Thèse n° 9570

On the projectivity of some moduli spaces of varieties

Présentée le 1er juillet 2022

Faculté des sciences de base
Chaire de géométrie algébrique
Programme doctoral en mathématiques

pour l'obtention du grade de Docteur ès Sciences

par

Quentin Arthur Frantisek POSVA

Acceptée sur proposition du jury

Prof. M. Viazovska, présidente du jury
Prof. Z. Patakfalvi, directeur de thèse
Prof. S. Schröer, rapporteur
Dr J. Witaszek, rapporteur
Prof. D. Wyss, rapporteur

École
polytechnique
fédérale
de Lausanne

2022

A mon frère.

Caliban:
Be not afraid: the isle is full of noises,
Sounds, and sweet airs, that give delight, and hurt not.
Sometimes a thousand twangling instruments
Will hum about mine ears; and sometimes voices,
That, if I then had waked after long sleep,
Will make me sleep again: and then, in dreaming,
The clouds methought would open and show riches
Ready to drop upon me; that, when I waked,
I cried to dream again.

Shakespeare
The Tempest, Acte III Scene 2.

Abstract

This thesis is constituted of the article [Pos22] and the preprints [Pos21c, Pos21b, Pos21a]. Their common theme is the moduli theory of algebraic varieties. In the first article I study the Chow–Mumford line bundle for families of uniformly K-stable Fano pairs, and I show it is ample when the family has maximal variation. The three preprints deal with a generalization to positive characteristic of Kollár's gluing theory for stable varieties. I generalize this theory to surfaces and threefolds. Then I apply it to study the abundance conjecture for surfaces, the topology of lc centers on threefolds, existence of semi-resolutions for surfaces, and gluing theory for families of surfaces in mixed characteristic.

Keywords

Algebraic geometry, algebraic variety, birational geometry, moduli theory, Fano variety, K-stability, Chow–Mumford line bundle, Kollár's gluing theory, stable variety, node, surfaces, threefolds, abundance, lc center, semi-resolution, positive characteristic, mixed characteristic.

Résumé

Cette thèse de doctorat est constituée de l'article [Pos22] et des pré-publications [Pos21c, Pos21b, Pos21a]. Le thème commun est la théorie des modules des variétés algébriques. Dans le premier article, j'étudie le fibré en droites de Chow–Mumford pour les familles de variétés de Fano logarithmiques uniformément K-stables, et je montre que ce fibré est ample si la famille est de variation maximale. Les trois pré-publications sont consacrées à une généralisation en caractéristique positive de la théorie de recollement de Kollár pour les variétés stables. Je développe une telle généralisation pour les surfaces et les solides; puis je l'applique à la conjecture d'abondance pour les surfaces, à la topologie des centres log canoniques sur les solides, à l'existence de semi-résolutions pour les surfaces, et à la théorie de recollement pour les familles de surfaces en caractéristique mixte.

Mots clés

Géométrie algébrique, variété algébrique, géométrie birationnelle, théorie des modules, variété de Fano, K-stabilité, fibré de Chow–Mumford, théorie de recollement de Kollár, variété stable, nœud, surfaces, solides, abondance, centre log canonique, semi-résolution, caractéristique positive, caractéristique mixte.

Remerciements

Maintenant tu es au bord de la
Méditerranée
Sous les citronniers qui sont en fleur
toute l'année
Avec tes amis tu te promènes en barque
L'un est Nissard il y a un Mentonasque
et deux Turbiasques
Nous regardons avec effroi les poulpes
des profondeurs
Et parmi les algues nagent les poissons
images du Sauveur

Apollinaire
Alcools, Zone.

J'éprouve de la fierté à parcourir les pages qui constituent cette thèse. Elles sont comme des offrandes à mon astuce et à ma pugnacité, en reconnaissance des rares instants où elles m'ont projeté au-delà des écueils où vient battre habituellement ma pensée. Pour un œil autre, ces pages ne témoignent que d'une étrange grammaire ; et si cet œil est formé à cette grammaire, il portera le regard de l'artisan, auquel n'échappe pas les faiblesses de l'instrument ou celles de la main qui le manie. Mais aucun œil ne verra les traces de mes interrogations, de mes joies incertaines, ni les embranchements des fausses pistes, ni les sources qui ont soudain jailli après des semaines d'errance ; aucun œil sinon le mien. J'y décèle encore les frémissements de ma pensée, le mouvement continu et répétitif d'une imagination au travail ; et je fais mien ces mots d'Ellul : *Ce n'est pas l'addition des fragments entendus, ce n'est pas le cheminement lent et tortueux d'un déroulement processif, ce n'est pas le CQFD trimophant d'une algèbre achevée — c'est l'illumination qui en un éclair fait apparaître le sens de tout le discours*[1].

De ces illuminations, je ne saurais rendre compte. Il faudrait d'abord dépeindre ces *clairières en feu de la Mathématique, [aux] vérités plus ombrageuses à notre approche que les encolures des bêtes fabuleuses*[2]. Puis évoquer les éclosions lentes et paresseuses des intuitions, l'enthousiasme des percées victorieuses ; et surtout les éclairs cruels de lucidité, le sentiment panique qui monte comme une eau noire, lorsque les idées prometteuses se révèlent défectueuses. Et puis témoigner des bourgeons qui surgissent malgré tout, des floraisons plus fortes que mes incertitudes. En somme, la thèse est une moisson dont on ne raconte pas les orages, les sécheresses ou les récoltes miraculeuses. Je finirai moi-même par oublier ces aléas, pour tout ordrer sous le prisme de l'évidence.

Mais la thèse n'est pas un travail solitaire, une tâche d'oiseau nocturne : la mienne a fleuri sous la lumière de rencontres et d'amitiés courtes ou durables. De ceci encore moins, cette thèse ne porte de traces visibles ; mais je peux aisément en témoigner. Je veux nommer et remercier les amies et les amis grâce auxquels mon doctorat fut égayé de rires, de discussions légères et profondes, de partages et de surprises. Que toutes et tous soient assurés de ma reconnaissance

[1] Jacques Ellul, *La parole humiliée*, p.36, La Table Ronde, 2014.
[2] Saint-John Perse, *Amers*, p.122, NRF Gallimard.

et de mon affection.

Je dédicace ce travail à **Ferdinand**, comme gratitude pour ta générosité, ton soutien et notre complicité. C'est une joie et une fierté renouvelées d'être ton frère : merci du fond du cœur. Merci à ma mère **Corinne** de m'avoir donné sans jamais hésiter ton soutien, ton écoute et ton conseil ; merci pour nos discussions passionnantes et pour les intérêts artistiques que tu as suscités en moi. Merci à mon père **Zdenek** pour ton indépendance d'esprit et ton amour de la vie. Merci à mes grands-parents **Eliane** et **Jacques** pour votre enthousiasme constant, votre générosité et vos magnifiques sourires.

Je voudrais aussi remercier **Théo** pour sa compréhension et ses dons de cuisinier ; **André** pour sa belle humeur, sa grande gueule et son amitié ; **Sandrine**, **Sylvain**, **Sarah**, **Hugo** et **Colin** pour leur intérêt et pour quelques repas de famille bien arrosés.

Je n'aurais pu mener ce travail sans la confiance et le soutien de mon directeur de thèse : merci **Zsolt** pour le soin que tu portes à accompagner tes doctorants. Ton contact m'a enrichi, mathématiquement et humainement.

Ma gratitude va également aux membres du groupe. **Maciek**, je te dis mon admiration, comme mathématicien et comme ami. **Christian**, **Javier**, **Marta** et **Stefano**, vos portes étaient toujours ouvertes et de conseils vous n'étiez pas avares : merci pour tout cela. **Fabio**, ton enthousiasme et ton sourire sont comme des soleils, ne les égare pas. Merci **Giulio** pour ton soutien et ton grand cœur. **Roberto**, je n'ai jamais réussi à te faire sourire, mais je n'oublierai pas ton aide précieuse. **Sina** et **Emmi**, je garde en tête nos voyages en conférence et nos discussions à bâtons rompus : merci d'avoir partagé avec moi la majeure partie de ces années. **Jeff**, merci pour ce que tu m'as appris, et j'espère vivement avoir d'autres occasions de travailler avec toi. Et à **Monique** toute ma reconnaissance, pour ton assistance indispensable.

Je salue mes amis mathématiciens et doctorants, que j'ai fréquentés à l'EPFL ou rencontrés en conférence. Tout d'abord, merci à **Arnaud** et à **Hugo** pour vos intelligences et pour la joie de votre compagnie : qu'il est beau de vous avoir pour amis. **Ophélie**, discuter avec toi c'est comme respirer le grand air, merci de ton amitié. Merci à **Emile**, **Gauthier**, **Guillaume**, **Léo**, **Martin**, **Maude**, **Moritz** et **Paride** : pour les cafés, les rires et les bières. C'était un plaisir toujours vif et renouvelé de vous côtoyer. Merci à **Moritz** pour ta bonne humeur et ton fabuleux sourire. Mes amitiés à **Gabriel**, au-delà de la distance et des occasions trop rares. Je remercie aussi **Anna**, **Pascal**, **Sokratis** et **Julia**, que j'ai fréquentés à intervalles réguliers, avec un plaisir toujours plus grand. Un grand merci et un grand coup de chapeau à **David** et à **Samuel**, dont l'amitié et le soutien n'ont jamais faibli. Merci à **Anne-Sophie** pour nos longues discussions agrémentées de sushis. Et merci à toi, **Zhixin**, qui as la tranquillité et l'élégance des pins au-dessus de la mer.

La musique n'a cessé de m'accompagner, et je veux aussi saluer ceux qui l'ont partagée avec moi. Merci à **Tim**, **Noémie**, **Rayan** — et à **Agnès** malgré tout — pour ces heures merveilleuses passées en quatuor, et pour vos personnalités si riches. Merci à **Stéphane** de m'avoir embarqué dans ses aventures. **Laurine**, merci de t'être échappée quelques fois avec un sourire radieux pour un peu de musique. Mes amitiés à **Bernard**, **Daniel** et **Manuel** pour nos joyeuses séances de quintette à intervalles irréguliers, et à **Françoise** pour m'avoir permis de dévaliser sa bibliothèque. Merci à **Vincent** pour l'émulation musicale. Et bien sûr mille mercis à **Gabrielle** et à **Katerina**, que j'ai trop rapidement perdues de vue : pour votre gentillesse, votre patience, votre musicalité, et pour tout ce que vous m'avez appris.

Je ne saurais oublier les musiciens de l'Orchestre de Saint-Maurice ! C'était comme une seconde maison, pleine de rires et de musique : merci de m'y avoir accueilli. **Ernst**, je ne sais comment exprimer ma reconnaissance pour ton enseignement et ta confiance ; merci de m'avoir sans cesse poussé à faire mieux. Mon admiration à **Annick**, **Laura T.** et à **Silke** pour votre énergie et votre créativité : j'étais heureux de vous côtoyer et de vous être de quelqu'aide. Mon amitié à **François D.**, qui a guidé mes premiers pas à l'orchestre et dont l'humour m'a toujours enchanté. Merci à la bande de contrebassistes — **Alexandre**, **François**, **Lionel** et

Mathias — dont la disponibilité et la bonne humeur n'ont jamais été prises en défaut. Merci à mes collègues violoncellistes — **Erine**, **Jean-Luc**, **Léana** et **Orféas** — pour leur patience lors des partielles et pour leur confiance. Merci à **Jeanne** et **Jules** pour leur complicité. Mes hommages à **Charlotte** pour son talent ; et mes amitiés sincères à **Laurène** et **Lydia**, que j'ai le plaisir de fréquenter depuis toutes ces années.

Merci à **Shyam** pour son amitié fidèle : ta distinction m'éblouit à chaque rencontre. Merci à mes colocataires **Antoine**, **Macha**, **Marion**, **Paula** et **John** pour les soirées, les éviers remplis et les gueules de bois. Merci à **Maxime** et **Maximilien** pour votre longue amitié : pardon de ne pas l'avoir honorée plus souvent. Merci aussi à **Emma**, un rayon de soleil que j'ai toujours plaisir à revoir. Merci bien sûr à **Laura V.**, que j'admire et à qui je dois tant. Et finalement un doux baiser à **Alexandra**, pour le bonheur inattendu que tu m'as offert.

Contents

Chapter 1

Introduction

> It seems that I must bid the Muse go pack,
> Choose Plato and Plotinus for a friend
> Until imagination, ear and eye,
> Can be content with argument and deal
> In abstract things; or be derided by
> A sort of battered kettle at the heel.
>
> W. B. Yeats
> *The Tower.*

This PhD thesis consists of four articles [Pos22, Pos21c, Pos21b, Pos21a] that I wrote during my doctoral studies at EPFL, between February 2018 and May 2022. The first article [Pos22] has been accepted for publication by the *Transactions of the AMS*, and the three other have been submitted to peer-reviewed journals. Compared to the versions available on the arXiv, I have made few changes: the common notations and terminology have been gathered in Chapter 2 and the technical background of [Pos21c, Pos21b, Pos21a] is presented in Chapter 4. The first article [Pos22] has little in common with the three other ones — with the exception of their author. My goal for the present introduction is therefore to sketch the mathematical theory from which they all sprouted, and then give an overview of the new results contained in this thesis. The reader will find more specific informations about each article in their original introduction at the beginning of the corresponding chapters.

1.1 Moduli theory

The motivation of these four articles can be traced back to MODULI THEORY. This theory is a central theme of algebraic geometry: given a set of algebraic objects $\mathfrak{F}$ sharing some common properties, is there a geometric space $\mathcal{M}_{\mathfrak{F}}$ whose points are in bijection with the elements of $\mathfrak{F}$, and whose geometry reflects the way these elements may vary?

A classical instance of this question is the moduli theory of Riemann surfaces. Riemann proved, already in the middle of the XIX$^{\text{th}}$ century, that the parameter space $\mathcal{M}_g$ of compact complex surfaces of genus $g \geq 2$ depends on $3g-3$ parameters. The Italian school of algebraic geometry studied degenerating families of curves and rational properties of $\mathcal{M}_g$ in the first half of the XX$^{\text{th}}$ century. Modern methods to study $\mathcal{M}_g$ were introduced by Mumford [Mum65] and latter by Deligne–Mumford [DM69], and the Kodaira dimension and topological invariants of $\mathcal{M}_g$ remain an active topic of research nowadays.

Depending on the nature of the objects in $\mathfrak{F}$, there might exist different and non-equivalent approaches to construct and study $\mathcal{M}_{\mathfrak{F}}$. In the case of Riemann surfaces, one can use either Teichmüller theory or Hodge theory to construct $\mathcal{M}_g$ (see [HM98, 2.C]). These two methods rely in an essential way on the analytic structure that a Riemann surface carries. But (compact) Riemann surfaces can be seen as complex algebraic curves, and one might rightfully insist to develop a moduli theory that relies only on algebraic tools. As a by-product, such a theory would apply to a broader class of curves, for example curves that are defined over finite fields, for which the help of complex analysis is not available. A purely algebraic approach to moduli theory was pioneered by Mumford with his Geometric Invariant Theory (GIT), first exposed in [Mum65] (see [MFK94] for the most recent edition). This approach has been extremely successful in several cases, including moduli of curves, of surfaces of general type [Gie77a], of canonically polarized manifolds [Vie95] and of vector bundles [NS64, Gie77b]. Although GIT is not discussed in this thesis, it has a fundamental feature that I want to highlight.

1.2 GIT approach to moduli theory

A typical situation for a GIT moduli is the following. Assume that $\mathfrak{F}$ is a set of isomorphism class of pairs (X, L), where X is a variety of fixed dimension and L a line bundle on X of fixed Hilbert polynomial $h(t)$, with the property that the linear system $|L|$ embeds X into $\mathbb{P}^N$ for $N = N(h(t))$ independent of X. According to a theorem of Grothendieck, there is a proper space $\mathbb{H} = \mathbb{H}(N, h(t))$ that parametrizes in the strongest possible sense the embedded subschemes $\mathcal{X} \subset \mathbb{P}^N$ for which the Hilbert polynomial of $\mathcal{O}(1)|_{\mathcal{X}}$ is $h(t)$. Amongst those $\mathcal{X}$ are the varieties $X \in \mathfrak{F}$ embedded using a basis of the linear system $|L|$. Usually the locus $\mathbb{H}_{\mathfrak{F}}$ of $\mathbb{H}$ corresponding to these embeddings is locally closed. Moreover the multiple choices of embeddings $X \subset \mathbb{P}^N$ corresponding to the many basis of $|L|$, are recovered as the orbit of one embedding under the action of $\mathrm{Aut}(\mathbb{P}^N, \mathcal{O}(1)) = \mathrm{PGL}(N+1)$ on $\mathbb{H}$. Therefore the quotient space $\mathbb{H}_{\mathfrak{F}}/\mathrm{PGL}(n+1)$, should it exist, would give a good notion of parameter space for $\mathfrak{F}$. However taking quotients by infinite groups is a delicate matter: GIT was designed to understand to which extent quotients exist in these situations. After a polarization is chosen on $\mathbb{H}$, the general theory identifies a subset $\mathbb{H}^{\mathrm{ss}} \subset \mathbb{H}$, called the locus of semi-stable points, for which a categorical quotient morphism

$$\pi^{\mathrm{ss}} \colon \mathbb{H}^{\mathrm{ss}} \longrightarrow \mathbb{H} /\!\!/ \mathrm{PGL}(N+1)$$

exists. It also identifies an open subset $\mathbb{H}^{\mathrm{s}} \subset \mathbb{H}^{\mathrm{ss}}$, called the locus of stable points, for which the restriction $\pi^{\mathrm{s}} = \pi^{\mathrm{ss}}|_{\mathbb{H}^{\mathrm{s}}}$ has all the properties one can reasonably expect from a quotient (for example, its fibers are exactly the orbits). In favourable cases it holds that $\mathbb{H}_{\mathfrak{F}} \subseteq \mathbb{H}^{\mathrm{s}}$, and by restricting π^{s} accordingly we obtain a parameter space for $\mathfrak{F}$. In fact, we usually enlarge the collection $\mathfrak{F}$ so that we have an equality $\mathbb{H}_{\mathfrak{F}} = \mathbb{H}^{\mathrm{s}}$.

Now let me explain the special feature given by GIT. It follows from the general theory that the quotient $\overline{\mathcal{M}}_{\mathfrak{F}} = \mathbb{H} /\!\!/ \mathrm{PGL}(N+1)$ is projective: it is therefore a compactification of $\mathcal{M}_{\mathfrak{F}} = \mathbb{H}^{\mathrm{s}}/\mathrm{PGL}(N+1)$, which sits inside $\overline{\mathcal{M}}_{\mathcal{F}}$ as an open subset. This compactification has a modular interpretation: while points of $\mathcal{M}_{\mathfrak{F}}$ correspond bijectively to elements of $\mathfrak{F}$, points in the boundary $\partial\overline{\mathcal{M}}_{\mathfrak{F}} = \overline{\mathcal{M}}_{\mathfrak{F}} \setminus \mathcal{M}_{\mathfrak{F}}$ can be interpreted as one-parameter semi-stable degenerations of elements of $\mathfrak{F}$.

The existence of a modular compactification $\mathcal{M}_{\mathfrak{F}} \subset \overline{\mathcal{M}}_{\mathfrak{F}}$ is the best situation we can hope for in moduli theory. Tools from cohomology, intersection theory and birational geometry can be used to understand the geometry of $(\overline{\mathcal{M}}_{\mathfrak{F}}, \partial\overline{\mathcal{M}}_{\mathfrak{F}})$: see for example [Has03, HH09, HH13]. Furthermore, the mere existence of a modular compactification implies that any one-parameter family of objects in $\mathfrak{F}$ can be filled-in uniquely (up to finite base-change) with semi-stable objects.

Unfortunately, while the GIT approach is successful for constructing the moduli space of curves (see [MFK94]), in many other situations it has serious shortcomings:

- If we use it to construct a moduli space of canonically polarized smooth varieties, then the choice of different pluricanonical polarizations $(X, L = \omega_X^{\otimes r})$ lead to an infinite collection of modular compactifications [WX14].

- ◦ The geometric meaning of GIT (semi-)stability is not easy to understand: consequently the points on the boundary usually correspond to objects that are singular and difficult to tell apart from unstable objects using only geometric criterions.
- ◦ In addition, If our collection $\mathfrak{F}$ contains varieties that are singular, it is not even clear that we can use the GIT approach! For surfaces already, it turns out that (asymptotic) GIT stability imposes strong restrictions on the multiplicity of the local rings [Sha81]: hence many interesting yet mildly singular surfaces violate these restrictions.

Therefore different methods are needed to construct moduli spaces of varieties in higher dimensions. They should be powerful enough to provide modular compactifications, and provide a good description of the objects at the boundary. Depending on the varieties we want to parametrize, we can devise various strategies: below I present the KSBA approach to the moduli of varieties of general type, and the K-stability approach to moduli of Fano varieties. These are the two moduli theories that I study in the articles forming this thesis.

1.3 Functorial approach to moduli theory

Both approaches have a common theoretical framework, which goes back to the article of Deligne and Mumford about the moduli space of curves [DM69]. Given a collection $\mathfrak{F}$ for which there is a meaningful notion of *families of objects in* $\mathfrak{F}$ *over a base scheme*, one considers the pseudo-functor

$$\mathbb{M}_{\mathfrak{F}} \colon (\text{Schemes})^{\text{op}} \longrightarrow (\text{Groupoids})$$

sending a scheme S to the groupoid category of families of objects in $\mathfrak{F}$ over S. We call $\mathbb{M}_{\mathfrak{F}}$ the MODULI FUNCTOR of $\mathfrak{F}$.

At first glance it seems difficult to think about $\mathbb{M}_{\mathfrak{F}}$ as an actual space. However by the Yoneda lemma, any scheme X induces a pseudo-functor $h_X = \mathrm{Hom}(\bullet, X)$, and many properties of X – such as separatedness, properness, smoothness, irreducibility – can be read from the pseudo-functor h_X. By analogy, we think of $\mathbb{M}_{\mathfrak{F}}$ as a space in a generalized sense, and investigate whether such scheme-like properties hold for $\mathbb{M}_{\mathfrak{F}}$. In practice, this means proving properties about families of objects in $\mathfrak{F}$: for example, properness of $\mathbb{M}_{\mathfrak{F}}$ amounts to show that, up to finite base-change, a one-parameter family of objects in $\mathfrak{F}$ has one and only one limit in $\mathfrak{F}$ (about valuative criterions for stacks, see [LMB00, §7]). Therefore the choice of elements we allow in $\mathfrak{F}$ is critical for the properness of $\mathbb{M}_{\mathfrak{F}}$: if we want $\mathfrak{F}$ to contain a certain class of smooth varieties, then we have to identify the correct class of singular degenerations of these varieties, so that $\mathbb{M}_{\mathfrak{F}}$ will be a proper pseudo-functor.

1.4 Moduli of KSBA stable varieties

Following this functorial approach, let me first focus on the case of canonically polarized varieties: these are the smooth varieties X for which the canonical line bundle $\omega_X = (\det TX)^*$ is ample. What is a good notion of degenerations for these varieties? The curve case was settled in the article of Deligne and Mumford, who identified the so-called stable curves as the correct class of degenerations, and showed that the associated moduli functor is proper. This definition was generalized in [KSB88], where stable surfaces were introduced. Shortly after it was shown that the moduli functor of stable complex surfaces is proper [Kol90, Ale94]. Similarly, KSBA STABLE VARIETIES can be defined in any dimension. It is now a theorem that in characteristic zero their moduli functor $\mathbb{M}_{\text{KSBA}}$ is a proper Deligne–Mumford stack [Kol21][1].

Now that we know that (KSBA) stability is a successful notion, you might rightfully ask for its definition. It contains three parts. The first one is a purely algebraic restriction on the type

[1]Using the method that I will describe in Section 1.5.1, one can show in addition that the coarse moduli space of $\mathbb{M}_{\text{KSBA}}$ (in the sense of Keel–Mori) is a projective variety [Fuj18, KP17, PX17]. Since I will not discuss the *projectivity* of the moduli space of stable varieties in this thesis, I will not elaborate on this topic.

of singularities (they should be at worst demi-normal, see Definition 4.2.0.2): it is important to notice that a stable variety is not necessarily normal. The second part is that the canonical sheaf should be ample. The third part is a geometric restriction on the singularities, in the spirit of the Minimal Model Program (MMP): a stable variety is at worst semi-log canonical (see Definition 4.2.0.4).

Let us pause for a second: why should the MMP be relevant in moduli theory? In the case of stable varieties, it was quickly realized that the log canonical condition on the singularities implies that the moduli functor is separated (see [Kol21, 1.27, 2.47-48]; but a truly illuminating proof is given in [Ben20, §3.1]). Thus it makes sense to restrict our attention to log canonical varieties, for which the tools of MMP and birational geometry are available. In fact, it turns out that the MMP is an extremely powerful tool: the recent breakthroughs in the moduli theory of stable varieties and of Fano varieties (see Section 1.5 below) would not have been possible without these techniques.

While it is now clear that log canonical varieties are interesting to consider, the attentive reader may have noticed that I defined stable varieties to be semi-log canonical. There is no mistake: we are forced to work with non-normal varieties to obtain a proper moduli space, and accordingly semi-log canonical is the non-normal version of log canonical. This creates serious technical challenges, because techniques of the MMP are usually not available on non-normal varieties (see [Kol11]). A solution is given by KOLLÁR'S GLUING THEORY [Kol13, §5], which establishes a dictionary between semi-log canonical singularities and their normalizations. This dictionary is an essential tool to prove the properness of $\mathbb{M}_{\mathrm{KSBA}}$.

We have so far considered stable varieties in characteristic zero, but the definition is also valid in positive characteristic. So what about the moduli functor of stable varieties in characteristic $p > 0$? For curves the theory of [DM69] is general enough to encompass stable curves in positive characteristic[2]. But already for surfaces, much less is known: the current state of our knowledge is summarized in [Pat17]. Many techniques from characteristic zero break down, and potential replacements were only developed recently. The article [Pos21c] is my attempt to adapt Kollár's gluing theory in positive characteristic, and [Pos21b, Pos21a] consist of applications to the geometry of surfaces in positive and mixed characteristics. My results will be presented in Section 1.6 below.

1.5 MODULI OF K-STABLE FANO VARIETIES

Let us now consider the moduli of Fano varieties over the complex numbers. Fano varieties are smooth complex projective varieties X with the property that their canonical line bundle $(\det TX)^*$ is anti-ample. If $\mathfrak{F}$ contains the class of Fano varieties, which degenerations should we allow so that $\mathbb{M}_{\mathfrak{F}}$ is proper? General deformation theory together with the Kodaira–Nakano vanishing theorem imply that the infinitesimal deformations of Fano varieties are very well-behaved. However their one-parameter deformations are much more complicated to understand, and for a long time the moduli theory of Fano varieties seemed out of reach.

Inspiration came from analytic complex geometry. Tian and Donaldson [Tia97, Don02] introduced the notion of K-(SEMI)STABILITY in their work about Kähler–Einstein metrics on Fano manifolds. It turned out that K-stability provides a good stability notion to construct proper moduli spaces of Fano manifolds, as demonstrated by [OSS16] and [LWX18]. In parallel, the definition of K-stability had been translated into purely algebraic terms [Fuj19, FO18], sparking hope for the construction of algebraic moduli spaces of Fano varieties[3].

[2] Actually the moduli theory of stable curves *à la* Deligne–Mumford works over $\mathbb{Z}$, in other words in the broadest possible generality.

[3] The algebraic definition of K-semistability is given in Section 3.2.2. Since we discussed the GIT approach to moduli theory in this introduction, we should remark that the original definition of K-semistability is an infinite dimensional generalisation of the Hilbert–Mumford criterion, which characterizes GIT semistability. In general K-semistability cannot be recovered as a limit of GIT semistability: counterexamples exist Fano varieties

This program was successfully completed very recently, in a series of articles culminating in [LXZ21] and [Xu20] (see Section 3.1 for a detailed bibliography, and [Xu21] for a complete survey). The moduli functor $\mathbb{M}_{\mathrm{Kss}}$ of K-semistable Fano varieties is not proper, but it can be approximated by a so-called GOOD MODULI SPACE M_{Kps} which is a projective algebraic space[4].

When I starting working on [Pos22], the properness and projectivity of M_{Kps} were still conjectures. I studied the question of projectivity: my results will be stated in Section 1.6. To motivate it, I will now explain what it concretely means to show that M_{Kps} is projective.

1.5.1 Projectivity of moduli spaces

Let me take a step back, and consider anew a moduli functor $\mathbb{M}_{\mathfrak{F}}$, represented by an algebraic space $M_{\mathfrak{F}}$ (either a coarse moduli space in the sense of Keel–Mori, either a good moduli space in the sense of Alper). Assume that $M_{\mathfrak{F}}$ is proper. To establish projectivity, we need to exhibit an ample line bundle. The general theory tells us that every family $f\colon X \to T$ of objects in $\mathfrak{F}$ is equipped with a moduli morphism $\mathbb{M}_{\mathfrak{F}}(f)\colon T \to M_{\mathfrak{F}}$, sending a point t to the moduli point of $f^{-1}(t)$. Hence a line $\mathcal{L}$ bundle on $M_{\mathfrak{F}}$ gives, for every such family $f\colon X \to T$, a functorial line bundle $\mathcal{L}(f)$ on T. But the converse is not true: a functorial family $\{\mathcal{L}(f) \mid f\colon X \to T\}$ descends to a line bundle on $M_{\mathfrak{F}}$ if and only if, for every object X of $\mathfrak{F}$, the automorphism group of X acts trivially on $\mathcal{L}(X)$ [Alp13, 10.3]. Nonetheless, assume that we are able to construct such a family $\mathcal{L} = \{\mathcal{L}(f)\}$. How do we prove it is ample?

We present a strategy that was developed by Kollár in [Kol90]. It relies on the Nakai–Moishezon criterion: $\mathcal{L}$ is ample if and only if, for every irreducible and proper sub-algebraic space $V \subset M_{\mathfrak{F}}$, the self-intersection $\mathcal{L}^{\dim V}$ is strictly positive. If we manage to find a surjective and (generically) finite cover $V' \to V$ that supports a family $f\colon X \to V'$ of objects in $\mathfrak{F}$ for which $\mathcal{L}(f)^{\dim V'} > 0$, then we are done. The difficulty is of course to find such families $f\colon X \to V'$ for which $\mathcal{L}(f)^{\dim V'} > 0$.

Kollár's key observation is the following. Assume that the family $f\colon X \to V'$ has *maximal variation*, in the sense that (generically) distinct points $v, v' \in V'$ parametrize non-isomorphic objects $X_v \ncong X_{v'}$. Furthermore, let L be a relative ample line bundle on X, and assume that the pushforward vector bundles f_*L^j on V' are semipositive for j sufficiently large. Then $\det f_*L^l$ is ample for l large enough: this is the content of Kollár's *ampleness lemma* [Kol90, §3]. If $\det f_*L^l = \mathcal{L}(f)$ then we have obtained that $\mathcal{L}$ is ample on $M_{\mathfrak{F}}$.

Kollár's strategy was successfully applied in several situations: Kollár considered the cases of the compactified Picard functor and of stable surfaces in [Kol90], and the method was generalized to stable varieties of any dimension in [Fuj18, KP17].

Now we specialize to the case where $\mathbb{M}_{\mathfrak{F}} = \mathbb{M}_{\mathrm{Kss}}$ is the moduli functor of K-semistable

[OSY12]. But there are several cases where the K-stable moduli space is the same as the GIT moduli space, or is closely related to it: see for example [LX19, Liu22, ADL21a, ADL21b].

[4] Let me give a short explanation. One defines several flavours of stability for Fano varieties: K-semistability, K-polystability and K-stability. K-semistability is the weaker notion, and we consider its moduli functor $\mathbb{M}_{\mathrm{Kss}}$. Unfortunately $\mathbb{M}_{\mathrm{Kss}}$ is not even separated: a given K-semistable variety can degenerate to several non-isomorphic K-semistable varieties. This corresponds to the existence of non-closed rational points on $\mathbb{M}_{\mathrm{Kss}}$. One would need to somehow collapse the degeneration equivalence classes to achieve separatedness.

There is a strong parallel with GIT. Indeed, using the notations of Section 1.2, the GIT quotient $\mathbb{H}^{ss} \to \mathbb{H} /\!\!/ \mathrm{PGL}(V)$ identifies the non-closed orbits whose closures intersect (the non-closed orbits are precisely those of strictly GIT semistable points). However no GIT interpretation of the moduli functor of K-semistable Fano varieties is known.

A solution was recently given by the powerful framework of *good moduli spaces* developed by Alper [Alp13]. We say that an Artin stack $\mathfrak{X}$ admits a good moduli space if there exists a morphism to an algebraic space $\mathfrak{X} \to \mathcal{X}$ with properties that emulate those of GIT quotients. In particular, the existence of a proper good moduli space for $\mathbb{M}_{\mathrm{Kss}}$ would be a satisfactory replacement for the properness of the moduli functor.

The existence of a separated good moduli space (in characteristic zero) is guaranteed by two technical conditions on the stack, called *S-completeness* and *Θ-reductivity* [AHLH19]. It is proved in [ABHLX20] that these properties hold for $\mathbb{M}_{\mathrm{Kss}}$. By construction the points of its good moduli space M_{Kps} parametrize K-polystable Fano varieties. The properness and projectivity of M_{Kss} were subsequently proved in [LXZ21, Xu20].

Fano varieties. A candidate for the polarization is given by the so-called CHOW–MUMFORD (CM) LINE BUNDLE. It was first considered in [Tia97] in connection to the stability of Fano manifolds. In the smooth case, curvature calculations with the Weil–Petersson metric show that the CM line bundle is ample [Sch12]: therefore the CM line bundle is expected to provide a polarization on the good moduli space M_{Kps}. This was recently confirmed in [XZ20, LXZ21]. But at the beginning of my PhD it was still a conjecture: the only available purely algebraic results were those of [CP21], where ampleness of the CM line bundle was established for the special class of uniformly K-stable Fano varieties[5]. My goal in [Pos22] was to generalize the main result of [CP21] to a more general set-up, as I will explain in Section 1.6.

Let me say a word about the method of [CP21]. It follows Kollár's strategy, but some twists are necessary. The reason is the following: given a family of Fano varieties $f\colon X \to T$, the natural relative polarization is given by relative anti-canonical sheaf $\omega_{X/T}^{-1}$. But $f_*\omega_{X/T}^{-j}$ is usually not semi-positive! Furthermore, the CM line bundle is not equal to $\det f_*\omega_{X/T}^{-j}$: rather, the CM line bundle corresponds (up to a sign) to the leading coefficient of the Knudsen–Mumford expansion of this determinant (see Section 3.4). Fortunately some controlled twist of $f_*\omega_{X/T}^{-j}$ is semipositive, the ampleness lemma still applies and gives the positivity of some twist of $\det f_*\omega_{X/T}^{-1}$, and the product trick of Viehweg can be used to deduce positivity of the CM line bundle: see [CP21, §1.7.3] for more precisions. The other parts of the strategy (descent of the CM line bundle to M_{Kps}, existence of families with maximal variation) also hold for K-semistable Fano varieties [CP21, §10].

1.6 MAIN RESULTS

It is high time to state the results of my thesis. I have chosen to give the precise statements, even though some of them are technical. Most of them are phrased in the language of PAIRS, that is commonly used in birational geometry. This notion appears naturally when we perform adjunction, when we use inductive arguments on the dimension, or when we consider families over a curve with the data of a special fiber. See Section 2.4 for our conventions.

1.6.1 Ampleness of the CM line bundle

The following results can be found in Chapter 3, which corresponds to [Pos22]. As hinted above, it concerns the Chow–Mumford line bundle of a family of Fano varieties. It can be defined as follows: let $f\colon (X, D) \to T$ be a morphism for which each fiber (X_t, D_t) is a log Fano variety of dimension n. Then the CM line bundle is defined by

$$\lambda_{f,D} = -f_*((-K_{X/T} - D)^{n+1}).$$

By definition it is an element of the Chow group of codimension one cycles on T, and one can show that it is actually a $\mathbb{Q}$-Cartier divisor (see Section 3.2.4). The main result of [Pos22] is the following:

Theorem 1 (Theorem 3.1.0.2)**.** *Let $f\colon (X, D) \to T$ be a flat morphism of relative dimension n with connected fibers from a normal projective pair to a normal projective variety, such that $-(K_{X/T} + D)$ is $\mathbb{Q}$-Cartier and f-ample. Assume that D does not contain any fibers.*

(a) BIGNESS*: If each fiber (X_t, D_t) is klt, the general geometric fibers $(X_{\bar{t}}, D_{\bar{t}})$ are uniformly K-stable, and the variation of f is maximal, then $\lambda_{f,D}$ is big.*

(b) AMPLENESS*: If all the geometric fibers $(X_{\bar{t}}, D_{\bar{t}})$ are uniformly K-stable, and the variation of f is maximal, then $\lambda_{f,D}$ is ample.*

[5]The definition is given in Section 3.2.2. It is *a priori* stronger than the other forms of stability, but it was conjectured that uniform K-stability is actually equivalent to K-stability. This has been confirmed (for Fano varieties) in [LXZ21, Theorem 1.6].

This generalizes [CP21, 1.9] to the logarithmic case, that is when $\Delta \neq 0$. The implication for moduli theory is the following: the CM line bundle gives a polarization on the normalization of any proper subspace of M_{Kps} parametrizing uniformly K-stable log Fano varieties. This result has been superseded since by [LXZ21, 1.3].

1.6.2 Geometry of surfaces in positive characteristic

Next I will state the main results about the geometry of surfaces and threefolds obtained in [Pos21c, Pos21a, Pos21a], which you can read in Chapter 5, Chapter 6 and Chapter 7. I begin with surfaces.

The first result generalizes Kollár's gluing theory to surfaces in positive characteristic:

Theorem 2 (Theorem 5.1.1.1)**.** *Let k be a field of positive characteristic.*

(a) If $\operatorname{char} k > 2$, then normalization gives a bijection

$$\begin{pmatrix} \textit{Slc surface pairs } (S, \Delta) \\ \textit{of finite type over } k \end{pmatrix} \xrightarrow{1:1} \begin{pmatrix} \textit{Lc surface pairs } (\bar{S}, \bar{D} + \bar{\Delta}) \\ \textit{of finite type over } k \\ \textit{plus an involution } \tau \textit{ of } (\bar{D}^n, \operatorname{Diff}_{\bar{D}^n} \bar{\Delta}) \\ \textit{that is generically fixed point free on every component.} \end{pmatrix}$$

(b) If $\operatorname{char} k = 2$, then normalization gives a bijection

$$\begin{pmatrix} \textit{Slc surface pairs } (S, \Delta) \\ \textit{of finite type over } k \end{pmatrix} \xrightarrow{1:1} \begin{pmatrix} \textit{Lc surface pairs } (\bar{S}, \bar{D}_{\mathrm{Gal}} + \bar{D}_{\mathrm{ins}} + \bar{\Delta}) \\ \textit{of finite type over } k \\ \textit{where } \bar{D}_{\mathrm{Gal}}, \bar{D}_{\mathrm{ins}} \textit{ and } \bar{\Delta} \textit{ have no common component,} \\ \textit{plus an involution } \tau \textit{ of } (\bar{D}^n_{\mathrm{Gal}}, \operatorname{Diff}_{\bar{D}^n_{\mathrm{Gal}}}(\bar{\Delta} + \bar{D}_{\mathrm{ins}})) \\ \textit{that is generically fixed point free on every component.} \end{pmatrix}$$

While the proof of Theorem 5.1.1.1 is a fairly easy implementation of Kollár's strategy (see [Kol13, §5.6-7]), I have tried to give a proof that is valid in the largest possible generality. In particular, there are no assumption on the base field k, and we work with surfaces that are essentially of finite type over k. The additional cases appearing in characteristic 2 are studied extensively (see Section 4.5).

Theorem 2 gives, in theory, a complete classification of slc surface germs. Indeed, germs of lc surface pairs with a non-empty reduced boundary are classified (see [Kol13, §3.3]), and our theorem says that any additional data of a log involution on the boundary, determines an slc singularity. So, at least when the characteristic is different from 2, we get a local picture that is similar to [Kol21, 2.21]. For this reason I have not tried to write down an exhaustive list.

An interesting consequence of Theorem 2 is the existence of semi-resolutions (see Definition 5.3.4.4) of demi-normal surfaces in characteristic $\neq 2$. This is certainly a folklore result, but I was unable to find in the literature a precise proof in positive characteristic[6]. Using gluing theory for surfaces, it is not difficult to show:

Theorem 3 (Theorem 5.3.4.8 and Proposition 5.3.4.12)**.** *Let S be a demi-normal surface that is essentially of finite type over an arbitrary field k of positive characteristic.*

(a) If $\operatorname{char} k \neq 2$, then S has an slc good semi-resolution.

(b) More generally, if S has only separable nodes, then there exists a proper birational morphism $f : T \to S$ such that

(i) T is slc 2-Gorenstein with regular conductor D_T;

(ii) f is an isomorphism over a big open subset of f;

[6] For example, Kollár states the existence of two-dimensional semi-resolutions in [Kol90, 4.2], and refers to [vS87, 1.4.3]. However the latter article deals with complex algebraic spaces. But to be fair, our gluing method to construct semi-resolutions is very similar to the method used in [vS87].

(iii) no component of D_T is f-exceptional;
(iv) each component of $\mathrm{Exc}(f)$ is regular and intersects D_T transversally, and $\mathrm{Exc}(f)$ has only normal crossings.

As demonstrated by Hacon and Xu in [HX16], Kollár's theory can be used to reduce the abundance conjecture on slc varieties, to the abundance conjecture on lc varieties. Building on their strategy and on the abundance on lc surfaces proved in [Tan20a], we obtain abundance on slc surfaces over arbitrary fields:

Theorem 4 (Theorem 7.1.0.1)**.** *Let (S, Δ) be an slc surface pair and $f\colon S \to B$ a projective morphism where B is quasi-projective over a field of positive characteristic. Assume that $K_S+\Delta$ is f-nef; then it is f-semi-ample.*

1.6.3 Geometry of threefolds in positive and mixed characteristics

I was able to extend Kollár's gluing theory to threefolds in positive characteristic, granted the characteristic is not too small:

Theorem 5 (Theorem 5.4.3.6)**.** *Let k be a perfect field of characteristic > 5. Then normalization gives a bijection*

$$\left(\begin{array}{c}\textit{Proper slc threefold pairs}\\ (X,\Delta)\ \textit{such that}\\ K_X+\Delta\ \textit{is ample}\end{array}\right) \xrightarrow{1:1} \left(\begin{array}{c}\textit{Proper lc threefold pairs } (\bar{X}, \bar{D}+\bar{\Delta})\\ \textit{plus an involution } \tau \textit{ of } (\bar{D}^n, \mathrm{Diff}_{\bar{D}^n}\bar{\Delta})\\ \textit{that is generically fixed point free on each component}\\ \textit{such that } K_{\bar{X}}+\bar{D}+\bar{\Delta}\ \textit{is ample.}\end{array}\right)$$

In comparison to Theorem 5.1.1.1, there are additional hypothesis about the base field and the projectivity of the threefolds. We cannot expect a local gluing theory in dimensions three and bigger, so the projectivity hypothesis is forced on us. The restrictions on the base field come from the MMP theorems that are currently available.

The implication of Theorem 5 for the moduli space of stable surfaces in mixed characteristic is the following:

Theorem 6 (Theorem 5.1.2.5)**.** *Let k be an algebraically closed field of characteristic > 5 and v a rational number. Let $\mathcal{N} \subset \overline{\mathcal{M}}_{2,v,k}$ be a closed sub-Artin-stack parametrizing families of stable surfaces for which slc adjunction and semi-stable reduction hold. Then $\mathcal{N}$ is proper (and furthermore projective by [Pat17]).*

The proof of Theorem 5 follows once again Kollár's strategy, but it is much more involved than the two-dimensional case. The conceptual hearth of the proof is the theory of sources and springs, developed in [Kol13, §4.4-5] for varieties in characteristic zero. We recover this theory in positive characteristic for threefolds:

Theorem 7 (Theorem 5.1.1.2)**.** *Let $f\colon (Y, \Delta_Y) \to (X, \Delta = \Delta^{=1} + \Delta^{<1})$ be crepant $\mathbb{Q}$-factorial dlt blow-up of a quasi-projective lc threefold pair over a perfect field of characteristic > 5. Let $Z \subset X$ be a lc center contained in $\Delta^{=1}$ with normalization $Z^n \to Z$.*

Let $(S, \Delta_S := \mathrm{Diff}^*_S \Delta_Y) \subset Y$ be a minimal lc center over Z, with Stein factorization $f^n_S\colon S \to Z_S \to Z^n$. Then:

(a) The crepant birational equivalence class of (S, Δ_S) over Z does not depend on the choice of S or Y. We call it the source of Z, and denote it by $\mathrm{Src}(Z, X, \Delta)$.
(b) The isomorphism class of Z_S over Z does not depend on the choice of S or Y. We call it the spring of Z, and denote it by $\mathrm{Spr}(Z, X, \Delta)$.
(c) (S, Δ_S) is dlt, $K_S + \Delta_S \sim_{\mathbb{Q},Z} 0$ and (S, Δ_S) is klt on the generic fiber above Z.
(d) The field extension $k(Z) \subset k(Z_S)$ is Galois and $\mathrm{Bir}^c_Z(S, \Delta_S) \twoheadrightarrow \mathrm{Gal}(Z_S/Z)$.

(e) *For $m > 0$ divisible enough, there are well-defined Poincaré isomorphisms*

$$\omega_Y^{[m]}(m\Delta_Y)|_S \cong \omega_S^{[m]}(m\Delta_S).$$

(f) *If $W \subset (\Delta^{=1})^n$ is an irreducible closed subvariety such that $n(W) = Z$, where $n\colon (\Delta^{=1})^n \to \Delta^{=1}$ is the normalization, then*

$$\mathrm{Src}(W, (\Delta^{=1})^n, \mathrm{Diff}_{(\Delta^{=1})^n} \Delta^{<1}) \overset{\mathrm{cbir}}{\sim} \mathrm{Src}(Z, X, \Delta)$$

and

$$\mathrm{Spr}(W, (\Delta^{=1})^n, \mathrm{Diff}_{(\Delta^{=1})^n} \Delta^{<1}) \cong \mathrm{Spr}(Z, X, \Delta).$$

The theory of sources and springs has some consequences for the topology of lc centers:

Theorem 8 (Theorem 5.1.2.4). *Let (X, Δ) be a quasi-projective slc threefold over a perfect field of characteristic > 5. Then:*

(a) *Intersections of lc centers are union of lc centers.*

(b) *Minimal lc centers are normal up to universal homeomorphism.*

An interesting consequence of Theorem 4 for threefolds over arbitrary fields is the following:

Theorem 9 (Theorem 7.1.1.2). *Let (X, Δ) be a projective $\mathbb{Q}$-factorial dlt threefold over an arbitrary field k of characteristic $p > 5$. Assume that $K_X + \Delta$ is nef. Then $(K_X + \Delta)|_{\Delta^{=1}}$ is semi-ample.*

Motivated by the recent progresses in the geometry of threefolds in mixed characteristic [Wit20, Wit21, TY21, BMP+21], I also studied the gluing technique and its consequences for families of surfaces over a mixed characteristic base. Theorem 5 extends with few changes to mixed characteristic:

Theorem 10 (Theorem 6.1.0.1). *Let R be a DVR of mixed characteristic with maximal ideal πR. Then normalization gives a bijection*

$$\left(\begin{array}{c}\textit{Threefold pairs } (X,\Delta)\\ \textit{flat and proper over } R\\ \textit{such that } (X, \Delta + X_\pi) \textit{ is slc}\\ \textit{and } K_X + \Delta \textit{ is ample}\end{array}\right) \xrightarrow{1:1} \left(\begin{array}{c}\textit{Threefold pairs } (\bar{X}, \bar{D} + \bar{\Delta})\\ \textit{flat and proper over } R\\ \textit{such that } (\bar{X}, \bar{D} + \bar{\Delta} + \bar{X}_\pi) \textit{ is lc}\\ \textit{and } K_{\bar{X}} + \bar{D} + \bar{\Delta} \textit{ is ample}\\ \textit{plus a generically fixed point free}\\ R\textit{-involution } \tau \textit{ of } (\bar{D}^n, \mathrm{Diff}_{\bar{D}^n} \bar{\Delta}).\end{array}\right)$$

I also obtained abundance on mixed characteristic families of surfaces:

Theorem 11 (Theorem 7.4.0.1). *Let S be an excellent regular one-dimensional scheme of mixed characteristic, $f\colon (X, \Delta) \to S$ a dominant flat projective morphism of relative dimension two. Assume that $(X, \Delta + X_s)$ is slc for every closed point $s \in S$, and that every fiber X_s is S_2. Then if $K_X + \Delta$ is f-nef, it is f-semi-ample.*

Finally, let me mention a result that I find quite satisfying: in the context of Theorem 10, the fibers of $X \to \operatorname{Spec} R$ are the quotients of the fibers of $\bar{X} \to \operatorname{Spec} R$. More precisely:

Theorem 12 (Theorem 6.1.0.4). *Let $(\mathcal{S}, \Delta) \to \operatorname{Spec} R$ be a projective family of surfaces over a DVR with residue characteristic ≥ 7, with normalization $\bar{\mathcal{S}}$. Assume that $(\mathcal{S}, \Delta + \mathcal{S}_\pi)$ is slc. Then the fibers of $\mathcal{S}$ are the quotients of the fibers of $\bar{\mathcal{S}}$.*

Further considerations about the properties of fibers of quotients are contained in Section 6.4.

Chapter 2

Notations and preliminaries

In this chapter I collect some notations and results that will be used through this thesis. I have tried to be consistent with the terminology across the different chapters. However, since the set-up varies from one chapter to another, I will recall at the beginning of each of them what the current assumptions are.

2.1 Bibliographic references

Whenever quoting a result or an argument from an article or a book, I have tried to indicate the precise reference within that article or that book. There are a few exceptions: when I quote the main result of an article, or when I give bibliographic details in the introduction.

I use the following convention, best explained by an example: [Kol13, §4.1] refers to chapter 4.1 in [Kol13], while [Kol13, 4.1] refers to the definition/lemma/proposition/theorem/etc 4.1 in [Kol13].

2.2 Varieties

Let k be the spectrum of a field or a discrete valuation ring (DVR). We work with schemes and morphisms over k.

A **variety** over k is a reduced equidimensional scheme that is separated of finite type over k. The varieties that appear in Chapter 3 are irreducible, but the ones appearing in the subsequent chapters might be reducible or even disconnected[1]. A **curve** (respectively a **surface**, respectively a **threefold**) is a variety of dimension one (respectively two, respectively three).

Fix a Noetherian k-scheme X. An open subset $U \subset X$ is **big** if it contains every codimension one point of X. More generally, if $f\colon X \to T$ is a morphism, then U is **relatively big** over T if $U_t \subset X_t$ is big for every $t \in T$.

Given a point $x \in X$ we denote by $k(x)$ the **residue field** at x. If $Z \subset X$ is an irreducible reduced closed subscheme, we denote by $k(Z)$ its **function field**: by definition it is the residue field at the generic point of Z.

The **normalization** of X is defined to be its relative normalization along the structural morphism $\bigsqcup_\eta \operatorname{Spec}(k(\eta)) \to X$ where η runs through the generic points of X. We usually denote by $\bar{X}$ or X^ν the normalization of X. If X is excellent, then the normalization morphism $\nu\colon \bar{X} \to X$ is finite. We say that X is **normal** if ν is an isomorphism (in which case X is necessarily reduced).

An étale morphism of pointed schemes $(Y, y) \to (X, x)$ is called **elementary** if it induces an isomorphism $k(y) \cong k(x)$.

[1] This is to include objects such are the normalization of $\operatorname{Spec} k[x, y, z]/(xyz)$ in our discussion.

2.3 Sheaves

We fix a variety X (in the sense given above) and $\mathcal{F}$ a coherent $\mathcal{O}_X$-module.

We say that $\mathcal{F}$ satisfies **Serre's condition** S_i, for $i \geq 0$, if the inequality $\operatorname{depth}_{\mathcal{O}_{X,x}} \mathcal{F}_x \geq \min\{i, \dim \mathcal{F}_x\}$ holds at all $x \in X$. Recall that X is normal if and only if it is regular in codimension one and $\mathcal{O}_X$ is S_2; and that X is Cohen–Macaulay if and only if $\mathcal{O}_X$ is $S_{\dim X}$.

We define the **dual** of $\mathcal{F}$ by $\mathcal{F}^* = \mathcal{H}om(\mathcal{F}, \mathcal{O}_X)$. There is a natural map $\mathcal{F} \to \mathcal{F}^{**}$; we say that $\mathcal{F}$ is **reflexive** if it is an isomorphism. Any reflexive sheaf on X is automatically S_1 [Sta, 0AV5].

Assume that $\mathcal{F}$ is S_1. Then it is S_2 if and only if $\mathcal{F} = j_*j^*\mathcal{F}$ for a big open subset $j\colon U \hookrightarrow X$ [Har07, 1.8].

If X is normal and $\mathcal{F}$ torsion-free, then the following are equivalent: $\mathcal{F}$ is reflexive, $\mathcal{F}$ is S_2, and $\mathcal{F} = j_*j^*\mathcal{F}$ for j as above.

2.4 Birational Geometry

Let X be a normal variety. We use the language of $\mathbb{Q}$-divisors on X, as presented in [KM98, 0.4]. We denote by K_X any $\mathbb{Z}$-Weil divisor on X associated to the invertible sheaf $\omega_{X_{\mathrm{reg}}}$.

A **pair** structure (X, Δ) on X is the additional data of a $\mathbb{Q}$-Weil divisor Δ whose coefficients belongs to $[0;1]$, such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier. The divisor Δ is sometimes called the **boundary** of the pair. Given Δ, we often consider the following divisors:

$$\Delta^{=1} = \lfloor \Delta \rfloor := \sum_{E\,:\,\operatorname{coeff}_E \Delta = 1} E, \qquad \Delta^{<1} := \Delta - \Delta^{=1}.$$

We follow the standard terminology of [Kol13, §2.1] for the birational geometry of pairs (X, Δ). In particular, we refer the reader to *loc. cit.* for the notions of **discrepancy** $a(\bullet; X, D)$ of divisors, and for those of **log canonical** (**lc**), **Kawamata log terminal** (**klt**), **divisorial log terminal** (**dlt**), **canonical** and **terminal** pairs.

In dimension two, there is a (seemingly) more general notion of **numerically lc surface pair**, defined in [Kol13, 2.27].

Let (X, Δ) be a pair. An **lc center** of (X, Δ) is a closed subset $Z \subset X$ that is the image of a divisor of discrepancy -1 with respect to (X, Δ). More precisely, there exists a proper birational morphism from a normal variety $f\colon Y \to X$ and a prime divisor $E \subset Y$ such that $a(E; X, \Delta) = -1$ and $f(E) = Z$. It is convenient to allow X itself to be considered as a lc center.

Let $(X, \Delta + D)$ be a pair, where D is a reduced divisor with normalisation D^n. Then there is a canonically-defined $\mathbb{Q}$-divisor $\operatorname{Diff}_{D^n} \Delta$ on D^n such that restriction on D^n induces an isomorphism $\omega_X^{[m]}(m\Delta + mD)|_{D^n} \cong \omega_{D^n}(m \operatorname{Diff}_{D^n} \Delta)$ for m divisible enough. Singularities of $(X, \Delta + D)$ along D and singularities of $(D^n, \operatorname{Diff}_{D^n} \Delta)$ are related by so-called **adjunction theorems**. We refer to [Kol13, §4.1] for fundamental theorems of adjunction theory.

2.5 Quotients by finite equivalence relations

Quotients by equivalence relations are useful for studying non-normal varieties: we will use this technique extensively in Chapter 5, Chapter 6 and Chapter 7. The theory of quotients by finite equivalence relations is developped in [Kol12] and [Kol13, §9]. For convenience, we recall the basic definitions and constructions that we will need.

Let S be a base scheme, and X, R two reduced S-schemes. An S-morphism $\sigma = (\sigma_1, \sigma_2)\colon R \to X \times_S X$ is a **set theoretic equivalence relation** if, for every geometric point $\operatorname{Spec} K \to S$, the induced map

$$\sigma(\operatorname{Spec} K)\colon \operatorname{Hom}_S(\operatorname{Spec} K, R) \to \operatorname{Hom}_S(\operatorname{Spec} K, X) \times \operatorname{Hom}_S(\operatorname{Spec} K, X)$$

is injective and an equivalence relation of the set $\mathrm{Hom}_S(\mathrm{Spec}\, K, X)$. We say in addition that $\sigma\colon R \to X \times_S X$ is **finite** if both $\sigma_i\colon R \to X$ are finite morphisms.

Assume that G is a groupoid, that is a category where all the arrows are isomorphisms. An action of G on X over S is a functor $F\colon G \to *_{\mathrm{Aut}_S X}$, where the target is the groupoid with one element induced by the abstract group $\mathrm{Aut}_S X$. Given such an action, for each $g \in \mathrm{Arrow}(G)$ we let $\Gamma(g) \subset X \times_S X$ be the graph of the S-automorphism $F(g)$. Then the union $\bigcup_g \Gamma(g) \subset X \times_S X$ is a set theoretic equivalence relation. Conversely, a set theoretic equivalence relation $R \subset X \times_S X$ is called a **groupoid** if it is of this form.

Suppose that $\sigma\colon R \hookrightarrow X \times_S X$ is a reduced closed subscheme. Then there is a minimal set theoretic equivalence relation generated by R: see [Kol13, 9.3]. Even if both $\sigma_i\colon R \to X$ are finite morphisms, the resulting relation may not be finite: achieving transitivity can create infinite equivalence classes.

A case we will frequently consider is the following: X is a normal variety , $D \subset X$ a reduced divisor with normalization $n\colon D^n \to D$ and $\tau\colon D^n \cong D^n$ an involution. The **equivalence relation induced by** τ is the smallest set theoretic equivalence relation $R(\tau) \to X \times_k X$ induced by the closure of the set of those $(x, y) \in X \times_k X$ such that

$$\exists x', y' \in D^n \text{ such that } n(x') = x,\ n(y') = y,\ \tau(x') = y'.$$

A central task will be to identify conditions on (X, D, τ) that guarantee that $R(\tau)$ is a finite equivalence relation.

Let $(\sigma_1, \sigma_2)\colon R \to X \times_S X$ be a finite set theoretic equivalence relation. A **geometric quotient** of this relation is an S-morphism $q\colon X \to Y$ such that

(a) $q \circ \sigma_1 = q \circ \sigma_2$,

(b) $(Y, q\colon X \to Y)$ is initial in the category of algebraic spaces for the property above: if $p\colon X \to Z$ is such that $p \circ \sigma_1 = p \circ \sigma_2$ there exists a unique $\phi\colon Y \to Z$ such that $p = \phi \circ q$; and

(c) q is finite.

Clearly the quotient (Y, q) is unique (up to unique isomorphism) if it exists. It may happen that Y exists only as an algebraic space.

Remark 2.5.0.1. It is standard to add an extra condition in the definition of a geometric quotient $q\colon X \to Y$ that the geometric fibers are the R-equivalence classes: more precisely that, for every geometric point $\mathrm{Spec}\, K \to S$, the fibers of $q_K\colon X_K(K) \to Y_K(K)$ are the $\sigma(R_K(K))$-equivalence classes of $X_K(K)$. But this condition turns out to be a consequence of the three other ones, see the proof of [Kol12, Lemma 17].

The most important result for us is that quotients by finite equivalence relation usually exist in positive characteristic:

Theorem 2.5.0.2 ([Kol12, Theorem 6, Corollary 48])**.** *If X is essentially of finite type over a field k of positive characteristic and $R \rightrightarrows X$ is a finite set theoretic equivalence relation, then the geometric quotient X/R exists and is a k-scheme.*

In Chapter 6 we will also consider equivalence relations in mixed characteristic. The question of the existence of geometric quotients in this situation is studied in [Wit20], see in particular [Wit20, 1.4].

We also record the following lemma:

Lemma 2.5.0.3. *Let X be a reduced Noetherian pure-dimensional scheme, $R \rightrightarrows X$ a finite equivalence relation for which there exists a finite quotient $q\colon X \to Y := X/R(\tau)$. Let L be a line bundle on Y, with pullback $L_X = q^*L$. Then L is equal to the subsheaf of q_*L_X formed those sections which are R-invariant.*

Proof. Denote by $\sigma_1, \sigma_2 \colon R \rightrightarrows X$ the two projection morphisms. By [Kol13, 9.10] we have

$$\mathcal{O}_Y = \ker \left[q_*\mathcal{O}_X \xrightarrow{\sigma_1^* - \sigma_2^*} (q \circ \sigma_i)_*\mathcal{O}_R \right].$$

Tensor this expression by L and use the projection formula to obtain the result. $\square$

Chapter 3

Positivity of the Chow–Mumford line bundle for families of log Fano varieties

This chapter corresponds to the article [Pos22][1].

Convention 3.0.0.1. We work over an algebraically closed field k of characteristic zero. Varieties over k are supposed to be irreducible.

3.1 Introduction

The notion of K-stability originates from complex analytic geometry. It was first formulated by Tian in [Tia97] to study the existence of Kähler–Einstein metrics on Fano manifolds, and later expressed in algebraic terms by Donaldson in [Don02]. The connection between K-stability and birational geometry, and in particular with the Minimal Model program (MMP), was first noticed several years later: Odaka showed in [Oda13] that K-stable Fano varieties are log terminal, and Li and Xu used methods from the MMP to approach questions related to K-stability [LX14]. These were the first steps of a purely algebraic K-stability theory of polarized varieties, with a particular emphasis on the study of K-stability of Fano varieties. Equivalent definitions of K-stability were afterwards formulated in terms of Ding invariant in [Ber16] and of valuation theory [Fuj19, FO18]. This established a solid ground to study K-stable Fano varieties with methods of birational geometry.

The algebraic geometers' interest for K-stable Fano varieties comes, amongst other reasons, from the possibility of constructing well-behaved moduli spaces. Indeed, after the work of several authors, K-stability appeared to be an adequate global stability condition to obtain compact coarse moduli spaces of Fano varieties. To wit, several compact moduli spaces of del Pezzo K-stable surfaces were constructed [OSS16], as well as the moduli space of smoothable K-polystable Fano varieties [LWX18]. These constructions however rely heavily on techniques from analytic geometry, and it was desirable to find purely algebraic constructions. This program became reality by combining the progress in the algebraic theory of K-stability mentioned above, with the recent breakthroughs in birational geometry (e.g. [BCHM10], [HMX14] and [Bir19]) and in abstract moduli theory ([Alp13] and [AHLH19]). Thanks to several recent works [Jia20, BX19, ABHLX20, BLX19, Xu20, CP21, XZ20, LXZ21], we have now a good understanding of the algebraic moduli functor of K-stable Fano varieties. This chapter contributes to the study of its compactness properties.

We shall now explain in more details what is known about the moduli functor of K-stable

[1]First published electronically in Transactions of the American Mathematical Society (Volume 375, Number 7, July 2022, DOI: `https://doi.org/10.1090/tran/8640`), published by American Mathematical Society.

Fano varieties, and what is our contribution. We refer to Section 3.2 for the relevant definitions regarding K-stability and birational geometry.

We consider the moduli functor $\mathcal{M}^{\mathrm{Kss}}_{n,v,c}$, where $c \in \mathbb{Q}_+$, sending a k-scheme S to the set

$$\mathcal{M}^{\mathrm{Kss}}_{n,v,c}(S) = \left\{ \begin{array}{c} \text{Families } (X, c\Delta) \to S \text{ where } (X, \Delta) \to S \text{ is a} \\ \text{family of log pairs, and for every } t \in T \text{ the log} \\ \text{fiber } (X_t, c\Delta_t) \text{ is a K-semistable log Fano pair} \\ \text{of dimension } n \text{ and volume } v. \end{array} \right\}$$

It was conjectured that this functor is represented by an Artin stack of finite type over k and admits a projective good moduli space $M^{\mathrm{Kps}}_{n,v,c}$ (in the sense of [Alp13]), whose closed points are in bijection with n-dimensional K-polystable $\mathbb{Q}$-Fano varieties of volume v. As hinted above, this conjecture is now verified, thanks to the work of several authors:

Theorem 3.1.0.1 ([Jia20, BX19, ABHLX20, BLX19, Xu20, XZ20, LXZ21]). *The moduli functor $\mathcal{M}^{Kss}_{n,v,c}$ is an Artin stack of finite type over k and admits a projective good moduli space $M^{Kps}_{n,v,c}$ whose k-points parametrize K-polystable $\mathbb{Q}$-Fano varieties of dimension n and volume v.*

At the time the first version of the present chapter was written, the above theorem was not yet proved entirely: the missing parts were the properness and projectivity of $M^{\mathrm{Kps}}_{n,v,c}$, which were latter settled through the proof of Finite Generation Conjecture in [LXZ21]. It was also conjectured, and has been verified in full generality in *op.cit.*, that the polarization on the moduli space is given by the so-called *Chow–Mumford (CM) line bundle*. Our work was part of the effort, together with [CP21] and [XZ20], to show that the CM line bundle is indeed a good candidate.

Before stating our result, let us define the CM line bundle (see also Section 3.2.4). We consider $f\colon (X, D) \to T$ a flat family of log pairs of relative dimension n, such that X and T are projective and normal, and $-(K_{X/T} + \Delta)$ is f-ample. We let

$$\lambda_{f,D} := -f_*((-(K_{X/T} + D))^{n+1}),$$

where f_* is the cycle-pushforward. Then $\lambda_{f,D}$ is a $\mathbb{Q}$-Cartier divisor on T, called the CM line bundle of the family $f\colon (X, D) \to T$. It has a good functorial behaviour (see Proposition 3.2.4.1) and therefore defines a $\mathbb{Q}$-line bundle λ on $\mathcal{M}^{\mathrm{Kss}}_{n,v,c}$. Better still, it descends to the good moduli space $M^{\mathrm{Kps}}_{n,v,c}$ in the sense that there exists a $\mathbb{Q}$-line bundle L on $M^{\mathrm{Kps}}_{n,v,c}$ whose pullback to $\mathcal{M}^{\mathrm{Kss}}_{n,v,c}$ is λ [CP21, Lemma 10.2].

Our main result, which is a step towards the ampleness of λ, reads as follows:

Theorem 3.1.0.2. *Let $f\colon (X, D) \to T$ be a flat morphism of relative dimension n with connected fibers from a normal projective pair to a normal projective variety, such that $-(K_{X/T}+D)$ is $\mathbb{Q}$-Cartier and f-ample. Assume that D does not contain any fibers.*

(a) Bigness*: If each fiber (X_t, D_t) is klt, the general geometric fibers $(X_{\bar{t}}, D_{\bar{t}})$ are uniformly K-stable, and the variation of f is maximal, then $\lambda_{f,D}$ is big.*

(b) Ampleness*: If all the geometric fibers $(X_{\bar{t}}, D_{\bar{t}})$ are uniformly K-stable, and the variation of f is maximal, then $\lambda_{f,D}$ is ample.*

(Here a general geometric fiber denotes the fiber along a geometric point $\operatorname{Spec} \overline{\Omega} \to U \subseteq X$*, where $U \subseteq X$ is a dense open subset and $\overline{\Omega}$ some algebraically closed field.)*

The case $D = 0$ of Theorem 3.1.0.2 was proved previously in [CP21, Theorem 1.9]. However that proof does not generalize to the case $D \neq 0$. The difficulty lies in that there exist non-isomorphic log Fano pairs whose underlying varieties are isomorphic, so a family of log Fano pairs $(X, D) \to T$ can be of maximal variation while the underlying family $X \to T$ is not. Thus special attention to the geometry of the boundary D is required. Our strategy of proof of Theorem 3.1.0.2 is explained in Section 3.1.1: it relies on a perturbative argument on the boundary.

After the first version of this chapter was put on ArXiv, new positivity results for the CM line bundle were proved in [XZ20]. The authors introduce the notion of *reduced uniform K-stability*, which generalise that of uniform K-stability, and they proved the analogue of Theorem 3.1.0.2 for families of reduced uniform K-stable log Fano pairs, see [XZ20, §7]. Their strategy to deal with the case $D \neq 0$ was inspired by ours.

Remark 3.1.0.3. In Theorem 3.1.0.2, one of our assumptions is that each fiber (X_t, D_t) of the family is a klt pair. This hypothesis is natural for applications to moduli space of K-stable Fano varieties, where the families we consider have klt fibers (see Theorem 3.2.2.3). However it might not be necessary, since in the case $D = 0$ we only need the general log fiber to be klt [CP21, Theorem 1.9.a].

3.1.1 Overview of the proof

The proof of the bigness statement is based on the following idea. Let $(X, D) \to T$ be a $\mathbb{Q}$-Gorenstein family of log Fano pairs of maximal variation with uniformly K-stable general geometric fibers. By [CP21, Theorem 1.8], we know that $\lambda_{f,D}$ is a pseudo-effective divisor. Assume that the components of D are $\mathbb{Q}$-Cartier. Then for a small perturbation D^ϵ of D, the perturbed family $(X, D^\epsilon) \to T$ has the same properties as the original one. Hence the perturbed CM line bundle λ_{f,D^ϵ} remains pseudo-effective. By understanding the variation of $\lambda_{f,D}$ into λ_{f,D^ϵ}, we will deduce that $\lambda_{f,D}$ belongs to the interior of the pseudo-effective cone. If the components of D are not $\mathbb{Q}$-Cartier, we use techniques from the Minimal Model Program (MMP) to run a similar analysis.

Curve base and $\mathbb{Q}$-Cartier coefficients.

The variation of $\lambda_{f,D}$ is easy to analyse when the base T is a smooth curve, and all the reduced components D^i of D are $\mathbb{Q}$-Cartier. It follows from the definition of the CM line bundle that

$$-\deg \lambda_{f,D} = (-K_{X/T} - D)^{n+1}, \quad n+1 = \dim X.$$

Let $D^\epsilon = D - \sum_i \epsilon_i D^i$ be a perturbed boundary. As explained above, the divisor λ_{f,D^ϵ} is pseudo-effective for small values of ϵ, which means $\deg \lambda_{f,D^\epsilon} \geq 0$. We calculate this degree as above:

$$\begin{aligned}
-\deg \lambda_{f,D^\epsilon} &= \left(-K_{X/T} - D + \sum_i \epsilon_i D^i\right)^{n+1} \\
&= (-K_{X/T} - D)^{n+1} + (n+1) \sum_i \epsilon_i (-K_{X/T} - D)^n \cdot D^i + O(\epsilon^2) \\
&= -\deg \lambda_{f,D} + (n+1) \sum_i \epsilon_i (-K_{X/T} - D)^n \cdot D^i + O(\epsilon^2)
\end{aligned}$$

Hence for small values of ϵ, the function $\epsilon \mapsto -\deg \lambda_{f,D^\epsilon}$ can be approximated by an affine polynomial with linear coefficients $(-K_{X/T} - D)^n \cdot D^i$. Assume that these first order derivatives $(-K_{X/T} - D)^n \cdot D^i$ are all positive. Then $\deg \lambda_{f,D}$ cannot be too small in comparison to them, for otherwise $\deg \lambda_{f,D^\epsilon} < 0$ for a small value of ϵ.

We estimate these first order derivatives using the so-called product trick, pioneered in the work of Viehweg [Vie83]. For positive integers $r_0, \ldots, r_N$, let

$$D^{(r_\bullet)} = X^{(r_0)} \times_T \left(D^1\right)^{(r_1)} \times_T \cdots \times_T \left(D^N\right)^{(r_N)}$$

and let L be the Cartier divisor on $D^{(r_\bullet)}$ given by the sum of the pullbacks of $-K_{X/T} - D$ restricted to the different factors. Then the self-intersection of L depends only on $(-K_{X/T} -$

$D)^{n+1}$, $(-K_{X/T}-D)^n \cdot D^i$ and r_i. On the other hand, if $r_\bullet$ is suitably chosen, we can infer some positivity of L from the positivity of the sheaf

$$\det\left(f_*\mathcal{O}_X(-K_{X/T}-D)\otimes\bigotimes_i f_*\mathcal{O}_{D^i}(-K_{X/T}-D)\right).$$

The positivity of this determinant sheaf is a consequence of the maximal variation assumption via Kollár's ampleness lemma. From the positivity of L, we deduce a positive lower bound for the first-order derivatives $(-K_{X/T}-D)^n \cdot D^i$.

It is useful in the argument to twist $-K_{X/T}-D$ with a sufficient multiple of $f^*\lambda_{f,D}$, since we obtain a nef divisor [CP21, Theorem 1.20]. This replacement has technical significance, but does not affect the strategy.

General case.

The CM line bundle behaves well with respect to base-change (Proposition 3.2.4.1). In particular, it holds that $\lambda_{f,D}\cdot C = \deg \lambda_{f_C,D_C}$ for a smooth curve C mapping to T. Hence if T has higher dimension, we can base-change over a general curve C, apply the previous case and obtain $\lambda_{f,D}\cdot C > 0$. However, this does not suffice to prove that $\lambda_{f,D}$ is big, as the boundary of the cone of movable curves of T need not be spanned by classes of movable irreducible curves. Nevertheless, this strategy still works if we keep a precise track of the positivity.

(a) First we need to estimate the derivatives $(-K_{X_C/C}-D_C)^n \cdot D_C^i$. We can construct $D^{(r_\bullet)}$ and L as before, conclude to some positivity of L and base-change to C. However the base-change $D^{(r_\bullet)}\times_T C$ might not be flat over a general curve C, which creates difficulties. Thus we construct the product from a suitable birational model of X (Notation 3.6.3.2). Then we use the ampleness lemma and the product trick to estimate the derivatives (Proposition 3.6.2.1 and Proposition 3.6.3.5).

(b) We can garantee that these derivatives do not simultaneously go to zero when the class $[C]$ gets closer to the boundary of the movable cone (Lemma 3.6.3.6). This is done using the theory of Knudsen–Mumford expansion, which is recalled in Section 3.4.

(c) Once we have a uniform control on the derivatives, we would like to perturb the boundary D. However the components D^i might not be $\mathbb{Q}$-Cartier. Using the techniques of the MMP, we produce a birational model W of X on which some components become $\mathbb{Q}$-Cartier, and such that the morphism $W \to T$ has good properties (see Proposition 3.5.0.4 for the precise statement). Then we are in position to perform the perturbation argument on W (Section 3.6.4) and conclude.

3.2 Preliminaries

3.2.1 Notations and conventions

A pair (X,D) is *Fano* if X is projective and $-K_X-D$ is ample. A pair (X,D) is *weak log Fano* if it is a klt projective pair such that $-K_X-D$ is big and nef. A pair (X,D) is *log Fano* if it is a klt Fano pair. We say that X is $\mathbb{Q}$-*Fano* if $(X,0)$ is log Fano.

A birational proper morphism $\pi\colon Y\to X$ between projective varieties is called *small* if the exceptional locus of π has codimension at least 2.

Definition 3.2.1.1 (General movable curves)**.** Let X be a projective variety. A *smooth curve* $C\to X$ is a non-constant morphism (not necessarily an embedding) from a projective smooth curve C to X. We say that a smooth curve $C\to X$ is a *general movable curve* if it is the normalization of a general curve in a family of curves covering X.

Let $Z \subsetneq X$ be a proper closed subset. When fixing a general movable curve, we can always assume that it is not contained in Z. By [Laz04b, 11.4.C], a $\mathbb{Q}$-Cartier divisor D on X is big (resp. pseudo-effective) if and only if $D \cdot C > 0$ (resp. $D \cdot C \geq 0$) for every general movable curve $C \to X$.

Definition 3.2.1.2 (Families of log Fano pairs)**.** A $\mathbb{Q}$-*Gorenstein family of log Fano pairs* $f\colon (X, D) \to T$ is the data of a flat projective morphism $f\colon X \to T$ between normal projective varieties, and of an effective Weil $\mathbb{Q}$-divisor D, such that

(a) the fibers of f are irreducible and normal,
(b) the support of D does not contain any fiber,
(c) (X_t, D_t) is klt for each $t \in T$ (the definition of the restricted divisor D_t is given in Section 3.2.3), and
(d) $-K_{X/T} - D$ is an f-ample $\mathbb{Q}$-Cartier divisor.

Definition 3.2.1.3 (Maximal variation)**.** Let $f\colon (X, D) \to T$ be a $\mathbb{Q}$-Gorenstein family of log Fano pairs. Then f has *maximal variation* if there is a non-empty dense open subset $V \subset T$ such that for every point $t \in V$, the set $\{t' \in V \mid (X_t, D_t) \cong (X_{t'}, D_{t'})\}$ is finite.

Notation 3.2.1.4 (Coefficient parts)**.** Let X be a normal variety and D a Weil $\mathbb{Q}$-divisor on X. For $c \in \mathbb{Q}$, the part of coefficient c of D is defined to be

$$D^{=c} := \sum_{\operatorname{coeff}_E D = c} E$$

where the sum runs through the set of prime Weil divisors E of X. We have $D = \sum_{c \in \mathbb{Q}} c D^{=c}$. For simplicity, if $\{c \in \mathbb{Q} \mid D^{=c} \neq 0\} = \{c_1, \ldots, c_m\}$, we let $D^i := D^{=c_i}$ so that $D = \sum_{i=1}^m c_i D^i$. We will also denote by D^i the corresponding reduced closed subscheme.

Notation 3.2.1.5 (Volumes)**.** Let D be a $\mathbb{Q}$-Cartier divisor on a proper scheme X. We denote its *volume* by $\operatorname{Vol}(D)$. We refer to [Laz04a, §2.2.C] for the definition and the properties of the volume.

Notation 3.2.1.6 (Intersection numbers)**.** Let D be a $\mathbb{Q}$-Cartier divisor on a proper equidimensional scheme X of dimension n. We denote by $D^n = (D \cdots D)$ its self-intersection. If D is ample, it holds that $\operatorname{Vol}(D) = D^n$.

If X is also reduced and $\mathcal{L}$ is a line bundle on X, by abuse of notation we denote by $\mathcal{L}$ the associated linear equivalence class of Cartier divisors (see [Liu02, Corollary 1.19]). Then it makes sense to write

$$\mathcal{L}^m \cdot D^{n-m} = (\underbrace{\mathcal{L} \cdots \mathcal{L}}_{m \text{ times}} \cdot \underbrace{D \cdots D}_{n-m \text{ times}}).$$

If $f\colon C \to X$ is a morphism from a smooth proper curve, we write

$$D \cdot C = \frac{1}{r} \deg_C f^* \mathcal{O}_X(rD),$$

where $r > 0$ is such that rD is Cartier.

Notation 3.2.1.7 (m-fold products)**.** Let $f\colon X \to T$ be a morphism of proper schemes. We denote by $X^{(m)}$ the m-times fiber product of X with itself over T. It comes with projection morphisms $p_i\colon X^{(m)} \to X$ for $i = 1, \ldots, m$ and the structural morphism $f^{(m)}\colon X^{(m)} \to T$. Given a line bundle $\mathcal{L}$ on X, or a Cartier divisor D on X, we write

$$\mathcal{L}^{(m)} := \bigotimes_{i=1}^m p_i^* \mathcal{L}, \quad D^{(m)} := \sum_{i=1}^m p_i^* D.$$

3.2.2 K-stability

In this section, we recall briefly one characterization of the δ-invariant for log Fano pairs, and its relation with K-stability. We refer to [Fuj19] for the algebraic definition of K-stability in terms of test configurations.

Consider a n-dimensional weak log Fano pair (X, D). Let E be a prime divisor over X, and $\pi\colon Y \to X$ be a smooth birational model on which E appears. We can write

$$K_Y \equiv_{\text{num}} \pi^*(K_X + D) + \sum_F a_{X,D}(F)F$$

where F runs through the prime divisors of Y. The *log discrepancy* of E with respect to (X, D) is

$$A_{X,D}(E) := a_{X,D}(E) + 1.$$

We also define the quantity

$$S_{X,D}(E) := \frac{1}{(-K_X - D)^n} \int_0^{+\infty} \mathrm{Vol}(\pi^*(-K_X - D) - xE)dx.$$

Definition 3.2.2.1 (Delta invariant)**.** Let (X, D) be a log Fano pair. The δ*-invariant* of (X, D) is given by

$$\delta(X, D) := \inf_E \frac{A_{X,D}(E)}{S_{X,D}(E)},$$

where E runs through the prime divisors over X.

Remark 3.2.2.2. The original definition of the delta invariant in [FO18] is formulated in terms of basis-type divisors of the anti-log-canonical linear system. However, the above one is more convenient for our purpose. The equivalence between the two definitions is proved in [BJ20, Theorem 4.4] in the case $D = 0$, and [CP21, Theorem 4.6] in the general logarithmic case.

The relation between the delta invariant and K-stability of log Fano pairs is given by the following theorem. In this article, we use this characterization as the definition of uniform K-stability. See [FO18, Theorems 1.1 and 2.1] and [BJ20, Theorem B] for the proof of equivalence.

Theorem 3.2.2.3. *Let (X, D) be a Fano pair.*

(a) (X, D) is K-semistable if and only if (X, D) is klt and $\delta(X, D) \geq 1$.

(b) (X, D) is uniformly K-stable if and only if (X, D) is klt and $\delta(X, D) > 1$.

The next result will be useful in Section 3.5.

Proposition 3.2.2.4. *Let (X, D) be a weak log Fano pair, and Γ an effective $\mathbb{Q}$-Cartier divisor supported on Supp(D). Assume that*

$$\inf_E \frac{A_{X,D}(E)}{S_{X,D}(E)} > 1,$$

where E runs through the divisors over X. Then for all rational $\epsilon > 0$ small enough,

$$\inf_E \frac{A_{X,D-\epsilon\Gamma}(E)}{S_{X,D-\epsilon\Gamma}(E)} > 1.$$

Proof. Replacing Γ by a small multiple, we may assume that there is an effective $\mathbb{Q}$-Cartier divisor Γ' on X such that $\Gamma + \Gamma' \in |-K_X - D|_{\mathbb{Q}}$. By assumption, there is an $a > 0$ such that $A_{X,D}(E) \geq (1 + a)S_{X,D}(E)$ for all divisors E. Choose $a' \in (0, a)$ and define the function

$$f(\epsilon) := (1 + a')(1 + \epsilon)^{n+1} \frac{(-K_X - D)^n}{(-K_X - D + \epsilon\Gamma)^n}, \quad \epsilon \in \mathbb{R}.$$

Since $\lim_{\epsilon \to 0} f(\epsilon) = 1 + a'$, we can fix $\epsilon_0 = \epsilon_0(a') > 0$ such that for all $\epsilon \in (0, \epsilon_0)$, we have $f(\epsilon) < 1 + a$. Since we assumed $\mathrm{Supp}(\Gamma) \subset \mathrm{Supp}(D)$, we can also arrange that $D - \epsilon_0 \Gamma$ is effective.

Now let E be a divisor over X, appearing on a birational model $\pi\colon Y \to X$. For any $\epsilon > 0$ and any $x \in \mathbb{R}_+$, observe that

$$\begin{aligned} \mathrm{Vol}(\pi^*(-K_X - D + \epsilon\Gamma) - xE) &\leq \mathrm{Vol}(\pi^*(-K_X - D + \epsilon(\Gamma + \Gamma')) - xE) \\ &= \mathrm{Vol}((1+\epsilon)\pi^*(-K_X - D) - xE). \end{aligned}$$

Integrating over x, we obtain

$$\begin{aligned} \int_0^\infty \mathrm{Vol}(\pi^*(-K_X - D + \epsilon\Gamma) - xE)dx &\leq \int_0^\infty \mathrm{Vol}((1+\epsilon)\pi^*(-K_X - D) - xE)dx \\ &\overset{y=(1+\epsilon)x}{=} (1+\epsilon)^{n+1} \int_0^\infty \mathrm{Vol}(\pi^*(-K_X - D) - yE)dy. \end{aligned}$$

Together with the definition of the functional $S_{X,\bullet}$, the above inequality implies

$$S_{X,D-\epsilon\Gamma}(E) \leq (1+\epsilon)^{n+1} \frac{(-K_X - D)^n}{(-K_X - D + \epsilon\Gamma)^n} S_{X,D}(E).$$

Now take $\epsilon < \epsilon_0$ and assume that $A_{X,D-\epsilon\Gamma}(E) \leq (1+a')S_{X,D-\epsilon\Gamma}(E)$. Then we have

$$\begin{aligned} A_{X,D}(E) \leq A_{X,D}(E) + \epsilon \cdot \mathrm{ord}_E(\Gamma) = A_{X,D-\epsilon\Gamma}(E) &\leq (1+a')S_{X,D-\epsilon\Gamma}(E) \\ &\leq f(\epsilon)S_{X,D}(E) \\ &< (1+a)S_{X,D}(E), \end{aligned}$$

and we have obtained a contradiction with the hypothesis. Thus $A_{X,D-\epsilon\Gamma}(E) > (1+a')S_{X,D-\epsilon\Gamma}$ for all $\epsilon < \epsilon_0$. Since ϵ_0 does not depend on E, the proof is complete. □

3.2.3 Base-change of divisors

Notation 3.2.3.1. In this subsection, we consider a flat morphism $f\colon X \to T$ between normal projective varieties, and an effective Weil $\mathbb{Q}$-divisor D on X, such that

(a) the fibers of f are connected and normal,

(b) the support of D does not contain any fiber.

Remark 3.2.3.2. If $(X, D) \to T$ is as in Notation 3.2.3.1, then cD is a Mumford divisor in the sense of [Kol19, Definition 1] for $c > 0$ divisible enough. Since T is reduced, cD is automatically K-flat over T [Kol19, Definition 2 and paragraph 6]. Thus $(X, D = \frac{1}{c} \cdot cD) \to T$ is a K-flat family in the sense of Kollár.

Lemma 3.2.3.3. *In the situation of Notation 3.2.3.1, each component of D dominates T.*

Proof. We may assume that D is irreducible. Since D does not contain any component of any fiber, the scheme-theoretic restriction $D \cap X_t$ has dimension at most $\dim f - 1$ for any $t \in T$. If D does not dominate T, then $\dim f(D) < \dim T$. But in this case, for a general point $t \in f(D)$ we have

$$\dim X - 1 = \dim D = \dim D \cap X_t + \dim f(D) < \dim f - 1 + \dim T = \dim X - 1,$$

which is a contradiction. □

Definition 3.2.3.4 (Divisorial pullbacks)**.** Let $U \subset X$ be the smooth locus of f. By assumption U is relatively big over T, thus $f(U) = T$ and U is a big open subset of X. By [Kol21, Theorem 4.21], every Weil divisor on X not containing any component of a fiber is Cartier over U.

Let $u: S \to T$ be a morphism from a normal variety S. We define the *divisorial pullback of D along u* as follows. The open set $U_S = U \times_T S$ is big in $X_S = X \times_S T$. Since $D|_U$ is $\mathbb{Q}$-Cartier, it pullbacks to a $\mathbb{Q}$-Cartier divisor $D|_{U_S}$ on U_S. We let the divisorial pullback D_{X_S} of D along u be the unique Weil $\mathbb{Q}$-divisor extending the $\mathbb{Q}$-Cartier divisor $D|_{U_S}$.

In particular, if $t \in T$ is a closed point, then D_{X_t} is the unique Weil $\mathbb{Q}$-divisor of X_t extending the $\mathbb{Q}$-Cartier divisor $D|_{U \cap X_t}$. For ease of notation, we will mostly write $D_t := D_{X_t}$.

It follows from this definition that there is a $\mathbb{Q}$-linear equivalence

$$K_{X_S/S} + D_{X_S} \sim_{\mathbb{Q}} v^*(K_{X/T} + D) \tag{2.3.4.a}$$

where $v: X_S \to X$ is the induced morphism, see [CP21, §2.4.1].

Lemma 3.2.3.5. *In the situation of Notation 3.2.3.1, let $S \to T$ be a morphism from a normal projective variety. If D is Cartier, then the divisorial pullback of D and the pullback of D as Cartier divisor along $\sigma: X_S \to X$ agree.*

Proof. If U is the smooth locus of $X \to T$, then $\sigma^* D$ represents D_{X_S} on U_S by definition. A Cartier divisor on a normal variety is determined in codimension one, and U_S is big. Thus $\sigma^* D$ represent the Weil divisor D_{X_S}. □

Lemma 3.2.3.6. *In the situation of Notation 3.2.3.1, there is a dense big open set $U \subseteq T$ over which all the possible unions of components of D (with the reduced structure) are flat.*

Proof. Let E be a union of components of D with the reduced structure. By generic flatness, the locus of T over which E is flat, say U_E, is dense open. Pick a codimension one point $t \in T$, and any $x \in E$ such that $f(x) = t$. The morphism $\mathcal{O}_{T,t} \to \mathcal{O}_{E,x}$ is flat if and only if the uniformizer π of $\mathcal{O}_{T,t}$ is sent to a non-zero-divisor of $\mathcal{O}_{E,x}$. Now if π is a zero-divisor in $\mathcal{O}_{E,x}$, then the components of X_t passing through x are contained in E. But by assumption X_t is irreducible and E does not contain any fiber, so this cannot happen. Thus $E \to T$ is flat at x. Since x is arbitrary, we conclude that $t \in U_E$. Therefore $U = \bigcap_E U_E$ is big. □

Lemma 3.2.3.7. *In the situation of Lemma 3.2.3.6, there is a dense open set $V \subseteq U$ such that:*

(a) for any $v \in V$ and $c \in \mathbb{Q}$, the divisorial restriction $(D^{=c})_v$ is equal to the coefficient part $(D_v)^{=c}$.

(b) for any $v \in V$ and $c \in \mathbb{Q}$, the scheme-theoretic fiber $D^{=c} \times k(v)$ is equal to the divisorial restriction $(D^{=c})_v$ with the reduced structure.

Proof. Base-changing if necessary, we may assume that $U = T$. Given a reduced Weil divisor E not containing any fiber, we claim that the divisorial restriction E_t is reduced for a general $t \in T$. In view of Definition 3.2.3.4, we may assume that E is Cartier. Since the claim is local on X, we may assume that E is actually principal, say cut out by $s \in \mathcal{O}(X)$. Then $\mathcal{O}_X/(s)$ is reduced and flat over T; thus its fiber $\mathcal{O}_X/(s) \otimes k(t)$ over a general $t \in T$ is reduced [Gro66, 12.2.1]. This means exactly that the divisor E_t is reduced. If E' is another reduced divisor not containing any fiber, such that E and E' have no common components, by applying the claim to $E + E'$ we see that the divisorial restrictions E_t and E'_t have no common component for a general $t \in T$. Let E run through the coefficient parts $D^{=c}$ of D to obtain the first assertion.

If we consider E as a reduced closed subscheme, the scheme-theoretic fiber $E \times k(t)$ has pure codimension one for all $t \in T$ [Har77, III.9.6] and is reduced for a general $t \in T$. Combining this and the first assertion, we obtain the second assertion. □

Corollary 3.2.3.8. *In the situation of Lemma 3.2.3.7, let $C \to T$ be a smooth curve whose image intersect V. Let $Z := X \times_T C$ and $D_Z = \sum_{c \in \mathbb{Q}} c(D_Z)^{=c}$ be the divisorial pullback of D. Then $(D_Z)^{=c}$ is the divisorial pullback of $D^{=c}$ for all $c \in \mathbb{Q}$.*

Proof. We have to check that the divisorial pullbacks of any two distinct coefficient parts $D^{=c}$ and $D^{=c'}$, have no component in common. Since these divisorial pullbacks are horizontal over C, this can be checked on a general fiber of $Z \to C$. Since C meets V, the result follows from Lemma 3.2.3.7. □

3.2.4 The CM line bundle

Let $f\colon X \to T$ be a flat projective morphism of relative dimension n between normal projective varieties, let D be an effective $\mathbb{Q}$-divisor on X such that $-(K_{X/T}+D)$ is $\mathbb{Q}$-Cartier and f-ample. Assume also that the fibers of f are irreducible and normal, and that $\mathrm{Supp}(D)$ does not contain any fiber. Then the *Chow–Mumford line bundle* of $f\colon (X,D)\to T$ is defined to be

$$\lambda_{f,D} := -f_*((-K_{X/T}-D)^{n+1})$$

where f_* denotes the pushforward of cycles. By [CP21, Proposition 3.7], $\lambda_{f,D}$ is a $\mathbb{Q}$-Cartier $\mathbb{Q}$-Weil divisor. It is compatible with base-change in the following sense:

Proposition 3.2.4.1. *In the above situation, let $\tau\colon S\to T$ be a morphism from a normal variety S. Let $f_S\colon X_S\to S$ be the induced morphism and D_S be the divisorial pullback in the sense of Section 3.2.3. Then $\tau^*\lambda_{f,D}=\lambda_{f_S,D_S}$.*

We refer to [CP21, §3] for the proof and more background.

3.3 Ampleness lemma

The next theorem, which is an elaboration of [Kol90, 3.9], will be useful to establish positivity properties of line bundles.

Let us fix our notations for the Grassmannians and the general linear groups over k. Given integers $w\ge q$, we let $\mathrm{Gr}_k(w,q)$ be the Grassmannian of q-dimensional quotients of a w-dimensional k-vector space. Given an integer n, we let $\mathrm{GL}_k(n)$ be the general linear group over k of degree n.

Theorem 3.3.0.1. *Let U be a normal variety that can be embedded as a big open subset of a projective variety. Let $W,Q_1,\dots,Q_s$ be vector bundles on U of respective ranks $w,q_1,\dots,q_s$. Assume that there exist morphisms $\phi_i\colon W\to Q_i$ for $i=1,\dots,s$ which are generically surjective. Assume also that the classifying map*

$$U(k)\to\left(\prod_{i=1}^{s}\mathrm{Gr}_k(w,q_i)\right)/\,\mathrm{GL}_k(w)$$

is finite-to-one on a dense subset of U. Then for any ample Cartier divisor B on U, there exists a positive integer $m>0$ and a non-zero morphism

$$\mathrm{Sym}^{mq}\left(\bigoplus_{i=1}^{s^2w} W\right)\longrightarrow \mathcal{O}_U(-B)\otimes\left(\bigotimes_{i=1}^{s}\det Q_i\right)^{\otimes m}$$

where $q=\sum_{i=1}^{s}q_i$.

Proof. The proof is contained in [KP17, §5]. More precisely, let Q_i' be the image of $\phi_i\colon W\to Q_i$, with corestricted morphisms $\phi_i'\colon W\twoheadrightarrow Q_i'$. There is a big open subset $U'\subset U$ over which Q_i' is locally free of rank q_i, for each i. Since the statement is about the existence of a non-zero map between two locally free sheaves, by reflexivity we may replace U by U' and assume that all Q_i' are locally free. Now let $W'=\oplus_{i=1}^s W$, $Q'=\oplus_{i=1}^s Q_i'$ and $\phi'=\oplus_{i=1}^s\phi_i'$. As explained in [KP17, Lemma 5.6], ϕ' is surjective over U, and there is a dense open set of U where the classifying map corresponding to ϕ' has finite fibers. Now we follow the proof of [KP17, Theorem 5.5] applied to $\phi' \colon W' \to Q'$. In this proof, the assumption of weak positivity is only used in the last three lines of proof; in particular, the equation (5.5.5) and the subsequent displayed isomorphisms hold without this assumption. Thus they apply to our setting: for some $m > 0$, there exists a non-zero morphism

$$\mathrm{Sym}^{m\cdot \mathrm{rk}(Q')}\left(\bigoplus_{i=1}^{\mathrm{rk}(W')} W'\right) \longrightarrow \mathcal{O}_U(-B) \otimes \left(\det Q'\right)^{\otimes m}.$$

Moreover, it follows from Lemma 3.3.0.2 below that we have an inclusion

$$\mathcal{O}_U(-B) \otimes (\det Q')^{\otimes m} \hookrightarrow \mathcal{O}_U(-B) \otimes \left(\bigotimes_{i=1}^{s} \det Q_i\right)^{\otimes m}.$$

The result follows by composing the two morphisms. □

Lemma 3.3.0.2. *Let U be a normal variety and $\alpha \colon Q' \hookrightarrow Q$ an inclusion of locally free sheaves. Assume that α is generically surjective. Then $\det(Q') \hookrightarrow \det(Q)$.*

Proof. Let $\mathcal{F}$ be the cokernel of α. Then $\mathcal{F}$ is torsion, and the determinant sheaf $\det(\mathcal{F}) = \det(Q) \otimes \det(Q')^{-1}$ is of the form $\mathcal{O}_U(E)$ for an effective divisor E. It follows that $\det(Q') \cong \mathcal{O}_U(-E) \otimes \det(Q)$ embeds into $\mathcal{O}_U \otimes \det(Q) \cong \det(Q)$. □

3.4 About the Knudsen–Mumford expansion

In this section, we give an alternative description of the CM line bundle, that will be useful to study its positivity properties.

To begin with, we recall a special case of [KM76, Theorem 4]. Let $f \colon X \to T$ be a projective morphism between Noetherian schemes of relative dimension n. We do not require that f is flat. Let A be an f-very ample Cartier divisor. Then there exist $\mathcal{M}_i \in \mathrm{Pic}(T)$ such that for every $q \gg 0$, there is an isomorphism

$$\det f_*\mathcal{O}_X(qA) \cong \det Rf_*\mathcal{O}_X(qA) \cong \bigotimes_{i=0}^{n+1} \mathcal{M}_i^{\otimes \binom{q}{i}}$$

We call this expression the *Knudsen-Mumford expansion* of $\mathcal{O}_X(A)$, and refer to the $\mathcal{M}_i$ as the coefficients of the expansion. This isomorphism is moreover functorial: if $S \to T$ is a morphism from a Noetherian scheme and A_S the pullback of A to $X \times_T S$, then it holds that

$$\det(f_S)_*\mathcal{O}_{X_S}(qA_S) \cong \bigotimes_{i=0}^{n+1} (\mathcal{M}_i)_S^{\otimes \binom{q}{i}}, \quad q \gg 0.$$

Now consider the particular case where $f \colon (X, D) \to T$ is a $\mathbb{Q}$-Gorenstein family of log Fano pairs. Let s be such that $s(-K_{X/T} - D)$ is very ample over T. Then one can show that

$$-s^{n+1}\lambda_{f,D} = \mathcal{M}_{n+1},$$

where $\mathcal{M}_{n+1}$ is the leading coefficient of the Knudsen-Mumford expansion of $s(-K_{X/T} - D)$. See [CP21, Proposition 3.7] for a proof.

The following proposition characterizes the numerical class of $\mathcal{M}_{n+1}$ in several situations: when A is nef, or when the morphism f has pleasant properties.

Proposition 3.4.0.1. *Let $f\colon X \to T$ be an equidimensional projective morphism of relative dimension n between Noetherian proper schemes (we do not require f to be flat). Let A be an f-very ample Cartier divisor on X, and $\mathcal{M}_{n+1}$ be the leading coefficient of the Knudsen-Mumford expansion of $\mathcal{O}_X(A)$.*

(a) *Assume that A is nef. For any smooth curve $C \to T$, it holds that $\mathcal{M}_{n+1} \cdot C = A_C^{n+1}$.*

(b) *Assume that A is nef and X is generically reduced. Let $X' \to X$ be the normalization morphism, $f'\colon X \to T$ be the induced morphism and A' be the pullback of A. Then $\mathcal{M}_{n+1} \cdot C = f'_*((A')^{n+1}) \cdot C$ for a general movable curve $C \to T$.*

(c) *Assume that T is normal and f is flat with normal fibers. Then for any smooth curve $C \to T$, we have $\mathcal{M}_{n+1} \cdot C = f_*(A^{n+1}) \cdot C = A_C^{n+1}$.*

Proof. Fix a smooth curve $C \to T$. In any case, since both qA and qA_C are relatively very ample for $q \gg 0$, both sheaves $f_*\mathcal{O}_X(qA)$ and $(f_C)_*\mathcal{O}_{X_C}(qA_C)$ are locally free with vanishing R^i, $i > 0$. It follows from the functoriality of the Knudsen-Mumford expansion that

$$\det\left[f_*\mathcal{O}_X(qA)\right] \cdot C = \deg \det\left[(f_C)_*\mathcal{O}_{X_C}(qA_C)\right], \quad q \gg 0. \tag{4.0.1.b}$$

With this set-up:

(a) Assume that A is nef. The left-hand side of equation (4.0.1.b) is given by

$$\sum_{i=0}^{n+1} \binom{q}{i} \mathcal{M}_i \cdot C = \frac{q^{n+1}}{(n+1)!}\mathcal{M}_{n+1} \cdot C + O(q^n),$$

where $\mathcal{M}_i$ are the Knudsen-Mumford coefficients of $\mathcal{O}_X(A)$. Now consider the right-hand side of the same equation. By Riemann-Roch, for q large enough we have

$$h^0(X, \mathcal{O}_{X_C}(qA_C)) = \deg \det\left[(f_C)_*\mathcal{O}_{X_C}(qA_C)\right] + \chi(C, \mathcal{O}_C) \cdot \mathrm{rk}\ (f_C)_*\mathcal{O}_{X_C}(qA_C).$$

Since A_C is nef, we have

$$h^0(X, \mathcal{O}_{X_C}(qA_C)) = \frac{q^{n+1}}{(n+1)!} A_C^{n+1} + O(q^n)$$

by [Kol96, VI.2.15]. Since the fibers of f are n-dimensional, the function

$$q \mapsto \mathrm{rk}\ (f_C)_*\mathcal{O}_{X_C}(qA_C)$$

is a polynomial in q of degree at most n. Hence

$$\deg \det\left[(f_C)_*\mathcal{O}_{X_C}(qA_C)\right] = \frac{q^{n+1}}{(n+1)!} A_C^{n+1} + O(q^n).$$

It follows by comparing the leading coefficients in (4.0.1.b) that $A_C^{n+1} = \mathcal{M}_{n+1} \cdot C$.

(b) Assume that A is nef and X generically reduced. The normalization morphism $X' \to X$ is finite, so A' is nef and relatively ample over T. Say that sA' is relatively very ample for some $s > 0$, and let $\mathcal{M}'_{n+1}$ be the leading coefficient of the Knudsen-Mumford polynomial of $\mathcal{O}_{X'}(sA')$. Since X is generically reduced, the normalization $X' \to X$ is an isomorphism away from a closed subset $Z \subsetneq X$. If $C \to T$ is a smooth curve which intersects $f(X - Z)$, the pullback morphism $(X')_C \to X_C$ is birational. Using the first assertion, we obtain that

$$\mathcal{M}_{n+1} \cdot C = A_C^{n+1} = (A')_C^{n+1} = s^{-n-1}\mathcal{M}'_{n+1} \cdot C.$$

By [CP21, Lemma A.2] it holds that $\mathcal{M}'_{n+1} = f'_*(sA')^{n+1}$, so the second assertion follows.

(c) Assume that T is normal and f is flat with normal fibers. Then both X and X_C are normal. It follows from [CP21, Lemma A.2] that for $q \gg 0$

$$\det f_*\mathcal{O}_X(qA) = \frac{q^{n+1}}{(n+1)!} f_*(A^{n+1}) + O(q^n)$$

and

$$\det(f_C)_*\mathcal{O}_{X_C}(qA_C) = \frac{q^{n+1}}{(n+1)!}(f_C)_*(A_C^{n+1}) + O(q^n)$$

in the Chow groups of T and C respectively. It follows that $\mathcal{M}_{n+1} = f_*(A^{n+1})$, and by intersecting with C that

$$\mathcal{M}_{n+1} \cdot C = f_*(A^{n+1}) \cdot C = \deg(f_C)_*(A_C^{n+1}) = A_C^{n+1}$$

as claimed.

□

3.5 Perturbation of families of K-stable log Fanos

Consider a $\mathbb{Q}$-Gorenstein family $(X, D) \to T$ of log Fano pairs of maximal variation, with uniformly K-stable general fibers. We show in Proposition 3.5.0.4 below that we can find a model of (X, D) with the same properties over T, and on which some components of the boundary D become $\mathbb{Q}$-Cartier.

We need a few preliminary lemmas. For the first one, we use the terminology of [BCHM10].

Lemma 3.5.0.1. *Let $f: X \to T$ be a projective morphism between quasi-projective normal varieties. Let D be an effective $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on X such that (X, D) is (klt) weak log Fano over T. Assume that $D_1, \ldots, D_m$ are effective $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisors on X with supports contained in the support of D. Then there exists a full-dimensional closed polytope $P \subset (\mathbb{R}_{\geq 0})^m$ with the following properties:*

(a) P contains the origin, and its interior $\mathrm{int}(P)$ is contained in $(\mathbb{R}_{>0})^m$; and

(b) for every rational vector $(\epsilon_1, \ldots, \epsilon_m) \in \mathrm{int}(P)$, the log canonical models of $(-K_X - D + \sum_i \epsilon_i D_i)$ over T have isomorphic underlying varieties.

Proof. Fix a general very ample divisor A on X, which has no component in common with D. Since X is of Fano type over T, there is a $\mathbb{Q}$-boundary Δ such that (X, Δ) is klt, Δ is big over T and $a(-K_X - D - A) \sim_{\mathbb{Q},T} K_X + \Delta$ for some small rational $a > 0$. Replacing A by a general member of its linear system, we may assume that $(X, \Delta + aA)$ is also klt [KM98, 5.17]. We have $K_X + \Delta + aA + \sum_i \epsilon_i D_i \sim_{\mathbb{Q},T} a(-K_X - D) + \sum_i \epsilon_i D_i$. So a log canonical model of $(X, \Delta + aA + \sum_i \epsilon_i D_i)$ over T is also a $(-K_X - D + \sum_i \frac{\epsilon_i}{a} D_i)$-log canonical model of X over T. Therefore it is equivalent to prove that: there is a full-dimensional closed polytope $P \subset (\mathbb{R}_{>0})^m$ containing the origin, such that for all $(\epsilon_1, \ldots, \epsilon_m) \in \mathrm{int}(P)$, the pairs $(X, \Delta + aA + \sum_i \epsilon_i D_i)$ have a log canonical model over T with isomorphic underlying varieties.

Let us write $A' := aA$ and define the affine cone $V := \Delta + \sum_i \mathbb{R}_+ D_i$ in $\mathrm{Weil}(X)_{\mathbb{R}}$. Since $(X, \Delta + A')$ is klt, there is an open Euclidean neighborhood $\mathcal{U}$ of $\Delta \in V$ such that for all $\Gamma \in \mathcal{U}$, the pair $(X, \Gamma + A')$ is klt. Also, since $K_X + \Delta + A' \sim_{\mathbb{Q},T} a(-K_X - D)$ is big over T, we may shrink $\mathcal{U}$ so that $K_X + \Gamma + A'$ is big over T for all $\Gamma \in \mathcal{U}$. With the notations of [BCHM10, 1.1.4], this implies that $\mathcal{U} \subset \mathcal{E}_{A',f}(V)$.

It follows from [BCHM10, Corollary 1.1.5 and Theorem E] that there are finitely birational contractions $\psi_i: X \dashrightarrow Z_i$ over T, $i = 1, \ldots, n$, and a decomposition

$$\mathcal{E}_{A',f}(V) = \bigcup_{i=1}^{n} \mathcal{W}_i$$

where each $\mathcal{W}_i = \mathcal{W}_{\psi_i,A',f}(V)$ is a rational polytope, such that for each $\Gamma \in \mathcal{W}_i$, the underlying variety of a weak log canonical model of $(X, \Gamma + A')$ over T is isomorphic to Z_i.

By [BCHM10, Theorem 1.2], for every $\Gamma \in \mathcal{U}$ the pair $(X, \Gamma + A')$ has a log canonical model over T. Relative log canonical models are in particular relative weak log canonical models. So we obtain that for any $\Gamma \in \mathcal{U} \cap \mathcal{W}_i$, the underlying variety of a log canonical model of $(X, \Gamma + A')$ is isomorphic to Z_i.

Since $\mathcal{E}_{A',f}(V)$ contains the open neighborhood of the origin of V, there must be a polytope $\mathcal{W}_i$ which is of full dimension in V and whose closure contains the origin of V. Thus we may find a closed full-dimensional polytope $P \subset (\mathbb{R}_{\geq 0})^m$ containing the origin, with non-empty interior $\mathrm{int}(P) \subset (\mathbb{R}_{>0})^m$ such that

$$\left(\bigcup_{(\epsilon_1,\ldots,\epsilon_m)\in \mathrm{int}(P)} \Delta + \sum_i \epsilon_i D_i \right) \subset \mathcal{U} \cap \mathcal{W}_i \quad \text{for some } i.$$

This finishes the proof. □

Lemma 3.5.0.2. *Let $(X, D) \to T$ be a $\mathbb{Q}$-Gorenstein family $(X, D) \to T$ of log Fano pairs of maximal variation. Write $D = \sum_i c_i D^i$ as in Notation 3.2.1.4. Then there is a rational number $r > 0$ such that for all i such that D^i is $\mathbb{Q}$-Cartier, and for all rational $\epsilon \in (-r; r)$, the family $(X, D + \epsilon D^i) \to T$ has maximal variation.*

Proof. Take $r = \min_{i\neq j}\{\frac{1}{2}|c_i - c_j|\}$. □

Lemma 3.5.0.3. *Let $f\colon (X, D) \to T$ be a flat equidimensional proper morphism from a normal pair to a smooth variety. Assume that every fiber (X_t, D_t) is klt. Then:*

(a) (X, D) is klt, and

(b) for any closed point $t \in T$, if $H_1, \ldots, H_d$ ($d = \dim T$) are general Cartier divisors in a base-point free linear system such that in a neighbourhood of t we have $\bigcap_i H_i = \{t\}$, then $(X, D + \sum_i f^*H_i)$ is dlt in a neighborhood of X_t.

Proof. Let $t \in T$ be a closed point, and let $H_1, \ldots, H_d$ ($d = \dim T$) be general Cartier divisors on T such that $\bigcap_i H_i = \{t\}$.

To begin with, we prove that (X, D) is klt. Indeed, we can choose the $H_1, \ldots, H_{d-1}$ in a general linear system, so the iterated hyperplane sections $X^m := \bigcap_{i=1}^m f^*H_i$ are normal varieties for $m \leq d-1$ [Sei50]. By inversion of adjunction, since (X_t, D_t) is assumed to be klt, we obtain that $(X^{d-1}, D|_{X^{d-1}} + f^*H_d)$ is plt along X_t [KM98, Theorem 5.50]. Hence $(X^{d-1}, D|_{X^{d-1}})$ is klt along X_t. We repeat this argument to obtain that (X, D) is klt along X_t. The choice of t was arbitrary, so we conclude that (X, D) is klt.

Now we think of $t \in T$ as a fixed point and claim that if the H_i are suitably chosen, then the pair $(X, D + \sum_{i=1}^d f^*H_i)$ is dlt in a neighborhood of X_t. Indeed, we can choose the H_m inductively with the property that

$$\text{for each } I \subseteq \{1, \ldots, m\}, \text{ the intersection } X^I := \bigcap_{i\in I} f^*H_i \text{ is irreducible and normal.}$$

Each of these conditions is satisfied for a general H_m passing through t, except for the condition on H_d that $\bigcap_{i=1}^d f^*H_i = X_t$ is irreducible and normal. But this is satisfied for any choice of H_d, since X_t is assumed to be irreducible and normal.

It follows from the choices of these H_i that $(X, D + \sum_{i=1}^d f^*H_i)$ is snc at every generic point of the X^I. If E is an exceptional divisor over X whose center $c_X(E)$ belongs to X_t but does not belong to the snc locus of $(X, D + \sum_i f^*H_i)$, then $c_X(E)$ defines a point of codimension ≥ 1 in X_t, and by adjunction we obtain that (X_t, D_t) is not klt, which is a contradiction. □

Proposition 3.5.0.4. *Let $f\colon (X,D) \to T$ be a $\mathbb{Q}$-Gorenstein family of log Fano pairs of maximal variation with uniformly K-stable general geometric fiber. Assume that T is smooth. Then there exists a positive number $r_{X,D} > 0$ with the following property. For every coefficient part $\Gamma := D^i$ of D (as in Notation 3.2.1.4), there exists a small proper birational morphism $\nu\colon W \to X$ such that:*

(a) the strict transform Γ_W of Γ is $\mathbb{Q}$-Cartier, and

(b) for any rational $0 < \epsilon < r_{X,D}$, the family $(W, D_W - \epsilon\Gamma_W) \to T$ is a $\mathbb{Q}$-Gorenstein family of log Fano pairs of maximal variation with uniformly K-stable general geometric fibers. (Here D_W denotes the strict transform of D.)

Proof. Since there are finitely many coefficient part of D, we only need to prove the result for a fixed Γ. First we construct $\nu\colon W \to X$.

- The pair (X,D) is klt by Lemma 3.5.0.3, so by [Kol13, Corollary 1.37] there is a small proper birational morphism $\mu\colon Y \to X$ where Y is a $\mathbb{Q}$-factorial projective variety. Denote by D_Y the strict transform of D, and Γ_Y the strict transform of Γ. We have

$$\mu^*(K_X + D) \sim_{\mathbb{Q}} K_Y + D_Y.$$

- For $\epsilon > 0$, run a $(-K_Y - D_Y + \epsilon\Gamma_Y)$-MMP over X to obtain a relative log canonical model. By Lemma 3.5.0.1, this model W is the same for all $0 < \epsilon \ll 1$. Denote by $p\colon Y \dashrightarrow W$ the induced morphism and $D_W := p_*D_Y, \Gamma_W := p_*\Gamma_Y$.

Our construction is pictured by the following diagram:

$$\begin{tikzcd}
(Y,D_Y,\Gamma_Y) \arrow[r, dashed, "p"] \arrow[d, "\mu"] & (W,D_W,\Gamma_W) \arrow[dl, "\nu"] \arrow[ddl, bend left, "g"] \\
(X,D,\Gamma) \arrow[d, "f"] & \\
T &
\end{tikzcd}$$

We must show that for smal rationall $\epsilon > 0$, the morphism $(W, D_W - \epsilon\Gamma_W) \to T$ is flat between normal projective varieties, of maximal variation, with (klt) log Fano fibers and uniformly K-stable general geometric fibers, and that Γ_W is $\mathbb{Q}$-Cartier. First we establish the global properties of W and Γ_W.

- The morphism $\nu\colon W \to X$ is small. Indeed, μ is small and p extracts no divisors. Notice that D_W is equal to the strict transform of D.
- Since W is the end product of an MMP, it is normal. Moreover, since ν is small and (X,D) is klt by Lemma 3.5.0.3, (W,D_W) is klt and hence W is Cohen-Macaulay.
- The $\mathbb{Q}$-divisor Γ_W is $\mathbb{Q}$-Cartier. Indeed, $p_*(-K_Y - D_Y + \epsilon\Gamma_Y) = -K_W - D_W + \epsilon\Gamma_W$ is $\mathbb{Q}$-Cartier by construction. It holds that $\nu^*(-K_X - D) = -K_W - D_W$, and so $-K_W - D_W$ is also $\mathbb{Q}$-Cartier. Therefore Γ_W is $\mathbb{Q}$-Cartier.
- We have

$$\epsilon\Gamma_Y \equiv_X -K_Y - (D_Y - \epsilon\Gamma_Y)$$

and thus

$$\epsilon\Gamma_W \equiv_X -K_W - (D_W - \epsilon\Gamma_W) \quad \text{is ample over } X.$$

Hence Γ_W is a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor which is ample over X. Furthermore,

$$\begin{aligned} -K_W - (D_W - \epsilon\Gamma_W) &= p_*(-K_Y - (D_Y - \epsilon\Gamma_Y)) \\ &= p_*(p^*\nu^*(-K_X - D) + \epsilon\Gamma_Y) \\ &= \nu^*(-K_X - D) + \epsilon\Gamma_W. \end{aligned}$$

Now by [Pat15, Lemma 2.4],

$$-K_W + g^*K_T = -K_{W/T}, \quad -K_X + f^*K_T = -K_{X/T}.$$

Hence

$$-K_{W/T} - (D_W - \epsilon\Gamma_W) = \nu^*(-K_{X/T} - D) + \epsilon\Gamma_W$$

and therefore $-K_{W/T} - D_W + \epsilon\Gamma_W$ is ample over T for $0 < \epsilon \ll 1$.

To study the properties of a fiber W_t, we may shrink T and work in a neighborhood of W_t. Let $H_1, \dots, H_d$ be general Cartier divisors such that $\bigcap_i H_i = \{t\}$.

- The fibers of ν are connected by construction. Since the fibers of f are irreducible, we deduce that g has connected fibers.
- Since ν is small, it gives a crepant morphism from $(W, D_W + \sum_{i=1}^d g^*H_i)$ to $(X, D + \sum_{i=1}^d f^*H_i)$. Moreover ν is an isomorphism above the snc locus of $(X, D + \sum_i f^*H_i)$ [Deb01, 1.40]. By Lemma 3.5.0.3 the pair $(X, D + \sum_i f^*H_i)$ is dlt. Thus $(W, D_W + \sum_{i=1}^d g^*H_i)$ is lc, and every lc center is contained in the locus where ν is an isomorphism. It follows that $(W, D_W + \sum_{i=1}^d g^*H_i)$ is also dlt.
- From [Kol13, 4.16] we deduce that every irreducible component of $W_t = \bigcap_{i=1}^d g^*H_i$ is normal of codimension d, and an lc center of $(W, D_W + \sum_{i=1}^d g^*H_i)$. Assume that W_t has at least two different components; by connectedness of the fibers of $W \to T$, the two components must intersect, and the intersection is a union of lc centers of $(W, D_W + \sum_{i=1}^d g^*H_i)$ [Kol13, 4.20.2]. Since (X, D) is klt by Lemma 3.5.0.3, we have $\lfloor D \rfloor = 0$ and thus $\lfloor D_W \rfloor = 0$. So by [Kol13, 4.16.1] the components of W_t are minimal lc centers, and we have reached a contradiction. Hence W_t is irreducible and normal of codimension d.
- Since W is Cohen-Macaulay, T smooth and the fibers W_t equidimensional, the morphism $g\colon W \to T$ is flat [Mat89, Theorem 23.1].
- Assume that some fiber W_t is contained in the support of D_W. Since W_t dominates X_t and D_W dominates D, we obtain that X_t is contained in the support of D, which is impossible. Thus $\mathrm{Supp}(D_W)$ contains no fibers of $W \to T$.

Finally we study the pairs $(W_t, (D_W - \epsilon\Gamma_W)_t)$.

- We know that $(W, D_W + \sum_i g^*H_i)$ is dlt with reduced boundary $\sum_i g^*H_i$. Combining adjunction and [Kol13, 4.16], we obtain that the pair $(W_t, (D_W)_t)$ is klt. Hence $(W_t, (D_W - \epsilon\Gamma_W)_t)$ is klt for every $0 < \epsilon < \mathrm{coeff}_{\Gamma_W} D_W$ [KM98, 2.27].
- Since μ is small, for a general $t \in T$ the morphism $W_t \to X_t$ is small and $(D_W)_t$ is the strict transform of D_t. Fix one such t for which (X_t, D_t) is uniformly K-stable. Then for every prime divisor E over W_t, we have

$$A_{W_t,(D_W)_t}(E) = A_{X_t,D_t}(E), \quad S_{W_t,(D_W)_t}(E) = S_{X_t,D_t}(E),$$

so by Theorem 3.2.2.3 we have

$$\inf_E \frac{A_{W_t,(D_W)_t}(E)}{S_{W_t,(D_W)_t}(E)} = \inf_E \frac{A_{X_t,D_t}(E)}{S_{X_t,D_t}(E)} = \delta(X_t, D_t) > 1.$$

Moreover $(W_t, (D_W)_t)$ is a weak log Fano pair. Thus we may apply Proposition 3.2.2.4 to obtain that

$$\delta(W_t, (D_W)_t - \epsilon\Gamma_t) = \inf_E \frac{A_{W_t,(D_W)_t - \epsilon\Gamma_t}(E)}{S_{W_t,(D_W)_t - \epsilon\Gamma_t}(E)} > 1,$$

for all $\epsilon > 0$ small enough depending on t. Since $(W_t, (D_W)_t - \epsilon\Gamma_t)$ is a log Fano pair, we conclude by Theorem 3.2.2.3 that $(W_t, (D_W)_t - \epsilon\Gamma_t)$ is uniformly K-stable for all $\epsilon = \epsilon(t) > 0$ small enough.

- By openness of the uniform K-stable locus [BL18, Theorem 6.8], we conclude that: for all rational $0 < \epsilon \ll 1$, the general fiber of the family $(W, D_W - \epsilon\Gamma_W) \to T$ is uniformly K-stable.
- For a general $t \in T$, the pair (X_t, D_t) is a log canonical model of $(W_t, (D_W)_t)$. Thus $(W_t, (D_W)_t) \cong (W_u, (D_W)_u)$ implies that $(X_t, D_t) \cong (X_u, D_u)$. Moreover by Lemma 3.5.0.2, for ϵ small enough, $(W_t, (D_W - \epsilon\Gamma_W)_t) \cong (W_u, (D_W - \epsilon\Gamma_W)_u)$ if and only if $(W_t, (D_W)_t) \cong (W_u, (D_W)_u)$. Therefore $(W, D_W - \epsilon\Gamma_W) \to T$ has maximal variation for $0 < \epsilon \ll 1$.

This shows that $(W, D_W - \epsilon\Gamma_W) \to T$ has the required properties for all rational numbers $0 < \epsilon \ll 1$. □

3.6 Proof of the main result

This section is devoted to the proof of Theorem 3.1.0.2, which we divide into several steps. See Section 3.1.1 for an overview of the strategy. In Section 3.6.1, we set-up the notational framework of the proof. We use the ampleness lemma in Section 3.6.2 to obtain the positivity of some relevant sheaf. The estimates of the derivatives using the product trick is obtained in Section 3.6.3, and the perturbation argument is given in Section 3.6.4.

3.6.1 General notations

Notation 3.6.1.1. Let T be a smooth variety and $f\colon (X, D = \sum_{i=1}^N c_i D^i) \to T$ be a $\mathbb{Q}$-Gorenstein family of log Fano pairs of maximal variation with uniformly K-stable general geometric fibers. Here D^i is the part of coefficient c_i, see Notation 3.2.1.4. We introduce the following additional notations, and shall use them for the rest of this section.

(a) Let $n := \dim X - \dim T$ and $v := ((-K_{X/T} - D)|_{X_t})^n$. We write $\delta := \delta(X_{\bar\eta}, D_{\bar\eta})$ where η is the generic point of T. Then δ is the value of $\delta(X_t, D_t)$ for a very general point $t \in T$ [CP21, Proposition 4.15].

(b) We let $\lambda_{f,D} := -f_*((-K_{X/T} - D)^{n+1})$ be the CM line bundle provided by the family $f\colon (X, D) \to T$.

(c) The restrictions of the morphism f to the support of the D^i (with the reduced structure) are denoted $f_i\colon D^i \to T$. We also write $D^0 = X$ and $f_0 = f$. By Lemma 3.2.3.3, each f_i is surjective.

(d) We fix a rational number $\alpha > \max\{1, \frac{\delta}{(\delta-1)v(n+1)}\}$. Notice that by [CP21, Theorems 1.8 and 1.20], for any $\alpha' \geq \alpha$ the $\mathbb{Q}$-Cartier divisor $-K_{X/T} - D + \alpha' f^*\lambda_{f,D}$ is nef.

Notation 3.6.1.2. In the situation of Notation 3.6.1.1, let $\iota\colon C \to T$ be a smooth curve. Form the Cartesian square

$$\begin{array}{ccc} Z & \xrightarrow{\sigma} & X \\ \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle f} \\ C & \xrightarrow{\iota} & T \end{array}$$

Note that Z is normal because h is flat and its fibers are normal. Let D_Z be the divisorial pullback of D (see Definition 3.2.3.4), with coefficient parts D_Z^i. According to (2.3.4.a) and to Proposition 3.2.4.1, we have

$$K_{Z/C} + D_Z \sim_{\mathbb{Q}} \sigma^*(K_{X/T} + D), \quad \iota^*\lambda_{f,D} = \lambda_{h,D_Z}. \tag{6.1.2.c}$$

3.6.2 Application of the ampleness lemma

Proposition 3.6.2.1. *In the situation of Notation 3.6.1.1, for q divisible enough, the line bundle*

$$\bigotimes_{i\geq 0} \det(f_i)_*\mathcal{O}_{D^i}(q(-K_{X/T} - D + 2\alpha f^*\lambda_{f,D}))$$

is big on T.

The argument of the proof is inspired by [CP21, §9.4] and by [KP17, Theorem 7.1.1].

Proof. We let $V \subseteq U$ be the open subsets of T given by Lemma 3.2.3.6 and Lemma 3.2.3.7. By the maximal variation assumption, shrinking V if necessary, we may assume that for any $t \in V$, there are only finitely many $t' \in V$ such that $(X_t, D_t) \cong (X_{t'}, D_{t'})$.

If r is a fixed integer divisible by the Cartier index of $-K_{X/T} - D + 2\alpha f^*\lambda_{f,D}$, for an arbitrary $d \in \mathbb{Z}$ we write

$$\mathcal{M}_d := \mathcal{O}_X(dr(-K_{X/T} - D + 2\alpha f^*\lambda_{f,D})), \quad \mathcal{M}_d^{D^i} := \mathcal{M}_d|_{D^i}.$$

We choose an integer $r \geq 2$ such that for every $d > 0$:
(a) $-dr(K_{X/T} + D)$ and $dr\alpha \cdot \lambda_{f,D}$ are Cartier;
(b) $\mathcal{M}_d$ is f-very ample;
(c) $R^j f_*\mathcal{M}_d = 0$ for all $j \geq 1$;
(d) for each $i \geq 1$: $(R^j(f_i)_*\mathcal{M}_d^{D^i})|_V = 0$ for all $j \geq 1$;
(e) for each $i \geq 1$: $f_*\mathcal{M}_1 \to (f_i)_*\mathcal{M}_1^{D^i}$ is surjective on the open set V.
These conditions imply that $f_*\mathcal{M}_1$ and $((f_i)_*\mathcal{M}_1^{D^i})|_V$ are locally free and compatible with base-change. In particular,
(e) if $s := \mathrm{rk}\ f_*\mathcal{M}_1$, then $s = h^0(X_t, \mathcal{M}_1|_{X_t})$ for all $t \in T$.
We may also assume that:
(f) the multiplication maps

$$\mathrm{Sym}^d f_*\mathcal{M}_1 \to f_*\mathcal{M}_d \quad \text{and} \quad \mathrm{Sym}^d(f_*\mathcal{M}_1)|_V \to ((f_i)_*\mathcal{M}_d^{D_i})|_V$$

are surjective.
Now that r is chosen, we can find $d > 0$ such that:
(g) For all $t \in T$, the kernel

$$K_t := \ker\left[\mathrm{Sym}^d H^0(\mathcal{M}_1|_{X_t}) \longrightarrow H^0(\mathcal{M}_d|_{X_t})\right]$$

generates $\mathcal{I}_t(d)$, where $\mathcal{I}_t$ is the ideal sheaf of X_t for the embedding

$$\varphi_{\mathcal{M}_1|_{X_t}} \colon X_t \hookrightarrow \mathbb{P}^{s-1}.$$

Here $\varphi_{\mathcal{M}_1|_{X_t}}$ is only defined up to the action of $\mathrm{GL}_k(s)$ on the target. Hence, writing $w := \mathrm{rk}\ \mathrm{Sym}^d f_*\mathcal{M}_1$ and $q_0 := \mathrm{rk}\ f_*\mathcal{M}_d$, we see that the orbit of K_t in $\mathrm{Gr}_k(w, q_0)/\,\mathrm{GL}_k(s)$ determines the projective embedding $\varphi_{\mathcal{M}_1|_{X_t}}$ of X_t up to linear automorphisms of $\mathbb{P}^{s-1}$.
(h) Similarly, for all $v \in V$ and $i \geq 1$, the kernel

$$K_v^{D^i} := \ker\left[\mathrm{Sym}^d H^0(\mathcal{M}_1|_{X_v}) \longrightarrow H^0(\mathcal{M}_d^{D^i}|_{(D^i)_v})\right]$$

generates $\mathcal{I}_{v,i}(d)$, where $\mathcal{I}_{v,i}$ is the ideal sheaf of $(D^i)_v$ for the embedding

$$\varphi_{\mathcal{M}_1^{D^i}|_{(D^i)_v}} \colon (D^i)_v \hookrightarrow X_v \xrightarrow{\varphi_{\mathcal{M}_1|_{X_v}}} \mathbb{P}^{s-1},$$

Here $\varphi_{\mathcal{M}_1^{D_i}|_{(D_i)_v}}$ is only defined up to the action of $\mathrm{GL}_k(s)$ on the target. Hence, writing $q_i := \mathrm{rk}\ (f_i)_*\mathcal{M}_d^{D^i}$, we see that the orbit of $K_v^{D^i}$ in $\mathrm{Gr}_k(w, q_i)/\,\mathrm{GL}_k(s)$ determines the projective embedding of $(D^i)_v$ up to linear automorphisms of $\mathbb{P}^{s-1}$.

Having choosen r and d with these properties, we let

$$W := \mathrm{Sym}^d(f_*\mathcal{M}_1)|_U, \quad Q_0 := (f_*\mathcal{M}_d)|_U, \quad Q_i := \left((f_i)_*\mathcal{M}_d^{D^i}\right)\Big|_U \ (i \geq 1).$$

The sheaves W and Q_0 are locally free over U by construction. Since D^i is reduced the invertible sheaf $\mathcal{M}_d^{D^i}$ satisfies the Serre condition S_1, thus its sections are supported on entire components, and each component of D^i dominates T by Lemma 3.2.3.3. It follows that for $i \geq 1$, the $\mathcal{O}_T$-modules $(f_i)_*\mathcal{M}_d^{D^i}$ are torsion-free. Since T is normal, the $(f_i)_*\mathcal{M}_d^{D^i}$ are therefore locally free at codimension one points of T. Since they are also locally free over V, we may restrict U, but keeping it a big open set, so that the Q_i become locally free for all $i \geq 0$.

By construction there are morphisms $W \to Q_i$ for $i \geq 0$, defined over U, which are surjective over V. We claim that the corresponding classifying map

$$\xi\colon U(k) \longrightarrow \left(\prod_{i\geq 0} \mathrm{Gr}_k(w, q_i)\right) / \, \mathrm{GL}_k(s) \quad \text{(here } \mathrm{GL}_k(s) \text{ acts diagonally)}$$

is finite-to-one over $V(k)$. Fix $t \in V(k)$. The discussion in points (g) and (h) above shows the following. For $t' \in V(k)$, the equality $\xi(t) = \xi(t')$ holds if and only if: X_t and $X_{t'}$ have isomorphic embeddings into $\mathbb{P}^{s-1}$, and under this isomorphism $(D^i)_t$ is sent to $(D^i)_{t'}$ for every $i \geq 1$. As explained at the beginning of the proof, there are only finitely many such t'. Thus ξ is finite-to-one on $V(k)$. Hence by Theorem 3.3.0.1, given an ample line bundle B on T, there is a non-zero morphism

$$\mathrm{Sym}^{mq}\left(\bigoplus_{j=1}^{l} \mathrm{Sym}^d(f_*\mathcal{M}_1)\right)\Bigg|_U \longrightarrow \mathcal{O}_U(-B_U) \otimes \left(\bigotimes_{i\geq 0} \left(\det(f_i)_*\mathcal{M}_d^{D^i}\right)^{\otimes m}\right)\Bigg|_U$$

for some integers $l, m > 0$ (the precise value of l is given in Theorem 3.3.0.1, but it is not important for our purpose). Since U is a big dense open subset of T, and since both sides are restrictions of locally free sheaves, by reflexivity this map extends to a non-zero morphism

$$\mathrm{Sym}^{mq}\left(\bigoplus_{j=1}^{l} \mathrm{Sym}^d(f_*\mathcal{M}_1)\right) \longrightarrow \mathcal{O}_T(-B) \otimes \left(\bigotimes_{i\geq 0} \left(\det(f_i)_*\mathcal{M}_d^{D^i}\right)^{\otimes m}\right) \tag{6.2.1.d}$$

As the right-hand side is a line bundle, this map is generically surjective.

Now let $\iota\colon C \to T$ be a general smooth curve. We use the notations of Notation 3.6.1.2. As $f_*\mathcal{M}_1$ is compatible with base-change (see the beginning of the proof), we obtain

$$\iota^* f_*\mathcal{M}_1 \cong h_*\mathcal{O}_Z(\sigma^* rM) \cong h_*\mathcal{O}_Z(r(-K_{Z/C} - D_Z + 2\alpha h^*\lambda_{h,D_Z})).$$

As C is general, the general geometric fiber of $(Z, D_Z) \to C$ is uniformly K-stable. Thus by [CP21, Theorem 1.20], the divisor $-K_{Z/C} - D_Z + 2\alpha h^*\lambda_{h,D_Z}$ is h-ample and nef. Moreover, since we can write

$$r(-K_{Z/C} - D_Z + 2\alpha h^*\lambda_{h,D_Z}) = K_{Z/C} + D_Z + \underbrace{(r+1)(-K_{Z/C} - D_Z + \alpha h^*\lambda_{h,D_Z}) + (r-1)\alpha h^*\lambda_{h,D_Z}}_{\text{nef and } h\text{-ample}},$$

we may apply [CP21, Proposition 6.4] to obtain that $\iota^* f_*\mathcal{M}_1$ is nef. Hence the pullback of $\mathrm{Sym}^{mq}\left(\bigoplus_{j=1}^{l} \mathrm{Sym}^d(f_*\mathcal{M}_1)\right)$ to C is also nef. By generality of C, the restriction of the morphism (6.2.1.d) to C is generically surjective, so we obtain that

$$\mathcal{O}_T(-B)\Big|_C \otimes \left(\bigotimes_{i\geq 0} \left(\det(f_i)_*\mathcal{M}_d^{D^i}\right)^{\otimes m}\right)\Bigg|_C \text{ is nef for a general movable } C \to T.$$

This shows that the line bundle $\mathcal{O}_T(-B) \otimes \left(\bigotimes_{i\geq 0} \left(\det(f_i)_*\mathcal{M}_d^{D^i}\right)^{\otimes m}\right)$ is pseudo-effective. Thus $\bigotimes_{i\geq 0} \det(f_i)_*\mathcal{M}_d^{D^i}$ is big, and we conclude by letting $q = rd$. □

3.6.3 Estimation of the derivatives

Notation 3.6.3.1. In the situation of Notation 3.6.1.1:

(a) We fix a positive integer q such that the divisor $q(-K_{X/T} - D + 2\alpha f^*\lambda_{f,D})$ is Cartier, and write
$$\mathcal{N} = \mathcal{O}_X(q(-K_{X/T} - D + 2\alpha f^*\lambda_{f,D})) \quad \text{and} \quad \mathcal{N}^{D^i} = \mathcal{N}|_{D^i}.$$

(b) According to Proposition 3.6.2.1, we may and will choose q such that
$$\bigotimes_{i\geq 0} \det(f_i)_*\mathcal{N}^{D^i} \text{ is a big line bundle.}$$

(c) If $C \to T$ is a smooth curve and $Z = X \times_T C$ as in Notation 3.6.1.2, then we let $\mathcal{N}_Z := \sigma^*\mathcal{N}$. Recall from (6.1.2.c) that $\mathcal{N}_Z \cong \mathcal{O}_Z(q(-K_{Z/C} - D_Z + 2\alpha h^*\lambda_{f,D_Z}))$, and this line bundle is nef if C meets the open locus of uniformly K-stable fibers [CP21, Theorem 1.20].

We aim to give a lower bound to the intersection numbers $(\mathcal{N}_Z)^{\dim D^i_Z} \cdot D^i_Z$. As explained in Section 3.1.1, the idea is to construct a product $D^{(r\bullet)}$ over T and then base-change over a general curve. In view of Lemma 3.7.0.5, we want the pullback of $D^{(r\bullet)}$ to be flat over that curve. Hence it would be convenient that the restricted morphisms $D^i \to T$ are flat already. To achieve this, we pass to a birational model of X. Unfortunately this makes the notation quite cumbersome.

Notation 3.6.3.2. In the situation of Notation 3.6.3.1. Let r_i be the generic rank of $(f_i)_*\mathcal{N}^{D^i}$ $(i = 0, \ldots, N)$.

(a) We let
$$D^{(r\bullet)} := \left(D^0\right)^{(r_0)} \times_T \cdots \times_T \left(D^N\right)^{(r_N)}.$$
The projection morphism from $D^{(r\bullet)}$ to the i^{th} D^j-factor is denoted by $p^{ij} \colon D^{(r\bullet)} \to D^j$. We denote
$$D^{(r\bullet)}_{\text{red}} := \left(D^{(r\bullet)}\right)_{\text{red}}, \quad D^{(r\bullet)}_{\text{norm}} := \text{normalization of } D^{(r\bullet)}_{\text{red}}.$$
We denote by $g \colon D^{(r\bullet)} \to T$, $g_{\text{red}} \colon D^{(r\bullet)}_{\text{red}} \to T$ and $g_{\text{norm}} \colon D^{(r\bullet)}_{\text{norm}} \to T$ the structural morphisms.

We define the line bundles
$$\mathcal{N}^{(r\bullet)} = \bigotimes_{i,j} \left(p^{ij}\right)^* \mathcal{N}^{D^j}$$
and
$$\mathcal{N}^{(r\bullet)}_{\text{red}} := \text{pullback of } \mathcal{N}^{(r\bullet)} \text{ to } D^{(r\bullet)}_{\text{red}}, \quad \mathcal{N}^{(r\bullet)}_{\text{norm}} := \text{pullback of } \mathcal{N}^{(r\bullet)} \text{ to } D^{(r\bullet)}_{\text{norm}}.$$

(b) Next we fix a small $\mathbb{Q}$-factorial proper model $\mu \colon Y \to X$. There exists one since X has a klt structure, see [Kol13, 1.37]. Denote by $D_Y = \sum_{i=1}^N c_i D^i_Y$ the strict transform of D. Let $\mu^i \colon D^i_Y \to D^i$ be the induced birational morphisms, with $\mu^0 = \mu$. We write
$$D^{(r\bullet)}_Y := \left(D^0_Y\right)^{(r_0)} \times_T \cdots \times_T \left(D^N_Y\right)^{(r_N)}.$$
The projection morphism from $D^{(r\bullet)}_Y$ to the i^{th} D^j_Y-factor is denoted by $p^{ij}_Y \colon D^{(r\bullet)}_Y \to D^j_Y$.

We define the line bundles
$$\mathcal{N}^{D^i}_Y := (\mu^i)^*\mathcal{N}^{D^i} \quad \text{and} \quad \mathcal{N}^{(r\bullet)}_Y = \bigotimes_{i,j} \left(p^{ij}_Y\right)^* \mathcal{N}^{D^j_Y}_Y.$$

(c) If $\iota\colon C \to T$ is a smooth curve, we denote by Y_C, respectively $D^i_{Y_C}, D^{(r_\bullet)}_{Y_C}, \mathcal{N}^{(r_\bullet)}_{Y_C}, \mu^i_C, p^{ij}_{Y_C}$, the scheme-theoretic pullbacks along ι of Y, respectively $D^i_Y, D^{(r_\bullet)}_Y, \mathcal{N}^{(r_\bullet)}_Y, \mu^i, p^{ij}_Y$. Notice that

$$D^{(r_\bullet)}_{Y_C} = D^{(r_\bullet)}_Y \times_T C = \left(D^0_{Y_C}\right)^{(r_0)} \times_C \cdots \times_C \left(D^N_{Y_C}\right)^{(r_N)}$$

and that the projection morphisms $D^{(r_\bullet)}_{Y_C} \to D^j_{Y_C}$ are exactly the $p^{ij}_{Y_C}$. Notice also that if $\mathcal{N}^{D^j_{Y_C}}$ is the pullback of $\mathcal{N}^{D^j_Y}$ along $D^j_{Y_C} \to D^j_Y$, then

$$\mathcal{N}^{(r_\bullet)}_{Y_C} \cong \bigotimes_{i,j} \left(p^{ij}_{Y_C}\right)^* \mathcal{N}^{D^j_{Y_C}}.$$

The construction of parts (a) and (b) is summarized by the following diagram, where the arrow $D^{(r_\bullet)}_Y \to D^{(r_\bullet)}_{\mathrm{red}}$ exists by Lemma 3.6.3.3(c) given below.

$$\begin{tikzcd}
\left(D^j_Y, \mathcal{N}^{D^j_Y}\right) \arrow[dd, "\mu^j"'] & \left(D^{(r_\bullet)}_Y, \mathcal{N}^{(r_\bullet)}_Y\right) \arrow[l, "p^{ij}_Y"'] \arrow[dd, "\mu^{(r_\bullet)}"'] \arrow[dr] & & \left(D^{(r_\bullet)}_{\mathrm{norm}}, \mathcal{N}^{(r_\bullet)}_{\mathrm{norm}}\right) \arrow[dl] \\
& & \left(D^{(r_\bullet)}_{\mathrm{red}}, \mathcal{N}^{(r_\bullet)}_{\mathrm{red}}\right) \arrow[dl] & \\
\left(D^j, \mathcal{N}^{D^j}\right) & \left(D^{(r_\bullet)}, \mathcal{N}^{(r_\bullet)}\right) \arrow[l, "p^{ij}"'] & &
\end{tikzcd}$$

Next we establish some properties of these product varieties and these product line bundles.

Lemma 3.6.3.3. *In the situation of Notation 3.6.3.2:*

(a) $D^{(r_\bullet)}$ *is equidimensional over* T*, and every component dominates* T*; moreover there is a big open set of* T *over which* $D^{(r_\bullet)}$ *is flat and reduced;*

(b) $D^{(r_\bullet)}_Y$ *is reduced, flat and equidimensional over* T*, and every components dominates* T*;*

(c) $\mu^{(r_\bullet)}\colon D^{(r_\bullet)}_Y \to D^{(r_\bullet)}$ *factors through* $D^{(r_\bullet)}_{red}$*;*

(d) $\mu^{(r_\bullet)}$ *is an isomorphism over each generic point of* $D^{(r_\bullet)}$*, and every component of* $D^{(r_\bullet)}_Y$ *dominates a component of* $D^{(r_\bullet)}$*;*

(e) $D^{(r_\bullet)}_{Y_C}$ *is flat equidimensional over* C*, and it is reduced if* C *is general movable.*

Proof. Assertion (a) is proved in [KP17, Lemma 7.11], and assertion (c) will follow immediately from assertion (b).

The pair (X, D) is klt by Lemma 3.5.0.3, so (Y, D_Y) is klt and hence Y is Cohen-Macaulay. The divisors D^i_Y are $\mathbb{Q}$-Cartier because Y is $\mathbb{Q}$-factorial, so each D^i_Y is also Cohen-Macaulay [KM98, 5.25]. Hence all the morphisms $D^i_Y \to T$ are flat [Mat89, Theorem 23.1]. This implies that the morphism $D^{(r_\bullet)}_Y \to T$ is flat. The fibers of $D^{(r_\bullet)}_Y \to T$ have the same dimension, so the morphism is equidimensional. If one component of $D^{(r_\bullet)}_Y$ does not dominate T, then it belongs to the non-flat locus, which is empty. The generic fiber if reduced, so Lemma 3.7.0.3 implies that $D^{(r_\bullet)}_Y$ is reduced. This proves the assertion (b), and assertion (e) is proved similarly.

To conclude, we must prove assertion (d). Let $V \subseteq X$ be the open subset over which $\mu\colon Y \to X$ is an isomorphism. Since X is normal, V is big. Thus $V_i := V \cap D^i$ is dense in D^i for all i, and $\mu^{(r_\bullet)}$ is an isomorphism over the open set $\mathcal{V} := V^{(r_0)} \times_T V_1^{(r_1)} \times_T \cdots \times_T V_N^{(r_N)} \subset D^{(r_\bullet)}$. Assume that a generic point η of $D^{(r_\bullet)}$ does not belong to $\mathcal{V}$. Then η belongs to a product of the form $\left(D^0\right)^{(r_0)} \times_T \cdots \times_T \left(D^N\right)^{(r_N)}$ with one factor D^i replaced by $D^i - V_i$. Such a product has dimension strictly smaller than $D^{(r_\bullet)}$ and so $\dim \overline{\{\eta\}} < \dim D^{(r_\bullet)}$, which contradicts equidimensionality. Thus $\eta \in \mathcal{V}$. Similarly, a component of $D_Y^{(r_\bullet)}$ that is contracted by $\mu^{(r_\bullet)}$ must belong to a product of the form $\left(D_Y^0\right)^{(r_0)} \times_T \cdots \times_T \left(D_Y^N\right)^{(r_N)}$ with one factor D_Y^i replaced by $D_Y^i - f_i^{-1} V_i$. Such a product has dimension strictly smaller than $D_Y^{(r_\bullet)}$, which contradicts the equidimensionality of $D_Y^{(r_\bullet)}$. So assertion (d) follows. □

Lemma 3.6.3.4. *In the situation of Notation 3.6.3.2,*

(a) $\mathcal{N}_{\mathrm{norm}}^{(r_\bullet)}$ is relatively ample over T and pseudo-effective;

(b) for a general movable curve $C \to T$, the line bundle $\mathcal{N}_{Y_C}^{(r_\bullet)}$ is nef.

Proof. Let $\mathcal{U} \subset T$ be a non-empty open subset with the property that for all $t \in T$, the pair (X_t, D_t) is uniformly K-stable. Let $C \to T$ be a smooth curve whose image intersects $\mathcal{U}$. Denote by $\mathcal{N}_Z$ the pullback of $\mathcal{N}$ on $Z = X \times_T C$, and $\mathcal{D}_C^i = D^i \times_C T$. Then by [CP21, Theorem 1.20], $\mathcal{N}_Z$ and $\mathcal{N}^{\mathcal{D}_C^i} = \mathcal{N}_Z|_{\mathcal{D}_C^i}$ are nef. Thus the product line bundle

$$\mathcal{N}_Z^{(r_\bullet)} = \bigotimes_{i,j} \left(p_C^{ij}\right)^* \mathcal{N}^{D_C^i} \quad \text{on } D^{(r_\bullet)} \times_T C$$

is nef. By definition $\mathcal{N}_{Y_C}^{(r_\bullet)}$ is a pullback of $\mathcal{N}_Z^{(r_\bullet)}$, so it is also nef. This proves the second assertion.

By construction, the line bundle $\mathcal{N}^{(r_\bullet)}$ is relatively ample over T. Let $\tilde{C} \to D^{(r_\bullet)}$ be a smooth curve that meets $g^{-1}\mathcal{U}$ (recall that $g\colon D^{(r_\bullet)} \to T$ is the structural morphism). If $\tilde{C}$ is contracted by g, then $\mathcal{N}^{(r_\bullet)} \cdot \tilde{C} > 0$ by relative ampleness. Otherwise, let $C \to T$ be the normalization of $g(\tilde{C})$. Then $\tilde{C} \to D^{(r_\bullet)}$ factors through $D^{(r_\bullet)} \times_T C$, on which the pullback of $\mathcal{N}^{(r_\bullet)}$ is nef. Thus $\tilde{C} \cdot \mathcal{N}^{(r_\bullet)} \geq 0$. This shows that $\mathcal{N}^{(r_\bullet)}$ is pseudo-effective. Since $\mathcal{N}_{\mathrm{norm}}^{(r_\bullet)}$ is the pullback of $\mathcal{N}^{(r_\bullet)}$ through the finite morphism $D_{\mathrm{norm}}^{(r_\bullet)} \to D^{(r_\bullet)}$, the first assertion follows. □

We are now ready to estimate the intersection numbers $(\mathcal{N}_Z)^{\dim D_Z^i} \cdot D_Z^i$ (where $Z = X \times_T C$). The first part of the proof is similar to the proof of [KP17, 7.1.1].

Proposition 3.6.3.5. *In the situation of Notation 3.6.3.2, there are an ample Cartier divisor A on some irreducible component P of $D_Y^{(r_\bullet)}$ and a rational number $e = e(X, D, q) > 0$ with the following property: for every general movable curve $C \to T$, letting A_C be the pullback of A to $P \times_T C$, it holds that*

$$e(N+1) \cdot \mathrm{Vol}(A_C) \leq \sum_{j=0}^{N} (\mathcal{N}_Z)^{\dim D_Z^j} \cdot D_Z^j.$$

In particular, for every general movable curve $C \to T$,

$$\exists\, j = j(C) \geq 0 \quad : \quad (\mathcal{N}_Z)^{\dim D_Z^j} \cdot D_Z^j \geq e \cdot \mathrm{Vol}(A_C). \tag{6.3.5.e}$$

Proof. By Lemma 3.6.3.3, the morphism $g\colon D^{(r_\bullet)} \to T$ is equidimensional and every component dominates T. Thus $g_{\mathrm{red}}\colon D_{\mathrm{red}}^{(r_\bullet)} \to T$ and $g_{\mathrm{norm}}\colon D_{\mathrm{norm}}^{(r_\bullet)} \to T$ are also equidimensional morphisms, and any components of $D_{\mathrm{red}}^{(r_\bullet)}$ or $D_{\mathrm{norm}}^{(r_\bullet)}$ dominates T.

By Lemma 3.6.3.3 and by the proof of Proposition 3.6.2.1, there is a big open subset $U \subseteq T$ over which $D^{(r_\bullet)}$ is flat and reduced, and the sheaves $(f_i)_*\left(\mathcal{N}^{D^i}\right)|_U$ are locally free. Let $U^0 := g^{-1}U$. Since U^0 is reduced, it embeds as an open subset of $D_{\mathrm{red}}^{(r_\bullet)}$. Therefore the open set U^0 is big in $D_{\mathrm{red}}^{(r_\bullet)}$ and meets every component, and so the preimage of U^0 in $D_{\mathrm{norm}}^{(r_\bullet)}$ is big and meets every component.

$$\begin{tikzcd}
 & (D^{(r\bullet)}_{\mathrm{norm}}, \mathcal{N}^{(r\bullet)}_{\mathrm{norm}}) \arrow[d] \\
 & (D^{(r\bullet)}_{\mathrm{red}}, \mathcal{N}^{(r\bullet)}_{\mathrm{red}}) \arrow[d] \\
U^0 \arrow[ur, hook] \arrow[r, hook] \arrow[d] & (D^{(r\bullet)}, \mathcal{N}^{(r\bullet)}) \arrow[d, "g"] \\
U \arrow[r, hook] & T
\end{tikzcd}$$

Let us write $\mathcal{L} := \bigotimes_{i \geq 0} \det(f_i)_* \mathcal{N}^{D^i}$. On U, there is an embedding

$$\mathcal{L}|_U \hookrightarrow \bigotimes_{i \geq 0} \bigotimes_{j=1}^{r_i} (f_i)_* \left(\mathcal{N}^{D^i}|_U \right) \cong (g|_{U^0})_* \left(\mathcal{N}^{(r\bullet)}|_{U^0} \right)$$

where the first arrow is given by the natural embedding of det into the appropriate tensor power, and the isomorphism is given by [KP17, Lemma 3.6]. By adjunction, we obtain a morphism

$$(g_{\mathrm{red}}|_{U^0})^* \mathcal{L}|_U = (g|_{U^0})^* \mathcal{L}|_U \longrightarrow \mathcal{N}^{(r\bullet)}|_{U^0} = \mathcal{N}^{(r\bullet)}_{\mathrm{red}}|_{U^0}. \tag{6.3.5.f}$$

Since U^0 dominates T, the map (6.3.5.f) is non-zero. We may pull back this map to the normalization $D^{(r\bullet)}_{\mathrm{norm}}$. Since the preimage U^0 in $D^{(r\bullet)}_{\mathrm{norm}}$ is big, by reflexivity this pullback morphism extends to a non-zero morphism

$$(g_{\mathrm{norm}})^* \mathcal{L} \longrightarrow \mathcal{N}^{(r\bullet)}_{\mathrm{norm}}$$

which induces a non-zero map

$$\mathcal{N}^{(r\bullet)}_{\mathrm{norm}} \otimes (g_{\mathrm{norm}})^* \mathcal{L} \longrightarrow \left(\mathcal{N}^{(r\bullet)}_{\mathrm{norm}} \right)^{\otimes 2}. \tag{6.3.5.g}$$

The line bundle $\mathcal{N}^{(r\bullet)}_{\mathrm{norm}}$ is relatively ample over T and pseudo-effective by Lemma 3.6.3.4. Moreover $(g_{\mathrm{norm}})^* \mathcal{L}$ is the pullback of a big divisor by Proposition 3.6.2.1. Thus the left-hand side of (6.3.5.g) is big on every component by Lemma 3.7.0.2. Hence $\mathcal{N}^{(r\bullet)}_{\mathrm{norm}}$ is big on at least one component. This implies that $\mathcal{N}^{(r\bullet)}_{\mathrm{red}}$ is big on at least one component.

By Lemma 3.6.3.3, $\mu^{(r\bullet)} \colon D_Y^{(r\bullet)} \to D^{(r\bullet)}$ is an isomorphism in codimension zero and factors through $D^{(r\bullet)}_{\mathrm{red}}$. Hence we obtain that $\mathcal{N}_Y^{(r\bullet)}$ is big on one component P of $D_Y^{(r\bullet)}$. So we may write

$$\left(\mathcal{N}_Y^{(r\bullet)}|_P \right)^{\otimes m} \cong \mathcal{O}_P(mA + mE)$$

where A is ample and E is effective on P, for some $m > 0$.

Now we fix a general movable curve $C \to T$ with the following properties. Firstly, no component of its preimage in P is contained in the support of E. Secondly, the line bundles $\mathcal{N}_Z$ and $\mathcal{N}_{Y_C}^{(r\bullet)}$ are nef (see Lemma 3.6.3.4). Thirdly, the induced morphisms $\mu_C^j \colon D^j_{Y_C} \to \mathcal{D}_Z^j := D^j \times_T C$ are birational for all j; this is acheviable since all $\mu^j : D_Y^j \to D^j$ are birational. Fourthly, for all j the product $\mathcal{D}_Z^j$ agrees with the j-coefficient part D_Z^j of D_Z in codimension one; this is achievable by combining Lemma 3.2.3.5 and Corollary 3.2.3.8. Finally, $D_{Y_C}^{(r\bullet)}$ is equidimensional and reduced (Lemma 3.6.3.3.(e)).

With such a curve $C \to T$ fixed, we write

$$\mathcal{O}_{P_C}(mA_C + mE_C) \cong \left(\mathcal{N}_{Y_C}^{(r\bullet)} \Big|_{P_C} \right)^{\otimes m} \cong \left(\bigotimes_{i,j} \left(p^{ij}_{Y_C} \right)^* \mathcal{N}^{D^j_{Y_C}} \right) \Big|_{P_C}^{\otimes m}.$$

Since A is ample, its pullback A_C is also ample. By our choice of C, the divisor E_C is effective. Thus
$$\operatorname{Vol}(A_C) \leq \operatorname{Vol}(A_C + E_C) = \left(\mathcal{N}^{(r_\bullet)}_{Y_C}\Big|_{P_C} \right)^{\dim P_C}.$$
It holds by equidimensionality that $\dim P_C = \dim D^{(r_\bullet)}_{Y_C}$. Using [Kol96, VI.2.7.3], we see that
$$\left(\mathcal{N}^{(r_\bullet)}_{Y_C} \right)^{\dim D^{(r_\bullet)}_{Y_C}} = \left(\mathcal{N}^{(r_\bullet)}_{Y_C}\Big|_{P_C} \right)^{\dim D^{(r_\bullet)}_{Y_C}} + \sum_{P'} \left(\mathcal{N}^{(r_\bullet)}_{Y_C}\Big|_{P'} \right)^{\dim D^{(r_\bullet)}_{Y_C}}$$
where P' runs through the component of $D^{(r_\bullet)}_{Y_C}$ not contained in P_C. Since $\mathcal{N}^{(r_\bullet)}_{Y_C}$ is nef, the sum over P' is non-negative, and therefore we obtain
$$\operatorname{Vol}(A_C) \leq \left(\mathcal{N}^{(r_\bullet)}_{Y_C} \right)^{\dim D^{(r_\bullet)}_{Y_C}} \tag{6.3.5.h}$$
On the other hand, by Lemma 3.7.0.5 we have:
$$\left(\mathcal{N}^{(r_\bullet)}_{Y_C} \right)^{\dim D^{(r_\bullet)}_{Y_C}} = \sum_{i=0}^{N} d_i \left(\mathcal{N}^{D^j_{Y_C}} \right)^{\dim D^j_{Y_C}} \prod_{j \neq i} \left(\mathcal{N}^{D^j_{Y_C}}_t \right)^{\dim D^j_{Y_C} - 1} \tag{6.3.5.i}$$
for some rational numbers $d_i = d_i(X, D, q) > 0$ and any closed point $t \in C$. The right-hand side of (6.3.5.i) can be simplified: observe that
$$\left(\mathcal{N}^{D^j_{Y_C}} \right)^{\dim D^j_{Y_C}} = \left(\mathcal{N}^{\mathcal{D}^j_Z} \right)^{\dim \mathcal{D}^j_Z} = \left(\mathcal{N}^{D^j_Z} \right)^{\dim D^j_Z} = (\mathcal{N}_Z)^{\dim D^j_Z} \cdot D^j_Z.$$
Indeed, the first equality holds because μ^i_C is birational, while the second equality holds because $\mathcal{D}^j_Z$ and D^j_Z are equal in codimension 1 and $\mathcal{N}$ has full support (see [Kol96, VI.2.7.3]). Similarly to the previous displayed equalities, we also have
$$\left(\mathcal{N}^{D^j_{Y_C}}_t \right)^{\dim D^j_{Y_C} - 1} = (\mathcal{N}^{D^j_{Y_C}})^{\dim D^j_{Y_C} - 1} \cdot (D_{Y_C})_t = (\mathcal{N}_Z)^{\dim D^j_Z - 1} \cdot (D^j_Z)_t$$
for $t \in C$ closed. Since $\mathcal{N}_Z$ is the pullback of a relatively ample line bundle over T, the $(\mathcal{N}_Z)^{\dim D^j_Z - 1} \cdot (D^j_Z)_t$ are positive. Moreover $\mathcal{N}_Z$ is nef so the quantities $(\mathcal{N}_Z)^{\dim D^j_Z} \cdot D^j_Z$ are non-negative. Therefore, setting $a = \max_j \left\{ (\mathcal{N}_Z)^{\dim D^j_Z - 1} \cdot (D^j_Z)_t \right\}$ and $b = \max_i \{d_i\}$, it follows from (6.3.5.h) and (6.3.5.i) that
$$\operatorname{Vol}(A_C) \leq \left(\mathcal{N}^{(r_\bullet)}_{Y_C} \right)^{\dim D^{(r_\bullet)}_{Y_C}} \leq ab \sum_{i=0}^{N} (\mathcal{N}_Z)^{\dim D^i_Z} \cdot D^i_Z.$$
Notice that ab depends only on (X, D) and q, so we put $e = (ab(N+1))^{-1}$ to conclude. ☐

We have now a lower bound on the derivatives $(\mathcal{N}_Z)^{\dim D^i_Z} \cdot D^i_Z$ in terms of the volume of the pullback divisor A_C. Of course, this depends on the curve $C \to T$, but it is possible to obtain some kind of uniformity. Indeed, the next lemma shows that $\operatorname{Vol}(A_C)$ cannot converge to zero when $[C]$ gets close to the boundary of the movable cone.

Lemma 3.6.3.6. *Let A be the ample $\mathbb{Q}$-Cartier divisor on the component P of $D^{(r_\bullet)}_Y$ given by Proposition 3.6.3.5. Then there exists a big $\mathbb{Q}$-Cartier divisor Ψ on T such that for a general movable curve $C \to T$, we have $\operatorname{Vol}(A_C) = \Psi \cdot C$.*

Proof. By Lemma 3.6.3.3 the scheme P is reduced and the morphism $P \to T$ is equidimensional, say of relative dimension d. So we may apply Proposition 3.4.0.1 to $(P, A) \to T$. Namely, let A' be the pullback of A to the normalization P' of P, and let $f' \colon P' \to T$ be the induced morphism. Then for a general smooth curve $C \to T$, we have $\operatorname{Vol}(A_C) = A_C^{d+1} = f'_*(A')^{d+1} \cdot C$. By Lemma 3.7.0.1, $f'_*(A')^{d+1}$ is big. We take $\Psi = f'_*(A')^{d+1}$. ☐

3.6.4 Variation of the boundary

Notation 3.6.4.1. In this subsection, we follow Notation 3.6.3.1 and let Ψ be the big $\mathbb{Q}$-Cartier divisor on T obtained in Lemma 3.6.3.6.

Given a general smooth curve $C \to T$, the inequality (6.3.5.e) gives a lower bound for some intersection number $(\mathcal{N}_Z)^{\dim D^j_Z} \cdot D^j_Z$. As explained in Section 3.1.1, we wish to derive a lower bound on $\lambda_{f,D} \cdot C$. The case $j = 0$, corresponding to $D^0 = X$, is the easiest.

Proposition 3.6.4.2 (Case $j = 0$)**.** *Let $C \to T$ be a smooth curve. Assume that $(\mathcal{N}_Z)^{n+1} \geq e \cdot (\Psi \cdot C)$ for some $e > 0$. Then $\lambda_{f,D} \cdot C \geq e_0 \cdot (\Psi \cdot C)$ for some rational number $e_0 = e_0(X, D, q, e) > 0$.*

Proof. Recall the fact that $\mathcal{N}_Z \cong \mathcal{O}_Z(q(-K_{Z/C} - D_Z + 2\alpha h^* \lambda_{f,D_Z}))$ (see Notation 3.6.3.1). We have

$$\begin{aligned}
\frac{1}{q^{n+1}} (\mathcal{N}_Z)^{n+1} &= (-K_{Z/C} - D_Z)^{n+1} + (n+1)2\alpha(h^* \lambda_{h,D_Z} \cdot (-K_{Z/C} - D_Z)^n) \\
&= \deg_C h_*((-K_{Z/C} - D_Z)^{n+1}) + (n+1)2\alpha \cdot \deg \lambda_{h,D_Z} \cdot ((-K_{Z/C} - D_Z)_t)^n) \\
&= \deg_C \lambda_{h,D_Z} \cdot [-1 + 2\alpha v(n+1)] \\
&= (\lambda_{f,D} \cdot C) \cdot \left[\underbrace{-1 + 2\alpha v(n+1)}_{>0 \text{ by choice of } \alpha} \right]
\end{aligned}$$

We let $e_0 = e(q^{n+1}[2\alpha v(n+1) - 1])^{-1}$ to obtain the desired inequality. $\square$

If $j > 0$, we wish to relate $(\mathcal{N}_Z)^{\dim D^j_Z} \cdot D^j_Z$ to the a first-order derivative of $\lambda_{f,D} \cdot C$ as the component D^j_Z is perturbed. Since D^j_Z might not be $\mathbb{Q}$-Cartier, we introduce a birational model where it is $\mathbb{Q}$-Cartier.

Notation 3.6.4.3. In the situation of Notation 3.6.4.1. By Proposition 3.5.0.4, we may fix $r_{X,D} \in (0; 1)$ with the property that for every coefficient part D^i of D, there is a small birational proper morphism $W_i \to X$ such that for all rational numbers $\epsilon \in (0; r_{X,D})$, the family $(W_i, D_{W_i} - \epsilon D^i_{W_i}) \to T$ is a $\mathbb{Q}$-Gorenstein family of log Fano pairs of maximal variation with uniformly K-stable general geometric fibers.

Fix an index $j > 0$ and any smooth curve $\iota \colon C \to T$. Write $\nu \colon W := W_j \to X$ and $\Gamma := D^j$. We let

$$V := W \times_X Z \cong W \times_T C.$$

Together with the notations of Notation 3.6.1.2 we obtain the diagram

$$\begin{tikzcd}
(V, D_V, \Gamma_V) \arrow[r, "\tau"] \arrow[d, "\mu"] \arrow[dd, bend right=60, "g"'] & (W, D_W, \Gamma_W) \arrow[d, "\nu"'] \arrow[dd, bend left=60, "f'"] \\
(Z, D_Z, D^j_Z) \arrow[r, "\sigma"] \arrow[d, "h"] & (X, D, \Gamma) \arrow[d, "f"'] \\
C \arrow[r, "\iota"] & T
\end{tikzcd}$$

where D_Z is the divisorial base-change of D, D^j_Z the c_j-coefficient part of D_Z, and where the other $\mathbb{Q}$-divisors are defined as follows:

(a) let D_W and Γ_W be the ν-strict transforms of respectively D and Γ;

(b) let D_V be defined on V by the equality $K_{V/C} + D_V = \tau^*(K_{W/T} + D_W)$;

(c) let Γ_V be defined on V by $\Gamma_V = \tau^* \Gamma_W$. In particular, it is $\mathbb{Q}$-Cartier.

Define the polynomial $F(C,j) \in \mathbb{R}[t]$ by

$$F(C,j)(t) := (-K_{V/C} - D_V + t\Gamma_V)^{\dim V} = (-K_{V/C} - D_V + t\Gamma_V)^{n+1}.$$

We will also use the ad hoc notation

$$\mathfrak{m}_2\left(\sum_{l=0}^{r} a_l t^l\right) := \max_{l\geq 2}\{|a_l|\} \quad \text{where} \quad \sum_l a_l t^l \in \mathbb{R}[t].$$

Lemma 3.6.4.4. *In the situation of Notation 3.6.4.3. If $C \to T$ is general movable, then for every $0 < \epsilon < r_{X,D}$, the family $(V, D_V - \epsilon\Gamma_V) \to C$ is a $\mathbb{Q}$-Gorenstein family of log Fanos of maximal variation with uniformly K-stable general general geometric fibers.*

Proof. For all rational $0 < \epsilon < r_{X,D}$, the family $(W, D_W - \epsilon\Gamma_W) \to T$ is a $\mathbb{Q}$-Gorenstein family of log Fanos of maximal variation with uniformly K-stable general geometric fibers. Since the fibers are normal and C is normal, V is also normal. If C meets the open locus where the fibers are uniformly K-stable and of maximal variation, then the statement holds. □

We need to relate the intersection products one can do on V, to the intersection products one can do on Z and on T. This is the purpose of the next three lemmas.

Lemma 3.6.4.5. *In the situation of Notation 3.6.4.3,*
(a) $\mu^*(K_{Z/C} + D_Z) \sim_{\mathbb{Q}} K_{V/C} + D_V$.
Moreover, if $C \to T$ is a general movable curve, then
(b) $\mu\colon V \to Z$ is small birational;
(c) D_V, Γ_V are the strict transforms of D_Z and D_Z^j respectively;
(d) Γ_V is the divisorial pullback of Γ_W and the c_j-coefficient part of D_V.

Proof. By (6.1.2.c) we have $\sigma^*(K_{X/T} + D) \sim_{\mathbb{Q}} K_{Z/C} + D_Z$, and since $\nu\colon W \to X$ is small we have $K_{W/T} + D_W \sim_{\mathbb{Q}} \nu^*(K_{X/T} + D)$. By definition of D_V, we obtain that $K_{V/C} + D_V \sim_{\mathbb{Q}} \mu^*(K_{Z/C} + D_Z)$. This proves part (a).

By Proposition 3.5.0.4, the morphism $W_t \to X_t$ is small birational for a general $t \in T$. So if C meets the open locus of such $t \in T$, the morphism $\mu\colon V \to Z$ is birational and small as well. In this case D_V, Γ_V are the strict transforms of D_Z and D_Z^j. By Corollary 3.2.3.8, if C is general movable then D_Z^j is the divisorial pullback of Γ, and the c_j-coefficient part of D_Z. So Γ_V is the c_j-coefficient part of D_V and the divisorial pullback of Γ_W. This proves parts $(b), (c)$ and (d). □

Lemma 3.6.4.6. *In the situation of Notation 3.6.4.3, if $C \to T$ is general movable, we have*

$$(-K_{V/C} - D_V)^{n+1} = (-K_{Z/C} - D_Z)^{n+1} \quad \textit{and} \quad (-K_{V/C} - D_V)^n \cdot \Gamma_V = (-K_{Z/C} - D_Z)^n \cdot D_Z^j.$$

Proof. We use Lemma 3.6.4.5. By part (a), it holds that $K_{V/C} + D_V \sim_{\mathbb{Q}} \mu^*(K_{Z/C} + D_Z)$. If $C \to T$ is general movable, μ is birational by part (b). So the first equality follows. By part (c), the morphism μ restricts to a birational morphism $\mu|_{\Gamma_V}\colon \Gamma_V \to D_Z^j$, and the second equality follows. □

Lemma 3.6.4.7. *In the situation of Notation 3.6.4.3, for each $j > 0$ there is a non-empty finite collection of $\mathbb{Q}$-Cartier divisors $\{\Upsilon_{j,l}\}_{l=2}^{n+1}$ on T such that $\mathfrak{m}_2(F(C,j)) = \max_{l\geq 2}\{|\Upsilon_{j,l}\cdot C|\}$ for any smooth curve $C \to T$.*

Proof. Fix an index $j > 0$ and let $(f'\colon W \to T, D_W, \Gamma_W)$ be as in Notation 3.6.4.3. The fibers of f' are normal, and for small positive values of ϵ, the $\mathbb{Q}$-Cartier divisor $-K_{W/T} - D_W + \epsilon\Gamma_W$ is relatively ample over T. Thus by Proposition 3.4.0.1, for any smooth curve $C \to T$ we have:

$$f'_*(-K_{W/T} - D_W + \epsilon\Gamma_W)^{n+1} \cdot C = (-K_{V/C} - D_V + \epsilon\Gamma_V)^{n+1} = F(C,j)(\epsilon).$$

Write $f'_*(-K_{W/T} - D_W + \epsilon\Gamma_W)^{n+1} = \sum_{l=0}^{n+1} \epsilon^l \Upsilon_{j,l}$ in the Chow ring of T. By linearity of the intersection product and of f'_*, one can describe $\Upsilon_{j,l}$ as a multiple of the pushforwards along f' of the intersection $(-K_{W/T} - D_W)^{n+1-l} \cdot \Gamma_W^l$. We obtain

$$\mathfrak{m}_2(F(C,j)) = \max_{l\geq 2}\{|\Upsilon_{j,l} \cdot C|\}$$

as claimed. Notice that the family $\{\Upsilon_{j,l}\}_{l=2}^{n+1}$ is non-empty since $n \geq 1$. □

We are now able to treat the case $j > 0$.

Proposition 3.6.4.8 (Case $j > 0$)**.** *Let $C \to T$ be a general movable smooth curve. Assume that $(\mathcal{N}_Z)^n \cdot D_Z^j \geq e \cdot (\Psi \cdot C)$ for some $j > 0$ and $e > 0$. Then there exists a rational number $e_1 = e_1(X, D, q, e) > 0$ such that $\lambda_{f,D} \cdot C \geq e_1 \cdot (\Psi \cdot C)$.*

Proof. By generality $C \to T$, we may and will assume that the results of Lemma 3.6.4.4, Lemma 3.6.4.5 and Lemma 3.6.4.6 hold. Thus we have

$$F(C,j)(0) = (-K_{Z/C} - D_Z)^{n+1} = -\deg_C \lambda_{h,D_Z} \tag{6.4.8.j}$$

and

$$F'(C,j)(0) = (n+1)(-K_{Z/C} - D_Z)^n \cdot D_Z^j. \tag{6.4.8.k}$$

A direct calculation gives

$$(\mathcal{N}_Z)^n \cdot D_Z^j = q^n(-K_{Z/C} - D_Z)^n \cdot D_Z^j + 2n\alpha \cdot \deg_C \lambda_{h,D_Z} \cdot \left(\mathcal{N}_t^{D^j}\right)^{n-1} \tag{6.4.8.l}$$

Combining (6.4.8.k), (6.4.8.l) and the hypothesis on $(\mathcal{N}_Z)^n \cdot D_Z^j$, we obtain that

$$F'(C,j)(0) \geq \frac{(n+1)e}{q^n}(\Psi \cdot C) - \frac{2n(n+1)\alpha}{q^n}\left(\mathcal{N}_t^{D^j}\right)^{n-1} \deg_C \lambda_{h,D_Z}. \tag{6.4.8.m}$$

On the other hand, for any rational $0 < \epsilon < r_{X,D}$, the family $(V, D_V - \epsilon\Gamma_V) \to C$ is a $\mathbb{Q}$-Gorenstein family of log Fano pairs of maximal variation with uniformly K-stable general geometric fibers. Thus

$$-\deg_C \lambda_{h,D_V-\epsilon\Gamma_V} = F(C,i)(\epsilon) \leq 0 \quad \forall\, \epsilon \in (0, r_{X,D}) \tag{6.4.8.n}$$

by [CP21, Theorem 1.8.a]. To conclude the proof, we are going to combine (6.4.8.m) and (6.4.8.n) to get a negativity condition on $F(C,i)(0) = \deg \lambda_{h,D_Z}$.

For convenience, let us write

$$\beta_0 = \sup\left\{\left(\mathcal{N}_t^{D^i}\right)^{n-1} \mid i > 0, t \in C(k)\right\}, \quad w = \Psi.C, \quad a = \frac{(n+1)e}{q^n}, \quad b = \frac{2n(n+1)\alpha}{q^n}\beta_0$$

(notice that, by generic flatness and Noetherianity, β_0 is finite and actually a maximum). Therefore (6.4.8.m) implies that

$$F'(C,j)(0) \geq aw - b\deg_C \lambda_{h,D_Z} \tag{6.4.8.o}$$

Assume that

$$\deg_C \lambda_{h,D_Z} \leq \frac{a}{2b}w. \tag{6.4.8.p}$$

Then the estimate (6.4.8.o) implies

$$F'(C,j)(0) \geq \frac{a}{2}w > 0.$$

To summarize, we know by (6.4.8.n) that $F(C,j)(t)$ must be negative in a neighborhood of $t=0$, and if $F(C,j)(0)$ is small we have a positive lower bound on its first derivative. This gives an upper bound on $F(C,j)(0)$. Indeed, we apply Lemma 3.7.0.6 with

$$G=\frac{a}{2}w,\quad H=\mathfrak{m}_2(F(C,j)),\quad l=r_{X,D},\quad d=n+1\geq 2,$$

and we obtain that $F(C,j)$ takes a strictly positive value on $[0;r_{X,D}/2)$ if

$$F(C,j)(0)>\max\left\{-\frac{ar_{X,D}}{2}w,-\frac{a^2}{4n}\frac{w^2}{\mathfrak{m}_2(F(C,j))}\right\},\tag{6.4.8.q}$$

where we set $\frac{1}{\mathfrak{m}_2(F(C,j))}=+\infty$ if $\mathfrak{m}_2(F(C,j))=0$. But if (6.4.8.q) holds, then we get a contradiction with (6.4.8.n). Thus either (6.4.8.p) fails, or (6.4.8.p) holds and (6.4.8.q) fails. This can be synthetized as

$$\deg_C\lambda_{h,D_Z}=-F(C,j)(0)\geq\min\left\{\frac{a}{2b}w,\frac{ar_{X,D}}{2}w,\frac{a^2}{4n}\frac{w^2}{\mathfrak{m}_2(F(C,j))}\right\}.\tag{6.4.8.r}$$

To conclude, we need to modify the right-hand side of (6.4.8.r) so that the only quantity that depends on C is $w=(\Psi.C)$. The only problematic term is $\frac{w^2}{\mathfrak{m}_2(F(C,j))}$: it can be dealt with using Lemma 3.6.4.7, as we explain now.

Let $\{\Upsilon_{r,s}\}_{r,s}$ be the collection of $\mathbb{Q}$-Cartier divisors on T given by Lemma 3.6.4.7 when considering every index $r>0$. Consider the function

$$\overline{\mathrm{Mov}}(T)_{\mathbb{R}}-\{\mathbf{0}\}\to\mathbb{R},\quad\gamma\mapsto\frac{\max_{r,s}|\Upsilon_{r,s}\cdot\gamma|}{\Psi\cdot\gamma}.$$

This function is well-defined since Ψ is big and hence defines a strictly positive functional on $\overline{\mathrm{Mov}}(T)_{\mathbb{R}}-\{\mathbf{0}\}$. It is also continuous and invariant under $\mathbb{R}_+^*$-scaling of its argument. So it admits a maximum which is strictly positive, since the numerator is not zero for all movable curves. Thus there exists $\beta_1>0$ such that

$$\frac{\mathfrak{m}_2(F(C,j))}{w}=\frac{\max_s|\Upsilon_{j,s}\cdot C|}{\Psi\cdot C}\leq\frac{\max_{r,s}|\Upsilon_{r,s}\cdot C|}{\Psi\cdot C}<\beta_1$$

for all general movable curve C and $j>0$. So (6.4.8.r) implies that

$$\deg_C\lambda_{h,D_Z}=-F(C,j)(0)\geq\min\left\{\frac{a}{2b},\frac{ar_{X,D}}{2},\frac{a^2}{4n\beta_1}\right\}\cdot w.$$

The quantity $e_1=\min\left\{\frac{a}{2b},\frac{ar_{X,D}}{2},\frac{a^2}{4n\beta_1}\right\}$ depends only on X,D,q and e. Therefore the proof is complete. $\square$

3.6.5 Proof of Theorem 3.1.0.2

Proof of point (c) of Theorem 3.1.0.2. Let $\tau\colon T'\to T$ be a resolution of singularities. Then the induced family $f_{T'}\colon(X_{T'},D_{T'})\to T'$ is again a $\mathbb{Q}$-Gorenstein family of log Fano pairs of maximal variation with general geometric uniformly K-stable fibers. The morphism τ is birational and $\tau^*\lambda_{f,D}=\lambda_{f_{T'},D_{T'}}$ by Proposition 3.2.4.1. So $\lambda_{f,D}$ is big if and only if $\lambda_{f_{T'},D_{T'}}$ is big. Thus we may assume that T is smooth to begin with. Let $C\to T$ be a general movable curve. By Proposition 3.6.3.5 and Lemma 3.6.3.6, the hypothesis of either Proposition 3.6.4.2 or Proposition 3.6.4.8 is fullfilled, with a constant e that depends only on X,D and q. Thus there is a constant $c=c(X,D,q)>0$ such that $\lambda_{f,D}\cdot C\geq c\cdot(\Psi\cdot C)$. As Ψ is big, the result follows. $\square$

Proof of point (d) of Theorem 3.1.0.2. By the Nakai-Moishezon theorem it is enough to prove that for all normal varieties V mapping finitely to T, we have $(\lambda_{f,D}|_V)^{\dim V} > 0$. Let $V' \to V$ be a resolution of singularities. By (6.1.2.c) and since $(\lambda_{f,D}|_V)^{\dim V} = (\lambda_{f,D}|_{V'})^{\dim V'}$, we may replace $f\colon (X,D) \to T$ by $f_{V'}\colon (X_{V'}, D_{V'}) \to V'$. By assumption all the closed fibers of $f_{V'}$ are uniformly K-stable, hence klt. So all the fibers of $f_{V'}$ are klt. Therefore we are in position to apply point (c) of Theorem 3.1.0.2. □

3.7 Appendix

We gather some technical results that were used in the chapter.

Lemma 3.7.0.1. *Let $f\colon X \to T$ be an equidimensional proper morphism of relative dimension n between projective schemes. Assume that T is smooth. Let A be an ample $\mathbb{Q}$-Cartier divisor on X. Then the cycle f_*A^{n+1} is $\mathbb{Q}$-Cartier and big. (Here f_* denotes the cycle-theoretic pushforward.)*

Proof. Since f_* is linear, we may replace A by a multiple and assume it is very ample. If $H_1, \dots, H_{n+1} \in |A|$ are general elements, then $f_*(H_1 \cap \cdots \cap H_{n+1})$ is a divisor on T, and it is Cartier as T is smooth. Since f_* preserves rational equivalence, we have $f_*(H_1 \cap \cdots \cap H_{n+1}) \in |f_*A^{n+1}|$. It follows that this linear system is base-point free and separates points. The result now follows from [KM98, 2.60]. □

Lemma 3.7.0.2. *Let $f\colon X \to T$ be proper morphism between normal projective k-schemes. Let A be a pseudo-effective relatively ample $\mathbb{Q}$-Cartier divisor on X, and B a big $\mathbb{Q}$-Cartier divisor on T. Then $A + f^*B$ is big on every component of X.*

Proof. We may assume that X is integral. Write $B \sim_{\mathbb{Q}} C + E$ where C is ample and E effective. Fix an ample divisor H on X. Choose $\epsilon' \in \mathbb{Q}^*_+$ small enough such that $\epsilon' A + f^*C$ is ample on X. Then choose $\epsilon \in \mathbb{Q}^*_+$ small enough such that $A + \epsilon H$ is effective, and $\epsilon' A + f^*C - (1-\epsilon')\epsilon H$ is still ample. We write

$$A + f^*B \sim_{\mathbb{Q}} f^*E + (1-\epsilon')(A + \epsilon H) + (\epsilon' A + f^*C - (1-\epsilon')\epsilon H)$$

so $A + f^*B$ is the sum of an effective and an ample $\mathbb{Q}$-divisors. By [KM98, 2.60], it is big. □

Lemma 3.7.0.3. *Let $f\colon X \to T$ be a flat morphism between Noetherian schemes. Assume that T is integral, and that the generic fiber of f is reduced. Then X is reduced.*

Proof. Let x be an associated point of X. By assumption, the local morphism $\mathcal{O}_{T,f(x)} \to \mathcal{O}_{X,x}$ is flat. If $f(x)$ is not the generic point η of T, then $\mathcal{O}_{T,f(x)}$ has dimension at least one, and so its maximal ideal contains a non-zero divisor. By flatness, the image of this element is also a non-zero divisor in the maximal ideal of $\mathcal{O}_{X,x}$. This contradicts the fact that x is an associated point, so $f(x) = \eta$. Now X_η is reduced, so x cannot be an embedded associated point. Therefore X is reduced. □

Lemma 3.7.0.4. *Let X_i be proper schemes of dimensions n_i ($i = 1, \dots, r$). Set $\mathcal{X} := X_1 \times_k \cdots \times_k X_r$, with projections p_i onto its factors. There is a positive rational number $c = c(n_1, \dots, n_r)$ with the following property: if L_i are Cartier divisors on X_i and $L := \sum_{i=1}^r p_i^* L_i$, then*

$$L^{\dim \mathcal{X}} = c \prod_{i=1}^r L_i^{n_i}.$$

Proof. By induction on r, it suffices to consider the case $r = 2$. In this case we have

$$L^{n_1+n_2} = \binom{n_1+n_2}{n_1} \cdot (p_1^*L_1)^{n_1} \cdot (p_2^*L_2)^{n_2} = \binom{n_1+n_2}{n_1} \cdot L_1^{n_1} \cdot L_2^{n_2},$$

as claimed. In the general case, the precise form of the constant is $c = \prod_{i=1}^r \binom{\sum_{k \geq i} n_k}{n_i}$. □

Lemma 3.7.0.5. *Let $X_i \to T$ be flat morphisms from proper schemes of dimension $1+n_i$ to a common smooth curve ($i = 1, \dots, r$). Set $\mathcal{X} := X_1 \times_T \cdots \times_T X_r$ with projections p_i onto its factors. Then there are positive rational numbers $d_i = d(n_1, \dots, n_r)$ with the following property: if L_i are Cartier divisors on X_i and $L := \sum_{i=1}^r p_i^* L_i$, then*

$$L^{\dim \mathcal{X}} = \sum_{i=1}^{r} d_i L_i^{n_i+1} \prod_{j \neq i} (L_j)_t^{n_j}$$

for any closed $t \in T$.

Proof. Notice that $\dim \mathcal{X} = 1 + \sum_j n_j$. Hence $L^{\dim \mathcal{X}}$ is a weighted sum of $(p_1^* L_1)^{i_1} \cdots (p_r^* L_r)^{i_r}$ with $\sum_j i_j = 1 + \sum_j n_j$. Such a term is zero as soon as $i_j > 1 + n_j$ for some j. On the other hand, by the pigeon-hole principle, at least one i_j is greater or equal to $1 + n_j$. Thus:

$$\begin{aligned} L^{1+\sum_j n_j} &= \sum_{i=1}^{r} \binom{1+\sum_j n_j}{1+n_i} (p_i^* L_i)^{1+n_i} \cdot \left(\sum_{j \neq i} p_j^* L_j \right)^{\sum_{j\neq i} n_j} \\ &= \sum_{i=1}^{r} \binom{1+\sum_j n_j}{1+n_i} L_i^{1+n_i} \cdot \left(\sum_{j \neq i} p_j^* L_j |_{p_i^{-1}(x_i)} \right)^{\sum_{j\neq i} n_j} \end{aligned}$$

where the second equality holds for any $x_i \in X_i$ by flatness of p_i. Notice that the fiber of $p_i \colon \mathcal{X} \to X_i$ above x_i is naturally isomorphic to the fiber product $\times_{j \neq i} (X_j)_{t_i}$ taken over $\operatorname{Spec} k$, where t_i is the image of x_i through $X_i \to T$. By flatness of $X_i \to T$, the intersection number $(L_i)_{t_i}^{n_i}$ does not depend on t_i. Applying Lemma 3.7.0.4, we get

$$L^{\dim \mathcal{X}} = \sum_{i=1}^{r} \binom{1+\sum_j n_j}{1+n_i} L_i^{1+n_i} c(n_1, \dots, \widehat{n_i}, \dots, n_r) \prod_{j \neq i} (L_j)_t^{n_j}$$

where $t \in T$ is any closed point. Put $d_i(n_1, \dots, n_r) := \binom{1+\sum_j n_j}{1+n_i} c(n_1, \dots, \widehat{n_i}, \dots, n_r)$ to conclude. $\square$

Lemma 3.7.0.6. *Let $G > 0, H \geq 0$ and $l \in (0;1)$ be positive real numbers, and $d \geq 2$ be an integer. Then for every choice of real numbers $a_0, \dots, a_d$ satisfying*

$$\max\left\{ -\frac{Gl}{4}, -\frac{G^2}{4H(d-1)} \right\} < a_0 \leq 0, \quad a_1 \geq G, \quad \text{and} \quad |a_i| \leq H \ \forall i \geq 2,$$

the polynomial $p(t) = \sum_{i=0}^d a_i t^i$ takes a strictly positive value in the interval $(0; l/2)$. (If $H = 0$ we set $\frac{G^2}{4H(d-1)} = +\infty$).

Proof. Let $a_0, \dots, a_d$ be real numbers satisfying the prescribed conditions. We have, for $0 < t < 1$:

$$\begin{aligned} p(t) &= a_0 + a_1 t + \sum_{i \geq 2} a_i t^i \\ &\geq a_0 + Gt - H(d-1)t^2. \end{aligned}$$

So it it enough to prove that $q(t) := a_0 + Gt - H' t^2$ takes a strictly positive value on $(0, l/2)$, with $H' := H(d-1)$. First consider the special case where $H = 0$. Then $a_0 > -Gl/4$, so $q(l/3) > Gl/12 > 0$. From now we assume that $H > 0$. We have to show that $q(t)$ has a real positive root $t_0 \in (0, l/2)$ such that $q'(t_0) > 0$. Real roots exist if

$$a_0 > \frac{-G^2}{4H'}.$$

which holds by assumption on a_0. Then the smallest positive root of $q(t)$ is

$$t_0 = \frac{G - \sqrt{G^2 + 4a_0 H'}}{2H'}.$$

Note that

$$q'(t_0) = G - 2H't_0 = \sqrt{G^2 + 4a_0 H'} > 0.$$

Hence we just have to verify that $t_0 < l/2$. This condition is equivalent to

$$G - lH' < \sqrt{G^2 + 4a_0 H'}.$$

This inequality is trivially satisfied if $G - lH' < 0$. If $G - lH' \geq 0$, then it is equivalent to

$$\frac{l(lH' - G)}{4} - \frac{Gl}{4} < a_0,$$

which holds because $l(lH' - G)/4 < 0$ and $-Gl/4 < a_0$ by assumption. Therefore $q(t)$ takes a strictly positive value in $(0, l/2)$, as desired. $\square$

Chapter 4

Nodes on algebraic varieties

This chapter corresponds to the preprint [Pos21c, §3].

Convention 4.0.0.1. In this chapter we work with excellent reduced schemes. For the applications given in Section 5.3.4 and Section 4.6, we work over a field of positive characteristic.

4.1 Introduction

This chapter is a technical prelude to the following ones. It introduces the class of **demi-normal schemes**: they are higher-dimensional analogues of nodal curves, and appear naturally in the moduli theory of canonically polarised varieties. There are more difficult to handle than normal varieties: in particular the tools from the MMP are usually not available (see [Kol11]). The technical solution that is proposed in [Kol13] is to establish a dictionary between a demi-normal singularity and its normalization, and to work with the latter. This strategy works best in the global setting, as we will see in Chapter 5. For now, we establish the local theory of demi-normal schemes and their normalizations.

This chapter is organized as follows. In Section 4.2 we give several characterizations and basic properties of demi-normal schemes. Their normalization is studied in Section 4.3: we distinguish between **separable and inseparable nodes**, and we give a precise structure theorem for the normalization morphism (Proposition 4.3.3.1). In Section 4.4 we give a method to construct demi-normal schemes, while in Section 4.5 we give a complete classification of inseparable nodes. Finally, we give an application of these methods to the geometry of demi-normal schemes in positive characteristic: in Section 4.6 we prove a demi-normal compactification theorem.

4.2 Demi-normal schemes

Let us define right away the singularities we will study in this chapter:

Definition 4.2.0.1 ([Kol13, 1.41])**.** A one-dimensional Noetherian local ring $(R, \mathfrak{m})$ is called a **node** if there exists a ring isomorphism $R \cong S/(f)$, where $(S, \mathfrak{n})$ is a regular two-dimensional local ring and $f \in \mathfrak{n}^2$ is an element that is not a square in $\mathfrak{n}^2/\mathfrak{n}^3$.

Definition 4.2.0.2. A locally Noetherian reduced scheme (or ring) is called **nodal** if its codimension one local rings are regular or nodal. It is called **demi-normal** if it is S_2 and nodal.

Let X be a reduced scheme with normalization $\pi\colon \bar{X} \to X$. The conductor ideal of the normalization is defined as

$$\mathfrak{I} := \mathcal{H}om_X(\pi_*\mathcal{O}_{\bar{X}}, \mathcal{O}_X).$$

It is an ideal in both $\mathcal{O}_X$ and $\mathcal{O}_{\bar{X}}$. We let

$$D := \operatorname{Spec}_X \mathcal{O}_X/\mathfrak{I}, \quad \bar{D} := \operatorname{Spec}_{\bar{X}} \mathcal{O}_{\bar{X}}/\mathfrak{I},$$

and call them the *conductor subschemes.*

Lemma 4.2.0.3. *Notations as above. Assume that X is demi-normal with a dualizing sheaf. Then:*

(a) D and $\bar{D}$ are reduced of pure codimension 1.

(b) If $\eta \in D$ is a generic point such that $\operatorname{char} k(\eta) \neq 2$, the morphism $\bar{D} \to D$ is étale of degree 2 in a neighborhood of η. If $\operatorname{char} k(\eta) = 2$, then $\bar{D} \to D$ might be purely inseparable.

(c) The dualizing sheaf of X is invertible in codimension one. In particular, X has a well-defined canonical divisor class K_X.

(d) Let Δ be a $\mathbb{Q}$-divisor on X with no component supported on D, and such that $K_X+\Delta$ is $\mathbb{Q}$-Cartier. If $\bar{\Delta}$ denotes the divisorial part of $\pi^{-1}(\Delta)$, then there is a canonical isomorphism

$$\pi^*\omega_X^{[m]}(m\Delta) \cong \omega_{\bar{X}}^{[m]}(m\bar{D} + m\bar{\Delta})$$

for m divisible enough.

Proof. Lemma 4.2.1.4 below shows that the dualizing sheaf of X is invertible in codimension one. The rest follows from [Kol13, 5.2,5.7]. □

The language of $\mathbb{Q}$-divisors and divisorial sheaves can be extended to demi-normal schemes, if we insist that no component of the divisors belong to the singular locus. We refer the reader to [Kol13, 5.6] for more precisions. This allow us to define a non-normal version of log canonical singularities as follows.

Definition 4.2.0.4. We say that (X, Δ) is a **semi-log canonical (slc) pair** if: X is demi-normal, Δ is a $\mathbb{Q}$-divisor with no components along D, $K_X + \Delta$ is $\mathbb{Q}$-Cartier, and the normalization $(\bar{X}, \bar{D} + \bar{\Delta})$ is an lc pair.

In most cases the conductor $\bar{D} \subset \bar{X}$ comes with an additional structure that remembers the morphism $\bar{D} \to D$:

Lemma 4.2.0.5. *Let X be a demi-normal scheme with normalization $\pi\colon \bar{X} \to X$. Assume that the morphism of conductors $\bar{D} \to D$ is étale over every generic point. Then:*

(a) the induced morphism of normalizations $\bar{D}^n \to D^n$ is the geometric quotient by a Galois involution τ;

(b) if $K_X + \Delta$ is $\mathbb{Q}$-Cartier, then τ is a log involution of $(\bar{D}^n, \operatorname{Diff}_{\bar{D}^n} \bar{\Delta})$.

Proof. Since $\bar{D} \to D$ is generically étale, the field extension $k(D) \subset k(\bar{D})$ is separable of degree 2, hence Galois. The non-trivial field automorphism gives a well-defined morphism τ on $\bar{D}^n$, and it is easy to see that $D^n = \bar{D}^n/\tau$ (see also Corollary 5.4.2.5 below). This proves the first point.

Assume that $K_X+\Delta$ is Cartier. Since $\tau\colon \bar{D}^n \to \bar{D}^n$ commutes with the projection to D, the pullback of $K_X+\Delta$ to $\bar{D}^n$ is τ-invariant. We conclude using Lemma 4.2.0.3 and adjunction. □

4.2.1 Characterization of nodes

Nodal singularities have been defined in Definition 4.2.0.1. We will need some equivalent formulations.

Lemma 4.2.1.1. *Let $(R, \mathfrak{m}, k)$ be a one-dimensional reduced local ring that is a quotient of a regular ring. Let T be its normalization, and $\mathfrak{n} \subset T$ be its Jacobson radical. Then R is nodal if and only if $\dim_k T/\mathfrak{n} = 2$ and $\mathfrak{n} \subset R$.*

Proof. The only difference with [Kol13, 1.41.1] is that we assume that R is the quotient of a regular ring that is not necessarily local. But the two conditions are equivalent: for if $\pi\colon B \twoheadrightarrow A$ is a surjective map of rings and A is local, then $\mathfrak{p} := \pi^{-1}(\mathfrak{m}_A)$ is a prime ideal, π factors through $B_\mathfrak{p}$ by the universal property of localization, and $B_\mathfrak{p}$ is regular if B is regular. □

Another characterization of nodes can be given in terms of semi-normality, which we define below. See [Kol13, §10.2] for more details.

Definition 4.2.1.2. Let X be an excellent scheme with normalization $\nu\colon X^\nu \to X$. For a point $x \in X$, let $x^\nu := \nu^{-1}(x)_{\mathrm{red}}$. The sections $s \in \nu_*\mathcal{O}_{X^\nu}$ that satisfy

$$s|_{x^\nu} \in \operatorname{im}[\nu^*\colon k(x) \to H^0(x^\nu, \mathcal{O}_{x^\nu})] \quad \forall x \in X$$

form a finite $\mathcal{O}_X$-algebra $\mathcal{O}'$. We say that X is **semi-normal** if $\mathcal{O}' = \mathcal{O}_X$.

Observation 4.2.1.3. Since $\mathcal{O}'$ is finite over $\mathcal{O}_X$, the locus where the map $\mathcal{O}_X \to \mathcal{O}'$ is an isomorphism, is open. Thus the semi-normal locus of an excellent scheme is open.

We will compare demi-normality and seminormality in Section 4.3.2, and show in Corollary 4.3.2.4 that demi-normality implies semi-normality.

Lemma 4.2.1.4. *Let $(R, \mathfrak{m}, k)$ be a one-dimensional excellent reduced semi-normal local ring that is a quotient of a regular ring. Then R is nodal if and only if it is Gorenstein.*

Proof. We elaborate [Kol13, 5.9.3]. Write $X = \operatorname{Spec} R$ and let $\bar{X}$ be the normalization. Then $\bar{X}$ is a regular scheme and $\bar{X} \to X$ is finite. Let $D \subset X$ and $\bar{D} \subset \bar{X}$ be the conductors subschemes. Then $\operatorname{Supp} D$ is the closed point $x \in X$, and $\operatorname{Supp} \bar{D}$ is its preimage in $\bar{X}$. By [LV81, Corollary 1.5.1], thanks to the semi-normality assumption, the conductor ideal is radical in $\mathcal{O}_{\bar{X}}$, hence also in $\mathcal{O}_X$, and therefore D and $\bar{D}$ are reduced.

Since R is S_1 and the quotient of a regular ring, X has a canonical sheaf ω_X. The 0-dimensional reduced schemes D and $\bar{D}$ also have canonical sheaves, which are invertible. The calculations of [Kol13, 9.3] show that there is an isomorphism of $k(x)$-vector spaces

$$\omega_X \otimes k(x) \cong \ker[\operatorname{Tr}\colon \omega_{\bar{D}} \to \omega_D]$$

where Tr the Grothendieck trace map.

Suppose that R is nodal. Then explicit computations show $\ker(\operatorname{Tr})$ is one-dimensional, see Lemma 4.3.0.5 below. By Nakayama's lemma, it follows that ω_X is generated by a single element. Since a canonical sheaf has full support, we see that ω_X is free of rank one, so R is Gorenstein.

Conversely, suppose that ω_X is invertible. Then $\ker(\operatorname{Tr})$ is one-dimensional, and as $\bar{D}$ is Gorenstein we obtain that $\dim_{k(x)} \omega_{\bar{D}} = \dim_{k(x)} H^0(\bar{D}, \mathcal{O}_{\bar{D}}) = 2$. In particular we have the following dichotomy:

(a) $\bar{D}$ is a single point. Thus $\bar{X}$ is local, and its maximal ideal is the conductor ideal, which is contained in R.

(b) $\bar{D}$ is supported on two distinct closed points, corresponding to two maximal ideals $\mathfrak{m}_1$ and $\mathfrak{m}_2$. Therefore $\mathfrak{m}_1 \cap \mathfrak{m}_2 = \sqrt{\mathfrak{m}_1 \cap \mathfrak{m}_2}$ is the conductor ideal, contained in R.

In both cases we apply Lemma 4.2.1.1 to see that R is nodal. □

4.3 Normalization of nodes

We take a look at the normalization of nodes, following [Tan16, §3]. Let $(R, \mathfrak{m})$ be a nodal Noetherian local ring, and pick a presentation $R \cong S/(f)$ as in Definition 4.2.0.1. We can choose a set of local parameters $\mathfrak{n} = (x, y)$ for S such that

$$f = ax^2 + bxy + cy^2 + g$$

for $a, b \in S$, $c \in S^\times$ and $g \in \mathfrak{n}^3$ [Tan16, 3.2, 3.3]. Denote by $\bar{x}, \bar{y}$ the images of x, y in R.

Proposition 4.3.0.1. *Notations as above. Assume that $(R, \mathfrak{m})$ is not an integral domain. Then:*

(a) the normalization of R is a product $T_1 \times T_2$ of local regular one-dimensional rings;
(b) $\mathfrak{m}$ is the conductor of the normalization;
(c) $R \hookrightarrow T_1 \times T_2$ induces the diagonal embedding

$$k(\mathfrak{m}) \hookrightarrow (T_1/\mathfrak{m}T_1) \times (T_2/\mathfrak{m}T_2) \cong k(\mathfrak{m}) \times k(\mathfrak{m}).$$

Assume now that $(R, \mathfrak{m})$ is an integral domain. Then:
(a) the normalization of $(R, \mathfrak{m})$ is given by

$$R \hookrightarrow R\left[\frac{\bar{y}}{\bar{x}}\right] =: T.$$

Moreover we have $T = R + R \cdot \frac{\bar{y}}{\bar{x}}$.
(b) the conductor of the normalization is $\mathfrak{m}$, and $\mathfrak{m}T = \bar{x}T$.
(c) We have

$$T/\mathfrak{m}T \cong \frac{k(\mathfrak{m})[Z]}{(\bar{a} + \bar{b}Z + \bar{c}Z^2)}$$

where $\bar{a}, \bar{b}, \bar{c}$ are the image of a, b, c through $S \to k(\mathfrak{m}) = R/\mathfrak{m}$, and Z is the image of $\frac{\bar{y}}{\bar{x}}$. In particular T is local with maximal ideal $\mathfrak{m}T$ as soon as $\bar{a} + \bar{b}Z + \bar{c}Z^2$ is irreducible in $k(\mathfrak{m})[Z]$.

Proof. These results are contained in the statements and the proofs of [Tan16, 3.4, 3.5]. □

Corollary 4.3.0.2. *Let $(R, \mathfrak{m})$ be a non-integral nodal ring and T be its normalization. Denote the conductors by $D := V(\mathfrak{m}) \subset \operatorname{Spec} R$ and $\bar{D} := V(\mathfrak{m}T) \subset \operatorname{Spec} T$. Then $\mathcal{O}_D$ is recovered as the fixed sub-ring of an involution on $\mathcal{O}_{\bar{D}}$.*

Proof. By Proposition 4.3.0.1, $\mathcal{O}_D \hookrightarrow \mathcal{O}_{\bar{D}} \cong \mathcal{O}_D \oplus \mathcal{O}_D$ is the diagonal embedding, so the involution is given by the permutation of direct summands. □

Corollary 4.3.0.3. *Let $(R, \mathfrak{m})$ be an integral nodal ring and T be its normalization. Denote the conductors by $D := V(\mathfrak{m}) \subset \operatorname{Spec} R$ and $\bar{D} := V(\mathfrak{m}T) \subset \operatorname{Spec} T$. Then:*
(a) $\bar{D} \to D$ is two-to-one if and only if $\bar{a} + \bar{b}Z + \bar{c}Z^2$ splits in $k(\mathfrak{m})[Z]$.
(b) $\bar{D} \to D$ is bijective and Galois if and only if $\bar{a} + \bar{b}Z + \bar{c}Z^2$ is irreducible and separable over $k(\mathfrak{m})$.
(c) $\bar{D} \to D$ is bijective and purely inseparable if and only if $\bar{a} + \bar{b}Z + \bar{c}Z^2$ is irreducible and inseparable over $k(\mathfrak{m})$.
Moreover exactly one of these cases occurs. In the first two cases, $\mathcal{O}_D$ is recovered as the fixed sub-ring of an involution on $\mathcal{O}_{\bar{D}}$.

Proof. We need to show that $\bar{a} + \bar{b}Z + \bar{c}Z^2$ cannot be a square in $k(\mathfrak{m})[Z]$. If this was the case, then $\mathcal{O}_{\bar{D}} = T/\mathfrak{m}T$ would not be reduced, which contradicts Lemma 4.2.0.3.

In the first case we have $T/\mathfrak{m}T \cong k(\mathfrak{m}) \oplus k(\mathfrak{m})$ in which $k(\mathfrak{m})$ embeds diagonally, and the involution is given by exchanging the direct summands.

In the second case $T/\mathfrak{m}T$ is a Galois extension of $k(\mathfrak{m})$ of degree 2, and the involution is the unique non-trivial element of the Galois group. □

Definition 4.3.0.4. A nodal Noetherian local ring $(R, \mathfrak{m})$ is an **inseparable node** if the condition of Corollary 4.3.0.3.(c) is satisfied. In the other cases, we call it a **separable node**.

We study the pluricanonical sections that descend the normalization morphism, aiming to give a slight generalization of [Kol13, 5.8, 5.18].

Lemma 4.3.0.5. *Notations as in Corollary 4.3.0.2 and Corollary 4.3.0.3. Then:*

(a) *The kernel of the Grothendieck trace map $\omega_{\bar{D}} \to \omega_D$ is one-dimensional over $k(\mathfrak{m})$.*

(b) *Assume $\bar{D} \to D$ is separable with induced involution τ on $\bar{D}$. Then the kernel of the Grothendieck trace map is the sub-module of τ-anti-invariant sections. (If $\operatorname{char} k(\mathfrak{m}) = 2$, we interpret τ-anti-invariant sections as the τ-invariant ones.)*

Proof. In any case, since we can take $\mathcal{O}_D$ and $\mathcal{O}_{\bar{D}}$ as the dualizing sheaves of D and $\bar{D}$, the Grothendieck trace map is obtained by applying $\operatorname{Hom}_{\mathcal{O}_D}(\bullet, \mathcal{O}_D)$ to the map $\iota\colon \mathcal{O}_D \to \mathcal{O}_{\bar{D}}$. We distinguish the possible cases, according to Corollary 4.3.0.2 and Corollary 4.3.0.3. For ease of notation, we write $k := k(\mathfrak{m}) = \mathcal{O}_D$.

(a) The morphism $\bar{D} \to D$ is two-to-one. Then $\mathcal{O}_{\bar{D}} = k \oplus k$, the involution τ exchanges the two summands, and $\bar{D} \to D$ corresponds to the diagonal embedding $\iota\colon k \to k \oplus k$. The Grothendieck trace is given by

$$\operatorname{Hom}(\iota)\colon \operatorname{Hom}_k(k \oplus k, k) \to \operatorname{Hom}_k(k, k), \quad \phi_{a,b} \mapsto \mu_{a+b}$$

where $\phi_{a,b}(x, y) = ax + by$ and $\mu_{a+b}(z) = (a + b)z$. Thus $\ker \operatorname{Hom}(\iota) = \{\phi_{a,-a} \mid a \in k\}$. Since τ acts on $\operatorname{Hom}_k(k \oplus k, k)$ by pre-composition, we see that $\ker \operatorname{Hom}(\iota)$ is the sub-module of τ-anti-invariant sections.

(b) The map $\bar{D} \to D$ is bijective and Galois. Then $\iota\colon k \to L := \mathcal{O}_{\bar{D}}$ is a Galois field extension of degree 2, and τ is the non-trivial element of $\operatorname{Gal}(L/k)$. As above, the Grothendieck trace map is given by

$$\operatorname{Hom}(\iota)\colon \operatorname{Hom}_k(L, k) \to \operatorname{Hom}_k(k, k), \quad \phi \mapsto \phi \circ \iota.$$

We distinguish two cases:

(i) If $\operatorname{char} k \neq 2$, then we can find an element $d \in k$ such that $L = k(\sqrt{d})$. Taking $\{1, \sqrt{d}\}$ as a k-basis of L, we see that $\ker \operatorname{Hom}(\iota) = \{\phi_{0,a} \mid a \in k\}$. Since $\tau(\sqrt{d}) = -\sqrt{d}$, we see that $\ker \operatorname{Hom}(\iota)$ is the sub-module of τ-anti-invariant elements.

(ii) Assume that $\operatorname{char} k = 2$ and consider the k-linear map

$$\mathfrak{t}\colon L \to k, \quad \mathfrak{t}(x) = x + \tau(x).$$

Since $L^\tau = k$, it holds $k \cdot \mathfrak{t} \subseteq \ker \operatorname{Hom}(\iota)$. Since $\operatorname{Hom}(\iota)$ is surjective and $\dim_k L = 2$, we have $\dim_k \ker \operatorname{Hom}(\iota) = 1$ and therefore $k \cdot \mathfrak{t} = \ker \operatorname{Hom}(\iota)$.

Since $\mathfrak{t}$ is τ-invariant, we obtain that $\ker \operatorname{Hom}(\iota)$ is included in the τ-invariant sub-module of $\operatorname{Hom}_k(L, k)$. By counting dimensions, it remains to show that there exists elements of $\operatorname{Hom}_k(L, k)$ that are not τ-invariant.

Let us exhibit such linear maps. We can write $L = k(\alpha)$. If $\tau(\alpha) = c\alpha$ with $c \in k$, then $(1 + c)\alpha \in k$, so either $c = 1$ and τ is the identity, either $\alpha \in k$. So $\{\alpha, \tau(\alpha)\}$ is a k-basis of L. In this basis, if a, b are distinct elements of k, the k-linear map $\phi_{a,b}$ is not τ-invariant.

(c) The morphism $\bar{D} \to \bar{D}$ is purely inseparable. Then $\iota\colon k \to L := \mathcal{O}_{\bar{D}}$ is a purely inseparable field extension of degree 2. Thus we can write $L = k(\sqrt{d})$ for some $d \in k \setminus k^2$. Taking $\{1, \sqrt{d}\}$ as a k-basis of L, we see that $\ker \operatorname{Hom}(\iota) = \{\phi_{0,a} \mid a \in k\}$, which is one-dimensional over k.

The proof is complete. □

Proposition 4.3.0.6. *Let X be an excellent demi-normal scheme with only separable nodes and normalization $(\bar{X}, \bar{D}, \tau)$, and Δ a $\mathbb{Q}$-divisor on X that has no common component with $\operatorname{Sing}(X)$. Then a section of $\omega_{\bar{X}}^{[m]}(m\bar{D} + m\bar{\Delta})$ descends to $\omega_X^{[m]}(m\Delta)$ if and only if its Poincaré residue at generic points of $\bar{D}$, taking values in $\omega_{\bar{D}}^{[m]}(m \operatorname{Diff}_{\bar{D}} \bar{\Delta})$, is τ-invariant and m is even, or is τ-anti-invariant and m is odd.*

Proof. We are dealing with reflexive sheaves, and the morphism $\pi\colon \bar{X} \to X$ is an isomorphism above $X \setminus D$. Thus the only question is above the generic points of D. In particular we may assume that $\Delta = 0$, that X is Cohen–Macaulay, semi-normal (see Observation 4.2.1.3) and that $\bar{X}$ is regular. Then according to [Kol13, 5.9], there is an exact sequence

$$0 \to \omega_X \to \pi_* \omega_{\bar{X}}(\bar{D}) \xrightarrow{\partial} \omega_D \to 0$$

with

$$\partial = \left(\omega_{\bar{X}}(\bar{D}) \xrightarrow{\mathcal{R}} \omega_{\bar{D}} \xrightarrow{\mathrm{Tr}} \omega_D \right)$$

where $\mathcal{R}$ is the Poincaré residue map, and Tr the Grothendieck trace map. Hence the result follows from Lemma 4.3.0.5. □

Corollary 4.3.0.7. *Let (X, Δ) be as in Proposition 4.3.0.6, with normalization $(\bar{X}, \bar{D} + \bar{\Delta}, \tau)$. Then the following are equivalent:*

(a) $\mathrm{Diff}_{\bar{D}^n} \bar{\Delta}$ *is τ-invariant, and*

(b) (X, Δ) *is slc outside a closed subset of codimension at least* 3.

Proof. The only difference between our statement and the statement of [Kol13, 5.18] is that we do not assume that $2 \in \mathcal{O}_X$ is invertible. This assumption is only needed in order to use [Kol13, 5.8], which is generalized to our setting by Proposition 4.3.0.6. □

Next we study inseparable nodes.

Example 4.3.0.8. Let k be any field of characteristic 2. Consider the ring $A := k[u, v, w]/(wv^2 - u^2)$. We claim that A has only nodal singularities in codimension one. Indeed, the singular locus of A is defined by $(u = v = 0)$, so the prime ideal $\mathfrak{p} := (u, v)$ is the only height one prime such that $A_\mathfrak{p}$ is not regular. Now

$$A_\mathfrak{p} \cong \frac{k[u, v, w]_{(u,v)}}{(wv^2 - u^2)}$$

is nodal according to Definition 4.2.0.1 since $wv^2 - u^2$ does not have a square root modulo $\mathfrak{p}^3 A_\mathfrak{p}$.

The normalization of A is given by the regular two-dimensional ring $B := k\left[t = \frac{u}{v}, v\right]$. Hence the normalization of $A_\mathfrak{p}$ is given by $B_{(v,tv)} = B_{\bar{\mathfrak{p}}}$, with $\bar{\mathfrak{p}} = vB$. The conductor of the normalization is generated by the element v.

Denote $D = V(\mathfrak{p}) \subset \operatorname{Spec} A$ and $\bar{D} = V(\bar{\mathfrak{p}}) \subset \operatorname{Spec} B$. Then $\pi\colon \operatorname{Spec} B \to \operatorname{Spec} A$ is an isomorphism above the complement of D. Moreover $\bar{D} \to D$ is given by the k-algebra morphism

$$A/\mathfrak{p} = k[w] \longrightarrow k[t] = B/\bar{\mathfrak{p}}, \quad w \mapsto t^2.$$

Since k has characteristic 2, we obtain that $A_\mathfrak{p}$ is an inseparable node.

In view of the gluing theory we want to develop, we should answer the following question: how to reconstruct the nodal surface A from its normalization B? Looking a the commutative diagram

$$\begin{tikzcd}
A \arrow[r, "v\mapsto v"', "u\mapsto tv,\ w\mapsto t^2"'] \arrow[d] & B \arrow[d] \\
A_\mathfrak{p} \arrow[r] \arrow[d] & B_{\bar{\mathfrak{p}}} \arrow[d] \\
k(A_\mathfrak{p}) = k(w) \arrow[r, "w\mapsto t^2"] & k(t) = k(B_{\bar{\mathfrak{p}}})
\end{tikzcd}$$

we see that A is the preimage of $k(t^2) \subset k(t)$ under the canonical map $B \to k(B_{\bar{\mathfrak{p}}})$. Hence the data of $A \subset B$ is equivalent to the data of B together with the degree 2 purely inseparable extension $k(w) \subset k(t)$.

Example 4.3.0.9. The calculations of Example 4.3.0.8 can be generalized to the case of $A' := k[u,v,w]_{(u,v)}/(f(w)v^2 - u^2)$, where $f(w) \in k(w)$ is not a square. In this case the data of A' is also recovered from its normalization $B' = k[t = u/v, v]_{(v)}$ and the purely inseparable degree 2 extension

$$k(w) \hookrightarrow k(t) \cong k\left(\sqrt{f(w)}\right), \quad w \mapsto t^2 = f(w).$$

Remark 4.3.0.10. Keeping the notations of Proposition 4.3.0.1, $(R,\mathfrak{m})$ is an inseparable node if and only if R is a domain, $k(\mathfrak{m})$ has characteristic 2, $\bar{b} = 0$ and $\bar{a}/\bar{c} \notin k(\mathfrak{m})^2$. In particular the residue field of $(R,\mathfrak{m})$ is not perfect. This implies that a nodal curve over a perfect field of characteristic 2, does not have any inseparable nodes.

On the other hand, there exist curves with inseparable nodes over imperfect fields. The calculations of Example 4.3.0.8 show that $k(w)[u,v]/(wv^2 - u^2)$ is one such.

While separable nodes can be reconstructed from their normalization using the induced involution, the following proposition shows that inseparable nodes can be reconstructed from their normalization using a method generalizing Example 4.3.0.8.

Proposition 4.3.0.11. *Let $(R,\mathfrak{m})$ be a Noetherian local ring which is an inseparable node. Let $(T, \mathfrak{m}_T = \mathfrak{m}T)$ be its normalization. Then R is the preimage in T of the subfield $k(\mathfrak{m}) \hookrightarrow k(\mathfrak{m}_T)$.*

Proof. We use the notations and results of Proposition 4.3.0.1. Then $T = R[\bar{y}/\bar{x}] = R + R\cdot\bar{y}/\bar{x}$, and $k(\mathfrak{m}) \hookrightarrow k(\mathfrak{m}_T)$ is given explicitly by the inclusion of constants

$$k(\mathfrak{m}) \subset \frac{k(\mathfrak{m})[Z]}{(\bar{a} + \bar{c}Z^2)} = k(\mathfrak{m}_T),$$

where Z is the image of $\bar{y}/\bar{x}$. An element $\alpha + \beta\cdot\bar{y}/\bar{x} \in T$, with $\alpha, \beta \in R$, reduces modulo $\mathfrak{m}_T$ to an element of $k(\mathfrak{m})$ if and only if $\beta \in \mathfrak{m}_T = \mathfrak{m}T$. But $\mathfrak{m}T = \bar{x}T$, so we see that R is equal to the preimage of $k(\mathfrak{m})$ in T. $\square$

4.3.1 Being nodal is an étale-local property

For varieties over a field of characteristic different from 2, one can show that a local ring $\mathcal{O}_{X,x}$ is nodal if and only if

$$\widehat{\mathcal{O}_{X,x}} \otimes_k \bar{k} \cong \bar{k}[[X,Y]]/(XY),$$

where $k = k(x)$ and $\bar{k}$ is an algebraic closure, see for example [Kol13, 1.41.2]. In particular, by Artin approximation, the property of being nodal is étale-local. In this subsection, we show that being an inseparable node is also an étale-local property.

Lemma 4.3.1.1. *Let $(R,\mathfrak{m})$ be a Noetherian local ring, with strict henselization $(R^{\mathrm{sh}}, \mathfrak{m}^{\mathrm{sh}})$. Then*

(a) R is Gorenstein if and only if R^{sh} is Gorenstein, and

(b) R is semi-normal if and only if R^{sh} is semi-normal.

(c) If R is the quotient of a regular ring, then so is R^{sh}.

(d) The square

$$\begin{array}{ccc} \operatorname{Spec}(R^{\mathrm{sh}})^{\nu} & \longrightarrow & \operatorname{Spec} R^{\mathrm{sh}} \\ \downarrow & & \downarrow \\ \operatorname{Spec} R^{\nu} & \longrightarrow & \operatorname{Spec} R \end{array}$$

is Cartesian, where $(\cdot)^{\nu}$ denotes normalization.

Proof. Recall that $\mathfrak{m}R^{\mathrm{sh}} = \mathfrak{m}^{\mathrm{sh}}$ and that $R \to R^{\mathrm{sh}}$ is faithfully flat. So by [Mat89, 23.4] we obtain the first equivalence, and the second one follows from [GT80, 1.8, 5.2, 5.3].

Assume that $R \cong Q/I$, where Q is a regular local ring and $I \subset Q$ an ideal. Then $R^{\mathrm{sh}} \cong Q^{\mathrm{sh}}/IQ^{\mathrm{sh}}$ by [Sta, 05WS]. Moreover Q^{sh} is regular because $Q \to Q^{\mathrm{sh}}$ is étale.

The property of the square is proved in [Sta, 0CBM]. $\square$

Corollary 4.3.1.2 (Being nodal is étale-local)**.** *Let $(R, \mathfrak{m})$ be an excellent reduced one-dimensional local ring that is a quotient of a regular ring, with strict henselization $(R^{sh}, \mathfrak{m}^{sh})$. Then $(R, \mathfrak{m})$ is a node if and only if $(R^{sh}, \mathfrak{m}^{sh})$ is a node.*

Proof. Combine Lemma 4.2.1.4 and Lemma 4.3.1.1. □

Corollary 4.3.1.3 (Being an inseparable node is étale-local)**.** *Let $(R, \mathfrak{m})$ be an excellent reduced one-dimensional local ring that is a quotient of a regular ring, with strict henselization $(R^{sh}, \mathfrak{m}^{sh})$. Then $(R, \mathfrak{m})$ is an inseparable node if and only if $(R^{sh}, \mathfrak{m}^{sh})$ is an inseparable node.*

Proof. The local ring $(R, \mathfrak{m})$ is an inseparable node if and only if it is semi-normal Gorenstein, its normalization R^ν is local and $R/\mathfrak{m} \to R^\nu/\mathfrak{m}R^\nu$ is a purely inseparable extension of degree 2. We know by Lemma 4.3.1.1 that R is semi-normal Gorenstein if and only if R^{sh} is, so we may assume R is a node and consider the other conditions.

Now consider the Cartesian diagram given in Lemma 4.3.1.1. Prime ideals of $(R^{\mathrm{sh}})^\nu$ corresponds to pairs $(\mathfrak{p} \in \operatorname{Spec} R^{\mathrm{sh}}, \mathfrak{q} \in \operatorname{Spec} R^\nu)$ with the property that $\mathfrak{p} \cap R = \mathfrak{q} \cap R$. Now let $\mathfrak{m}'$ be a maximal ideal of $(R^{\mathrm{sh}})^\nu$. Since R is excellent, the normalization $R \to R^\nu$ is finite, and therefore the pullback $R^{\mathrm{sh}} \to (R^{\mathrm{sh}})^\nu$ is also finite. Hence $\mathfrak{m}' \cap R^{\mathrm{sh}}$ is maximal and therefore equal to $\mathfrak{m}^{\mathrm{sh}}$. Since $\mathfrak{m}^{\mathrm{sh}}$ is the unique prime ideal of R^{sh} lying above $\mathfrak{m}$, we see that maximal ideals of $(R^{\mathrm{sh}})^\nu$ corresponds to pairs $(\mathfrak{m}^{\mathrm{sh}}, \mathfrak{q})$ with $\mathfrak{q} \cap R = \mathfrak{m}$. Thefore $(R^{\mathrm{sh}})^\nu$ is local if and only if R^ν is local.

Assume this is the case, and base-change along $R \to R/\mathfrak{m}$. As $\mathfrak{m}R^{\mathrm{sh}} = \mathfrak{m}^{\mathrm{sh}}$, we obtain the push-out square

$$\begin{tikzcd}
A := (R^{\mathrm{sh}})^\nu/\mathfrak{m}^{\mathrm{sh}}(R^{\mathrm{sh}})^\nu & k^{\mathrm{sh}} := R^{\mathrm{sh}}/\mathfrak{m}^{\mathrm{sh}} \arrow[l] \\
k' := R^\nu/\mathfrak{m}R^\nu \arrow[u] & k := R/\mathfrak{m} \arrow[l] \arrow[u]
\end{tikzcd}$$

Since R^ν is local, by Proposition 4.3.0.1 we have that k' is a field of the form

$$k' = k[Z]/p(Z),$$

where $p(Z) \in k[Z]$ has degree 2. Thus

$$A \cong k^{\mathrm{sh}}[Z]/p(Z).$$

Assume that k' is purely inseparable over k. Then the roots of $p(Z)$ belongs to a purely inseparable extension of k. Since k^{sh} a separable closure of k, we obtain that $p(Z) \in k^{\mathrm{sh}}[Z]$ is irreducible and inseparable, and so $k^{\mathrm{sh}} \subset A$ is a purely inseparable field extension of degree 2. Conversely, if A is a degree 2 inseparable field extension of k^{sh}, then $p(Z)$ is inseparable over k and therefore $k \subset k'$ is purely inseparable of degree 2. □

Remark 4.3.1.4. All the results of this section hold with the Henselization instead of the strict Henselization, and the proofs are the same.

4.3.2 Demi-normality, seminormality and weak normality

In this subsection we compare the definition of demi-normality, seminormality and weak normality. For the definitions of **seminormal** and **weakly normal** factorizations, we refer to [Kol96, Appendix I.7.2] (in the case of seminormality, this generalises Definition 4.2.1.2). For us, the relevant property is the following: a finite surjective morphism of reduced schemes $Z \to X$ is seminormal (resp. weakly normal) if, given a factorization $Z \to X' \to X$ where $X' \to X$ is an homeomorphism inducing isomorphisms on residues fields (resp. an homeomorphism inducing purely inseparable extension of residue fields), it holds that $X' = X$.

Lemma 4.3.2.1. *Consider a pushout diagram of reduced $\mathbb{F}_p$-schemes*

$$\begin{tikzcd} X_1 \arrow[r, hook] \arrow[d, "q"] & X_2 \arrow[d, "p"] \\ X_3 \arrow[r, hook] & Y \end{tikzcd}$$

where the horizontal arrows are closed embeddings, p is finite surjective and an isomorphism over every generic point of X, and q is finite surjective. Then Y is weakly normal in X_2 if and only X_3 is weakly normal in X_1.

Proof. We may assume that all schemes are affine. We use the following characterization of weak normality for $\mathbb{F}_p$-schemes: a finite surjective morphism $Z \to Z'$ is weakly normal if and only if $\mathcal{O}_{Z'}$ is p-closed in $\mathcal{O}_Z$ [Yan83, Corollary to Theorem 1].

Assume that Y is weakly normal in X_2. Pick an element $s \in \mathcal{O}_{X_1}$ such that $s^p \in \mathcal{O}_{X_3}$. Choose any lift $u \in \mathcal{O}_{X_2}$ of s. Then the pair $(u^p, s^p) \in \mathcal{O}_{X_2} \times \mathcal{O}_{X_3}$ glues to an element of $\mathcal{O}_Y$ that is a p-th power of u. Therefore $u \in \mathcal{O}_Y$, and its restriction to X_3 is precisely s.

Conversely, assume that X_3 is weakly normal in X_1. Pick an element $v \in \mathcal{O}_{X_2}$ such that $v^p \in \mathcal{O}_Y$. If $t \in \mathcal{O}_{X_1}$ is the restriction of v to X_1, then $t^p \in \mathcal{O}_{X_3}$. Thus $t \in \mathcal{O}_{X_3}$ already. Therefore the pair $(v, t) \in \mathcal{O}_{X_2} \times \mathcal{O}_{X_3}$ glues to $v \in \mathcal{O}_Y$. □

Next let us recall the structure theorems for seminormal and weakly normal ring extensions, given in [Tra70] and [Yan83]. Let $A \subset B$ be a finite extension of rings, $\mathfrak{p}$ be a prime ideal of A and $\mathfrak{p}_1, \dots, \mathfrak{p}_n$ the prime ideals of B lying over $\mathfrak{p}$. We have a natural map $k(\mathfrak{p}) \to \prod_i k(\mathfrak{p}_i)$. Consider the pullback diagram of rings

$$\begin{tikzcd} A^+ \arrow[r] \arrow[d] & B \arrow[d] \\ k(\mathfrak{p}) \arrow[r] & \prod_{i=1}^n k(\mathfrak{p}_i). \end{tikzcd}$$

We call A^+ the **gluing of** B **over** $\mathfrak{p} \subset A$. It is an intermediate extension of $A \subset B$, with a unique prime ideal $\mathfrak{p}^+$ over $\mathfrak{p} \subset A$ whose residue field is $k(\mathfrak{p}^+) = k(\mathfrak{p})$.

Furthermore, let k' be the largest subfield of $\prod_i k(\mathfrak{p}_i)$ which is a purely inseparable extension of $k(\mathfrak{p})$. Consider the pullback diagram

$$\begin{tikzcd} A^* \arrow[r] \arrow[d] & B \arrow[d] \\ k' \arrow[r] & \prod_{i=1}^n k(\mathfrak{p}_i). \end{tikzcd}$$

We call A^* the **weak gluing of** B **over** $\mathfrak{p} \subset A$. It is an intermediate extension of $A \subset B$, with a unique prime ideal $\mathfrak{p}^*$ over $\mathfrak{p} \subset A$ whose residue field is $k(\mathfrak{p}^*) = k'$.

Theorem 4.3.2.2 ([Tra70, Theorem 2.1] and [Yan83, Theorem 3])**.** *Let $A \subset B$ be a finite extension of Noetherian rings. Then A is seminormal (resp. weakly normal) in B if and only if then there exists a finite sequence of finite extensions*

$$A = A_0 \subset A_1 \subset \cdots \subset A_n = B$$

where A_i is the gluing (resp. the weak gluing) of A_{i+1} over a prime ideal of A_i.

Lemma 4.3.2.3. *Let $(R, \mathfrak{m})$ be a nodal ring. Then R is seminormal in its normalization R^ν, and R is the gluing of R^ν over $\mathfrak{m} \subset R$. Furthermore, R is weakly normal in R^ν if and only if R is a separable node.*

Proof. By Lemma 4.2.0.3 the conductor of $R \subset R^\nu$ is a radical ideal of R^ν. Thus by [Kol96, Appendix I.7.2.5] we obtain that R is seminormal in R^ν if and only if $k(\mathfrak{m})$ is seminormal in $R^\nu/\mathfrak{m}R^\nu$. The latter is trivially true.

Let R^+ be the gluing of R^ν over $\mathfrak{m} \subset R$. By construction $R \subset R^+$ induces an homeomorphism of spectra and isomorphisms on residue fields. Thus $R = R^+$ by seminormality.

In particular we have a pullback diagram

$$\begin{array}{ccc} R & \longrightarrow & R^\nu \\ \downarrow & & \downarrow \\ k(\mathfrak{m}) & \longrightarrow & R/\mathfrak{m}R. \end{array} \tag{3.2.3.a}$$

By Proposition 4.3.0.1, $R/\mathfrak{m}R$ is a two-dimensional $k(\mathfrak{m})$-vector space, and a purely inseparable field extension if and only if R is an inseparable node. Thus R is the weak gluing of R^ν above $\mathfrak{m} \subset R$ if and only if R is a separable node. □

Corollary 4.3.2.4. *Let X be a demi-normal scheme with normalization $\pi\colon \bar{X} \to X$. Then π is seminormal. It is weakly normal if and only if X has only separable nodes.*

Proof. Since X is reduced and satisfies the condition S_2, this follows at once from Lemma 4.3.2.3 and from the characterization of seminormality and weak normality in terms of functions explained in [Yan83, §1]. □

Corollary 4.3.2.5. *Let X be a demi-normal scheme with normalization $\pi\colon \bar{X} \to X$. A section $s \in \pi_*\mathcal{O}_{\bar{X}}$ belongs to $\mathcal{O}_X$ if and only if, for all codimension one point $\eta \in X$, we have*

$$s|_{\bar{X}_\eta} \in \operatorname{im}\left[k(\eta) \to H^0(\bar{X}_\eta, \mathcal{O}_{\bar{X}_\eta})\right].$$

Proof. Let $s \in \pi_*\mathcal{O}_{\bar{X}}$ be a section satisfying the restriction condition of the statement. Looking at the pullback diagram (3.2.3.a), with $R = \mathcal{O}_{X,\eta}$, we deduce that $s \in \mathcal{O}_{X,\eta}$. Since X is S_2 and η is arbitrary, we obtain that $s \in \mathcal{O}_X$. The converse is clear. □

4.3.3 Structure of the normalization morphism

Using the constructions to be described in the next two sections, we can now give a precise structure result for the normalization of demi-normal schemes:

Proposition 4.3.3.1. *Let X be a demi-normal variety and $\pi\colon \bar{X} \to X$ its normalization. Then we have a factorization*

$$\pi = \left(\bar{X} \xrightarrow{\nu} \tilde{X} \xrightarrow{F} X\right)$$

where

(a) ν is the geometric quotient of $\bar{X}$ by the finite set-theoretic equivalence relation induced by the separable nodes of X (see Section 4.4);

(b) F is the purely inseparable gluing induced by the inseparable nodes of X (see Section 4.5), and $F = \mathrm{id}$ if $2 \in \mathcal{O}_X^*$.

Moreover ν is weakly normal, and F is seminormal but not weakly normal.

Proof. Let $D = D_G + D_I \subset X$ be the conductor where D_G is the divisor corresponding to separable nodes and D_I is the divisor corresponding to inseparable nodes. Write accordingly $\bar{D} = \bar{D}_G + \bar{D}_I$ the conductor in $\bar{X}$. By Lemma 4.2.0.5 the morphism $\bar{D}_G^n \to \bar{D}_G$ is Galois with involution τ. The morphism π is finite and τ-invariant, so the quotient $\nu\colon \bar{X} \to \bar{X}/R(\tau) =: \tilde{X}$ exists and factors π. The scheme $\tilde{X}$ is a demi-normal variety with normalization $\bar{X}$ (see Proposition 4.4.0.1).

The purely inseparable morphism $\bar{D}_I \to D_I$ gives a collection of degree 2 purely inseparable field extensions $k(\eta_i) \subset k(\bar{\eta}_i)$, where η_i runs through the generic points of D_I and $\bar{\eta}_i$ through the corresponding generic points of $\bar{D}_I$. Since ν is an isomorphism at the generic points of $\bar{D}_I$, we may apply Construction 4.5.0.2 to these field extensions to obtain the purely inseparable $F\colon \tilde{X} \to X'$. Notice that $\mathcal{O}_{X'} \subset \mathcal{O}_{\tilde{X}}$ is an equality at every codimension one point of X' that is not a generic point of $\nu(\bar{D}_I)$.

We claim that $X' = X$. By the universal property of the quotient the morphism $\pi\colon \bar{X} \to X$ factors through $\tilde{X}$, say

$$\begin{tikzcd} \bar{X} \arrow[rr, "\pi"] \arrow[dr, "\nu"'] & & X \\ & \tilde{X} \arrow[ur, "r"'] & \end{tikzcd}$$

Both π and ν are finite, so by [Sta, 01YQ] and [AT51, Theorem 1] we see that r is also finite. By the proof of [Kol13, 5.3] the morphism r is an isomorphism above every codimension one point of X that is not a generic point of $D_I = r(\nu(\bar{D}_I))$. Combining with Proposition 4.3.0.11 we obtain that for every codimension one point η of X, we have $\mathcal{O}_{X,\eta} = (r_*\mathcal{O}_{X'})_\eta$ as subrings of $(r_*\mathcal{O}_{\tilde{X}})_\eta$. Since $\mathcal{O}_X$ and $r_*\mathcal{O}_{X'}$ are S_2 [KM98, 5.4] we deduce that $\mathcal{O}_X = r_*\mathcal{O}_{X'}$. In other words $(r\colon \tilde{X} \to X) = (F\colon \tilde{X} \to X')$. □

4.4 Construction of separable nodes

In this subsection we show how separable nodes appear when quotienting by finite equivalence relations.

Proposition 4.4.0.1. *Let X be a Noetherian scheme with disjoint and normal irreducible components of finite type over a scheme k, $D \subset X$ a reduced divisor with normalization $n\colon D^n \to D$ and a generically fixed point free involution $\tau\colon D^n \to D^n$. Let $R \subset X \times X$ be the equivalence relation induced by τ. Assume that either*

- *k is a field of positive characteristic and R is finite, or*
- *there exists a finite morphism $X \to X'$ which is R-invariant.*

Then:

(a) the finite geometric quotient $\pi\colon X \to X/R$ exists;

(b) X/R is demi-normal of finite type over k, $\pi(D)$ is its conductor subscheme, π is the normalization morphism and is étale over the codimension one points of X/R;

(c) if X is proper over k, then so is X/R;

(d) the involution on D^n induced by $D \to \pi(D)$ is precisely τ.

Proof. It follows from Theorem 2.5.0.2 (in case k is a field of positive characteristic) or from [Kol13, 9.10] (in case a finite R-invariant morphism exists) that the geometric quotient $\pi\colon X \to X/R =: Y$ exists and the quotient morphism is finite, see Section 2.5. Since R restricts to the identity on $X \setminus D$, the restriction $\pi|_{X\setminus D}$ is an isomorphism onto its image. Since π is finite, it must therefore be the normalization morphism. If X is of finite type over k, then so is Y by [Kol12, Theorem 41.2].

If X is proper over k, then Y is also proper over k by [Sta, 09MQ,03GN].

For the rest of the proof, let us remark that taking the quotient commutes with flat base-change [Kol13, 9.11], thus we are free to localize on Y to perform our arguments.

Next we to show that Y is demi-normal. First of all, since Y is of finite type over k, it is excellent and each local ring of Y is a quotient of a regular ring. We claim that Y is reduced. Indeed, since X is reduced then the two compositions

$$R \rightrightarrows X \xrightarrow{\pi} Y$$

factor uniquely through Y_{red}, and by the universal property of categorical quotients [Kol12, Definition 4] it follows that $Y = Y_{\mathrm{red}}$. (A similar argument shows that π must be dominant, so Y is irreducible if and only if X is).

We show that Y is nodal. For this we may localize at a codimension one point of Y. Then the set-up is the following: $Y = \operatorname{Spec} A$ is local with maximal ideal $\mathfrak{n}$, $X = \operatorname{Spec} B$ is semi-local with maximal ideals $\mathfrak{m}_i$, the map $D \to \pi(D)$ corresponds to $A/\mathfrak{n} \hookrightarrow \bigoplus_i B/\mathfrak{m}_i$, and since D is normal the involution τ becomes a non-trivial involution of $\bigoplus_i B/\mathfrak{m}_i$. Notice that the natural map $\pi\colon B \to \bigoplus_i B/\mathfrak{m}_i$ is surjective by the Chinese remainder theorem. Now consider the commutative square

$$\begin{tikzcd} A \arrow[r, hook] \arrow[d, two heads] & B \arrow[d, two heads, "\pi"] \\ K := A/\mathfrak{n} \arrow[r, hook] & L := \bigoplus_i B/\mathfrak{m}_i \end{tikzcd}$$

According to [Kol13, 9.10], a section $s \in B$ belongs to A if and only if $\pi(s) = \tau(\pi(s))$. So we deduce that the fixed subring $L^{\langle\tau\rangle}$ is equal to K, and that the Jacobson radical of B belongs to A. We claim that $\dim_K L = 2$. Since the fibers of $X \to Y$ are the R-equivalence classes, we see that B has at most two maximal ideals, and we consider these two cases separately:

(a) B is local. Then $K \subset L$ is a field extension. Since $L^{\langle\tau\rangle} = K$, we deduce that the extension is Galois of degree 2.

(b) B has two maximal ideals. Then $L = L_1 \oplus L_2$, in which K embedds diagonally, and τ exchanges the two components via an isomorphism $L_1 \cong L_2$. Identifying L_2 with L_1 via this isomorphism, we see that τ acts on $L_1 \oplus L_1$ via $(x, y) \mapsto (y, x)$. In particular the diagonal is fixed, so it must be equal to (the diagonal embedding) of K. Since $L_1 \oplus L_1$ is of dimension 2 over its diagonal, we have $\dim_K L = 2$.

By Lemma 4.2.1.1 it follows that Y is nodal. This also proves that the extension of function fields given by $\pi\colon D \to \pi(D)$ is precisely $k(D)^{\langle\tau\rangle} \subset k(D)$, so the last statement holds.

To show that Y is demi-normal, by virtue of Lemma 4.2.1.4 it remains to show that Y is S_2. This follows from Lemma 4.4.0.2 below. □

Lemma 4.4.0.2. *Let X be a reduced equidimensional scheme and $D \subset X$ a reduced scheme of pure codimension one. Let τ be an involution of D^n, such that $R(\tau) \rightrightarrows X$ is a finite set theoretic equivalence relation, and such that the geometric quotient $p\colon X \to Y := X/R(\tau)$ exists as a scheme. If X is S_2, then so is Y.*

Proof. Since X is reduced, the morphism $p\colon X \to Y$ factors through Y_{red}. It follows by the universal property of p that $Y_{\mathrm{red}} \to Y$ is an isomorphism, so Y is reduced and in particular S_1. Thus by [Har07, 1.8], it is sufficient to show that for every $U \subset Y$ open and $Z \subset U$ closed of codimension ≥ 2, the restriction map $\mathcal{O}_Y(U) \to \mathcal{O}_Y(U \setminus Z)$ is surjective. The quotient is Zariski-local, so we may assume that $U = Y$.

Take $s \in \mathcal{O}(Y \setminus Z)$. Since X is S_2 and $p^{-1}(Z)$ has codimension ≥ 2, the section p^*s extends to a global section over X. By construction $p^*s|_{D^n}$ is τ-invariant on an open dense subset, hence it is globally τ-invariant. Thus p^*s descends to a global section of Y [Kol13, 9.10] which is an extension of s. □

4.5 Construction of inseparable nodes

The goal of this section is to characterize completely demi-normal varieties with only inseparable nodes in terms of their normalizations. This is achieved in Theorem 4.5.0.6 below. The main ingredient is a construction of inseparable nodes which we present now. It can be seen as a global equivalent of the *weak gluing of rings* described in [Yan83, §3], that we recalled in Section 4.3.2: it is however more convenient for us to give a complete treatment of the construction.

Convention 4.5.0.1. In this subsection, we let X be a Noetherian equidimensional reduced scheme defined over a field k of characteristic 2.

Construction 4.5.0.2. Let $D = \sum_{i=1}^n D_i$ be a reduced Weil divisor on X. Let $k_i := k(D_i)$ be the function field D_i. Assume that for each i, we have an intermediate extension $k \subset k_i' \subsetneq k_i$ such that $k_i' \subset k_i$ is purely inseparable of degree 2. We construct a subsheaf $\mathcal{A} = \mathcal{A}(k_1', \dots, k_n')$ of $\mathcal{O}_X$ as follows: if $U \subset X$ is open, we let

$$\mathcal{A}(U) := \{s \in \mathcal{O}_X(U) \mid s(\eta_i) \in k_i' \ \forall \eta_i \in U\}$$

where η_i is the generic point of D_i, and $s(\eta_i)$ denotes the image of s through the canonical map $\mathcal{O}_X(U) \to k_i$. This defines a presheaf, which is easily seen to be a subsheaf of $\mathcal{O}_X$.

Proposition 4.5.0.3. *Let $X, D, k_i' \subset k_i$ and $\mathcal{A}$ be as in Construction 4.5.0.2. Then:*

(a) $\mathcal{O}_X^2 \subset \mathcal{A} \subset \mathcal{O}_X$*;*

(b) $\mathcal{A}$ *is a sheaf of k-algebras and* $\operatorname{Spec} \mathcal{A} := (|X|, \mathcal{A})$ *is a reduced equidimensional scheme;*

(c) $\operatorname{Spec} \mathcal{A}$ *is Noetherian if X is excellent or F-finite;*

(d) if X is S_2 and $\operatorname{Spec} \mathcal{A}$ *is Noetherian, then* $\operatorname{Spec} \mathcal{A}$ *is S_2;*

(e) if X is excellent (resp. F-finite, resp. locally of finite type over k, resp. proper over k), then so is $\operatorname{Spec} \mathcal{A}$*;*

(f) the morphism $\pi \colon X \to$ $\operatorname{Spec} \mathcal{A}$ *is an affine integral birational universal homeomorphism, and it is finite if X is excellent;*

(g) X and $\operatorname{Spec} \mathcal{A}$ *have the same normalization.*

Proof. It is clear that $\mathcal{A}$ is a sheaf of k-sub-algebras of $\mathcal{O}_X$. Since k_i/k_i' is inseparable of degree 2, we have $(k_i)^2 \subseteq k_i'$ for each i and it follows that $\mathcal{O}_X^2 \subseteq \mathcal{A}$. To show that $\operatorname{Spec} \mathcal{A}$ is a scheme, we may assume that $X = \operatorname{Spec} R$ is affine Noetherian. Then $|\operatorname{Spec} \mathcal{A}| = |X|$, and it is sufficient to show that $\mathcal{A}(X_f) = \mathcal{A}(X)_{f^2}$ for $f \in R$. The containment $\supseteq$ is clear. For the converse one, let $s/f^n \in \mathcal{A}(X_f) \subset R_f$ with $s \in R$. Then $\bar{s}/\bar{f}^n \in k_i'$, so $\bar{f}^n \bar{s} \in k_i'$ as $\bar{f}^{2n} \in k_i'$. Hence $f^n s \in \mathcal{A}(X)$, so

$$\frac{s}{f^n} = \frac{f^n s}{f^{2n}} \in \mathcal{A}(X)_{f^2}.$$

Hence $\operatorname{Spec} \mathcal{A}$ is an integral scheme and the structural morphism $\pi \colon \operatorname{Spec} X \to \operatorname{Spec} \mathcal{A}$ factors the Frobenius morphism of X. In particular π is an affine integral universal homeomorphism.

It is clear that π is an isomorphism away from the support of D. Since π is integral, the normalization of X and $\operatorname{Spec} \mathcal{A}$ are the same.

Assume that X is locally of finite type over k. Then $\operatorname{Spec} \mathcal{A}$ is locally of finite type over k^2 by [Kol12, Theorem 41.2]. Similarly, if X is proper over k then by [Sta, 09MQ,03GN] it follows that $\operatorname{Spec} \mathcal{A}$ is proper over k.

Assume that X is S_2. Since $\operatorname{Spec} \mathcal{A}$ is S_1 and $X \to \operatorname{Spec} \mathcal{A}$ is an homeomorphism, it follows easily from the criterion given in [Har07, 1.8] that $\operatorname{Spec} \mathcal{A}$ is S_2, provided it is Noetherian.

Now assume that R is excellent. Since R is reduced, we have an abstract ring isomorphism $R \cong R^2$, and so R^2 is also excellent. An excellent ring is Nagata [Sta, 07QV], so $A := \mathcal{A}(X) \subset \operatorname{Frac}(R)$ is a finite R^2-module. Therefore A is Noetherian and excellent [Sta, 07QU]. Applying the same argument for $A \subset R$, we obtain that R is finite over A. In particular π is finite.

If R is F-finite, then it is a finite R^2-module. Since R^2 is Noetherian, we get that R is a Noetherian R-module. As $A \subset R$ is an R^2-submodule, we obtain that A is a finite R^2-module, and hence a Noetherian ring. Moreover $R^4 \to R^2$ is a finite extension and factors through A^2, so $A^2 \to R^2$ is a finite extension. Hence $A^2 \to R^2 \to A$ is a composition of finite extensions, so it is finite, which means that A is F-finite. $\square$

Proposition 4.5.0.4. *Notations as in Proposition 4.5.0.3. Assume that X is excellent, normal at the generic points of D and demi-normal elsewhere. Then* $\operatorname{Spec} \mathcal{A}$ *is demi-normal with inseparable nodes at the generic points of D.*

Proof. Let $\eta \in X$ be a generic point of D, and $k'_\eta \subset k_\eta := k(\eta)$ the inseparable degree 2 sub-extension fixed at the beginning. By assumption $\mathcal{O}_{X,\eta}$ is a DVR. We know that

$$\mathcal{O}_{X,\eta}^2 \subset \mathcal{A}_\eta \subset \mathcal{O}_{X,\eta}$$

and since X is excellent, the extension $\mathcal{O}_{X,\eta}^2 \subset \mathcal{A}_\eta$ is finite. In particular there is a sujective map of rings

$$\mathcal{O}_{X,\eta}^2[x_1, \dots, x_n] \twoheadrightarrow \mathcal{A}_\eta$$

and $\mathcal{O}_{X,\eta}^2[x_1, \dots, x_n]$ is a regular ring. This shows that $\mathcal{A}_\eta$ is a quotient of a regular ring. Moreover $\mathcal{A}_\eta \to \mathcal{O}_{X,\eta}$ is the normalization. By construction $\mathfrak{m}_{X,\eta} = \mathfrak{m}_{\mathcal{A},\eta} \subset \mathcal{A}_\eta$ and $\mathcal{A}_\eta/\mathfrak{m}_{\mathcal{A},\eta} = k'_\eta$, so

$$\dim_{\mathcal{A}_\eta/\mathfrak{m}_{\mathcal{A},\eta}} \mathcal{O}_{X,\eta}/\mathfrak{m}_{X,\eta} = \dim_{k'_\eta} k_\eta = 2.$$

Therefore $\eta \in \operatorname{Spec} \mathcal{A}$ is nodal by Lemma 4.2.1.1, and it is an inseparable node by definition. On the other hand, $\operatorname{Spec} \mathcal{A}$ is demi-normal at codimension one points that are not in the image of D, since π is an isomorphism in a neighborhood of such points. Since the S_2 property of X descends to $\operatorname{Spec} \mathcal{A}$, we obtain that $\operatorname{Spec} \mathcal{A}$ is demi-normal. □

Proposition 4.5.0.5. *Notations as in Proposition 4.5.0.3. Assume that X is excellent, normal at the codimension one points of D and demi-normal elsewhere. Let Δ be a $\mathbb{Q}$-Weil divisor on X that shares no common component with D. Assume that $K_X + D + \Delta$ is $\mathbb{Q}$-Cartier. Then:*

(a) if $\Delta_{\mathcal{A}} := \pi_*\Delta$, then $K_{\operatorname{Spec}\mathcal{A}} + \Delta_{\mathcal{A}}$ is $\mathbb{Q}$-Cartier;

(b) $K_X + D + \Delta = \pi^*(K_{\operatorname{Spec}\mathcal{A}} + \Delta_{\mathcal{A}})$.

Proof. The Gorenstein locus $j : U \subset \operatorname{Spec} \mathcal{A}$ is open [GM78, 1.5], and by Proposition 4.5.0.4 and Lemma 4.2.1.4 it contains every codimension one point. By [Kol13, 5.7], we have $\pi^* K_{\operatorname{Spec}\mathcal{A}} = K_X + D$ above every codimension one point of $\operatorname{Spec} \mathcal{A}$. Since Δ and D have no common components, we may shrink U (but keeping it a big open subset) and assume that $\pi^{-1}(U) \cap D \cap \operatorname{Supp} \Delta$ is empty. As π is an isomorphism outside D, we obtain that $K_{\operatorname{Spec}\mathcal{A}} + \Delta_{\mathcal{A}}$ is $\mathbb{Q}$-Cartier over U and

$$\pi_U^*(K_{\operatorname{Spec}\mathcal{A}} + \Delta_{\mathcal{A}})|_U \sim_{\mathbb{Q}} (K_X + D + \Delta)|_{\pi^{-1}U}.$$

By [Kee99, Lemma 1.4.3], the pullback map $\pi^* \colon \operatorname{Pic}(\operatorname{Spec}\mathcal{A})[1/2] \to \operatorname{Pic}(X)[1/2]$ is an isomorphism, and this statement localizes on X. Since $K_X + D + \Delta$ is $\mathbb{Q}$-Cartier, for m divisible enough there is a Cartier divisor L on $\operatorname{Spec} \mathcal{A}$, unique up to isomorphism, such that $\pi^* L = m(K_X + D + \Delta)$. By uniqueness over U we have

$$L_U \sim_{\mathbb{Q}} m(K_{\operatorname{Spec}\mathcal{A}} + \Delta_{\mathcal{A}})|_U.$$

Since $\mathcal{O}(L)$ and $\mathcal{O}(m(K_{\operatorname{Spec}\mathcal{A}} + \Delta_{\mathcal{A}}))$ are reflexive and U is big, we obtain that

$$L \sim_{\mathbb{Q}} m(K_{\operatorname{Spec}\mathcal{A}} + \Delta_{\mathcal{A}})$$

which proves that $K_{\operatorname{Spec}\mathcal{A}} + \Delta_{\mathcal{A}}$ is $\mathbb{Q}$-Cartier and pullbacks (as $\mathbb{Q}$-divisor) to $K_X + D + \Delta$. □

Theorem 4.5.0.6. *Let k be a field of characteristic 2. Let $\mathbf{P}$ be any (or none) of the following properties: F-finite, locally of finite type over k, proper over k.*

Then normalization gives a one-to-one correspondence

$$(\operatorname{char} k = 2) \qquad \left(\begin{array}{c} \textit{Demi-normal excellent reduced} \\ \textit{equidimensional schemes } Y \textit{ over } k \\ \textit{with only inseparable nodes} \\ \textit{satisfying } \mathbf{P} \end{array} \right) \xrightarrow{1:1} \left(\begin{array}{c} \textit{Triples } \left(X, \sum_i D_i, k \subset k'_i \subset k(D_i)\right) \\ \textit{where } X \textit{ is a normal excellent} \\ \textit{scheme over } k \textit{ satisfying } \mathbf{P}, \\ \sum_i D_i \textit{ is a reduced Weil divisor,} \\ k'_i \subset k(D_i) \textit{ are degree 2} \\ \textit{inseparable extensions over } k \end{array} \right)$$

whose inverse is given by Construction 4.5.0.2. This correspondence specializes to

$$(\mathrm{char}\, k = 2) \qquad \begin{pmatrix} \textit{Demi-normal surface} \\ \textit{pairs } (S, \Delta) \textit{ over } k \\ \textit{with only inseparable nodes} \end{pmatrix} \xrightarrow{1:1} \begin{pmatrix} \textit{Normal surface} \\ \textit{pairs } (\tilde{S}, \tilde{D} + \tilde{\Delta}) \textit{ over } k \end{pmatrix}$$

where S is proper if and only if $\tilde{S}$ is proper, and $K_S + \Delta$ is ample if and only if $K_{\tilde{S}} + \tilde{D} + \tilde{\Delta}$ is ample.

Proof. Given $\left(X, \sum_i D_i, k \subset k_i' \subset k(D_i)\right)$ as in the right-hand side, Construction 4.5.0.2 gives a k-scheme Y. By Proposition 4.5.0.3 and Proposition 4.5.0.4, Y satisfies the claimed properties. This gives a map $\Phi\colon \{(X, \sum_i D_i, k_i' \subset k(D_i))\} \to \{Y\}$.

Conversely, let Y be as in the left-hand side with conductor $D \subset Y$. Let $Y^\nu \to Y$ be its normalization with conductor $D^\nu \subset Y^\nu$. Since the normalization is finite, the finiteness properties of Y ascend to Y^ν. Then $(Y^\nu, D^\nu, k(D) \subset k(D^\nu))$ is a triplet as in the right-hand side. This gives a map $\nu\colon \{Y\} \to \{(X, \sum_i D_i, k_i' \subset k(D_i))\}$.

The composition $\nu \circ \Phi$ is the identity by Proposition 4.5.0.3 and Proposition 4.5.0.4. To check that $\Phi \circ \nu(Y) = Y$, observe that both sides can be described by their structural sheaves on the topological space $|Y|$, and that both $\mathcal{O}_{\Phi\circ\nu(Y)}$ and $\mathcal{O}_Y$ are S_2 subsheaves of $\mathcal{O}_{Y^\nu}$. Thus it suffices to show that $\mathcal{O}_{\Phi\circ\nu(Y)} = \mathcal{O}_Y$ at codimension one points. Equality is obvious on the normal locus. On the demi-normal locus, it follows from Proposition 4.3.0.11.

Now let us consider the surface case. We claim that if $\tilde{S}$ is a normal surface over k and $\tilde{D} \subset \tilde{S}$ is a prime Weil divisor, then there is a unique purely inseparable degree 2 sub-extension $k' \subset k(\tilde{D})$ containing k. Since $\mathrm{trdeg}_k\, k(\tilde{D}) = 1$, there is indeed a unique such sub-extension, given by the relative Frobenius of $k(D)$ over k: see the proof of [Sta, 0CCY]. By Proposition 4.5.0.5, $K_S + \Delta$ is $\mathbb{Q}$-Cartier if and only if $K_{\tilde{S}} + \tilde{D} + \tilde{\Delta}$ is $\mathbb{Q}$-Cartier, and the first one pullbacks to the second one. Since the normalization $\tilde{S} \to S$ is finite, the ampleness statement is immediate. □

4.6 Application: Embeddings of demi-normal varieties

Let us prove the following theorem, which is a positive characteristic analog of the main result of [Ber14].

Theorem 4.6.0.1. *Let U be a demi-normal scheme that is separated and of finite type over a scheme B that is of finite type and separated over an arbitrary field of positive characteristic. Then there exists a demi-normal $\overline{U}$ that is proper over B, and a commutative diagram*

$$\begin{tikzcd} U \arrow[rr, hook, "j"] \arrow[dr] & & \overline{U} \arrow[dl] \\ & B & \end{tikzcd}$$

where j is a dominant open embedding. In addition:

(a) We can arrange so that the singular codimension one points of $\overline{U}$ are all contained in U.

(b) If U is quasi-projective over B, then we can arrange so that $\overline{U}$ is projective over B.

Proof. By Nagata compactification [Sta, 0F41] we can find a proper V over B such that U embeds as a dense open B-subscheme of V. Let $n\colon V^n \to V$ be the normalization. Let $\bar{D}$ be the closure of the conductor $D_{U^n} \subset U^n$. Since U is demi-normal we can write $\bar{D} = \bar{D}_G + \bar{D}_I$, we have an involution τ of $\bar{D}_G^n$ over V, and for every generic point $\eta \in \bar{D}_I$ the data of a purely inseparable degree 2 sub-extension of $k(\eta)$. Notice that $R(\tau) \rightrightarrows V^n$ is finite, since it arises from a normalization. Thus we may apply Proposition 4.4.0.1 and Theorem 4.5.0.6 to obtain a finite morphism $V^n \to \overline{U}$ where $\overline{U}$ is demi-normal. The image of U^n is open dense and isomorphic to U, and the codimension one singular points of $\overline{U}$ are contained in U by construction.

We show that $\overline{U}$ is proper over B. By Proposition 4.3.3.1, the morphism $V^n \to \overline{U}$ can be decomposed as

$$V^n \xrightarrow{q_1} \tilde{U} \xrightarrow{q_2} \overline{U}$$

where q_1 is the quotient by $R(\tau)$, and q_2 factors the Frobenius. Since τ commutes with the projection to V on the dense open subset $D^n_{U^n}$, it does globally, and so $V^n \to V$ factors as $V^n \to \tilde{U} \xrightarrow{h} V$. The construction of $\mathcal{O}_{\overline{U}} \subset \mathcal{O}_{\tilde{U}}$ is given by Construction 4.5.0.2 and it is easy to verify that $\mathcal{O}_V \subset h_*\mathcal{O}_{\tilde{U}}$. Thus $\tilde{U} \to V$ factors through $\overline{U}$. Since $V^n \to V$ and $V^n \to \overline{U}$ are finite, by [Sta, 01YQ] and [AT51, Theorem 1] we see that $\overline{U} \to V$ is finite. In particular $\overline{U}$ is proper over B.

Finally, assume that U is quasi-projective over B. Then instead of an arbitrary proper compactification, we can simply take V to be a projective closure of U over B. Let H be a Cartier divisor on V that is ample over B. Its pullback $H_{\overline{U}}$ is also ample over B, thus $\overline{U}$ is projective over B. □

Chapter 5

Gluing theory for slc surfaces and threefolds in positive characteristic

This chapter corresponds to the preprint [Pos21c, §4-5].

Convention 5.0.0.1. We work over a field k of positive characteristic. The assumptions on k vary through the chapter, and will be precised at the appropriate places.

5.1 Introduction

Semi-log canonical (slc) varieties were first introduced by Kollár and Shepherd-Barron in [KSB88] to study the moduli functor of surfaces of general type over the complex numbers. They showed that slc singularities appear on the special fibers of stable degenerations of smooth surfaces of general type, and that the moduli functor of complex slc surfaces of general type is coarsely represented by a separated algebraic space. It was proved a few years later that this coarse moduli space is in fact projective [Kol90, Ale94]. These results showed the importance of canonically polarized slc varieties (also called stable varieties) for the compactification of the moduli functor of smooth varieties of general type. Indeed, it has been shown that the moduli functor $\overline{\mathcal{M}}_{n,v}$ of stable varieties (of any fixed dimension n and volume v) over a field of characteristic zero, is coarsely represented by a projective algebraic space of finite type over that field [Vie95, HK04, Kar00, AH11, Kol13, Fuj18, HMX18, KP17]. See the forthcoming book [Kol21] for an exposition of the proof and of many related results.

The study of slc singularities and the construction of the compact coarse moduli space of $\overline{\mathcal{M}}_{n,v}$ owe much to the recent development of the Minimal Model Program (MMP), see e.g. [Kar00, HMX18, Kol13]. Here a technical issue seems to arise: the methods of the MMP work best for normal varieties while they might fail for normal crossing singularities [Kol11], and slc singularities are not normal as they might have nodes in codimension one. An effective way to solve this issue is to study the normalization, which has lc singularities, and then descend the information along the normalization morphism. This idea is exploited to prove the valuative criterion of properness of $\overline{\mathcal{M}}_{n,v}$, see Section 5.5 for details, and has been fruitful to study abundance of stable varieties, [Fli92, §12] and [HX13, HX16, FG14]. These results rely on a theory that allows one to go back and forth between slc varieties and their normalization in a systematic way. The main goal of this article is to establish such a theory for surfaces and threefolds in positive characteristic.

5.1.1 Gluing theory for slc varieties

Let X be an slc variety, with normalization $\nu\colon \bar{X} \to X$, and conductor divisors $D \subset X$ and $\bar{D} \subset \bar{X}$. The morphism $\bar{D} \to D$ generically looks like the normalization of the node, hence it is generically Galois of degree 2 (at least in characteristic $\neq 2$). This induces a Galois involution τ on the normalization of $\bar{D}$. Then one should be able to reconstruct X by gluing together the points that belong to the same τ-equivalence classes. This is rather easy to accomplish with the theory of quotients by finite equivalence relations. What is difficult is to decide which triplets $(\bar{X}, \bar{D}, \tau)$ arise as normalization of slc varieties. In the case of stable varieties over a field of characteristic zero, the answer is given by Kollár's gluing theory, developed in [Kol13]. An overview of this theory is given in Section 5.2.

In this article, we establish a gluing theory for slc surfaces and threefolds in positive characteristic. As the case of surfaces and the case of threefolds are quite different, with respect to both methods and results, we shall present them separately.

Gluing for surfaces.

For surfaces we prove the following theorem:

Theorem 5.1.1.1 (Theorem 5.3.1.1 and Theorem 5.3.2.1, see also Theorem 5.3.3.1)**.** *Let k be a field of positive characteristic.*

(a) If $\operatorname{char} k > 2$, then normalization gives a bijection

$$\begin{pmatrix} \textit{Slc surface pairs } (S, \Delta) \\ \textit{of finite type over } k \end{pmatrix} \xrightarrow{1:1} \begin{pmatrix} \textit{Lc surface pairs } (\bar{S}, \bar{D} + \bar{\Delta}) \\ \textit{of finite type over } k \\ \textit{plus an involution} \tau \textit{ of } (\bar{D}^n, \operatorname{Diff}_{\bar{D}^n} \bar{\Delta}) \\ \textit{that is generically fixed point free on every component }. \end{pmatrix}$$

(b) If $\operatorname{char} k = 2$, then normalization gives a bijection

$$\begin{pmatrix} \textit{Slc surface pairs } (S, \Delta) \\ \textit{of finite type over } k \end{pmatrix} \xrightarrow{1:1} \begin{pmatrix} \textit{Lc surface pairs } (\bar{S}, \bar{D}_{\mathrm{Gal}} + \bar{D}_{\mathrm{ins}} + \bar{\Delta}) \\ \textit{of finite type over } k \\ \textit{where } \bar{D}_{\mathrm{Gal}}, \bar{D}_{\mathrm{ins}} \textit{ and } \bar{\Delta} \textit{ have no common component,} \\ \textit{plus an involution} \tau \textit{ of } (\bar{D}^n_{\mathrm{Gal}}, \operatorname{Diff}_{\bar{D}^n_{\mathrm{Gal}}}(\bar{\Delta} + \bar{D}_{\mathrm{ins}})) \\ \textit{that is generically fixed point free on every component.} \end{pmatrix}$$

Here it is understood that the divisors $\bar{D}, \bar{D}_{\mathrm{Gal}}, \bar{D}_{\mathrm{ins}}$ are reduced and that the divisors $\bar{\Delta}$ have rational coefficients in $(0;1]$.

We emphasize that the surface pairs in the above theorem are not assumed to be proper over the base field. Actually, we also prove a semi-local version of the above theorem in Theorem 5.3.3.1. This seems to be a special feature of the lc surface case: compare with [Kol13, 5.13] and the counterexamples in [Kol13, §9.4].

If $\operatorname{char} k > 2$ the proof follows from the results of [Kol13, §5.3] and the observation that the equivalence relation on $\bar{S}$ generated by τ is finite by adjunction and dimensional reasons — with the caveat that in the semi-local situation, the construction of the inverse map to the normalization process is slightly indirect. The case $\operatorname{char} k = 2$ is more interesting, as a special case of normalization appears: the morphism between the conductors $\bar{D} \to D$ might be purely inseparable, see Example 4.3.0.8. To analyse these cases, we build on Theorem 4.5.0.6 that we proved in the previous chapter.

Gluing for threefolds.

Gluing theory for threefold pairs (X, Δ) is more challenging. We follow the strategy of Kollár. As the proof of the two-dimensional case hints at, we would like to have a theory of adjunction for lc centers that have codimension greater than one (see Remark 5.3.1.2). Instead of looking directly at an lc center $Z \subset X$, Kollár's idea is to consider a crepant $\mathbb{Q}$-factorial dlt blow-up (Y, Δ_Y), and a strata of $\Delta_Y^{=1}$ that is minimal over Z. The following theorem shows that the crepant birational class of that strata serves as an higher codimension version of adjunction:

Theorem 5.1.1.2 (Theorem 5.4.2.18 and Corollary 5.4.2.25)**.** *Let $f\colon (Y,\Delta_Y) \to (X, \Delta = \Delta^{=1} + \Delta^{<1})$ be crepant $\mathbb{Q}$-factorial dlt blow-up of a quasi-projective lc threefold pair over a perfect field of characteristic > 5. Let $Z \subset X$ be a lc center contained in $\Delta^{=1}$ with normalization $Z^n \to Z$.*

Let $(S, \Delta_S := \mathrm{Diff}^*_S \Delta_Y) \subset Y$ be a minimal lc center over Z, with Stein factorization $f^n_S\colon S \to Z_S \to Z^n$. Then:

(a) The crepant birational equivalence class of (S,Δ_S) over Z does not depend on the choice of S or Y. We call it the ***source*** *of Z, and denote it by $\mathrm{Src}(Z,X,\Delta)$.*

(b) The isomorphism class of Z_S over Z does not depend on the choice of S or Y. We call it the ***spring*** *of Z, and denote it by $\mathrm{Spr}(Z,X,\Delta)$.*

(c) (S,Δ_S) is dlt, $K_S + \Delta_S \sim_{\mathbb{Q},Z} 0$ and (S,Δ_S) is klt on the generic fiber above Z.

(d) The field extension $k(Z) \subset k(Z_S)$ is Galois and $\mathrm{Bir}^c_Z(S,\Delta_S) \twoheadrightarrow \mathrm{Gal}(Z_S/Z)$.

(e) For $m > 0$ divisible enough, there are well-defined Poincaré isomorphisms

$$\omega_Y^{[m]}(m\Delta_Y)|_S \cong \omega_S^{[m]}(m\Delta_S).$$

(f) If $W \subset (\Delta^{=1})^n$ is an irreducible closed subvariety such that $n(W) = Z$, where $n\colon (\Delta^{=1})^n \to \Delta^{=1}$ is the normalization, then

$$\mathrm{Src}(W, (\Delta^{=1})^n, \mathrm{Diff}_{(\Delta^{=1})^n} \Delta^{<1}) \overset{\text{cbir}}{\sim} \mathrm{Src}(Z,X,\Delta)$$

and

$$\mathrm{Spr}(W, (\Delta^{=1})^n, \mathrm{Diff}_{(\Delta^{=1})^n} \Delta^{<1}) \cong \mathrm{Spr}(Z,X,\Delta).$$

This theorem is analog to [Kol13, 4.45] and indeed our proof is similar to Kollár's one. Notice however that we only consider lc centers that are contained in the reduced boundary: see Remark 5.4.2.26 for what we can say about the other lc centers. The main step of the proof is to study the relation between the lc centers of (Y,Δ_Y) that are minimal above Z. This is done in Section 5.4.2, using the notion of weak $\mathbb{P}^1$-links (see Definition 5.4.2.6), which is a slight generalization of [Kol13, 4.36]. The proofs are once again similar to those of Kollár, but some complications arise: for example the analog of [Kol13, 4.37] in positive characteristic requires some thought, since the torsion-freeness of some higher pushforward is not available. This is bypassed in the proof of Proposition 5.4.2.12 using MMP and connectedness theorems.

Once the theory of higher codimension adjunction is established, we apply it to the gluing problems for lc threefolds. We obtain:

Theorem 5.1.1.3 (Theorem 5.4.3.6)**.** *Let k be a perfect field of characteristic > 5. Then normalization gives a bijection*

$$\begin{pmatrix} \textit{Proper slc threefold pairs} \\ (X,\Delta) \textit{ such that} \\ K_X + \Delta \textit{ is ample} \end{pmatrix} \xrightarrow{1:1} \begin{pmatrix} \textit{Proper lc threefold pairs } (\bar{X}, \bar{D} + \bar{\Delta}) \\ \textit{plus an involution } \tau \textit{ of } (\bar{D}^n, \mathrm{Diff}_{\bar{D}^n} \bar{\Delta}) \\ \textit{that is generically fixed point free on each component} \\ \textit{such that } K_{\bar{X}} + \bar{D} + \bar{\Delta} \textit{ is ample.} \end{pmatrix}$$

The method of proof is similar to [Kol13, 5.33, 5.37, 5.38]. Let us emphasize that quotients are actually easier to construct in positive characteristic than in characteristic zero, thanks to [Kol12, Theorem 6]: one only has to prove that the gluing relation is finite. The theory of sources and springs gives an appropriate set-up to study the gluing problem, but contrarily to the surface case, finiteness is non-trivial. In characteristic zero, the proof of finiteness relies on the theory of pluricanonical representations (also called B-representations by some authors), see [Kol13, §10.5]. This is well-understood in characteristic zero, but to my knowledge completely open in general over a field of positive characteristic. Fortunately the gluing problem for threefolds involves only simple cases, which are discussed in Section 5.4.1.

Before presenting some applications of this theory, let us say a word about the generalization to higher dimensions. The theory of sources and springs for threefolds (except for the Galois property) ultimately relies on existence of crepant $\mathbb{Q}$-factorial blow-up for threefolds and on the MMP for surfaces: recent developments in birational geometry, e.g. [Wal18, HW19, BK21], make us confident that these tools will soon be available in one dimension higher at least. Two other aspects of the proof are more problematic: the Galois property and pluricanonical representations. For threefolds, we prove the Galois property using the classification of surface lc singularities to make sure that inseparable extensions do not appear (see Step 3 in the proof of Theorem 5.4.2.18). A finer approach is needed in higher dimension. As for pluricanonical representations, already the case of surfaces of intermediate Kodaira dimension is challenging.

5.1.2 Applications

Let us present some applications of our theorems.

Geometry of demi-normal varieties

The study of nodal singularities leads to several interesting consequence for the geometry of demi-normal schemes.

In characteristic 2, the normalization of a demi-normal scheme $\bar{X} \to X$ might create a conductor morphism $\bar{D} \to D$ that is purely inseparable. To understand these cases, which we baptise *inseparable nodes*, we analyse in detail the normalization theory of nodes in Section 4.3. It turns out that inseparable nodes in any dimension are completely determined by the extension of functions fields given by $\bar{D} \to D$, namely:

Theorem 5.1.2.1 (Theorem 4.5.0.6). *Let k be a field of characteristic 2. Then normalization gives a bijection*

$$\begin{pmatrix} \textit{Demi-normal varieties over } k \\ \textit{with only inseparable nodes} \end{pmatrix} \xrightarrow{1:1} \begin{pmatrix} \textit{Triples } \left(X, \sum_i D_i, k \subset k_i' \subset k(D_i)\right) \\ \textit{where } X \textit{ is a normal variety over } k, \\ \sum_i D_i \textit{ is a reduced Weil divisor,} \\ k_i' \subset k(D_i) \textit{ are degree } 2 \\ \textit{inseparable extensions over } k \end{pmatrix}$$

whose inverse is given by Construction 4.5.0.2.

We also study the étale theory of inseparable nodes in Section 4.3.1 and show that the distinction between separable and inseparable nodes holds étale-locally.

The study of the gluing formalism in Section 4.4 and Section 4.5 can be applied to the following compactification property:

Theorem 5.1.2.2 (Theorem 4.6.0.1). *Let X be a demi-normal scheme that is separated and of finite type over a field k of positive characteristic. Then X embedds as an open subset of a demi-normal scheme that is proper over k.*

This is a positive characteristic analog of the main result of [Ber14], and the proof is similar in spirit.

It is possible to study slc surfaces by means of partial resolutions, as it is done in [KSB88, §4] for complex surfaces. While we do not use this approach, we record that gluing theory can be used to prove the existence of partial resolutions in positive characteristic $\neq 2$:

Theorem 5.1.2.3 (Theorem 5.3.4.8). *Let S be a demi-normal surface over an arbitrary field of characteristic $\neq 2$. Then S has an slc good semi-resolution (see Definition 5.3.4.4).*

See Remark 5.3.4.11 and Proposition 5.3.4.12 for some partial results in characteristic 2.

Lc centers of threefolds.

The theory of sources and springs has the following consequence for the topology of lc centers:

Theorem 5.1.2.4 (Corollary 5.4.2.16 and Corollary 5.4.2.17)**.** *Let (X, Δ) be a quasi-projective slc threefold over a perfect field of characteristic > 5. Then:*

(a) Intersections of lc centers are union of lc centers.

(b) Minimal lc centers are normal up to universal homeomorphism.

Let us mention that the question of normality of plt centers has been studied extensively. There are examples of non-normal plt centers on threefolds in characteristic 2 [CT19]. More generally, if the characteristic is too small compared to the dimension, there are examples of non-normal plt centers in arbitrary big dimensions [Ber19, Theorem 1.1]. On the positive side, plt centers on threefolds are normal in characteristic > 5 [HX15, Theorem 3.11, Proposition 4.1] and normal up to universal homeomorphism in general [HW19, Theorem 1.2] (see also [GNT19, §3.2]).

Moduli space of stable surfaces.

We also give an application of the gluing theory of threefolds, to the moduli theory of stable surfaces in positive characteristic. By contrast to the characteristic zero theory, not much is known about the moduli functor of stable varieties in positive characteristic beyond the classical case of stable curves. For example, the existence of a coarse moduli space in arbitrary dimension is not known. Nevertheless, the moduli functor $\overline{\mathcal{M}}_{2,v,k}$ of stable surfaces over an algebraically closed field k of characteristic > 5 is known to be represented by a separated algebraic space of finite type over k [Pat17, Corollary 9.8]. Modulo a weak version of semi-stable reduction and a technical adjunction condition (denoted respectively by **(SSR)** and **(S2)**, see Section 5.5 for details), we prove the valuative criterion of properness for $\overline{\mathcal{M}}_{2,v,k}$. Combining with the main result of [Pat17] we obtain:

Theorem 5.1.2.5 ([Pat17] and Theorem 5.5.0.5)**.** *Let k be an algebraically closed field of characteristic > 5 and v a rational number. Assume that conditions **(S2)** and **(SSR)** hold. Then $\overline{\mathcal{M}}_{2,v,k}$ admits a projective coarse moduli space.*

5.2 Introduction to Kollár's gluing theory

We give a short introduction to Kollár's gluing theory. Let X be a demi-normal variety defined over a field k of arbitrary characteristic, with normalization $\pi\colon \bar{X} \to X$. Let $D \subset X$ and $\bar{D} \subset \bar{X}$ be as in Section 4.2. If $\operatorname{char} k \neq 2$, we have seen in Lemma 4.2.0.5 that $\bar{D}^n$ comes with an involution τ and that the different $\operatorname{Diff}_{\bar{D}^n}(0)$ is τ-invariant. Hence normalization gives a map

$$(\operatorname{char} k \neq 2) \quad \begin{pmatrix} X \text{ demi-normal and} \\ \text{proper over } k \end{pmatrix} \longrightarrow \begin{pmatrix} \text{Proper normal pair } (\bar{X}, \bar{D}) \text{ with} \\ \text{an involution } \tau \text{ of } (\bar{D}^n, \operatorname{Diff}_{\bar{D}^n}(0)) \end{pmatrix} \tag{2.0.0.a}$$

Kollár's gluing theory constructs, under additional assumptions, an inverse map to (2.0.0.a). More precisely, Kollár shows [Kol13, 5.13] that normalization of demi-normal proper varieties induces a bijection

$$(\operatorname{char} k = 0) \quad \begin{pmatrix} \text{Proper slc pairs} \\ (X, \Delta) \text{ such that} \\ K_X + \Delta \text{ is ample} \end{pmatrix} \xrightarrow{1:1} \begin{pmatrix} \text{Proper lc pairs } (\bar{X}, \bar{D} + \bar{\Delta}) \text{ plus} \\ \text{a generically fixed point free} \\ \text{involution } \tau \text{ of } (\bar{D}^n, \operatorname{Diff}_{\bar{D}^n} \bar{\Delta}) \\ \text{such that } K_{\bar{X}} + \bar{D} + \bar{\Delta} \text{ is ample.} \end{pmatrix} \tag{2.0.0.b}$$

We sketch how the inverse map is constructed. Let $(\bar{X}, \bar{D}, \tau)$ be a triplet such as in the right-hand side of (2.0.0.b) (we take $\bar{\Delta} = 0$ for simplicity). There are two questions to solve: how to construct X, and how to show that K_X is $\mathbb{Q}$-Cartier.

(a) **Construction of X.** Similarly to a nodal curve that can be reconstructed from its normalization by gluing two points, the variety X should be obtained by identifying the points of $\bar{D}$ that are conjugate under τ. Since the normalization $\bar{X} \to X$ ought to be finite, the induced equivalence relation $R(\tau) \subset \bar{X} \times \bar{X}$ should be finite. However τ is defined on $\bar{D}^n$ and it is not clear that the equivalence classes on $\bar{D} \subset \bar{X}$ are finite. Indeed, they need not be if one drops the hypothesis that $(\bar{X}, \bar{D})$ is log canonical or that τ respects the log canonical stratification of $(\bar{D}^n, \mathrm{Diff}_{\bar{D}^n} 0)$ (see [Kol13, 5.17]).

Kollár's solution is to take advantage of the log canonical stratification and to proceed by induction on the lc centers of $(\bar{X}, \bar{D})$. While the singularities of the lc centers $Z \subset \bar{X}$ of high codimension can be complicated, the picture is more transparent if we take a dlt crepant blow-up $(Y, D_Y) \to (\bar{X}, \bar{D})$. Then the lc centers of $(\bar{X}, \bar{D})$ are images of the strata of $D_Y^{=1}$, and one shows [Kol13, 4.45] that the crepant birational class of a minimal stratum (S, D_S) above a fixed lc center Z of $(\bar{X}, \bar{D})$, is independant of the choice of the resolution Y and of the choice of S. Then one observes [Kol13, 5.36–37] that the equivalence classes generated by τ on Z are governed by the group of birational crepant self-maps of (S, D_S), more precisely by the representation of this group on the global sections of the invertible sheaf associated to $K_S + \Delta_S$. Then one uses the theory of pluricanonical representations [Kol13, §10.5] to obtain finiteness.

The relation between (S, D_S) and the Stein factorization Z_S of $S \to Z$ is subtle. One can think of (S, D_S) as a higher codimension version of adjunction for divisors. Kollár coined the term **source** for the crepant birational class of (S, D_S) and the term **spring** for the isomorphism class of Z_S, and studied them extensively in [Kol13].

In characteristic zero, finiteness of $R(\tau)$ is not sufficient to guarantee the existence of the quotient $\bar{X}/R(\tau)$. However it is sufficient in positive characteristic [Kol12, Theorem 6], so we will not elaborate on this issue.

(b) **Descent of $K_{\bar{X}} + \bar{D}$.** Once X is constructed, we would like to descend the $\mathbb{Q}$-Cartier divisor $K_{\bar{X}} + \bar{D}$. Kollár's strategy is to descend the total space T of $K_{\bar{X}} + \bar{D}$. Indeed, the equivalence relation $R(\tau)$ lifts to an equivalence relation R_T on T, which is shown to be finite using a strategy similar to the one explained above [Kol13, 5.38].

5.3 Gluing theory for surfaces

5.3.1 *Gluing theory for surfaces in characteristic* > 2

In this section we prove the analog of [Kol13, 5.13] for surfaces in positive characteristic different from 2.

Theorem 5.3.1.1. *Let k be a field of characteristic > 2. Then normalization gives a one-to-one correspondence*

$$(\mathrm{char}\, k > 2) \qquad \begin{pmatrix} \text{Slc surface pairs } (S, \Delta) \\ \text{of finite type over } k \end{pmatrix} \xrightarrow{1:1} \begin{pmatrix} \text{Lc surface pairs } (\bar{S}, \bar{D} + \bar{\Delta}) \\ \text{of finite type over } k \\ \text{plus an involution } \tau \text{ of } (\bar{D}^n, \mathrm{Diff}_{\bar{D}^n} \bar{\Delta}) \text{ that is} \\ \text{generically fixed point free on every component.} \end{pmatrix}$$

Moreover, S is proper if and only if $\bar{S}$ is proper, and $K_S + \Delta$ is ample if and only if $K_{\bar{S}} + \bar{D} + \bar{\Delta}$ is ample.

Proof. Given $(\bar{S}, \bar{D} + \bar{\Delta}, \tau)$, we claim that the equivalence relation $R(\tau) \subset \bar{S} \times_k \bar{S}$ is finite. It suffices to show that the equivalence classes of the points on $\bar{D}$ are finite. By [Kol13, 2.35], the closed subset $\Sigma = \mathrm{Supp}\, \mathrm{Diff}_{\bar{D}^n} \bar{\Delta}$ is equal to the preimage through $\bar{D}^n \to \bar{D}$ of the set of points $s \in \bar{D}$ such that: $s \in D \cap \mathrm{Supp}(\bar{\Delta})$, or $\bar{D}$ is singular at s, or $\bar{S}$ is singular at s. Since τ

is an involution of the pair $(\bar{D}^n, \mathrm{Diff}_{\bar{D}^n} \bar{D})$, we have $\tau(\Sigma) = \Sigma$. Moreover $n^{-1}(n(\Sigma)) = \Sigma$ and $n\colon \bar{D}^n - \Sigma \longrightarrow \bar{D} - n(\Sigma)$ is an isomorphism. Therefore:

(a) if $s \in \bar{D} - n(\Sigma)$, then the equivalence class of s is equal to $\{s, (n \circ \tau)(s')\}$ where $s' \in \bar{D}^n$ is the unique point above s;

(b) if $s \in n(\Sigma)$ then the equivalence class of s is contained in $n(\Sigma)$, which is finite.

This proves our claim. By Proposition 4.4.0.1, the geometric quotient $\pi\colon \bar{S} \to \bar{S}/R(\tau) =: S$ exists. It is a demi-normal surface over k, and is proper if and only if $\bar{S}$ is proper.

Set $\Delta := \pi_*\bar{\Delta}$. Lemma 4.3.0.5 shows that the proof of [Kol12, 5.18] is also valid over imperfect fields. Thus $K_S+\Delta$ is $\mathbb{Q}$-Cartier. Since π is an isomorphism from $\bar{S}-\bar{D}$ onto $S-\pi(\bar{D})$ and Δ has no components along $\pi(\bar{D})$, it follows from Lemma 4.2.0.3 that $\pi^*(K_S + \Delta) \sim_{\mathbb{Q}} K_{\bar{S}} + \bar{D} + \bar{\Delta}$.

This gives a map $\Phi\colon \{(\bar{S}, \bar{D} + \bar{\Delta}, \tau)\} \to \{(S, \Delta)\}$, while the nornalization gives a map ν in the other direction. We have $\nu \circ \Phi = \mathrm{Id}$ by Proposition 4.4.0.1, and $\Phi \circ \nu = \mathrm{Id}$ by Proposition 4.3.3.1. Therefore the theorem is proved. □

Remark 5.3.1.2. The key point of the proof of Theorem 5.3.1.1 is to find the τ-invariant proper subset $\Sigma \subset \bar{D}^n$ with the property that $D^n - \Sigma \longrightarrow \bar{D} - n(\Sigma)$ is an isomorphism. Its existence is given by the fact that τ preserves the lc stratification of $(\bar{D}^n, \mathrm{Diff}_{\bar{D}^n} \bar{\Delta})$. The fact that $\dim \bar{S} = 2$ then implies that $n(\Sigma)$ is finite, from which we deduce finiteness of the equivalence relation. In particular we do not need any positivity assumption on $K_{\bar{S}} + \bar{D} + \bar{\Delta}$ to conclude.

If $\dim \bar{S} > 2$ then the proof hints at an inductive process on the strata of $(\bar{D}^n, \mathrm{Diff}_{\bar{D}^n} \bar{\Delta})$. This seems dubious at first, since we have no control on the singularities of the higher codimension strata. In characteristic zero, the theory of source and springs of lc centers [Kol13, §4.5] provides a well-behaved replacement for the strata. In Section 5.4.2, we will develop a similar theory for threefolds in positive characteristic. However, it seems that positivity assumptions are needed to ensure that finiteness holds in higher codimension.

Claim 5.3.1.3. *The map $\Phi\colon \{(\bar{S}, \bar{D}+\bar{\Delta}, \tau)\} \to \{(S, \Delta)\}$ in the proof of Theorem 5.3.1.1 is also defined if* $\mathrm{char}\, k = 2$. *Indeed, [Kol13, 2.35] and Proposition 4.4.0.1 hold in every positive characteristic, and descent of the log canonical divisor holds in dimension two by Corollary 4.3.0.7.* ◊

5.3.2 Gluing theory for surfaces in characteristic 2

We prove the analog of [Kol13, 5.13] for surfaces in characteristic 2. It is given by a combination of Theorem 4.5.0.6 and the method of Theorem 5.3.1.1.

Theorem 5.3.2.1. *Let k be a field of characteristic* 2. *Then normalization gives a one-to-one correspondence*

$$(\mathrm{char}\, k = 2) \qquad \begin{pmatrix} \textit{Slc surface pairs } (S, \Delta) \\ \textit{of finite type over } k \end{pmatrix} \xrightarrow{1:1} \begin{pmatrix} \textit{Lc surface pairs } (\bar{S}, \bar{D}_{\mathrm{Gal}} + \bar{D}_{\mathrm{ins}} + \bar{\Delta}) \\ \textit{of finite type over } k \\ \textit{plus a generically fixed point free} \\ \textit{component, plus an involution} \\ \tau \textit{ of } (\bar{D}^n_{\mathrm{Gal}}, \mathrm{Diff}_{\bar{D}^n_{\mathrm{Gal}}}(\bar{\Delta} + \bar{D}_{\mathrm{ins}})) \textit{ that is} \\ \textit{generically fixed point free on every component.} \end{pmatrix}$$

Moreover, S is proper if and only if $\bar{S}$ is proper, and $K_S + \Delta$ is ample if and only if $K_{\bar{S}} + \bar{D}_{\mathrm{Gal}} + \bar{D}_{\mathrm{ins}} + \bar{\Delta}$ is ample.

Proof. We construct an inverse map in two steps. First of all, thanks to Claim 5.3.1.3 we can apply the method of Theorem 5.3.1.1 to the triplet $(\bar{S}, \bar{D}_{\mathrm{Gal}} + \bar{\Gamma}, \tau)$ where $\bar{\Gamma} := \bar{D}_{\mathrm{ins}} + \bar{\Delta}$. We obtain an slc pair $(\tilde{S}, \tilde{\Gamma} = \tilde{D}_{\mathrm{ins}} + \tilde{\Delta})$. Then, we can apply Theorem 4.5.0.6 to it and obtain a slc pair (S, Δ). This gives a map

$$(\bar{S}, \bar{D}_{\mathrm{Gal}} + \bar{D}_{\mathrm{ins}} + \bar{\Delta}, \tau) \xrightarrow{\Phi_{\mathrm{Gal}}} (\tilde{S}, \tilde{D}_{\mathrm{ins}} + \tilde{\Delta}) \xrightarrow{\Phi_{\mathrm{ins}}} (S, \Delta).$$

It follows from Proposition 4.3.3.1 that $\Phi_{\mathrm{ins}} \circ \Phi_{\mathrm{Gal}}$ is an inverse map to the normalization. The statements about properness and ampleness follow from the corresponding statement in Theorem 4.5.0.6 and Claim 5.3.1.3. $\square$

5.3.3 Gluing theory for germs of surfaces

Our gluing theorems for surfaces, Theorem 5.3.1.1 and Theorem 5.3.2.1, are formulated for surfaces that are quasi-projective. It is natural to ask whether similar statements hold for germs of surfaces. If $(\bar{S}, \bar{D} + \bar{\Delta})$ is a germ of lc surface with a log involution τ on $(\bar{D}^n, \operatorname{Diff} \bar{\Delta})$, then the argument in the proof of Theorem 5.3.1.1 applies verbatim to show that $R(\tau) \rightrightarrows \bar{S}$ is a finite equivalence relation, and [Kol12, Theorem 6] applies to schemes that are essentially of finite type over k. So we obtain a quotient $\bar{S}/R(\tau)$. However I am not aware of an Eakin–Nagata type theorem for rings that are essentially of finite type over k, so it is unclear what type of algebraic space the quotient $\bar{S}/R(\tau)$ is. It turns out that it is a germ of variety, but our proof is somewhat roundabout. The details are given in the next theorem.

Theorem 5.3.3.1. *Let k be a field of positive characteristic. Then:*

(a) If $\operatorname{char} k > 2$, then normalization gives a one-to-one correspondence

$$\left(\begin{array}{c}\textit{Slc semi-local affine}\\ \textit{surface pairs } (S, \Delta)\\ \textit{essentially of finite type over } k\end{array}\right) \xrightarrow{1:1} \left(\begin{array}{c}\textit{Lc semi-local affine}\\ \textit{surface pairs } (\bar{S}, \bar{D} + \bar{\Delta})\\ \textit{essentially of finite type over } k\\ \textit{plus an involution } \tau \textit{ of } (\bar{D}^n, \operatorname{Diff}_{\bar{D}^n} \bar{\Delta}) \textit{ that is}\\ \textit{generically fixed point free on every component.}\end{array}\right)$$

(b) If $\operatorname{char} k = 2$, then normalization gives a one-to-one correspondence

$$\left(\begin{array}{c}\textit{Slc semi-local affine}\\ \textit{surface pairs } (S, \Delta)\\ \textit{essentially of finite type over } k\end{array}\right) \xrightarrow{1:1} \left(\begin{array}{c}\textit{Lc semi-local affine}\\ \textit{surface pairs } (\bar{S}, \bar{D}_{\mathrm{Gal}} + \bar{D}_{\mathrm{ins}} + \bar{\Delta})\\ \textit{essentially of finite type over } k\textit{, plus an involution}\\ \tau \textit{ of } (\bar{D}^n_{\mathrm{Gal}}, \operatorname{Diff}_{\bar{D}^n_{\mathrm{Gal}}}(\bar{\Delta} + \bar{D}_{\mathrm{ins}})) \textit{ that is}\\ \textit{generically fixed point free on every component.}\end{array}\right)$$

Moreover, in both cases S is local if and only if all closed points of $\bar{S}$ belong to the same $R(\tau)$-equivalence class.

Proof. We show that, starting with an lc semi-local affine pair $(\operatorname{Spec} \mathcal{O}, D_{\mathcal{O}} + \Delta_{\mathcal{O}})$ essentially of finite type over k with an involution τ, we can produce an slc semi-local affine pair $(\operatorname{Spec} \mathcal{O}', \Delta_{\mathcal{O}'})$ essentially of finite type over k and a finite morphism $\operatorname{Spec} \mathcal{O} \to \operatorname{Spec} \mathcal{O}'$ that is the quotient by $R(\tau)$. Then we show that Construction 4.5.0.2 preserves the property of being essentially of finite type over k. The proof that the combination of these operations is the inverse of the normalization is exactly the same as in Theorem 5.3.1.1 and Theorem 5.3.2.1.

Hence let $\mathcal{O}$ be a semi-local ring of equi-dimension two that is essentially of finite type over k, $D_{\mathcal{O}}$ a reduced divisor and $\Delta_{\mathcal{O}}$ an effective $\mathbb{Q}$-divisor on $\operatorname{Spec} \mathcal{O}$, with no component in common. Since $\mathcal{O}$ is essentially of finite type over k, there exists a scheme $\bar{S}$ of finite type over k, such that $\operatorname{Spec} \mathcal{O}$ is a subscheme of $\bar{S}$. Replacing $\bar{S}$ by the closure of $\operatorname{Spec} \mathcal{O}$, we may assume that $\bar{S}$ is two-dimensional.

Assume furthermore that $(\operatorname{Spec} \mathcal{O}, D_{\mathcal{O}} + \Delta_{\mathcal{O}})$ is lc. Then $\operatorname{Spec} \mathcal{O}$ belongs to the normal locus of $\bar{S}$, so taking the normalization we may assume that $\bar{S}$ is normal. Taking an open subset, we may actually assume that $\bar{S}$ is regular away from the closed points of $\operatorname{Spec} \mathcal{O}$.

Let $\bar{D}$ and $\bar{\Delta}$ be the closures (as $\mathbb{Q}$-divisors) of $D_{\mathcal{O}}$ and $\Delta_{\mathcal{O}}$ in $\bar{S}$. We may shrink $\bar{S}$ until $\bar{D}$ is regular away from the closed points of $\operatorname{Spec}\mathcal{O}$, and $\bar{D}\cap\bar{\Delta}$ is supported on the closed points of $\operatorname{Spec}\mathcal{O}$. Then $K_{\bar{S}}+\bar{D}+\bar{\Delta}$ is $\mathbb{Q}$-Cartier, $(\bar{S},\bar{D}+\bar{\Delta})$ is lc and $\operatorname{Diff}_{\bar{D}^n}\bar{\Delta}=\operatorname{Diff}_{D^n_{\mathcal{O}}}\Delta_{\mathcal{O}}$ by [Kol13, 2.35].

Let τ be a generically fixed point free involution of $(D^n_{\mathcal{O}},\operatorname{Diff}_{D^n_{\mathcal{O}}}\Delta_{\mathcal{O}})$. Then τ extends to an involution $\bar{\tau}$ on the projective regular model $\mathcal{D}$ of $D^n_{\mathcal{O}}$. Since $\bar{D}^n$ is a dense open subset of $\mathcal{D}$, the set $\mathcal{D}-\bar{D}^n$ is finite. As $\bar{\tau}$ is an involution and $D^n_{\mathcal{O}}$ is τ-stable, we can find finitely many closed points $x_1,\dots,x_n\in\bar{D}^n-D^n_{\mathcal{O}}$ such that $\bar{D}^n-\{x_1,\dots,x_n\}$ is $\bar{\tau}$-stable. Since $\bar{D}$ is regular away from $D_{\mathcal{O}}$, the x_i correspond to unique closed points of $\bar{S}$. Therefore after shrinking $\bar{S}$, we may assume that $\bar{\tau}$ gives an involution of $\bar{D}^n$. Moreover, since $\operatorname{Diff}_{D^n_{\mathcal{O}}}\Delta_{\mathcal{O}}=\operatorname{Diff}_{\bar{D}^n}\bar{\Delta}$, we see that $\bar{\tau}$ is a generically fixed point free involution of the pair $(\bar{D}^n,\operatorname{Diff}_{\bar{D}^n}\bar{\Delta})$.

Thus we may apply Theorem 5.3.1.1 and Claim 5.3.1.3 to obtain a finite quotient $q\colon\bar{S}\to\bar{S}/R(\bar{\tau})=:S$. Moreover the pair $(S,\Delta:=q_*\bar{\Delta})$ is an slc surface pair of finite type over k and q is the normalization morphism.

To simplify the rest of the discussion, we reduce to the case where S and $\bar{S}$ are affine. Let $\Sigma\subset S$ be the finite reduced subscheme supported on the image of the closed points of $\operatorname{Spec}\mathcal{O}$. There exists an affine open subscheme $S'\subset S$ containing Σ (because S satisfies the Chevalley–Kleiman property, see [Kol13, 9.28]). Then $q^{-1}S'$ is affine open and contains $\operatorname{Spec}\mathcal{O}$, and $q^{-1}S'\to S'$ is the quotient by the induced equivalence relation on $q^{-1}S'$ [Kol13, 9.11].

So say $S=\operatorname{Spec}A$, $\bar{S}=\operatorname{Spec}\bar{A}$ and $\Sigma=V(\mathfrak{m}_1)\cup\dots\cup V(\mathfrak{m}_n)$. Then $T:=A-\bigcup_i\mathfrak{m}_i$ is a multiplicatively closed set, $T^{-1}A$ is a semi-local ring essentially of finite type over k with maximal ideals $\mathfrak{m}_i$ and $T^{-1}\bar{A}$ is the normalization of $T^{-1}\bar{A}$ inside $\operatorname{Frac}A=\operatorname{Frac}\bar{A}$. On the other hand, $\mathcal{O}$ is a fraction ring of $\bar{A}$, say $\mathcal{O}=U^{-1}\bar{A}$. Since the maximal ideals of $U^{-1}\bar{A}$ are precisely those above the $\mathfrak{m}_i$'s, we see that $U^{-1}\bar{A}$ is a further localization of $T^{-1}\bar{A}$ and that these two rings have the same maximal ideals. But this implies that $T^{-1}\bar{A}=U^{-1}\bar{A}$ already. Applying [Kol13, 9.11] once again we obtain that $\operatorname{Spec}\mathcal{O}\to\operatorname{Spec}T^{-1}A$ is the quotient by the finite equivalence relation $R(\tau)$, and by construction $(\operatorname{Spec}T^{-1}A,\Delta_{T^{-1}A})$ is a semi-local slc surface pair essentially of finite type over k.

It remains to show that Construction 4.5.0.2 applied to a semi-local ring $\mathcal{O}$ essentially of finite type over k, produces a semi-local ring essentially of finite type over k. The construction output is a ring $\mathcal{A}$ and successive finite extensions $\mathcal{O}^2\subset\mathcal{A}\subset\mathcal{O}$ which shows that $\mathcal{A}$ is semi-local. Since $\mathcal{O}$ is essentially of finite type over k, so is $\mathcal{O}^2$, and as $\mathcal{O}^2\subset\mathcal{A}$ is a finite extension we obtain that $\mathcal{A}$ is essentially of finite type over k. This completes the proof. □

5.3.4 Application: Semi-resolutions of demi-normal surfaces

It might be difficult to study some aspects of a demi-normal scheme X in terms of its normalization $\nu\colon\bar{X}\to X$, since $\nu_*\mathcal{O}_{\bar{X}}\neq\mathcal{O}_X$ and ν is not an isomorphism in codimension one. Instead, one may try to resolve singularities in codimension two only. In characteristic zero, there is a good notion of such partial resolutions for demi-normal surfaces [vS87, KSB88]. We work out the case of demi-normal surfaces in positive characteristic.

Definition 5.3.4.1. A germ of surface $(s\in S)$ is called a **normal crossing point**, respectively a **pinch point**, if there is a finite étale morphism $\widehat{\mathcal{O}_{S,s}}\to\mathcal{O}'$ such that $\mathcal{O}'\cong k[[x,y,z]]/(xy)$, respectively $\mathcal{O}'\cong k[[x,y,z]]/(x^2-zy^2)$, where k is some field.

A surface (essentially of finite type over a field) is called **semi-smooth** if every closed point is either regular, normal crossing or a pinch point.

Remark 5.3.4.2. The étale base-change $\mathcal{O}_{S,s}\to\mathcal{O}'$ may be non-elementary (that is, it may not induce an isomorphism of residue fields). But if $k(s)$ is algebraically closed, $\widehat{\mathcal{O}_{S,s}}\to\mathcal{O}'$ is necessarily elementary. Since $\widehat{\mathcal{O}_{S,s}}$ is Henselian and $\mathcal{O}'$ is assumed to be local, we deduce that $\mathcal{O}'\cong\widehat{\mathcal{O}_{S,s}}$. In other words, we can omit the base-change in the definition if we work over an algebraically closed field.

Lemma 5.3.4.3. *Let S be a semi-smooth surface. Then:*

(a) *S is a local complete intersection scheme (that is, every completed local ring $\widehat{\mathcal{O}_{S,s}}$ is the quotient of a regular ring by a regular sequence), and in particular S is Cohen–Macaulay and Gorenstein;*

(b) *S is demi-normal;*

(c) $\mathrm{Sing}(S)$ *(with its reduced structure) is the conductor subscheme, and it is regular and of pure dimension one;*

(d) *the normalization of S is regular with regular conductor.*

Proof. For the first property, since the local complete intersection locus is open by [GM78, 3.3], it is sufficient to show the local rings of S at closed points are local complete intersections. This property descends and ascends étale morphisms [Sta, 09Q7]. Thus it suffices to show that the models $k[x,y,z]/(xy)$ and $k[x,y,z]/(x^2-zy^2)$ are local complete intersections (near the origin), which is clear.

We claim that S is also semi-normal. Since the semi-normal locus is open (Observation 4.2.1.3), it is sufficient to check that the closed points belong to it. Combining [GT80, Theorem 1.6] and [Gro65, 7.8.3.vii], it is sufficient to check that condition on the local models $k[x,y,z]/(xy)$ and $k[x,y,z]/(x^2-zy^2)$. This is easily seen, e.g. using [GT80, Corollary 2.7.vii]. Now it follows by Lemma 4.2.1.4 that S is demi-normal.

If $f\colon X\to Y$ is an étale or a regular morphism of Noetherian schemes, then $\mathcal{O}_{X,x}$ is regular if and only if $\mathcal{O}_{Y,f(x)}$ is regular [Mat89, 23.7], and the completion of an excellent local ring is a regular morphism [Gro65, 7.8.3.v]. Moreover the regular locus is open [Gro65, 7.8.3.iv], and the normalization commutes with the completion [Gro65, 7.8.3.vii]. Thus it is sufficient to prove the last two properties for the spectra of $k[x,y,z]/(xy)$ and $k[x,y,z]/(x^2-zy^2)$ near the origin. This is clear in both cases (see Example 4.3.0.8 for a study of the second singularity in characteristic 2). □

Definition 5.3.4.4. A proper birational morphism of demi-normal surfaces $f\colon T\to S$ is called a **semi-resolution** if:

(a) T is semi-smooth;

(b) no component of D_T, the conductor divisor of T, is f-exceptional;

(c) f is an isomorphism over a big open subset of S.

We say that f is a **good semi-resolution** if in addition

(d) $\mathrm{Exc}(f)$ has regular components which intersect with only double points, and $\mathrm{Exc}(f)\cup D_T$ has at most triple points.

In characteristic zero, it is well-known that demi-normal surfaces admit good semi-resolutions. We prove that it is also true for demi-normal surfaces in characteristic $\neq 2$. Our proof is similar in spirit to [vS87, §1.4].

Lemma 5.3.4.5. *Let K be a field of characteristic $\neq 2$ and τ a non-trivial non-necessarily K-linear involution of $K[[t]]$ such that $\tau(K)=K$. Then there exists a uniformizer $s\in(t)\setminus(t^2)$ such that $\tau(s)=-s$.*

Proof. We have $\tau(t)=\lambda t$ with $\lambda\in K[[t]]^\times$. If the constant term of λ is not equal to 1, then $1-\lambda\in K[[t]]^\times$ and so the K-linear ring map defined by $t\mapsto t-\tau(t)$ is an automorphism, thus we may take $s=t-\tau(t)$.

From now on assume that the constant term of λ is 1. If τ does not act as the identity on K, choose $\alpha\in K\setminus K^\tau$. If $\alpha=\tau(\alpha)+1$ then $\alpha=\tau(\tau(\alpha))=\tau(\alpha-1)=\tau(\alpha)-1$, a contradiction since the characteristic is different from 2. Therefore $\alpha-\tau(\alpha)\lambda$ is invertible with constant term different from 1. Thus we apply the argument of the previous paragraph with αt in place of t.

Finally assume that τ is K-linear. By the current assumption on λ we have $\tau(t)=t+O(t^2)$, and so $\tau(t^k)=t^k+O(t^{k+1})$. We define a Cauchy sequence (t_n) such that $t_n-\tau(t_n)\in O(t^{n+1})$.

Take $t_1 := t$. If $t_n - \tau(t_n) = at^{n+1} + O(t^{n+2})$ with $a \in K$, the element $t_{n+1} := t_n - \frac{a}{2}t^{n+1}$ is a valid choice. Then $t_\infty := \lim_n t_n$ is a uniformizer that satisfies $t_\infty = \tau(t_\infty)$. Hence τ is the identity, and we excluded this case in the hypothesis. Thus the proof is complete. □

Lemma 5.3.4.6. *Let S be a regular surface of finite type over an arbitrary field k of positive characteristic $\neq 2$, $D \subset S$ a reduced divisor with regular support and $\tau\colon D \cong D$ a non-trivial involution. Then $S/R(\tau)$ exists and is a semi-smooth surface whose conductor subscheme is the image of D.*

Proof. We elaborate the last paragraph of [Kol13, 1.43]. First of all, it is clear that $R(\tau)$ is finite, thus $U := S/R(\tau)$ exists as a scheme of finite type over k and $\pi\colon S \to U$ is the normalization. By Proposition 4.4.0.1, U is demi-normal with conductor $\pi(D)$. By [Kol13, 9.13, 9.30] the square

$$\begin{tikzcd} D \arrow[r, hook] \arrow[d, "q"] & S \arrow[d, "\pi"] \\ D/\langle\tau\rangle \arrow[r, hook] & U \end{tikzcd}$$

is a push-out. Geometric quotients by finite group actions preserve normality, so $\pi(D)$ is normal, therefore a regular curve.

To study the singularities of U, we may localize at a closed point $u \in U$ that belongs to $\pi(D)$. Then we may assume that $U = \operatorname{Spec}\mathcal{O}$ is local and that $S = \operatorname{Spec} A$. Since $D \to \pi(D)$ is a $\mathbb{Z}/2\mathbb{Z}$-quotient, only two cases can happen.

(a) A has exactly two maximal ideals. Since $\hat{\mathcal{O}} \cong \widehat{\mathcal{O}^h}$ [Sta, 06LJ], we might without loss of generality base-change along an elementary étale morphism $\operatorname{Spec}\mathcal{O}' \to \operatorname{Spec}\mathcal{O}$ and assume that $A = A_1 \oplus A_2$, where both A_i are local. Let $f_i \in A_i$ be the local equation of D, and $\tau\colon A_1/(f_1) \cong A_2/(f_2)$ be the involution. Since $A_1/(f_1)$ is regular local of dimension one, there exists $g_1 \in A_1$ such that $(f_1, g_1) = \mathfrak{m}_{A_1}$. Let $g_2 \in A_2$ be any lift of $\tau(g_1 + (f_1))$, it also holds that $(f_2, g_2) = \mathfrak{m}_{A_2}$.

The push-out description of $\mathcal{O} \subset A_1 \oplus A_2$ shows that $\mathfrak{m}_{\mathcal{O}} = \mathcal{O} \cap (\mathfrak{m}_{A_1} \oplus \mathfrak{m}_{A_2})$. Hence we see that $x := (f_1, 0), y := (0, f_2), z := (g_1, g_2)$ generate $\mathfrak{m}_{\mathcal{O}}$, with relation $xy = 0$. Thus $\hat{\mathcal{O}} \cong k[[x, y, z]]/(xy)$, where k is the residue field of $\mathcal{O}$.

(b) A is local, in other words u is the image of a τ-fixed point. Let $f \in A$ be the local equation of D, and $\tau\colon A/(f) \cong A/(f)$ be the involution. We may work with the completions, as $\hat{\mathcal{O}}$ is the preimage in $\hat{A}$ of the τ-invariant elements of $\hat{A}/(f)$.

The action of τ descends to the residue field K of A, and extends to the completion of $A/(f)$. Notice that the restriction of τ on the coefficient field $K \subset \hat{A}/(f)$ is precisely the action of τ on the residue field. The residue field of $\mathcal{O}$ is the fixed subfield $K' := K^\tau$. If K' is algebraically closed, notice that necessarily $K = K'$.

The completion $\hat{A}/(f)$ is isomorphic to $K[[t]]$. By Lemma 5.3.4.5 we may assume that $\tau(t) = -t$.

If $K' = K$, then $\hat{\mathcal{O}} = K[[f, fg, g^2]] \cong K[[x, y, z]]/(x^2z - y^2)$, where $g \in \hat{A}$ is any lift of $t \in \hat{A}/(f)$. This is a pinch point.

If $K' \subsetneq K$ then $K = K'(\gamma)$ where $\gamma^2 = c \in K$ and $\tau(\gamma) = -\gamma$. A monomial $(a + b\gamma)t^i$, with $a, b \in K'$, is τ-invariant if and only if i is even and $b = 0$, or i is odd and $a = 0$. Thus if $g \in \hat{A}$ is any lift of t, we have

$$\hat{\mathcal{O}} = K'[[f, fg, g^2]] + \gamma g \cdot K'[[g^2]] + \gamma \cdot K'[[f, fg, fg^2, fg^3, \dots]].$$

Let

$$x := g^2,\ y := \gamma g,\ z := f,\ v := fg,\ w := \gamma f,$$

then we have the presentation

$$\hat{\mathcal{O}} \cong \frac{K'[[x,y,z,v,w]]}{(y^2+cx, yw+cv, cz^2+w^2, xz^2-v^2)} \cong \frac{K'[[y,z,w]]}{(cz^2+w^2)}.$$

The finite étale extension $\hat{\mathcal{O}} \subset \hat{\mathcal{O}}[T]/(T^2+c)$ shows that $\mathcal{O}$ is a normal crossing point. This completes the proof. $\square$

Remark 5.3.4.7. In the situation of Lemma 5.3.4.6, notice that if the base field k is algebraically closed, then the normal crossing points of U are the image of the non-τ-fixed points of D, and the pinch points of U are the images of the τ-fixed points of D.

Theorem 5.3.4.8. *Let S be a demi-normal surface that is essentially of finite type over an arbitrary field k of positive characteristic $\neq 2$. Then S has an slc good semi-resolution.*

Proof. It follows from Lemma 4.2.1.4, Observation 4.2.1.3, [Gro65, 7.8.3.iv] and [GM78, 1.5] that the demi-normal locus is open on excellent schemes, thus we can realize S as the localization of a demi-normal surface of finite type over k. Hence we may assume that S is of finite type over k to begin with.

Let $\nu\colon (\bar{S}, D_{\bar{S}}) \to S$ be the normalization morphism. By [Kol13, 2.25] there exists a proper birational morphism $\bar{f}\colon \bar{T} \to \bar{S}$, where $\bar{T}$ is regular, $D_{\bar{T}} := \bar{f}_*^{-1} D_{\bar{S}}$ is regular, the components of $\operatorname{Exc}(\bar{f})$ are regular and $\operatorname{Exc}(\bar{f}) \cup D_{\bar{T}}$ has normal crossings.

The normalization $\bar{S} \to S$ induces an involution τ on $D^n_{\bar{S}} = D_{\bar{T}}$. So we get a finite equivalence relation $R(\tau) \rightrightarrows \bar{T}$. We can blow-up a few more points to ensure that the restriction of $R(\tau)$ to each irreducible component of $E = \operatorname{Exc}(\bar{f})$ is the identity. There is a minimal way to do so: let us call $\bar{T} \to (\bar{S}, D_{\bar{S}})$ the **minimal τ-log resolution**. Now let $q\colon \bar{T} \to T$ be the quotient morphism. By the universal property of the quotient, there is a unique morphism $f\colon T \to S$ such that the square

$$\begin{array}{ccc} \bar{T} & \xrightarrow{\bar{f}} & \bar{S} \\ \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle \nu} \\ T & \xrightarrow{f} & S \end{array}$$

commutes. Combining Proposition 4.4.0.1 and [Sta, 09MQ, 03GN] we see that f is proper, and clearly it is birational.

By Lemma 5.3.4.6 the surface T is semi-smooth. As ω_T pullbacks to $\omega_{\bar{T}}(D_{\bar{T}})$, we see that T is slc. Moreover each component of $q(E) = \operatorname{Exc}(f)$ is still regular. Since we glue along an involution, $q(E) \cup D_T$ are at worst triple points and $q(E)$ has at most double points.

Finally there is a big open subset $U \subset S$ such that both $\nu^{-1}(U)$ and $D_{\bar{S}} \cap \nu^{-1}(U)$ are regular, and $f(\operatorname{Exc}(f)) \cap U$ is empty. Then f is an isomorphism over U. $\square$

Example 5.3.4.9. Let us illustrate this semi-resolution procedure with the triple point $S = (xyz = 0) \subset \mathbb{A}^3$. The normalization $\bar{S}$ is a disjoint union of three planes, and the conductor is the union of the coordinate axis on each plane. Let $\bar{T} = \bigsqcup_{i=0}^2 \bar{T}_i$ be the blow-up of $\bar{S}$ at the three origins, and $L_i^1, L_i^2 \subset \bar{T}_i$ be the transforms of the coordinate axis. Then the semi-resolution $T \to S$ is obtained by gluing L_i^1 along L_{i+1}^2, where the index is taken modulo 3.

Proposition 5.3.4.10. *Let S be as in Theorem 5.3.4.8.*

(a) If $f\colon T \to S$ is a semi-resolution then $f_*\mathcal{O}_T \cong \mathcal{O}_S$,

(b) Grauert–Riemenschneider vanishing holds: $R^1 f_*\omega_T = 0$.

(c) There exists a minimal good semi-resolution of S.

Proof. By definition f is an isomorphism over a big open subset of S, thus the inclusion $\mathcal{O}_S \subseteq f_*\mathcal{O}_T$ is an equality in codimension one. Since $\mathcal{O}_S$ is S_2, equality holds everywhere.

Let $\pi\colon \bar{T} \to T$ be the normalization. The trace map $\pi_*\omega_{\bar{T}} \to \omega_T$ is given by the evaluation at 1

$$\pi_*\omega_{\bar{T}} = \mathrm{Hom}_T(\pi_*\mathcal{O}_{\bar{T}}, \omega_T) \xrightarrow{\mathrm{ev}_1} \omega_T.$$

Since the characteristic is different from 2, the normalization is étale over codimension one points by Lemma 4.2.0.3. Thus the trace map is injective in codimension one. The pushforward $\pi_*\omega_{\bar{T}}$ is S_2 and ω_T is torsion-free, so the trace map is injective. Thus we obtain an exact sequence

$$0 \to \pi_*\omega_{\bar{T}} \to \omega_T \to \mathcal{Q} \to 0,$$

where $\mathcal{Q}$ is supported on the divisor $D_T = \mathrm{Sing}(T)$. Pushing forward along $f\colon T \to S$ and using that π is finite, we obtain the exact sequence

$$R^1(f\circ\pi)_*\omega_{\bar{T}} \cong R^1 f_*(\pi_*\omega_{\bar{T}}) \to R^1 f_*\omega_T \to R^1 f_*\mathcal{Q} \cong R^1(f|_{D_T})_*\mathcal{Q}.$$

We have $R^1(f\circ\pi)_*\omega_{\bar{T}} = 0$ by [Kol13, 10.4] and $R^1(f|_{D_T})_*\mathcal{Q} = 0$ since $f|_{D_T}$ is finite on its image by assumption on f. Thus $R^1 f_*\omega_T = 0$.

Finally, let $\bar{S} \to S$ be the normalization, $\bar{T}_m \to \bar{S}$ the minimal τ-log resolution of $(\bar{S}, D_{\bar{S}})$ and $f_m\colon T_m \to S$ be the semi-resolution obtained from $\bar{T}_m$, as in the proof of Theorem 5.3.4.8. If $f\colon T \to S$ is a good semi-resolution, then using Lemma 5.3.4.3 we see that $\bar{T} \to (\bar{S}, D_{\bar{S}})$ is a τ-log resolution. By minimality of $\bar{T}_m$ the morphism $\bar{T} \to \bar{S}$ factors through $\bar{T}_m$. Since $\pi\colon \bar{T} \to T$ is a quotient, it is easy to see that we obtain a commutative diagram

$$\begin{tikzcd} \bar{T} \arrow[r] \arrow[d, "\pi"] & \bar{T}_m \arrow[r] \arrow[d] & \bar{S} \arrow[d] \\ T \arrow[r, "h"] & T_m \arrow[r, "f_m"] & S \end{tikzcd}$$

So both $f_m \circ h$ and f give a factorization of $\bar{T} \to \bar{S} \to S$ through T. Using again that $\bar{T} \to T$ is a quotient, we deduce that $f_m \circ h = f$. This shows minimality of T_m amongst good semi-resolutions. □

Remark 5.3.4.11. There are two obstacles to extend Theorem 5.3.4.8 in characteristic 2. First of all we must choose whether or not the semi-resolution $T \to S$ is an isomorphism over the inseparable nodes of S. If we want a local isomorphism, we must apply Construction 4.5.0.2 to a regular curve on $\bar{T}$. Then it seems difficult to provide a formal description of the local rings at the closed points of the inseparable-nodal locus of T.

The second difficulty stems from the failure of Lemma 5.3.4.5 in characteristic 2, as demonstrated by the $\mathbb{F}_2$-linear involution of $\mathbb{F}_2[[t]]$ given by $t \mapsto (1+t^2+t^3+\dots)t$. Thus the treatment of τ-fixed points in the proof of Lemma 5.3.4.6 becomes problematic. In principle one could use Artin–Schreier theory to classify involutions on one-dimensional power series, see e.g. the example at the end of [Art75] for the linear cases. This leads to surface singularities that are not normal crossing or pinch points, for example $(x^2 + zy^2 + xyz^r = 0) \subset \mathbb{A}^3$ for $r \geq 1$.

We can nonetheless prove a weaker semi-resolution statement for surfaces with only separable nodes:

Proposition 5.3.4.12. *Let S be a demi-normal surface with only separable nodes over an arbitrary field. Then there exists a proper birational morphism $f\colon T \to S$ such that*

(a) T is slc 2-Gorenstein with regular conductor $D_T = \mathrm{Sing}(T)$;

(b) f is an isomorphism over a big open subset of S;

(c) no component of D_T is f-exceptional;

(d) each component of $\mathrm{Exc}(f)$ is regular, $\mathrm{Exc}(f)$ has at worst double points and $\mathrm{Exc}(f) \cup D_T$ at worst triple points.

Proof. We repeat the construction of Theorem 5.3.4.8. The only part of the proof of that does not extend in our generality is the study of the ring-theoretic singularities of T. By Proposition 4.4.0.1, T is a demi-normal surface. Moreover $(\bar{T}, D_{\bar{T}})$ is lc and $\omega_{\bar{T}}(D_{\bar{T}})|_{D_{\bar{T}}} = \omega_{D_{\bar{T}}}$ is τ-invariant. Hence we may apply the proof of [Kol13, 5.38], which works over any field as explained in Corollary 4.3.0.7, and get that $\omega_T^{[2]}$ is invertible. Thus T is 2-Gorenstein and slc. $\square$

5.4 Gluing theory for threefolds

5.4.1 Preliminary results

In this first section some results that will be needed for the proof of the gluing theorem for threefolds. While most of them are well-known, we include proofs to conveniently reference them.

Some results about surfaces and threefolds.

We gather some facts about the geometry of surface and threefold pairs.

Fact 5.4.1.1 (Birational Geometry of excellent surfaces)**.** *Resolutions of singularities (resp. log resolutions) exist for excellent surfaces (resp. excellent surface pairs) [Kol13, 2.25]. Moreover, the MMP works for quasi-projective lc surface pairs over an arbitrary field [Tan18].*

Fact 5.4.1.2 (Resolutions for threefolds)**.** *Resolutions of singularities exist for quasi-projective threefolds over a perfect field [CP08, 2.1]. Moreover, if (X, Δ) is a quasi-projective threefold pair, then there exists a snc log resolution $(X', \Delta') \to (X, \Delta)$ that is an isomorphism above the snc locus of (X, Δ) [CP08, 2.1,4.1].*

Corollary 5.4.1.3. *If (S, Δ) is a numerically dlt (resp. numerically terminal) surface pair over an arbitrary field, then S is $\mathbb{Q}$-factorial (resp. regular).*

Proof. The numerically terminal case is shown in [Kol13, 2.29]. To prove the numerically dlt case, we can apply [KM98, 4.11] once we know that [KM98, 4.10] applies as well. It is the case, since log resolutions are available, negative-definiteness of contracted curves holds by [Kol13, 10.1] and the Base-point freeness theorem is established with a sufficient level of generality in [Tan18, 4.2]. $\square$

Fact 5.4.1.4 (Inversion of adjunction for threefolds)**.** *Log canonical inversion of adjunction holds for quasi-projective threefold pairs over a perfect field of characteristic > 5 [Pat17, Lemma 3.3] (the proof there is given for an algebraically closed field, but extends easily to the case of a perfect field).*

Next we gather some results about fibrations of surfaces.

Lemma 5.4.1.5. *Let $f: S \to T$ be a proper flat morphism from a normal surface onto a normal curve with connected fibers, over a arbitrary field. Let E be a $\mathbb{Q}$-divisor on S that is vertical over T. Then $E^2 \leq 0$, with equality if and only if E is a weighted sum of fibers of f.*

Proof. By [B01, 2.6], the result holds if S is regular. It is assumed there that the base field k is algebraically closed, but this is not necessary: the Néron–Severi theorem holds over any field [SGA71, Exp. XIII, Théorème 5.1], the Grothendieck–Riemann–Roch formula [SGA71, Exp. VIII, Théorème 3.6] holds for S since $S \to k$ is a complete intersection morphism, and it reduces to the usual formula by the arguments of [Har77, Appendix A, 4.1.2]. The rest of the proof in [B01] is linear algebra.

In general, let $\pi: S' \to S$ be a (minimal) resolution of singularities. Then $f \circ \pi: S' \to T$ has connected fibers and $(\pi^* E)^2 = E^2$. Moreover E is a weighted sum of fibers if and only if E is, so the lemma is proved. $\square$

Lemma 5.4.1.6. *Let $f\colon S \to T$ be a proper flat morphism from a normal surface onto a normal curve, such that $-K_S$ is f-ample and $f_*\mathcal{O}_S = \mathcal{O}_T$, over an arbitrary field of characteristic $p > 2$. Then the general geometric fiber of f is normal.*

Proof. To begin with, we show that the generic fiber of f is geometrically normal, assuming it is geometrically integral. Let $\eta \in T$ be the generic point, then S_η is a normal scheme of finite type over $k(\eta)$. Since $f_*\mathcal{O}_S = \mathcal{O}_T$, the field $k(\eta)$ is algebraically closed in $k(S_\eta)$. Then if Y is the normalization of the reduced structure of $S_{\overline{\eta}}$, with canonical morphism $\pi\colon Y \to S_\eta$, by [PW18, Theorem 1.1] there is an effective $\mathbb{Z}$-Weil divisor C on Y such that

$$K_{S_{\overline{\eta}}} + (p-1)C \sim \pi^* K_{S_\eta}. \tag{4.1.6.c}$$

Since $S_{\overline{\eta}}$ is reduced by the assumption we made, by [PW18, Theorem 1.2] we may choose C so that $(p-1)C$ is equal to the divisorial part of the conductor of the normalization $Y \to S_{\overline{\eta}}$. Notice that $S_{\overline{\eta}}$ satisfies the property S_2, since it is preserved by field-extension [Gro65, 6.4.2] and S_η satisfies it. Thus $S_{\overline{\eta}}$ is normal if and only if it is R_1, so we can choose $C = 0$ if and only if $S_{\overline{\eta}}$ is normal.

Suppose $S_{\overline{\eta}}$ is not normal. Then we may assume that $C > 0$. Since $-K_{S_\eta}$ is ample over $k(\eta)$, its pullback $-\pi^*K_{S_\eta}$ is ample over $\overline{k(\eta)}$. By (4.1.6.c) we deduce that $-K_{S_{\overline{\eta}}} - (p-1)C$ is ample. Since $C > 0$, we deduce that $-K_{S_{\overline{\eta}}}$ is already ample, and therefore $S_{\overline{\eta}} \cong \mathbb{P}_{\overline{k(\eta)}}$. Looking again at the equation (4.1.6.c) and taking degrees, we see that $\deg(p-1)C < 2$. Since $C > 0$, this condition is satisfied only if $p = 2$.

Now we show that S_η is geometrically integral. Since $f_*\mathcal{O}_S = \mathcal{O}_T$, the field $k(\eta)$ is algebraically closed in $k(S_\eta)$. Therefore the function field of T is separable over the function field of S [B0̆1, 7.2]. It follows from [Gro65, 4.5.9, 4.6.1] that S_η is geometrically integral.

Finally, by [Gro66, 12.2.4] the set of points $t \in T$ such that the fiber S_t is geometrically normal, is open. $\square$

Lemma 5.4.1.7. *Let $f\colon S \to T$ be a proper flat morphism from an integral surface to an integral curve. Let D be a prime $\mathbb{Q}$-Cartier divisor such that $\deg_{k(t)} D|_{S_t} = 1$ for every $t \in T$. Assume that S is normal, or more generally that the Cartier locus of D dominates T. Then $f|_D\colon D \to T$ is a birational morphism, and an isomorphism if T is normal.*

Proof. Notice that S_t is proper for every $t \in T$, so the intersection product $D.S_t$ is well-defined (see [Kol96, Appendix A.2]). Let us assume first that D is Cartier. Then by [Kol96, A.2.8] (applied with $X = S_t$ and $F = \mathcal{O}_{S_t}$) we have

$$1 = \deg_{k(t)} D|_{S_t} = \sum_{x \in D \cap S_t} \mathrm{length}_{k(t)} \left(\mathcal{O}_{S_t,x}/(\xi_x)\right)$$

where ξ_x is the restriction to S_t of the local equation of D at x. This implies that D meets S_t at a single point, and that the canonical map $\mathcal{O}_T \to (f|_D)_*\mathcal{O}_D$ is surjective after tensoring by $k(t)$. So $f|_D$ is proper and quasi-finite, therefore finite, and $(f|_D)_*\mathcal{O}_D$ is a finite $\mathcal{O}_T$-module. By Nakayama's lemma, we obtain that $\mathcal{O}_T \to (f|_D)_*\mathcal{O}_D$ is surjective. But it is also injective, so it is an isomorphism. So $f|_D$ is an isomorphism.

If D is only Cartier over a dense open subset of T, we obtain that $f|_D\colon D \to T$ is birational. Assume that T is normal: then the composition $D^n \to D \to T$ is a birational morphism of normal curves, hence an isomorphism, so actually $D^n = D \cong T$. $\square$

Lemma 5.4.1.8. *Let (S, Δ) be a dlt surface pair and $g\colon S \to T$ be a birational proper morphism onto a (non-necessarily normal) surface, over a perfect field. Assume that $K_S + \Delta \sim_{\mathbb{Q},g} 0$. Then $R^1 g_*\mathcal{O}_S(-\Delta^{=1}) = 0$.*

Proof. By flat base-change we may assume that the base-field is algebraically closed. Since S is normal, there is a factorization $g^\nu\colon S \to T^\nu$ through the normalization $\nu\colon T^\nu \to T$. Since ν is finite, for any coherent sheaf $\mathcal{F}$ on S we have $R^1(\nu \circ g)_*\mathcal{F} \cong \nu_* R^1 g^\nu_* \mathcal{F}$. Therefore we may assume that T is normal.

Let $\varphi\colon S' \to (S, \Delta)$ is a minimal log resolution. Since $(S, \Delta^{<1})$ is klt, we can write $\varphi^*(K_S + \Delta^{<1}) = K_{S'} + \Delta'$ where $\Delta' \geq 0$ and $\lfloor \Delta' \rfloor = 0$.

We notice that since S is smooth in a neighborhood of $\Delta^{=1}$, the projection formula yields $\varphi_*\mathcal{O}_{S'}(-\varphi^*\Delta^{=1}) = \mathcal{O}_S(-\Delta^{=1})$ (such projection formula holds in greater generality, see for example [Sak84, 2.1, 2.4]).

By assumption, there is a $\mathbb{Q}$-Cartier divisor N on T such that $-\Delta^{=1} \sim_{\mathbb{Q}} K_S + \Delta^{<1} + g^*N$. It follows that

$$-\varphi^*\Delta^{=1} \sim_{\mathbb{Q}} K_{S'} + \Delta' + \varphi^*(g^*N).$$

We can write $\Delta' = V_\varphi + H_\varphi$, where V_φ is φ-exceptional and H_φ has no φ-exceptional components. Then H_φ is φ-nef and $\lfloor V_\varphi \rfloor = 0$. So by [KK, 2.2.5], we obtain that $R^i\varphi_*\mathcal{O}_{S'}(-\varphi^*\Delta^{=1}) = 0$ for $i > 0$. A similar argument with the composition $g' = g \circ \varphi\colon S' \to T$ in place of φ shows that $R^1 g'_*\mathcal{O}_{S'}(-\varphi^*\Delta^{=1}) = 0$. As $\varphi_*\mathcal{O}_{S'}(-\varphi^*\Delta^{=1}) = \mathcal{O}_S(-\Delta^{=1})$, the Leray spectral sequence for $g' = g \circ \varphi$ gives that

$$0 = R^1 g'_*\mathcal{O}_{S'}(-\varphi^*\Delta^{=1}) \cong R^1 g_*\mathcal{O}_S(-\Delta^{=1})$$

and the proof is complete. $\square$

Crepant birational maps.

Definition 5.4.1.9. Let $f\colon (X', \Delta') \to (X, \Delta)$ be a birational proper morphism of pairs. We say that f is **crepant** if $K_{X'} + \Delta' = f^*(K_X + \Delta)$.

Definition 5.4.1.10. Let (X, Δ) be a pair. A **crepant dlt blow-up** of (X, Δ) is a projective birational morphism $f\colon Y \to X$ such that

(a) Y is $\mathbb{Q}$-factorial, and
(b) $(Y, f^{-1}_*\Delta + E)$ is a dlt pair, where E is the sum of all f-exceptional divisors with coefficients 1, and
(c) $K_Y + f^{-1}_*\Delta + E \sim_{\mathbb{Q}} f^*(K_X + \Delta)$.

The following fact will be crucial for our study of lc centers on threefolds.

Fact 5.4.1.11. *Crepant dlt blow-ups exist for lc surface pairs over an arbitrary field (see Fact 5.4.1.1), and for quasi-projective lc threefold pairs over a perfect field of characteristic > 5 [HNT20, 3.6].*

Definition 5.4.1.12. More generally, a birational map $f\colon (X', \Delta') \dashrightarrow (X, \Delta)$ of pairs is **crepant** if there is a normal variety Y, a (non-necessarily effective) $\mathbb{Q}$-divisor Δ_Y and a commutative diagram

$$\begin{tikzcd} & (Y, \Delta_Y) \arrow[dl, "v'"'] \arrow[dr, "v"] & \\ (X', \Delta') \arrow[rr, dashed, "f"] & & (X, \Delta) \end{tikzcd}$$

where v, v' are proper, such that

$$(v')^*(K_{X'} + \Delta') = K_Y + \Delta_Y = v^*(K_X + \Delta).$$

The set of crepant birational self-map of (X, Δ) with the composition forms a group, denoted $\mathrm{Bir}^c(X, \Delta)$.

If X is endowed with a morphism $X \to Z$, we let $\mathrm{Bir}^c_Z(X, \Delta)$ be the subgroup of crepant birational self-maps over Z.

Lemma 5.4.1.13. *Let $\phi\colon (S,\Delta) \dashrightarrow (S',\Delta')$ be a crepant birational map between two excellent surface pairs over an arbitrary field. Then:*

(a) (S,Δ) is klt if and only if (S',Δ') is klt;

(b) more generally, there is a bijection between the connected components of $\mathrm{Nklt}(S,\Delta)$ *and those of* $\mathrm{Nklt}(S',\Delta')$.

Proof. An equivalent definition of being crepant is that $a(E;S,\Delta) = a(E;S',\Delta')$ for every prime divisor E of $k(S) = k(S')$ [Kol13, 2.32.2] so the first statement holds. To prove the other one, let (Y,Δ_Y) be a crepant snc resolution of ϕ. Then $\mathrm{Nklt}(S,\Delta)$, respectively $\mathrm{Nklt}(S',\Delta')$, is the image of $\lfloor \Delta_Y^{\geq 0} \rfloor$ through $Y \to S$, respectively through $Y \to S'$. By [Kol13, 2.36], the fibers of

$$\lfloor \Delta_Y^{\geq 0} \rfloor \to \mathrm{Nklt}(S,\Delta), \quad \lfloor \Delta_Y^{\geq 0} \rfloor \to \mathrm{Nklt}(S',\Delta')$$

are connected, thus each morphism induces a bijection between the connected components of the target and those of the source. The result follows. $\square$

We will frequently encounter pairs (X,Δ) together with a proper morphism onto a normal variety $X \to Z$, satisfying the condition $K_X + \Delta \sim_{\mathbb{Q},Z} 0$. It will be useful to understand the crepant birational Z-maps of such pairs.

Lemma 5.4.1.14. *Let (X,Δ) be a pair, $f\colon X \to Z$ a proper surjective morphism onto a normal variety such that $K_X + \Delta \sim_{\mathbb{Q},f} 0$. Let X' be a normal variety with a proper surjective morphism $f'\colon X' \to Z$, and $\phi\colon X \dashrightarrow X'$ a birational Z-map. Then:*

(a) There exists a unique $\mathbb{Q}$-divisor Δ' on X' such that (X',Δ') is a pair and $\phi\colon (X,\Delta) \dashrightarrow (X',\Delta')$ is crepant;

(b) $K_{X'} + \Delta' \sim_{\mathbb{Q},f'} 0$;

(c) If ϕ^{-1} does not extract divisors, then $\Delta' = \phi_*\Delta$.

Proof. Uniqueness of Δ' is clear, we prove its existence. We can find a commutative diagram

$$\begin{tikzcd} & Y \arrow[dl, "u"'] \arrow[dr, "v"] & \\ X \arrow[rr, dotted, "\phi"] \arrow[dr, "f"'] & & X' \arrow[dl, "f'"] \\ & Z & \end{tikzcd}$$

where Y is normal, u and v are birational, $u_*\mathcal{O}_Y = \mathcal{O}_X$ and $v_*\mathcal{O}_Y = \mathcal{O}_{X'}$. Write $K_Y + \Gamma = u^*(K_X + \Delta)$. We claim that $\Delta' := v_*\Gamma$ is a valid choice. By commutativity of the diagram, there is a $\mathbb{Q}$-Cartier divisor N on Z such that $(f' \circ v)^*N \sim_{\mathbb{Q}} K_Y + \Gamma$. Fix a canonical divisor $K_{X'}$ on X' such that $v_*K_Y = K_{X'}$. We have:

$$(f')^*N \sim_{\mathbb{Q}} v_*(f' \circ v)^*N \sim_{\mathbb{Q}} v_*(K_Y + \Gamma) = K_{X'} + \Delta'$$

which shows that $K_{X'} + \Delta'$ is $\mathbb{Q}$-Cartier. Now $\pm((K_Y+\Gamma) - v^*(K_{X'}+\Delta'))$ are v-exceptional and $\mathbb{Q}$-linearly trivial over Z. In particular both are v-nef, and by the negativity lemma [KM98, 3.39] we deduce that $K_Y + \Gamma = v^*(K_{X'} + \Delta')$. This shows that $\phi\colon (X,\Delta) \dashrightarrow (X',\Delta')$ is crepant.

If ϕ^{-1} does not extract divisors, then any u-exceptional divisor is also v-exceptional, so $v_*\Gamma = \phi_*\Delta$. $\square$

Pluricanonical representations in low dimensions.

The key to the gluing theorems in Section 5.4.3 is a finiteness result in the theory of pluricanonical representations. See [Kol16, Theorem 7] and [Kol13, §10.5] for what is known in characteristic 0. I am not aware of similar general results in positive characteristic. Fortunately, for the gluing of threefolds there are only a few easy cases to consider, which we discuss below. The results are probably well-known, we give proofs for convenience.

Proposition 5.4.1.15. *Let (C, E) be a proper dlt curve over a perfect field k such that $K_C + E \sim_{\mathbb{Q}} 0$. Then $\operatorname{im}\left[\operatorname{Bir}^c_k(C, E) \to \operatorname{Aut}_k H^0(C, \omega_C^m(mE))\right]$ is finite, for m divisible enough.*

Proof. We may extend the scalars along an algebraic closure of k, and assume it is algebraically closed. By assumption on E, the curve C is smooth proper of genus 0 or 1. Moreover $\operatorname{Bir}^c_k(C, E) = \operatorname{Aut}_k(C, E)$.

(a) If C has genus 1, then $\omega_C \sim 0$ and so $E = 0$. Fix a closed point $o \in C$, and consider the elliptic curve (C, o). The 1-dimensional k-vector space $H^0(C, \omega_C)$ is generated by a differential δ with the property that $t_c^*\delta = \delta$ for any $c \in C$, where $t_c \colon C \to C$ is the translation by C [Sil09, III.5.1]. Thus for any $\tau \in \operatorname{Aut}(C)$ we have
$$\tau^*\delta = \tau^* t^*_{-\tau(o)}\delta$$
and $t_{-\tau(o)} \circ \tau \in \operatorname{Aut}(C, o)$. Thus we only need to show that $\operatorname{Aut}(C, o)$ acts as a finite group on $H^0(C, \omega_C)$. But $\operatorname{Aut}(C, o)$ is already a finite group [Sil09, III.10.1].

(b) If C has genus 0, then $C \cong \mathbb{P}^1_k$. If $\operatorname{Supp}(E)$ contains at least three points, then $\operatorname{Aut}(\mathbb{P}^1_k, E)$ is finite. If E is the sum of two distinct reduced points, we may choose coordinates x, y such that $E = [0; 1] + [1; 0]$. Then $\operatorname{Aut}_k(\mathbb{P}^1, E)$ sits in an exact sequence
$$1 \to \operatorname{Aut}_k(\mathbb{A}^1_k, 0) \to \operatorname{Aut}_k(\mathbb{P}^1, E) \to \operatorname{Bij}(E) \to 1$$
so it suffices to show that $\operatorname{Aut}(\mathbb{A}^1_k, 0) = \{x \mapsto ax \mid a \in k^*\}$ acts finitely on the 1-dimensional k-vector space $H^0(\mathbb{P}^1_k, \omega_{\mathbb{P}^1_k}(E))$. This vector space is generated by dx/x, which is invariant through $x \mapsto ax$, thus the action of $\operatorname{Aut}_k(\mathbb{A}^1_k, 0)$ is actually trivial. $\square$

Proposition 5.4.1.16. *Let (C, E) be a dlt curve over a perfect field k. Assume that $K_C + E$ is ample. Then $\operatorname{Aut}_k(C, E)$ is finite.*

Proof. As above, we may assume that k is algebraically closed. If $C \cong \mathbb{P}^1_k$ then $\operatorname{Supp}(E)$ contains at least three points, so the log automorphism group is finite. If $g(C) = 1$ then $\operatorname{Supp}(E)$ contains at least one point, so the log automorphism group is finite. Finally if $g(C) \geq 2$ then $\operatorname{Aut}_k(C)$ is already finite. $\square$

Lemma 5.4.1.17. *Let (S, Δ) be a projective lc surface pair over an algebraically closed field k. Assume that $K_S + \Delta$ is big. Then $\operatorname{Aut}_k(S, \mathcal{O}_S(K_S + \Delta))$ is finite.*

Proof. This actually holds in any dimension. The argument follows the proof of [PZ20, 10.1], with the following modifications: the inequality in (10.1.b) becomes strict (by our bigness assumption); the first sentence after (10.1.c) reads *as (X, Δ) is lc, all coefficients of Γ are smaller or equal to* 1; and the inequality in the last displayed equation is not strict. $\square$

Proposition 5.4.1.18. *Let $f \colon (S, \Delta) \to T$ be a proper morphism with $f_*\mathcal{O}_S = \mathcal{O}_T$ between normal projective surfaces over a perfect field k. Assume that (S, Δ) is a dlt pair and that $K_S + \Delta \sim_{\mathbb{Q}} f^*L$, where L is ample on T. Then for m divisible enough,*
$$\operatorname{im}\left[\operatorname{Bir}^c_k(S, \Delta) \to \operatorname{Aut}_k H^0(T, L^{\otimes m})\right]$$
is finite.

Proof. We may extend to the algebraic closure of k, and assume it is algebraically closed. Since L is ample, we may assume that $L^{\otimes m}$ is very ample and thus $\mathrm{Aut}_k H^0(T, L^{\otimes m}) \subseteq \mathrm{Aut}_k(T)$.

It follows from the assumptions that $f\colon S \to T$ is a birational morphism. So $K_S + \Delta$ is big and nef. Therefore the canonical model $(S_{\mathrm{can}}, \Delta_{\mathrm{can}})$ of (S, Δ) exists, and it is given by

$$\psi\colon S \dashrightarrow S_{\mathrm{can}} := \mathrm{Proj} \sum_r H^0(S, rm(K_S + \Delta)), \quad \Delta_{\mathrm{can}} := \psi_*\Delta$$

where m is sufficiently divisible. Since $f_*\mathcal{O}_S = \mathcal{O}_T$, we have $H^0(S, rm(K_S+\Delta)) = H^0(T, L^{\otimes rm})$. As L is ample, we deduce that $T \cong S_{\mathrm{can}}$ and that we can identify ψ with f. Writing $\Delta_T = f_*\Delta$, we have that $K_T + \Delta_T$ is $\mathbb{Q}$-Cartier and (T, Δ_T) is lc.

We claim that $f\colon (S, \Delta) \to (T, \Delta_T)$ is crepant. Indeed, we have $K_T + \Delta_T \sim_{\mathbb{Q}} f_*f^*L = L$, so $(K_S + \Delta) - f^*(K_T + \Delta_T)$ is $\mathbb{Q}$-linearly equivalent to 0 and exceptional over T. The negativity lemma then implies that $K_S + \Delta = f^*(K_T + \Delta_T)$.

Take $\tau \in \mathrm{Bir}^c(S, \Delta)$ and let $\beta(\tau) \in \mathrm{Aut}(T)$ be the induced automorphism. Then we have a have a commutative diagram

$$\begin{tikzcd} S \arrow[r, dotted, "\tau"] \arrow[d, "f"'] & S \arrow[d, "f"] \\ T \arrow[r, "\beta(\tau)"] & T \end{tikzcd}$$

and therefore $\beta(\tau)^*\mathcal{O}_T(K_T + \Delta_T) \cong \mathcal{O}_T(K_T + \Delta_T)$. Thus $\mathrm{Bir}^c_k(S, \Delta) \to \mathrm{Aut}_k(T)$ factorizes through $\mathrm{Aut}_k(T, \mathcal{O}_T(K_T + \Delta_T))$, which is finite by Lemma 5.4.1.17. $\square$

Some remarks on fields of definition.

Let k be a (not necessarily perfect) field, k^s a separable closure and X a k-scheme. We say that a k^s-sub-scheme $W \subset X_{k^s}$ is defined over k if there exists a sub-k-scheme $\mathcal{W} \subset X$ such that $W = \mathcal{W}_{k^s}$ (as sub-schemes of X_{k^s}).

Lemma 5.4.1.19. *If $\mathcal{Y}$ is a proper reduced connected k-scheme, then $H^0(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$ is a finite field extension of k.*

Proof. The structure morphism $f\colon \mathcal{Y} \to k$ is proper, so $f_*\mathcal{O}_{\mathcal{Y}}$ is a coherent k-module. Hence $H^0(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$ is a finite k-algebra. In particular it is Artinian. It is reduced by hypothesis, thus it is a finite direct product of field extensions of k. By considering the Stein factorization of f, we see that there is a bijection between these direct factors and the connected components of $\mathcal{Y}$. Since $\mathcal{Y}$ is connected, there can be only one direct summand. Therefore $H^0(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$ is a field. $\square$

Lemma 5.4.1.20. *Let $\mathcal{Y}$ be a proper connected k-scheme, and assume that $H^0(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}) = k$. Then $\mathcal{Y}$ is geometrically connected.*

Proof. It is sufficient to show that $\mathcal{Y}_{k^s}$ is connected. By flat base-change we have $H^0(\mathcal{Y}_{k^s}, \mathcal{O}_{\mathcal{Y}_{k^s}}) = H^0(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}) \otimes_k k^s = k^s$. If $\mathcal{Y}_{k^s} = \bigsqcup_{i=1}^d Y^i$ is the decomposition into connected components, then

$$H^0(\mathcal{Y}_{k^s}, \mathcal{O}_{\mathcal{Y}_{k^s}}) = \bigoplus_{i=1}^d H^0(Y^i, \mathcal{O}_{Y^i})$$

where each $H^0(Y^i, \mathcal{O}_{Y^i})$ is a finite k^s-vector space. Considering the dimensions as k^s-vector spaces, we see that $d = 1$. $\square$

Lemma 5.4.1.21. *Let X be a k-scheme and $\mathcal{Y} \subset X$ an connected reduced proper sub-k-scheme. Assume that $H^0(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$ is separable over k. Then every connected component of $\mathcal{Y}_{k^s}$ is defined over the Galois closure of $H^0(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})/k$.*

Proof. Let us write $l := H^0(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$, which by assumption and Lemma 5.4.1.19 we can consider as a subfield of k^s. Let $L \subset k^s$ be the Galois closure of l/k and let $\mathcal{Y}_L = \bigsqcup_i \mathcal{Y}_L^i$ be the decomposition into connected components. By separability $H^0(\mathcal{Y}_L, \mathcal{O}_{\mathcal{Y}_L}) = l \otimes_k L$ is a reduced Artinian L-algebra, thus a direct product of finitely many field extensions of L. By considering the Stein factorization of $\mathcal{Y}_L \to L$, we see that in fact these field extensions are the $H^0(\mathcal{Y}_L^i, \mathcal{O}_{\mathcal{Y}_L^i})$. On the other hand, we have an inclusion $l \otimes_k L \hookrightarrow L \otimes_k L$, and since L is Galois over k we have an L-algebra isomorphism $L \otimes_k L \cong \bigoplus_{\mathrm{Gal}(L/k)} L$. This induces inclusions $H^0(\mathcal{Y}_L^i, \mathcal{O}_{\mathcal{Y}_L^i}) \subseteq L$. Hence $H^0(\mathcal{Y}_L^i, \mathcal{O}_{\mathcal{Y}_L^i}) = L$ for every i. By Lemma 5.4.1.20 the proof is complete. □

Corollary 5.4.1.22. *Let $\mathcal{Y} \subset X$ be a connected reduced proper k-sub-scheme. Assume that $H^0(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$ is separable over k. Then the number of connected components of $\mathcal{Y}_{k^s}$ is equal to $\dim_k H^0(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$.*

Proof. By Lemma 5.4.1.21 and its proof, there is a finite Galois extension L/k with the following property: if $\mathcal{Y}_L = \bigsqcup_{i=1}^d Y^i$ is the decomposition into connected components, then $H^0(\mathcal{Y}_L, \mathcal{O}_{\mathcal{Y}_L}) = \bigoplus_{i=1}^d H^0(Y^i, \mathcal{O}_{Y^i})$ and $H^0(Y^i, \mathcal{O}_{Y^i}) = L$ for each i. It follows from Lemma 5.4.1.20 that the number of connected components of $\mathcal{Y}_{k^s}$ is equal to d. On the other hand

$$d = \dim_L H^0(\mathcal{Y}_L, \mathcal{O}_{\mathcal{Y}_L}) = \dim_L H^0(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}) \otimes_k L = \dim_k H^0(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$$

so the result follows. □

Lemma 5.4.1.23. *Let X be quasi-projective k-scheme, K/k a Galois extension and $W \subset X_K$ a closed reduced sub-k-scheme. Assume that W is stable under the action of a non-trivial subgroup H of $G := \mathrm{Gal}(K/k) \circlearrowright X_K$. Then W is defined over the sub-field K^H.*

Proof. By Artin's lemma, the extension $K^H \subset K$ is Galois with Galois group H. Replacing k by K^H, we may assume that W is stable under the action of G. The canonical morphism $\pi \colon X_K \to X$ is the quotient by G. Let $Y := \pi(W)$ be the reduced closed image of W. Since W is G-invariant and the fibers of π are the G-orbits, we see that $\mathrm{Supp}(W) = \mathrm{Supp}(Y_K)$. Since W and Y_K are reduced, we deduce that $W = Y_K$. □

5.4.2 Sources of lc centers

In this section we develop the theory of sources for lc centers of lc threefold pairs over a perfect field of characteristic > 5. Our approach follows closely Kollár's original one [Kol13, §4].

Notation 5.4.2.1. Let (X, Δ) be a quasi-projective lc threefold pair defined over a perfect field k of characteristic > 5, and $(Y, \Delta_Y) \to (X, \Delta)$ be a crepant dlt $\mathbb{Q}$-factorial blow-up (which exists by Fact 5.4.1.11).

Our program is the following:

(a) In Section 5.4.2, we observe that the lc centers of dlt $\mathbb{Q}$-factorial pairs (Y, Δ_Y) are the strata of the reduced boundary, analogously to the characteristic 0 case. This allows us to define higher codimension adjunction for dlt pairs.

(b) In Section 5.4.2, we compare the fibrations between lc centers obtained from $(Y, \Delta_Y) \to (X, \Delta)$.

(c) In Section 5.4.2, we define springs and sources for lc centers on the reduced boundary of threefold pairs.

Higher Poincaré residues.

In characteristic 0, the use of crepant blow-ups is motivated by the very simple structure of lc centers on a dlt pair [Kol13, 4.16]. Using the recent results of [ABL20], we can extend this result to positive characteristic. The same result was obtained in [DH16, 2.2] with other methods.

Proposition 5.4.2.2. *Let (Y, Δ) be a $\mathbb{Q}$-factorial dlt pair of dimension ≤ 3 over a perfect field of characteristic $p > 5$. Write $\Delta = \Delta^{<1} + \sum_i D_i$ with each D_i prime. Then*

(a) The lc centers of (Y, Δ) are exactly the irreducible components of the intersections of the D_i's.

(b) Every irreducible component of such an intersection is normal of the expected codimension.

(c) Let $Z \subset Y$ be an lc center of (Y, Δ). If D_i is $\mathbb{Q}$-Cartier and does not contain Z, then every irreducible component of $D_i|_Z$ is $\mathbb{Q}$-Cartier.

(d) Each $(D_i, \mathrm{Diff}_{D_i}(\Delta - D_i))$ is dlt.

Proof. Since Y is $\mathbb{Q}$-factorial, the pair $(Y, (1-\epsilon)\Delta^{=1} + \Delta^{<1})$ is klt. Hence by [ABL20, Corollary 1.3], if D is any Weil divisor on Y then $\mathcal{O}_Y(D)$ is CM. Thus the proof of [Kol13, 4.16] applies verbatim. □

This implies the existence of higher-dimensional Poincaré residue maps as in [Kol13, 4.18-19].

Corollary 5.4.2.3. *Let (Y, Δ) be as above and Z an lc center of (Y, Δ). Then there exists a canonically-defined $\mathbb{Q}$-Cartier divisor $\mathrm{Diff}^*_Z \Delta$ on Z such that:*

(a) $(Z, \mathrm{Diff}^*_Z \Delta)$ is dlt;

(b) If m is even and $m(K_Y + \Delta)$ is Cartier, there is a canonical isomorphism

$$\mathcal{R}^m_{Y,Z} \colon \omega_Y^{[m]}(m\Delta)|_Z \cong \omega_Z^{[m]}(m \,\mathrm{Diff}^*_Z \Delta).$$

(c) If $W \subset Z$ is a lc center of (Y, Δ), then W is also a lc center of $(Z, \mathrm{Diff}^*_Z \Delta)$ and

$$\mathrm{Diff}^*_W \Delta = \mathrm{Diff}^*_W(\mathrm{Diff}^*_Z \Delta).$$

Remark 5.4.2.4. If $Z \subset \Delta^{=1}$ is a prime divisor then we have $\mathrm{Diff}^*_Z \Delta = \mathrm{Diff}_Z(\Delta - Z)$, which is arguably a conflict of notations. When working on dlt pairs we will only use the Diff^*-notation, hence no confusion should arise.

Corollary 5.4.2.5. *Let $(Y_1, \Delta_1), (Y_2, \Delta_2)$ be surface pairs over an arbitrary field, or threefold pairs over a perfect field of characteristic > 5. Assume that (Y_1, Δ_1) is $\mathbb{Q}$-factorial dlt.*

Let $\phi \colon (Y_1, \Delta_1) \dashrightarrow (Y_2, \Delta_2)$ be a crepant birational map. Assume that $S_1 \subset Y_1$ is an lc center of (Y_1, Δ_1) and that ϕ is a local log isomorphism at the generic point of S_1. Then $S_2 := \phi_* S_1$ is an lc center of (Y_2, Δ_2), and ϕ restricts to a crepant birational map $\phi|_{S_1} \colon (S_1, \mathrm{Diff}^*_{S_1} \Delta_1) \dashrightarrow (S_2^n, \mathrm{Diff}_{S_2^n} \Delta_2)$.

Proof. By induction on the strata of $\Delta_1^{=1}$, we may assume that S_1 is a divisor. Since ϕ is a log isomorphism at the generic point of S_1, we obtain that S_2 is a component of $\Delta_2^{=1}$ and a lc center of (Y_2, Δ_2).

Consider a resolution

$$\begin{tikzcd} & W \arrow[dl, "u"'] \arrow[dr, "v"] & \\ Y_1 \arrow[rr, dotted, "\phi"] & & Y_2 \end{tikzcd}$$

Let $S_W \subset W$ be the strict transform of S_1, with normalization $n\colon S_W^n \to S_W$. Since ϕ is a local isomorphism at the generic point of S_1, v maps S_W^n to S_2^n and we have a commutative diagram of birational maps

$$\begin{tikzcd} & S_W^n \arrow[dl, "u|_{S_W^n}"'] \arrow[dr, "v|_{S_W^n}"] & \\ S_1 \arrow[rr, dotted, "\phi|_{S_1}"] & & S_2^n \end{tikzcd}$$

Since ϕ is crepant and a local log isomorphism at S_1, we can write

$$u^*(K_{Y_1} + \Delta_1) = K_W + S_W + \Gamma = v^*(K_{Y_2} + \Delta_2)$$

where $\mathrm{coeff}_{S_W} \Gamma = 0$. Thus by adjunction

$$\begin{aligned} (u|_{S_W^n})^*(K_{S_1} + \mathrm{Diff}^*_{S_1} \Delta_1) &= n^* u^*(K_{Y_1} + \Delta_1) \\ &= n^*(K_W + S_W + \Gamma) \\ &= K_{S_W^n} + \mathrm{Diff}_{S_W^n} \Gamma \end{aligned}$$

and similarly $(v|_{S_W^n})^*(K_{S_2^n} + \mathrm{Diff}_{S_2^n} \Delta_2) = K_{S_W^n} + \mathrm{Diff}_{S_W^n} \Gamma$. This implies that the birational map $\phi|_{S_1} \colon (S_1, \mathrm{Diff}^*_{S_1} \Delta_1) \dashrightarrow (S_2^n, \mathrm{Diff}_{S_2^n} \Delta_2)$ is crepant. □

Geometry of lc lcenters.

Let $(Y, \Delta_Y) \to (X, \Delta)$ be as in Notation 5.4.2.1, $Z \subset X$ an lc center of (X, Δ) and $S, S' \subset Y$ lc centers of (Y, Δ_Y) that are minimal for the property of dominating Z. In characteristic 0, a crucial observation due to Kollár [Kol16] is that S and S' are birational, and that the birational map arises from a special structure which he calls a *standard* $\mathbb{P}^1$*-link*. In this section, we prove some analogous statements in positive characteristic.

We begin by defining the $\mathbb{P}^1$-links.

Definition 5.4.2.6 (Weak standard $\mathbb{P}^1$-link)**.** A **weak standard** $\mathbb{P}^1$**-link** is a $\mathbb{Q}$-factorial pair $(T, W_1 + W_2 + E)$ together with a proper morphism $\pi\colon T \to W$ such that

(a) $K_T + W_1 + W_2 + E \sim_{\mathbb{Q},\pi} 0$,

(b) W_1 and W_2 are disjoint and normal,

(c) both $\pi\colon W_i \to W$ are isomorphisms, and

(d) $\mathrm{red}(T_w) \cong \mathbb{P}^1_{k(w)}$ for every $w \in W$.

Remark 5.4.2.7. In Kollár's definition of standard $\mathbb{P}^1$-link [Kol13, 4.36], it is assumed that $(T, W_1 + W_2 + E)$ is plt. Let us justify our deviation. The use of (weak) $\mathbb{P}^1$-links is motivated by Proposition 5.4.2.9 below, which is analogous to [Kol13, 4.37]. In both cases, during the proof we find a weak $\mathbb{P}^1$-link $(T, W_1 + W_2 + E) \to W$ such that (T, E) is klt. Using cyclic covers, we can see that $(T, W_1 + W_2 + E)$ is locally the quotient of a product $(\tilde{W} \times \mathbb{P}^1, \tilde{W} \times [0;1] + \tilde{W} \times [1;0] + E_{\tilde{W}} \times \mathbb{P}^1)$. By inversion of adjunction, this product is plt. In characteristic 0 this implies that the quotient is plt, but this is not necessarily the case in positive characteristic.

However, our weaker assumption on standard links is sufficient for our purpose, as we shall see.

Lemma 5.4.2.8. *Let* $\pi\colon (T, W_1 + W_2 + E) \to W$ *be a weak standard* $\mathbb{P}^1$*-link over a field of characteristic* > 2*. Then* $\mathrm{Supp}(E)$ *is a union of fibers and there is a log isomorphism* $(W_1, \mathrm{Diff}_{W_1} E) \cong (W_2, \mathrm{Diff}_{W_2} E)$ *that commutes with the projections onto* W*.*

Proof. A general geometric fiber F is normal by Lemma 5.4.1.6, and by assumption K_F is anti-ample. In particular $2 = -K_T \cdot F = (W_1 + W_2 + E) \cdot F$. Since $W_i \cdot F = 1$, we deduce that the support of E is a union of fibers. Since $W_i \to W$ are isomorphisms, the projections provide an isomorphism $\phi\colon W_1 \cong W_2$ that commutes with the projections $\pi_i\colon W_i \to W$. The morphism $\phi\colon (W_1, \mathrm{Diff}_{W_1} E) \to (W_2, \mathrm{Diff}_{W_2} E)$ is crepant by Lemma 5.4.1.14. □

Proposition 5.4.2.9. *Let (S, Δ) be a quasi-projective klt surface pair defined over a field of characteristic > 2, with a projective morphism $f\colon S \to Z$. Let D be an effective $\mathbb{Z}$-divisor that dominates Z, such that $K_S + \Delta + D \sim_{\mathbb{Q},f} 0$ and $(S, D + \Delta)$ is dlt. Pick a point $z \in Z$ such that $f^{-1}(z)$ is connected but $f^{-1}(z) \cap D$ is disconnected.*

Then there exists an elementary étale morphism $(z' \in Z') \to (z \in Z)$ and a proper morphism $W \to Z'$ such that $(S, \Delta + D) \times_Z Z' = (S', \Delta' + D')$ is birational to a weak standard $\mathbb{P}^1$-link over W.

Proof. Given an elementary étale neighborhood $(z' \in Z') \to (z \in Z)$, we form the Cartesian diagram

$$\begin{tikzcd} (S', D' + \Delta') \arrow[r] \arrow[d, "f'"] & (S, D + \Delta) \arrow[d, "f"] \\ Z' \arrow[r] & Z \end{tikzcd}$$

We can choose $Z' \to Z$ such that the different connected components of $(f')^{-1}(z') \cap D'$ are contained in different connected components of D'. Notice that (S', Δ') is klt, $(S', D' + \Delta')$ is dlt, $K_{S'} + D' + \Delta' \sim_{\mathbb{Q},f'} 0$ and $(f')^{-1}(z')$ is connected.

By [Tan18, Theorem 1.1], we may run an MMP for (S', Δ') over Z'. Since $K_{S'} + \Delta' \sim_{\mathbb{Q},f'} -D'$ is not pseudo-effective on the generic fiber of f', the MMP terminates with a Fano contraction $p\colon (S^*, \Delta^*) \to W$ over Z'. We picture our construction as follows:

$$\begin{tikzcd} (S', D' + \Delta') \arrow[rr, dashed, "\varphi"] \arrow[d, "f'"] & & (S^*, D^* + \Delta^*) \arrow[d, "p"] \arrow[dll, "g"'] \\ Z' & & W \arrow[ll] \end{tikzcd}$$

where $D^* = \varphi_* D$ and $\Delta^* = \varphi_* \Delta$. (Since S' is a surface, φ is actually a morphism.)

By construction we have $K_{S^*} + D^* + \Delta^* \sim_{\mathbb{Q},g} 0$, so in particular $K_{S^*} + D^* + \Delta^* \sim_{\mathbb{Q},p} 0$. Moreover $\varphi\colon (S', D' + \Delta') \dashrightarrow (S^*, D^* + \Delta^*)$ is crepant, so it follows from Lemma 5.4.1.13 that $g^{-1}(z') \cap D^*$ is still disconnected.

Since p is a Fano contraction, there is a component $D_1^* \subset D^*$ that has positive intersection with the contracted ray inducing $p\colon S^* \to W$. Thus D_1^* is p-ample. Now there is another component $D_2^* \subset D^*$ that is disjoint from D_1^*. Take a curve C which intersects D_2^* and is contained in a fiber of p. The intersection $C \cdot D_2^*$ cannot be negative, otherwise the fiber would be included in D_2^*, and D_1^* and D_2^* would not be disjoint. So D_2^* also has positive intersection with the contracted ray, and is also p-ample.

So D_1^* and D_2^* are disjoint and have positive intersection with every curve contracted by p. Assume that a fiber F of p has dimension at least 2: then $F \cap D_1^*$ must be at least 1-dimensional, hence it intersects D_2^*, contradiction. Thus the fibers of p have dimension 1. Since (S^*, Δ^*) is a klt surface, it is $\mathbb{Q}$-factorial and K_{S^*} is $\mathbb{Q}$-Cartier. As $K_{S^*} + \Delta^*$ is p-anti-ample, it follows that K_{S^*} is p-anti-ample. By Lemma 5.4.1.6, a general geometric fiber F of p is smooth rational. As

$$F \cdot (\Delta^* + D^*) = -K_{S^*} \cdot F = 2,$$

we deduce via Lemma 5.4.1.7 that D_1^* and D_2^* are sections of p and that the other components of $\Delta^* + D^*$ are vertical over W. If there was another component of D^* whose image under g contained z', then that component would also contain $g^{-1}(z)$, and thus $g^{-1}(z) \cap D^*$ would not be disconnected. Thus we can shrink Z' and assume that $D^* = D_1^* + D_2^*$.

By [Tan18, Theorem 1.3], we have $R^1 p_* \mathcal{O}_{S^*} = 0$, and thus every reduced fiber is a tree of $\mathbb{P}^1$. By intersecting with the D_i^* we see these trees must be irreducible.

Since S^* is a normal surface, W is a normal curve, and as D_i^* are sections of p we deduce that they are normal.

Thus $p\colon (S^*, D_1^* + D_2^* + \Delta^*) \to W$ is a weak standard $\mathbb{P}^1$-link. This proves the proposition. $\square$

Definition 5.4.2.10 (Weak $\mathbb{P}^1$-links)**.** Let $f\colon (Y, \Delta_Y) \to (X, \Delta)$ be as in Notation 5.4.2.1, and let $Z_1, Z_2 \subset Y$ be two lc centers. We say that Z_1 and Z_2 are **directly weakly $\mathbb{P}^1$-linked** if there exists an lc center $W \subseteq Y$ (we allow $W = Y$) such that $f(Z_1) = f(Z_2) = f(W)$ and $(W, \mathrm{Diff}_W^* \Delta_Y)$ is crepant birational to a $\mathbb{P}^1$-link with the Z_i as the two sections, and the base of the $\mathbb{P}^1$-link factorizes the morphism to X.

We say that Z_1 and Z_2 are **weakly $\mathbb{P}^1$-linked** if there is a sequence of lc centers $Z_1', \dots, Z_m'$ with $Z_1 = Z_1'$ and $Z_2 = Z_m'$, such that Z_i' and Z_{i+1}' are directly weakly $\mathbb{P}^1$-linked.

Lemma 5.4.2.11. *Let $f\colon (Y, \Delta_Y) \to (X, \Delta)$ be as in Notation 5.4.2.1, and let $S_1, S_2 \subset Y$ be two weakly $\mathbb{P}^1$-linked lc centers. Then:*

(a) $(S_1, \mathrm{Diff}_{S_1}^* \Delta_Y)$ and $(S_2, \mathrm{Diff}_{S_2}^* \Delta_Y)$ are crepant birational over X, and

(b) $(S_1, \mathrm{Diff}_{S_1}^* \Delta_Y)$ is klt if and only if $(S_2, \mathrm{Diff}_{S_2}^* \Delta_Y)$ is klt.

Proof. We may assume that S_1 and S_2 are directly weakly $\mathbb{P}^1$-linked. Then there is a lc center $T \subset Y$ containing both S_i's, a weak standard $\mathbb{P}^1$-link $(T', W_1 + W_2 + E) \to W$ and a crepant birational map $\phi\colon (T, \mathrm{Diff}_T^* \Delta_Y) \dashrightarrow (T', W_1 + W_2 + E)$ such that the map $T \dashrightarrow T' \to W$ factors the morphism $T \to X$, and such that ϕ maps birationally S_i to W_i.

By Corollary 5.4.2.5, ϕ induces crepant birational X-maps $(S_i, \mathrm{Diff}_{S_i}^* \Delta_Y) \dashrightarrow (W_i, \mathrm{Diff}_{W_i} E)$. Composing these maps with the log isomorphism $(W_1, \mathrm{Diff}_{W_1} E) \cong (W_2, \mathrm{Diff}_{W_2} E)$ over W given by Lemma 5.4.2.8, we obtain that the $(S_i, \mathrm{Diff}_{S_i}^* \Delta_Y)$ are crepant birational over X.

The second assertion follows immediately from the first one. $\square$

Proposition 5.4.2.12. *Let $f\colon (Y, \Delta_Y) \to (X, \Delta)$ be as in Notation 5.4.2.1. Fix $T \subset X$ an lc center of (X, Δ). Then the lc centers of (Y, Δ_Y) that are minimal for dominating T, are weakly $\mathbb{P}^1$-linked to each other.*

Proof. We prove the following more precise result:

Claim 5.4.2.13. *Given a point $x \in X$, an lc center $Z \subset Y$ of (Y, Δ_Y) that is minimal for the property $x \in f(Z)$, an lc center $W \subset Y$ such that $x \in f(W)$, then there exists an lc center $Z_W \subset W$ such that Z and Z_W are weakly $\mathbb{P}^1$-linked.*

By Lemma 5.4.2.11 this claim implies the result, taking x to be the generic point of T.

First we prove our claim holds after passing to an elementary étale neighborhood of $(x \in X)$. Then we will show that the weak $\mathbb{P}^1$-link descends along the étale base-change.

By [NT20, 1.2], the fibers of

$$\mathrm{Supp}\, \Delta_Y^{=1} = \mathrm{Nklt}(Y, \Delta_Y) \longrightarrow X$$

are geometrically connected. After passing to an elementary étale neighborhood of $(x \in X)$, we may assume that $\Delta_Y^{=1} = \sum_i \Delta_i$, that each Δ_i has a connected fiber above x and that every lc center of (Y, Δ_Y) intersects $f^{-1}(x)$. Relabelling the Δ_i is necessary, we assume that $Z \subset \Delta_1, W \subset \Delta_r$ and $\Delta_i \cap \Delta_{i+1} \cap f^{-1}(x) \neq \emptyset$ for all $i = 1, \dots, r-1$.

Now we prove that $Z \subset \Delta_1$ is $\mathbb{P}^1$-linked (after the étale base-change) to an lc center contained in $\Delta_1 \cap \Delta_2$. For ease of notation, we let $S := \Delta_1$, $D := \mathrm{Diff}_S^* \Delta_Y$, $E := \Delta_1 \cap \Delta_2$. Then (S, D) is dlt, admits a projective morphism $g = f|_S\colon S \to X$ with connected fibers such that $K_S + D \sim_{\mathbb{Q},g} 0$. Notice that $E \subset \mathrm{Supp}\, D^{=1}$ and that $Z \subset \mathrm{Supp}\, D^{=1} \cap g^{-1}(x)$. We distinguish a few cases according to the dimension of $g(S)$.

(a) Suppose that $\dim g(S) = 2$. Then $R^1 g_* \mathcal{O}_S(-D^{=1}) = 0$ by Lemma 5.4.1.8, so the natural map $g_* \mathcal{O}_S \to g_* \mathcal{O}_{D^{=1}}$ is surjective. If $D^{=1} \cap g^{-1}(x)$ was disconnected, then we would get a section of $g_* \mathcal{O}_S$ that vanishes on one connected component of $g^{-1}(x) \cap D^{=1}$, and is identically 1 on another. But $g^{-1}(x)$ is connected and sections of $\mathcal{O}_{g^{-1}(x)_{\mathrm{red}}}$ are supported on entire components, so we get a contradiction. Thus $D^{=1} \cap g^{-1}(x)$ is connected.

The scheme $D^{=1} \cap g^{-1}(x)$ is either 0-dimensional or 1-dimensional. If it is 0-dimensional then $D^{=1} \cap g^{-1}(x) = Z$ and we are done.

Assume that $D^{=1} \cap g^{-1}(x)$ is 1-dimensional. Then Z is an intersection point of two components of $D^{=1}$ above $g^{-1}(x)$. If Z is the only such point, then we are done. If not, then by adjunction we easily see that each component of $D^{=1} \cap g^{-1}(x)$ must support two such points. In particular these points are $k(x)$-points, and the components of $D^{=1} \cap g^{-1}(x)$ are isomorphic to $\mathbb{P}^1_{k(x)}$.

(b) Suppose that $\dim g(S) = 1$. First we consider the case where x is the generic point of $C := g(S)$. Then $D^{=1} \cap g^{-1}(x)$ is the set of generic points of those components of $D^{=1}$ that dominate C. If it is connected, then $D^{=1} \cap g^{-1}(x) = Z$ and we are done. If $D^{=1} \cap g^{-1}(x)$ is not connected, then since $D^{=1}$ dominates C we may use Proposition 5.4.2.9 to obtain that, after possibly a further elementary étale base-change, $D^{=1} \cap g^{-1}(x)$ has two connected components that are $\mathbb{P}^1$-linked to each other.

Now assume that $x \in C$ is a closed point. If $D^{=1} \cap g^{-1}(x)$ is connected, we can apply the same analysis as in the previous point. If $D^{=1} \cap g^{-1}(x)$ is disconnected and $D^{=1}$ dominates C, then we can use Proposition 5.4.2.9.

Finally, if $D^{=1}$ does not dominate C, we show that $D^{=1} \cap g^{-1}(x)$ is connected, hence all cases have been covered. Indeed, if $D^{=1}$ does not dominate C then it is supported on fibers, and $K_S + D^{<1} \sim_{\mathbb{Q},g} -D^{=1}$ is pseudo-effective on the generic one. By [Tan18] we can run a $(K_S + D^{<1})$-MMP over C, which terminates with a birational model S' of S, say

$$\begin{tikzcd}
(S, D^{<1}, D^{=1}) \arrow[rr, dashed, "\varphi"] \arrow[dr, "g"'] & & (S', \Delta' := \varphi_* D^{<1}, \Gamma' := \varphi_* D^{=1}) \arrow[dl, "g'"] \\
& C &
\end{tikzcd}$$

such that $K_{S'} + \Delta' \sim_{\mathbb{Q},g'} -\Gamma'$ is g'-nef. (The map φ is actually a morphism.) Notice that:

- By Lemma 5.4.1.14 the map $\varphi\colon (S, D) \dashrightarrow (S', \Delta' + \Gamma')$ is crepant;
- Since $K_{S'} + \Delta' + \Gamma' \sim_{\mathbb{Q},g'} 0$ and Γ' is vertical over C, we see that $(K_{S'} + \Delta') \cdot (g')^* M = 0$ for any $\mathbb{Q}$-Cartier divisor M on C;
- $\Gamma' \neq 0$. For if φ would contract $D^{=1}$, then looking at in intermediate step of the MMP we may assume that $D^{=1}$ is irreducible. But then $(K_S + D^{<1}) \cdot D^{=1} = -(D^{=1})^2$ is non-negative by Lemma 5.4.1.5, which is a contradiction.

Now we claim that $\Gamma' \cap (g')^{-1}(x)$ is connected for any $x \in C'$. By Lemma 5.4.1.5 we have $(\Gamma')^2 \leq 0$, with equality if and only if Γ' is a weighted sum of fibers. In any case we can write

$$-\Gamma' \sim_{\mathbb{Q}} (g')^* N + E$$

where N is $\mathbb{Q}$-Cartier on C and E is an effective $\mathbb{Q}$-divisor supported on the fibers. Hence we have

$$0 \geq (\Gamma')^2 = (K_{S'} + \Delta') \cdot (-\Gamma') = (K_{S'} + \Delta') \cdot ((g')^* N + E) \geq 0,$$

thus $(\Gamma')^2 = 0$. Since $g' \colon S' \to C$ has connected fibers, we deduce that every $\Gamma' \cap (g')^{-1}(x)$ is connected.

Since $(S, D^{<1})$ is klt and φ is a $(K_S + D^{<1})$-MMP, the pair (S', Δ') is also klt. In particular $\mathrm{Nklt}(S', \Delta' + \Gamma') = \Gamma'$. On the other hand, since (S, D) is dlt we have $\mathrm{Nklt}(S, D) = D^{=1}$. Therefore, since $\varphi\colon (S, D) \dashrightarrow (S', \Delta' + \Gamma')$ is a crepant C-map, by Lemma 5.4.1.13 we have a bijection between the connected components of $D^{=1} \cap g^{-1}(x)$ and the connected components of $\Gamma' \cap (g')^{-1}(x)$. In particular $D^{=1} \cap g^{-1}(x)$ is connected.

(c) Finally, suppose that $\dim g(S) = 0$. If $D^{=1} \cap g^{-1}(x)$ is connected, we apply the same analysis as in the first point. If $D^{=1} \cap g^{-1}(x)$ is disconnected, we can use Proposition 5.4.2.9.

It remains to show that the weak $\mathbb{P}^1$-link descends the étale base-change. More precisely, let $Z_i \subset Y$ be the lc centers that are minimal for the property $x \in f(Z_i)$. We have proved that after an elementary étale base-change $(x' \in X') \to (x \in X)$, the $Z_i' := Z_i \times_X X'$ are weakly $\mathbb{P}^1$-linked. In particular they have the same image in X', and thus the Z_i have the same image in X, say V. Let $v \in X$ be the generic point of V. Notice that the lc centers of (Y, Δ_Y) that are minimal above v are exactly the Z_i's. Thus we can refine the earlier statement: there is a elementary étale neighborhood $(\tilde{v} \in \tilde{X}) \to (v \in X)$ such that the $\tilde{Z}_i := Z_i \times_X \tilde{X}$ are $\mathbb{P}^1$-linked. Since $k(\tilde{v}) = k(v)$, the morphisms $\tilde{Z}_i \to Z_i$ are generical isomorphisms. Thus the weak $\mathbb{P}^1$-links between the $\tilde{Z}_i$'s descend to weak $\mathbb{P}^1$-links between the Z_i's. □

Remark 5.4.2.14. The same argument shows that Proposition 5.4.2.12 also holds for an arbitrary quasi-projective lc surface pair (S, Δ) over a field and a crepant dlt blow-up $(S', \Delta') \to (S, \Delta)$. This can also be shown by considering the minimal resolution of (S, Δ).

Corollary 5.4.2.15. *Let $f\colon (Y, \Delta_Y) \to (X, \Delta)$ be as in Notation 5.4.2.1, and $S \subset Y$ a lc center of (Y, Δ_Y). Denote by*

$$S \xrightarrow{f_S} X_S \xrightarrow{\pi} X$$

the Stein factorization of $f|_S$. If $Z \subset X$ is an lc center of (X, Δ), then every irreducible component of $\pi^{-1}Z \subset X_S$ is the image of an lc center of $(S, \Delta_S := \mathrm{Diff}^*_S \Delta_Y)$.

Proof. Choose a minimal lc center $V \subset Y$ above Z. By Claim 5.4.2.13, there is a minimal lc center $V' \subseteq Y$ that dominates Z and that is contained in S. By adjunction for dlt pairs, $(V', \mathrm{Diff}^*_{V'} \Delta_Y)$ is an lc center of (S, Δ_S). Then $f_S(V') \subset X_S$ is one of the irreducible components of $\pi^{-1}Z$.

Now let $\eta \in X$ be the generic point of Z. After passing to an elementary étale neighborhood of $(\eta \in X)$, we can assume that S is the union of irreducible components S_j where each $f^{-1}(\eta) \cap S_j$ is connected. The previous argument show that the irreducible components of $\pi^{-1}Z$ are images of lc centers of the S_j, and these lc centers descend to S by [Kol13, 2.15]. □

Corollary 5.4.2.16. *Let (X, Δ) be a quasi-projective slc threefold pair over a perfect field of characteristic > 5. Then any intersection of lc centers is a union of lc centers.*

Proof. The normalization $(\bar{X}, \bar{D} + \bar{\Delta}) \to (X, \Delta)$ is crepant and $(\bar{X}, \bar{D} + \bar{\Delta})$ is lc. Thus we may assume that (X, Δ) is lc. Let $Z, Z' \subset X$ be lc centers, and pick a point $x \in Z \cap Z'$. If $f\colon (Y, \Delta_Y) \to (X, \Delta)$ is a crepant dlt blow-up, then by Claim 5.4.2.13 we can find a minimal lc center W of (Y, Δ_Y) above x whose image $f(W)$ is contained in $Z \cap Z'$. Since $f(W)$ is an lc center of (X, Δ), the result is proved. □

Corollary 5.4.2.17. *Let (X, Δ) be a quasi-projective slc threefold pair over a perfect field of characteristic > 5. Then the minimal lc centers of (X, Δ) are normal up to universal homeomorphism.*

Proof. Let $Z \subset X$ be a minimal lc center of (X, Δ), and $z \in Z$ be any point. Choose an étale neighborhood $(z' \in Z') \to (z \in Z)$. After shrinking it if necessary, we may assume that it is a standard étale neighborhood [Sta, 02GI, 02GU] and thus there is an étale neighborhood $(z' \in X') \to (z \in X)$ such that $Z' = Z \times_X X'$. Hence Z' is an lc center of (X', Δ') and it is connected. If it was reducible, then the intersection of its components would be a union of lc centers by Corollary 5.4.2.16, and their images in X would also be lc centers [Kol13, 2.15]. This contradicts the minimality of Z. Thus $(z' \in Z')$ is irreducible.

By [Sta, 0BQ4] we obtain that Z is geometrically unibranch. So the normalization morphism $Z^n \to Z$ is universally bijective [Sta, 0C1S]. Since it is surjective and universally closed, we obtain that that $Z^n \to Z$ is a universal homeomorphism. $\square$

Springs and sources for the reduced boundary.

We are now able to define sources of lc centers, analogously to [Kol13, §4.5]. The theory of sources should be thought of as higher codimension version of adjunction for divisors. However we only define sources for lc centers that are contained in the reduced boundary: see Remark 5.4.2.26 below.

Theorem 5.4.2.18. *Let $f\colon (Y, \Delta_Y) \to (X, \Delta)$ be as in Notation 5.4.2.1. Let $Z \subset X$ be a lc center contained in $\Delta^{=1}$ with normalization $n\colon Z^n \to Z$.*

Let $(S, \Delta_S := \mathrm{Diff}^*_S \Delta_Y) \subset Y$ be a minimal lc center above Z, with Stein factorization $f^n_S\colon S \to Z_S \to Z^n$. Then:

(a) Uniqueness of sources. *The crepant birational class of (S, Δ_S) over Z does not depend on the choice of S. We call it the **source** of Z, and denote it by $\mathrm{Src}(Z, Y, \Delta_Y)$.*

(b) Uniqueness of springs. *The isomorphism class of Z_S over Z does not depend on the choice of S. We call it the **spring** of Z, and denote it by $\mathrm{Spr}(Z, Y, \Delta_Y)$.*

(c) Crepant log structure. *(S, Δ_S) is dlt, $K_S + \Delta_S \sim_{\mathbb{Q},Z} 0$ and (S, Δ_S) is klt on the generic fiber above Z.*

(d) Galois property. *The field extension $k(Z) \subset k(Z_S)$ is Galois and the morphism $\mathrm{Bir}^c_Z(S, \Delta_S) \to \mathrm{Gal}(Z_S/Z)$ is surjective.*

(e) Poincaré residue map. *For $m > 0$ divisible enough, there are well-defined isomorphisms*

$$\omega_Y^{[m]}(m\Delta_Y)|_S \cong \omega_S^{[m]}(m\Delta_S)$$

and

$$n^*(f_*(\omega_Y^{[m]}(m\Delta_Y))) \cong ((f^n_S)_*\omega_S^{[m]}(m\Delta_S))^{\mathrm{Bir}^c_Z(S,\Delta_S)}.$$

(f) Birational invariance. *Let $(Y', \Delta_{Y'})$ be another crepant dlt blow-up of (X, Δ). Then*

$$\begin{aligned} \mathrm{Src}(Z, Y, \Delta_Y) &\overset{\mathrm{cbir}}{\cong} \mathrm{Src}(Z, Y', \Delta_{Y'}), \\ \mathrm{Spr}(Z, Y, \Delta_Y) &\cong \mathrm{Spr}(Z, Y', \Delta_{Y'}) \end{aligned}$$

over Z. In particular, we may write $\mathrm{Src}(Z, X, \Delta)$ for the source, and $\mathrm{Spr}(Z, X, \Delta)$ for the spring of $Z \subset X$.

Proof. For clarity, we divide the proof into four steps and several claims.

Step 1: Sources, springs and invariance. If $S' \subset Y$ is another minimal lc center above Z, then by Proposition 5.4.2.9 and Lemma 5.4.2.11 there is a crepant birational map $(S, \Delta_S) \dashrightarrow (S', \Delta_{S'})$ over Z given by a composition of weak direct $\mathbb{P}^1$-links. Since $Z_S = \mathrm{Spec}_{Z^n} f_*\mathcal{O}_S$, we obtain a Z-isomorphism $Z_S \cong Z_{S'}$. This shows the uniqueness of (S, Δ_S) up to crepant birational maps over Z, and the uniqueness of Z_S up to isomorphisms over Z.

The fact that (S, Δ_S) is dlt follows from Corollary 5.4.2.3, and that fact that it is klt on the generic fiber above Z holds by Proposition 5.4.2.2 and the fact that S is minimal.

To obtain birational invariance, we apply [Kol13, 4.44] to a common log resolution of (Y, Δ_Y) and $(Y', \Delta_{Y'})$. (The proof of [Kol13, 4.44] uses [Kol13, 10.45.2], which we replace by Fact 5.4.1.2).

The following observation will be crucial for the rest of the proof:

Claim 5.4.2.19. *If $\dim Z = 2$ then $\dim \mathrm{Src}(Z, Y, \Delta_Y) = 2$. If $\dim Z \leq 1$ then $\dim \mathrm{Src}(Z, Y, \Delta_Y) \leq 1$.*

Proof. Indeed, if Z is a divisor, then S can be chosen to be the strict transform Z_Y of Z. If Z is a curve or a single point, contained in a component Γ of $\Delta^{=1}$, then by Corollary 5.4.2.15 we can choose S to be contained in Γ_Y. Since Γ_Y is 2-dimensional and not f-exceptional, S is at most 1-dimensional. Since the dimension of S depends only on Z, our claim is proved. $\Diamond$

STEP 2: POINCARÉ RESIDUE. To obtain the Poincaré residue map, let $m > 0$ be even and sufficiently divisible such that $\omega_Y^{[m]}(m\Delta_Y) \sim f^*L$ for some line bundle L on X. By Corollary 5.4.2.3 we have isomorphisms

$$\mathcal{R}^m_{Y,S} \colon f^*L|_S \cong \omega_Y^{[m]}(m\Delta_Y) \xrightarrow{\sim} \omega_S^{[m]}(m\Delta_S).$$

However the choice of S is in general not unique. The following claim shows how the maps $\mathcal{R}^m_{Y,S}$ relate for different choices of S.

Claim 5.4.2.20. *Let $(S', \Delta_{S'})$ be another minimal lc center above Z. Then there is a crepant birational map $\phi\colon (S, \Delta_S) \dashrightarrow (S', \Delta_{S'})$ such the diagram*

$$\begin{tikzcd} \omega_Y^{[m]}(m\Delta_Y) \arrow[r, "\sim"] \arrow[d, "\mathcal{R}^m_{Y,S}"] & f^*L \arrow[r, "\sim"] & \omega_Y^{[m]}(m\Delta_Y) \arrow[d, "\mathcal{R}^m_{Y,S'}"] \\ \omega_S^{[m]}(m\Delta_S) \arrow[rr, "\phi^*"] & & \omega_{S'}^{[m]}(m\Delta_{S'}) \end{tikzcd} \tag{4.2.20.d}$$

is commutative.

Proof. Indeed, we may assume that S and S' are directly weakly $\mathbb{P}^1$-linked. Then there is a lc center $T \subset Y$ containing both S and S', which is birational to the total space of a weak standard $\mathbb{P}^1$-link $T' \to W$ whose sections map birationally to S and S'. Moreover, the map $T \dashrightarrow W$ factors the morphism $T \to f(T)$. The induced projections $S \dashrightarrow W \dashleftarrow S'$ are birational, and induce a birational map $\phi\colon S \dashrightarrow S'$, which we claim is the map we are looking for.

Since ϕ is obtained from a $\mathbb{P}^1$-link, by Lemma 5.4.2.11 it is crepant.

To prove the commutativity of the diagram, since $\mathcal{R}^m_{T,S} \circ \mathcal{R}^m_{Y,T} = \mathcal{R}^m_{Y,S}$, we may assume that $Y = T$. In this case, note that $S, S' \subset \Delta_Y^{=1}$. Moreover we are dealing with torsion-free sheaves, so it is enough to check commutativity generically. Thus we may assume that we have a standard weak $\mathbb{P}^1$-link $X \to W$ factorizing f, with sections S and S'. Localizing at the generic point of W, we may furthermore assume that W is the spectrum of a field L and that $X = \mathbb{P}^1_L$ [Har10, 25.3]. In this case $\Delta_Y = S + S'$ and we may choose coordinates x, y on X such that $S = [0; 1]$ and $S' = [1; 0]$. Then a generator of $H^0(\mathbb{P}^1_L, \omega_{\mathbb{P}^1_L}(S + S'))$ is dx/x, and

$$\mathcal{R}^1_{Y,S}(dx/x) = 1, \quad \mathcal{R}^1_{Y,S'}(dx/x) = -1$$

while ϕ^* is the identity map on L. Thus (4.2.20.d) indeed commutes for m even. $\Diamond$

If we think about (S, Δ_S) as a crepant birational class, then Claim 5.4.2.20 shows that we can define a Poincaré residue map $\mathcal{R}^m_{Y,S} \colon \omega_Y^{[m]}(m\Delta_Y) \to \omega_S^{[m]}(m\Delta_S)$, up to the action of the group $\mathrm{Bir}^c_Z(S, \Delta_S)$ on the target. We can remedy to this ambiguity using the following claim.

Claim 5.4.2.21. *The group $\mathrm{Bir}^c_Z(S, \Delta_S)$ acts on $\omega_S^{[m]}(m\Delta_S)$ as a finite group of roots of unity in $k(Z)$.*

Proof. We are dealing with torsion-free sheaves and with a group action that commutes with the projection to Z. So to understand the action, we can localize over the generic point of Z. Then we obtain a proper $k(Z)$-pair $(S_{k(Z)}, \Delta_{S_{k(Z)}})$ such that $\omega^{[m]}_{S_{k(Z)}}(m\Delta_{S_{k(Z)}})$ is trivial, and we must show that the action of $\mathrm{Bir}^c_{k(Z)}(S_{k(Z)}, \Delta_{S_{k(Z)}})$ on the 1-dimensional $k(Z)$-vector space $H^0(S_{k(Z)}, \omega^{[m]}_{S_{k(Z)}}(m\Delta_{S_{k(Z)}}))$is finite. If $\dim S = \dim Z$ then $S_{k(Z)}$ is the spectrum of a finite field extension of $k(Z)$, and so $\mathrm{Aut}_{k(Z)}(S_{k(Z)})$ is finite. By Claim 5.4.2.19, the only case left is when $Z = \{x\}$ is a closed point of X and (S, Δ_S) is a proper 1-dimensional klt pair over $k(x)$ such that $\omega^{[m]}_S(m\Delta_S)$ is trivial. Since $k(x)$ is a perfect field, it follows from Proposition 5.4.1.15 that $\mathrm{Bir}^c_{k(x)}(S, \Delta_S)$ acts finitely on the 1-dimensional vector space $H^0(S, \omega^{[m]}_S(m\Delta_S))$.

So in every case $\mathrm{Bir}^c_Z(S, \Delta_S)$ acts finitely on the generic stalk of $\omega^{[m]}_S(m\Delta_S)$, hence through the multiplication with some r^{th}-root of unity in $k(Z)$. $\Diamond$

Replacing m by mr, the action becomes trivial and the ambiguity about the Poincaré residue map disappears.

Step 3: the Galois property. We wish to prove that the field extension $k(Z_S)/k(Z)$ is Galois.

The case where Z is a divisor is the easiest, we treat it first. By [NT20, 1.2] there is a unique lc center S above Z (namely, its strict transform) and the morphism $S \to Z$ has connected fibers in a neighborhood of the generic point of Z. Thus $S \to Z$ is birational, $\mathrm{Bir}^c_Z(S, \Delta_S)$ is trivial and $k(Z_S) = k(Z)$.

From now on, we assume that $\dim Z \leq 1$. To prove that the finite morphism $Z_S \to Z$ induces a Galois extension on the function fields, we may localize at the generic point of Z. Then the situation is the following: $(x \in X, \Delta)$ is a local lc pair of dimension 2 or 3, $(Y, \Delta_Y) \to (X, \Delta)$ is a crepant dlt blow-up and $(S, \Delta_S) \subset Y$ is an lc center of dimension ≤ 1 such that the morphism $S \to X$ factorizes through the closed point, and S is minimal for this property.

Claim 5.4.2.22. *In this set-up, the fields of definition of the geometric connected components of S are the same, and it is a Galois extension of $k(x)$.*

Proof. Indeed, let K^s be a separable closure of $k(x)$. Then $(Y, \Delta_Y) \times_{k(x)} K^s$ is a dlt $\mathbb{Q}$-factorial pair, and every component of S_{K^s} is an lc center that is minimal. Let W be one of them, with field of definition F. Then every lc center containing W is also defined over F [Kol13, 4.17], and so any lc center that is weakly $\mathbb{P}^1$-linked to W is also defined over F. By Proposition 5.4.2.12 (see Remark 5.4.2.14) we obtain that all the components of S_{K^s} are defined over F. If σ is an element of the Galois group of the Galois closure of $k(x) \subset F$, then W^σ is a minimal lc center defined over the conjugate field F^σ. Therefore $F^\sigma = F$, so we see that $k(x) \subset F$ is Galois. $\Diamond$

Claim 5.4.2.23. *$H^0(S, \mathcal{O}_S)$ is a separable field extension of $k(x)$.*

Proof. If $(x \in X)$ has dimension 3, this holds because $k(x)$ is perfect (since in this case we localized at a closed point). If $(x \in X)$ has dimension 2, then $(x \in X, \Delta)$ is a local lc surface singularity with the property that $\Delta^{=1} \neq 0$. (By [Kol13, 2.28], these $(x \in X)$ are rational surface singularities.) The possible dual graphs of the minimal resolutions (T, Γ) of such pairs are classified, see for example [Kol13, 3.31]. Inspecting them, we see that if $C \subset T$ is an exceptional proper curve above x, then $\dim_{k(x)} H^0(C, \mathcal{O}_C) \leq 4$, and if C' is another exceptional proper curve, then $C \cdot C' = \mathrm{length}_{k(x)} C \cap C' \leq 4$. Hence if $E \subset T$ is a lc place of $(x \in X, \Delta)$ then $H^0(E, \mathcal{O}_E)$ is a separable extension of $k(x)$ provided that $\mathrm{char}\, k(x) \geq 5$. $\Diamond$

Claim 5.4.2.24. *$H^0(S, \mathcal{O}_S)$ is a Galois extension of $k(x)$.*

Proof. By Lemma 5.4.1.21 and Claim 5.4.2.23, every connected component of S_{K^s} is already defined over the Galois closure K of $k(x) \subset H^0(S, \mathcal{O}_S)$. On the other hand, the field of definition of these components is a Galois extension of $k(x)$ by Claim 5.4.2.22. So we deduce that K is the field of definition of every connected component of S_{K^s}.

It remains to show that $H^0(S, \mathcal{O}_S) = K$. Consider the Cartesian diagram

$$\begin{tikzcd} S & S_K = \bigcup_i S_K^{(i)} \arrow[l] \arrow[d] \\ k(x) \arrow[from=u] & K \arrow[l] \end{tikzcd}$$

where the $S_K^{(i)}$ are the irreducible components of S_K. As before, $(Y, \Delta_Y) \times_{k(x)} K$ is a dlt pair above K, and each $S_K^{(i)}$ is an lc center. Thus the intersection of the $S_K^{(i)}$ are also lc centers. By minimality of S, we obtain that the $S_K^{(i)}$ are disjoint. So the $S_K^{(i)}$ are also the connected components of S_K. Since K is Galois over $k(x)$, the Galois group $G = \mathrm{Gal}(K/k(x))$ acts on S_K. Assume that some $S_K^{(j)}$ is stable under the action of a non-trivial subgroup of G. Then by Lemma 5.4.1.23, $S_K^{(j)}$ would be defined over a proper sub-extension K' of $k(x) \subset K$. Then every $S_K^{(i)}$ would be defined over K', which is a contradiction with the previous paragraph. Thus G acts freely on the set of $S_K^{(i)}$'s. The action is also transitive, since $S_K \to S$ is the geometric quotient by G.

By flat base-change, we have

$$H^0(S_K, \mathcal{O}_{S_K}) = H^0(S, \mathcal{O}_S) \otimes_{k(x)} K$$

and the G-action is given by the action on K. On the other hand, since S_K is the disjoint union of the $S_K^{(i)}$ and G permutes them freely, we also have

$$H^0(S_K, \mathcal{O}_{S_K}) = \prod_{\sigma \in G} K$$

where G acts by permuting the factors. Taking G-invariants, we obtain

$$H^0(S, \mathcal{O}_S) = H^0(S, \mathcal{O}_S) \otimes_{k(x)} K^G = \left(H^0(S, \mathcal{O}_S) \otimes_{k(x)} K\right)^G = \left(\prod_{\sigma \in G} K\right)^G = K$$

as desired. $\Diamond$

Step 4: Galois group and crepant birational maps. It remains to show that every element of the Galois group $\mathrm{Gal}(Z_S/Z)$ is induced by a crepant birational self-map of (S, Δ_S).

There is nothing to prove if $\dim Z = 2$, as noticed at the beginning of the previous step.

From now on assume that $\dim Z \leq 1$. We have proved in the previous step that $K := k(Z_S)$ is Galois over $k(Z)$. Let W be a component of $S \times_{k(Z)} K$ and pick $\sigma \in \mathrm{Gal}(K/k(Z))$. The proof of Claim 5.4.2.24 shows that W is defined over K, so it has a conjugate $W^\sigma \subset S \times_{k(Z)} K$. Fix a weak $\mathbb{P}^1$-link between W and W^σ inside $Y \times_X K$: then the $\mathrm{Gal}(K/k(Z))$-orbit of this link descends to an element of $\mathrm{Bir}^c_{k(Z)}((S, \Delta_S) \times_Z k(Z))$, and in turn this crepant birational map induces σ on $H^0(S \times_{k(Z)} K, \mathcal{O}) = K$.

This proves that Galois automorphisms of $k(Z) \subset k(Z_S)$ are induced by birational self-maps of S which are generically $(K_S + \Delta_S)$-crepant over Z. We need to show that these maps are crepant, not only generically crepant. If $\dim Z = 0$ there is nothing to show, so assume $\dim Z = 1$. Then we make the following observation: Z is contained in a component

D of $\Delta^{=1}$ and by adjunction any preimage of Z in D^n is a codimension one lc center of $(D^n, \mathrm{Diff}_{D^n}(\Delta - D))$. By the classification of lc surface singularities [Kol13, 2.31], we see that D is regular at the generic point of Z. Hence the strict transform of Z must appear in $\lfloor \mathrm{Diff}_{D^n}(\Delta - D) \rfloor$, and by adjunction we get a natural boundary Γ on Z^n. Moreover the equality $\dim S = \dim Z$ implies $S = Z_S$, so $f_S^n \colon S \to Z^n$ is a finite morphism. By the definition of (S, Δ_S) we must have

$$(f_S^n)^*(K_{Z^n} + \Gamma) = K_S + \Delta_S. \tag{4.2.24.e}$$

On the other hand, the fact that $S = Z_S$ shows that the birational self-maps of S we found are just the Galois automorphisms of Z_S over Z^n. Since Δ_S can be defined by (4.2.24.e), we see that Galois automorphisms are crepant.

The proof is complete. □

Corollary 5.4.2.25 (Adjunction)**.** *Let $(X, D+\Delta)$ be a quasi-projective lc threefold pair above a perfect field of characteristic > 5, where D is a reduced divisor with normalization $n \colon D^n \to D$. Let $Z \subset D$ be an lc center of $(X, D+\Delta)$, and $Z_D \subset D^n$ be an irreducible variety such that $n(Z_D) = Z$. Then:*

(a) Z_D is an lc center of $(D^n, \mathrm{Diff}_{D^n} \Delta)$,

(b) there is a commutative diagram

$$\begin{tikzcd} \mathrm{Src}(Z_D, D^n, \mathrm{Diff}^*_{D^n} \Delta) \arrow[r, dashed, "\mathrm{cbir}"] \arrow[d] & \mathrm{Src}(Z, X, D+\Delta) \arrow[d] \\ D^n \arrow[r, "n"] & D \end{tikzcd}$$

(c) There is an isomorphism $\mathrm{Spr}(Z_D, D^n, \mathrm{Diff}^*_{D^n} \Delta) \cong \mathrm{Spr}(Z, X, D+\Delta)$.

Proof. By enlarging Δ, we may assume that D is irreducible. Choose a $\mathbb{Q}$-factorial dlt crepant blow-up (Y, Δ_Y) of $(X, D+\Delta)$ and let D_Y be the strict transform of D. The morphism $D_Y \to D$ is birational, so its Stein factorization is precisely

$$D_Y \xrightarrow{\sim} D^n \to D,$$

which implies that $(D_Y, \mathrm{Diff}^*_{D_Y} \Delta_Y) = (D^n, \mathrm{Diff}_{D^n} \Delta)$. By Corollary 5.4.2.15 there is a lc center $W \subset D_Y$ of (Y, Δ_Y) that dominates (actually is equal to) $Z_D \subset D^n$. By Proposition 5.4.2.2 W is also an lc center of $(D_Y, \mathrm{Diff}^*_{D_Y} \Delta_Y)$, which proves the first item. Moreover W is a representative for both $\mathrm{Src}(Z_D, D^n, \mathrm{Diff}_{D^n} \Delta)$ and $\mathrm{Src}(Z, Y, \Delta_Y)$, and the second item follows by uniqueness of the source up to crepant birational map. The third item is a consequence of the second item. □

Remark 5.4.2.26. The same method gives sources, springs, crepant log structure, adjunction and birational invariance of arbitrary lc centers of (X, Δ). However:

(a) The Galois property is problematic: if Z is 1-dimensional and not contained in any component of $\Delta^{=1}$, then $\mathcal{O}_{X,Z}$ might by an elliptic or a cusp singularity, and the degrees of the exceptional curves might be arbitrarily large [Kol13, 3.27.3]. For example, we can take a product of such singularities with $\mathbb{P}^1$. So inseparable extensions might appear.

(b) The Poincaré residue map can be defined, but it is not clear how to get rid of the $\mathrm{Bir}^c_Z(S, \Delta_S)$-ambiguity. Indeed, in constrast to Claim 5.4.2.19, one more case can show up, namely $\dim S = 2$ and $\dim Z = 1$, and we were not able to show finiteness of representations with sufficient generality in this case.

5.4.3 Gluing theorems for threefolds in characteristic > 5

We now prove the gluing theorems for lc threefolds. Our proofs follow closely those of Kollár [Kol13, §5].

Lemma 5.4.3.1. *Let (X, Δ) and (X', Δ') be quasi-projective lc threefold pairs over a perfect field of characteristic > 5, and $\phi \colon (X, \Delta) \cong (X', \Delta')$ a log isomorphism. Let $Z \subset X$ be an lc center of (X, Δ). Then:*

(a) $Z' := \phi(Z)$ is an lc center of (X', Δ'), and

(b) we have a commutative square

$$\begin{tikzcd} \mathrm{Src}(Z, X, \Delta) \arrow[r, dotted, "\mathrm{cbir}"] \arrow[d] & \mathrm{Src}(Z', X', \Delta') \arrow[d] \\ Z \arrow[r, "\phi"] & Z' \end{tikzcd}$$

Proof. Since ϕ is a log isomorphism, Z' is an lc center of (X', Δ'). The second point follows from the birational invariance property proved in Theorem 5.4.2.18. The source is defined up to crepant birational map over the lc center, so there is no ambiguity about the commutativity of the diagram. $\square$

First we show that the geometric quotient exists.

Proposition 5.4.3.2. *Let $(X, D + \Delta)$ be a projective lc threefold pair over a perfect field k of characteristic > 5. Let $\tau \colon (D^n, \mathrm{Diff}_{D^n} \Delta) \cong (D^n, \mathrm{Diff}_{D^n} \Delta)$ be a generically fixed point free involution. Assume that $K_X + D + \Delta$ is ample. Then the induced equivalence relation $R(\tau) \rightrightarrows X$ is finite, the geometric quotient $X/R(\tau)$ exists, and X/R is proper over k.*

Proof. By Theorem 2.5.0.2, the geometric quotient exists as soon as $R := R(\tau)$ is finite. Notice that, by construction, a point of $\mathrm{Supp}(D)$ and a point of $X \setminus \mathrm{Supp}(D)$ cannot be in the same equivalence class. Moreover, R restricts to the identity relation away from $\mathrm{Supp}(D)$. Thus we only need to prove that R is finite on $\mathrm{Supp}(D)$.

For every lc center Z of $(X, D+\Delta)$ contained in D, let $Z^0 := Z \setminus$(lower dimensional lc centers) and let $\mathrm{Spr}^0(Z, X, D + \Delta)$ be the preimage of Z^0 through the finite morphism $\mathrm{Spr}(Z, X, D + \Delta) \to Z$. Set

$$p \colon \mathrm{Spr}^0(X, D + \Delta, \subseteq D) := \bigsqcup_{Z \subseteq D} \mathrm{Spr}^0(Z, X, D + \Delta) \longrightarrow X,$$

it is a quasi-finite morphism mapping surjectively onto D.

The equivalence relation $R \rightrightarrows D$ pullbacks to an equivalence relation $(p \times p)^* R \rightrightarrows \mathrm{Spr}^0(X, D + \Delta, \subseteq D)$ that commutes with the projections to D, and since p is surjective onto D it is sufficient to show that $(p \times p)^* R$ is finite. We describe how the generators of R pullbacks to $\mathrm{Spr}^0(X, D + \Delta, \subseteq D)$. To make book-keeping easier, we let $\{Z_j\}_j$ be the set of lc centers of $(X, D + \Delta)$ contained in D.

By the Galois property of Theorem 5.4.2.18, over the normal locus of Z_j^0 the morphism $\mathrm{Spr}^0(Z_j, X, D+\Delta) \to Z_j^0$ is the quotient by the Galois group $G_j := \mathrm{Gal}(\mathrm{Spr}(Z_j, X, D+\Delta)/Z_j)$. Thus the preimage of the diagonal $Z_j^0 \times Z_j^0$ under p is a union of the graphs of the G_j-action, together with other components that do not dominate Z_j^0 (their images are contained in the locus where Z_j^0 is not normal).

Next we understand the pullback of τ.

Claim 5.4.3.3. *Let $Z_{jh} \subset D^n$ be a subvariety dominating $Z_j \subset X$. Let $Z_l := n(\tau(Z_{jh}))$. Then Z_{jh} is a lc center of $(D^n, \mathrm{Diff}_{D^n} \Delta)$. Moreover* *(a) τ induces a crepant birational map*

$$\tilde{\tau}_{jhl}\colon \operatorname{Src}(Z_j, X, D+\Delta) \overset{\text{cbir}}{\sim} \operatorname{Src}(Z_l, X, D+\Delta)$$

determined up to the left and right action of $\operatorname{Bir}^c_{Z_j} \operatorname{Src}(Z_j, X, D+\Delta)$ *and* $\operatorname{Bir}^c_{Z_l} \operatorname{Src}(Z_l, X, D+\Delta)$*;*

(b) $\tilde{\tau}_{jhl}$ induces an isomorphism

$$\tau_{jhl}\colon \operatorname{Spr}^0(Z_j, X, D+\Delta) \cong \operatorname{Spr}^0(Z_l, X, D+\Delta)$$

determined up to left and right multiplication by G_j and G_l.

Proof. By Corollary 5.4.2.25, Z_{jh} is an lc center of $(D^n, \operatorname{Diff}_{D^n}\Delta)$ and

$$\operatorname{Src}(Z_{jh}, D^n, \operatorname{Diff}_{D^n}\Delta) \overset{\text{cbir}}{\sim} \operatorname{Src}(Z_j, X, D+\Delta). \tag{4.3.3.f}$$

By Lemma 5.4.3.1 the automorphism τ of $(D^n, \operatorname{Diff}_{D^n}\Delta)$ induces a crepant birational map between $\operatorname{Src}(Z_{jh}, D^n, \operatorname{Diff}_{D^n}\Delta)$ and $\operatorname{Src}(\tau(Z_{jh}), D^n, \operatorname{Diff}_{D^n}\Delta)$. Then by (4.3.3.f) we obtain crepant birational maps

$$\tilde{\tau}_{jkl}\colon \operatorname{Src}(Z_j, X, D+\Delta) \overset{\text{cbir}}{\sim} \operatorname{Src}(Z_l, X, D+\Delta).$$

Since $\tilde{\tau}_{jhl}$ preserves the non-klt locus and since $\operatorname{Spr}^0(Z_j, X, D+\Delta)$ is precisely the image of the klt locus of $\operatorname{Src}(Z_j, X, D+\Delta)$, the $\tilde{\tau}_{jhl}$ descend to an isomorphism

$$\tau_{jhl}\colon \operatorname{Spr}^0(Z_j, X, D+\Delta) \cong \operatorname{Spr}^0(Z_l, X, D+\Delta)$$

as claimed. However τ_{jhl} is not uniquely defined, since $\tilde{\tau}_{jhl}$ is determined up to left and right multiplication by $\operatorname{Bir}^c_{Z_j} \operatorname{Src}(Z_j, X, D+\Delta)$ and $\operatorname{Bir}^c_{Z_l} \operatorname{Src}(Z_l, X, D+\Delta)$. By Theorem 5.4.2.18 we obtain that τ_{jhl} is determined up to left and right multiplication by G_j and G_l. ◊

Since $n(n^{-1}Z_j) = \bigcup_i D_i \cap Z_j$ and each $D_i \cap Z_j$ is a union of lc centers by Corollary 5.4.2.16, we see that each component of $n^{-1}Z_j$ dominates a lc center of $(X, D+\Delta)$. So, thanks to Claim 5.4.3.3, we have found all the generators of $(p \times p)^*R$.

To show that $(p \times p)^*R$ is finite, by [Kol13, 9.55] it is sufficient to show that it is finite over the generic point of every Z_j^0. Therefore we may assume that $(p \times p)^*R$ is the groupoid generated by the G_j and the τ_{jhl}, and the stabilizer of $\operatorname{Spr}^0(Z_j, X.D+\Delta)$ is generated by the sets $\{\tau^{-1}_{jh'l} G_l \tau_{jhl}\}_{h,h',l}$.

The Galois property of Theorem 5.4.2.18 shows that G_j is a subgroup of

$$\operatorname{Aut}^s_k \operatorname{Spr}(Z_j, X, D+\Delta) := \operatorname{im}\left[\operatorname{Bir}^c_k \operatorname{Src}(Z_j, X, D+\Delta) \to \operatorname{Aut}_k \operatorname{Spr}(Z_j, X, D+\Delta)\right].$$

By Claim 5.4.3.3, the $\tau^{-1}_{jh'l} G_l \tau_{jhl}$ are also subgroups of $\operatorname{Aut}^s_k \operatorname{Spr}(Z_j, X, D+\Delta)$. To complete the proof we will show that these groups of k-automorphisms are finite. This is where the ampleness assumption on $K_X + D + \Delta$ will be used.

(a) If $\dim Z_j = 2$ then $\operatorname{Src}(Z_j, X, D+\Delta) \to Z_j$ is birational and therefore $\operatorname{Aut}^s_k(Z_j, X, D+\Delta)$ is finite by Proposition 5.4.1.18.

(b) If $\dim Z_j = 1$ then $\dim \operatorname{Src}(Z_j, X, D+\Delta) = 1$ by Claim 5.4.2.13. Hence $\operatorname{Src}(Z_j, X, D+\Delta)$ is a 1-dimensional pair of general type, since its log canonical divisor is the pullback of the ample divisor $K_X + D + \Delta$ through the composition of finite morphisms

$$\operatorname{Src}(Z_j, X, D+\Delta) \to Z_j \hookrightarrow X.$$

Thus $\operatorname{Bir}^c \operatorname{Src}(Z_j, X, D+\Delta)$ is finite by Proposition 5.4.1.16.

(c) If $\dim Z_j = 0 = \dim \operatorname{Src}(Z_j, X, D+\Delta)$ then finiteness is clear.

(d) The only case left, according to Claim 5.4.2.19, is $\dim Z_j = 0$ and $\dim \operatorname{Src}(Z_j, X, D+\Delta) = 1$. Then $\operatorname{Src}(Z_j, X, D+\Delta)$ is a Calabi-Yau curve and finiteness follows from Proposition 5.4.1.15.

Thus X/R exists and is a k-scheme. Since $X \to X/R$ is finite, X/R is proper over k [Sta, 09MQ,03GN]. This completes the proof. □

Remark 5.4.3.4. With the notations of the proof of Proposition 5.4.3.2, let us observe that the stabiliser of $\operatorname{Spr}(Z_j^0, X, D+\Delta)$ is not contained in the smaller group

$$\operatorname{Aut}_{Z_j} \operatorname{Spr}(Z_j, X, D+\Delta) := \operatorname{im}\left[\operatorname{Bir}^c_{Z_j} \operatorname{Src}(Z_j, X, D+\Delta) \to \operatorname{Aut}_{Z_j} \operatorname{Spr}(Z_j, X, D+\Delta)\right]$$

even though G_j belongs to it and each τ_{jhl} commutes with the projections to Z_j and Z_l. Indeed the stabiliser is generated by the groups $\tau^{-1}_{jh'l} G_l \tau_{jhl}$ where it may happen that $h \neq h'$. In this case the corresponding automorphisms of $\operatorname{Spr}(Z_j, X, D+\Delta)$ may not commute with the projection to Z_j. This happens if Z_j is dominated by several lc centers of $(D^n, \operatorname{Diff}_{D^n} \Delta)$ whose images under τ dominate the same lc center $Z_l \subset X$.

Notice that $\operatorname{Aut}_{Z_j} \operatorname{Spr}(Z_j, X, D+\Delta)$ is a finite group since $\operatorname{Spr}(Z_j, X, D+\Delta) \to Z_j$ is a finite morphism. On the other hand, it is not clear that $\operatorname{Aut}^s_k \operatorname{Spr}(Z_j, X, D+\Delta)$ should be finite, and this is where the ampleness assumption comes into the picture.

Now we show that the log canonical divisor descends to the geometric quotient.

Proposition 5.4.3.5. *Let $(X, D+\Delta)$ be a quasi-projective lc threefold pair over a perfect field k of characteristic > 5. Let $\tau\colon (D^n, \operatorname{Diff}_{D^n} \Delta) \cong (D^n, \operatorname{Diff}_{D^n} \Delta)$ be an involution. Assume that $R(\tau) \rightrightarrows X$ is finite, let $q\colon X \to Y := X/R(\tau)$ be the geometric quotient, and let $\Delta_Y := q_*\Delta$. Then $K_Y + \Delta_Y$ is $\mathbb{Q}$-Cartier.*

Proof. By Corollary 4.3.0.7 the $\mathbb{Q}$-divisor $K_Y + \Delta_Y$ is $\mathbb{Q}$-Cartier in codimension 2. Hence we may localize over a closed point of Y, and assume that Y is local with closed point y, and that $K_Y + \Delta_Y$ is $\mathbb{Q}$-Cartier on $Y^0 := Y \setminus \{y\}$. Since $q\colon X \to Y$ is an isomorphism above the away from $q(D)$, we may assume that y belongs to the nodal locus. Since an $\mathcal{O}_Y$-module is locally free if and only if it is locally free after an étale base-change, we may base-change along the strict henselization of Y, and assume that $k(y)$ is separably closed. Since k is perfect, we may therefore assume that $k(y)$ is algebraically closed.

We are going to descend the total space of a multiple of $K_X + D + \Delta$ to Y, and use the theory of Seifert bundles (see [Kol13, §9.3]) to conclude that it defines a line bundle on Y.

Choose an even integer $m > 0$ such that $m\Delta$ is a $\mathbb{Z}$-divisor, and both $\omega_X^{[m]}(mD + m\Delta)$ and $\omega_{Y^0}^{[rm]}(rm\Delta_Y|_{Y^0})$ are invertible sheaves. We consider the $\mathbb{A}^1$-bundle over X given by

$$X_L := \operatorname{Spec}_X \sum_{r\geq 0} \omega_X^{[rm]}(rmD + rm\Delta) \xrightarrow{p} X.$$

Set $D_L := p^{-1}D$ and $\Delta_L := p^{-1}\Delta$. Clearly $(X_L, D_L + \Delta_L)$ is lc. Since $X_L \to X$ is an $\mathbb{A}^1$-bundle, we see that the normalization $D^n_L \to D_L$ is equal to $D^n \times_D D_L$. By adjunction and functoriality of the relative spectrum, this gives the alternative description

$$D^n_L = \operatorname{Spec}_{D^n} \sum_{r\geq 0} \omega_{D^n}^{[rm]}(rm \operatorname{Diff}_{D^n} \Delta).$$

The fiber product description shows that $\operatorname{Diff}_{D^n_L} \Delta_L = p^* \operatorname{Diff}_{D^n} \Delta$ and that the lc centers of $(D^n_L, \operatorname{Diff}_{D^n_L} \Delta_L)$ are the preimages of the lc centers of $(D^n, \operatorname{Diff}_{D^n} \Delta)$. As $\operatorname{Diff}_{D^n} \Delta$ is τ-invariant, the relative spectrum description shows that τ lifts to an involution τ_L of the pair $(D^n_L, \operatorname{Diff}_{D^n_L} \Delta_L)$.

Now we wish to show that the induced equivalence relation $R_L := R(\tau_L) \rightrightarrows X_L$ is finite, so that we can form the quotient X_L/R_L.

Denote by $p'\colon Y_L^0 \to Y^0$ the total space of the invertible sheaf $\omega_{Y^0}^{[rm]}(rm\Delta_Y|_{Y^0})$, $X^0 := q^{-1}Y^0$ and $X_L^0 := (p \circ q)^{-1}Y^0$. Then we have a natural finite morphism of $\mathbb{A}^1$-bundles $q'\colon X_L^0 \to Y_L^0$ making the diagram

$$\begin{array}{ccc} X_L^0 & \xrightarrow{p} & X^0 \\ \Big\downarrow{\scriptstyle q'} & & \Big\downarrow{\scriptstyle q} \\ Y_L^0 & \xrightarrow{q'} & Y^0 \end{array}$$

commutative. Since X_L^0 is the total space of the line bundle $\omega_{X^0}^{[m]}(mD|_{X^0} + m\Delta|_{X^0})$ which descends to Y^0, we have that $Y_L^0 = X_L^0/R_L^0$ where R_L^0 is the restriction of R_L to X_L^0 [Kol13, 9.48]. Therefore R_L^0 is finite, and we only need to prove the finiteness of R_L over the complement of X_L^0.

Let $x_1, \ldots, x_s \in X$ be the preimages of y. Since y belongs to the nodal locus of Y, we have $x_1, \ldots, x_s \in D$. If none of the x_i are lc centers of $(X, D + \Delta)$, then every lc center of $(X_L, D_L + \Delta_L)$ intersects X_L^0 and therefore R_L is finite by [Kol13, 9.55].

Assume that one of the x_i is an lc center. Then every x_i is an lc center: for the x_i form an equivalence class of τ, by adjunction one of them corresponds to an lc center of $(D^n, \mathrm{Diff}_{D^n} \Delta)$ and τ permutes these lc centers.

Since $k(y)$ is algebraically closed, we have $k(x_i) = k(y)$ for each i. The fiber of $p\colon X_L \to X$ above x_i is the spectrum of the symmetric algebra of the 1-dimensional $k(y)$-vector space

$$V_i := \omega_X^{[m]}(mD + m\Delta) \otimes_{\mathcal{O}_X} k(x_i)$$

and $X_L \setminus X_L^0 = \bigcup_i V_i$. If $x_i' \in D^n$ is a preimage of x_i, then the data of $\tau(x_i') = x_j'$ defines an isomorphism $\tau_{ijl}\colon V_i \to V_j$ (the index l accounts for the fact that x_i might have several preimages in D^n). The collection of isomorphisms $\{\tau_{ijl}\colon V_i \to V_j\}$ generates a groupoid, and R_L is finite if and only if each group $\mathrm{Stab}(V_i) \subset \mathrm{Aut}(V_i)$ of all possible compositions $\tau_{ij_1l_1} \circ \cdots \circ \tau_{j_nil_n}\colon V_i \to V_i$, is finite.

To show this property, consider the sources $(S_i, \Delta_i) := \mathrm{Src}(x_i, X, D + \Delta)$. These are proper Calabi-Yau varieties over $k(y)$. If we pullback $\omega_X^{[m]}(mD + m\Delta)$ to a crepant dlt blow-up, restrict it to (a model of) S_i and take global sections, we obtain V_i. Thus the Poincaré residue maps constructed in Theorem 5.4.2.18 give canonical isomorphisms

$$V_i \cong H^0(S_i, \omega_{S_i}^{[m]}(m\Delta_i)).$$

Moreover Claim 5.4.3.3 shows that each isomorphism $\tau_{ijl}\colon V_i \to V_j$ is induced by a crepant birational map $\phi_{ijl}\colon (S_j, \Delta_j) \dashrightarrow (S_i, \Delta_i)$. Hence we conclude that

$$\mathrm{Stab}(V_i) \subseteq \mathrm{im}\left[\mathrm{Bir}^c(S_i, \Delta_i) \to \mathrm{Aut}_{k(y)} H^0(S_i, \omega_{S_i}^{[m]}(m\Delta_i))\right]. \tag{4.3.5.g}$$

Now observe that the S_i are at most 1-dimensional by Claim 5.4.2.19, and therefore by Proposition 5.4.1.15 the right-hand side in (4.3.5.g) is finite.

It follows that R_L is finite, and thus the quotient X_L/R_L exists. By [Kol13, 9.48] the complement of the zero section is a Seifert bundle over Y, and by [Kol13, 9.53] this implies that its define a line bundle on Y. By construction this line bundle is equal to $\omega_{Y^0}^{[m]}(m\Delta_Y|_{Y^0})$ over Y^0. Hence $\omega_Y^{[m]}(m\Delta_Y)$ is invertible, as was to be shown. □

Theorem 5.4.3.6. *Let k be a perfect field of characteristic > 5. Then normalization gives a one-to-one correspondence*

$$(\operatorname{char} k > 5) \quad \begin{pmatrix} \text{Proper slc threefold pairs} \\ (X, \Delta) \text{ such that} \\ K_X + \Delta \text{ is ample} \end{pmatrix} \xrightarrow{1:1} \begin{pmatrix} \text{Proper lc threefold pairs } (\bar{X}, \bar{D} + \bar{\Delta}) \\ \text{plus a generically fixed point free} \\ \text{involution } \tau \text{ of } (\bar{D}^n, \operatorname{Diff}_{\bar{D}^n} \bar{\Delta}) \\ \text{such that } K_{\bar{X}} + \bar{D} + \bar{\Delta} \text{ is ample.} \end{pmatrix}$$

Proof. Given $(\bar{X}, \bar{D} + \bar{\Delta}, \tau)$ as in the right-hand side, by Proposition 5.4.3.2 the equivalence relation $R(\tau)$ is finite and we can form the geometric quotient $q \colon \bar{X} \to X := \bar{X}/R(\tau)$. Set $\Delta := q_* \bar{\Delta}$. By Proposition 4.4.0.1 the scheme X is demi-normal, and by Proposition 5.4.3.5 the $\mathbb{Q}$-divisor $K_X + \Delta$ is $\mathbb{Q}$-Cartier. Therefore (X, Δ) is slc. This gives a map $\{(\bar{X}, \bar{D} + \bar{\Delta}, \tau)\} \to \{(X, \Delta)\}$. It is an inverse to the normalization map by Proposition 4.4.0.1 and [Kol13, 5.3]. $\square$

5.5 Application to the moduli theory of stable surfaces

In this section we apply the theory of gluing to the moduli functor of stable varieties. Our discussion will be conditional, since some technical results are not known yet in positive characteristic.

Let k be an algebraically closed field. We define families of stable log varieties and the moduli functor of stable varieties following [Pat17].

Definition 5.5.0.1. A projective connected pure-dimensional k-scheme X together with a $\mathbb{Q}$-divisor is a **stable log pair** if (X, D) is slc and $K_X + D$ is and ample. If $D = 0$, we simply say that X is a **stable variety**.

Let T be a k-scheme. A **family of pairs** over T is a flat morphism of k-schemes $X \to T$ together with a $\mathbb{Q}$-divisor D on X such that: for every $t \in T$, none of the irreducible components of X_t is contained in $\operatorname{Supp} D$, and none of the irreducible components of $X_t \cap \operatorname{Supp} D$ is contained in $\operatorname{Sing} X_t$. This allows us to define a restricted divisor D_t on X_t.

Let T be a k-scheme. A **family of stable log pairs** over T is a family of pairs $f \colon (X, D) \to T$ such that $K_{X/T} + D$ is $\mathbb{Q}$-Cartier and the geometric fiber $(X_{\bar{t}}, D_{\bar{t}})$ is a stable log pair for every $t \in T$.

The **moduli functor** $\overline{\mathcal{M}}_{n,v,k}$, where $v \in \mathbb{Q}_+$, is defined on Sch_k by the values

$$\overline{\mathcal{M}}_{n,v,k}(T) = \left\{ \begin{array}{c|l} X & 1) \ f \text{ is a flat morphism of } k\text{-schemes,} \\ \Big\downarrow f & 2) \ \left(\omega_{X/T}^{[m]}\right)_S \cong \omega_{X_S/S}^{[m]} \text{ for every } S \to T \text{ and } m \in \mathbb{N}, \\ & 3) \text{ for every } t \in T, X_{\bar{t}} \text{ is a stable variety of dimension } n \\ T & \quad \text{with } \operatorname{Vol}(K_{X_{\bar{t}}}) = v. \end{array} \right\}$$

and for $T' \to T$, the corresponding map $\overline{\mathcal{M}}_{n,v}(T) \to \overline{\mathcal{M}}_{n,v}(T')$ is given by pullbacks.

Remark 5.5.0.2. There are subtle differences between families of stable varieties and families parametrized by the functor $\overline{\mathcal{M}}_{n,v,k}$:

(a) At least when $n = 2$, if $(X \to T) \in \overline{\mathcal{M}}_{2,v,k}(T)$ and T is normal, then $X \to T$ is a family of stable surfaces, see [Pat17, Lemma 2.3].

(b) If $(X \to T) \in \overline{\mathcal{M}}_{n,v,k}(T)$, then $X \to T$ need not be a family of stable varieties, see [Pat17, Remark 1.6].

Remark 5.5.0.3. We have only defined the moduli functor in the boundary-free case. To define a moduli functor of stable log pairs we need a good notion of family of divisors above arbitrary bases. A good notion, at least in characteristic 0, is developed in [Kol19]. However, to avoid technical difficulties, we will restrict ourselves to the boundary-free case in what follows. This way, the only pairs we will have to deal with are the one arising from the normalization of demi-normal varieties.

From now on we consider the case of stable surfaces (that is, $n = 2$) over an algebraically closed field k of characteristic $p > 5$. Then it is known that $\overline{\mathcal{M}}_{2,v,k}$ is a separated Artin stack of finite type over k with finite diagonal [Pat17, Theorem 9.7]. We discuss the valuative criterion of properness using the methods of [Kol21, §2.4].

We are interested in the following situation. Let T be an affine one-dimensional regular scheme of finite type over k, $t \in T$ a closed point and $T^0 := T \setminus \{t\}$. Suppose we are given a family $(f^0 \colon X^0 \to T^0) \in \overline{\mathcal{M}}_{2,v}(T^0)$. Then we are looking for a finite morphism $\pi \colon T' \to T$ and a family $(f' \colon X' \to T') \in \overline{\mathcal{M}}_{2,v}(T')$ such that the pullback family $f^0 \times_T \pi \colon X^0 \times_T T' \to T'$ is isomorphic to $X' \to T'$ over $\pi^{-1}T^0$.

The method of [Kol21, §2.4] can be outlined as follows: establish a dictionary between the fiberwise properties and the global properties of a family of stable varieties, and apply methods of the MMP on $X^0 \to T^0$ to produce the desired completed family. In general, the scheme X^0 is only demi-normal, and the MMP works best for normal varieties: here the gluing theory developed in [Kol13] is useful to go back and forth between slc varieties and their normalizations.

One must solve several problems to carry out this program in positive characteristic:

(a) The dictionary between fiberwise and global properties works well in one direction. Suppose $f \colon (X, D) \to T$ is a family of stable log surfaces over a one-dimensional normal base. Since T and every fiber X_t are S_2, we see that X is S_2. Points of codimension one of X that do not dominate T are regular, and those which dominate T are G_1 since the generic fiber is demi-normal. Thus X is demi-normal. Now inversion of adjunction implies that $(X, D + X_t)$ is slc for every $t \in T$.

Notice that if $(f \colon X \to T) \in \overline{\mathcal{M}}_{2,v,k}(T)$ and $(\bar{X}, \bar{D}) \to X$ is the normalization, then it follows from Remark 5.5.0.2 and Lemma 4.2.0.3 that the morphism $(\bar{X}, \bar{D}) \to T$ is a family of stable log surfaces (with $\bar{D}$ dominating T if X is not already normal).

Problems appear with the converse implication. Assume that $(X, D + X_t)$ is slc. Then the deformation theory of nodes show that X_t is G_1 [Kol13, 2.33], and adjunction would then imply that (X_t, D_t) is slc, provided that X_t is S_2. That X_t is S_2 in characteristic 0 follows from a non-trivial result of Alexeev (see [Kol13, 7.21]). This is not known at the moment in positive characteristic, so let us formulate the condition

(S2) If $(X, D) \to T$ is a flat family of geometrically reduced surface pairs over a one-dimensonal normal base such that $(X, D + X_t)$ is slc for every $t \in T$, then every X_t is S_2.

(b) To produce the completed family, one first extends $X^0 \to T^0$ to a flat family $X_1 \to T$. In general the central fiber is not even reduced: so one looks for a base-change along a finite $T' \to T$ such that the fibers of $X_1' := X_1 \times_T T' \to T'$ are reduced. If we find one such base-change, we have to make sure that $(X_1', (X_1')_{t'})$ is still slc for every $t' \in T'$. In positive characteristic, this is a problem if $T' \to T$ is wildly ramified or inseparable, see [Kol13, 2.14.5-6]. So we formulate the following condition of semi-stable reduction:

(SSR) Let $X \to T$ is a flat morphism where X is a regular threefold and T a one-dimensional curve. Let E be a reduced effective divisor on X such that $(X, E + \operatorname{red} X_t)$ is snc for every closed $t \in T$. Then there exists a finite morphism $T' \to T$ such that: if Y is the normalization of $X \times_T T'$ and E_Y is the pullback divisor, then every closed fiber $Y_{t'}$ is reduced and every $(Y, E_Y + Y_{t'})$ is lc.

Modulo these two conditions, we can prove the valuative criterion of properness for $\overline{\mathcal{M}}_{2,v,k}$, following the method of [Kol21, 2.49].

Lemma 5.5.0.4. *Let $(X, D + \Delta) \to T$ be a family of stable log surfaces over a normal one-dimensional base, where D is a reduced divisor with normalization $n \colon D^n \to D$. Then every* *componenent of D^n dominates T and $(D^n, \mathrm{Diff}_{D^n}\Delta) \to T$ is a (disjoint union of) families of stable log curves.*

Proof. By definition of families of pairs, every irreducible component of D dominates T. In particular every component of D^n dominates T and $D^n \to T$ is flat. So the fibers $D^n_t := (D^n)_t$ are of pure dimension one. By inversion of adjunction the pair $(X, D + \Delta + X_t)$ is slc. Passing to the normalization (recall that X is normal at the generic points of D), we may assume it is lc. Since X_t is Cartier we have

$$\mathrm{Diff}_{D^n}(\Delta + X_t) = \mathrm{Diff}_{D^n}(\Delta) + X_t|_{D^n} = \mathrm{Diff}_{D^n} + D^n_t.$$

By adjunction, this implies that $(D^n, \mathrm{Diff}_{D^n}\Delta + D^n_t)$ is lc. Since D^n is a surface, classification of surface lc singularities show that D^n_t is Gorenstein. Moreover, as D^n is S_2 (it is normal), the D^n_t are S_1. Thus D^n_t is a demi-normal curve, and by adjunction we deduce that $(D^n, \mathrm{Diff}_{D^n}\Delta) \to T$ is a family of log curves. As

$$K_{D^n/T} + \mathrm{Diff}_{D^n}\Delta = n^*(K_{X/T} + D + \Delta),$$

we obtain that $K_{D^n/T} + \mathrm{Diff}_{D^n}\Delta$ is ample over T. □

Theorem 5.5.0.5. *We work over an algebraically closed field k of characteristic > 5. Assume that the conditions **(S2)** and **(SSR)** hold. Then $\overline{\mathcal{M}}_{2,v,k}$ is proper.*

Proof. We consider anew an affine one-dimensional regular k-scheme of finite type T, a closed point $t \in T$ and a family $(f_0 \colon X^0 \to T^0 := T \setminus \{t\}) \in \overline{\mathcal{M}}_{2,v,k}(T^0)$. Since separatedness holds, we have to show that there exists a finite morphism $\pi \colon T' \to T$ such that the pullback family $f^0 \times_T \pi \colon X^0 \times_T T' \to \pi^{-1}T^0$ extends to a family in $\overline{\mathcal{M}}_{2,v,k}(T')$. By Remark 5.5.0.2, we may think of f^0 as a family of stable surfaces.

Step 1: Normal case. First assume that X^0 is actually normal. We can extend f^0 to a flat morphism $f_1 \colon X_1 \to T$. Let $g_1 \colon Y_1 \to X_1$ be a log resolution with $E_1 := \mathrm{Exc}(g_1)$ such that $(Y_1, E_1 + \mathrm{red}(Y_1)_t)$ is snc for every $t \in T$. Such a resolution exists for threefolds: the proof is the same as in [Kol13, 10.46], using Fact 5.4.1.2 at the appropriate places.

By **(SSR)**, there is a finite morphism $\pi \colon T' \to T$ such that: if Y_2 is the normalization of $Y_1 \times_T T'$ with induced morphism $f_2 \colon Y_2 \to T'$ and E_2 is the pullback divisor of E_1, then every fiber $(Y_2)_{t'}$ is reduced and $(Y_2, E_2 + (Y_2)_{t'})$ is lc for every closed $t' \in T'$.

By [Wal18], the family $(Y_2, E_2) \to T'$ admits a relative canonical model $(X_{\mathrm{can}}, E_{\mathrm{can}}) \to T'$. Notice that the pullback family $X^0 \times_T T' \to \pi^{-1}T^0$ is a relative canonical model of $(Y_2, E_2) \to T'$ over a dense open susbset of T'. By uniqueness of canonical models, these two families are generically isomorphic; by separatedness, the isomorphism extends to the whole $\pi^{-1}T^0$ (see [Pat17, Lemma 9.4] for the technical statement), and this implies that $E_{\mathrm{can}} = 0$. Using [Kol13] we see that the pair $(X_{\mathrm{can}}, (X_{\mathrm{can}})_{t'})$ is lc for every closed $t' \in T'$. The condition **(S2)** then ensures that $(X_{\mathrm{can}} \to T') \in \overline{\mathcal{M}}_{2,v,k}(T')$.

Step 2: Demi-normal case. Now we consider the case where X^0 is only demi-normal. Let $(\bar{X}^0, \bar{D}^0) \to T^0$ be the normalization, with induced involution τ^0. Then we can apply the same argument as before (all the results we used are available for log pairs). We obtain a finite morphism $\pi \colon T' \to T$ and a family of surface pairs $\bar{f}_{\mathrm{can}} \colon (\bar{X}_{\mathrm{can}}, \bar{D}_{\mathrm{can}}) \to T'$ which extends the pullback of $(\bar{X}^0, \bar{D}^0) \to T^0$ and such that $(\bar{X}_{\mathrm{can}}, \bar{D}_{\mathrm{can}} + (\bar{X}_{\mathrm{can}})_{t'})$ is lc for every closed $t' \in T'$.

By Lemma 5.5.0.4, the morphism $(\bar{D}^n_{\mathrm{can}}, \mathrm{Diff}_{\bar{D}^n_{\mathrm{can}}}(0)) \to T'$ is a family of stable log curves. Hence by [Pat17, Lemma 9.4], the pullback of the involution τ^0 extends to an involution τ_{can} on $(\bar{D}^n_{\mathrm{can}}, \mathrm{Diff}_{\bar{D}^n_{\mathrm{can}}}(0))$.

By [Kol21, 2.14, 2.15.3], none of the lc centers of $(\bar{X}_{\mathrm{can}}, \bar{D}_{\mathrm{can}})$ are disjoint from $\bar{X}^0 \times_T T'$. Moreover, since $R(\tau^0) \rightrightarrows \bar{X}^0$ is a finite equivalence relation and $T' \to T$ is finite as well, then

$R(\tau_{\mathrm{can}}) \rightrightarrows \bar{X}_{\mathrm{can}}$ is finite on $\bar{f}_{\mathrm{can}}^{-1}(\pi^{-1}T^0)$. Thus [Kol13, 9.55] implies that $R(\tau_{\mathrm{can}})$ is finite. Thus there exists a geometric quotient $\bar{X}_{\mathrm{can}} \to X_{\mathrm{can}}$. Since τ^0 commutes with the projection to T, the involution τ_{can} commutes with the projection to T', and therefore $\bar{f}_{\mathrm{can}}$ factorises through a morphism $f_{\mathrm{can}} \colon X_{\mathrm{can}} \to T'$.

By Proposition 4.4.0.1 the scheme X_{can} is demi-normal, and by Proposition 5.4.3.5 the divisor $K_{X_{\mathrm{can}}}$ is $\mathbb{Q}$-Cartier. Since $(X_{\mathrm{can}}, (X_{\mathrm{can}})_{t'})$ pullbacks to $(\bar{X}_{\mathrm{can}}, \bar{D}_{\mathrm{can}} + (\bar{X}_{\mathrm{can}})_{t'})$, we see that $(X_{\mathrm{can}}, (X_{\mathrm{can}})_{t'})$ is slc for every closed $t' \in T'$. Since $\bar{X}^0 \times_T T' \to \bar{X}^0$ is flat and the formation of geometric quotient commutes with flat base-change [Kol13, 9.11], we see that the pullback of X^0 along $\pi^{-1}T^0 \to T^0$ is isomorphic to $f_{\mathrm{can}}^{-1}(\pi^{-1}T^0)$. The condition **(S2)** ensures that $(f_{\mathrm{can}} \colon X_{\mathrm{can}} \to T') \in \overline{\mathcal{M}}_{n,v,k}(T')$. This finishes the proof. $\square$

Chapter 6

Gluing theory for families of slc surfaces in mixed characteristic

This chapter corresponds to the preprint [Pos21b].

Convention 6.0.0.1. We work with varieties over a discrete valuation ring $(R, \mathfrak{m}_R = (\pi))$, and denote by p the characteristic of the residue field $k(\pi) = R/(\pi)$.

6.1 Introduction

Semi-log canonical (slc) singularities play a central role in the moduli theory of canonically polarized varieties in characteristic zero: they appear at the boundary of the compact moduli space of stable varieties [Kol21]. It is expected that slc varieties play a similar role in positive and mixed characteristics.

Recent breakthroughs in the Minimal Model Program (MMP) for threefolds in mixed characteristic [BMP+21, TY21] have led to advances in the moduli theory of stable surfaces in mixed characteristic. The corresponding moduli stack is known to be a separated Artin stack with finite diagonal and of finite type [BMP+21, Theorem I]. Properness is not known yet, except for some specific subspaces [BMP+21, Theorem J].

In characteristic zero, properness of the moduli stack of stable varieties is proved through a delicate use of semi-stable reductions and MMP methods: see [Kol21, §2.4] for a presentation of the proof. In addition to semi-stable reduction and MMP, that proof needs a technical gluing statement. Indeed, we consider a pointed curve $(t \in T)$ and a stable family $X^0 \to T^0 = T \setminus \{t\}$, and we need to complete the family over T, possibly after a finite base-change $T' \to T$. The variety X^0 is only demi-normal in general, and the MMP might fail for demi-normal varieties [Fuj14, Example 5.4]. Thus we normalize X^0 and try to complete the family $(X^0)^n \to T^0$. If we succeed, we need to de-normalize the completed family. In characteristic zero, this is achieved through Kollár's gluing theory [Kol13, §5], that gives a dictionary between slc stable varieties and their normalizations.

In [Pos21c] I have extended this dictionary to surfaces and threefolds in positive characteristic, and given applications to the properness of the moduli space of stable surfaces in positive characteristic (see also [Pat17] for related results about this moduli space). In this paper, I study the gluing statement necessary for the proof of properness of the moduli stack of stable surfaces in mixed characteristic along the lines sketched above.

The tailor-made statement is our main result:

Theorem 6.1.0.1 (see Proposition 6.3.1.8 and Proposition 6.3.2.1)**.** *Let R be a DVR of mixed* *characteristic with maximal ideal πR. Then normalization gives a bijection*

$$\left(\begin{array}{c}\textit{Threefold pairs } (X,\Delta)\\ \textit{flat and proper over } R\\ \textit{such that } (X,\Delta+X_\pi) \textit{ is slc}\\ \textit{and } K_X+\Delta \textit{ is ample}\end{array}\right) \xrightarrow{1:1} \left(\begin{array}{c}\textit{Threefold pairs } (\bar{X},\bar{D}+\bar{\Delta})\\ \textit{flat and proper over } R\\ \textit{such that } (\bar{X},\bar{D}+\bar{\Delta}+\bar{X}_\pi) \textit{ is lc}\\ \textit{and } K_{\bar{X}}+\bar{D}+\bar{\Delta} \textit{ is ample}\\ \textit{plus a generically fixed point free}\\ R\textit{-involution } \tau \textit{ of } (\bar{D}^n, \mathrm{Diff}_{\bar{D}^n}\bar{\Delta}).\end{array}\right)$$

Let us comment on the condition that $(X,\Delta+X_\pi)$ is slc. Since we think of the family of surfaces $(X,\Delta)\to \operatorname{Spec} R$ as an element in moduli stack of stable surfaces, it would be natural to ask for each fiber to be slc. In characteristic zero, this is equivalent to $(X,\Delta+X_\pi)$ being log canonical (lc) [Kol21, Theorem 2.4]. The local-to-global direction relies on inversion of adjunction, which is known in dimension three in positive and mixed characteristics provided that the non-zero characteristics are >5 ([Pat17, Lemma 3.3] and [BMP+21, Corollary 10.1]). On the other hand, there is a difficult step in the proof of the global-to-local direction, namely that $(X,\Delta+X_\pi)$ being slc implies that X_π is S_2: in characteristic zero this follows from a theorem of Alexeev [Kol13, 7.21]. A similar result is not yet proved in positive and mixed characteristic, so the slc condition on $(X,\Delta+X_\pi)$ is *a priori* weaker.

We prove our gluing statement for this weaker stability assumption. We also prove in Proposition 6.4.2.6 that, under the assumption $\operatorname{char} R/(\pi)\geq 7$, if $(\bar{X}_\pi, \mathrm{Diff}_{\bar{X}_\pi}(\bar{D}+\bar{\Delta}))$ is slc then the pair $(X_\pi, \mathrm{Diff}_{X_\pi}\Delta)$ is also slc. However the converse statement seems more difficult without an Alexeev-type statement (see Remark 6.4.2.8). Thus we cannot yet formulate an analog of Theorem 6.1.0.1 for the stronger stability assumption.

Let us say a word about the proof of Theorem 6.1.0.1. We construct an inverse map to the normalization process: $\bar{X}$ should be the normalization of an slc quotient $\bar{X}/R(\tau)$. It turns out that the main difficulty is the existence of the quotient, for then it is easy to show that it has an slc structure, see Proposition 6.3.2.1.

To show that the quotient exists, thanks to a result of Witaszek [Wit20, Theorem 1.4] and to Kollár's theory [Kol13, 5.13], we only need to show that the equivalence relations induced by τ on the special fiber X_π is finite. To achieve this, we relate $\bar{D}^n_\pi$ to a divisor contained in the round-down of $\mathrm{Diff}_{X^n_\pi}(\bar{D}+\bar{\Delta})$, and show that the relation on $\bar{X}_\pi$ comes from a log involution on the reduced boundary of the $k(\pi)$-pair $(\bar{X}^n_\pi, \mathrm{Diff}_{X^n_\pi}(\bar{D}+\bar{\Delta})$. This is done using adjunction and classification of codimension two lc singularities. Then we are in position to apply the results from [Pos21c] and obtain finiteness. We also obtain a conditional finiteness result in case $\dim \bar{X}=4$: see Proposition 6.3.1.8 for the precise statement.

We emphasize that all our results are independent of $\dim \bar{X}$ and of the properties of the positive characteristic residue field $k(\pi)$ — except for Proposition 6.3.1.8, which shows that the quotient indeed exists in several situations.

In the second part of the paper, we study the fibers of $X=\bar{X}/R(\tau)$ over $\operatorname{Spec} R$. The involution τ induces an involution on both fibers of $\bar{D}^n\to\operatorname{Spec} R$, and therefore we can take the quotient of both fibers of $\bar{X}\to\operatorname{Spec} R$. We ask whether these quotients are equal to the fibers of X. This commutativity property is always true for the generic fiber (Lemma 6.4.1.1). For the special fiber, we first study the general case of the normalization of a demi-normal scheme over $\operatorname{Spec} R$ and give some sufficient conditions (see Proposition 6.4.1.5 and Proposition 6.4.1.6). In particular, we prove that if $\bar{D}$ is normal and $\operatorname{char} R/(\pi)>2$ then the commutativity property always holds.

Then we turn to the case that interests us the most, when $(X,\Delta+X_\pi)$ is slc. We are able to refine the method of Proposition 6.4.1.6 to prove that commutativity holds for families of slc surfaces under mild hypothesis:

Theorem 6.1.0.2 (Theorem 6.4.1.9)**.** *Suppose $(X, \Delta + X_\pi)$ is slc of dimension 3, proper over* $\operatorname{Spec} R$ *and that* $\operatorname{char} R/(\pi) > 2$*. If $\bar{D}_\pi$ is reduced, the special fiber of X is the quotient of the special fiber of $\bar{X}$.*

We remark that $\bar{D}_\pi$ is the scheme-theoretic intersection of two lc centers of $(\bar{X}, \bar{D} + \bar{\Delta} + \bar{X}_\pi)$. It is a question of independent interest whether intersections of lc centers are reduced. This is true in characteristic zero [Kol13, 7.8], but not much is known in mixed (or positive) characteristics. Nonetheless, we can prove that the hypothesis of Theorem 6.1.0.2 holds in relative dimension two and residue characteristic large enough:

Theorem 6.1.0.3 (Proposition 6.4.1.10)**.** *Let $(\bar{X}, \bar{D} + \bar{\Delta} + \bar{X}_\pi)$ be a lc threefold pair that is projective over* $\operatorname{Spec} R$*. Assume that* $\operatorname{char} R/(\pi) \geq 7$*. Then $\bar{D}_\pi$ is reduced.*

The proof relies on the technique of [Kol13, 7.8] and on the results of [BK21]. Putting together Theorem 6.1.0.2 and Theorem 6.1.0.3 we obtain:

Theorem 6.1.0.4. *Let $(\mathcal{S}, \Delta) \to$* $\operatorname{Spec} R$ *be a projective family of surfaces over a DVR with residue characteristic ≥ 7, with normalization $\bar{\mathcal{S}}$. Assume that $(\mathcal{S}, \Delta + \mathcal{S}_\pi)$ is slc. Then the fibers of $\mathcal{S}$ are the quotients of the fibers of $\bar{\mathcal{S}}$.*

A corollary of Theorem 6.1.0.4 is the fact, already mentioned above, that $(\mathcal{S}_\pi, \operatorname{Diff}_{\mathcal{S}_\pi} \Delta)$ is slc if $(\bar{\mathcal{S}}_\pi, \operatorname{Diff}_{\bar{\mathcal{S}}_\pi}(\bar{D}+\bar{\Delta}))$ is slc. The main difficulty is to show that $\mathcal{S}_\pi$ is S_2, which follows here from the commutativity of fibers and quotients. We take this problem as an opportunity to study the Serre properties of a demi-normal quotient. In view of [Kol13, 10.18], it is not surprising that there is an interplay between the Serre properties of the quotient, and those of its normalization and of the conductor subschemes. The precise statement is given in Proposition 6.4.2.4.

6.2 Preliminaries

If X is an R-scheme, we denote by X_π the scheme-theoretic special fiber. We usually assume that X is flat over $\operatorname{Spec} R$: if in addition X is pure-dimensional, then every irreducible component of X_π has dimension $\dim X - 1$ by [GW20, 14.97].

We will use many times the coarse classification of codimension two lc singularities [Kol13, 2.31]: if (X, Δ) is an lc pair and $x \in X$ a codimension two point, then $\lfloor \Delta \rfloor$ is either regular or nodal at x. Notice that the proof given in [Kol13, 2.31] holds in the generality of normal excellent schemes.

We will also use the following fact about depths: if $t \in \mathcal{O}_X$ is a non-invertible non-zero divisor and $x \in \operatorname{Supp}(\mathcal{O}_X/t\mathcal{O}_X)$, then $\mathcal{O}_X$ is S_i at x if and only if $\mathcal{O}_X/t\mathcal{O}_X$ is S_{i-1} at x [Bou07, §1 n.4 Proposition 7].

6.3 Gluing for families of surfaces

The goal of this section is the proof of Theorem 6.1.0.1. We consider the following situation (where, compared to Theorem 6.1.0.1, we modify slightly our notations):

Notation 6.3.0.1. Let R be a DVR of mixed characteristic $(0, p > 0)$, let $X \to \operatorname{Spec} R$ be a flat proper morphism with connected fibers from an equidimensional normal scheme X of any dimension, let D be an effective reduced Weil divisor on X and Δ an effective $\mathbb{Q}$-Weil divisor, and assume that $(X, D + \Delta + X_\pi)$ is log canonical. In particular, X_π is reduced and has no component contained in the support of $D + \Delta$.

We denote by $\bar{D} \to D$ the normalization of D, by $\bar{D}_\pi$ the special fiber of $\bar{D} \to \operatorname{Spec} R$ and by $\bar{D}_\pi^n$ its normalization.

We also assume that there exists an involution τ of the log pair $(\bar{D}, \operatorname{Diff}_{\bar{D}} \Delta)$ over $\operatorname{Spec} R$.

6.3.1 Existence of the quotient

Lemma 6.3.1.1. *Let $Z \to \operatorname{Spec} R$ be a morphism and τ an involution of Z over $\operatorname{Spec} R$. Then τ restricts to involutions of Z_π, $\operatorname{red}(Z_\pi)$ and $Z \otimes_R \operatorname{Frac}(R)$.*

Proof. The ideal of Z_π is $\pi\mathcal{O}_Z$, so by looking at the exact sequence

$$0 \to \pi\mathcal{O}_Z \to \mathcal{O}_Z \to \mathcal{O}_{Z_\pi} \to 0$$

we see that τ restricts to an involution of Z_π if and only if $\tau(\pi) \subseteq \pi\mathcal{O}_Z$. But $\tau(\pi) = \pi$, so this is immediate. So τ descends to Z_π, and thus to $\operatorname{red}(Z_\pi)$. Since $\tau(\pi) = \pi$, it also holds that τ descends to an involution of $\mathcal{O}_Z \otimes_R R[1/\pi] = \mathcal{O}_{Z\otimes_R \operatorname{Frac}(R)}$. □

Lemma 6.3.1.2. *Let $(X, D + \Delta) \to \operatorname{Spec} R$ be as in Notation 6.3.0.1. If E is a divisor over X whose center $c_X(E)$ belongs to the special fiber X_π, then $a(E; X, D + \Delta) \geq 0$.*

Proof. The proof is the same as [Kol21, 2.14]. Let E be a divisor over X, appearing on a proper birational model $\pi \colon Y \to X$. Write $b_E := \operatorname{coeff}_E \pi^* X_\pi$. Since $\pi^* X_\pi$ is Cartier and effective, b_E is a non-negative integer. If $c_X(E) \subset X_\pi$ then b_E is actually a positive integer. Then:

$$-1 \leq a(E; X, D + \Delta + X_\pi) = a(E; X, D + \Delta) - b_E$$

so $a(E; X, D + \Delta) \geq 0$. □

Lemma 6.3.1.3. *In the situation of Notation 6.3.0.1, every irreducible component of D dominates $\operatorname{Spec} R$, and the irreducible components of $D \cap X_\pi$ have dimension $\dim D - 1$.*

Proof. Since X is flat over $\operatorname{Spec} R$, every component of X_π is a divisor by [GW20, 14.97]. Since $(X, D + \Delta + X_\pi)$ is lc, no component of the boundary has coefficient > 1. Thus D does not contain any component of X_π, and so every component of D dominates $\operatorname{Spec} R$ and so $D \to \operatorname{Spec} R$ is flat [Har77, III.9.7]. Applying [GW20, 14.97] again yields the result. □

Lemma 6.3.1.4. *The special fiber $\bar{D}_\pi$ of $\bar{D} \to \operatorname{Spec} R$ is reduced.*

Proof. Since $\bar{D}$ is normal and $\bar{D}_\pi$ is an hypersurface, $\bar{D}_\pi$ is S_1. Thus we only need to show that $\bar{D}_\pi$ is generically reduced.

The generic points of $\bar{D}_\pi$ dominates the generic points of the intersection $D \cap X_\pi$, and by Lemma 6.3.1.3 these points have codimension two in X. Since $(X, D + \Delta + X_\pi)$ is lc, the classification of codimension two lc singularities [Kol13, 2.32] shows that around the generic points of $D \cap X_\pi$, the divisors D and X_π are regular and meet transversally. Hence $\bar{D} \to D$ is an isomorphism around those generic points, with $\bar{D}_\pi$ isomorphic to the regular $D \cap X_\pi$. This shows that $\bar{D}_\pi$ is generically reduced. □

Lemma 6.3.1.5. *The involution τ on $(\bar{D}, \operatorname{Diff}_{\bar{D}} \Delta)$ induces an involution σ of the lc pair $(\bar{D}^n_\pi, \Gamma)$, where Γ is defined by the adjunction formula $(K_{\bar{D}} + \operatorname{Diff}_{\bar{D}}(\Delta) + \bar{D}_\pi)|_{\bar{D}^n_\pi} = K_{\bar{D}^n_\pi} + \Gamma$.*

Moreover, the equivalence relation $R_{X_\pi}(\sigma) \rightrightarrows X_\pi$ is equal to the restriction of the equivalence relation $R_X(\tau) \rightrightarrows X$ to X_π.

Proof. By Lemma 6.3.1.1 and Lemma 6.3.1.4 the involution τ preserves the Cartier divisor $\bar{D}^n_\pi$ and descends to an involution σ' on $\bar{D}_\pi$. By the universal property of normalization, we obtain an involution σ of the normalization $\bar{D}^n_\pi$ that makes the diagram

$$\begin{tikzcd} \bar{D}^n_\pi \arrow[r, "\sigma"] \arrow[d] & \bar{D}^n_\pi \arrow[d] \\ \bar{D}_\pi \arrow[r, "\sigma'"] & \bar{D}_\pi \end{tikzcd} \tag{3.1.5.a}$$

commutative.

Since the $\mathbb{Q}$-Cartier divisors $K_{\bar{D}} + \operatorname{Diff}_{\bar{D}} \Delta$ and $\bar{D}_\pi$ are τ-invariant, so is their sum. Hence the pullback of $K_{\bar{D}} + \operatorname{Diff}_{\bar{D}} + \bar{D}^n_\pi$ to $\bar{D}^n_\pi$ is σ-invariant.

The pair $(\bar{D}^n_\pi, \Gamma)$ is lc by adjunction.

It is clear from the construction that $R(\sigma') \rightrightarrows X_\pi$ is equal to the restriction of $R_X(\tau)$ to X_π, so we only need to compare $R_{X_\pi}(\sigma)$ and $R_{X_\pi}(\sigma')$. Since (3.1.5.a) is commutative, we see that the two equivalence relations are the same. □

Lemma 6.3.1.6. *The special fiber X_π is reduced, regular at the generic points of $D \cap X_\pi$ and at worst nodal at other codimension one points. Moreover if $X^n_\pi \to X_\pi$ is the normalization morphism then:*

(a) it is an isomorphism over the generic points of $D \cap X_\pi$,

(b) the pair $(X^n_\pi, \operatorname{Diff}_{X^n_\pi}(D + \Delta))$ is lc, and

(c) the strict transform of $D \cap X_\pi$ is contained in $\lfloor \operatorname{Diff}_{X^n_\pi}(D + \Delta) \rfloor$.

Proof. The hypersurface X_π of X is S_1. It is regular at the generic points of $D \cap X_\pi$ and at worst nodal at other codimension one points by [Kol13, 2.32]. To decide whether the strict transform of $D \cap X_\pi$ appears in $\lfloor \operatorname{Diff}_{X^n_\pi}(D + \Delta) \rfloor$ is a local question around the generic points of $D \cap X_\pi$. So the first and third points hold. The pair $(X^n_\pi, \operatorname{Diff}_{X^n_\pi}(D + \Delta))$ is lc by adjunction. □

Lemma 6.3.1.7. *Denote by $E \subset \lfloor \operatorname{Diff}_{X^n_\pi}(D + \Delta) \rfloor$ the strict transform of $D \cap X_\pi$, and by (E^n, Θ) the lc pair obtained from $(X^n_\pi, \operatorname{Diff}_{X^n_\pi}(D + \Delta))$ by adjunction. Then there exists a log isomorphism $f\colon (E^n, \Theta) \cong (\bar{D}^n_\pi, \Gamma)$ such that the diagram*

$$\begin{tikzcd} E^n \arrow[rr, "f"] \arrow[dr] & & \bar{D}^n_\pi \arrow[dl] \\ & X & \end{tikzcd} \tag{3.1.7.b}$$

commutes.

Proof. As observed in Lemma 6.3.1.4 and Lemma 6.3.1.6, the two morphisms

$$E \to D \cap X_\pi \leftarrow \bar{D}_\pi$$

are birational and finite. This induces finite birational morphisms

$$E^n \to (D \cap X_\pi)^n \leftarrow \bar{D}^n_\pi$$

of normal schemes, so these must be isomorphisms. Thus we obtain an isomorphism $f\colon E^n \cong \bar{D}^n_\pi$ that commutes with the morphisms to X. Since the divisors Θ and Γ are defined by adjunction, we see that f is a log isomorphism. □

Proposition 6.3.1.8. *In the situation of Notation 6.3.0.1, assume that X is proper over* $\operatorname{Spec} R$ *and that $K_X + D + \Delta$ is ample over* $\operatorname{Spec} R$. *Assume also that*

(a) $\dim X = 3$, *or*

(b) $\dim X = 4$, *and X_π is S_2 in a neighborhood of $D \cap X_\pi$ and the residue field of R is perfect of characteristic > 5.*

Then the quotient $X/R(\tau)$ exists as a demi-normal scheme that is flat and proper over $\operatorname{Spec} R$.

Proof. By [Wit20, 1.4], the geometric quotient $X/R(\tau)$ exists as an algebraic space as soon as $R(\tau)$ is finite and the geometric quotient $X_\mathbb{Q}/R(\tau)_\mathbb{Q}$ exists. If it exists, it is a scheme by [Kol12, Corollary 48].

Lemma 6.3.1.1 implies that τ induces an involution $\tau_\mathbb{Q}$ on the generic fiber $(\bar{D}_\mathbb{Q}, \operatorname{Diff}_{\bar{D}_\mathbb{Q}}(\Delta_\mathbb{Q}))$ and $R(\tau_\mathbb{Q}) = R(\tau)|_{X_\mathbb{Q}}$. Since $K_X + D + \Delta$ is ample over $\operatorname{Spec} R$, we see that $(X_\mathbb{Q}, D_\mathbb{Q} + \Delta_\mathbb{Q})$

is a projective lc pair over a field of characteristic zero with ample log canonical divisor a log involution $\tau_{\mathbb{Q}}$ on $(\bar{D}_{\mathbb{Q}}, \mathrm{Diff}_{\bar{D}_{\mathbb{Q}}}(\Delta_{\mathbb{Q}}))$. By [Kol13, 5.13] the quotient $X_{\mathbb{Q}}/R(\tau)_{\mathbb{Q}}$ exists.

We still have to show that $R(\tau)$ is finite. Since it respects the fibration to $\operatorname{Spec} R$, we only need to show that $R(\tau)|_{X_\pi}$ is finite. By Lemma 6.3.1.5 it is equivalent to show that $R_{X_\pi}(\sigma) \rightrightarrows X_\pi$ is finite. By Lemma 6.3.1.7 and the commutativity of (3.1.7.b), we may transport σ to an involution of (E^n, Θ). We are now in situation to apply Theorem 5.3.1.1, Claim 5.3.1.3 and Proposition 5.4.3.2, which show that $R_{X_\pi^n}(\sigma)$ is finite.

To go from X_π^n to X_π, observe that the equivalence generated by σ on X_π^n, respectively on X_π, is trivial away from the support of E, respectively away from the support of $D \cap X_\pi$. Now X_π is R_1 is a neigborhood of $D \cap X_\pi$ by Lemma 6.3.1.7. If it is also S_2 then $X_\pi^n \to X_\pi$ is an isomorphism in a neighborhood of E, and we deduce that $R_{X_\pi}(\sigma)$ is finite.

If $\dim X = 3$, then we do not need the fact that X_π is S_2. In this case X_π is a reduced surface that is regular in codimension one and such that $\omega_{X_\pi}^{[m]}(m(D+\Delta)|_{X_\pi})$ is invertible for m sufficiently divisible (this follows from adjunction along $X_\pi \subset X$), hence we can perform adjunction along $E^n \to X_\pi$ even if X_π is not normal (see [Kol13, 4.2]). The crucial point is that E^n is a curve, so the points of E^n where $E^n \to D \cap X_\pi$ is not an isomorphism, are contained in $\operatorname{Supp} \Theta$ by [Kol13, 4.5.1]. Thus the the proof of Theorem 5.3.1.1 is also valid in this situation: the finite closed subset $\Sigma := \operatorname{Supp} \Theta$ is σ-invariant and $E^n \setminus \Sigma \to E \setminus n(\Sigma)$ is an isomorphism. Thus we obtain finiteness.

Flatness of $X/R(\tau)$ over $\operatorname{Spec} R$ follows from [Har77, III.9.7].

It remains to show that $X/R(\tau)$ is demi-normal and proper over $\operatorname{Spec} R$. This follows from Proposition 4.4.0.1. □

Remark 6.3.1.9. More generally, the proof of Proposition 6.3.1.8 applies to $(X, D+\Delta, \tau)$ as soon as we have a gluing theorems for stable lc varieties of dimension $\dim X - 1$ above the residue field $k(\pi)$ of R, and that the special fiber X_π is S_2.

6.3.2 Descent of the log canonical sheaf

We show that in the situation of Proposition 6.3.1.8, the log canonical $\mathbb{Q}$-Cartier divisor descends to the quotient. We can actually show it in any dimension, as soon as the quotient exists:

Proposition 6.3.2.1. *Let $(Y, \Delta_Y) \to \operatorname{Spec} R$ be a demi-normal flat $\operatorname{Spec} R$-scheme, with induced normalization $(X, \Delta + D, \tau)$. If $(X, D + \Delta + X_\pi)$ is lc and $\mathrm{Diff}_{D^n}(\Delta)$ is τ-invariant, then $K_Y + \Delta_Y$ is $\mathbb{Q}$-Cartier.*

Proof. First base-change over the generic point of $\operatorname{Spec} R$. Then [Kol13, 5.38] shows that $K_Y + \Delta_Y$ is $\mathbb{Q}$-Cartier on the generic fiber.

So the closed locus where $K_Y + \Delta_Y$ is not $\mathbb{Q}$-Cartier, if not empty, is contained in the special fiber Y_π. By Lemma 6.3.1.2 and the fact that $(X, D+\Delta) \to (Y, \Delta_Y)$ is crepant, we see that no lc center of (Y, Δ_Y) is contained in Y_π. Thus we may apply the first part of the proof of [Kol13, 5.38], which is valid in our setting, and conclude that $K_Y + \Delta_Y$ is also $\mathbb{Q}$-Cartier along the special fiber. □

Proof of Theorem 6.1.0.1. If (X, Δ) is as in the left-hand side of Theorem 6.1.0.1, we claim its normalization is a triplet $(\bar{X}, \bar{D}+\bar{\Delta}, \tau)$ as on the right-hand side. We indeed have a generically fixed point free involution τ: this follows from Lemma 4.2.0.5 since by Lemma 6.3.1.3 the generic points of $\bar{D}$ have characteristic zero residue fields. The other properties clearly hold.

Let $(\bar{X}, \bar{D}+\bar{\Delta}, \tau)$ be as in the right-hand side of Theorem 6.1.0.1. By Proposition 6.3.1.8 the quotient $X := \bar{X}/R(\tau)$ exists, it is a demi-normal scheme flat and proper over $\operatorname{Spec} R$. By Proposition 6.3.2.1 we obtain that (X, Δ) is slc.

This defines a map in the opposite direction as the normalization. They are inverse to each other by [Kol13, 5.3] and the fact that $\bar{X}$ is the normalization of X. □

6.4 Fibers of the quotient

In the previous section, we have shown that the quotient of the family exists in several situations. In this section, we study the fibers of the quotient whenever it exists. We will use the following notations:

Notation 6.4.0.1. Let R be a DVR of mixed characteristic $(0, p)$, let $Y \to \operatorname{Spec} R$ be a flat separated morphism of finite type from a demi-normal scheme Y. (We do not assume any properness property relatively to $\operatorname{Spec} R$.) Let $p\colon (X, D, \tau) \to Y$ be the normalization morphism. Then D is reduced of pure codimension one and we let $\bar{D} \to D$ be its normalization.

Since τ is an R-involution on $\bar{D}$, by Lemma 6.3.1.1 it restricts to an involution τ_π of $\bar{D}_\pi$ and $\tau_\mathbb{Q}$ of $\bar{D}_\mathbb{Q}$. It is clear that $R_{X_\pi}(\tau_\pi) = R_X(\tau)|_{X_\pi}$ and that $R_{X_\mathbb{Q}}(\tau_\mathbb{Q}) = R_X(\tau)|_{X_\mathbb{Q}}$. Since $R(\tau) \rightrightarrows X$ is finite, it follows that $R(\tau_\pi) \rightrightarrows X\pi$ and $R(\tau_\mathbb{Q}) \rightrightarrows X_\mathbb{Q}$ are finite.

We do not assume systematically that Y has an slc structure. When we do, we let Δ_Y be the boundary on Y, and Δ its strict transform on X.

6.4.1 Commutativity of fibers and quotients

We let $p\colon (X, D, \tau) \to Y \to \operatorname{Spec} R$ be as in Notation 6.4.0.1. We investigate to which extent the quotients $X_\mathbb{Q}/R(\tau_\mathbb{Q})$ and $X_\pi/R(\tau_\pi)$ are comparable to the fibers $Y_\mathbb{Q}$ and Y_π.

Lemma 6.4.1.1. $Y_\mathbb{Q} = X_\mathbb{Q}/R(\tau_\mathbb{Q})$.

Proof. This follows immediately from [Kol13, 9.11] since $Y_\mathbb{Q} \to Y$ is flat. □

Lemma 6.4.1.2. *The quotient $Z := X_\pi/R(\tau_\pi)$ exists, and $X_\pi \to Y_\pi$ factorizes through a finite birational universal homeomorphism $Z \to Y_\pi$.*

Proof. The quotient exists as a scheme by [Kol12, Theorem 6, Corollary 48]. By the universal property of the quotient, $X_\pi \to Y_\pi$ factors through a morphism $q\colon Z \to Y_\pi$.

$$\begin{tikzcd} X_\pi \arrow[r, hook] \arrow[d] & X \arrow[dd, "p"] \\ Z \arrow[d, "q"] & \\ Y_\pi \arrow[r, hook] & Y \end{tikzcd} \tag{4.1.2.c}$$

Using [Kol13, 9.2] for $X_\pi \to Z$ and $X \to Y$, we see that $q(\operatorname{Spec} K)\colon Z(\operatorname{Spec} K) \to Y_\pi(\operatorname{Spec} Z)$ is a bijection for every geometric point $\operatorname{Spec} K \to \operatorname{Spec} k(\pi)$. Thus q is a universal homeomorphism by [Gro60, 3.5.3-5]. It is finite birational since $X_\pi \to Y_\pi$ is so. □

Let us study more precisely the morphism $Z \to Y_\pi$. The question is flat-local on Y by [Kol13, 9.11], so in particular we may assume that every scheme appearing in (4.1.2.c) is affine, and work with sections of structural sheaves as if they were global sections.

We introduce the following sub-sheaves:

$$\mathcal{O}_X^+ := \{s \in \mathcal{O}_X \mid s|_{\bar{D}} \text{ is } \tau\text{-invariant}\}, \quad \mathcal{O}_D^+ := \{s \in \mathcal{O}_D \mid s|_{\bar{D}} \text{ is } \tau\text{-invariant}\}$$

and

$$\mathcal{O}_{X_\pi}^+ := \{s \in \mathcal{O}_X \mid s|_{\bar{D}_\pi} \text{ is } \tau_\pi\text{-invariant}\}, \quad \mathcal{O}_{D_\pi}^+ := \{s \in \mathcal{O}_{D_\pi} \mid s|_{\bar{D}_\pi} \text{ is } \tau_\pi\text{-invariant}\}.$$

They fit into the following commutative diagram:

$$\begin{tikzcd} \mathcal{O}_{D_\pi}^+ & \mathcal{O}_D^+ \arrow[l] \\ \mathcal{O}_{X_\pi}^+ \arrow[u] & \mathcal{O}_X^+ \arrow[l] \arrow[u] \end{tikzcd} \tag{4.1.2.d}$$

Claim 6.4.1.3. *The quotient $W(\pi) := D_\pi / R_{D_\pi}(\tau_\pi)$ exists as a scheme, and $\mathcal{O}_{W(\pi)} = \mathcal{O}^+_{D_\pi}$.*

Proof. The quotient exists as a scheme by [Kol13, 9.10] applied to $D_\pi \to Y_\pi$, and the second assertion follows from the same reference. $\Diamond$

Claim 6.4.1.4. $\mathcal{O}_Z = \mathcal{O}^+_{X_\pi}$.

Proof. Indeed, by [Kol12, Proposition 25] the diagram

$$\begin{tikzcd} D_\pi \arrow[r, hook] \arrow[d] & X_\pi \arrow[d] \\ W(\pi) = D_\pi / R_{D_\pi}(\tau_\pi) \arrow[r, hook] & Z \end{tikzcd}$$

is a universal push-out. This implies that

$$\mathcal{O}_Z = \mathcal{O}_{X_\pi} \times_{\mathcal{O}_{D_\pi}} \mathcal{O}_{W(\pi)} = \mathcal{O}_{X_\pi} \times_{\mathcal{O}_{D_\pi}} \mathcal{O}^+_{D_\pi} = \mathcal{O}^+_{X_\pi}$$

as claimed. $\Diamond$

Proposition 6.4.1.5. *$Z \to Y_\pi$ is an isomorphism if and only if the restriction map $\mathcal{O}^+_X \to \mathcal{O}^+_{X_\pi}$ appearing in (4.1.2.d) is surjective.*

Proof. By [Kol13, 9.10] we have $\mathcal{O}_Y = \mathcal{O}^+_X$. Combining this with the previous claim and the commutative diagram (4.1.2.c) we obtain

$$\begin{tikzcd} \mathcal{O}^+_{X_\pi} & \mathcal{O}^+_X \arrow[l, "\alpha"'] \\ \mathcal{O}_Z \arrow[u, "="] & \\ \mathcal{O}_{Y_\pi} \arrow[u, hook] & \mathcal{O}_Y \arrow[l, two heads] \arrow[uu, "="'] \end{tikzcd}$$

It is easy to see that α is surjective if and only if $\mathcal{O}_{Y_\pi} \hookrightarrow \mathcal{O}_Z$ is bijective. $\square$

Proposition 6.4.1.6. *Assume that:*

(a) $\tau(\mathcal{O}_D) \subseteq \mathcal{O}_D$
(b) $\mathcal{O}_{D_\pi} \to \mathcal{O}_{\bar{D}_\pi}$ *is injective,*
(c) $D \to \operatorname{Spec} R$ *is flat, and*
(d) $p = \operatorname{char} R/(\pi) \neq 2$.

Then $Z = Y_\pi$.

Proof. We prove that $\mathcal{O}^+_X \to \mathcal{O}^+_{X_\pi}$ is surjective. Take $s \in \mathcal{O}^+_{X_\pi}$ and any lift $t \in \mathcal{O}_X$. Write $v = t|_D$. By hypothesis $\tau(v) \in \mathcal{O}_D$ and by construction $\tau(v) - v$ vanishes when restricted to $\bar{D}_\pi$. Since $\mathcal{O}_{D_\pi} \to \mathcal{O}_{\bar{D}_\pi}$ is injective, we see that $v - \tau(v)$ belongs to the ideal $I_{D_\pi} = \pi \cdot \mathcal{O}_D$. Thus we can write $\tau(v) = v + a\pi$ for some $a \in \mathcal{O}_D$. Since τ is an R-involution,

$$v = \tau^{\circ 2}(v) = v + (a + \tau(a))\pi.$$

By flatness π is not a zero-divisor in $\mathcal{O}_D$, so we have $a = -\tau(a)$. Let $b \in \mathcal{O}_X$ be any lift of a. Since $2 \in \mathcal{O}_X$ is invertible we can form the element $t' := t + \frac{b}{2}\pi$. Then $t' \in \mathcal{O}^+_X$ and $t'|_{X_\pi} = t|_{X_\pi} = s$, as desired. $\square$

The case of locally stable families.

We are mainly interested in the case where $(Y, \Delta_Y + Y_\pi)$ is slc for some divisor Δ_Y. In this case the conditions of Proposition 6.4.1.6 are not necessarily met. To wit, consider the following example.

Example 6.4.1.7. Let $Y = \operatorname{Spec} R[x,y,z]/(xyz)$. Since Y is an hypersurface, it is Cohen–Macaulay and Gorenstein. Then its normalization X is the union of three copies of $\mathbb{A}^2_R$, with the morphism $X \to Y$ given by:

$$\begin{array}{ccccccc} R[x,y,z]/(xyz) & \longrightarrow & R[u_1,u_2] & \oplus & R[v_1,v_2] & \oplus & R[w_1,w_2] \\ x & \mapsto & u_1 & \oplus & v_2 & \oplus & 0 \\ y & \mapsto & u_2 & \oplus & 0 & \oplus & w_1 \\ z & \mapsto & 0 & \oplus & v_1 & \oplus & v_2 \end{array}$$

The conductor D is given by $V(u_1u_2) \sqcup V(v_1v_2) \sqcup V(w_1w_2) \subset X$. Notice that (Y, Y_π) is slc: for it is easily checked using inversion of adjunction that $(X, D + X_\pi)$ is lc.

The involution τ on $\bar{D}$ is given by three isomorphisms of lines, namely

$$\tau = \left[(\mathbb{A}^1_{u_1} \cong \mathbb{A}^1_{v_2},\ u_1 \mapsto v_2), \quad (\mathbb{A}^1_{v_1} \cong \mathbb{A}^1_{w_2},\ v_1 \mapsto w_2), \quad (\mathbb{A}^1_{u_2} \cong \mathbb{A}^1_{w_1},\ u_2 \mapsto w_1) \right].$$

The picture on the special fiber is exactly the same, except that R is replaced with its residue field $k(\pi)$.

The involution τ does not descend to D, since otherwise the three origins would belong to the same orbit.

On the other hand, $\mathcal{O}_X^+$ is the set of $f(s(u_1,v_2), s'(v_1,w_2), s''(u_2,w_1))$ where $f \in R[X,Y,Z]$ and s, s', s'' run through the symmetric polynomials in two variables. Similarly for $\mathcal{O}_{X_\pi}^+$, except that we take $f \in k(\pi)[X,Y,Z]$. In particular we see that $\mathcal{O}_X^+ \to \mathcal{O}_{X_\pi}^+$ is surjective.

An easy application of Proposition 6.4.1.6 is the following:

Proposition 6.4.1.8. *Suppose that $p \neq 2$ and that $(Y, \Delta_Y + Y_\pi)$ is slc. Then:*

(a) $Z \to Y_\pi$ is an isomorphism in codimension one.

(b) If Y_π is S_2, then $Z \to Y_\pi$ is an isomorphism.

(c) If D is normal, then $Z \to Y_\pi$ is an isomorphism.

Proof. Let $(X, \Delta + D + X_\pi)$ be the normalization. By [Kol13, 2.32] we see that D is normal in codimension one along D_π, thus $\tau(\mathcal{O}_D) \subseteq \mathcal{O}_D$ holds in a neighbourhood of the generic points of D_π.

We consider the pullback morphism $\mathcal{O}_{D_\pi} \to \mathcal{O}_{\bar{D}_\pi}$. Recall that $\bar{D}_\pi$ is reduced (Lemma 6.3.1.4) and $\bar{D}_\pi \to D_\pi$ is dominant. Thus $\mathcal{O}_{D_\pi} \to \mathcal{O}_{\bar{D}_\pi}$ is injective if and only if D_π is reduced. By [Kol13, 2.32], we know that D_π is generically reduced. If D is normal, then D_π is S_1 and thus reduced everywhere.

Hence we deduce that there is an open subset $U \subset Y$ such that X_U contains every codimension one point of X_π and such that the conditions of Proposition 6.4.1.6 are satisfied for the normalization $(X_U, D_U, \tau_U) \to U$. It follows that $Z_U \to Y_\pi \cap U$ is an isomorphism. This proves the first point, and the second point follows easily. If D is normal we can take $U = Y$, thus obtaining the third point. □

In the surface case, a finer analysis of the singularities of D yields a stronger statement under mild hypothesis:

Theorem 6.4.1.9. *Suppose that $k(\pi)$ is perfect of characteristic $p \neq 2$ and that $(Y, \Delta_Y + Y_\pi)$ is slc of dimension 3 and proper over* $\operatorname{Spec} R$. *If the scheme-theoretic intersection D_π is reduced, then $Z \to Y_\pi$ is an isomorphism.*

Before proving this theorem, let us comment on the reducedness of D_π. It is is the scheme-theoretic intersection of two lc centers of $(X, \Delta + D + X_\pi)$. In characteristic zero, intersections of lc centers are reduced [Kol13, 7.8]. In mixed characteristic, we can prove the following.

Proposition 6.4.1.10. *If $(Y, \Delta_Y + Y_\pi)$ is slc projective over $\operatorname{Spec} R$, $\dim Y = 3$ and $k(\pi)$ is perfect of characteristic $p \geq 7$, then D_π is reduced.*

Proof. First assume that $(X, \Delta + D + X_\pi)$ is dlt. Then the result holds by [BK21, Theorem 19 and subsequent paragraph].

In the general case, let $\varphi\colon (X', \Delta' + D' + E' + X'_\pi) \to (X, \Delta + D + X_\pi)$ be a $\mathbb{Q}$-factorial crepant dlt model. Such a model exists: log resolutions exist for pairs of dimension three over the spectrum of R [BMP$^+$21, 2.12], and combining [BMP$^+$21, Theorem F] with the arguments of [Kol13, 1.35-1.36] we can run a MMP to produce a model with the desired properties. We claim that $R^1\varphi_*\mathcal{O}(-D' - X'_\pi) = R^1\varphi_*\mathcal{O}(-D') \otimes \mathcal{O}(-X_\pi) = 0$ along $\operatorname{Supp}(X_\pi)$. This follows from [BK21, Proposition 7] applied to $\varphi\colon X' \to X$, the dlt pair $(X', X'_\pi + (\Delta' + E'))$ and the $\mathbb{Z}$-divisor $-D'$, since:

(a) $-D' \sim_{\varphi,\mathbb{Q}} K_{X'} + X'_\pi + E' + \Delta'$,
(b) $\mathcal{O}(-D' - mX'_\pi)$ is S_3 for every $m \geq 1$ by [BK21, Theorem 19],
(c) $-X'_\pi$ is relatively nef over X, since it is relatively trivial,
(d) strong Grauert–Riemenschneider vanishing holds for $(X'_\pi)^n \to X_\pi$ since this is a birational morphism of excellent surfaces (this can be deduced from [Kol13, 10.4]).

Therefore pushing forward along φ the exact sequence

$$0 \to \mathcal{O}(-D' - X'_\pi) \to \mathcal{O}_{X'} \to \mathcal{O}_{D' \cup X'_\pi} \to 0,$$

we obtain that $\varphi_*\mathcal{O}_{X'} = \mathcal{O}_X \to \varphi_*\mathcal{O}_{D' \cup X'_\pi}$ is surjective in a neighbourhood of X_π. This map factors through $\mathcal{O}_{D \cup X_\pi}$, so we deduce that

$$\varphi_*\mathcal{O}_{D' \cup X'_\pi} = \mathcal{O}_{D \cup X_\pi}.$$

We are now in position to apply the argument of [Kol13, 7.8] to conclude that $D \cap X_\pi$ is reduced. □

The following lemma will be useful for the proof of Theorem 6.4.1.9.

Lemma 6.4.1.11. *Let $X = \operatorname{Spec} A$ be an affine demi-normal scheme with normalization $\bar{X} = \operatorname{Spec} \bar{A}$. Let $\eta \in X$ be a node, and assume that η has two preimages $\xi, \xi' \in \bar{X}$. Let $V = \operatorname{Spec} \bar{A}/\mathfrak{p}_\xi$ and $V' = \operatorname{Spec} \bar{A}/\mathfrak{p}_{\xi'}$. Then:*

(a) The natural maps $k(\eta) \to k(\xi)$ and $k(\eta) \to k(\xi')$ are isomorphisms. We denote by $\phi\colon k(\xi') \cong k(\xi)$ the induced isomorphism.
(b) Let $v \in \bar{A}$. If V is normal, then the element $\operatorname{res}_\xi(v) - \phi(\operatorname{res}_{\xi'}(v)) \in k(\xi)$ extends to a regular function on V.
(c) If both V and V' are normal, then ϕ extends to an isomorphism $V \cong V'$.

Proof. The first point follows from Proposition 4.3.0.1. To prove the other two, set $S = \operatorname{Spec} A/\mathfrak{p}_\eta$ and let Γ be the main component of the fiber product $V \times_S V'$, equipped with its reduced structure. Its generic point is (ξ, ξ') and its function field is $k(\Gamma) = k(\xi) \otimes_{k(\eta)} k(\xi')$. By construction the projection $\mathrm{pr}_1\colon \Gamma \to V$ is finite, thus closed; since its image contains the generic point ξ of V, it is surjective. By the first point we see that pr_1 induces an equality of functions fields, so it is birational. Similarly $\mathrm{pr}_2\colon \Gamma \to V'$ is finite surjective birational.

Under the identification $k(\xi) \otimes_{k(\eta)} k(\xi') = k(\xi)$ induced by ϕ, we have

$$\mathrm{pr}_1^*(v) - \mathrm{pr}_2^*(v)|_{k(\Gamma)} = \operatorname{res}_\xi(v) - \phi(\operatorname{res}_{\xi'}(v)).$$

Assume that V is normal. Then by Zariski's Main Theorem the finite surjective birational morphism $\mathrm{pr}_1 \colon \Gamma \to V$ is an isomorphism. Thus the rational function $\mathrm{res}_\xi(v) - \phi(\mathrm{res}_{\xi'}(v))$ extends to a regular function on V.

Finally, if V and V' are normal, then the composition $\mathrm{pr}_2 \circ \mathrm{pr}_1^{-1} \colon V \to V'$ gives the isomorphism extending ϕ. $\square$

Proof of Theorem 6.4.1.9. We assume from now on that D_π is reduced. The first important observation is that the divisor D has mild singularities:

Claim 6.4.1.12. *The surface D is demi-normal with singular codimension one points mapping to the generic point η of* $\operatorname{Spec} R$.

Proof. By hypothesis D_π is S_1. Since it is a hypersurface of D and since π is not a zero-divisor in $\mathcal{O}_D$, we deduce that D is S_2 in a neighbourhood of D_π. Since the S_2 locus is open and $D \to \operatorname{Spec} R$ is proper, in particular closed, we deduce that D is S_2 everywhere.

Let $\xi \in D$ be a codimension one point. Then $\xi \in X$ is a codimension two point, and by [Kol13, 2.31] we see that D is at worse nodal at ξ. Thus D is demi-normal. Assume that ξ is the generic point of an irreducible component of D_π. Then ξ belongs to the intersection $X_\pi \cap D$, and by [Kol13, 2.31] again we see that ξ is a regular point of D. Therefore, if ξ is a singular point of D, it maps to the generic point of $\operatorname{Spec} R$. $\lozenge$

Claim 6.4.1.13. *Write $\lfloor \mathrm{Diff}_{\bar{D}_\eta}(\Delta) \rfloor = \sum_\alpha p_\alpha$, where each $p_\alpha \in \bar{D}$ is a codimension one point with ideal $\mathfrak{p}_\alpha \subset \mathcal{O}_{\bar{D}}$. Then:*

(a) the collection $\{p_\alpha\}$ contains all the preimages of the nodes of D, and

(b) the $V(p_\alpha) = \operatorname{Spec} \mathcal{O}_{\bar{D}}/\mathfrak{p}_\alpha$ are normal and pairwise disjoint subschemes of $\bar{D}$.

Proof. Recall that $(\bar{D}, \mathrm{Diff}_{\bar{D}}(\Delta) + \bar{D}_\pi)$ is an lc surface by hypothesis. If $m \in \bar{D}$ is a preimage of a node of D, a local calculation shows that the divisors $V(m)$ appear with coefficient one in $\mathrm{Diff}_{\bar{D}}(\Delta)$, see for example [Kol13, 2.31.2]. If $z \in V(p_\alpha) \cap V(p_\beta)$ then z belongs to the special fiber $\bar{D}_\pi$. However this contradicts [Kol13, 2.31.2]. So the $V(p_\alpha)$ are pairwise disjoint.

By [Kol13, 2.31.2] again, the intersection $V(p_\alpha) \cap \bar{D}_\pi$ is regular. We deduce that the one-dimensional scheme $V(p_\alpha)$ is normal around its special fiber. It is also normal along its generic fiber. Therefore it is normal everywhere. $\lozenge$

By [Kol12, Proposition 25] the diagram

$$\begin{tikzcd} D \arrow[r, hook] \arrow[d] & X \arrow[d, "p"] \\ W = D/R_D(\tau) \arrow[r, hook] & Y \end{tikzcd}$$

is a universal pushout. Therefore the diagram

$$\begin{tikzcd} D_\pi \arrow[r, hook] \arrow[d] & X_\pi \arrow[d, "p_\pi"] \\ W_\pi \arrow[r, hook] & Y_\pi \end{tikzcd} \tag{4.1.13.e}$$

is also a pushout.

Claim 6.4.1.14. *Every scheme appearing in (4.1.13.e) is reduced.*

Proof. By assumption D_π is reduced, and X_π is reduced by Lemma 6.3.1.6. Assume for the moment that Y_π is reduced. If $0 \neq s \in \mathrm{nil}(\mathcal{O}_{W_\pi})$, then by the pushout property $(s, 0) \in \mathcal{O}_{W_\pi} \times \mathcal{O}_{X_\pi}$ gives a non-zero nilpotent element of Y_π, a contradiction. Thus W_π is reduced.

To show that $\mathcal{O}_{Y_\pi}$ is reduced, consider the exact sequence

$$0 \to \mathcal{O}_Y \to p_*\mathcal{O}_X \to \mathcal{Q} \to 0$$

and tensor it by $R/\pi R$ over R. If $\operatorname{Tor}_1^R(R/\pi R, \mathcal{Q}) = 0$ then $\mathcal{O}_{Y_\pi}$ embeds into $(p_\pi)_*\mathcal{O}_{X_\pi}$. Since the latter is a sheaf of reduced algebras, this would entail that $\mathcal{O}_{Y_\pi}$ is reduced as well. Now assume that $\operatorname{Tor}_1^R(R/\pi R, \mathcal{Q}) \neq 0$. Then there is $q \in p_*\mathcal{O}_X$ such that $q \notin \mathcal{O}_Y$ but $\pi q \in \mathcal{O}_Y$. But then $q\pi|_{\bar{D}}$ is τ-invariant. Since $\pi|_{\bar{D}}$ is τ-invariant and $\bar{D}$ is flat over $\operatorname{Spec} R$, it must hold that $q|_{\bar{D}}$ is τ-invariant. Therefore $q \in \mathcal{O}_Y$, which is a contradiction. Hence $\operatorname{Tor}_1^R(R/\pi R, \mathcal{Q}) = 0$. $\diamondsuit$

We have seen in Lemma 6.4.1.2 that $Z \to Y_\pi$ is a finite universal homeomorphism. Hence if $X_\pi \to Y_\pi$ is weakly normal, then $Z \to Y_\pi$ is an isomorphism (see Section 4.3.2). Therefore, by Claim 6.4.1.14 and Lemma 4.3.2.1, we obtain:

Observation 6.4.1.15. To prove Theorem 6.4.1.9 it is sufficient to prove that $D_\pi \to W_\pi$ is weakly normal.

The situation can be slightly simplified:

Claim 6.4.1.16. *We may (and will) assume that:*

(a) $k(\pi)$ is algebraically closed,

(b) any node $n \in D$ has two preimages $m, m' \in \bar{D}$, and

(c) we have canonical equalities $k(\eta) = k(m) = k(n) = k(m')$.

Proof. We are studying whether $D_\pi \to W_\pi$ is weakly normal. This property descends faithfully flat covers [Yan83, Corollary to Proposition 1]. Moreover étale base-changes commute with normalizations and preserve slc singularities. Thus we can base-change along the strict henselization $\operatorname{Spec} R^{\mathrm{sh}} \to \operatorname{Spec} R$ and assume that $k(\pi)$ is algebraically closed.

Let $n \in D$ be a node, and let $\bar{n} \in \bar{D}$ be a preimage. By Claim 6.4.1.12 both n and $\bar{n}$ maps to the generic point η of $\operatorname{Spec}(R)$. Then $V = V(\bar{n}) \subset \bar{D}$ is one-dimensional, it is normal by Claim 6.4.1.13, and the morphism $V \to \operatorname{Spec} R$ is flat and proper. Therefore $V \to \operatorname{Spec} R$ is finite, and by flatness we have

$$\operatorname{length}_{k(\eta)} V_\eta = \operatorname{length}_{k(\pi)} V_\pi.$$

Since V_η is the spectrum of the field $k(\bar{n})$, by the valuative criterion of properness we deduce that the special fiber V_π is connected. We have seen in the proof of Claim 6.4.1.13 that V_π is reduced. Therefore V_π is the spectrum of a finite field extension of $k(\pi)$. By the assumption on $k(\pi)$ we obtain that $\operatorname{length}_{k(\pi)} V_\pi = 1$, and therefore $V_\eta \cong k(\eta)$. This implies that $k(n) = k(\bar{n})$. Now if $\bar{n}$ is the unique preimage of n, then $k(n) \subset k(\bar{n})$ is a field extension of degree 2 (see Proposition 4.3.0.1), which is a contradiction. Thus n has two preimages $m, m' \in \bar{D}$, with equalities of residue fields $k(\eta) = k(n) = k(m) = k(m')$. $\diamondsuit$

By [Yan83, Corollary to Theorem 1], weak normality of $D_\pi \to W_\pi$ is equivalent to the following property: if $x \in \mathcal{O}_{D_\pi}$ is such that $x^p \in \mathcal{O}_{W_\pi}$, then $x \in \mathcal{O}_{W_\pi}$ already.

Let $v \in \mathcal{O}_D$ be any lift of x. Arguing as in the proof of Proposition 6.4.1.6, we find that $\tau(v) = v + b\pi$, where $b \in \mathcal{O}_{\bar{D}}$ is such that $\tau(b) = -b$. Replacing v by $v + \frac{b}{2}\pi$, we may assume that v is τ-invariant, but at the cost that v only belongs to $\mathcal{O}_{\bar{D}}$. However we retain the property that the reduction of v modulo π belongs to $\mathcal{O}_{D_\pi}$.

The strategy is now the following: building on Corollary 4.3.2.5, descend $v + c\pi$ to $\mathcal{O}_D$ for a well-chosen $c \in \mathcal{O}_{\bar{D}}^+$. If we manage to do so, we are done, for then $v + c\pi \in \mathcal{O}_D^+ = \mathcal{O}_W$ restricts to $x \in \mathcal{O}_{W_\pi}$.

Thus we reduce to prove the following claim:

Claim 6.4.1.17. *Under the assumptions of Claim 6.4.1.16, given $v \in \mathcal{O}_{\bar{D}}^+$ such that $v|_{\bar{D}_\pi} \in \mathcal{O}_{D_\pi}$, there exists $c \in \mathcal{O}_{\bar{D}}^+$ such that $v + c\pi \in \mathcal{O}_D$.*

To fix the ideas, let us consider two simple cases. We denote by η the generic point of $\operatorname{Spec} R$.

Example 6.4.1.18. Assume that $(X, D+\Delta)$ is plt on the generic fiber. Then $D_\eta = \bar{D}_\eta$, and in this situation the condition of Corollary 4.3.2.5 holds automatically. So v descends to $\mathcal{O}_D$.

Example 6.4.1.19. Assume that D_η has a single node n and X_η is regular along D_η. This node is not contained in the support of Δ, so let us assume for simplicity that $\Delta = 0$. Then $\mathrm{Diff}_{\bar{D}_\eta}(0) = m + m'$.

Denote by $\phi\colon k(m') \xrightarrow{\sim} k(m)$ the isomorphism on residue fields given by Lemma 6.4.1.11. We say that v *has the same residues at m and m'* if $\mathfrak{r}_n(v) := \mathrm{res}_m(v) - \phi(\mathrm{res}_{m'}(v)) = 0$.

(a) If v has the same residues at m and m' (this happens for example if $\tau(m) = m'$, since v is τ-invariant) then v descends to $\mathcal{O}_D$ by Corollary 4.3.2.5.

(b) Assume that v has not the same residues at m and m'. Then $\mathfrak{r}_n(v)$ is a rational function on $V(m)$. As in Lemma 6.4.1.11, if $\Gamma \subset V(m) \times_{V(n)} V(m')$ is the main component with reduced structure then

$$\mathrm{pr}_1^*(v) - \mathrm{pr}_2 = \mathfrak{r}_n(v) \quad \text{in} \quad k(\Gamma) = k(m) \otimes_{k(n)} k(m').$$

Since $V(m)$ is normal by Claim 6.4.1.13, the projection $\mathrm{pr}_1\colon \Gamma \to V(m)$ is an isomorphism. We deduce that $\mathfrak{r}_n(v)$ is regular on $V(m)$. Moreover $\mathrm{pr}_1^*(v) - \mathrm{pr}_2^*(v)$ vanishes modulo π, since $v|_{\bar{D}_\pi}$ belongs to $\mathcal{O}_{D_\pi}$. So we obtain that $\mathfrak{r}_n(v) = \bar{c}\pi$ for some $\bar{c} \in \mathcal{O}_{V(m)}$.

Since $\mathrm{Diff}(0)$ is τ-invariant, we have $\tau(m) = m$ and $\tau(m') = m'$. The isomorphism $\phi\colon k(m') \cong k(m)$ is τ-equivariant (this follows for example from Claim 6.4.1.16), and since v is τ-invariant we deduce that $\bar{c}\pi$ is τ-invariant. Therefore $\bar{c}$ is also τ-invariant. Now consider the restriction map

$$\varphi\colon \mathcal{O}_{\bar{D}} \longrightarrow \mathcal{O}_{V(m)} \oplus \mathcal{O}_{V(m')}.$$

By Claim 6.4.1.13 and Lemma 6.4.1.20 below there is $c \in \mathcal{O}_{\bar{D}}^+$ such that $\varphi(c) = (\bar{c}, 0)$. By construction $v - \bar{c}\pi$ is τ-invariant and descends to $\mathcal{O}_D$ thanks to Corollary 4.3.2.5.

In the second example we used the following lemma:

Lemma 6.4.1.20. *Let A be a Noetherian ring acted on by a finite group G, such that $|G| \in A^\times$. Let $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$ be a G-invariant set of prime ideals, with the property that $\mathfrak{p}_i + \mathfrak{p}_j = A$ for any $i \neq j$. Then the natural map*

$$\varphi^G\colon A^G \longrightarrow \left(\bigoplus_{i=1}^n A/\mathfrak{p}_i\right)^G$$

is surjective.

Proof. By the Chinese remainder theorem the natural restriction map

$$\varphi\colon A \longrightarrow \bigoplus_{i=1}^n A/\mathfrak{p}_i$$

is surjective. It is easily seen to be G-equivariant: we denote by φ^G the induced map between the G-invariant subrings. Take any G-invariant element $\mathbf{a} \in \bigoplus_i A/\mathfrak{p}_i$, and choose a lift $a \in A$. Then $a_\mu := \frac{1}{|G|}\sum_{g\in G} g(a)$ is G-invariant and satisfies $\varphi^G(a_\mu) = \mathbf{a}$. This shows that φ^G is also surjective. $\Diamond$

Now let us treat the general case. Let $n_1, \dots, n_r \in D_\eta$ be the nodes of D. Notice that $\mathrm{Sing}(D_\eta) = \{n_1, \dots, n_r\}$ since D_η is a nodal curve. Let $m_i, m_i' \in \bar{D}_\eta$ be the two preimages of n_i. For any such preimage m, we define $V(m)$ as in Claim 6.4.1.13. We also let $\phi_i \colon k(m_i) \cong k(m_i')$ be the canonical isomorphism of residue fields over the node n_i (see Lemma 6.4.1.11).

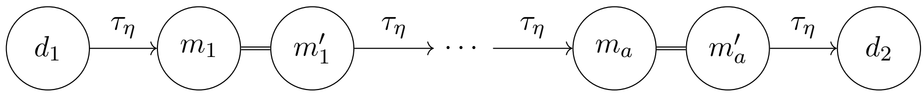
We can write

$$\lfloor \mathrm{Diff}_{\bar{D}_\eta}(\Delta) \rfloor = \left(\sum_{i=1}^{r} m_i + m_i' \right) + \sum_{j=1}^{s} d_j,$$

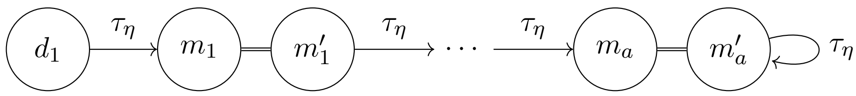
where the points $d_j \in \bar{D}_\eta$ belong to the locus where $\bar{D}_\eta \to D_\eta$ is an isomorphism. Since τ preserves $\mathrm{Diff}_{\bar{D}}(\Delta)$, on the generic fiber τ_η preserves $\lfloor \mathrm{Diff}_{\bar{D}_\eta}(\Delta) \rfloor$. Therefore the collection $\mathcal{P} = \{m_i, m_i', d_j\}_{i,j}$ is a τ_η-invariant subset of $\bar{D}_\eta$. It is also a union of equivalence classes for the relation given by the fibers of $\bar{D}_\eta \to D_\eta$.

The equivalence relation generated by τ_η and by the fibers of $\bar{D}_\eta \to D_\eta$ gives a partition $\mathcal{P} = \bigsqcup_l \mathcal{P}_l$ into classes that are of four possible types (up to relabelling):

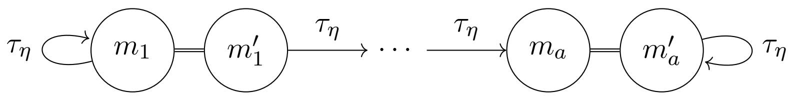
Type T_0:



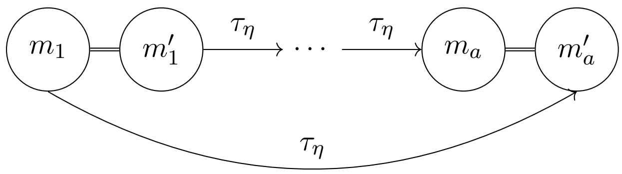
Type T_1:



Type T_2:



Type $T_\circlearrowleft$:



By Claim 6.4.1.16, the structural map $k(\eta) \hookrightarrow k(m)$ is an equality for any $m \in \{m_i, m_i'\}$. Thus if $\tau_\eta(m_i') = m_j$ then the induced isomorphism on residue fields $\tau_\eta \colon k(m_i') \cong k(m_j)$ is the unique isomorphism that makes the diagram

$$\begin{tikzcd} k(m_i') \arrow[rr, dotted, "\tau_\eta"] \arrow[dr, "\cong"'] & & k(m_j) \arrow[dl, "\cong"] \\ & k(\eta) & \end{tikzcd}$$

commute.

Now consider $v \in \mathcal{O}^+_{\bar{D}}$. We write $\mathrm{res}_p(v)$ for the image of v in the residue field of any $p \in \mathcal{P}$. As in Example 6.4.1.19, for any i we can write

$$\mathrm{res}_{m_i'}(v) - \phi_i(\mathrm{res}_{m_i}(v)) = \gamma_i \pi \quad \text{for some } \gamma_i \in \mathcal{O}_{V(m_i')}.$$

In general these differences are non-zero, and we look for a correction term that is τ-invariant. Take an equivalence class $\mathcal{P}_l \subset \mathcal{P}$, and say that the preimages of nodes contained in $\mathcal{P}_l$ are $m_1, m_1', \dots, m_a, m_a'$. Up to relabelling we may assume that

$$\tau_\eta(m_1') = m_2, \quad \tau_\eta(m_2') = m_3, \quad \dots \quad \tau_\eta(m_{a-1}') = m_a,$$

or in picture that

$$\begin{tikzcd} m_1 \arrow[r, equal] & m_1' \arrow[r, "\tau_\eta"] & \cdots \arrow[r, "\tau_\eta"] & m_a \arrow[r, equal] & m_a' \end{tikzcd}$$

so that the values $\tau_\eta(m_1), \tau_\eta(m_a')$ determine if $\mathcal{P}_l$ is of type T_0, T_1, T_2 or $T_\circlearrowleft$. For $i = 1, \dots, a$ we define inductively $c_i \in \mathcal{O}_{V(m_i')}$ by

$$c_1 = \gamma_1, \quad c_{i+1}\pi = \phi_{i+1}(\tau_\eta(c_i\pi)) - \gamma_{i+1}\pi \quad (i \leq a-1). \tag{4.1.20.f}$$

We think of c_{i+1} as a correction term for the residue of v at the point m_{i+1}'. Indeed, by construction we have

$$\phi_{i+1}(\mathrm{res}_{m_{i+1}}(v) + \tau_\eta(c_{i+1}\pi)) = \mathrm{res}_{m_{i+1}'}(v) + c_{i+1}\pi. \tag{4.1.20.g}$$

Claim 6.4.1.21. *These $c_i \in \mathcal{O}_{V(m_i')}$ exist.*

Proof. We check this inductively on i. It is clear for $i = 1$. Assuming that $c_i \in \mathcal{O}_{V(m_i')}$, it is sufficient to show that

$$\phi_{i+1}(\tau_\eta(c_i)) - \gamma_{i+1} \in k(m_{i+1}) \text{ is a regular function on } V(m_{i+1}).$$

Since γ_{i+1} is a regular function, we reduce to show that $\phi_{i+1}(\tau_\eta(c_i))$ is a regular function on $V(m_{i+1})$. Now τ_η is the extension of $\tau\colon V(m_i') \cong V(m_{i+1})$, and by Lemma 6.4.1.11 ϕ_{i+1} extends to an isomorphism $V(m_{i+1}) \cong V(m_{i+1}')$. Since c_i is regular on $V(m_i')$ we deduce that $\phi_{i+1}(\tau_\eta(c_i))$ is regular on $V(m_{i+1}')$. $\Diamond$

For each $\mathcal{P}_l$, we record the above previous construction using a vector that we call $\mathbf{c}_{\mathcal{P}_l}$:

(a) If $\mathcal{P}_l$ is of type T_2 or $T_\circlearrowleft$ we let

$$\mathbf{c}_{\mathcal{P}_l} = (0, c_1\pi, \tau(c_1\pi), \dots, \tau(c_{a-1}\pi), c_a\pi)) \in \bigoplus_{i=1}^{a} \mathcal{O}_{V(m_i)} \oplus \mathcal{O}_{V(m_i')}.$$

(b) If $\mathcal{P}_l$ is of type T_1 we let

$$\mathbf{c}_{\mathcal{P}_l} = (0, 0, c_1\pi, \tau(c_1\pi), \dots, \tau(c_{a-1}\pi), c_a\pi) \in \mathcal{O}_{V(d_1)} \oplus \bigoplus_{i=1}^{a} \mathcal{O}_{V(m_i)} \oplus \mathcal{O}_{V(m_i')}.$$

(c) If $\mathcal{P}_l$ is of type T_0 we let

$$\mathbf{c}_{\mathcal{P}_l} = (0, c_1\pi, \tau(c_1\pi), \dots, \tau(c_{a-1}\pi), c_a\pi, \tau(c_a\pi)) \in \mathcal{O}_{V(d_1)} \oplus \left(\bigoplus_{i=1}^{a} \mathcal{O}_{V(m_i)} \oplus \mathcal{O}_{V(m_i')} \right) \oplus \mathcal{O}_{V(d_2)}.$$

Claim 6.4.1.22. *Each vector $\mathbf{c}_{\mathcal{P}_l}$ is τ-invariant.*

Proof. It suffices to check τ_η-invariance at the generic points. This is clear if $\mathcal{P}_l$ is of type T_0, T_1 or T_2 (recall that, under the assumptions of Claim 6.4.1.16, τ_η acts trivially on the residue field of a fixed point). If $\mathcal{P}_l$ is of type $T_\circlearrowleft$, then we only have to check that $\tau_\eta(c_a\pi) = 0$. For simplicity, identify the residue field of every point of $\mathcal{P}_l$ with $k(\eta)$. Then by construction we have

$$\begin{aligned} \mathrm{res}_{m_1}(v) &= \mathrm{res}_{m_1'}(v) + c_1\pi \\ &= \mathrm{res}_{m_2}(v) + \tau_\eta(c_1\pi) \\ &= \dots \\ &= \mathrm{res}_{m_{a-1}'}(v) + c_{a-1}\pi \\ &= \mathrm{res}_{m_a}(v) + \tau_\eta(c_{a-1}\pi). \end{aligned}$$

On the other hand, since $\tau(m_1) = m'_a$, by τ-invariance of v we have $\mathrm{res}_{m_1}(v) = \mathrm{res}_{m'_a}(v)$. In particular
$$\tau_\eta(c_{a-1}\pi) = \mathrm{res}_{m'_a}(v) - \mathrm{res}_{m_a}(v) = \gamma_a \pi,$$
and by (4.1.20.f) this implies that $c_a\pi = 0$. $\Diamond$

We can finally conclude the proof of Theorem 6.4.1.9. Consider the concatenated vector $\mathbf{c} = (\mathbf{c}_{\mathcal{P}_l})_l$. Up to permutation of its entries, it belongs to $\bigoplus_{p\in\mathcal{P}} \mathcal{O}_{V(p)}$. By Claim 6.4.1.22 it is τ-invariant. Therefore by Claim 6.4.1.13 and Lemma 6.4.1.20, there exists $c \in \mathcal{O}^+_{\bar{D}}$ that reduces to $\mathbf{c}$. The element $v + c\pi$ is τ-invariant and by (4.1.20.g) it satisfies the descent condition of Corollary 4.3.2.5. This proves Claim 6.4.1.17, and Theorem 6.4.1.9 follows as already observed. $\square$

6.4.2 Serre properties of the fibers

In this subsection we consider the Serre conditions S_r on the special fiber Y_π. Since we always assume $Y \to \operatorname{Spec} R$ to be flat, π is not a zero-divisor in $\mathcal{O}_Y$ and so
$$Y_\pi \text{ is } S_r \iff Y \text{ is } S_{r+1} \text{ along } Y_\pi.$$
The analog equivalence holds for $X_\pi \subset X$, since X is automatically flat over $\operatorname{Spec} R$, by [Har77, III.9.7] and the fact that $p\colon X \to Y$ is finite. Similarly, an analog equivalence holds for D and $D/R_D(\tau)$ as soon as one of them (equivalently both of them) is flat over $\operatorname{Spec} R$.

Remark 6.4.2.1. By [Gro65, 7.8.6.iii] the S_i loci of excellent schemes are open. If $Y \to \operatorname{Spec} R$ is proper (or simply a closed morphism), a closed subset that is disjoint from Y_π must be empty. In this case the previous equivalence becomes: Y_π is S_r if and only if Y is S_{r+1}, and similarly for X, D and $D/R_D(\tau)$ under suitable hypothesis.

To approach the Serre properties of Y, we rely on the fact that it can be described as a universal push-out [Kol12, Proposition 25]. The main technical tool is the following lemma.

Lemma 6.4.2.2. *Consider a commutative square of complexes of abelian groups*
$$\begin{tikzcd} A^\bullet \arrow[r, "\alpha^\bullet"] \arrow[d, "\beta^\bullet"] & B^\bullet \arrow[d, "q^\bullet"] \\ C^\bullet \arrow[r, "p^\bullet"] & D^\bullet \end{tikzcd}$$
which is a pullback diagram at every degree. Then for every j there exists a natural map
$$\xi^j\colon h^j(A^\bullet) \to h^j(B^\bullet) \times_{h^j(D^\bullet)} h^j(C^\bullet)$$
which is surjective. If $h^{j-1}(D^\bullet) = 0$ and p^{j-2} is surjective, then ξ^j is bijective.

Proof. Let us embark for some diagram-chasing in

$$
\begin{tikzcd}
 & & A^{j-1} \arrow[d] \arrow[ddll] \arrow[ddrr] & & & \\
 & & A^j \arrow[dl, "\alpha^j"'] \arrow[dr, "\beta^j"] & & & \\
B^{j-1} \arrow[r] \arrow[ddrr, "q^{j-1}"'] & B^j \arrow[dr] & & C^j \arrow[dl] & C^{j-1} \arrow[l] \arrow[ddll] & C^{j-2} \arrow[l] \arrow[dddlll, "p^{j-2}"] \\
 & & D^j & & & \\
 & & D^{j-1} \arrow[u] & & & \\
 & & D^{j-2} \arrow[u] & & &
\end{tikzcd}
$$

We denote the differential maps of $A^\bullet$ by $d_A^j\colon A^j \to A^{j+1}$, and similarly for the other complexes.

If $x \in \ker(d_A^j)$, then clearly $\alpha^j(x) \in \ker(d_B^j)$ and $\beta^j(x) \in \ker(d_C^j)$. This defines a map

$$\xi_0^j\colon \ker(d_A^j) \to \ker(d_B^j) \times_{\ker(d_D^j)} \ker(d_C^j).$$

We claim that ξ_0^j is surjective. Indeed, a pair $(s,t) \in \ker(d_B^j) \times_{\ker(d_D^j)} \ker(d_C^j)$ gives, by the pullback property, a unique element $x \in A^j$ such that $\alpha^j(x) = s$ and $\beta^j(x) = t$. Since

$$\alpha^{j+1}(d_A^j x) = d_B^j(\alpha^j(x)) = 0, \quad \beta^{j+1}(d_A^j x) = d_B^j(\beta^j(x)) = 0$$

the pullback property ensures that $d_A^j(x) = 0$. Thus ξ_0^j is surjective.

It is easy to see that ξ_j' descends to a surjective map

$$\xi^j\colon h^j(A^\bullet) \to h^j(B^\bullet) \times_{h^j(D^\bullet)} h^j(C^\bullet).$$

Now let us discuss the injectivity of ξ^j. Let $x \in \ker(d_A^j)$ be such that $\alpha^j(x) = d_B^{j-1}(y)$ and $\beta^j(x) = d_C^{j-1}(z)$ for some $y \in B^{j-1}$ and $z \in C^{j-1}$. We investigate whether $x \in \operatorname{im}(d_A^{j-1})$. The element $\partial := q^{j-1}(y) - p^{j-1}(z)$ is non-zero in general, but it belongs to $\ker(d_D^{j-1})$. Thus if $h^{j-1}(D^\bullet) = 0$, there exists $\partial' \in D^{j-2}$ such that $d_D^{j-2}(\partial') = \partial$. If p^{j-2} is surjective, there exists $\delta' \in C^{j-2}$ such that $p^{j-2}(\delta') = \partial'$. Set $z_1 := z + d_C^{j-2}(\delta')$. We have

$$p^{j-1}(z_1) = p^{j-1}(z) + \delta = q^{j-1}(y),$$

thus by the pullback property the pair (y, z_1) corresponds to a unique $x' \in A^{j-1}$. Since $\alpha^j(d_A^{j-1}(x')) = d_B^{j-1}(y) = \alpha^j(x)$ and $\beta^j(d_A^{j-1}(x')) = d_C^{j-1}(z_1) = d_C^{j-1}(z) = \beta^j(x)$, the pullback property shows that $x = d_A^{j-1}(x')$. This concludes the proof. $\square$

Lemma 6.4.2.3. *Let A be a Noetherian semi-local ring with Jacobson ideal $I = \mathfrak{m}_1 \cap \cdots \cap \mathfrak{m}_n$. Let M be an A-module. Then $H_I^r(M) = 0$ if and only if $H^r_{\mathfrak{m}_i A_{\mathfrak{m}_i}}(M_{\mathfrak{m}_i}) = 0$ for every $\mathfrak{m}_i$.*

Proof. Since the localization maps $A \to A_{\mathfrak{m}_i}$ are flat, we have isomorphism s

$$H_I^r(M) \otimes_A A_{\mathfrak{m}_i} \cong H^r_{IA_{\mathfrak{m}_i}}(M_{\mathfrak{m}_i}) \quad \text{for every } i.$$

Since the $\mathfrak{m}_i$ are pairwise coprime, we have $I = \mathfrak{m}_1 \cdots \mathfrak{m}_n$ and therefore $IA_{\mathfrak{m}_i} = \mathfrak{m}_i A_{\mathfrak{m}_i}$. Thus the result follows from the fact that a module is trivial if and only if its localization at every maximal ideal is trivial. $\square$

Proposition 6.4.2.4. *Let $(X, D, \tau) \to Y$ be as in Notation 6.4.0.1 (flatness of $Y \to \operatorname{Spec} R$ is not necessary). Assume in addition that D is S_r.*

(a) If X and $D/R_D(\tau)$ are S_{r+1}, then Y is S_{r+1}.

(b) If D is S_{r+1}, then the converse holds.

Remark 6.4.2.5. As the proof will show, the statement of Proposition 6.4.2.4 is local on Y. That is, the statement holds if we replace adequately *being* S_i with *being* S_i *at* $y \in Y$, respectively with *being* S_i *along* $p^{-1}(y) \subset X$.

Proof. For simplicity, we write $\mathcal{W} = D/R_D(\tau)$. Then by [Kol12, Proposition 25] the diagram

$$\begin{tikzcd} D \arrow[r, hook] \arrow[d] & X \arrow[d, "p"] \\ \mathcal{W} \arrow[r, hook] & Y \end{tikzcd}$$

is a universal push-out and $\mathcal{W} \hookrightarrow Y$ is a closed embedding. Thus if $y \in Y$ is a point with $\mathfrak{m}_y = (f_1, \dots, f_n) \subset \mathcal{O}_y := \mathcal{O}_{Y,y}$, then the diagrams

$$\begin{tikzcd} (\mathcal{O}_{D\times\mathcal{O}_y})_{f_{i_0}\cdots f_{i_s}} & (\mathcal{O}_{X\times\mathcal{O}_y})_{f_{i_0}\cdots f_{i_s}} \arrow[l] \\ (\mathcal{O}_{\mathcal{W},y})_{f_{i_0}\cdots f_{i_s}} \arrow[u] & (\mathcal{O}_y)_{f_{i_0}\cdots f_{i_s}} \arrow[l, "p^{i_0,\dots,i_s}"'] \arrow[u] \end{tikzcd} \tag{4.2.5.h}$$

are pullback diagrams of semi-local rings. Notice that $p^{i_0,\dots,i_s}$ is always surjective, since $\mathcal{W} \hookrightarrow Y$ is a closed embedding.

For $\mathcal{A} \in \{\mathcal{O}_y, \mathcal{O}_{\mathcal{W},y}, \mathcal{O}_{X\times\mathcal{O}_y}, \mathcal{O}_{D\times\mathcal{O}_y}\}$ we denote by $\check{C}^\bullet(\mathbf{f}; \mathcal{A})$ the alternating Čech complex associated to $\mathcal{A}$ and to $\mathbf{f} = \{f_1, \dots, f_n\}$. Recall that

$$\check{C}^\bullet(\mathbf{f}; \mathcal{A}) = \left[0 \to \mathcal{A} \to \bigoplus_i \mathcal{A}_{f_i} \to \bigoplus_{i_0<i_1} \mathcal{A}_{f_{i_0} f_{i_1}} \to \cdots \to \mathcal{A}_{f_1\cdots f_n} \to 0 \right]$$

with differential defined as alternated sums of the natural localization maps. Thus it is easy to see that the diagrams (4.2.5.h) induce a commutative square of complexes

$$\begin{tikzcd} \check{C}^\bullet(\mathbf{f}; \mathcal{O}_{D\times\mathcal{O}_y}) & \check{C}^\bullet(\mathbf{f}; \mathcal{O}_{X\times\mathcal{O}_y}) \arrow[l] \\ \check{C}^\bullet(\mathbf{f}; \mathcal{O}_{W,y}) \arrow[u] & \check{C}^\bullet(\mathbf{f}; \mathcal{O}_y) \arrow[l, two heads, "p^\bullet"'] \arrow[u] \end{tikzcd}$$

where $p^\bullet$ is surjective and the square is a pullback at every degree. Now the cohomology of $\check{C}(\mathbf{f}; \mathcal{A})$ is equal to the local cohomology of $\mathcal{A}$ along $\mathfrak{m}_y\mathcal{A}$ [ILL+07, Theorem 7.13]. By the first part of Lemma 6.4.2.2, we get a surjective map

$$H^j_y(\mathcal{O}_y) \twoheadrightarrow H^j_{X_y}(\mathcal{O}_{X\times\mathcal{O}_y}) \times_{H^j_{D_y}(\mathcal{O}_{D\times\mathcal{O}_y})} H^j_y(\mathcal{O}_{W,y}).$$

Now assume that D is S_r. Then $H^j_z(\mathcal{O}_{D,z}) = 0$ for every $j < r$ and $z \in D$. If z maps to $y \in Y$, then $\mathcal{O}_{D,z}$ is the localization of $\mathcal{O}_{D\times\mathcal{O}_y}$ at some maximal ideal. Moreover $\mathfrak{m}_y\mathcal{O}_{D,z}$ is equal (up to taking its radical, to which the local cohomology is insensitive) to the Jacobson radical of $\mathcal{O}_{D\times\mathcal{O}_y}$. Thus $H^j_{D_y}(\mathcal{O}_{D\times\mathcal{O}_y}) = 0$ for every $j < r$ by Lemma 6.4.2.3. Therefore by the second part of Lemma 6.4.2.2, we have an isomorphism

$$H^j_y(\mathcal{O}_y) \cong H^j_{X_y}(\mathcal{O}_{X\times\mathcal{O}_y}) \times H^j_y(\mathcal{O}_{W,y}) \quad \forall j < r. \tag{4.2.5.i}$$

Similarly to above the group $H^j_{X_y}(\mathcal{O}_{X\times\mathcal{O}_y})$ is equal to the j^{th} local cohomology group of the semi-local ring $\mathcal{O}_{X\times\mathcal{O}_y}$ along its Jacobson radical.

Thus we see that if X and $\mathcal{W}$ are S_{r+1}, then by (4.2.5.i) we have $H^j_y(\mathcal{O}_y) = 0$ for every $j < r$. This proves the first point. Conversely, if D and Y are S_{r+1}, then the isomorphisms (4.2.5.i) hold for all $j \leq r$, and thus X and W are S_{r+1}. □

For the S_2 property, we have a stronger statement in the slc surface case:

Proposition 6.4.2.6. *Assume that $(Y, \Delta_Y + Y_\pi)$ is slc 3-dimensional and proper over* $\operatorname{Spec} R$, *D_π is reduced and $p = \operatorname{char} R/(\pi) \neq 2$. If $(X_\pi, \operatorname{Diff}_{X_\pi}(\Delta + D))$ if slc then $(Y_\pi, \operatorname{Diff}_{Y_\pi} \Delta_Y)$ is also slc.*

Proof. Since D_π is reduced, D is S_2 along D_π. By [Kol13, 2.31] it is also R_1 in a neighbourhood of D_π. Restricting over an open subset of Y that contains its special fiber, we may assume that D is normal.

By Theorem 6.4.1.9 the special fiber Y_π is the quotient of X_π by the equivalence relation generated by the involution τ_π on the divisor D_π. If X_π is S_2, then Lemma 4.4.0.2 shows that $X_\pi/R_{X_\pi}(\tau_\pi)$ is also S_2.

We show that Y_π is at worst nodal in codimension one. By [Kol13, 2.31] we know that X_π is at worst nodal in codimension one, so it is sufficient to consider the codimension one points of Y_π that belong to the image of D_π. These are precisely the images of the generic points of D_π. By [Kol13, 2.31] again, we know that X_π is normal in a neighbourhood of the generic points of D_π. Thus we may assume that X_π is normal, and the nodal property follows from Proposition 4.4.0.1.

Therefore Y_π is demi-normal, and we can perform adjunction along $Y_\pi \subset Y$ (see [Kol13, 4.2]). We obtain a pair $(Y_\pi, \operatorname{Diff}_{Y_\pi} \Delta_Y)$. From the diagram (4.1.2.c) we see that its normalization is $(X^n_\pi, \operatorname{Diff}_{X^n_\pi}(\Delta + D))$, which is lc by assumption. □

Remark 6.4.2.7. The result of Proposition 6.4.2.6 is also valid in equicharacteristic 0, without any restriction on the dimension. The difficult part is to show that Y_π is S_2: this follows from the general result [Kol13, 7.21].

Remark 6.4.2.8. The converse direction of Proposition 6.4.2.6 seems more difficult. The question is whether Y_π being S_2 implies that X_π is also S_2. The only question is along D_π, thus we may localize over a point of $p(D_\pi)$.

Let W be the quotient of X^n_π by $R(\tau_\pi)$. We have a commutative diagram

$$\begin{tikzcd} X^n_\pi \arrow[r, "n"] \arrow[d, "q"] & X_\pi \arrow[d, "p"] \\ W \arrow[r, "m"] & Y_\pi \end{tikzcd}$$

where m is a finite morphism. Since X_π is reduced it is S_2 in codimension one. So there is a big open subset $U \subset Y_\pi$ such that n_U, and therefore m_U, are isomorphisms. Hence $\mathcal{O}_{Y_\pi} \subset m_*\mathcal{O}_W$ is an equality over U. Since $\mathcal{O}_{Y_\pi}$ is S_2 and $m_*\mathcal{O}_W$ is torsion-free, we obtain by [Sta, 0AV9] that $\mathcal{O}_{Y_\pi} = m_*\mathcal{O}_W$. In other words $W = Y_\pi$ in a neighbourhood of $p(D_\pi)$.

However, this equality alone does not imply that X_π is S_2 in a neighbourhood of $D \cap X_\pi$. Here is a counterexample. Consider $\mathcal{S}^n = \mathbb{A}^2_{x,y}$ over a field k, the two lines $L_0 = (y = 0), L_1 = (y = 1)$ and the two points $s_0 = (0,0) \in L_0, s_1 = (0,1) \in L_1$. Let $\mathcal{S}$ be the surface obtained from $\mathcal{S}^n$ by gluing s_0 and s_1 together, and $\mathcal{T}$ be the surface obtained from $\mathcal{S}^n$ by gluing L_0 and L_1 along the involution $L_0 \ni (x,0) \longleftrightarrow (x,1) \in L_1$. Then $\mathcal{T}$ is demi-normal, it is also a quotient of $\mathcal{S}$ and we have a commutative diagram

$$\begin{tikzcd} \mathcal{S}^n \arrow[rr] \arrow[dr] & & \mathcal{S} \arrow[dl] \\ & \mathcal{T} & \end{tikzcd}$$

However $\mathcal{S}$ is R_1 but not S_2.

Chapter 7

Abundance for slc surfaces

This chapter corresponds to the preprint [Pos21a].

Convention 7.0.0.1. We work with varieties over an arbitrary field k of positive characteristic, except in Section 7.4 where we work over a excellent base scheme S. We use the same terminology in both cases.

7.1 Introduction

The Minimal Model Program (MMP) predicts that a variety with mild singularities X admits a birational model X' such that either $K_{X'}$ is nef, or such that there is a fibration $X' \to Y$ whose general fiber is a Fano variety. In the first case, the MMP is completed by the Abundance conjecture: if $K_{X'}$ is nef, then it should also be semi-ample.

In the case of surfaces, both the MMP and the Abundance conjecture are established in many cases. For smooth surfaces over the complex numbers, this goes back to the work of the Italian school at the beginning of the twentieth century and to the subsequent work of Kodaira, although the results were formulated in different terms: see [Mat02, §1] for an exposition of these results. These classical methods were extended by Mumford [Mum69] to surfaces over algebraically closed fields of positive characteristic. Since then, MMP and Abundance were proved more generally for log canonical surface pairs over the complex numbers by Fujino [Fuj12] and over algebraically closed fields of positive characteristic by Tanaka [Tan14].

The work of Kollár and Shepherd-Barron on the moduli space of canonically polarized smooth complex surfaces [KSB88] has demonstrated that in order to have a good moduli theory of such surfaces, we should consider the larger class of so-called semi-log canonical (slc) surfaces. Hence it is natural to ask whether the MMP and the Abundance theorem can be extended to that class of surfaces. As a matter of fact, the usual MMP does not work (see [Fuj14, Example 5.4] and [Kol11]). On the other hand, Abundance holds for slc surface pairs in characteristic zero by [Fli92, §8 and §12] and over algebraically closed fields of positive characteristic by [Tan16].

The purpose of this article is to extend the Abundance theorem to slc surfaces over any field of positive characteristic. We prove:

Theorem 7.1.0.1. *Let (S, Δ) be an slc surface pair and $f\colon S \to B$ a projective morphism where B is quasi-projective over a field of positive characteristic. Assume that $K_S + \Delta$ is f-nef; then it is f-semi-ample.*

Let us sketch the proof in the case B is the spectrum of a field. Abundance holds over arbitrary fields for lc surface pairs by the work of Tanaka [Tan20a]. Thus if $(\bar{S}, \bar{D} + \bar{\Delta})$ is the normalization of (S, Δ), since $K_S + \Delta$ pullbacks to $K_{\bar{S}} + \bar{D} + \bar{\Delta}$ the latter is semi-ample. We have to find a way to descend semi-ampleness along the normalization.

Our strategy is similar to the one of [HX16]. Let τ be the involution of $\bar{D}^n$ induced by the normalization, and let $\varphi\colon \bar{S} \to \bar{T}$ be the fibration given by a sufficiently divisible multiple of $K_{\bar{S}} + \bar{D} + \bar{\Delta}$. One shows that the set theoretic equivalence relation on $\bar{T}$ induced by $(\varphi, \varphi \circ \tau)\colon \bar{D}^n \rightrightarrows \bar{T}$ is finite using finiteness of $\mathbf{B}$-representations. It follows that the quotient $T := \bar{T}/(\bar{D}^n \rightrightarrows \bar{T})$ exists and similar arguments show that the hyperplane divisor of $\bar{T}$ descends to T. Then it is not difficult to show that the composition $\bar{S} \to \bar{T} \to T$ factors through S and that a multiple of $K_S + \Delta$ is the pullback of the hyperplane divisor of T.

In [HX16] the authors use the theory of sources and springs of a crepant log structure developed by Kollár (see [Kol13, §4.3]) to prove finiteness of the equivalence relation and descent of the hyperplane divisor. We have not placed our proof on such axiomatic ground, since the few cases of crepant log structures $\bar{S} \to \bar{T}$ that arise in our situation can be quite explicitly described. However our proof is an illustration of Kollár's theory: the technical details are easier, yet we encounter its main steps and subtleties. We highlight the correspondences in Remark 7.3.1.15.

There is another approach to slc abundance, developed in characteristic zero by Fujino [Fuj99, Fuj00] and used by Tanaka to prove slc abundance for surfaces over algebraically closed fields of positive characteristic [Tan16]. This approach is actually closely related to that of Hacon and Xu, and to Kollár's theory of crepant log structures: the finiteness of $\mathbf{B}$-representations plays a crucial role (see [Fuj99, Conjecture 4.2]), and the geometric properties of $\bar{S} \to \bar{T}$ that are relevant in Fujino's approach (see [Fuj99, Proposition 3.1]) can be understood in terms of sources, springs and $\mathbb{P}^1$-links (see [Kol13, §4.3]).

The set-up of [HX16] may also be applied to the relative setting $f\colon S \to B$. However, instead of adapting all the previous steps to this relative setting, we choose to reduce to the absolute case as in [Tan20a]. The main step is to compactify both S and B while preserving the properties of S and f. This is achieved using a carefully chosen MMP and some gluing theory.

7.1.1 Applications

We give two applications of Theorem 7.1.0.1. The first one is about families of slc surfaces in mixed characteristic. In positive characteristic, relative semi-ampleness is a property of fibers by [CT20]. Recent work of Witaszek [Wit21] shows that a similar statement holds for relative semi-ampleness in mixed characteristic. Combining our main result with abundance for threefolds in characteristic zero, we therefore obtain:

Theorem 7.1.1.1 (Theorem 7.4.0.1)**.** *Let S be an excellent regular one-dimensional scheme of mixed characteristic, $f\colon (X, \Delta) \to S$ a dominant flat projective morphism of relative dimension two. Assume that $(X, \Delta + X_s)$ is slc for every closed point $s \in S$, and that every fiber X_s is S_2.*

Then if $K_X + \Delta$ is f-nef, it is f-semi-ample.

The second application is about dlt threefolds of general type over arbitrary fields:

Theorem 7.1.1.2 (Theorem 7.3.3.4)**.** *Let (X, Δ) be a projective $\mathbb{Q}$-factorial dlt threefold over an arbitrary field k of characteristic $p > 5$. Assume that $K_X + \Delta$ is nef. Then $(K_X + \Delta)|_{\Delta^{=1}}$ is semi-ample.*

This theorem is a generalization of [Wal18, Theorem 1.3], which is a key step for the existence of good minimal models for lc threefolds over algebraically closed fields of characteristic $p > 5$ [Wal18, Theorem 1.1]. In a forthcoming note, I plan to combine Theorem 7.3.3.4, the techniques of [Wal18] and the tools of [DW19], to obtain the existence of good minimal models for lc threefolds over imperfect fields.

7.2 Preliminaries

Let X be a k-variety and D be a $\mathbb{Q}$-divisor. We denote by $\mathrm{Aut}_k(X, D)$ the group of k-automorphisms σ of X with the property that $\sigma(D) = D$. Similarly, if L is a line bundle then $\mathrm{Aut}_k(X, L)$ is the group of k-automorphisms σ of X such that $\sigma^* L \cong L$.

Let $C \subset X$ be a proper k-curve and L be a Cartier divisor on X. Then the intersection number $L \cdot C$ can be calculated with respect to k or to $K = H^0(C, \mathcal{O}_C)$. If necessary we distinguish between the two numbers by writing $L \cdot_k C$ and $L \cdot_K C$.

If a scheme fails to be satisfy the property S_2, in many cases it has a finite alteration that is S_2:

Proposition 7.2.0.1. *Let X be a reduced equidimensional excellent scheme. Then the locus U where X is S_2 is an open subset with $\mathrm{codim}_X(X \setminus U) \geq 2$, and there exists a morphism $g \colon X' \to X$ such that*

(a) X' is S_2 and reduced,

(b) g is finite and an isomorphism precisely above U, and

(c) the normalization $X^n \to X$ factorizes through g.

*We call $g \colon X' \to X$, the **$\mathbf{S_2}$-fication** of X.*

Proof. The morphism $g \colon X' \to X$ is the one given by [Gro65, 5.10.16] (see [Gro65, 5.10.13] for the definition of the $Z^{(2)}$ appearing there). The first two items also follow from [Gro65, 5.10.16] granted that g is finite, which holds by [Gro65, 5.11.1]. The fact that g factors the normalization follows from finiteness of g and from [Sta, 035Q]. □

7.2.1 Preliminary results

Proposition 7.2.1.1. *Let C be a regular projective curve over k, and D a boundary such that $K_C + D$ is ample. Then $\mathrm{Aut}_k(C, D)$ is finite.*

Proof. We may replace D by $\lceil D \rceil = \sum_{i=1}^m p_i$. Since $K_C + D$ is ample and preserved by the elements of $\mathrm{Aut}_k(C, D)$, we can describe the latter group as the group of k-points of the linear algebraic group

$$G := \left\{ \Phi \in \mathrm{PGL}_k\, H^0(C, m(K_C + D)) \mid \Phi(D) = D \right\}, \quad m \text{ divisible enough.}$$

The tangent space of G at the identity morphism is given by $H^0(C, T_C \otimes I_D)$ [Deb01, §2.9], which is trivial since $T_C \otimes I_D = \mathcal{O}(-K_C - D)$ is anti-ample. It follows that G is a finite group scheme, and thus $\mathrm{Aut}_k(C, D)$ is a finite group. □

Lemma 7.2.1.2. *Let (S, Δ) be a dlt surface. Then S is $\mathbb{Q}$-factorial and the irreducible components of $\lfloor \Delta \rfloor$ are normal.*

Proof. The $\mathbb{Q}$-factorial property is proved in [Tan18, 4.11]. Hence to show that the components of $\lfloor \Delta \rfloor$ are normal, we may assume that $\Delta = \lfloor \Delta \rfloor$ is irreducible. Then we can repeat the proof of [KM98, 5.51], using [Tan18, 3.2] instead of [KM98, 2.68]. □

The next two results study the pluricanonical representations on regular curves of genus zero.

Lemma 7.2.1.3. *Let X be a proper variety over k and L a line bundle on X. Write $K := H^0(X, \mathcal{O}_X)$. If the natural representation $\rho_K \colon \mathrm{Aut}_K(X, L) \to \mathrm{GL}_K\, H^0(X, L)$ has finite image, then so does $\rho_k \colon \mathrm{Aut}_k(X, L) \to \mathrm{GL}_k\, H^0(X, L)$.*

Proof. If $\varphi \in \mathrm{Aut}_k(X)$, then $\varphi^* = \rho_k(\varphi)\colon K \to K$ is a k-linear field automorphism. This gives a partition

$$\mathrm{Aut}_k(X,L) = \bigsqcup_{\sigma \in \mathrm{Aut}_k(K)} \mathrm{Aut}_k^\sigma(X,L)$$

which is finite since $k \subset K$ is a finite field extension. Notice that if $\varphi \in \mathrm{Aut}_k^\sigma(X,L)$, then $\varphi^{-1} \in \mathrm{Aut}_k^{\sigma^{-1}}(X,L)$.

For each σ, fix an element $\varphi_\sigma \in \mathrm{Aut}_k^\sigma(X,L)$ (if that subset is not empty). If we have another element $\psi \in \mathrm{Aut}_k^\sigma(X,L)$ then the automorphism

$$\rho_k(\varphi_\sigma^{-1} \circ \psi)\colon H^0(X,L) \to H^0(X,L)$$

is K-linear. Therefore we get a map

$$\iota_\sigma\colon \mathrm{Aut}_k^\sigma(X,L) \to \mathrm{GL}_K H^0(X,L), \quad \psi \mapsto \rho_k(\varphi_\sigma^{-1} \circ \psi).$$

Since $\varphi_\sigma^{-1} \circ \psi \in \mathrm{Aut}_K(X,L)$, we see that $\mathrm{im}(\iota_\sigma) \subseteq \mathrm{im}(\rho_K)$. Thus $\mathrm{im}(\iota_\sigma)$ is finite. Moreover

$$|\,\mathrm{im}(\iota_\sigma)| = |\rho_k(\varphi_\sigma^{-1}) \circ \mathrm{im}(\rho_k|_{\mathrm{Aut}_k^\sigma(X,L)})| = |\,\mathrm{im}(\rho_k|_{\mathrm{Aut}_k^\sigma(X,L)})|$$

so $\rho_k|_{\mathrm{Aut}_k^\sigma(X,L)}$ has finite image. Since

$$\mathrm{im}(\rho_k) = \bigcup_{\sigma \in \mathrm{Aut}_k(K)} \mathrm{im}(\rho_k|_{\mathrm{Aut}_k^\sigma(X,L)})$$

the lemma is proved. $\square$

Proposition 7.2.1.4. *Let C be a regular proper curve of genus zero over an arbitrary field k, and E an effective $\mathbb{Q}$-divisor such that $K_C + E \sim_{\mathbb{Q}} 0$. Then for m divisible enough, the natural representation $\mathrm{Aut}_k(C, mE) \to \mathrm{GL}_k(H^0(C, \omega_C^m(mE))$ has finite image.*

Proof. By Lemma 7.2.1.3 we may replace k by $H^0(C, \mathcal{O}_C)$ to prove the result. If C is smooth over k, then we may assume that k is algebraically closed and the result was proved in Proposition 5.4.1.15. So for the rest of the proof, we assume that C is non-smooth over k. By [Tan20b, 9.8, 9.10] it holds that $\mathrm{char}\, k = 2$ and we can find degree 2 purely inseparable extensions $k \subset l \subset k'$ such that $C_l = C \otimes_k l$ is integral with non-isomorphic normalization $B \cong \mathbb{P}^1_{k'}$:

$$\begin{tikzcd} B \arrow[d, "\nu"] \arrow[dr, "g"] & \\ C_l \arrow[r, "f"'] & C \end{tikzcd}$$

Moreover there is $P \in B(k')$ such that $K_B + P \sim g^* K_C$. Thus if $m > 0$ is such that mE is a $\mathbb{Z}$-divisor, then $mK_B + mP + g^*(mE) \sim g^*(K_C + mE)$. Since C_l is reduced we have an inclusion $\mathcal{O}_{C_l} \subset \nu_* \mathcal{O}_B$. Tensoring with $f^* \omega_C^m(mE)$ and using the projection formula, we obtain an inclusion $f^* \omega_C^m(mE) \subset \nu_* g^* \omega_C^m(mE)$. Taking global section, we get a sequence of inclusions

$$H^0(C, \omega_C^m(mE)) \subset H^0(C, \omega_C^m(mE)) \otimes_k l \subset H^0(B, \omega_B^m(mP + mg^{-1}(E))). \tag{2.1.4.a}$$

On the other hand, extending scalars along $k \subset l$ gives a natural map

$$f^*\colon \mathrm{Aut}_k(C, mE) \to \mathrm{Aut}_l(B, mP + mg^{-1}(E)),$$

whose image respects the flag (2.1.4.a). Therefore it is sufficient to show that the action of $\mathrm{Aut}_l(B, mP + mg^{-1}(E))$ on $H^0(B, \omega_B^m(mP + mg^{-1}(E)))$ is finite. By Lemma 7.2.1.3, it is sufficient to prove finiteness after replacing l by k'. By [Tan20b, 9.8] we may choose the extensions $k \subset l \subset k'$ so that P does not belong to the support of $g^{-1}(E)$. Thus the support of $mP + mg^{-1}(E)$ contains at least two points, and we may apply the usual argument (see Proposition 5.4.1.15). $\square$

The next proposition summarized useful results contained in the proofs of Proposition 5.4.2.9 and Proposition 5.4.2.12.

Proposition 7.2.1.5. *Let $\varphi\colon S \to T$ be a projective morphism from a surface S to a variety T of dimension ≤ 1, with $\varphi_*\mathcal{O}_S = \mathcal{O}_T$. Let Θ, Υ be divisors such that $(S, \Theta + \Upsilon)$ is dlt, $(\Theta + \Upsilon)^{=1} = \Theta$ and $K_S + \Theta + \Upsilon \sim_{\mathbb{Q},\varphi} 0$. Let $z \in T$ be a closed point and W, W' be lc centers of $(S, \Theta + \Upsilon)$ that are minimal for the property that their image through φ is equal to z. Then:*

(a) There exists a log isomorphism $(W, \mathrm{Diff}^*(\Theta + \Upsilon)) \cong (W', \mathrm{Diff}^*(\Theta + \Upsilon))$.

(b) If moreover $\dim T = 1$ and W is a genus one curve, then $W = W'$.

Proof. Since $(S, \Theta + \Upsilon)$ is dlt its lc centers are the strata of Θ and we can perform adjunction in any codimension. In this situation we write the different with Diff^* (see Corollary 5.4.2.3). Thus we are interested in the strata of Θ that are contained in $\Theta \cap \varphi^{-1}(z)$.

First we prove the existence of the log isomorphisms. First assume that $\Theta \cap \varphi^{-1}(z)$ is connected. There is nothing to show if it is 0-dimensional. If it is 1-dimensional, it is a chain of regular proper curves, and the minimal lc centers are the intersections of these curves. If there is more that one minimal lc center, then by adjunction it is easy to verify that they are $k(z)$ points.

We show that if Θ does not dominate T, then $\Theta \cap \varphi^{-1}(z)$ is connected. We follow the method of the last part of the proof of Proposition 5.4.2.12, so we only sketch the argument. Notice that $\dim T = 1$. We run a $(K_S + \Upsilon)$-MMP over T, which ends with a birational model [Tan18]:

$$\begin{tikzcd}
(S, \Theta + \Upsilon) \arrow[rr, "g"] \arrow[dr, "\varphi"'] & & (S', \Theta' + \Upsilon') \arrow[dl, "\varphi'"] \\
& T &
\end{tikzcd}$$

Then g is crepant by Lemma 5.4.1.14, $K_{S'} + \Theta' + \Upsilon'$ is φ'-nef, $(\Theta')^2 \leq 0$ by Lemma 5.4.1.5 and we can write $\Theta' = (\varphi')^*N + E$ for $E > 0$ vertical and a $\mathbb{Q}$-divisor N on T. Taking in account that $K_{S'} + \Upsilon'$ is numerically equivalent to the vertical divisor $-\Theta'$, we obtain

$$0 \geq (\Theta')^2 = (K_{S'} + \Upsilon') \cdot ((\varphi')^*N + E) \geq 0$$

and thus Θ' is a reduced fiber by Lemma 5.4.1.5. Therefore $\Theta' \cap (\varphi')^{-1}(z)$ is connected, and since g is crepant it follows that $\Theta \cap \varphi^{-1}(z)$ is also connected by Lemma 5.4.1.13.

To conclude the first point, we need to produce log isomorphisms in the case $\Theta \cap \varphi^{-1}(z)$ is not connected and Θ does dominate T. We follow the method of Proposition 5.4.2.9; once again we only sketch the argument. Base-changing along an étale morphism $(z' \in T') \to (z \in T)$ with $k(z) = k(z')$, we may assume that the different components of $\Theta \cap \varphi^{-1}(z)$ belong to different components of Θ, and that they are all horizontal over T. We run a $(K_S + \Upsilon)$-MMP over T, it terminates with a Fano contraction [Tan18]:

$$\begin{tikzcd}
(S, \Theta + \Upsilon) \arrow[r, "f"] \arrow[d, "\varphi"] & (S'', \Theta'' + \Upsilon'') \arrow[d, "p"] \\
T & B \arrow[l]
\end{tikzcd}$$

Notice that f is crepant by Lemma 5.4.1.14. By Lemma 5.4.1.13 the morphism f induces a bijection between connected components of Θ and Θ''. There is a component $D_1 \subset \Theta''$ that is p-ample. By assumption there is another component $D_2 \subset \Theta''$ that is disjoint from D_1. Take a curve C that intersects D_2 and is contracted by p: then $D_2 \cdot C < 0$ cannot happen, for otherwise $D_1 \cap D_2 \neq \emptyset$. Thus D_2 is also p-ample. A similar argument shows that the fibers of

p are one-dimensional. Hence if η is the generic point of B, then S''_η is a regular proper curve with $k(\eta) = H^0(S''_\eta, \mathcal{O}_{S''_\eta})$. Since

$$0 = (K_{S''} + \Theta'' + \Upsilon'')|_{S''_\eta},$$

we see that $D_1|_{S''_\eta}$ and $D_2|_{S''_\eta}$ are $k(\eta)$-points, and therefore $D_1 \to B$ and $D_2 \to B$ are isomorphisms. We also deduce from the equality that, up to shrink T around z, we have $\Theta'' = D_1 + D_2$. Therefore we obtain log isomorphisms on S'', and since f is crepant we get log isomorphisms on S (see for example Corollary 5.4.2.5).

Let us now prove the second point. If Θ does not dominate T then we have seen that $\Theta \cap \varphi^{-1}(z)$ is connected, and by adjunction it must be equal to W. Assume that Θ dominates T. We run a few steps of the $(K_S + \Upsilon)$-MMP over T, and stop when the transform of Θ intersects the transform of W. This does happen eventually, since W is contained in $\varphi^{-1}(z)$ and that $K_S + \Upsilon \sim_{\mathbb{Q},\varphi} -\Theta$. Since each step of the MMP is crepant for $(S, \Theta + \Upsilon)$ Lemma 5.4.1.14, it follows that $\mathrm{Diff}^*_W(\Theta + \Upsilon) \neq 0$, which contradicts adjunction. Thus $\Theta \cap \varphi^{-1}(z)$ actually does not contain a genus one curve. □

7.3 Abundance in positive characteristic

7.3.1 Absolute case

We begin with abundance in the absolute case:

Theorem 7.3.1.1. *Let (S_0, Δ_0) be a projective slc surface pair over an arbitrary field k of positive characteristic. Assume that $K_{S_0} + \Delta_0$ is nef; then it is semi-ample.*

For the duration of the proof we fix (S_0, Δ_0) and let $(S, D + \Delta)$ be its normalization. Write $D = D_G + D_I$, where D_G is the preimage of the separable nodes of S_0, and D_I is the preimage of the inseparable ones. Let τ be the induced log involution of $(D^n_G, \mathrm{Diff}_{D^n_G}(\Delta + D_I))$.

We emphasize that S_0 is not assumed to be irreducible. Thus $(S, \Delta + D) = \bigsqcup_{i=1}^N (S_i, \Delta_i + D_i)$ is the disjoint union of its normal irreducible components.

We divide the proof in several steps.

Reduction to separable nodes.

This step is only necessary if $\operatorname{char} k = 2$. Applying Proposition 4.3.3.1, we get a factorization

$$(S, D + \Delta) \longrightarrow (S', D'_I + \Delta') \xrightarrow{\mu} (S_0, \Delta_0)$$

where μ is finite purely inseparable and $(S', D'_I + \Delta')$ is slc with only separable nodes. By [CT20, 2.11.3], if $K_{S'} + D'_I + \Delta' = \mu^*(K_{S_0} + \Delta_0)$ is semi-ample then so is $K_{S_0} + \Delta_0$. Thus it suffices to study $(S', D'_I + \Delta')$ and so we may assume that S_0 has only separable nodes.

Quotienting the fibration.

The pullback $K_S + D + \Delta$ is nef by assumption, so it is semi-ample by [Tan20a]. Choose $m > 0$ even such that $m(K_S + D + \Delta)$ is base-point free, and let $\varphi \colon S \to T$ be the corresponding fibration onto a normal projective variety. We let H be the hyperplane Cartier divisor on T with the property that $\varphi^*\mathcal{O}(H) = \mathcal{O}(m(K_S + D + \Delta))$.

Since S_0 has only separable nodes, it is the geometric quotient of its normalization S by the finite equivalence relation induced by the involution τ on $(D^n, \mathrm{Diff}_{D^n} \Delta)$. This equivalence relation is generated by the two morphisms $(\iota, \iota' = \iota \circ \tau) \colon D^n \rightrightarrows S$. Let $\psi \colon D^n \to E$ be the fibration corresponding to the base-point free divisor $m(K_S + D + \Delta)|_{D^n}$. Then we have a diagram

$$\begin{tikzcd} D^n \arrow[r, shift left, "\iota"] \arrow[r, shift right, "\iota'"'] \arrow[d, "\psi"] & S \arrow[d, "\varphi"] \\ E \arrow[r, shift left, "j"] \arrow[r, shift right, "j'"'] & T \end{tikzcd} \tag{3.1.1.b}$$

where (ψ, j) (resp. (ψ, j')) is the Stein factorization of $\varphi \circ \iota$ (resp. of $\varphi \circ \iota'$). The two morphisms $(j, j')\colon E \rightrightarrows T$ are finite, and they generate a pro-finite equivalence relation on T. (We only care about the reduced image of this relation in $T \times_k T$: see the comment after [Kol13, 9.1].)

We claim that this relation is actually finite (so that the quotient exists by Theorem 2.5.0.2), and that H descends to the quotient. We prove both claims below: for the moment assume they hold. Let $q\colon T \to T_0 := T/(E \rightrightarrows T)$ be the quotient. Since the two compositions $q \circ \varphi \circ \iota$ and $q \circ \varphi \circ \iota'$ are equal, we obtain a morphism $\varphi_0\colon S_0 \to T_0$ such that the diagram

$$\begin{tikzcd} S \arrow[r, "\varphi"] \arrow[d, "n"] & T \arrow[d, "q"] \\ S_0 \arrow[r, "\varphi_0"] & T_0 \end{tikzcd} \tag{3.1.1.c}$$

commutes. Moreover there is an ample Cartier divisor H_0 on T_0 such that $q^*H_0 = H$.

Claim 7.3.1.2. $\varphi_0^*\mathcal{O}(H_0) = \mathcal{O}(m(K_{S_0} + \Delta_0))$. *In particular $K_{S_0} + \Delta_0$ is semi-ample.*

Proof. Tensoring the inclusion $\mathcal{O}_{S_0} \subset n_*\mathcal{O}_S$ by $\varphi_0^*\mathcal{O}(H_0)$ and using the projection formula, we obtain

$$\varphi_0^*\mathcal{O}_{T_0}(H_0) \subset n_*\mathcal{O}_S(m(K_S + D + \Delta)).$$

By commutativity of (3.1.1.c) and the definition of φ and q, for $s \in \mathcal{O}(H_0)$ we see that its pullback $\varphi_0^*s \in n_*\mathcal{O}(m(K_S + D + \Delta))$ is a log pluricanonical section whose restriction to D^n is τ-invariant. Thus $\varphi_0^*\mathcal{O}(H_0) \subseteq \mathcal{O}(m(K_{S_0} + \Delta_0))$ by Proposition 4.3.0.6. Conversely, by Lemma 2.5.0.3 a section t of $q_*\mathcal{O}(H)$ belongs to $\mathcal{O}(H_0)$ if and only if $j^*t = (j')^*t$. Looking at the diagram (3.1.1.b), we see that this condition is equivalent to $\iota^*\varphi^*t = (\iota')^*\varphi^*t$, which by Proposition 4.3.0.6 means that $\varphi^*t \in \mathcal{O}(m(K_{S_0} + \Delta_0))$. We have obtained the inverse inclusion $\mathcal{O}(m(K_{S_0} + \Delta_0)) \subseteq \varphi_0^*\mathcal{O}(H_0)$, which concludes the proof. ♢

Remark 7.3.1.3. In the diagram (3.1.1.c), the Stein factorisation of $S_0 \to T_0$ need not be demi-normal.

For example, consider the product $T := E \times \mathbb{P}^1$ of an elliptic curve with a rational smooth curve over an algebraically closed field, with projection morphisms $p_{\mathbb{P}^1}$ and p_E onto its factors. Let Δ_T be the sum of one section of $p_{\mathbb{P}^1}$ and of three distinct sections of p_E. Then $K_T + \Delta_T$ is ample and (T, Δ_T) is dlt. Let $\varphi\colon S \to T$ be the blow-up of two distinct points p and q that are 0-dimensional strata of Δ, and let E_p, E_q be the corresponding φ-exceptional divisors. Then $\varphi^*(K_T + \Delta_T) = K_S + \Delta_S + E_p + E_q$. So $\varphi\colon S \to T$ is the ample model of $(S, \Delta_S + E_p + E_q)$. Let $\tau\colon E_p \cong E_q$ be an isomorphism that sends $\Delta_S|_{E_p}$ to $\Delta_S|_{E_q}$. Then the quotient $r\colon S \to S_0 := S/R(\tau)$ exists, $(S_0, \Delta_{S_0} + F)$ is slc with normalization $(S, \Delta_S + E_p + E_q)$, where $F = r(E_p) = r(E_q)$. On the other hand, the induced involution $E \rightrightarrows T$ is given by $p \sim q$, and so the fibration given by $|m(K_{S_0} + \Delta_{S_0} + F)|$ is $S_0 \to T_0 := T/(p \sim q)$. However, T_0 is not demi-normal since it is not S_2.

Finiteness.

It remains to show finiteness and descent, and we begin by the former. It is convenient to reduce to the case where (each component) of $(S, \Delta + D)$ is dlt:

Claim 7.3.1.4. *In order to show that the equivalence relation induced by $E \rightrightarrows T$ is finite, we may assume that $(S, D + \Delta)$ is dlt.*

Proof. Indeed, let $\phi\colon (S_{\mathrm{dlt}}, D_{\mathrm{dlt}} + \Delta_{\mathrm{dlt}} + E) \to (S, D + \Delta)$ be a crepant dlt blow-up where $D_{\mathrm{dlt}} = \phi_*^{-1}D$ and $E = \mathrm{Exc}(\phi)$ [Tan18, 4.7, 4.8]. Then $K_{S_{\mathrm{dlt}}} + D_{\mathrm{dlt}} + \Delta_{\mathrm{dlt}} + E$ is semi-ample, and the corresponding fibration is just $\varphi \circ \phi\colon S_{\mathrm{dlt}} \to S \to T$. Moreover $(D^n, \mathrm{Diff}_{D^n} \Delta) = (D^n_{\mathrm{dlt}}, \mathrm{Diff}_{D^n_{\mathrm{dlt}}}(\Delta_{\mathrm{dlt}} + E))$, so we recover the involution τ on the dlt model. Notice that D^n_{dlt} is just the disjoint union of its irreducible components by Lemma 7.2.1.2. ♢

Now let us write $T = \bigsqcup_{i \geq 1} T_i$ and $\varphi = \prod_i \varphi_i$ where $\varphi_i\colon S_i \to T_i$ is the fibration given by $m(K_{S_i} + \Delta_i + D_i)$. Let $\kappa_i := \kappa(S_i, \Delta_i + D_i) \geq 0$ be the respective Kodaira dimensions, it holds that $\dim T_i = \kappa_i$. We let $\mathfrak{C}$ be the collection of irreducible components of D^n. For $\Gamma \in \mathfrak{C}$ we write $\Delta_\Gamma := \mathrm{Diff}_\Gamma(\Delta + D - \Gamma)$.

The following claim follows immediately from the construction.

Claim 7.3.1.5. *The two morphisms $(j, j')\colon E \rightrightarrows T$ come from an involution B^τ on E defined as follows: if $\tau(\Gamma) = \Gamma'$, then $B^\tau\colon \psi(\Gamma) \cong \psi(\Gamma')$ is induced by the isomorphism $\tau^*\colon H^0(\Gamma', m(K_{\Gamma'} + \Delta_{\Gamma'})) \cong H^0(\Gamma, m(K_\Gamma + \Delta_\Gamma))$.* ♢

It is possible that two components of D^n are conjugated under τ but do not belong to the same irreducible component of S. However we have the following:

Claim 7.3.1.6. *A component Γ of D^n is φ-vertical if and only if $\tau(\Gamma)$ is. Moreover, Γ is non-φ-vertical if and only if $K_\Gamma + \Delta_\Gamma$ is ample.*

Proof. The one-dimensional component Γ is φ-vertical if and only if $\psi(\Gamma)$ is a point. Moreover $\psi(\Gamma)$ is a point if and only if $K_\Gamma + \Delta_\Gamma$ has Kodaira dimension zero. Since τ sends $K_\Gamma + \Delta_\Gamma$ to $K_{\tau(\Gamma)} + \Delta_{\tau(\Gamma)}$, we obtain the result. ♢

We need to understand the non-φ-vertical components of D^n, and how they relate. The informations we need are given by the next three claims.

Claim 7.3.1.7. *Let $(S_i, \Delta_i + D_i)$ be such that $\kappa_i = 2$. Then $(T_i, (\varphi_i)_*(\Delta_i + D_i))$ is the log canonical model of $(S_i, \Delta_i + D_i)$. If Γ is a non-φ_i-vertical irreducible component of D_i, then Γ is the normalization of a component of $(\varphi_i)_* D_i$.* ♢

Claim 7.3.1.8. *Let $(S_i, \Delta_i + D_i)$ be such that $\kappa_i = 1$ and assume the non-φ-vertical sub-curve Θ of $D_i + \Delta_i^{=1}$ is non-empty. Then either:*

(a) Θ is irreducible and $\varphi_i|_\Theta$ is an isomorphism;

(b) Θ is irreducible and $\varphi_i|_\Theta$ has degree 2;

(c) $\Theta = \Gamma_1 + \Gamma_2$ and each $\varphi_i|_{\Gamma_j}$ is an isomorphism.

Proof. For simplicity, we drop the index i for the duration of the proof. Let η be the generic point of T. Then the generic fiber $F = S_\eta$ is a regular curve. Since $\varphi_* \mathcal{O}_S = \mathcal{O}_T$ we obtain $H^0(F, \mathcal{O}_F) = k(\eta)$. We have

$$\deg_{k(\eta)} K_F = \deg_{k(\eta)}(-\Delta - D)|_F \leq -\Theta|_F < 0.$$

Thus $h^0(F, \omega_F) = h^1(F, \mathcal{O}_F) = 0$. By [Kol13, 10.6] we deduce that $\deg_{k(\eta)} K_F = -2$. Hence $\deg_{k(\eta)} \Theta|_F \leq 2$, which means that Θ has at most two irreducible components. If Γ_i is an irreducible component such that $\deg_{k(\eta)} K|_F = 1$, then $\varphi|_{\Gamma_i}\colon \Gamma_i \to T$ is a finite birational morphism of normal curves, hence an isomorphism. ♢

Claim 7.3.1.9. *In the situation of Claim 7.3.1.8, if $\Theta \to T_i$ is separable of degree 2, then there exists a non-trivial log involution of $(\Theta^n, \mathrm{Diff}_{\Theta^n}(\Delta + D - \Theta))$ over T_i.*

Proof. Assume first that $\Theta = \Theta^n$ is irreducible. Then $\varphi_i|_\Theta$ is Galois of degree 2 and induces an involution $\xi\colon \Theta \cong \Theta$ over T_i. We claim that ξ preserves the line bundle $\mathcal{O}(m(K_\Theta+\Delta_\Theta))$. Indeed, fix any global meromorphic form $\omega \in H^0(S_i, \mathcal{O}(m(K_{S_i} + D_i + \Delta_i))$ and take $s \in H^0(T_i, \mathcal{O}(H))$ such that $\varphi_i^* s = \omega$. Then since ξ commutes with $\varphi_i|_\Theta$ we have

$$\omega|_\Theta = (\varphi_i|_\Theta)^* \, s = (\varphi_i|_\Theta \circ \xi)^* \, s = \xi^* \omega|_\Theta.$$

Since the global sections of $\mathcal{O}(m(K_S + D + \Delta))$ generate, our claim is proved.

Now assume that $\Theta = \Gamma_1 + \Gamma_2$ and that both $\varphi|_{\Gamma_i}$ are isomorphisms. Then one proves as above that

$$\xi := \varphi_i|_{\Gamma_2}^{-1} \circ \varphi_i|_{\Gamma_1} \colon (\Gamma_1, \Delta_{\Gamma_1}) \longrightarrow (\Gamma_2, \Delta_{\Gamma_2})$$

is a log isomorphism. ◊

We are ready to show finiteness of the relation generated by $E \rightrightarrows T$.

Claim 7.3.1.10. *The equivalence relation defined by $(j, j')\colon E \rightrightarrows T$ is finite.*

Proof. We study the pullback of the equivalence relation $R(B^\tau) \rightrightarrows T$ through the finite structural morphism $j\colon E \to T$. If we can show that this equivalence relation on E is finite, it will follow that the equivalence relation $R(B^\tau) \rightrightarrows T$ is finite. This pullback is the equivalence relation generated by two types of pre-relations on E:

(a) the isomorphism $B^\tau\colon E \cong E$;

(b) the fibers of $j\colon E \to T$.

Outside a 0-dimensional closed subset $Z \subset j(E)$, the fibers of j are either singletons or of order 2. In that second case they are the fibers of the separable degree 2 morphisms $\Theta \to T_i$, as in Claim 7.3.1.9. These fibers are the orbits of the log involutions $\xi\colon \Theta^n \cong \Theta^n$ described in that same claim. Under the identification $\psi(\Theta^n) = \operatorname{Spec} H^0(\Theta^n, m(K_{\Theta^n} + \Delta_{\Theta^n}))$, we can describe this involution as follows: ξ induces an automorphism ξ^* of $H^0(\Theta^n, m(K_{\Theta^n} + \operatorname{Diff}_{\Theta^n}(\Delta + D - \Theta)))$, inducing in turn an automorphism B^ξ of $\psi(\Theta^n)$. We can extend B^ξ to an automorphism of the whole E by declaring it to be the identity on the other components.

Let us study first the relation generated by the group of automorphisms $G = \langle B^\tau, \{B^\xi\}\rangle$ of E. We claim that $G :=\leq \operatorname{Aut}_k(E)$ is finite. Actually, since the map

$$G' = \langle \tau, \{\xi\}\rangle \longrightarrow G, \quad \phi \mapsto B^\phi$$

satisfies $B^\phi \circ B^{\phi'} = B^{\phi\circ\phi'}$, it suffices to show that $G' \subset \operatorname{Aut}_k(D^n, \operatorname{Diff}_{D^n} \Delta)$ is finite. Furthermore, it is enough to show that the G'-stabilizer of each $\Gamma \in \mathfrak{C}$ is finite. Recall that the ξ's are isomorphisms between non-φ-vertical curves. Thus by Claim 7.3.1.8 the G'-stabilizer of a φ-vertical Γ is of order at most two. On the other hand, the G'-stabilizer of a non-φ-vertical Γ is contained in $\operatorname{Aut}_k(\Gamma, \Delta_\Gamma)$, which is finite by Proposition 7.2.1.1 since $K_\Gamma + \Delta_\Gamma$ is ample.

Now we must also declare to be equivalent those points that belong to the fiber above the points of Z: this means merging some G-orbits together. For the moment it is sufficient to know that the set Z is finite (we will describe it more precisely in Observation 7.3.1.11 below). Since the new relations we must add on E are supported on $G \cdot j^{-1}(Z) \times_k G \cdot j^{-1}(Z)$, which is finite over k, we obtain that the pullback of $R(B^\tau)$ on E is finite. ◊

Analysis of the special set.

Before proceeding to the descent of the line bundle H, we study the special set Z considered during the proof of Claim 7.3.1.10. Recall that it is the finite set of those $z \in j(E)$ such that $j^{-1}(z)$ is not contained in a single G-orbit.

Observation 7.3.1.11. The $z \in Z$ are of three types:

(a) $z \in T_i$ with $\dim T_i = 2$. By Claim 7.3.1.7 z is a singular point of $(\varphi_i)_* D_i$. Thus by [Kol13, 2.31.1] it is a node of $(\varphi_i)_* D_i$, with two preimages p and q in D_i.

(b) $z \in T_i$ with $\dim T_i = 1$. By Claim 7.3.1.8 it follows that z is the contraction of a φ-vertical component of D_i.

(c) $z \in T_i$ with $\dim T_i = 0$. Then z is the contraction of a φ-vertical component of D_i.

We note that in any case z is the image of an lc center W_z of $(S, \Delta + D)$.

Claim 7.3.1.12. *For each z as above, let W_z and W'_z be two lc centers of $(S, \Delta + D)$ that are minimal for the property that $\varphi(W_z) = z = \varphi(W'_z)$. Then there is a log isomorphism $(W_z, \mathrm{Diff}^*_{W_z}(\Delta + D)) \cong (W'_z, \mathrm{Diff}^*_{W'_z}(\Delta + D))$.*

Proof. If $z \in T_i$ with $\dim T_i = 2$, then by Claim 7.3.1.7 there is a proper curve C passing through p and q that is contracted by φ_i. By [Kol13, 2.31.2], the curve C belongs to the reduced boundary $\Delta_i^{=1} + D_i$. Since $(K_{S_i} + \Delta_i + D_i) \cdot C = 0$ it follows by the adjunction formula that $k(p) = H^0(C, \mathcal{O}_C) = k(q)$.

If $z \in T_i$ with $\dim T_i \leq 1$ we apply Proposition 7.2.1.5 with $\Theta = D_i + \Delta_i^{=1}$ and $\Upsilon = \Delta_i^{<1}$. ♢

Claim 7.3.1.13. *Moreover, in the case $z \in T_i$ with $\dim T_i = 1$, if one minimal W_z above z is a curve of genus one, then it is the unique minimal center over z.*

Proof. This follows immediately from Proposition 7.2.1.5 with $\Theta = D_i + \Delta_i^{=1}$ and $\Upsilon = \Delta_i^{<1}$. ♢

Descent.

To conclude we must show that the Cartier divisor H descends along the quotient $q\colon T \to T_0 = T/R(B^\tau)$. There is a useful reduction step we will make:

Claim 7.3.1.14. *To descend H, we may assume that $\dim T_i \geq 1$ for all i.*

Proof. Assume that we have found a line bundle H_0 on T_0 such that $(q^* H_0)|_{T_i} \cong H|_{T_i}$ for every $i \geq 1$ such that $\dim T_i \geq 1$. If $\dim T_j = 0$, then we trivially have $(q^* H_0)|_{T_j} \cong \mathcal{O}_{T_j} \cong H|_{T_i}$. So H_0 is the line bundle we are looking for. □

For the rest of the proof, we use the method of [Kol13, 5.38], and keep the notations of Claim 7.3.1.10.

Let $T_H := \mathrm{Spec}_T \sum_{r \geq 0} H^0(T, rH)$ be the total space of H, and similarly

$$E_H := T_H \times_T E = \bigsqcup_{\Gamma \in \mathfrak{C}} \mathrm{Spec}_T \sum_{r \geq 0} H^0(\Gamma, rm(K_\Gamma + \Delta_\Gamma)) \xrightarrow{j_H} T_H.$$

Since τ is a log isomorphism of $(D^n, \mathrm{Diff}_{D^n} \Delta)$, the involution $B^\tau\colon E \cong E$ lifts to an involution $B^\tau_H\colon E_H \cong E_H$. The Cartier divisor H descends to the quotient $T/R(B^\tau)$ if the equivalence relation $R(B^\tau_H) \rightrightarrows T_H$ is finite (see [Kol13, 9.48, 9.53]).

As in the proof of Claim 7.3.1.10, we consider the pullback of $R(B^\tau_H) \rightrightarrows T_H$ to E_H. It is generated by two types of pre-relations:

(a) the fibers of the structural morphism $j_H\colon E_H \to T_H$, and

(b) the isomorphisms $B^\phi_H\colon E_H \cong E_H$ induced by the $B^\phi\colon E \cong E$, where $\phi \in G \subset \mathrm{Aut}_k(E)$. More precisely, each $\phi\colon E \cong E$ induce an automorphism of the graded section ring $\bigoplus_{\Gamma \in \mathfrak{C}} \sum_{r \geq 0} H^0(\Gamma, rm(K_\Gamma + \Delta_\Gamma))$, inducing in turn the automorphism B^ϕ_H of E_H.

We have seen in Claim 7.3.1.10 that the group $G = \langle B^\tau, \{B^\xi\} \rangle \subset \mathrm{Aut}_k(E)$ is finite. Therefore $G_H := \langle B^\tau_H, \{B^\xi_H\} \rangle$ is also finite, and so the G_H-orbits on T_H are finite.

Now we must take in account the fibers of $j_H\colon E_H \to T_H$ which are not single G_H-orbits. The new relations we get are supported on the fibers of $E_H \to T$ over the closed finite subset $Z \subset j(E)$. By Claim 7.3.1.12 we see that the new relations come from some isomorphisms between the W_z's, inducing isomorphisms between the section rings of the Cartier divisors $m(K_S + \Delta + D)|_{W_z}$, where W_z runs through the minimal lc centers of $(S, \Delta + D)$ over the points z of Z. As in [Kol13, 5.38], it suffices to show that the pluricanonical representations

$$\mathrm{Aut}_k(W_z, \mathrm{Diff}^*_{W_z}(\Delta + D)) \longrightarrow \mathrm{GL}_k\, H^0(W_z, m(K_{W_z} + \mathrm{Diff}^*_{W_z}(\Delta + D)) \tag{3.1.14.d}$$

are finite.

If a minimal lc center W_z above z is a genus one curve, then by Claim 7.3.1.14 and Observation 7.3.1.11 we may assume that $W_z \subset S_i$ with $\kappa_i = 1$. Then by Claim 7.3.1.13 we see that $j^{-1}(z)$ is contained in a G-orbit. Thus we may assume that the minimal lc centers W_z above z are either 0-dimensional, or genus zero curves.

If W_z is 0-dimensional then it is the spectrum of a finite field extension of k, and thus $\mathrm{Aut}_k(W_z)$ is finite. If W_z is a genus zero curve then finiteness of pluricanonical representations (3.1.14.d) is proved in Proposition 7.2.1.4. This concludes the proof. □

Remark 7.3.1.15. The discussion above is a simple illustration of some key features of Kollár's theory of sources and springs for crepant log structures:

(a) In Kollár's terminology, $\varphi\colon (S, \Delta + D) \to T$ is a crepant log structure, the components of D are the sources (of their images in T) and the components of E are the springs (of their images in T).

(b) In Claim 7.3.1.9, in case $\Theta = \Gamma_1 + \Gamma_2$, the fact that $(\Gamma_1, \Delta_{\Gamma_1})$ and $(\Gamma_2, \Delta_{\Gamma_2})$ are log-isomorphic to each other corresponds to the uniqueness of the source up to crepant birational map [Kol13, 4.45.1].

(c) In Claim 7.3.1.9, in case $\Theta \to T_i$ is a separable double cover, the fact that the extension of function fields is Galois and that the Galois involution can be realised by a log-automorphisms of (Θ, Δ_Θ) corresponds to the Galois property of springs [Kol13, 4.45.5].

(d) In Claim 7.3.1.9, in case $\Theta = \Gamma_1 + \Gamma_2$, we have that $(S_i, \Gamma_1 + \Gamma_2 + (\Delta_i - \Theta)) \to T$ is a weak $\mathbb{P}^1$-link (see Definition 5.4.2.10). Indeed, the general fiber $F = \varphi_i^{-1}(t)$ is an integral Gorenstein proper curve of genus 0 over $k(t)$ with $H^0(F, \mathcal{O}_F) = k(t)$ and has an invertible sheaf $\Gamma_1|_F$ of degree $\Gamma_1 \cdot_{k(t)} F = 1$. Thus $F \cong \mathbb{P}^1_{k(t)}$ by [Sta, 0C6U].

(e) To show finiteness and descent, we have reduced both times to a question about representation of a group of log automorphisms of E on the space of pluricanonical sections of H. This corresponds to the crucial role that pluricanonical representations have in [Kol13, 5.36-38]. Our case is easily manageable, since the groups of log automorphisms that appear are finite to begin with, so finiteness of representations is automatic.

7.3.2 Relative case

We prove abundance in the relative setting. We deduce the relative version from the absolute version, following the strategy of [Tan20a].

Assumption 7.3.2.1. Let (S_0, Δ_0) be a slc surface, $f_0\colon S_0 \to B_0$ a projective morphism where B_0 is quasi-projective over a field k of positive characteristic. We assume that $K_{S_0} + \Delta_0$ is f_0-nef.

We aim to show that $K_{S_0} + \Delta_0$ is f_0-semi-ample.

Reduction to separable nodes.

As in Section 7.3.1, we reduce to the case where S_0 has only separable nodes. The proof is similar, so we omit it.

Reduction to the projective case.

Next we reduce to case of a projective base. Since B_0 is quasi-projective over k, it embeds as a dense open subset of a projective k-scheme B. We look for a projective slc compactification of (S_0, Δ_0) over B. There is a commutative diagram

$$\begin{tikzcd} S_0 \arrow[r, hook, "j"] \arrow[d, "f_0"] & S \arrow[d, "f"] \\ B_0 \arrow[r, hook] & B \end{tikzcd}$$

where f is projective and j is a dense open embedding. Since S_0 is already projective over B_0, we may assume that $f^{-1}(B_0) = S_0$. Using the method of Theorem 4.6.0.1, we may furthermore assume that the singular codimension one points of S are contained in S_0. By out first reduction step, we may and will assume that these nodes are separable.

Let Δ be the closure in S of Δ_0, and let $(\bar{S}, \bar{\Delta} + \bar{D})$ be the normalization of (S, Δ). Then $S_0^n = \bar{S}_{B_0}$. By hypothesis on S, $\bar{D}$ is the closure of the conductor divisor of $S_0^n \to S_0$. Thus the normalization of its dense open subset $\bar{D} \cap S_0^n$ is equipped with an involution τ. Since $\bar{D}^n$ is a normal projective curve, τ extends uniquely to an involution of $\bar{D}^n$.

By repeatedly blowing-up some closed points on the complement of S_0^n and taking the strict transforms of our divisors, we may achieve the following:

(a) the scheme $\bar{S}$ is regular at the points of the boundary $Z := \bar{S} \setminus S_0^n$,

(b) $\bar{\Delta} \cap \bar{D}$ is contained in S_0^n, and

(c) if $\bar{E}$ is the divisorial part of Z, then $\mathrm{Supp}(\bar{\Delta}) + \bar{D} + \bar{E}$ is simple normal crossing in a neighbourhood of Z.

In particular $(\bar{S}, \bar{\Delta} + \bar{D})$ is lc. However $K_{\bar{S}} + \bar{\Delta} + \bar{D}$ might not be nef over B. We can run a $K_{\bar{S}} + \bar{\Delta} + \bar{D}$-MMP over B: but in order to denormalize, we ultimately want to recover an action of τ on the *log pair* obtained using adjunction on the pushforward of $\bar{D}$. It is not obvious that this additional data is preserved by the MMP, so let us prove it in the next claim.

Claim 7.3.2.2. *There exists a birational morphism over B*

$$\varphi \colon (\bar{S}, \bar{\Delta} + \bar{D}) \longrightarrow (\bar{S}', \bar{\Delta}' + \bar{D}')$$

with $\bar{\Delta}' = \varphi_* \bar{\Delta}$ and $\bar{D}' = \varphi_* \bar{D}$, such that

(a) $(\bar{S}', \bar{\Delta}' + \bar{D}')$ is lc and $K_{\bar{S}'} + \bar{\Delta}' + \bar{D}'$ is nef over B;

(b) $\varphi \colon S_0^n \to \varphi(S_0^n)$ is an isomorphism of open subsets;

(c) $\bar{D}^n \cong (\bar{D}')^n$, so we can transport the involution τ to $(D')^n$,

(d) the induced $R(\tau) \rightrightarrows \bar{S}'$ is a finite equivalence relation, and

(e) τ preserves the pullback of $K_{\bar{S}'} + \bar{\Delta}' + \bar{D}'$ to $(\bar{D}')^n$.

Proof. Consider an irreducible curve $C \subset Z$ vertical over B, and assume that $(K_{\bar{S}} + \bar{\Delta}' + \bar{D}') \cdot C < 0$. We make the following two observations:

- By [Tan18] we can run a $(K_{\bar{S}} + \bar{\Delta}' + \bar{D}')$-MMP which, by the generic nefness property of $K_{\bar{S}} + \bar{\Delta}' + \bar{D}'$, ends with a birational minimal model. This MMP eventually contracts C, so it follows that $C^2 < 0$ [Kol13, 10.1].
- Since C is not a component of $\bar{\Delta} + \bar{D}$ we have $K_{\bar{S}} \cdot C < 0$ (the intersection makes sense, since $\bar{S}$ is regular in a neighbourhood of C).

If we were working with a regular surface over an algebraically closed field it would follow that C is a (-1)-curve and we would apply Castelnuovo's theorem to contract it. It turns out that this picture carries over to our setting. Indeed, let $K = H^0(C, \mathcal{O}_C)$. Since C is regular, by the adjunction formula we have

$$-2\chi(\mathcal{O}_C) = \deg_K K_C = (K_{\bar{S}} + C) \cdot_K C < 0. \tag{3.2.2.e}$$

We also claim that $\deg_K C^2 = -h^0(C, \mathcal{O}_C)$. We have $0 > K_{\bar{S}} \cdot_K C = \deg_K K_C - C^2$ which implies that $\deg_K K_C < 0$ and thus $h^1(C, \mathcal{O}_C) = 0$. By [Kol13, 10.6] we obtain that $\deg_K K_C = -2$. Looking again at (3.2.2.e), we find that $h^0(C, \mathcal{O}_C) = -1 = \deg_K C^2$, as claimed.

By a generalisation of Castelnuovo's criterion [Lip69, 27.1], the facts that $\chi(\mathcal{O}_C) > 0$ and $-h^0(\mathcal{O}_C) = C^2$ imply that there exists a proper morphism $\varphi\colon \bar{S} \to \bar{S}_1$ onto a normal surface $\bar{S}_1$ that is projective over B, with the property that $\mathrm{Exc}(\varphi_1) = C$, and $\bar{S}_1$ is regular at $\varphi_1(C)$, and $\varphi_1\colon \bar{S} \setminus C \to \bar{S}_1 \setminus \varphi_1(C)$ is an isomorphism. Taking strict transforms of our divisors, we see that $(\bar{S}_1, \bar{\Delta}_1 + \bar{D}_1)$ is lc, and snc along the boundary $\varphi_1(Z)$, and that $K_{\bar{S}_1} + \bar{\Delta}_1 + \bar{D}_1$ is nef outside $\varphi_1(Z)$. Therefore the same analysis for negative curves on $\bar{S}_1$.

Continuing this way, we obtain a finite number of birational proper contractions

$$\bar{S} \xrightarrow{\varphi_1} \bar{S}_1 \xrightarrow{\varphi_2} \dots \xrightarrow{\varphi_r} \bar{S}_r = \bar{S}'$$

such that $K_{\bar{S}'} + \bar{\Delta}' + \bar{D}'$ is nef over B. We claim that $(\bar{S}', \bar{\Delta}' + \bar{D}')$ has the desired properties. Let $\varphi\colon \bar{S} \to \bar{S}'$ denote the composite morphism.

The first two ones are clear. Since we have not contracted any component of $\bar{D}$, the morphism $\bar{D} \to \bar{D}'$ induces an isomorphism on normalizations. Thus we can transport τ. The relation $R(\tau) \rightrightarrows \bar{S}'$ is finite over $\varphi(S_0^n) \cong S_0^n$, since it is the relation that arises from the normalization of S_0. The set $\bar{D}' \cap \varphi(S_0^n)$ is $R(\tau)$-stable, and its complement $\bar{D}' \setminus \varphi(S_0^n)$ is finite, so we deduce that $R(\tau) \rightrightarrows \bar{S}'$ is finite.

To prove the last property, it is sufficient to show that the divisor $\mathrm{Diff}(\bar{\Delta}')$ on $(\bar{D}')^n$ is contained in the preimage of $\varphi(S_0^n)$. This is equivalent to say that the intersection $\bar{\Delta}' \cap \bar{D}'$ is contained in $\varphi(S_0^n)$, since in a neighbourhood of $\varphi(Z)$ the curve $\bar{D}'$ is regular. Actually, it suffices to prove that every $\bar{\Delta}_i \cap \bar{D}_i$ lies above B_0, and so we reduce to the case $r = 1$. Say that the unique irreducible curve contracted by φ_1 is C; then C intersects both $\bar{D}$ and $\bar{\Delta}$. Writing once again $K = H^0(C, \mathcal{O}_C)$, the analysis of the first paragraph gives that

$$\begin{aligned}(K_{\bar{S}'} + \bar{\Delta}' + \bar{D}') \cdot_K C &= (\bar{\Delta}' + \bar{D}') \cdot_K C + \deg_K K_C - \deg_K C^2 \\ &> 1 - 2 + 1 \\ &= 0\end{aligned}$$

which is a contradiction. This completes the proof. $\diamondsuit$

By the two last items of Claim 7.3.2.2 and by Theorem 5.1.1.1 we can de-normalize the pair $(\bar{S}', \bar{\Delta}' + \bar{D}')$ along the gluing data given by $R(\tau) \rightrightarrows \bar{S}'$. We obtain a slc surface pair (S', Δ'), which contains (S_0, Δ_0) as an open subset, and such that $K_{S'} + \Delta'$ is nef over B. The proof that S' is projective over B is similar to the analog statement in Theorem 4.6.0.1.

Claim 7.3.2.3. *With the notations as above, it is sufficient to show that $K_{S'} + \Delta' + E'$ is semi-ample over B.*

Proof. More generally, consider a proper scheme morphism $g\colon X \to T$, a line bundle L on X, and open subschemes $T_0 \subset T$ and $V \subset X_{T_0}$:

$$\begin{tikzcd} V \arrow[rr, hook, "i"] \arrow[dr, "h"'] & & X_{T_0} \arrow[dl, "g_{T_0} =: m"] \\ & T_0 & \end{tikzcd}$$

We claim that if the functorial morphism $\epsilon_L\colon g^*g_*L \to L$ is surjective, then so is $\epsilon_{L_V}\colon h^*h_*L_V \to L_V$. Write $L_0 := L_{X_{T_0}}$. We have a commutative diagram

$$\begin{tikzcd} i^*m^*m_*L_0 \arrow[r, "\alpha"] \arrow[d, "\beta"] & i^*m^*m_*i_*i^*L_0 \arrow[d, "\cong"] \\ i^*L_0 = L_V & h^*h_*L_V \arrow[l, "\epsilon_{L_V}"] \end{tikzcd}$$

where $\alpha = i^*m^*m_*(L_0 \to i_*i^*L_0)$ and $\beta = i^*\epsilon_{L_0}$. If ϵ_L is surjective then ϵ_{L_0} is surjective [CT20, 2.12], so β is surjective and by commutativity of the diagram we deduce that ϵ_{L_V} is surjective.

We apply this result to $(X, T, T_0, V, L) = (S', B, B_0, S_0, \mathcal{O}(r(K_{S'} + \Delta' + E')))$ with r divisible enough to prove the claim. $\diamondsuit$

Conclusion of the proof.

We assume from now on that B_0 is projective over k, and (S_0, Δ_0) slc projective over B_0 with normalization $(\bar{S}_0, \bar{\Delta}_0 + \bar{D})$. Let A be an Cartier ample divisor on B_0. By [Tan20a, 4.11] the divisor $K_{\bar{S}_0} + \bar{\Delta}_0 + \bar{D} + n^*f_0^*(mA)$ is nef for m large enough. Thus $K_{S_0} + \Delta_0 + f_0^*(mA)$ is nef, hence semi-ample by Theorem 7.3.1.1. In particular it is f_0-semi-ample, say that $L := \mathcal{O}(r(K_{S_0} + \Delta_0 + f_0^*(mA)))$ defines a morphism over B_0. Tensoring the surjection $f_0^*(f_0)_*L \to L$ by $f_0^*\mathcal{O}(-mA)$ we see that $K_{S_0} + \Delta_0$ is f_0-semi-ample.

This completes the proof of Theorem 7.1.0.1. $\square$

7.3.3 Applications to threefolds

Lemma 7.3.3.1 (see [DW19, 6.17])**.** *Let (X, B) be a plt pair defined over an arbitrary field k. Let $l \subset k$ be a subfield and (X_l, B_l) a pair such that $(X_l, B_l) \otimes_l k = (X, B)$. Then (X_l, B_l) is plt.*

Proof. Since $\mu\colon X \to X_l$ is faithfully flat, X_l is normal [Sta, 033G]. Now let $f_l\colon Y_l \to X_l$ be a birational proper morphism with $K_{Y_l} + B_{Y_l} = f_l^*(K_{X_l} + B_l)$. We must show that the coefficients of B_{Y_l} along exceptional divisors are strictly less than 1. Let $(f\colon Y \to X) := (f_l\colon Y_l \to X_l) \otimes_l k$. As in the proof of [DW19, 6.17] one sees that Y is integral. Let $Y^n \to Y$ be its normalization:

$$\begin{tikzcd} Y^n \arrow[d, "n"] \arrow[dr, "\nu"] & \\ Y \arrow[r, "\mu_Y"] \arrow[d, "f"] & Y_l \arrow[d, "f_l"] \\ X \arrow[r, "\mu"] & X_l \end{tikzcd}$$

Then

$$K_{Y^n} + B_{Y^n} = n^*f^*(K_X + B) = \nu^*(K_{Y_l} + B_{Y_l}).$$

Let E_l be a prime divisor on Y_l that is exceptional over X_l, and E the strict transform of $E_l \otimes_l k$. Then E is exceptional over X, thus $\operatorname{coeff}_E B_{Y_n} < 1$ by the plt condition on (X, B). Since $\operatorname{coeff}_E B_{Y_n} \geq \operatorname{coeff}_{E_l} B_{Y_l}$, we deduce that $\operatorname{coeff}_{E_l} B_{Y_l} < 1$. This shows that (X_l, B_l) is plt. $\square$

Lemma 7.3.3.2. *Let (X, Δ) be a $\mathbb{Q}$-factorial dlt threefold defined over an arbitrary field k of characteristic $p > 5$. If $E \subset \Delta^{=1}$ is an irreducible component, then E is normal.*

Proof. The first paragraph of the proof of [Kol13, 4.16] works over any field, and shows that the lc centers of a dlt pair are amongst the strata of its reduced boundary. Therefore (X, E) is plt in a neighbourhood of E.

Assume that k is F-finite. Then we obtain that E is normal by [DW19, 6.3].

If k is not F-finite, then we can find a subfield $l \subset k$ that is finitely generated over $\mathbb{F}_p$ and such that (X, E) is defined over l, say $(X_l, E_l) \otimes_l k = (X, E)$. By Lemma 7.3.3.1 the pair (X_l, E_l) is plt, and by [DW19, 6.18] X_l is $\mathbb{Q}$-factorial (it is assumed there that l must be infinite: since k is not F-finite it is not contained in $\overline{\mathbb{F}_p}$, and so we may indeed choose l infinite). Then E_l is normal. This implies that E is normal. Indeed, if it is not, then we can find a finitely generated intermediate extension $l \subset l' \subset k$ such that $E_l \otimes_l l'$ is not normal. But

l' is still F-finite and we could have worked with $(X_l, E_l) \otimes_l l'$ to begin with, thus we get a contradiction. $\square$

Lemma 7.3.3.3. *Let (X, Δ) be a $\mathbb{Q}$-factorial dlt threefold defined over an arbitrary field k of characteristic $p > 5$. Let $\pi\colon Z \to \Delta^{=1}$ be the S_2-fication. Then π is a finite universal homeomorphism and Z is demi-normal.*

Proof. The S_2-fication is characterized in Proposition 7.2.0.1. In particular π is finite and an isomorphism above the codimension one points. Thus by [Kol13, 2.31] and the fact that Z is S_2, we obtain that Z is demi-normal.

It remains to show that π is a universal homeomorphism. It is surjective since it factors the normalization, and universally closed since it is finite. Thus it suffices to show that π is injective on geometric points. This is proved as in [Wal18, 5.1], using Lemma 7.3.3.2 instead of [Wal18, 2.11]. $\square$

Theorem 7.3.3.4. *Let (X, Δ) be a projective $\mathbb{Q}$-factorial dlt threefold over an arbitrary field k of characteristic $p > 5$. Assume that $K_X + \Delta$ is nef. Then $(K_X + \Delta)|_{\Delta^{=1}}$ is semi-ample.*

Proof. Let $\pi\colon Z \to \Delta^{=1}$ be the S_2-fication. Then Z is demi-normal by Lemma 7.3.3.3, by adjunction $(Z, K_Z + \operatorname{Diff}_Z \Delta^{<1})$ is a projective slc surface and $K_Z + \operatorname{Diff}_Z \Delta^{<1} = (K_X + \Delta)|_Z$ is nef. Then by Theorem 7.3.1.1 $(K_X + \Delta)|_Z$ is semi-ample. By Lemma 7.3.3.3 the morphism π is a universal homeomorphism, hence it factors a k-Frobenius of Z, and we deduce that $(K_X + \Delta)|_{\Delta^{=1}}$ is also semi-ample [CT20, 2.11.3]. $\square$

7.4 Abundance for surfaces in mixed characteristic

Theorem 7.4.0.1. *Let S be an excellent regular one-dimensional scheme of mixed characteristic, $f\colon (X, \Delta) \to S$ a dominant flat projective morphism of relative dimension two. Assume that $(X, \Delta + X_s)$ is slc for every closed point $s \in S$, and that every fiber X_s is S_2.*

Then if $K_X + \Delta$ is f-nef, it is f-semi-ample.

Proof. By abundance for slc threefolds in characteristic 0 [Fli92, HX16], $K_X + \Delta$ is semi-ample on the generic fiber. By assumption (X_s, Δ_s) is slc, thus by Theorem 7.3.1.1 we see that $K_X + \Delta$ is semi-ample on every closed fiber. We conclude by [Wit21, Theorem 1.2]. $\square$

Chapter 8

Future work

Et je m'étonne alors qu'il ait fallu
Ce temps, et cette peine. Car les fruits
Régnaient déjà dans l'arbre. Et le soleil
Illuminait déjà le pays du soir.
Je regarde les hauts plateaux où je puis vivre,
Cette main qui retient une autre main rocheuse,
Cette respiration d'absence qui soulève
Les masses d'un labour d'automne inachevé.

Y. Bonnefoy
Pierre écrite, Le Dialogue d'Angoisse et de Désir, I.

To conclude this thesis, I will sketch some possible developments of the results I have obtained.

8.1 Improvements

Some results obtained in [Pos21c] and [Pos21b] may be improved. For example:

(a) The theory of sources and springs for threefolds obtained in Section 5.4, more precisely in Theorem 5.4.2.18, is limited to lc centers contained in the reduced boundary $\Delta^{=1}$. It would be interesting to remove this condition, or even to work more generally with crepant log structures (of dimension ≤ 2) as Kollár does [Kol13, §4.4-5].

(b) In characteristic zero, sources and springs may be used to obtain normality of minimal lc centers [Kol13, 4.20]. On threefolds in positive characteristic, I have only obtained normality up to universal homeomorphisms (Theorem 5.1.2.4): the last step of Kollár's proof dramatically fails in positive characteristic (see [Kol13, 10.26-31]). Nonetheless we expect normality of minimal lc centers in sufficiently large characteristic (see [HX15, Theorem 3.11, Proposition 4.1]).

(c) I wonder whether the commutativity result Theorem 6.1.0.4 is true in higher dimensions, and locally (that is, when $\mathcal{S}$ is not proper over $\operatorname{Spec} R$).

- The proof in relative dimension two relies on the disjointness property proved in Claim 6.4.1.13. In higher dimension this property usually fails (there are already counter-examples for products), and so it is not clear that we can construct a correction term as in Claim 6.4.1.22.
- The local statement can be reduced to the global statement if we have existence of

log canonical closures as in [HX13]. Such a result might be reachable with the MMP tools developed in [BMP+21, TY21].

8.2 Further developments

Finally, let me indicate some questions and ideas that grew out of this thesis.

(a) Moduli theory of stable varieties in characteristic zero makes important use of semi-stable reduction [KKMSD73]. This is not known in positive characteristic, and thus it is essential to test it on a number of examples, in particular those which are special to positive characteristic (infinitesimal quotients, varieties with non-reduced automorphism group, etc.).

(b) In characteristic zero, some foundational aspects of moduli theory of pairs have only been worked out recently [Kol19]. These questions are still open in positive characteristic: thus it is interesting to compare the competing definitions of families of surface pairs, in particular over non-reduced bases.

(c) In characteristic zero, we have a complete picture of deformations of slc surface singularities (see [Kol21, §2.2]). Difficulties arise in positive characteristic, due to inseparability phenomenons. I hope that the tools developed in this thesis (for example Theorem 6.1.0.4) can be useful in this context.

(d) Applications of Kollár's theory of sources and springs to gluing statements rely heavily on finiteness of pluricanonical representations [Kol13, §10.5]. In characteristic zero, the proof takes a detour through topology. It is interesting – while not at all obvious – to find a similar approach in positive characteristic, using some cohomology theory.

(e) In characteristic zero, some topological properties of lc centers are captured by the Du Bois property [Kol13, §6]. It would be worth – but once again not obvious at all – to find similar common cohomological properties of lc centers in positive characteristic.

Chapter 9

Bibliography

[ABHLX20] Jarod Alper, Harold Blum, Daniel Halpern-Leistner, and Chenyang Xu. Reductivity of the automorphism group of K-polystable Fano varieties. *Invent. Math.*, 222(3):995–1032, 2020.

[ABL20] Emelie Arvidsson, Fabio Bernasconi, and Justin Lacini. On the Kawamata–Viewheg vanishing for log del Pezzo surfaces in positive characteristic. *ArXiv e-print, arXiv:2002.00679v1*, 2020.

[ADL21a] Kenneth Ascher, Kristin DeVleming, and Yuchen Liu. K-moduli of curves on a quadric surface and K3 surfaces. 2021. ArXiv eprint, arXiv:2006.06816v2. To appear in J. Inst. Math. Jussieu.

[ADL21b] Kenneth Ascher, Kristin DeVleming, and Yuchen Liu. K-stability and birational models of moduli of quartic K3 surfaces. 2021. ArXiv eprint, arXiv:2108.06848v1.

[AH11] Dan Abramovich and Brendan Hassett. Stable varieties with a twist. In *Classification of algebraic varieties*, EMS Ser. Congr. Rep., pages 1–38. Eur. Math. Soc., Zürich, 2011.

[AHLH19] Jarod Alper, Daniel Halpern-Leistern, and Jochen Heinloth. Existence of moduli spaces for algebraic stacks. 2019. ArXiv e-print, arXiv:1812.01128v3.

[Ale94] Valery Alexeev. Boundedness and K^2 for log surfaces. *Internat. J. Math.*, 5(6):779–810, 1994.

[Alp13] Jarod Alper. Good moduli spaces for Artin stacks. *Ann. Inst. Fourier (Grenoble)*, 63(6):2349–2402, 2013.

[Art75] M. Artin. Wildly ramified $Z/2$ actions in dimension two. *Proc. Amer. Math. Soc.*, 52:60–64, 1975.

[AT51] Emil Artin and John T. Tate. A note on finite ring extensions. *J. Math. Soc. Japan*, 3:74–77, 1951.

[BCHM10] Caucher Birkar, Paolo Cascini, Christopher D. Hacon, and James McKernan. Existence of minimal models for varieties of log general type. *J. Amer. Math. Soc.*, 23(2):405–468, 2010.

[Ben20] Olivier Benoist. Réduction stable en dimension supérieure [d'après Kollár, Hacon-Xu, ...]. *Astérisque*, (422, Séminaire Bourbaki. Vol. 2018/2019. Exposés 1151–1165):291–326, 2020.

[Ber14] Jeremy Berquist. Embeddings of Demi-Normal Varieties. *ArXiv e-print, arXiv:1411.2264v1*, 2014.

[Ber16] Robert J. Berman. K-polystability of $\mathbb{Q}$-Fano varieties admitting Kähler-Einstein metrics. *Invent. Math.*, 203(3):973–1025, 2016.

[Ber19] Fabio Bernasconi. Non-normal purely log terminal centres in characteristic $p \geqslant 3$. *Eur. J. Math.*, 5(4):1242–1251, 2019.

[Bir19] Caucher Birkar. Anti-pluricanonical systems on Fano varieties. *Ann. of Math. (2)*, 190(2):345–463, 2019.

[BJ20] Harold Blum and Mattias Jonsson. Thresholds, valuations, and K-stability. *Adv. Math.*, 365:107062, 57, 2020.

[BK21] Fabio Bernasconi and János Kollár. Vanishing theorems for threefolds in characteristic $p > 5$. *ArXiv e-print, arXiv:2012.08343v2*, 2021.

[BL18] Harold Blum and Yuchen Liu. Openess of uniform k-stability in families of q-fano varieties. *ArXiv e-print, arXiv:1808.09070v2. To appear in Ann. Sci. Ec. Norm. Supér.*, 2018.

[BLX19] Harold Blum, Yuchen Liu, and Chenyang Xu. Openess of K-semistability for Fano varieties. *ArXiv e-print, arXiv:1907.02408v2*, 2019.

[BMP+21] Bhargav Bhatt, Linquan Ma, Zsolt Patakfalvi, Karl Schwede, Kevin Tucker, Joe Waldron, and Jakub Witaszek. Globally +-regular varieties and the minimal model program for threefolds in mixed characteristic. *ArXiv e-print, arXiv:2012.15801v2*, 2021.

[Bou07] N. Bourbaki. *Éléments de mathématique. Algèbre commutative. Chapitre 10.* Springer-Verlag, Berlin, 2007. Reprint of the 1998 original.

[B01] Lucian Bădescu. *Algebraic surfaces.* Universitext. Springer-Verlag, New York, 2001. Translated from the 1981 Romanian original by Vladimir Maşek and revised by the author.

[BX19] Harold Blum and Chenyang Xu. Uniqueness of K-polystable degenerations of Fano varieties. *Ann. of Math. (2)*, 190(2):609–656, 2019.

[CP08] Vincent Cossart and Olivier Piltant. Resolution of singularities of threefolds in positive characteristic. I. Reduction to local uniformization on Artin-Schreier and purely inseparable coverings. *J. Algebra*, 320(3):1051–1082, 2008.

[CP21] Giulio Codogni and Zsolt Patakfalvi. Positivity of the CM line bundle for families of K-stable klt Fano varieties. *Invent. Math.*, 223(3):811–894, 2021.

[CT19] Paolo Cascini and Hiromu Tanaka. Purely log terminal threefolds with non-normal centres in characteristic two. *Amer. J. Math.*, 141(4):941–979, 2019.

[CT20] Paolo Cascini and Hiromu Tanaka. Relative semi-ampleness in positive characteristic. *Proc. Lond. Math. Soc. (3)*, 121(3):617–655, 2020.

[Deb01] Olivier Debarre. *Higher-dimensional algebraic geometry.* Universitext. Springer-Verlag, New York, 2001.

[DH16] Omprokash Das and Christopher D. Hacon. On the adjunction formula for 3-folds in characteristic $p > 5$. *Math. Z.*, 284(1-2):255–269, 2016.

[DM69] P. Deligne and D. Mumford. The irreducibility of the space of curves of given genus. *Inst. Hautes Études Sci. Publ. Math.*, (36):75–109, 1969.

[Don02] S. K. Donaldson. Scalar curvature and stability of toric varieties. *J. Differential Geom.*, 62(2):289–349, 2002.

[DW19] Omprokash Das and Joe Waldron. On the log minimal model program for 3-folds over imperfect fields. *ArXiv e-print, arXiv:1911.04394v2*, 2019.

[FG14] Osamu Fujino and Yoshinori Gongyo. Log pluricanonical representations and the abundance conjecture. *Compos. Math.*, 150(4):593–620, 2014.

[Fli92] *Flips and abundance for algebraic threefolds.* Société Mathématique de France, Paris, 1992. Papers from the Second Summer Seminar on Algebraic Geometry held at the University of Utah, Salt Lake City, Utah, August 1991, Astérisque No. 211 (1992) (1992).

[FO18] Kento Fujita and Yuji Odaka. On the K-stability of Fano varieties and anticanonical divisors. *Tohoku Math. J. (2)*, 70(4):511–521, 2018.

[Fuj99] Osamu Fujino. On abundance theorem for semi log canonical threefolds. *Proc. Japan Acad. Ser. A Math. Sci.*, 75(6):80–84, 1999.

[Fuj00] Osamu Fujino. Abundance theorem for semi log canonical threefolds. *Duke Math. J.*, 102(3):513–532, 2000.

[Fuj12] Osamu Fujino. Minimal model theory for log surfaces. *Publ. Res. Inst. Math. Sci.*, 48(2):339–371, 2012.

[Fuj14] Osamu Fujino. Fundamental theorems for semi log canonical pairs. *Algebr. Geom.*, 1(2):194–228, 2014.

[Fuj18] Osamu Fujino. Semipositivity theorems for moduli problems. *Ann. of Math. (2)*, 187(3):639–665, 2018.

[Fuj19] Kento Fujita. A valuative criterion for uniform K-stability of $\mathbb{Q}$-Fano varieties. *J. Reine Angew. Math.*, 751:309–338, 2019.

[Gie77a] D. Gieseker. Global moduli for surfaces of general type. *Invent. Math.*, 43(3):233–282, 1977.

[Gie77b] D. Gieseker. On the moduli of vector bundles on an algebraic surface. *Ann. of Math. (2)*, 106(1):45–60, 1977.

[GM78] Silvio Greco and Maria Grazia Marinari. Nagata's criterion and openness of loci for Gorenstein and complete intersection. *Math. Z.*, 160(3):207–216, 1978.

[GNT19] Yoshinori Gongyo, Yusuke Nakamura, and Hiromu Tanaka. Rational points on log Fano threefolds over a finite field. *J. Eur. Math. Soc. (JEMS)*, 21(12):3759–3795, 2019.

[Gro60] A. Grothendieck. Éléments de géométrie algébrique. I. Le langage des schémas. *Inst. Hautes Études Sci. Publ. Math.*, (4):228, 1960.

[Gro65] A. Grothendieck. Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas. II. *Inst. Hautes Études Sci. Publ. Math.*, (24):231, 1965.

[Gro66] A. Grothendieck. Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas. III. *Inst. Hautes Études Sci. Publ. Math.*, (28):255, 1966.

[GT80] S. Greco and C. Traverso. On seminormal schemes. *Compositio Math.*, 40(3):325–365, 1980.

[GW20] Ulrich Görtz and Torsten Wedhorn. *Algebraic geometry I. Schemes—with examples and exercises.* Springer Studium Mathematik—Master. Springer Spektrum, Wiesbaden, [2020] ©2020. Second edition [of 2675155].

[Har77] Robin Hartshorne. *Algebraic geometry.* Springer-Verlag, New York-Heidelberg, 1977. Graduate Texts in Mathematics, No. 52.

[Har07] Robin Hartshorne. Generalized divisors and biliaison. *Illinois J. Math.*, 51(1):83–98, 2007.

[Har10] Robin Hartshorne. *Deformation theory*, volume 257 of *Graduate Texts in Mathematics.* Springer, New York, 2010.

[Has03] Brendan Hassett. Moduli spaces of weighted pointed stable curves. *Adv. Math.*, 173(2):316–352, 2003.

[HH09] Brendan Hassett and Donghoon Hyeon. Log canonical models for the moduli space of curves: the first divisorial contraction. *Trans. Amer. Math. Soc.*, 361(8):4471–4489, 2009.

[HH13] Brendan Hassett and Donghoon Hyeon. Log minimal model program for the moduli space of stable curves: the first flip. *Ann. of Math. (2)*, 177(3):911–968, 2013.

[HK04] Brendan Hassett and Sándor J. Kovács. Reflexive pull-backs and base extension. *J. Algebraic Geom.*, 13(2):233–247, 2004.

[HM98] Joe Harris and Ian Morrison. *Moduli of curves*, volume 187 of *Graduate Texts in Mathematics.* Springer-Verlag, New York, 1998.

[HMX14] Christopher D. Hacon, James McKernan, and Chenyang Xu. ACC for log canonical thresholds. *Ann. of Math. (2)*, 180(2):523–571, 2014.

[HMX18] Christopher D. Hacon, James McKernan, and Chenyang Xu. Boundedness of moduli of varieties of general type. *J. Eur. Math. Soc. (JEMS)*, 20(4):865–901, 2018.

[HNT20] Kenta Hashizume, Yusuke Nakamura, and Hiromu Tanaka. Minimal model program for log canonical threefolds in positive characteristic. *Math. Res. Lett.*, 27(4):1003–1054, 2020.

[HW19] Christopher Hacon and Jakub Witaszek. On the relative Minimal Model Program for threefolds in low characteristics. *ArXiv e-print, arXiv:1909.12872v2*, 2019.

[HX13] Christopher D. Hacon and Chenyang Xu. Existence of log canonical closures. *Invent. Math.*, 192(1):161–195, 2013.

[HX15] Christopher D. Hacon and Chenyang Xu. On the three dimensional minimal model program in positive characteristic. *J. Amer. Math. Soc.*, 28(3):711–744, 2015.

[HX16] Christopher D. Hacon and Chenyang Xu. On finiteness of B-representations and semi-log canonical abundance. In *Minimal models and extremal rays (Kyoto, 2011)*, volume 70 of *Adv. Stud. Pure Math.*, pages 361–377. Math. Soc. Japan, [Tokyo], 2016.

[ILL+07] Srikanth B. Iyengar, Graham J. Leuschke, Anton Leykin, Claudia Miller, Ezra Miller, Anurag K. Singh, and Uli Walther. *Twenty-four hours of local cohomology*, volume 87 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2007.

[Jia20] Chen Jiang. Boundedness of $\mathbb{Q}$-Fano varieties with degrees and alpha-invariants bounded from below. *Ann. Sci. Éc. Norm. Supér. (4)*, 53(5):1235–1248, 2020.

[Kar00] Kalle Karu. Minimal models and boundedness of stable varieties. *J. Algebraic Geom.*, 9(1):93–109, 2000.

[Kee99] Seán Keel. Basepoint freeness for nef and big line bundles in positive characteristic. *Ann. of Math. (2)*, 149(1):253–286, 1999.

[KK] János Kollár and Sándor Kovács. Birational geometry of log surfaces. *Available at https://sites.math.washington.edu/~kovacs/pdf/BiratLogSurf.pdf*.

[KKMSD73] G. Kempf, Finn Faye Knudsen, D. Mumford, and B. Saint-Donat. *Toroidal embeddings. I.* Lecture Notes in Mathematics, Vol. 339. Springer-Verlag, Berlin-New York, 1973.

[KM76] Finn Faye Knudsen and David Mumford. The projectivity of the moduli space of stable curves. I. Preliminaries on "det" and "Div". *Math. Scand.*, 39(1):19–55, 1976.

[KM98] János Kollár and Shigefumi Mori. *Birational geometry of algebraic varieties*, volume 134 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 1998. With the collaboration of C. H. Clemens and A. Corti, Translated from the 1998 Japanese original.

[Kol90] János Kollár. Projectivity of complete moduli. *J. Differential Geom.*, 32(1):235–268, 1990.

[Kol96] János Kollár. *Rational curves on algebraic varieties*, volume 32 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, 1996.

[Kol11] János Kollár. Two examples of surfaces with normal crossing singularities. *Sci. China Math.*, 54(8):1707–1712, 2011.

[Kol12] János Kollár. Quotients by finite equivalence relations. In *Current developments in algebraic geometry*, volume 59 of *Math. Sci. Res. Inst. Publ.*, pages 227–256. Cambridge Univ. Press, Cambridge, 2012. With an appendix by Claudiu Raicu.

[Kol13] János Kollár. *Singularities of the Minimal Model Program*, volume 200 of *Cambridge Tracts in Mathematics*. 2013.

[Kol16] János Kollár. Sources of log canonical centers. In *Minimal models and extremal rays (Kyoto, 2011)*, volume 70 of *Adv. Stud. Pure Math.*, pages 29–48. Math. Soc. Japan, [Tokyo], 2016.

[Kol19] János Kollár. Families of divisors. *ArXiv e-print, arXiv:1910.00937v1*, 2019.

[Kol21] János Kollár. Families of varieties of general type. *Available at https://web.math.princeton.edu/~kollar/FromMyHomePage/modbook.pdf*, 2021. With the collaboration of K. Altmann and S. Kovács.

[KP17] Sándor J. Kovács and Zsolt Patakfalvi. Projectivity of the moduli space of stable log-varieties and subadditivity of log-Kodaira dimension. *J. Amer. Math. Soc.*, 30(4):959–1021, 2017.

[KSB88] J. Kollár and N. I. Shepherd-Barron. Threefolds and deformations of surface singularities. *Invent. Math.*, 91(2):299–338, 1988.

[Laz04a] Robert Lazarsfeld. *Positivity in algebraic geometry. I*, volume 48 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, 2004. Classical setting: line bundles and linear series.

[Laz04b] Robert Lazarsfeld. *Positivity in algebraic geometry. II*, volume 49 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, 2004. Positivity for vector bundles, and multiplier ideals.

[Lip69] Joseph Lipman. Rational singularities, with applications to algebraic surfaces and unique factorization. *Inst. Hautes Études Sci. Publ. Math.*, (36):195–279, 1969.

[Liu02] Qing Liu. *Algebraic geometry and arithmetic curves*, volume 6 of *Oxford Graduate Texts in Mathematics*. Oxford University Press, Oxford, 2002. Translated from the French by Reinie Erné, Oxford Science Publications.

[Liu22] Yuchen Liu. K-stability of cubic fourfolds. 2022. ArXiv eprint, arXiv:2007.14320v2. To appear in J. Reine Angew. Math.

[LMB00] Gérard Laumon and Laurent Moret-Bailly. *Champs algébriques*, volume 39 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, 2000.

[LV81] John V. Leahy and Marie A. Vitulli. Seminormal rings and weakly normal varieties. *Nagoya Math. J.*, 82:27–56, 1981.

[LWX18] Chi Li, Xiaowei Wang, and Chenyang Xu. Quasi-projectivity of the moduli space of smooth Kähler-Einstein Fano manifolds. *Ann. Sci. Éc. Norm. Supér. (4)*, 51(3):739–772, 2018.

[LX14] Chi Li and Chenyang Xu. Special test configuration and K-stability of Fano varieties. *Ann. of Math. (2)*, 180(1):197–232, 2014.

[LX19] Yuchen Liu and Chenyang Xu. K-stability of cubic threefolds. *Duke Math. J.*, 168(11):2029–2073, 2019.

[LXZ21] Yuchen Liu, Chenyang Xu, and Ziquan Zhuang. Finite generation for valuations computing stability thresholds and applications to K-stability. 2021. ArXiv e-print, arXiv:2102.09405v2.

[Mat89] Hideyuki Matsumura. *Commutative ring theory*, volume 8 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, second edition, 1989. Translated from the Japanese by M. Reid.

[Mat02] Kenji Matsuki. *Introduction to the Mori program*. Universitext. Springer-Verlag, New York, 2002.

[MFK94] D. Mumford, J. Fogarty, and F. Kirwan. *Geometric invariant theory*, volume 34 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (2) [Results in Mathematics and Related Areas (2)]*. Springer-Verlag, Berlin, third edition, 1994.

[Mum65] David Mumford. *Geometric invariant theory*. Ergebnisse der Mathematik und ihrer Grenzgebiete, (N.F.), Band 34. Springer-Verlag, Berlin-New York, 1965.

[Mum69] David Mumford. Enriques' classification of surfaces in char p. I. In *Global Analysis (Papers in Honor of K. Kodaira)*, pages 325–339. Univ. Tokyo Press, Tokyo, 1969.

[NS64] M. S. Narasimhan and C. S. Seshadri. Stable bundles and unitary bundles on a compact Riemann surface. *Proc. Nat. Acad. Sci. U.S.A.*, 52:207–211, 1964.

[NT20] Yusuke Nakamura and Hiromu Tanaka. A Witt Nadel vanishing theorem for threefolds. *Compos. Math.*, 156(3):435–475, 2020.

[Oda13] Yuji Odaka. The GIT stability of polarized varieties via discrepancy. *Ann. of Math. (2)*, 177(2):645–661, 2013.

[OSS16] Yuji Odaka, Cristiano Spotti, and Song Sun. Compact moduli spaces of del Pezzo surfaces and Kähler-Einstein metrics. *J. Differential Geom.*, 102(1):127–172, 2016.

[OSY12] Hajime Ono, Yuji Sano, and Naoto Yotsutani. An example of an asymptotically Chow unstable manifold with constant scalar curvature. *Ann. Inst. Fourier (Grenoble)*, 62(4):1265–1287, 2012.

[Pat15] Zsolt Patakfalvi. Semi-negativity of Hodge bundles associated to du bois families. *J. Pure Appl. Algebra*, 219:5387–5393, 2015.

[Pat17] Zsolt Patakfalvi. On the projectivity of the moduli space of stable surfaces in char > 5. *ArXiv e-print, arXiv:1710.03818v3*, 2017.

[Pos21a] Quentin Posva. Abundance for slc surfaces over arbitrary fields. *ArXiv e-print in preparation*, 2021.

[Pos21b] Quentin Posva. Gluing for stable families of surfaces in mixed characteristic. *ArXiv e-print, arXiv:2110.09845v1*, 2021.

[Pos21c] Quentin Posva. Gluing theory for slc surfaces and threefolds in positive characteristic. *ArXiv e-print, arXiv:2110.09387*, 2021.

[Pos22] Quentin Posva. Positivity of the CM line bundle for K-stable log Fanos. *Trans. of the AMS*, 375(7):4943–4978, 2022.

[PW18] Zsolt Patakfalvi and Joe Waldron. Singularities of General Fibers and the LMMP. *ArXiv e-print, arXiv:1708.04268v2*, 2018.

[PX17] Zsolt Patakfalvi and Chenyang Xu. Ampleness of the CM line bundle on the moduli space of canonically polarized varieties. *Algebr. Geom.*, 4(1):29–39, 2017.

[PZ20] Zsolt Patakfalvi and Maciej Zdanowicz. On the Beauville–Bogomolov decomposition in characteristic $p \geq 0$. *ArXiv e-print, arXiv:1912.12742v2.*, 2020.

[Sak84] Fumio Sakai. Weil divisors on normal surfaces. *Duke Math. J.*, 51(4):877–887, 1984.

[Sch12] Georg Schumacher. Positivity of relative canonical bundles and applications. *Invent. Math.*, 190(1):1–56, 2012.

[Sei50] A. Seidenberg. The hyperplane sections of normal varieties. *Trans. Amer. Math. Soc.*, 69:357–386, 1950.

[SGA71] *Théorie des intersections et théorème de Riemann-Roch.* Lecture Notes in Mathematics, Vol. 225. Springer-Verlag, Berlin-New York, 1971. Séminaire de Géométrie Algébrique du Bois-Marie 1966–1967 (SGA 6), Dirigé par P. Berthelot, A. Grothendieck et L. Illusie. Avec la collaboration de D. Ferrand, J. P. Jouanolou, O. Jussila, S. Kleiman, M. Raynaud et J. P. Serre.

[Sha81] Jayant Shah. Stability of two-dimensional local rings. I. *Invent. Math.*, 64(2):297–343, 1981.

[Sil09] Joseph H. Silverman. *The arithmetic of elliptic curves*, volume 106 of *Graduate Texts in Mathematics.* Springer, Dordrecht, second edition, 2009.

[Sta] The Stacks Project. `https://stacks.math.columbia.edu`.

[Tan14] Hiromu Tanaka. Minimal models and abundance for positive characteristic log surfaces. *Nagoya Math. J.*, 216:1–70, 2014.

[Tan16] Hiromu Tanaka. Abundance theorem for semi log canonical surfaces in positive characteristic. *Osaka J. Math.*, 53(2):535–566, 2016.

[Tan18] Hiromu Tanaka. Minimal model program for excellent surfaces. *Ann. Inst. Fourier (Grenoble)*, 68(1):345–376, 2018.

[Tan20a] Hiromu Tanaka. Abundance theorem for surfaces over imperfect fields. *Math. Z.*, 295(1-2):595–622, 2020.

[Tan20b] Hiromu Tanaka. Invariants of algebraic varieties over imperfect fields. *ArXiv e-print, arXiv:1903.10113v3*, 2020.

[Tia97] Gang Tian. Kähler-Einstein metrics with positive scalar curvature. *Invent. Math.*, 130(1):1–37, 1997.

[Tra70] Carlo Traverso. Seminormality and Picard group. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (3)*, 24:585–595, 1970.

[TY21] Teppei Takamatsu and Shou Yoshikawa. Minimal model program for semi-stable threefolds in mixed characteristic. *ArXiv e-print, arXiv:2012.07324v2*, 2021.

[Vie83] Eckart Viehweg. Weak positivity and the additivity of the Kodaira dimension for certain fibre spaces. In *Algebraic varieties and analytic varieties (Tokyo, 1981)*, volume 1 of *Adv. Stud. Pure Math.*, pages 329–353. North-Holland, Amsterdam, 1983.

[Vie95] Eckart Viehweg. *Quasi-projective moduli for polarized manifolds*, volume 30 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)].* Springer-Verlag, Berlin, 1995.

[vS87] Duco van Straten. *Weakly normal surface singularities and their improvements.* PhD thesis, Leiden, 1987.

[Wal18] Joe Waldron. The LMMP for log canonical 3-folds in characteristic $p > 5$. *Nagoya Math. J.*, 230:48–71, 2018.

[Wit20] Jakub Witaszek. Keel's base point free theorem and quotients in mixed characteristic. *ArXiv e-print, arXiv:2002.11915v1*, 2020.

[Wit21] Jakub Witaszek. Relative semiampleness in mixed characteristic. *ArXiv e-print, arXiv:2106.06088v1*, 2021.

[WX14] Xiaowei Wang and Chenyang Xu. Nonexistence of asymptotic GIT compactification. *Duke Math. J.*, 163(12):2217–2241, 2014.

[Xu20] Chenyang Xu. A minimizing valuation is quasi-monomial. *Ann. of Math. (2)*, 191(3):1003–1030, 2020.

[Xu21] Chenyang Xu. K-stability of Fano varieties: an algebro-geometric approach. *EMS Surv. Math. Sci.*, 8(1-2):265–354, 2021.

[XZ20] Chenyang Xu and Ziquan Zhuang. On positivity of the CM line bundle on K-moduli spaces. *Ann. of Math. (2)*, 192(3):1005–1068, 2020.

[Yan83] Hiroshi Yanagihara. Some results on weakly normal ring extensions. *J. Math. Soc. Japan*, 35(4):649–661, 1983.

Chapter 10

Curriculum Vitae

Informations personnelles

nom	Quentin Arthur Frantisek Posva
date de naissance	7 septembre 1994
nationalité	Suisse
email	quentin.posva@epfl.ch ou q.posva@bluewin.ch

Education

	Ecole Polytechnique Fédérale de Lausanne (EPFL)
02.2018 – 07.2022	Doctorat en mathématiques
	Superviseur : Zsolt Patakfalvi
	Laboratoire : Chair of Algebraic Geometry
09.2016 – 02.2018	Master en mathématiques
09.2013 – 06.2016	Bachelor en matématiques
	09.2015 – 06.2016: Echange à l'Université de Waterloo (Canada)