partial_t^{k+2} \partial_y^\beta G \right)(s,y) = \partial_t^{k+2} \left(t^{|\beta|} (\partial_y^\beta F)(ty) \right)(s) \\ = \sum_{l=0}^{k+2} \binom{k+2}{l} \partial_t^l \left(t^{|\beta|} \right)(s) \partial_t^{k+2-l} \left((\partial_y^\beta F)(ty) \right)(s),$$

with

$$\partial_t^{k+2-l} \left((\partial_y^\beta F)(ty) \right)(s) = \sum_{|\gamma| = k+2-l} y^\gamma \left(\partial_y^{\beta+\gamma} F \right)(sy).$$

We easily bound

$$\left| \partial_t^l \left(t^{|\beta|} \right)(s+\theta(y)) \right| \le c_{l,\beta} (s+|h|)^{\max(|\beta|-l,0)},$$

and use the trick of multiplying and dividing by $(s+\theta(y))^{|\alpha+\gamma|}$ to get

$$\left| y^{\alpha+\gamma} \left(\partial_y^{\beta+\gamma} F \right)((s+\theta(y))y) \right| \\ \le \frac{1}{(s-|h|)^{|\alpha+\gamma|}} \left| ((s+\theta(y))y)^{\alpha+\gamma} \left(\partial_y^{\beta+\gamma} F \right)((s+\theta(y))y) \right| \\ \le \frac{1}{(s-|h|)^{|\alpha+\gamma|}} \left\| F \right\|_{\alpha+\gamma, \beta+\gamma}.$$

Therefore,

$$\left\| \left(\frac{\frac{\partial^k G}{\partial t^k}(s+h,\cdot) - \frac{\partial^k G}{\partial t^k}(s,\cdot)}{h} - \frac{\partial^{k+1} G}{\partial t^{k+1}}(s,\cdot) \right) \right\|_{\alpha,\beta} \\ \le |h| \sum_{l=0}^{k+2} \binom{k+2}{l} c_{l,\beta} (s+|h|)^{\max(|\beta|-l,0)} \sum_{|\gamma|=k+2-l} \frac{\|F\|_{\alpha+\gamma,\beta+\gamma}}{(s-|h|)^{|\alpha+\gamma|}},$$

which converge to 0 as $h \to 0$. □

Remark that the last fraction $(s-|h|)^{-|\alpha+\gamma|}$ is another signal that no derivative exists at $s=0$. In the present case of interest $t \mapsto \mathcal{F}\Gamma(t) \cdot \mathcal{F}\psi$, the multiplication by the Schwartz function $\mathcal{F}\psi$ greatly improves the situation.

Lemma D.9. *For any $\phi \in \mathcal{S}(\mathbb{R}^d)$, the map $t \mapsto \phi \cdot G(t,\cdot)$ is in the space $C^\infty([0,\infty), \mathcal{S}(\mathbb{R}^d))$. Furthermore, its value at $t = 0$ is*

$$\phi \cdot G(0,\cdot) = \phi \cdot \int_{\mathbb{R}^d} f(x)\,dx.$$

Proof. In the present case, observe that $\phi \cdot G(0,\cdot) \in \mathcal{S}(\mathbb{R}^d)$. We use a similar argument as in the proof of the previous lemma. Let us fix $s > 0$ and h sufficiently small, as well as any integer $k \in \mathbb{N}$ and multi-indices $\alpha, \beta \in \mathbb{N}^d$. Then, by the mean value theorem, there exists $\theta = \theta(y,\delta)$ such that $|\theta| \leq |h|$ and

$$\begin{aligned} y^\alpha \partial_y^\beta &\left[\phi(y) \left(\frac{\frac{\partial^k G}{\partial t^k}(s+h,y) - \frac{\partial^k G}{\partial t^k}(s,y)}{h} - \frac{\partial^{k+1} G}{\partial t^{k+1}}(s,y) \right) \right] \\ &= h \cdot y^\alpha \sum_{\delta \leq \beta} \binom{\beta}{\delta} (\partial_y^{\beta-\delta}\phi)(y) \left(\partial_t^{k+2} \partial_y^\delta G \right)(s+\theta, y) \\ &= h \cdot y^\alpha \sum_{\delta \leq \beta} \binom{\beta}{\delta} (\partial_y^{\beta-\delta}\phi)(y) \sum_{l=0}^{k+2} \binom{k+2}{l} \partial_t^l \left(t^{|\delta|} \right)(s+\theta) \\ &\qquad \times \sum_{|\gamma|=k+2-l} y^\gamma \left(\partial_y^{\delta+\gamma} F \right)((s+\theta)y) \end{aligned}$$

As before, we bound

$$\left| \partial_t^l \left(t^{|\delta|} \right)(s+\theta) \right| \leq c_{l,\delta} (s+|h|)^{\max(|\delta|-l,0)}.$$

In the present case, we use the fact that F is a Schwartz function to only bound

$$\left| \left(\partial_y^{\delta+\gamma} F \right)((s+\theta)y) \right| \leq \|F\|_{0,\delta+\gamma}.$$

Thus,

$$\begin{aligned} &\left\| \phi \left(\frac{\frac{\partial^k G}{\partial t^k}(s+h,\cdot) - \frac{\partial^k G}{\partial t^k}(s,\cdot)}{h} - \frac{\partial^{k+1} G}{\partial t^{k+1}}(s,\cdot) \right) \right\|_{\alpha,\beta} \\ &\qquad \leq |h| \sum_{\delta \leq \beta} \sum_{l=0}^{k+2} c_{l,\delta} (s+|h|)^{\max(|\delta|-l,0)} \sum_{|\gamma|=k+2-l} \|\phi\|_{\alpha+\gamma,\beta-\delta} \|F\|_{0,\delta+\gamma}, \end{aligned}$$

which converge to 0 as $h \to 0$. We can also see that this derivation is valid at the point $s = 0$ and $h \searrow 0$. □

We are now ready to prove the last remaining part of Theorem D.1.

Proposition D.10. *For any Schwartz function ψ, the function $t \mapsto \Gamma(t) * \psi$ belongs to $C^\infty([0,\infty), \mathcal{S}(\mathbb{R}^d))$. Furthermore, its k-th order derivative at $t = 0$ is given by $\Delta^k \psi$.*

Proof. We will first prove that the map of interest is $C^\infty((0,\infty),\mathcal{S}(\mathbb{R}^d))$ using the previous lemmas. Then, we prove that it can be continuously extended up to $t=0$, i.e., it is in $C^0([0,\infty),\mathcal{S}(\mathbb{R}^d))$. Finally, the fact that Γ satisfies the heat equation enables to conclude that all derivatives can be continuously extended up to $t=0$.

If we set $f(x)=(4\pi)^{-d/2}\exp\left\{-x^2/4\right\}$ and $g(t,x)=t^{-d}f(x/t)$, we can rewrite

$$\Gamma(t,x)=(4\pi t)^{-d/2}\exp\left\{-x^2/4t\right\}=g(\sqrt{t},x).$$

Using the fact that $t\mapsto\sqrt{t}$ is $C^\infty((0,\infty))$ together with Lemmas D.5, D.7, and D.8, we can conclude that $t\mapsto\Gamma(t)$ is $C^\infty((0,\infty),\mathcal{S}(\mathbb{R}^d))$.

Using the fact that $t\mapsto\sqrt{t}$ is continuous at $t=0$, we can conclude that the map $t\mapsto\mathcal{F}\Gamma(t)\cdot\mathcal{F}\psi=\mathcal{F}(g(\sqrt{t}))\cdot\mathcal{F}\psi$ can be continuously extended at $t=0$, thanks to Lemmas D.5 and D.9. Its value at $t=0$ is given by $\mathcal{F}\psi$ since $\int_{\mathbb{R}^d}f(x)\,dx=1$. Lemma D.7 implies that $t\mapsto\Gamma(t)*\psi$ belongs to $C^0([0,\infty),\mathcal{S}(\mathbb{R}^d))$ and its value at $t=0$ is given by ψ, i.e. $\Gamma(t)*\psi\to\psi$ in $\mathcal{S}(\mathbb{R}^d)$, as $t\searrow 0$.

Using the fact that Γ is a solution to the heat equation, we get

$$\frac{\partial(\Gamma*\psi)}{\partial t}(t)=\frac{\partial\Gamma}{\partial t}(t)*\psi=\Delta\Gamma(t)*\psi=\Gamma(t)*\Delta\psi.$$

Since $\Delta\psi$ is a Schwartz function, we can conclude as before that $\Gamma(t)*\Delta\psi\to\Delta\psi$ in $\mathcal{S}(\mathbb{R}^d)$, as $t\searrow 0$. In a similar way,

$$\frac{\partial^k(\Gamma*\psi)}{\partial t^k}(t)=\Gamma(t)*\Delta^k\psi\longrightarrow\Delta^k\psi\quad\text{in }\mathcal{S}(\mathbb{R}^d),\ \text{as }t\searrow 0,$$

which concludes the proof. □

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Experiences

Swiss Institute of Technology, Lausanne

Ph.D. student (autonomous research in stochastic PDEs) 2016-Now
Main assistant (FAQ organisation, proctoring and grading exams) 2014-Now

Mathemactial outreach

Open doors EPFL and schools discovery day 2018-2019
Entertaining workshops for kids and parents (patience and teamwork)

Education

Master ès Sciences in Mathematics 2013-2015

Swiss Institute of Technology, Lausanne
Thesis research in probability: "Walsh's integration theory and an application to stochastic heat equations"

Bachelor ès Sciences in Mathematics 2009-2013

Swiss Institute of Technology, Lausanne
Waterloo University (Canada) - exchange programm (2011-2012)

Technical skills

Supervise a team, teach, reach objectives, respect deadlines, assess skills, organize, learn autonomously

Languages

French	mother tong
English	advanced, level C1 (European standard)
German	intermediate, level B2 (European standard)

Softwares

C++, Latex, Matlab, Mac-Office, MS-Office, R, Python

Activities

Gymnastic club, Lausanne 2014-Now

Gymnastic instructor (weekly courses, leadership)
Instructor certified "Jeunesse et Sport" (2015 and 2017)

Humanitarian project, South Africa 2013

Promote mathematical curiosity in depressed areas with didactic constructions

Hobbies

Apparatus gymnastics (yearly participation to the Swiss championship, performance) 1997-Now