COMPOSITIO MATHEMATICA

On the Kawamata–Viehweg vanishing theorem for log del Pezzo surfaces in positive characteristic

Emelie Arvidsson, Fabio Bernasconi and Justin Lacini

FOUNDATION
COMPOSITIO
MATHEMATICA

LONDON
MATHEMATICAL
SOCIETY
EST. 1865

On the Kawamata–Viehweg vanishing theorem for log del Pezzo surfaces in positive characteristic

Emelie Arvidsson, Fabio Bernasconi and Justin Lacini

Abstract

We prove the Kawamata–Viehweg vanishing theorem for surfaces of del Pezzo type over perfect fields of positive characteristic $p > 5$. As a consequence, we show that klt threefold singularities over a perfect base field of characteristic $p > 5$ are rational. We show that these theorems are sharp by providing counterexamples in characteristic 5.

1. Introduction

Fano varieties and their log generalizations are one of the basic building blocks of algebraic varieties according to the Minimal Model Program (MMP). In recent years a lot of progress has been made in the study of Fano varieties in characteristic 0, notably [Bir19, Bir21]. Less is known about Fano varieties in positive characteristic. It is conjectured that Kodaira type vanishing theorems, which in general fail in positive characteristic, hold for Fano varieties of a given dimension in large enough characteristic [Tot19, Open questions]. This conjecture has been confirmed in dimension 2 by [CTW17], where the authors prove the existence of an integer $p_0 > 0$ such that surfaces of del Pezzo type over perfect fields of characteristic $p > p_0$ satisfy the Kawamata–Viehweg vanishing theorem. However, an explicit value for p_0 cannot be obtained via the methods of [CTW17], and in this paper we show that $p_0 = 5$ is both necessary and sufficient.

THEOREM 1.1 (see Theorem 4.1). *Let X be a surface of del Pezzo type over a perfect field k of characteristic $p > 5$. Let D be a Weil divisor on X and suppose that there exists an effective $\mathbb{Q}$-divisor Δ such that (X, Δ) is a Kawamata log terminal (klt) pair and $D - (K_X + \Delta)$ is big and nef. Then*

$$H^i(X, \mathcal{O}_X(D)) = 0 \text{ for all } i > 0.$$

The crucial ingredient in the proof of Theorem 1.1 is establishing the following liftability theorem for klt surfaces whose canonical divisor is not pseudo-effective.

THEOREM 1.2 (see Theorem 3.4). *Let k be an algebraically closed field of characteristic $p > 5$. Let X be a klt projective surface over k such that K_X is not pseudo-effective. Then there exists a log resolution $\mu \colon V \to X$ such that $(V, \mathrm{Exc}(\mu))$ lifts to characteristic 0 over a smooth base.*

In recent years, it has become clear that the understanding of Fano varieties is crucial for the study of klt singularities, in particular in positive characteristics. In characteristic 0, Elkik

proved that klt singularities are always rational (see [Elk81]). This, however, fails in positive characteristic (see [CT19, Ber21a, Tot19, Yas19]), and the question is intertwined with the failure of vanishing theorems on Fano varieties. In [HW19a] the authors prove that klt threefold singularities are rational in large characteristic $p > p_0$ where p_0 is large enough to assure the validity of the Kawamata–Viehweg vanishing theorem for surfaces of del Pezzo type. As a consequence of their work and Theorem 1.1, we can therefore provide an effective bound on the characteristic p for which klt threefold singularities are rational.

COROLLARY 1.3 (see [HW19a, Theorem 1.1]). *Let k be a perfect field of characteristic $p > 5$. Let (X, Δ) be a klt threefold over k. Then X has rational singularities. Moreover, if X is $\mathbb{Q}$-factorial and D is a Weil divisor on X then $\mathcal{O}_X(D)$ is Cohen–Macaulay.*

On the other hand, Theorem 1.1 and Corollary 1.3 have several applications to the birational geometry of threefolds. We mention two: the finiteness of the local étale fundamental group of threefold klt singularities (see [CSK20, Theorems A, B and C]) and a refined version of the base-point-free theorem for klt threefolds (see [Ber21b]). The corollaries below were previously known to hold in large characteristic; we now know them to be true in characteristic $p > 5$.

COROLLARY 1.4 (see [CSK20, Theorems A, B]). *Let k be an algebraically closed field of characteristic $p > 5$. Let (X, Δ) be a klt threefold over k. Let $x \in X$ be a closed point. Then the local étale fundamental group of X at x is finite. Moreover, there exists a quasi-étale cover $\widetilde{X} \to X$ with prime-to-p degree such that every quasi-étale cover of $\widetilde{X}$ is étale.*

COROLLARY 1.5 (see [Ber21b, Theorem 1.1]). *Let k be a perfect field of characteristic $p > 5$. Let (X, Δ) be a quasi-projective klt threefold log pair and let $\pi \colon X \to Z$ be a projective contraction morphism of quasi-projective normal varieties over k. Let L be a π-nef Cartier divisor on X such that:*

(i) *$\dim(Z) \geq 1$ or $\dim(Z) = 0$ and $\nu(L) \geq 1$;*
(ii) *$nL - (K_X + \Delta)$ is a π-big and π-nef $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor for some $n > 0$.*

Then there exists $m_0 > 0$ such that mL is π-free for all $m \geq m_0$.

In §5 we construct counterexamples in characteristic 5 to Theorem 1.1 and Corollary 1.3 building upon a previous example found by the third author.

THEOREM 1.6 (see Proposition 5.2 and Theorem 5.4). *Let k be an algebraically closed field of characteristic $p = 5$. Then:*

(i) *there exist a klt del Pezzo surface T of Picard rank 1 and an ample $\mathbb{Q}$-Cartier divisor A such that $H^1(T, \mathcal{O}_T(-A)) \neq 0$;*
(ii) *there exists a $\mathbb{Q}$-factorial klt threefold singularity which is not Cohen–Macaulay.*

This, together with the examples found in [CT19] and [Ber21a], shows that our assumption on the characteristic in Theorem 1.1 and Corollary 1.3 is optimal.

1.1 Sketch of the proof

We now overview the main techniques and proofs in this paper. In §3 we prove Theorem 1.2. In order to do this, we first show that we can reduce to the case of such surfaces admitting a Mori fibre space structure (Lemma 2.9). We prove the case of Mori fibre space onto a curve (Proposition 3.2), while the case of klt del Pezzo surfaces of Picard rank 1 has been settled by the third author in his PhD thesis [Lac20].

In §4 we use Theorem 1.2 to prove Theorem 1.1, where the property of the surface being del Pezzo type plays a fundamental role. By running a Δ-MMP we construct a birational morphism $\pi\colon X \to Y$ which contracts a boundary $N \le \Delta$ such that $D - (K_X + N)$ is ample and such that the support of N coincides with the exceptional locus of π. Since Y remains a surface of del Pezzo type, we may apply Theorem 3.4 to construct a log resolution $(V, \mathrm{Exc}(\mu))$ of Y that lifts to characteristic 0 over a smooth base. Since N was contracted by π, after a sequence of blow-ups of points of V, we may find a log resolution of the pair (X, N) lifting to characteristic 0 over a smooth base. Then we can prove the desired vanishing theorem by applying the logarithmic Deligne–Illusie vanishing theorem (see Lemma 2.10).

2. Preliminaries

2.1 Notation

In this paper k denotes a perfect field of characteristic $p > 0$. For $n > 0$, we denote by $W_n(k)$ (respectively, $W(k)$) the ring of Witt vectors of length n (respectively, the ring of Witt vectors).

We say that X is a *variety over k* or a *k-variety* if X is an integral scheme that is separated and of finite type over k. We say that X is a *curve* over k or a *k-curve* (respectively, a *surface* over k or a *k-surface*, or a *threefold* over k) if X is a k-variety of dimension 1 (respectively, 2, or 3).

We say (X, Δ) is a *log pair* if X is normal variety, Δ is an effective $\mathbb{Q}$-divisor and $K_X + \Delta$ is $\mathbb{Q}$-Cartier. We say that a pair (X, Δ) is *log smooth* if X is regular and the support of Δ is simple normal crossing (snc). We refer to [Kol13] for the basic definitions in birational geometry and of the singularities appearing in the MMP (e.g. klt) and to [GNT19, Remark 2.7] for birational geometry over perfect fields. We say that a variety is $\mathbb{Q}$-factorial if every Weil divisor is $\mathbb{Q}$-Cartier. We recall that klt surface pairs are $\mathbb{Q}$-factorial (see, for example, [Tan15, Fact 3.4]), so we will often discuss Weil divisors on klt surface pairs without mentioning that they are $\mathbb{Q}$-Cartier.

Given a proper birational morphism $\pi\colon X \to Y$ between normal varieties, we denote by $\mathrm{Exc}(\pi)$ the exceptional locus of π. Given a variety X and a Weil divisor D, we say that $f\colon Y \to X$ is a *log resolution* of (X, D) if $\mathrm{Exc}(\pi)$ has pure codimension 1 and the pair $(Y, \mathrm{Exc}(\pi) + \pi_*^{-1}D)$ is log smooth.

2.2 Surfaces of del Pezzo type

In this subsection we collect some properties of surfaces of del Pezzo type.

Definition 2.1. Let k be a field. We say that a normal k-projective surface X is a *surface of del Pezzo type* if there exists an effective $\mathbb{Q}$-divisor Δ such that (X, Δ) is klt and $-(K_X + \Delta)$ is ample. The pair (X, Δ) is said to be a log del Pezzo pair. We say X is a *klt del Pezzo surface* if the pair $(X, 0)$ is log del Pezzo.

Surfaces of del Pezzo type behave nicely from the birational perspective, as the MMP algorithm can be run for any divisor on them and they form a closed class under birational contraction.

Lemma 2.2. *Let k be a perfect field. Let X be a surface of del Pezzo type over k. Then X is a Mori dream space, that is, given a $\mathbb{Q}$-divisor D on X we can run a D-MMP which terminates.*

Proof. Let Δ be an effective $\mathbb{Q}$-divisor on X such that (X, Δ) is a log del Pezzo pair. Choose a sufficiently small rational number $\varepsilon > 0$ such that $A = \varepsilon D - (K_X + \Delta)$ is ample. By [GNT19, Lemma 2.8], there exists an effective $\mathbb{Q}$-divisor A' such that $A' \sim_{\mathbb{Q}} A$ and $(X, \Delta + A')$ is a

klt pair. We conclude by noting that a D-MMP coincides with a $(K_X + \Delta + A')$-MMP, which exists and terminates by [Tan18, Theorem 1.1]. □

LEMMA 2.3. *Let k be a field. Let X a surface of del Pezzo type over k. Let $\pi \colon X \to Y$ be a birational k-morphism onto a normal surface Y. Then Y is a surface of del Pezzo type.*

Proof. See [BT22, Lemma 2.9]. □

2.3 Lifting to characteristic 0 over a smooth base

We now collect some notions on liftability we will use in this paper. In this subsection we fix k to be an algebraically closed field. We start by recalling the definition of liftability to characteristic 0 (see [CTW17, Definition 2.15]) and to the ring of Witt vectors.

DEFINITION 2.4. Let $(X, D = \sum_{i=1}^{r} D_i)$ be a log smooth pair over k. A *lifting* of (X, D) to a scheme T consists of

(i) a smooth and separated morphism $\mathcal{X}$ over T,
(ii) effective Cartier divisors $\mathcal{D}_i$ in $\mathcal{X}$ such that for every subset $J \subseteq \{1, \ldots, r\}$ the scheme-theoretic intersection $\bigcap_{j \in J} \mathcal{D}_j$ is smooth over T,
(iii) a morphism $\alpha \colon \mathrm{Spec}(k) \to T$,

such that the base change of the schemes $\mathcal{X}, \mathcal{D}_1, \ldots, \mathcal{D}_r$ via α is isomorphic to $X, D_1, \ldots, D_n$, respectively.

If T is smooth and separated over $\mathrm{Spec}(\mathbb{Z})$ (respectively, $T = \mathrm{Spec}(W(k))$), then we say that $(\mathcal{X}, \mathcal{D})$ is *a lifting of (X, D) to characteristic 0 over a smooth base* (respectively, *a lifting to the ring of Witt vectors*).

We relate the notions of liftability to characteristic 0 and liftability to the ring of Witt vectors.

PROPOSITION 2.5. *Let (X, D) be a log smooth pair over k. Then (X, D) admits a lifting to $W(k)$ if and only if (X, D) admits a lifting to characteristic 0 over a smooth base.*

Proof. Suppose that (X, D) admits a lifting $(\mathcal{X}, \mathcal{D})$ to characteristic 0 over a smooth base T. It is sufficient to show that $\alpha \colon \mathrm{Spec}(k) \to T$ lifts to a section $\widetilde{\alpha} \colon \mathrm{Spec}(W(k)) \to T$ since then the base change of $(\mathcal{X}, \mathcal{D})$ along $\widetilde{\alpha}$ is a lifting of (X, D) to $W(k)$. The base change $T \times_{\mathbb{Z}} \mathrm{Spec}(W(k)) \to \mathrm{Spec}(W(k))$ is a smooth morphism and $W(k)$ is a complete ring (and thus Henselian). Therefore the section $\alpha \colon \mathrm{Spec}(k) \to T$ lifts to a section $\mathrm{Spec}(W(k)) \to T$ (see [EGAIV$_4$, Théorème 18.5.17]).

Let $(\mathcal{X}, \mathcal{D})$ be a lifting of (X, D) to $W(k)$. As $\mathrm{Frac}(W(k))$ is a field of characteristic 0, $\mathrm{Spec}(\mathrm{Frac}(W(k))$ is geometrically regular at $\mathbb{Q}$ by [Sta, Tag 038U]. This means that $\mathrm{Spec}(W(k)) \to \mathrm{Spec}(\mathbb{Z})$ is flat with geometrically regular fibres and the morphism $\mathrm{Spec}(W(k)) \to \mathrm{Spec}(\mathbb{Z})$ is regular (see [Sta, Tag 07R7]). By a theorem of Popescu (see [CdeJ02, Theorem 1.3]), the ring $W(k)$ is therefore a direct limit of smooth $\mathbb{Z}$-algebras. By [EGAIV$_3$, Théorème 11.2.6] flatness is preserved under spreading out and thus there exists a smooth and separated $\mathbb{Z}$-scheme T such that $\mathcal{X}$ and all possible intersection of $\mathcal{D}_i$, for $1 \leq i \leq n$, are defined and smooth over T. Thus we conclude that (X, D) admits a lifting to characteristic 0 over a smooth base. □

Remark 2.6. As a consequence let us note that if (X, D) admits a lifting to characteristic 0 over a smooth base then it admits a lifting over $W_2(k)$ (cf. [CTW17, Remark 2.16]).

We are interested in understanding how liftability is preserved under blow-ups at smooth points. We therefore introduce the following terminology.

Definition 2.7. Let $(X, D = \sum_{i=1}^n D_i)$ be a log smooth surface pair over k. Let x be closed point of X and let $J \subseteq \{1, \ldots, n\}$ be the subset of maximal cardinality for which $x \in \bigcap_{j \in J} D_j$ (with the convention that $\bigcap_{j \in \emptyset} D_j = X$). Let $(\mathcal{X}, \mathcal{D} = \sum_{i=1}^n \mathcal{D}_i)$ be a lifting of (X, D) to a scheme T. A *lifting* $\overline{x} \subset \mathcal{X}$ of x is a subscheme of $\mathcal{X}$ smooth over T such that $\overline{x} \times_T \mathrm{Spec}(k) = x$. We say that a lifting $\overline{x}$ is *compatible with the snc structure* of $(\mathcal{X}, \mathcal{D})$ if $\overline{x} \subseteq \cap_{j \in J} \mathcal{D}_j$.

By possibly changing the base of the lifting, we may always find a lifting of a point to characteristic 0 compatible with the snc structure.

Lemma 2.8. *Let $(X, D = \sum_{i=1}^n D_i)$ be a log smooth surface pair over k. Let x be a closed point of X. Suppose that (X, D) lifts to characteristic 0 over a smooth base. Then there exists a lifting of (X, D) over a smooth base such that x lifts compatibly with the snc structure.*

Proof. Let $(\mathcal{X}, \mathcal{D})$ be a lifting of X over a $\mathbb{Z}$-smooth and separated scheme T. Then by Proposition 2.5 there exists a lifting $(\mathcal{X}, \mathcal{D})$ to $W(k)$. Let $J \subseteq \{1, \ldots, n\}$ be the subset of maximal cardinality for which $x \in \bigcap_{j \in J} D_j$. Consider the following diagram.

$$\begin{array}{ccccc} \bigcap_{j \in J} D_j & \longrightarrow & \bigcap_{j \in J} \mathcal{D}_j & \longrightarrow & \mathcal{X} \\ {\scriptstyle s}\uparrow\downarrow & & \downarrow & \swarrow & \\ \mathrm{Spec}(k) & \longrightarrow & \mathrm{Spec}(W(k)) & & \end{array}$$

Since $\bigcap_{j \in J} \mathcal{D}_j$ is smooth over $W(k)$, by [EGAIV$_4$, Theórème 18.5.17] there exists a lifting $\overline{s} \colon \mathrm{Spec}(W(k)) \to \bigcap_{j \in J} \mathcal{D}_j$. This shows the existence of a lifting of x compatible with the snc structure over $W(k)$. By using Popescu's theorem and a spreading out argument as in the proof of Proposition 2.5, we conclude the existence of a lifting of (X, D) to characteristic 0 over a smooth base such that x lifts compatibly with the snc structure. $\square$

We now show that liftability is preserved under blow-ups at smooth centres.

Proposition 2.9. *Let $(X, D = \sum_{i=1}^n D_i)$ be a log smooth surface over k which admits a lifting to characteristic 0 over a smooth base. Let x be a closed point of X and let $\pi \colon X' \to X$ be the blow-up at the point x. Then $(X', \pi_*^{-1} D \cup \mathrm{Exc}(\pi))$ admits a lifting to characteristic 0 over a smooth base.*

Proof. Let us choose $(\mathcal{X}, \mathcal{D} = \sum_{i=1}^n \mathcal{D}_i)$ a lifting of (X, D) over a smooth and separated scheme T over $\mathrm{Spec}(\mathbb{Z})$ such that x lifts compatibly with the snc structure, whose existence is guaranteed by Lemma 2.8. Let $\overline{\pi} \colon \mathcal{X}' \to \mathcal{X}$ be the blow-up along $\overline{x}$. Then $\mathcal{X}'$ is a lifting of X' because $\overline{x}$ is smooth and thus, by [Liu02, § 8, Theorem 1.19], $\mathcal{X}'$ is smooth over T and the exceptional divisor $\mathcal{E}$ is a $\mathbb{P}^1$-bundle over $\overline{x}$. We conclude that $(\mathcal{X}', \overline{\pi}_*^{-1} \mathcal{D} \cup \mathcal{E})$ is a lifting of $(X', \pi_*^{-1} D \cup \mathrm{Exc}(\pi))$ over T. $\square$

Apart from its intrinsic interest, the condition of liftability to characteristic 0 over a smooth base is important for its application to vanishing theorems (in fact, for our purposes liftability to $W_2(k)$ would be sufficient). We will apply the machinery developed by Deligne and Illusie (see [DI87] and [Hara98] for a logarithmic version) to prove vanishing theorems for liftable surfaces in positive characteristic, of which we recall the following special case.

Lemma 2.10 (cf. [CTW17, Lemma 6.1]). *Let k be a perfect field of characteristic $p > 2$. Let (X, Δ) be a two-dimensional projective klt pair over k. Suppose that there exists a log resolution $\mu \colon V \to X$ of (X, Δ) such that $(V, \mathrm{Exc}(\mu) \cup \mu_*^{-1}(\mathrm{Supp}\Delta))$ lifts to $W_2(k)$. Let D be a $\mathbb{Z}$-divisor on X such that $D - (K_X + \Delta)$ is ample. Then $H^i(X, \mathcal{O}_X(D)) = 0$ for any $i > 0$.*

3. Liftability of klt surfaces whose canonical divisor is not pseudo-effective

In this section we prove that the minimal resolution along with the exceptional locus of a projective klt surface whose canonical divisor K_X is not pseudo-effective admits a lifting to characteristic 0 over a smooth base. First we prove this for klt surfaces admitting a Mori fibre space structure onto a curve (see Proposition 3.2). We then use the classification of klt del Pezzo surfaces of Picard rank 1 to deduce the general result.

3.1 Liftability of Mori fibre spaces

We start by discussing liftability of klt Mori fibre spaces onto a curve.

LEMMA 3.1. *Let k be an algebraically closed field of characteristic $p > 0$. Let C be a projective smooth curve and let $\pi\colon X \to C$ be a minimal ruled surface over k. Let F_i be the fibres over closed points $x_i \in C$ for $1 \leq i \leq n$. Then $(X, \sum_{i=1}^n F_i)$ admits a lifting to $W(k)$.*

Proof. By [Hart77, V, Proposition 2.2] there exists a vector bundle E of rank 2 on C such that $\pi\colon X \to C$ is isomorphic to $p\colon \mathbb{P}_C(E) \to C$. The curve C admits a formal lifting $\mathcal{C}$ to $W(k)$ because the obstruction to the lifting lies in the $H^2(C, T_C)$ by [Ill71, Proposition 3.1.5, p. 248], which vanishes for dimension reasons. Analogously, the obstruction for lifting E to a vector bundle $\mathcal{E}$ over $\mathcal{C}$ lies in $H^2(C, \mathcal{E}nd(E))$ by [Ill71, Proposition 3.1.5, p. 248], which also vanishes. Thus we have a lifting $\mathbb{P}_\mathcal{C}(\mathcal{E}) \to \mathcal{C}$. Let $x_1, \ldots, x_n$ be the closed points on C such that $\pi^* x_i = F_i$. By formal smoothness, we can lift $x_1, \ldots, x_n$ to $\mathcal{C}$ and thus the fibres $F_1, \ldots, F_n$ admit a formal lifting $\mathfrak{F}_1, \ldots, \mathfrak{F}_n$ over $W(k)$. Therefore $(\mathbb{P}_\mathcal{C}(\mathcal{E}), \sum_{i=1}^n \mathfrak{F}_i)$ is a formal lifting of $(X, \sum_{i=1}^n F_i)$ to the formal spectrum $\mathrm{Spf}(W(k))$. Since $H^2(X, \mathcal{O}_X) = H^0(X, K_X) = 0$ by Serre duality, the pair is algebraizable by [FGI + 05, Corollary 8.5.6 and Corollary 8.4.5]. □

PROPOSITION 3.2. *Let k be an algebraically closed field of characteristic $p > 0$. Let $\pi\colon X \to C$ be a projective morphism such that:*

(i) *X is a projective surface over k with klt singularities;*
(ii) *C is a smooth projective curve;*
(iii) *$\pi_* \mathcal{O}_X = \mathcal{O}_C$;*
(iv) *$-K_X$ is π-ample.*

Let $\mu\colon V \to X$ be the minimal resolution. Then the pair $(V, \mathrm{Exc}(\mu))$ is log smooth and admits a lifting to characteristic 0 over a smooth base.

Proof. Let us note that the pair $(V, \mathrm{Exc}(\mu))$ is log smooth by the classification of klt surface singularities (see [Kol13, §3.3]). Let us denote by $\pi'\colon V \to C$ the induced fibration. We run a K_V-MMP over C. We have the following diagram.

$$\begin{array}{ccc} V & \xrightarrow{g} & Y \\ {\scriptstyle \pi'}\downarrow & \swarrow {\scriptstyle f} & \\ C & & \end{array}$$

Note that $f\colon Y \to C$ is a minimal ruled surface. Every irreducible component E of $\mathrm{Exc}(\mu)$ is contained in some fibre F_i of π'. Let $F_1, \ldots F_n$ be the minimal collection of distinct fibres such that $\mathrm{Supp}(\mathrm{Exc}(\mu)) \subset \mathrm{Supp}(\sum_{i=1}^n F_i)$. Let $x_i \in C$ be the images of F_i for $i = 1, \ldots, n$. Note that we have the inclusion $\mathrm{Supp}(g(F_i)) \subset \mathrm{Supp}(F_i^Y)$ where F_i^Y is the fibre over x_i of $Y \to C$. The pair $(Y, \sum_{i=1}^n F_i^Y)$ admits a lifting to $W(k)$ by Lemma 3.1 and admits a lifting to characteristic 0 over a smooth base by Proposition 2.5. Since $\sum_{i=1}^n F_i^Y$ is a reduced snc divisor and $g\colon V \to Y$

755

can be factored as a sequence of blow-ups of smooth points, the pair $(X, \operatorname{Exc}(g) + \sum_{i=1}^{n} g_*^{-1} F_i)$ admits a lifting to characteristic 0 over a smooth base by Proposition 2.9. $\qquad\square$

3.2 General case

The third author classified all klt del Pezzo surfaces of Picard rank 1 in characteristic $p > 3$ in his PhD thesis [Lac20]. As a consequence of the classification he was able to prove the existence of a log resolution that lifts to characteristic 0 for klt del Pezzo surfaces of Picard rank 1 defined over algebraically closed fields of characteristic $p > 5$. This serves as the main ingredient for our results on liftability of surfaces of del Pezzo type.

THEOREM 3.3 (see [Lac20, Theorem 7.2]). *Let k be an algebraically closed field of characteristic $p > 5$. Let S be a klt del Pezzo surface of Picard rank 1 over k. Then there exists a log resolution $\mu\colon V \to S$ such that $(V, \operatorname{Exc}(\mu))$ lifts to characteristic 0 over a smooth base.*

With the result above, we are now ready to prove the main result of this section.

THEOREM 3.4. *Let k be an algebraically closed field of characteristic $p > 5$. Let X be a klt projective surface over k such that K_X is not pseudo-effective. Then there exists a log resolution $\mu\colon V \to X$ such that $(V, \operatorname{Exc}(\mu))$ lifts to characteristic 0 over a smooth base.*

Proof. Let $g\colon X \to Y$ be a K_X-MMP. There are two cases to distinguish.

(i) Y is a klt del Pezzo surface of Picard rank 1. Then Y admits a log resolution $\mu\colon V \to Y$ such that $(V, \operatorname{Exc}(\mu))$ lifts to characteristic 0 over a smooth base by Theorem 3.3.
(ii) Y is a surface admitting a Mori fibre space structure $\pi\colon Y \to C$ onto a curve C. Then Y admits a log resolution $\mu\colon V \to Y$ such that $(V, \operatorname{Exc}(\mu))$ lifts to characteristic 0 over a smooth base by Proposition 3.2.

In both cases, Y admits a log resolution $\mu\colon V \to Y$ such that $(V, \operatorname{Exc}(\mu))$ lifts to characteristic 0 over a smooth base. Consider the induced rational map $\mu'\colon V \dashrightarrow X$ and let $\pi\colon V' \to V$ be a log resolution of the indeterminacy locus of μ'. Since the morphism $\pi\colon V' \to V$ can be factored as a composition of blow-ups at points, the pair $(V', \pi_*^{-1}\operatorname{Exc}(\mu) + \operatorname{Exc}(\pi))$ lifts to characteristic 0 over a smooth base by Proposition 2.9. The commutative diagram below illustrates the situation.

$$\begin{array}{ccc} X & \xleftarrow{\tilde\mu} & (V', \pi_*^{-1}\operatorname{Exc}(\mu) + \operatorname{Exc}(\pi)) \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle \pi} \\ Y & \xleftarrow{\mu} & (V, \operatorname{Exc}(\mu)) \end{array}$$

We have that the support of $\operatorname{Exc}(\tilde\mu)$ is contained in the support of $\pi_*^{-1}\operatorname{Exc}(\mu) + \operatorname{Exc}(\pi)$. Therefore $\tilde\mu\colon (V', \operatorname{Exc}(\tilde\mu)) \to X$ is a log resolution that lifts to characteristic 0 over a smooth base. $\qquad\square$

Remark 3.5. One cannot drop the assumption on K_X being not pseudo-effective and extend the previous result to the class of birationally ruled surfaces with klt singularities. Indeed, for every prime $p > 0$, Cascini and Tanaka construct examples of klt rational surfaces Y of Picard rank 1 over $\mathbb{F}_p$ for which no log resolution along with the exceptional divisors admits a lifting to $W_2(k)$ (see [CT18, Corollary 3.3]). In their examples the canonical divisor K_Y is ample if $p > 2$.

As a direct application of the previous result, we can prove the Kodaira vanishing theorem for Weil divisors for the class of klt surfaces whose canonical divisor is not pseudo-effective.

COROLLARY 3.6. *Let k be a perfect field of characteristic $p > 5$. Let X be a projective surface over k with klt singularities such that K_X is not pseudo-effective. Then, for an ample Weil divisor D on X, we have $H^i(X, \mathcal{O}_X(K_X + D)) = 0$ for all $i > 0$.*

Proof. We can assume k is algebraically closed by passing to the algebraic closure. Then X is klt and $(D + K_X) - K_X$ is ample. By Theorem 3.4 and Remark 2.6, X has a log resolution that lifts to $W_2(k)$, therefore the result follows from Lemma 2.10. $\square$

Remark 3.7. Let us note that Corollary 3.6 was first proven in [Arv21, Theorem D] using different techniques.

4. Kawamata–Viehweg vanishing theorem in characteristic $p > 5$

Using the results on liftability obtained in §3, we are now ready to prove the main result of this paper.

THEOREM 4.1. *Let X be a surface of del Pezzo type over a perfect field k of characteristic $p > 5$. Let D be a Weil divisor on X and suppose that there exists a boundary Δ such that (X, Δ) is a klt pair and $D - (K_X + \Delta)$ is big and nef. Then*

$$H^i(X, \mathcal{O}_X(D)) = 0 \text{ for all } i > 0.$$

Proof. By taking the base change to the algebraic closure, we may assume k to be algebraically closed. By possibly perturbing the boundary Δ we may assume that $D - (K_X + \Delta) = A$, where A is ample. Since X is a Mori dream space by Lemma 2.2, we can run a Δ-MMP $g: X \to Y$, which terminates with $\Delta_Y := g_*\Delta$ being a nef $\mathbb{Q}$-Cartier divisor. Thus we have $\Delta = g^*\Delta_Y + N$, where $N \geq 0$ is effective and exceptional by the negativity lemma [KM98, Lemma 3.39]. So we have that $D - (K_X + N) = A + g^*\Delta_Y$ is an ample divisor. Moreover, since $N \leq \Delta$, we have that (X, N) is klt.

By Lemma 2.3, Y is a surface of del Pezzo type and therefore there exists a log resolution $\mu: V \to Y$ such that $(V, \mathrm{Exc}(\mu))$ lifts to characteristic 0 over a smooth base by Theorem 3.4.

Let us consider the induced rational map $\mu': V \dashrightarrow X$ and let $\pi: V' \to V$ be a sequence of blow-ups at smooth points such that the induced birational map $\widetilde{\mu}: V' \dashrightarrow X$ is a morphism. The commutative diagram below illustrates the situation:

$$
\begin{array}{ccc}
X & \xleftarrow{\;\widetilde{\mu}\;} & (V', \pi_*^{-1}\mathrm{Exc}(\mu) + \mathrm{Exc}(\pi)) \\
\downarrow{\scriptstyle g} & & \downarrow{\scriptstyle \pi} \\
Y & \xleftarrow{\;\mu\;} & (V, \mathrm{Exc}(\mu))
\end{array}
$$

We have that the support of $\mathrm{Exc}(\widetilde{\mu})$ and the support of $\widetilde{\mu}_*^{-1}N$ are contained in the support of $\pi_*^{-1}\mathrm{Exc}(\mu) + \mathrm{Exc}(\pi)$. Therefore $\widetilde{\mu}: (V', \mathrm{Exc}(\widetilde{\mu}) + \mathrm{Supp}(\widetilde{\mu}_*^{-1}N))$ lifts to characteristic 0 over a smooth base, and thus to $W_2(k)$ by Remark 2.6. By Lemma 2.10, the result follows. $\square$

Remark 4.2. Motivated by Theorem 3.4, one could ask whether the Kawamata–Viehweg vanishing theorem still holds for the larger class of klt surfaces X whose canonical bundle is not pseudo-effective, at least in large enough characteristic. However, this is false in general, as shown by the counterexamples in [CT18, Theorem 3.1] and [Xie10, Theorem 3.1].
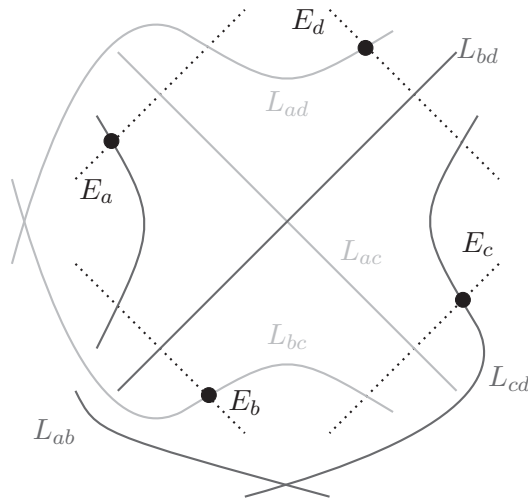
Figure 1. Configuration of curves on S_1 and images of exceptional locus of $\pi\colon S_2 \to S_1$.

5. Counterexamples in characteristic $p = 5$

It is well known that the Kawamata–Viehweg vanishing theorem may fail in positive characteristic. Counterexamples for klt del Pezzo surfaces were found in [CT19] and [Ber21a] in characteristic $p = 2$ and $p = 3$, respectively. As a consequence, in these characteristics there are threefold klt singularities which are not rational. The third author exhibited in [Lac20, Example 7.6] an example of a Picard rank 1 klt del Pezzo surface in characteristic 5 that does not admit a log resolution that lifts to characteristic 0 over a smooth base. In this section we show that this example also violates the Kawamata–Viehweg vanishing theorem. In particular, Theorem 1.1 and Corollary 1.3 are sharp.

5.1 Construction

We fix an algebraically closed field k of characteristic $p = 5$ and briefly recall [Lac20, Example 7.6 and B.12]. Consider the following four points in $\mathbb{P}^2_k$:

$$a = [-1, 1, 1], \quad b = [-1, -1, 1], \quad c = [1, -1, 1], \quad d = [1, 1, 1].$$

Let L_{ab} be the line connecting a and b, and similarly define L_{ac}, L_{ad}, L_{bc}, L_{bd} and L_{cd}. Consider the cubic curves $C_0 = L_{ad} + L_{ac} + L_{bc}$ and $C_\infty = L_{ab} + L_{bd} + L_{cd}$.

The base locus of the pencil of curves $\{C_t = C_0 + tC_\infty\}_t$ consists of the points a, b, c, d, counted with multiplicity 2, and of the point $[0, 0, 1]$, counted with multiplicity 1. Now resolve the base locus at the points a, b, c and d. This is done in two steps. First, let S_1 be the surface obtained by blowing-up once at a, b, c and d. We denote by E_a the exceptional divisor obtained by the first blow-up over a, and we use the analogous notation for the other points. The configuration is illustrated in Figure 1, where the strict transform of C_0 is drawn in grey, the strict transform of C_∞ is drawn in dark grey and the exceptional divisors are drawn as dotted lines.

For simplicity, we use the same notation for a curve and its strict transform under a birational morphism. Notice that, as drawn in the picture, the lines L_{ad} and L_{bc} meet at infinity, and the same is true for L_{ab} and L_{cd}. Next, let $\pi\colon S_2 \to S_1$ be the surface obtained by blowing up the points $E_a \cap L_{ab}$, $E_b \cap L_{bc}$, $E_c \cap L_{cd}$ and $E_d \cap L_{ad}$ (in Figure 1 these points are drawn as black circles). We call F_a, F_b, F_c and F_d the respective exceptional divisors. Notice that on S_2 the sets $\Gamma_1 = \{E_a, L_{ad}, L_{bc}, E_c\}$ and $\Gamma_2 = \{E_d, L_{cd}, L_{ab}, E_b\}$ are chains of four (-2) curves.
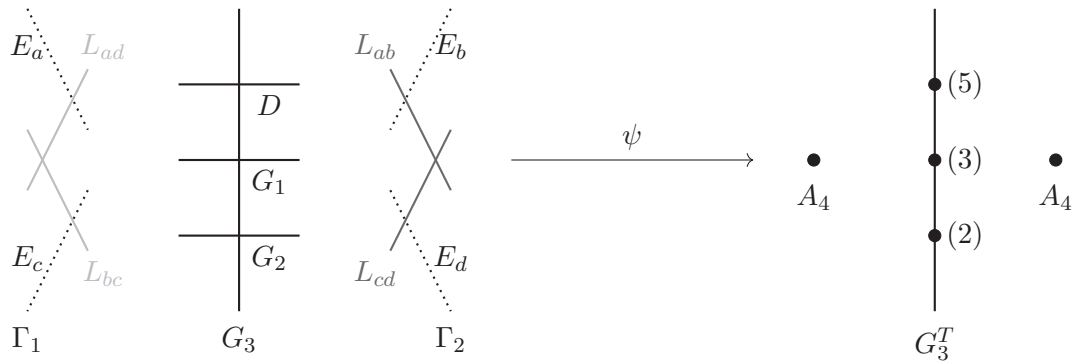
758

Figure 2. Arrangement of curves on V and the type of singularities on T.

Since we started from $\mathbb{P}^2_k$ and performed eight blow-ups, we get that $K^2_{S_2} = K_{\mathbb{P}^2_k} - 8 = 1$. The computations in [Lac20, Example B.12] show that, since the characteristic of k is 5, there is a rational cuspidal curve $D = C_2$ in the pencil $\{C_t\} \subset |-K_{\mathbb{P}^2}|$. Note that the strict transform of D is disjoint from Γ_1 and Γ_2 and that $D \in |-K_{S_2}|$. Starting from S_2, take a log resolution of the cusp of D by blowing up three times. We get a surface V with three exceptional rational curves G_1, G_2, G_3 and the following intersection numbers:

$$G_1^2 = -3, \quad G_2^2 = -2, \quad G_3^2 = -1, \quad G_1 \cdot G_2 = 0, \quad G_1 \cdot G_3 = 1, \quad G_2 \cdot G_3 = 1.$$

Note, moreover, that

$$D^2 = -5, \quad D \cdot F_a = 1, \quad D \cdot F_b = 1, \quad D \cdot G_1 = 0, \quad D \cdot G_2 = 0, \quad D \cdot G_3 = 1, \quad K_V^2 = -2.$$

The configuration of the curves on V is illustrated in Figure 2. We say that a singular point p on a surface is of type (n) if the exceptional divisor of its minimal resolution is a smooth rational curve of degree $-n$.

Finally, we consider the birational morphism $\psi \colon V \to T$ which contracts exactly the curves $\Gamma_1, \Gamma_2, G_1, G_2$ and D also illustrated in Figure 2. We denote $F_a^T, F_b^T, F_c^T, F_d^T$ and G_3^T the image via ψ of F_a, F_b, F_c, F_d and G_3, respectively.

Lemma 5.1. T is a surface of Picard rank 1 with only klt singularities.

Proof. The singularities on T are two A_4 points, one A_1 point, one (3) point and one (5) point, so they are klt. As V has Picard rank 12, it is clear that T has Picard rank 1. Since the Picard rank is 1, it is sufficient to compute the intersection of an effective curve with the canonical class to verify T is del Pezzo. By projection formula

$$K_T \cdot G_3^T = \psi^*(K_T) \cdot G_3 = \left(K_V + \tfrac{1}{3}G_1 + \tfrac{3}{5}D\right) \cdot G_3 = -1 + 1/3 + 3/5 < 0,$$

as desired. $\square$

5.2 A counterexample to the Kawamata–Viehweg vanishing theorem

In the following we construct a counterexample to the Kodaira vanishing theorem for a Weil divisor A on T. The choice of the specific Weil divisor was motivated by the fact that it passes through all the singular points and has maximal Cartier index at each point.

759

E. ARVIDSSON, F. BERNASCONI AND J. LACINI

Using the notation of §5.1, we start with the following computations:

$$\psi^* F_a^T = F_a + \tfrac{4}{5}E_a + \tfrac{3}{5}L_{ad} + \tfrac{2}{5}L_{bc} + \tfrac{1}{5}E_c + \tfrac{3}{5}E_b + \tfrac{6}{5}L_{ab} + \tfrac{4}{5}L_{cd} + \tfrac{2}{5}E_d + \tfrac{1}{5}D,$$

$$\psi^* F_b^T = F_b + \tfrac{3}{5}E_c + \tfrac{6}{5}L_{bc} + \tfrac{4}{5}L_{ad} + \tfrac{2}{5}E_a + \tfrac{1}{5}E_d + \tfrac{2}{5}L_{cd} + \tfrac{3}{5}L_{ab} + \tfrac{4}{5}E_b + \tfrac{1}{5}D,$$

$$\psi^* G_3^T = G_3 + \tfrac{1}{3}G_1 + \tfrac{1}{2}G_2 + \tfrac{1}{5}D.$$

PROPOSITION 5.2. *Let $A = G_3^T + F_a^T - F_b^T$. Then A is an ample $\mathbb{Q}$-Cartier Weil divisor and*

$$H^1(T, \mathcal{O}_T(-A)) \neq 0.$$

Proof. We first verify that A is ample. We introduce $B := -A = F_b^T - F_a^T - G_3^T$ and we compute its pullback to V:

$$\psi^* B = F_b - F_a - G_3 - \tfrac{2}{5}E_a - \tfrac{3}{5}L_{ab} - \tfrac{2}{5}L_{cd} + \tfrac{2}{5}E_c - \tfrac{1}{5}E_d$$
$$+ \tfrac{1}{5}L_{ad} + \tfrac{4}{5}L_{bc} + \tfrac{1}{5}E_b - \tfrac{1}{3}G_1 - \tfrac{1}{2}G_2 - \tfrac{1}{5}D.$$

Since T has Picard rank 1 and $B \cdot F_a^T = \psi^* B \cdot F_a = -\tfrac{1}{5}$ by the projection formula, we conclude B is anti-ample.

We now verify that $H^1(T, \mathcal{O}_T(B)) \neq 0$, using the Riemann–Roch theorem for surfaces. To this end we write

$$\lfloor \psi^* B \rfloor = F_b - F_a - G_3 - E_a - L_{ab} - L_{cd} - E_d - G_1 - G_2 - D,$$

and we compute its intersection numbers with ψ-exceptional divisors:

$$\lfloor \psi^* B \rfloor \cdot L_{ab} = 0, \quad \lfloor \psi^* B \rfloor \cdot L_{bc} = 1, \quad \lfloor \psi^* B \rfloor \cdot L_{cd} = 0, \quad \lfloor \psi^* B \rfloor \cdot L_{ad} = -1,$$
$$\lfloor \psi^* B \rfloor \cdot E_a = 1, \quad \lfloor \psi^* B \rfloor \cdot E_b = 0, \quad \lfloor \psi^* B \rfloor \cdot E_c = 0, \quad \lfloor \psi^* B \rfloor \cdot E_d = 1,$$
$$\lfloor \psi^* B \rfloor \cdot D = 4, \quad \lfloor \psi^* B \rfloor \cdot G_1 = 2, \quad \lfloor \psi^* B \rfloor \cdot G_2 = 1.$$

As a consequence one sees that $\lfloor \psi^* B \rfloor - (K_V + \tfrac{1}{3}G_1 + \tfrac{3}{5}D + \tfrac{1}{2}L_{ad}) = \lfloor \psi^* B \rfloor - (\psi^* K_T + \tfrac{1}{2}L_{ad})$ is ψ-nef and ψ-big. Thus, by the relative Kawamata–Viehweg vanishing theorem for surfaces (see [Tan18, Theorem 3.3]), we deduce $R^i \psi_* \mathcal{O}_V(\lfloor \psi^* B \rfloor) = 0$ for $i > 0$ and therefore $h^i(V, \lfloor \psi^* B \rfloor) = h^i(T, \mathcal{O}_T(B))$ by the Leray spectral sequence. To apply Riemann–Roch we need the self-intersection of $\lfloor \psi^* B \rfloor$. To this end we also compute the following intersection numbers:

$$\lfloor \psi^* B \rfloor \cdot F_a = -2, \quad \lfloor \psi^* B \rfloor \cdot F_b = -2, \quad \lfloor \psi^* B \rfloor \cdot G_3 = -2.$$

A straightforward computation now shows $\lfloor \psi^* B \rfloor^2 = -7$, and therefore by the Riemann–Roch theorem we conclude that

$$\chi(V, \lfloor \psi^* B \rfloor) = 1 + \tfrac{1}{2}(\lfloor \psi^* B \rfloor^2 - K_V \cdot \lfloor \psi^* B \rfloor) = 1 + \tfrac{1}{2}(-7 + 3) < 0,$$

thus showing that $h^1(V, \lfloor \psi^* B \rfloor) = h^1(T, \mathcal{O}_T(B)) \neq 0$. □

The following corollary shows that the lower bound on the characteristic for the liftability of a log resolution of a klt del Pezzo surface to the ring of second Witt vectors in Theorem 3.4 is optimal.

COROLLARY 5.3. *The surface T does not admit a log resolution $\mu \colon V \to T$ such that the log pair $(V, \mathrm{Exc}(\mu))$ lifts to $W_2(k)$.*

Proof. Since $H^1(T, \mathcal{O}_T(-A)) = H^1(T, \mathcal{O}_T(K_T + A)) \neq 0$, the surface T cannot admit a log resolution lifting to $W_2(k)$ by Lemma 2.10. □

760

5.3 A klt threefold singularity which is not rational

With the same notation as in §5.2, we consider the ample $\mathbb{Q}$-Cartier Weil divisor A on the Picard rank 1 klt del Pezzo surface T. Let

$$X := C_a(T, \mathcal{O}_T(A)) = \mathrm{Spec}_k \bigoplus_{m \geq 0} H^0(T, \mathcal{O}_T(mA))$$

be the affine cone over T induced by A. Let us denote by v the vertex of the cone, that is, the closed subscheme defined by the ideal $\bigoplus_{m \geq 1} H^0(T, \mathcal{O}_T(mA))$.

THEOREM 5.4. *The variety X is a normal $\mathbb{Q}$-factorial klt threefold which is not Cohen–Macaulay (and thus not rational).*

Proof. By [Ber21a, Proposition 2.4, (2)], X is $\mathbb{Q}$-factorial. By inversion of adjunction (see [HW19b, Corollary 1.5]), we can apply the proof of [Ber21a, Proposition 2.5, (3)] to conclude that X has klt singularities. Using [Ber21a, Proposition 2.6], we have

$$H_v^2(X, \mathcal{O}_X) \simeq \bigoplus_{m \in \mathbb{Z}} H^1(T, \mathcal{O}_T(mA)) \neq 0,$$

by Proposition 5.2. Thus we conclude that X is not Cohen–Macaulay. $\qquad\square$

REFERENCES

Arv21   E. Arvidsson, *On the Kodaira vanishing theorem for log del Pezzo surfaces in positive characteristic*, Math. Z. **297** (2021), 1–16.

Ber21a   F. Bernasconi, *Kawamata–Viehweg vanishing fails for log del Pezzo surfaces in characteristic* 3, J. Pure Appl. Algebra **225** (2021), Paper No. 106727.

Ber21b   F. Bernasconi, *On the base point free theorem for klt threefolds in large characteristic*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **22** (2021), 583–600.

BT22   F. Bernasconi and H. Tanaka, *On del Pezzo fibrations in positive characteristic*, J. Inst. Math. Jussieu **21** (2022), 197–239.

Bir19   C. Birkar, *Anti-pluricanonical systems on Fano varieties*, Ann. of Math. (2) **190** (2019), 345–463.

Bir21   C. Birkar, *Singularities of linear systems and boundedness of Fano varieties*, Ann. of Math. (2) **193** (2021), 347–405.

CSK20   J. Carvajal-Rojas, A. Stäbler and J. Kollár, *On the local étale fundamental group of KLT threefold singularities*, Preprint (2020), arXiv:2004.07628.

CT18   P. Cascini and H. Tanaka, *Smooth rational surfaces violating Kawamata–Viehweg vanishing*, Eur. J. Math. **4** (2018), 162–176.

CT19   P. Cascini and H. Tanaka, *Purely log terminal threefolds with non-normal centres in characteristic two*, Amer. J. Math. **141** (2019), 941–979.

CTW17   P. Cascini, H. Tanaka and J. Witaszek, *On log del Pezzo surfaces in large characteristic*, Compos. Math. **153** (2017), 820–850.

CdeJ02   B. Conrad and A. J. de Jong, *Approximation of versal deformations*, J. Algebra **255** (2002), 489–515.

DI87   P. Deligne and L. Illusie, *Relèvements modulo p^2 et décomposition du complexe de de Rham*, Invent. Math. **89** (1987), 247–270.

EGAIV$_3$ A. Grothendieck, *Eléménts de Géométrie Algébrique. IV. Étude locale des schémas et des morphismes de schémas, Troisième partie*, Publ. Math. Inst. Hautes Études Sci. **28** (1966).

EGAIV$_4$ A. Grothendieck, *Eléménts de Géométrie Algébrique. IV. Étude locale des schémas et des morphismes de schémas, Quatrieme partie*, Publ. Math. Inst. Hautes Études Sci. **32** (1967).

Elk81   R. Elkik, *Rationalité des singularités canoniques*, Invent. Math. **64** (1981), 1–6.

FGI$^+$05   B. Fantechi, L. Göttsche, L. Illusie, S. L. Kleiman, N. Nitsure and A. Vistoli, *Fundamental algebraic geometry. Grothendieck's FGA explained*, Mathematical Surveys and Monographs, vol. 123 (American Mathematical Society, Providence, RI, 2005).

GNT19   Y. Gongyo, Y. Nakamura and H. Tanaka, *Rational points on log Fano threefolds over a finite field*, J. Eur. Math. Soc. **21** (2019), 3759–3795.

HW19a   C. D. Hacon and J. Witaszek, *On the rationality of Kawamata log terminal singularities in positive characteristic*, Algebr. Geom. **6** (2019), 516–529.

HW19b   C. D. Hacon and J. Witaszek, *On the relative minimal model program for threefolds in low characteristic*, Peking Math. J., to appear. Preprint (2019), arXiv:1909.12872.

Hara98   N. Hara, *A characterization of rational singularities in terms of injectivity of Frobenius maps*, Amer. J. Math. **120** (1998), 981–996.

Hart77   R. Hartshorne, *Algebraic geometry*, Graduate Texts in Mathematics, vol. 52 (Springer, New York, 1977).

Ill71   L. Illusie, *Complexe cotangent et déformations. I*, Lecture Notes in Mathematics, vol. 239 (Springer, Berlin, 1971).

Kol13   J. Kollár, *Singularities of the minimal model program*, With a collaboration of Sándor Kovács. Cambridge Tracts in Mathematics, vol. 200 (Cambridge University Press, Cambridge, 2013).

KM98   J. Kollár and S. Mori, *Birational geometry of algebraic varieties*, Cambridge Tracts in Mathematics, vol. 134 (Cambridge University Press, Cambridge, 1998).

Lac20   J. Lacini, *On rank one log del Pezzo surfaces in characteristic different from two and three*, Preprint (2020), arXiv:2005.14544v3.

Liu02   Q. Liu, *Algebraic geometry and arithmetic curves*, Oxford Graduate Texts in Mathematics, vol. 6 (Oxford University Press, Oxford, 2002). Translated from the French by Reinie Erne.

Sta   The Stacks Project Authors, *Stacks Project*, https://stacks.math.columbia.edu.

Tan15   H. Tanaka, *The X-method for klt surfaces in positive characteristic*, J. Algebraic Geom. **24** (2015), 605–628.

Tan18   H. Tanaka, *Minimal model program for excellent surfaces*, Ann. Inst. Fourier (Grenoble) **68** (2018), 345–376.

Tot19   B. Totaro, *The failure of Kodaira vanishing for Fano varieties, and terminal singularities that are not Cohen–Macaulay*, J. Algebraic Geom. **28** (2019), 751–771.

Xie10   Q. Xie, *Counterexamples to the Kawamata–Viehweg vanishing on ruled surfaces in positive characteristic*, J. Algebra **324** (2010), 3494–3506.

Yas19   T. Yasuda, *Discrepancies of p-cyclic quotient varieties*, J. Math. Sci. Univ. Tokyo **26** (2019), 1–14.

Emelie Arvidsson   emmiarwidsson@ias.edu

School of Mathematics, Institute for Advanced Study, Princeton, NJ 08540, USA

Fabio Bernasconi   fabio.bernasconi@epfl.ch

École Polytechnique Fédérale de Lausanne, Chair of Algebraic Geometry (Bâtiment MA), Station 8, CH-1015 Lausanne, Switzerland

Justin Lacini   jlacini@ku.edu

Department of Mathematics, University of Kansas, 643 Snow Hall, Lawrence, KS 66046, USA