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Optimal matching of random parts

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ABSTRACT

This paper examines the minimization of the cost for an expected random production output, given an assembly of finished goods from two random inputs, matched in two categories. We describe the optimal input portfolio, first using the standard normal approximation of the binomial classification distributions, and second using a tight concave envelope instead of the exact output objective. The latter approach yields closed-form expressions for the factor demands and total costs which are linear in the expected output and which approximate the solution to the original minimum-cost matching problem for sufficiently large production batches. A key structural insight is that depending on the ratio of input prices, one of the inputs should be considered as "critical component" while the other assumes the role of a "buffer component." As long as the cost ratio does not reach a critical threshold, which is proportional to the ratio of the grade-attainment likelihoods, the relative composition of the optimal input portfolio remains largely invariant. A numerical study confirms the practicality of the envelope approach, both as a seed for a numerical solution of the exact optimality conditions and as an approximate solution in closed-form.

So every beast finds a mate[1]

HONORÉ DE BALZAC

1. Introduction

Firms produce finished goods from diverse inputs. For physical goods, the input components usually have to be assembled in fixed proportions, corresponding to the functionality and geometry specified in the product design. The problem is that the characteristics of the incoming parts is generally random, and to properly fit together when assembled, components of different types (e.g., a base plate and a screw) must therefore be compatible, that is, of the same "grade". The required matching of input grades implies that the firm's yield in finished products is *ex ante* random, as it depends on the result of its internal matching of components. For a lack of perfect matching, this means that not all of the available inputs can be used productively. The question then arises, how many pieces to order of each kind, so as to minimize the cost for a given expected number of (possibly weighted) same-grade matches?

Practical applications for input matching abound. For example, when building electric circuits for high-end audio equipment, the components for the stereo channels must have close characteristics to avoid perceptible side-to-side differences in the stereo image. Similarly, in order to achieve a defined resonance frequency in the movement of a mechanical watch, the respective characteristics of the balance spring and the balance wheel must be of matching grades.[2] When the unit prices of inputs are different, as would be the case for balance springs versus balance wheels (by at least a factor of 10), it is best to order the cheaper parts in excess so as to decrease the number of unmatched expensive parts.

In this paper, we formulate the "minimum-cost matching problem" for the binary fitting of parts with random characteristics that are independent and identically distributed, and naturally described by binomial distributions. We first determine the optimal input portfolio solution using the standard normal approximation of the binomial distribution. Then we introduce an envelope substitution of the objective function which leads to closed-form expressions for the optimal vector of inputs and the cost function. The comparative statics of the solution to the envelope optimization problem highlight the two mutually exclusive roles of inputs as *either* "critical components" *or* "buffer components". The latter needs to be acquired in relative excess to the former—with their relative proportions in the optimal

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[1] From: *The Maid of Thilouse* (originally *La pucelle de Thilhouze*, published in *Les contes drôlatiques*, 1832).

[2] The balance spring is characterized by its stiffness (as measured by the spring coefficient $\varkappa$ [in N m/rad]) while the defining property of a balance wheel is its moment of inertia (denoted by I [in kg m^2]). The quotient of these characteristics defines the oscillation period $T = \sqrt{2\pi}(I/\varkappa)$ (e.g., $1/T = 4$ Hz).

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portfolio being fairly insensitive to the level of output and to (small) variations of factor prices.

1.1. Literature

The perfect complementarity of goods in production functions, where inputs are transformed in constant proportions into outputs, was introduced as a simplifying assumption by Leontief (1941) to analyze a large economy. However, such perfect complementarity does arise naturally in assembly processes where the required number of parts for each input (per unit of output) is fixed.[3] Generalizations to situations where the minimum-cost combinations may vary according to the size of inputs are summarized by Fandel (1991) who also discusses the possibility of combining several assembly processes (ibid., p. 120). Production sets akin to those with combinable assembly processes appear in our problem as a tight bound of the (non-Leontief) smooth production surface with stochastic matching. The presence of uncertainty in input–output production functions is recognized by Daughety (1982) who emphasizes that in a stochastic setting cost becomes a function of expected output, not actual output. The underlying assumption, as a generalization of the notions of separability by Leontief (1947) and Sono (1961), is that the production function is stochastically separable, in the sense that the marginal rates of technical substitution are in fact deterministic. In our case, while the envelope of ordered parts is deterministic, the inputs to the matching categories are random. Since the sorting of inputs into different grades or categories is part of the production process, the associated production function is therefore stochastic: its realization relies on an ex-ante unobserved characteristic of the input. This produces a degree of substitutability across different input types (e.g., screws and base plates) provided each is available in positive quantities. That is, each additional unit of input ensures against a contingency where an extra matching part is available, so that an additional unit of output can be produced.

Production uncertainties in agriculture were examined by Pope and Chavas (1994), while Malikov et al. (2015) estimate the production technology of banks by taking into account their optimal provision of uncertain inputs to achieve a given output-attainment goal in expectation. A state-contingent approach to stochastic production is pursued by Chambers and Quiggin (2002) where the states would tend to influence input–output conversion as a whole. By contrast, we are interested here in the optimal matching of stationary random inputs, which realize in different quality categories. The present study is the first to determine the cost (as a function of its expected output) and the optimal input portfolio, for a risk-neutral firm that engages in binomial sorting and matching of corresponding input grades to produce its output.[4]

1.2. Outline

Section 2 describes the model primitives and introduces the minimum-cost matching problem. In Section 3, we determine the solution using the standard normal approximation to the binomial distribution. Section 4 pursues an alternative approach by finding cost-minimizing inputs of an upper envelope of the firm's objective function, which leads to a closed-form solution.

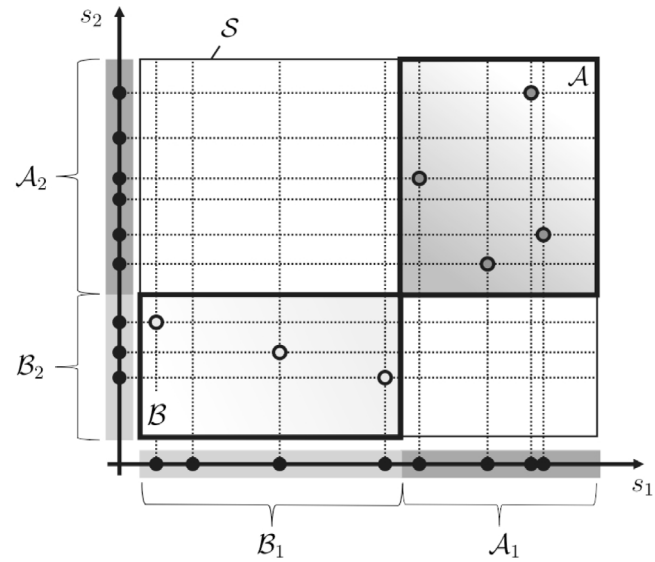


Fig. 1. Classification of $m = 8$ type-1 and $n = 9$ type-2 parts with characteristics (s_1, s_2) in $\mathcal{S} = \mathcal{S}_1 \times \mathcal{S}_2$, and random matching within grade-A and grade-B categories, respectively.

The latter approximates the solution to the original problem for sufficiently large production runs, thus revealing its structural properties. Numerically the envelope input approximation is shown to perform well in a variety of settings. Section 5 concludes.

2. Problem

Consider two types of parts to be matched. Each unit of a certain type $i \in \{1, 2\}$ has a random characteristic S_i with realizations s_i in the measurable space $\mathcal{S}_i \subset \mathbb{R}^{d_i}$, for some integer dimension $d_i \geq 1$, which has at least two elements (to exclude trivialities). A type-i part is called of "category A" (or "grade A") if its characteristic lies in the (nonempty) set $\mathcal{A}_i \subset \mathcal{S}_i$; otherwise it is called of "category B" (or "grade B") with a characteristic in the (nonempty) complement $\mathcal{B}_i = \mathcal{S}_i \setminus \mathcal{A}_i$. Two parts of different types are "matched" if they are of the same category.[5] It is clear that different types of parts, such as balance springs and balance wheels, may have quite different characteristics, measured in different units (cf. footnote 2). In watchmaking, for example (given suitable thresholds for the characteristics), balance springs may be either of "high stiffness" (grade A) or "low stiffness" (grade B), while balance wheels could be similarly classified into a "high inertia" (grade A) or "low inertia" (grade B) category. After randomly matching parts of corresponding grades, the number of assemblies for each grade is given by the minimum number of parts available from either type. Thus, the finished assemblies would be either grade A (with joint characteristics in $\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2$) or grade B (with joint characteristics in $\mathcal{B} = \mathcal{B}_1 \times \mathcal{B}_2$),[6] as illustrated in Fig. 1.

Let $p \in (0, 1)$ denote the probability that a randomly selected type-1 part is of grade A, and let $q \in (0, 1)$ be the probability that a randomly selected type-2 part is of grade A.[7] Furthermore,

[3] Here we assume that matches can occur only in equal proportions, one-to-one. For example, if four screws are required for each base plate, it is sufficient to consider bundles of four screws as units of the first ("type-1") input and base plates as units for the second ("type-2") input, so as to return to the equal-proportions assumption without any significant loss of generality.

[4] Lee et al. (1990) analyze the tuning of (normal) input distributions under two-bin component matching to maximize the output yield in the long-run; the problem is different and remains unconcerned with input unit costs.

[5] The classification in the physical space $\mathcal{S}_i$ of the type-i parts characteristics is important for connecting our model to the real world; see Weber (2021) for a minimum-error partition of $\mathcal{S}_i \subset \mathbb{R}$ into $\ell \geq 2$ matching classes.

[6] Here and in the following discussion we abstract from potential sorting and assembly errors, which can be all but eliminated using appropriate separation, labeling, and verification techniques.

[7] The degenerate cases where $p \in \{0, 1\}$ or $q \in \{0, 1\}$ are excluded, to keep the discussion interesting.

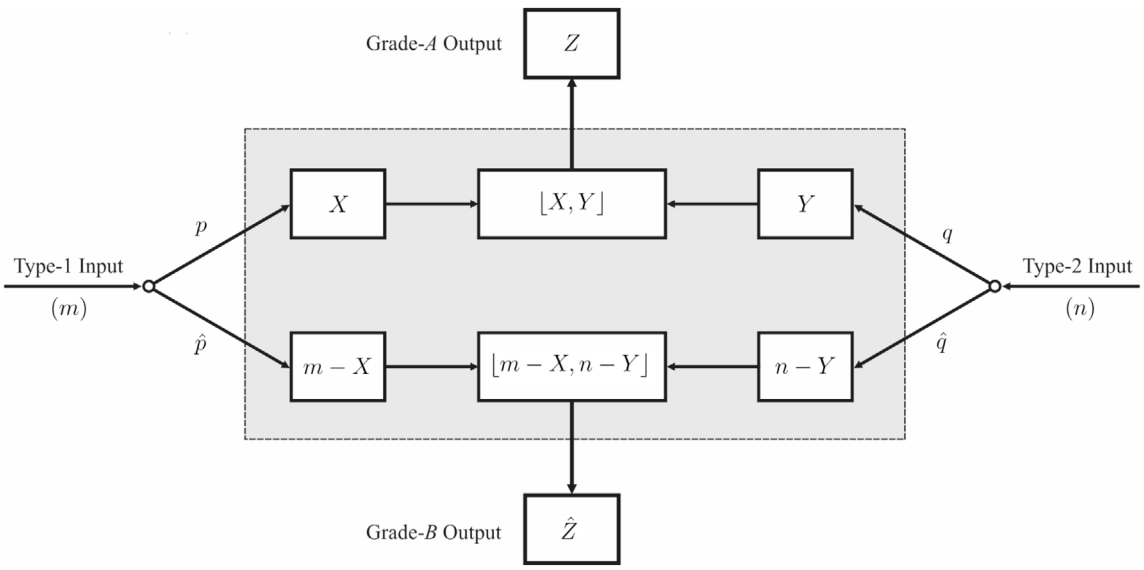


Fig. 2. Classification and conversion of stochastic inputs into grade-A and grade-B outputs.

given input quantities of m type-1 units and n type-2 units, let X describe the random number of type-1 grade-A units and Y the random number of type-2 grade-A units. Both follow a binomial distribution, so that

$$\text{Prob}(X = x|m) = \binom{m}{x} p^x (1-p)^{m-x}, \quad x \in \{0, \dots, m\}, \tag{1}$$

and

$$\text{Prob}(Y = y|n) = \binom{n}{y} q^y (1-q)^{n-y}, \quad y \in \{0, \dots, n\}. \tag{2}$$

Given the stochastic inputs X and Y, conditional on the selected input (m, n), the *output of matched grade-A parts* is $Z \triangleq \lfloor X, Y \rfloor$ while the *output of matched grade-B parts* amounts to $\hat{Z} \triangleq \lfloor m - X, n - Y \rfloor$.[8] Fig. 2 illustrates the assembly of the two types of inputs into the respective outputs. Without loss of generality, we assume that the firm cares more about grade-A matches than about grade-B matches, at least weakly. Such a weak preference ordering can always be ensured, by switching grade labels if necessary. The risk-neutral firm's *expected total output* becomes

$$U(m, n) = \mathbb{E}\left[Z + \rho \hat{Z} \middle| m, n \right], \tag{3}$$

where the coefficient $\rho \in [0, 1]$ denotes the relative weight of B-grade output to A-grade output. As may be intuitively clear, if more inputs are provided, the expected output must also go up.

Lemma 1 (*Nonsatiation*)**.** *The expected output $U(m, n)$ is increasing for $m, n > 0$.*

Given (positive) unit costs c_1 and c_2 for type-1 and type-2 components, respectively, the *minimum-cost matching problem* is to find the smallest expenditure,

$$C(u, c) = \min_{m,n \geq 0} \{c_1\, m + c_2 n\}, \quad \text{s.t.} \quad U(m, n) \geq u, \tag{*}$$

for a given expected output $u \geq 0$, where $c \triangleq (c_1, c_2)$. The minimum-cost matching problem (*) is well-posed in the sense that a feasible (or "admissible") solution (satisfying the *output-achievement constraint* $U(m, n) \geq u$) is guaranteed to exist.

[8] For convenience, we write $\lfloor x, y \rfloor$ for $\min\{x, y\}$ and $\lceil x, y \rceil$ for $\max\{x, y\}$, given any real numbers x and y.

Remark 1 (*Preview*)**.** Our solution to the minimum-cost matching problem (*) involves a "double approximation", in the sense that we first replace the discrete binomial distributions by a suitable smooth approximation; cf. Section 3.1. This yields a continuously differentiable output function, defined for all nonnegative inputs. Since the employed normal approximation of the binomial distribution is extremely good for any type of practically relevant quantities, *we continue to denote the smooth objective approximation by U and refer the reader to (*) whenever we speak of the (original) "minimum-cost matching problem".* A second approximation then replaces the aforementioned smooth output approximation by an envelope output $\bar{U} \geq U$ for which a convex "envelope optimization problem" (**) is obtained; cf. Sections 4.1 and 4.2.

Lemma 2 (*Existence*)**.** *For all $(u, c) \in \mathbb{R}_+ \times \mathbb{R}^2_{++}$, an (admissible) solution $(m^*(u, c), n^*(u, c))$ to the minimum-cost matching problem (*) does exist.*

The following result shows that it is possible to rescale the output measure nonlinearly to different units of measurement, as long as the corresponding transformation is monotonic.

Lemma 3 (*Invariance*)**.** *Let $\varphi : \mathbb{R}_+ \to \mathbb{R}$ be an increasing function, and let $\hat{U}(m, n) \triangleq \varphi(U(m, n))$, for all $m, n \geq 0$. Furthermore, let $\alpha > 0$. Then the "invariance property",*

$$\hat{C}(\varphi(u), \alpha c) = \alpha C(u, c), \tag{4}$$

holds for all $(u, c) \in \mathbb{R}_+ \times \mathbb{R}^2_{++}$, where

$$\hat{C}(\hat{u}, \hat{c}) \triangleq \min_{m,n \geq 0} \left\{ \hat{c}_1\, m + \hat{c}_2 n \right\}, \quad \text{s.t.} \quad \hat{U}(m, n) \geq \hat{u},$$

for any $\hat{u} \geq \varphi(0)$ and any $\hat{c} = (\hat{c}_1, \hat{c}_2)$ with $\hat{c}_1, \hat{c}_2 > 0$.

We have therefore established that optimal cost is homogeneous of degree 1 with respect to a scaling of the cost vector c. Thus, global inflation of factor prices has the same effect on the optimal cost as changing the payment currency for factor inputs. More generally, Lemma 3 can be used to include other types of objective function, as noted next.

Remark 2 (*Profit Maximization*)**.** The invariance property in Lemma 3 broadens the scope of minimum-cost matching, for example, to profit maximization (when viewed as minimum-cost

revenue maximization). Indeed, let r_i be the retail price (unit revenue) and MC_j the (constant) marginal cost of an assembled product of grade $j \in \{A, B\}$.[9] By setting $\hat{U}(m,n) = (r_A - \mathrm{MC}_A)\mathbb{E}[Z|m,n] + (r_B - \mathrm{MC}_B)\mathbb{E}[\hat{Z}|m,n]$, assuming the positive margins $r_A - \mathrm{MC}_A \geq r_B - \mathrm{MC}_B > 0$ (without loss of generality), one obtains Eq. (4) with $\varphi(u) = (r_A - \mathrm{MC}_A)u$ and $\rho = (r_B - \mathrm{MC}_B)/(r_A - \mathrm{MC}_A) \in (0, 1]$.

Lemma 4 (*Solution Homogeneity*)**.** *Any solution $(m^*, n^*)(u, c)$ to the minimum-cost matching problem (*) is invariant (or, homogeneous of degree zero) with respect to cost scaling. That is,*

$$(m^*, n^*)(u, c) = (m^*, n^*)(u, \alpha c),$$

for all $\alpha > 0$, given any $(u, c) \in \mathbb{R}_+ \times \mathbb{R}^2_{++}$.

Solutions to minimum-cost matching problems depend only on the cost ratio $\gamma = c_1/c_2$ and the output level u, not on the individual values of the factor costs, c_1 and c_2. Thus, there is no "input-substitution effect" as long as a cost shock leaves γ unchanged.

3. Solution

In an actual assembly situation, the components of the input vector (m, n) must be natural numbers. Yet, in order to be able to use differential calculus and simple algebra without paying attention to the integer requirement, any input vector $(m, n) \in \mathbb{R}^2_+$ is deemed *admissible*. The associated rounding errors are small and for virtually all practical purposes insignificant.

3.1. Normal approximation

To obtain an analytically convenient objective function $U(m, n)$, one needs a smooth version of the binomial distribution,[10] defined for all $(m, n) \in \mathbb{R}^2_+$. We employ here the "normal approximation" of the binomial distribution, which—although analytically only moderately tractable—yields good numerical results.[11] More specifically, we approximate the binomial distributions of X and Y by a standard-normal distribution, as

$$\mathrm{Prob}(X \leq x|m) \approx F(x|m) \triangleq \Phi\left(\frac{x - \mu_x(m)}{\sigma_x(m)}\right) \tag{5}$$

and

$$\mathrm{Prob}(Y \leq y|n) \approx G(y|n) \triangleq \Phi\left(\frac{y - \mu_y(n)}{\sigma_y(n)}\right), \tag{6}$$

where $\mu_x(m) \triangleq \mathbb{E}[X|m] = mp$, $\sigma_x^2(m) \triangleq \mathbb{E}\left[(X - \mu_x(m))^2 \middle| m\right] = mp\hat{p}$ are the first two moments of the binomial distribution of X, and similarly, $\mu_y(n) \triangleq nq$, $\sigma_y^2(n) = nq\hat{q}$ denote the first and second moments of Y, respectively, using the abbreviations $\hat{p} = 1 - p$ and $\hat{q} = 1 - q$. The cumulative distribution function (cdf) of the standard normal distribution is given by

$$\Phi(\xi) = \int_{-\infty}^{\xi} \phi(\zeta)\, d\zeta = \frac{1}{2}\left[1 + \mathrm{erf}(\xi/\sqrt{2})\right], \quad \xi \in \mathbb{R},$$

where

$$\phi(\xi) = \frac{\exp\left(-\xi^2/2\right)}{\sqrt{2\pi}}, \quad \xi \in \mathbb{R},$$

is the corresponding probability density function (pdf).

Remark 3 (*Approximation Error*)**.** In terms of absolute approximation error, the Berry–Esseen theorem (Berry, 1941; Esseen, 1942) guarantees limited absolute deviations, so

$$\sup_{x \in [0,m]} |\mathrm{Prob}\,(X \leq x|m) - F(x|m)| \leq \frac{K(p^2 + \hat{p}^2)}{\sigma_x(m)}$$

and

$$\sup_{y \in [0,n]} |\mathrm{Prob}\,(Y \leq y|n) - G(y|n)| \leq \frac{K(q^2 + \hat{q}^2)}{\sigma_y(n)},$$

with $K = 0.4748$ as the currently best estimate by Shevtsova (2011) (which is strictly above the theoretical lower bound for K of $(\sqrt{10} + 3)/(6\sqrt{2\pi}) \approx 0.4097$; Esseen, 1956). Provided that

$$m > \kappa^2 \frac{\lceil p, \hat{p} \rceil}{\lfloor p, \hat{p} \rfloor} \quad \text{and} \quad n > \kappa^2 \frac{\lceil q, \hat{q} \rceil}{\lfloor q, \hat{q} \rfloor},$$

the normal approximation of the binomial distribution has an error in the order of the tail probabilities of realizations away from the mean further than κ standard deviations.

Henceforth, we consider the firm's output objective in Eq. (3) with respect to the normal approximation of the binomial distribution, so

$$U(m, n) = \int_0^m \left[\int_0^n \left(\lfloor x, y \rfloor + \rho \lfloor m - x, n - y \rfloor\right) dG(y|n)\right] dF(x|m), \tag{7}$$

for all $(m, n) \in \mathbb{R}^2_+$. In particular, *the minimum-cost matching problem (*) is solved with respect to the smooth (approximate) output function U in Eq. (7)*; cf. Remark 1. For example, if $m, n \geq 200$ and $p, q \in [0.2, 0.8]$, then the approximation error in Remark 3 cannot exceed 6%.

As already alluded to in Lemma 1, which was formulated for the original binomial input distribution, increasing inputs in the normal approximation leads to a right-shift of the input distributions, also termed *first-order stochastic dominance* (FOSD).[12]

Lemma 5 (*Stochastic Dominance*)**.** *$\hat{m} > m$, $\hat{n} > n$ implies that $F(\cdot|\hat{m}) \succeq_{FOSD} F(\cdot|m)$ on $[0, \hat{m}]$, and $G(\cdot|\hat{n}) \succeq_{FOSD} G(\cdot|n)$ on $[0, \hat{n}]$.*

An increase of inputs produces an FOSD-shift of components available for matching, which in turn cannot decrease the number of matched parts (of either grade) in expectation. This captures the simple intuition that more input (of any type) is good for output, at least weakly.

3.2. Output distribution

The assembly system converts the input distributions for the different part types and quality-grades into grade-specific output distributions, as shown in Fig. 3. Given the normal approximations $F(\cdot|m)$ and $G(\cdot|n)$ for the random variables X and Y, the distribution $H(\cdot|m, n)$ of grade-A matches $Z = \lfloor X, Y \rfloor$ can be written as the probabilistic complement of the product of the associated survival likelihoods $(1 - F(\cdot|m))$ and $(1 - G(\cdot|n))$.

[9] The marginal cost MC_j would generally include all production costs directly related to a finished product which contains the matched components of grade j.

[10] In the simplest case where m, n are natural numbers, directly evaluating binomial coefficients pushes the limits of precision (e.g., as soon as $m, n \geq 100$), and it tends to require significant memory and computation time when using a recursive algorithm to avoid overflow (e.g., by precalculating the entries of Pascal's triangle).

[11] Ilienko (2013) provides a continuous version of the binomial distribution, usable also for small values of m, n.

[12] Given two cdf's α, β defined on the common domain $\mathcal{D}$ (which is taken to be a nonempty measurable subset of a finite-dimensional Euclidean space), α first-order stochastically dominates β (i.e., $\alpha \succeq_{\mathrm{FOSD}} \beta$) if and only if $\alpha \leq \beta$ on $\mathcal{D}$ (Lehmann, 1954).

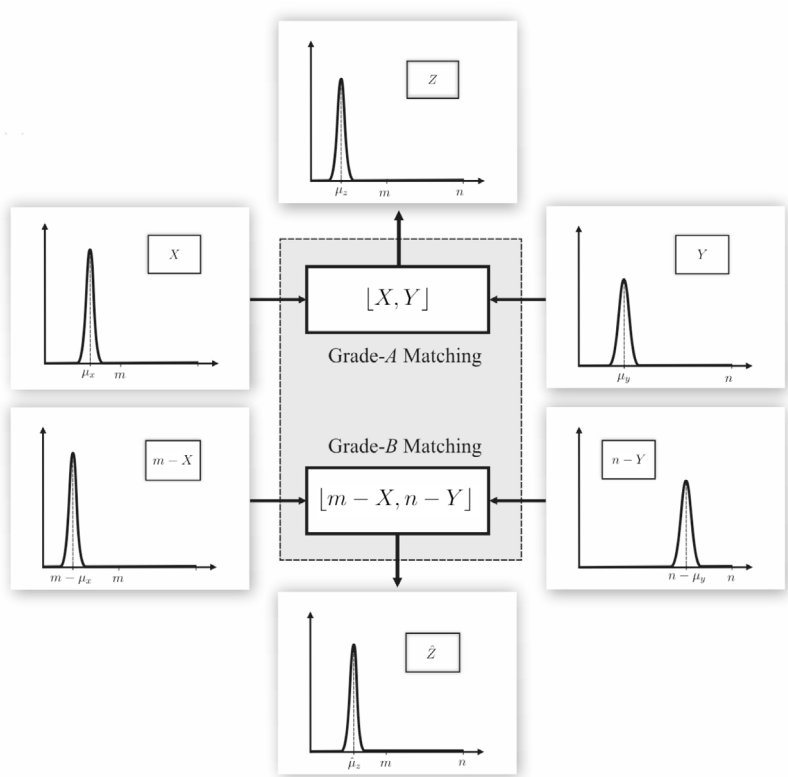


Fig. 3. Input and output distributions with Grade-Based matching.

Lemma 6 (*Grade-A Distribution*)**.** *The grade-A output Z follows the cdf $H(\cdot|m,n)$, where*

$$H(z|m,n) = 1 - \big(1 - F(z|m)\big)\big(1 - G(z|n)\big),$$

for all $z \in [0, \lfloor m, n \rfloor]$.

Simply put, Lemma 6 states that the probability of the grade-*A* output not exceeding a given value z is equal to the probability that not both component types exceed z in terms of their respective grade-*A* yields.

Lemma 7 (*Grade-A Output*)**.** *The expected number of category-A matches is*

$$\mu_z(m,n) = mp\,\Phi\left(\frac{nq - mp}{\sigma(m,n)}\right) + nq\,\Phi\left(\frac{mp - nq}{\sigma(m,n)}\right) - \sigma(m,n)\,\phi\left(\frac{nq - mp}{\sigma(m,n)}\right),$$

where $\sigma(m,n) = \sqrt{mp\hat{p} + nq\hat{q}}$ corresponds to the standard deviation of the difference of the statistically independent assembly inputs.

Noting that $\Phi\left((mp - nq)/\sigma(m,n)\right) = 1 - \Phi\left((nq - mp)/\sigma(m,n)\right)$, one can interpret the expected amount of grade-*A* output as a convex combination of both types of expected grade-*A* inputs (with each respective weight going down as the other component type becomes relatively more available), diminished by a likelihood of mismatch times the dispersion $\sigma(m,n)$.

Remark 4. When the expected inputs for both component types are the same (i.e., $mp = nq$), then the two weights are also the same, and the expected number of category-*A* matches is

$$\mu_z(m, m(p/q)) = mp - \sqrt{mp\left(\hat{p} + \hat{q}\right)/(2\pi)};$$

the latter increases in the probabilities p, q of individual components' realizing as grade *A*.

Consider now the number of category-*B* matches, $\hat{Z} = \lfloor m - X, n - Y \rfloor$, which follows the cdf $\hat{H}(\hat{z}|m,n)$ for $\hat{z}$ in $[0, \lfloor m, n \rfloor]$.

Lemma 8 (*Grade-B Distribution*)**.** *The grade-B output $\hat{Z}$ follows the cdf $\hat{H}(\cdot|m,n)$, where*

$$\hat{H}(\hat{z}|m,n) = 1 - F(m - \hat{z}|m)\,G(n - \hat{z}|n),$$

for all $\hat{z} \in [0, \lfloor m, n \rfloor]$.

The probability that the grade-*B* output does not exceed the given value $\hat{z}$ is equal to the likelihood that there are *either* more than $m - \hat{z}$ type-1 grade-*A* parts *or* more than $n - \hat{z}$ type-2 grade-*A* parts (or both). The counterfactual would mean that the grade-*A* output is small, implying in turn that the grade-*B* output exceeds $\hat{z}$, precisely the case to be excluded in the computation of $\hat{H}(\hat{z}|m,n)$. Analogous to Lemma 7, it is now possible to compute the expected grade-*B* output.

Lemma 9 (*Grade-B Output*)**.** *The expected number of grade-B matches is*

$$\hat{\mu}_z(m,n) = m\hat{p}\,\Phi\left(\frac{n\hat{q} - m\hat{p}}{\sigma(m,n)}\right) + n\hat{q}\,\Phi\left(\frac{m\hat{p} - n\hat{q}}{\sigma(m,n)}\right) - \sigma(m,n)\,\phi\left(\frac{n\hat{q} - m\hat{p}}{\sigma(m,n)}\right),$$

where $\sigma(m,n)$ is given in Lemma 7.

Again, we obtain that the expected grade-*B* output can be viewed as a convex combination of its component parts in expectation, less than a mismatch adjustment—the latter being proportional to the standard deviation of the difference (or sum) of the two grade-*B* inputs.

3.3. Optimality conditions

Taking into account the expected grade-A output in Lemma 7 and the expected grade-B output in Lemma 9, the firm's objective, that is, its *production function*, in Eq. (7) becomes

$$U(m,n) = \mu_z(m,n) + \rho\,\hat{\mu}_z(m,n), \tag{8}$$

for all $(m,n) \in \mathbb{R}^2_+$, under the normal approximation. This expected total output is increasing in the inputs, echoing the nonsatiation result in Lemma 1.

Lemma 10 (*Monotonicity*)**.** *$\partial_m U(m,n) > 0$ and $\partial_n U(m,n) > 0$, for all $(m,n) \in \mathbb{R}^2_+$.*

Since increasing input produces increasing output, at any solution of the minimum-cost matching problem (*), the output-attainment constraint must be binding.

Lemma 11 (*Output Efficiency*)**.** *Any solution $(m^*(u,c), n^*(u,c))$ to the minimum-cost matching problem (*) is such that*

$$U(m^*(u,c), n^*(u,c)) = u, \tag{9}$$

for all $(u,c) \in \mathbb{R}_+ \times \mathbb{R}^2_{++}$.

Condition (9) ensures that no money is wasted to produce any expected output beyond the required level u. This fixes an iso-output curve where $U(m,n) = u$. Along this curve, the relative effectiveness of the component types in terms of contributing to an expected unit of a finished good generally varies. Corner solutions where only one type of good is procured are never viable because they would surely not lead to any assemblies at all. The next result establishes that at an interior optimum the ratio of marginal productivities $\partial_m U/\partial_n U$ must be equal to the ratio of the input costs $\gamma = c_1/c_2$, so as to optimally balance the input portfolio in view of maximizing output.

Theorem 1. *A solution $(m^*(u,c), n^*(u,c))$ to the minimum-cost matching problem (*), with the expected output $U(m,n)$ in Eq. (7), satisfies Eq. (9) and*

$$\frac{\partial_m U(m^*(u,c), n^*(u,c))}{\partial_n U(m^*(u,c), n^*(u,c))} = \frac{c_1}{c_2} = \gamma, \tag{10}$$

for any $(u,c) \in \mathbb{R}_+ \times \mathbb{R}^2_{++}$, where the marginal product of additional inputs is given by

$$\begin{aligned}\partial_m U(m,n) = {} & p\Phi\left(\tfrac{nq-mp}{\sigma(m,n)}\right) - \tfrac{p\hat{p}/2}{\sigma(m,n)}\,\phi\left(\tfrac{nq-mp}{\sigma(m,n)}\right) \\ & + \rho\left[\hat{p}\Phi\left(\tfrac{n\hat{q}-m\hat{p}}{\sigma(m,n)}\right) - \tfrac{p\hat{p}/2}{\sigma(m,n)}\,\phi\left(\tfrac{n\hat{q}-m\hat{p}}{\sigma(m,n)}\right)\right],\end{aligned}$$

and

$$\begin{aligned}\partial_n U(m,n) = {} & q\Phi\left(\tfrac{mp-nq}{\sigma(m,n)}\right) - \tfrac{q\hat{q}/2}{\sigma(m,n)}\,\phi\left(\tfrac{nq-mp}{\sigma(m,n)}\right) \\ & + \rho\left[\hat{q}\Phi\left(\tfrac{m\hat{p}-n\hat{q}}{\sigma(m,n)}\right) - \tfrac{q\hat{q}/2}{\sigma(m,n)}\,\phi\left(\tfrac{n\hat{q}-m\hat{p}}{\sigma(m,n)}\right)\right],\end{aligned}$$

respectively, for any $(m,n) \in \mathbb{R}^2_{++}$, with $\sigma(m,n) = \sqrt{mp\hat{p} + nq\hat{q}}$.

Although there is no closed-form solution for the optimality conditions (9) and (10), it is possible to connect $(m^*(u,c), n^*(u,c))$ to the optimal expenditure (cost) $C(u,c)$ and establish comparative statics as well as symmetry properties. This is accomplished below. In Section 4, we show that a simplified "envelope objective" leads to a closed-form solution with very intuitive and readily interpretable results, at a relative error that decreases in the expected output u.

Remark 5 (*Nonconcavity*)**.** The output objective $U(m,n)$ is generally nonconcave in the input vector (m,n). To see this, it is sufficient to set $\rho = 0$ and consider the determinant of the Hessian (i.e., the matrix D^2U of second derivatives of U):

$$\begin{aligned}\det D^2U &= \det\begin{bmatrix}\partial^2_{mm}U & \partial^2_{mn}U \\ \partial^2_{mn}U & \partial^2_{nn}U\end{bmatrix} \\ &= -\frac{p^2q^2(2-p-q)^2}{8\,\sigma^4(m,n)}\phi^2\left(\frac{nq-mp}{\sigma(m,n)}\right) < 0,\end{aligned}$$

for all $m,n > 0$. This value corresponds to the product of the eigenvalues of the Hessian, thus implying that D^2U has eigenvalues of different signs and thus cannot be negative semidefinite, establishing the generic nonconcavity of U. For this reason, it is desirable to have a good initial seed for a numerical solution to the nonlinear first-order optimality conditions (9)–(10). This issue is addressed in Section 4, where a high-quality approximate solution is given in closed form.

3.4. Solution properties

The solution of the minimum-cost matching problem is the gradient of the cost function.

Lemma 12 (*Cost Gradient; Shephard, 1953*)**.** *Let $m^*(u,c)$ and $n^*(u,c)$ be a solution to the minimum-cost matching problem (*). Then $m^*(u,c) = \partial C(u,c)/\partial c_1$ and $n^*(u,c) = \partial C(u,c)/\partial c_2$. Moreover, the marginal cost of additional expected output is*

$$\frac{\partial C(u,c)}{\partial u} = \frac{c_1}{\partial_m U(m^*(u,c), n^*(u,c))} = \frac{c_2}{\partial_n U(m^*(u,c), n^*(u,c))},$$

for all $u > 0$ and all $c = (c_1, c_2)$ with $c_1, c_2 > 0$.

The last result (the first part of which corresponds to Shephard's lemma; Shephard, 1953) implies that it is possible to infer a cost-optimizing firm's actions by observing expenditure variations (e.g., reported in accounting reports) in response to shocks in the input prices (which are often publicly observable, at least by industry insiders). At the optimal input, the marginal cost of increasing the expected output is equal to the ratio of the input cost c_i and the "marginal productivity" of the corresponding input (with respect to increasing output). This corresponds to a unit cost which is generally different from the average cost, reflecting the firm's cost sensitivity exactly at the (binding) output-attainment constraint, due to the output-efficiency property in Lemma 11.

An additional solution property is obtained as a consequence of Young's theorem about the equality of cross-partial derivatives of smooth functions (Apostol, 1974, Thm. 12.13, p. 360), namely a symmetric effect of unit-cost shocks of type-i inputs on the optimal input quantity of type-j inputs, where $i \neq j$.

Lemma 13 (*Solution Symmetry*)**.** *Let $m^*(u,c)$ and $n^*(u,c)$ be a solution to the minimum-cost matching problem (*). Then*

$$\frac{\partial m^*(u,c)}{\partial c_2} = \frac{\partial n^*(u,c)}{\partial c_1},$$

for all $u > 0$ and all $c = (c_1, c_2)$ with $c_1, c_2 > 0$.

These properties carry over when using a high-quality approximate solution, which provides further insights about the nature of cost-minimizing inputs in a stochastic assembly situation.

4. Envelope optimization

The firm's objective function in Eq. (8), although smooth, is not amenable to finding an analytical solution to the minimum-cost matching problem (*). Thus, instead of approximating the

optimality conditions in Theorem 1,[13] we introduce in Section 4.1 the "envelope output" as alternative objective function. Besides providing an approximately optimal input, which in itself can be used as a high-quality initial seed for a numerical solution of the optimality conditions in Theorem 1, the closed-form expressions allow for structural conclusions leading to managerial insights (discussed in Section 5). In Section 4.2, the solution to the corresponding envelope optimization problem is obtained analytically. Section 4.3 provides a theoretical approximation guarantee, as a function of the expected production volume. In Section 4.4, we define a performance comparison in terms of the relative output error (and the closely related relative average-cost error). An extensive numerical experiment indicates that the approximation performs well over a wide range of parameters, when taken as a surrogate input to the original minimum-cost matching problem.

4.1. Envelope output

A piecewise linear substitute for the firm's nonlinear objective function in Eq. (7) is the *envelope output*, defined as

$$\bar{U}(m,n) = \lfloor mp, nq \rfloor + \rho \lfloor m\hat{p}, n\hat{q} \rfloor, \tag{11}$$

which is a tight upper bound for U.

Lemma 14 (*Envelope Bound*)**.** *For any $(m,n) \in \mathbb{R}^2_+$ it is $U(m,n) \leq \bar{U}(m,n)$.*

In other words, no expected input of a given grade can ever be less than the expected output for that grade. The reason is Jensen's inequality: since the minimum is a concave function (and a positive linear combination of two concave functions remains concave), the expectation of the minimum in Eq. (3) cannot exceed the minimum of the expectations in Eq. (11); see also Schaefer (1976). In particular, the absolute deviation between the output objective U and the envelope objective $\bar{U}$,

$$R(m,n) \triangleq |U(m,n) - \bar{U}(m,n)| = \bar{U}(m,n) - U(m,n) \geq 0,$$
$$m, n \geq 0,$$

describes the envelope approximation error. The following result, which follows from Lemmas 7 and 9, establishes that this approximation error is sublinear in the inputs.

Lemma 15 (*Approximation Error*)**.** *For any $(m,n) \in \mathbb{R}^2_{++}$ it is*

$$\frac{R(m,n)}{\sigma(m,n)} \leq \frac{1+\rho}{\sqrt{2\pi}},$$

where $\sigma(m,n) = \sqrt{mp\hat{p} + nq\hat{q}}$.

For $p = q = 1/2 = \hat{p} = \hat{q}$ and $m = n > 0$, the preceding error bound is tight. In an asymmetric parameter configuration, the bound is somewhat conservative because it is derived under the (generally infeasible) premise that the contingencies $mp = nq$ and $m\hat{p} = n\hat{q}$ apply simultaneously. The risk-pooling effect implied by the sublinearity means that the relative approximation error becomes arbitrarily small, as long as the input size (or equivalently, the expected-output objective) is sufficiently large; cf. Lemma 16 in Section 4.3.

4.2. Optimal envelope inputs

Consider now the (*minimum-cost*) *envelope optimization problem*, which consists of determining the smallest expenditure for a given envelope output $\bar{u} \geq 0$,

$$\bar{C}(\bar{u}, c) = \min_{m,n \geq 0} \{c_1 m + c_2 n\}, \quad \text{s.t.} \quad \bar{U}(m,n) \geq \bar{u}. \tag{**}$$

It turns out that the solution to this modified optimization problem is simple, since the cost objective is linear and the envelope-output attainment constraint is piecewise linear by construction. Let $\gamma = c_1/c_2$ and $\gamma_0 = p/(\rho\hat{q})$. Similarly, let $\hat{\gamma} = c_2/c_1$ and $\hat{\gamma}_0 = q/(\rho\hat{p})$.

Table 1
Solution to the envelope optimization problem for $p \neq q$.

	$p < q$		$p > q$	
	$\gamma < \gamma_0$	$\gamma > \gamma_0$	$\hat{\gamma} < \hat{\gamma}_0$	$\hat{\gamma} > \hat{\gamma}_0$
$\bar{m}^*$	$\frac{q}{p}\frac{\bar{u}}{q+\rho\hat{q}}$	$\frac{\bar{u}}{p+\rho\hat{p}}$	$\frac{\bar{u}}{p+\rho\hat{p}}$	$\frac{\hat{q}}{\hat{p}}\frac{\bar{u}}{q+\rho\hat{q}}$
$\bar{n}^*$	$\frac{\bar{u}}{q+\rho\hat{q}}$	$\frac{\hat{p}}{\hat{q}}\frac{\bar{u}}{p+\rho\hat{p}}$	$\frac{p}{q}\frac{\bar{u}}{p+\rho\hat{p}}$	$\frac{\bar{u}}{q+\rho\hat{q}}$
	$\gamma = \gamma_0,\ \lambda \in [0,1]$		$\hat{\gamma} = \hat{\gamma}_0,\ \lambda \in [0,1]$	
$\bar{m}^*$	$\lambda\left(\frac{q}{p}\frac{\bar{u}}{q+\rho\hat{q}}\right) + (1-\lambda)\left(\frac{\bar{u}}{p+\rho\hat{p}}\right)$		$\lambda\left(\frac{\bar{u}}{p+\rho\hat{p}}\right) + (1-\lambda)\left(\frac{\hat{q}}{p}\frac{\bar{u}}{q+\rho\hat{q}}\right)$	
$\bar{n}^*$	$\lambda\left(\frac{\bar{u}}{q+\rho\hat{q}}\right) + (1-\lambda)\left(\frac{\hat{p}}{q}\frac{\bar{u}}{p+\rho\hat{p}}\right)$		$\lambda\left(\frac{p}{q}\frac{\bar{u}}{p+\rho\hat{p}}\right) + (1-\lambda)\left(\frac{\bar{u}}{q+\rho\hat{q}}\right)$	

Theorem 2. *Let $p, q \in (0,1)$. For $p \neq q$, the solution $(\bar{m}^*, \bar{n}^*)$ of the envelope optimization problem (**) is given in Table 1; for $p = q$, it is $\bar{m}^* = \bar{n}^* = \bar{u}/(p + \rho\hat{p})$.*

Considering the nontrivial case where $p \neq q$, the optimal envelope inputs $\bar{m}^*$ and $\bar{n}^*$ are generically different. Their values depend on whether the cost ratio γ (resp., $\hat{\gamma}$) attains a threshold which corresponds to the ratio of the corresponding grade-attainment probabilities $p/\hat{q}$ (resp., $q/\hat{p}$) when the firm cares equally about the output grades (i.e., when $\rho = 1$) or a correspondingly larger number $p/(\rho\hat{q})$ (resp., $q/(\rho\hat{p})$) when the firm differentiates between output grades (i.e., when $\rho \in (0,1)$). Which particular threshold (γ or $\hat{\gamma}$) is used depends on whether $p < q$ or else $p > q$.

For $p < q$, if the cost ratio γ lies below the threshold γ_0, then the cost of the first (type-1) input is relatively small, leading to an optimal input ratio $\bar{m}^*/\bar{n}^* = q/p > 1$ favoring that input. If the cost ratio exceeds the threshold, then the optimal input ratio $\bar{m}^*/\bar{n}^* = \hat{q}/\hat{p} < 1$ favors the second (type-2) input. If the cost ratio happens to be equal to the threshold, the firm becomes indifferent between all the input ratios in the interval $[\hat{q}/\hat{p}, q/p]$, so in particular ordering the same amount of both component types is also optimal. Due to its singular nature, this knife-edge case is practically not very interesting, and the firm would usually want to operate in a mode with *either* type-1 excess *or* type-2 excess. For $p > q$, the situation is anti-symmetric, and the optimal input ratio depends on whether the (inverse) cost ratio $\hat{\gamma}$ lies above its threshold $\hat{\gamma}_0$ or not. Thus, if $\hat{\gamma} > \hat{\gamma}_0$, then $\bar{m}^*/\bar{n}^* = \hat{q}/\hat{p} > 1$, and conversely: if $\hat{\gamma} < \hat{\gamma}_0$, then $\bar{m}^*/\bar{n}^* = q/p < 1$. In the marginal case where $\hat{\gamma} = \hat{\gamma}_0$, the firm would be indifferent over all input ratios between q/p and $\hat{q}/\hat{p}$. Substituting the optimal envelope inputs in (**) yields the cost $\bar{C}(\bar{u}, c)$ of attaining any given envelope-output objective $\bar{u} > 0$ given the cost vector c.

Corollary 1. *Let $p, q \in (0,1)$. The optimal cost $\bar{C}(\bar{u}, c)$ in (**) is given in Table 2.*

Note that $\bar{C}$ is continuous in all parameters. More importantly, it increases linearly in the envelope-output objective $\bar{u}$, and it is affine (and increasing) in the unit costs c_1 and c_2 for type-1 and type-2 inputs, respectively. The envelope cost can be viewed as the potential of a vector field, producing the vector of optimal envelope inputs as gradient of $\bar{C}$ with respect to the unit-cost vector c.

[13] Such an approach tends to produce results that are not very useful, since the latter are either too crude (e.g., after linearization) or not analytically tractable (e.g., after a higher-order multidimensional Taylor expansion).

Table 2
Optimal cost $\bar{C}(\bar{u}, c)$ in (**), for all $\bar{u} \geq 0$ and $c = (c_1, c_2) \geq 0$.

	$p \leq q$		$p \geq q$	
	$\gamma \leq \gamma_0$	$\gamma \geq \gamma_0$	$\hat{\gamma} \leq \hat{\gamma}_0$	$\hat{\gamma} \geq \hat{\gamma}_0$
$\bar{C}(\bar{u}, c)$	$\left(\frac{\left(\frac{q}{p}\right)c_1+c_2}{q+\rho\hat{q}}\right)\bar{u}$	$\left(\frac{c_1+\left(\frac{\hat{p}}{\hat{q}}\right)c_2}{p+\rho\hat{p}}\right)\bar{u}$	$\left(\frac{c_1+\left(\frac{p}{q}\right)c_2}{p+\rho\hat{p}}\right)\bar{u}$	$\left(\frac{\left(\frac{\hat{q}}{\hat{p}}\right)c_1+c_2}{q+\rho\hat{q}}\right)\bar{u}$

Table 3
Minimum output $\underline{u}(\varepsilon)$ for a relative approximation error $\varepsilon > 0$ at $(\bar{m}^*, \bar{n}^*)$.

	$p \leq q$		$p > q$	
	$\gamma \leq \gamma_0$	$\gamma > \gamma_0$	$\hat{\gamma} \leq \hat{\gamma}_0$	$\hat{\gamma} > \hat{\gamma}_0$
$\underline{u}(\varepsilon)$	$\frac{\hat{p}+\hat{q}}{2\pi}\frac{q+\rho\hat{q}}{q}\left(\frac{1+\rho}{\varepsilon}\right)^2$	$\frac{p+q}{2\pi}\frac{p+\rho\hat{p}}{\hat{p}}\left(\frac{1+\rho}{\rho\varepsilon}\right)^2$	$\frac{\hat{p}+\hat{q}}{2\pi}\frac{p+\rho\hat{p}}{p}\left(\frac{1+\rho}{\varepsilon}\right)^2$	$\frac{p+q}{2\pi}\frac{q+\rho\hat{q}}{\hat{q}}\left(\frac{1+\rho}{\rho\varepsilon}\right)^2$

Corollary 2. *Let $p, q \in (0, 1)$. The solution $(\bar{m}^*, \bar{n}^*)$ of the envelope optimization problem (**) is given by the gradient of the optimal cost $\bar{C}$:*

$$\bar{m}^*(\bar{u}, c) = \frac{\partial \bar{C}(\bar{u}, c)}{\partial c_1} \quad \text{and} \quad \bar{n}^*(\bar{u}, c) = \frac{\partial \bar{C}(\bar{u}, c)}{\partial c_2}, \tag{12}$$

for all $\bar{u} > 0$ and all $c = (c_1, c_2)$ with $c_1, c_2 > 0$, $\gamma \neq \gamma_0$, and $\hat{\gamma}_0 \neq \hat{\gamma}_0$.

The last result provides solution symmetry for the envelope optimization problem (**), analogous to Lemma 12 for the original minimum-cost matching problem (*), with the expected output given by Eq. (7). That is, the slope of the optimal type-1 input $\bar{m}^*$ with respect to the unit cost c_2 is the same as the slope of the optimal type-2 input $\bar{n}^*$ with respect to the unit cost c_1.

4.3. Approximating the solution of the minimum-cost matching problem

Lemma 15, together with Theorem 2, implies that the absolute deviation of the envelope output from the actual expected output is maximal at precisely the solution of the envelope optimization problem. Furthermore, this approximation error grows only sub-linearly in the size of the input, which means that in terms of the *relative approximation error*,

$$r(m, n) = \frac{|U(m, n) - \bar{U}(m, n)|}{U(m, n)} = \frac{R(m, n)}{U(m, n)}, \quad m, n > 0, \tag{13}$$

the quality of the approximation increases as the expected output of the process goes up.

Lemma 16 (*Solution Approximation*)**.** *There exists a finite minimum output $\underline{u}(\varepsilon)$, specified in Table 3, such that*[14]

$$\bar{u} \geq \underline{u}(\varepsilon) \quad \Rightarrow \quad r(\bar{m}^*, \bar{n}^*) \leq \varepsilon,$$

for all $\varepsilon > 0$.

By the definition of the relative approximation error in Eq. (13) the preceding result immediately implies

$$u = \bar{u} \geq \underline{u}(\varepsilon) \quad \Rightarrow \quad \frac{\bar{u}}{1+\varepsilon} \leq U(\bar{m}^*, \bar{n}^*) \leq \bar{u} = \bar{U}(\bar{m}^*, \bar{m}^*) = U(m^*, n^*) = u,$$

for all $\varepsilon > 0$, where $(m^*(u, c), n^*(u, c))$ solves (*), while $(\bar{m}^*(\bar{u}, c), \bar{n}^*(\bar{u}, c))$ solves (**). But this means that for a given expected output u in the minimum-cost matching problem (*), one can simply set $\bar{u} = u$ in the envelope optimization problem (**), determine the closed-form solution $(\bar{m}^*(\bar{u}, c), \bar{n}^*(\bar{u}, c))$ according to Table 1, and use it as an approximate solution for (*). For any $\varepsilon > 0$, this procedure guarantees an expected output of at least $u/(1+\varepsilon)$ given any goal $u \geq \underline{u}(\varepsilon)$. Thus, *the envelope optimization problem (**) effectively approximates the original cost-minimization problem (*), provided the expected output is sufficiently large.*

4.4. Performance evaluation

Notwithstanding the rather conservative lower output bounds in Table 3, to evaluate the performance of the suggested envelope solution "in the field", we first introduce several practical performance measures for a given expected output and then conduct a numerical experiment.

Performance measures. To gauge the performance of the approximate solution $(\bar{m}^*, \bar{n}^*)$ in comparison with the exact solution (m^*, n^*), we first determine the relative loss in expected output when using $(\bar{m}^*, \bar{n}^*)$ instead of (m^*, n^*). That is, for a given probability tuple (p, q), cost vector $c = (c_1, c_2)$, and expected output u, we compute the minimal cost $C^*(u, c)$. Then we determine an "equivalent envelope-output objective" $\bar{u}(u, c)$, implicitly defined as

$$\bar{C}(\bar{u}(u, c), c) = C^*(u, c),$$

which generates the same expenditure as solution to the envelope optimization problem (**). Because of the linearity of the envelope cost in $\bar{u}$ (cf. Corollary 1), the average envelope cost $\overline{\mathrm{AC}}(c) = \bar{C}(\bar{u}, c)/\bar{u}$ does not depend on $\bar{u}$, leading to an expression for the equivalent envelope-output objective:

$$\bar{u}(u, c) = \frac{C^*(u, c)}{\overline{\mathrm{AC}}(c)}. \tag{14}$$

The corresponding solution of the envelope optimization problem (**) produces the expected output

$$\hat{U}(u, c) = U(\bar{m}^*(\bar{u}, c), \bar{n}^*(\bar{u}, c))\big|_{(\bar{u},c)=(\bar{u}(u,c),c)} \in (0, u],$$

at the same cost as the optimal solution of the original minimum-cost matching problem (*) which is required to attain the expected output u. And in general the approximate solution $(\bar{m}^*, \bar{n}^*)$ does not perform as well, so $\hat{U}(u, c) \leq u$. The situation is illustrated in Fig. 4. Overall, this gives rise to the *relative output error*,

$$e(u, c) = \frac{|u - \hat{U}(u, c)|}{u} \in [0, 1). \tag{15}$$

The absolute value is used to prevent negative results, as the solution to (*) needs to be obtained numerically leading to imprecise values C^* and $\hat{U}$ within the tolerances of the chosen algorithms. On the other hand, the average cost per part of expected output (or "unit cost"),

$$\mathrm{AC}(u, c) = \frac{C^*(u, c)}{u},$$

increases (when using the approximate solution) to

$$\widehat{\mathrm{AC}}(u, c) = \frac{C^*(u, c)}{\hat{U}(u, c)} = \frac{\mathrm{AC}(u, c)}{1 - e(u, c)}.$$

But this in turn implies the *relative unit-cost error*,

$$\hat{e}(u, c) = \frac{\left|\widehat{\mathrm{AC}}(u, c) - \mathrm{AC}(u, c)\right|}{\mathrm{AC}(u, c)} = \frac{e(u, c)}{1 - e(u, c)}, \tag{16}$$

which always exceeds the relative output error $e(u, c)$ (at least weakly).

[14] For $\rho = 0$, the *relevant* entries of Table 3 do remain finite.

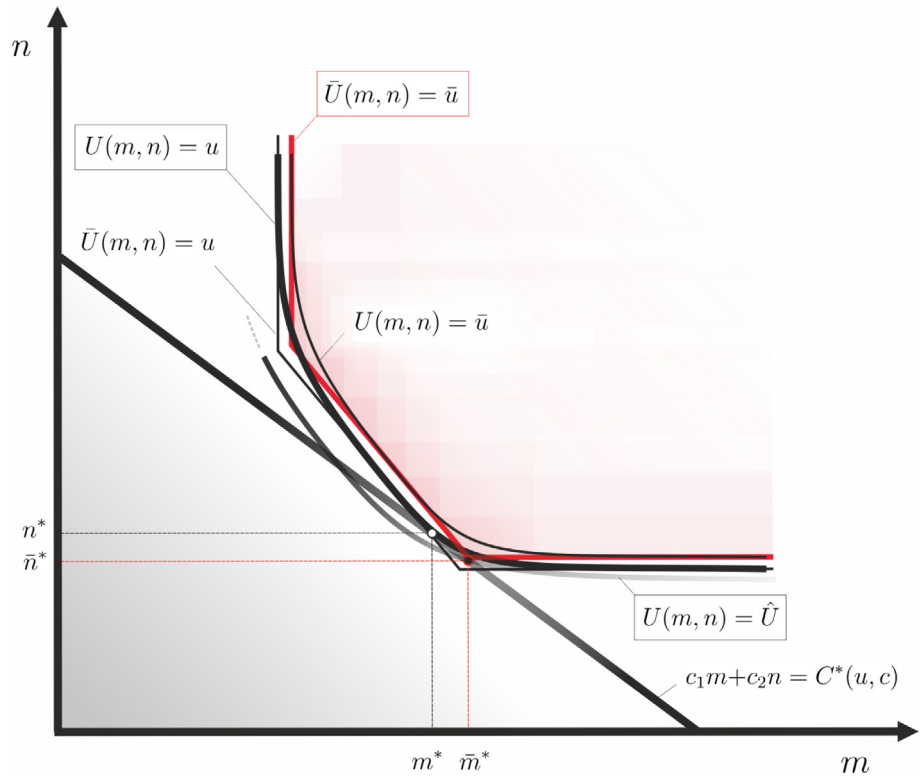


Fig. 4. Performance comparison.

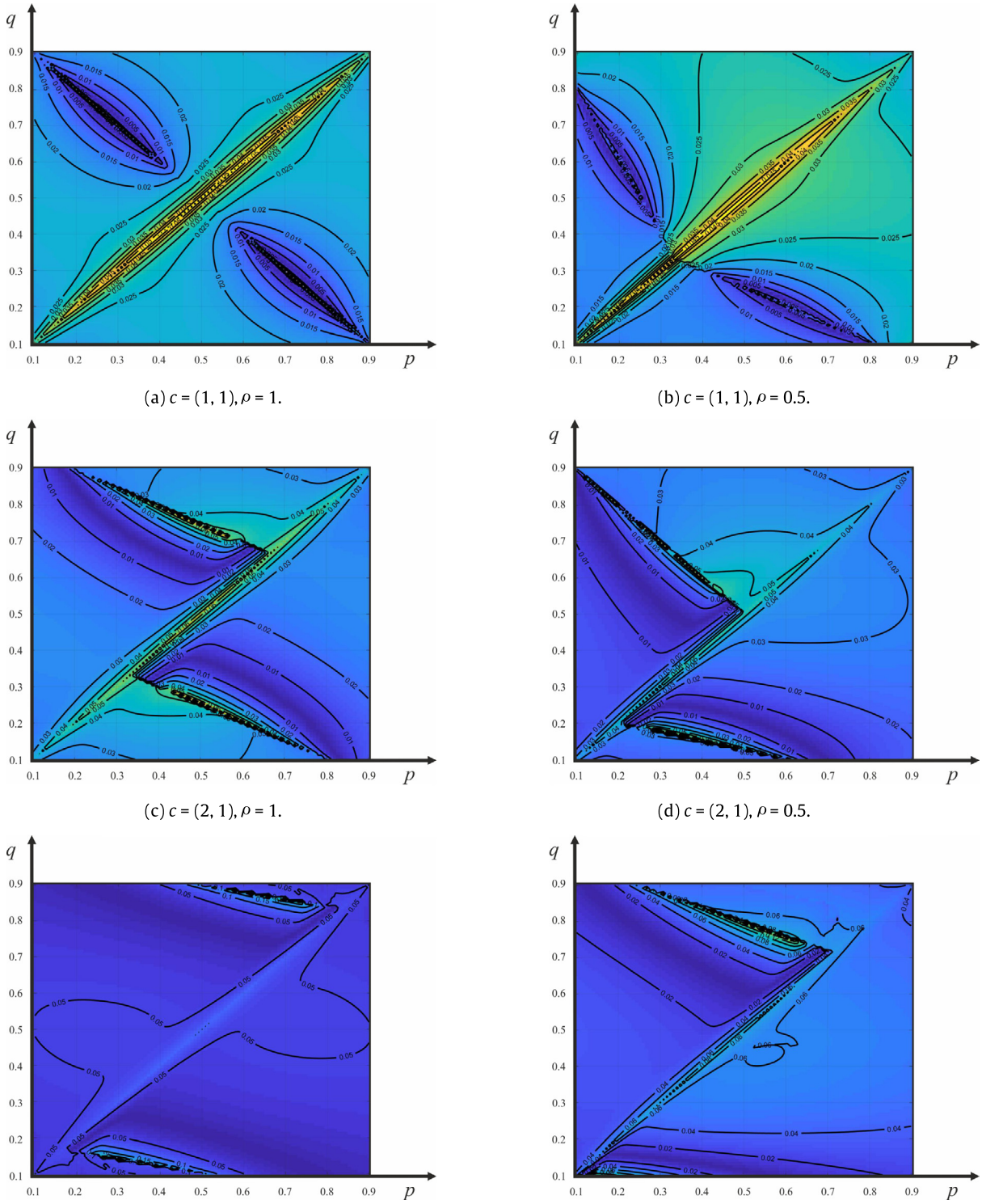
Numerical experiment. Since the relative errors, e and $\hat{e}$ in Eqs. (15) and (16), are quite close, we restrict our attention to a computation of the former, for the fixed target output level $u = 100$, cost tuples $c = (c_1, c_2)$ in the set $\{(1, 1), (2, 1), (5, 1)\}$, and a weight ρ for grade-B goods relative to grade-A goods with values in $\{0.5, 1\}$. For larger values of u, the performance of the envelope solution is even better, since—as a result of more risk pooling with larger outputs—the coefficients of variation in the relevant distributions decrease. For smaller values of u the error somewhat increases, but at that point we leave industrial scale and below $u = 50$ even the standard normal approximation of the binomial distribution becomes an area of concern. The conservative approximation bounds in Table 3 are quite large by comparison. For example, given $\varepsilon = 10\%$ (resp., $\varepsilon = 5\%$) and $c = (1, 1)$ with $\rho = 1$, we find from Lemma 16 that $\underline{u}(\varepsilon) \approx 127$ (resp., $\underline{u}(\varepsilon) \approx 509$), whereas the actual numerical performance is considerably better. Finally, we reiterate here that the main point of the solution to the envelope optimization problem (**) is *not* that it should necessarily be considered as a substitute of the (numerical) solution of the original minimum-cost matching problem (*), which can easily be obtained, but rather that its excellent numerical performance and theoretical approximation properties serve as a reasonable justification to glean structural solution properties from its explicit expressions.

The results of our numerical experiment are summarized in Fig. 5a–f which provide contour-plots of the relative output error, for p, q between 10% and 90%, and are indexed by the aforementioned values of parameters c and ρ. In the equal-cost case (with $c_1 = c_2 = 1$), depicted in Figs. 5(a) and 5(b), the relative error is largest (but not more than 5%) along the 45-degree line where the grade-A attainment probabilities p and q are the same, as this tends to align the cost envelope and the cost line (corresponding to a slight counterclockwise turn of the iso-output curves[15] in Fig. 4) thus producing a greater numerical instability in the numerical solution of (*). This theme (of greater error along regions of alignment between the cost frontier and iso-output curves) continues, albeit in a less apparent way, throughout Figs. 5(c)–5(f) as well. Since the implied relative output error of the solution to the envelope optimization problem (**) stays in the order of 5% throughout, the performance of the proposed approximation appears quite satisfying for many applications (where the dispersion of the underlying problem parameters, such as p, q and/or c_1, c_2, may well be higher, e.g., in the order of 10% or more).

5. Conclusion

Assembly situations often involve matching components that have random characteristics (e.g., specified in terms of mechanical or electromagnetic properties), thus implying different possible "grades" when ordering parts of a given type (e.g., balance springs or balance wheels in mechanical watchmaking). A valid assembly needs to respect that components of different types must be of the same grade, for instance, to reach an allowable band of resonance frequencies for assembled watch oscillators. Additional quantities of one input provide insurance against imbalances in the available same-grade matches across different component types. The optimal input portfolio (which satisfies the conditions in Theorem 1) determines the firm's minimal cost. In Eq. (8) we have provided an explicit form of the firm's production function with component sorting and matching. An important conclusion from our analysis is that the role of providing this insurance is clearly assigned: One input type can be considered as the "critical component", thus relegating the status of the other to serve as "buffer component". The latter provides a

[15] Assuming equal scales on both axes, the fact that the iso-cost line in Fig. 4 has been drawn for $c_1 < c_2$ (whereas in our numerical experiment we use $c_1 \geq c_2$) does not affect the response of the iso-output curves to changes in the grade-A attainment probabilities p and q, since both U and $\bar{U}$ are independent of the input costs.



(a) $c = (1, 1), \rho = 1$.

(b) $c = (1, 1), \rho = 0.5$.

(c) $c = (2, 1), \rho = 1$.

(d) $c = (2, 1), \rho = 0.5$.

(e) $c = (5, 1), \rho = 1$.

(f) $c = (5, 1), \rho = 0.5$.

Fig. 5. Relative output error $e(u, c)$, for $u = 100$ and $0.1 \leq p, q \leq 0.9$.

guarantee, albeit imperfect, against the possibility of excess in critical components. This distinction is locally insensitive to parameter changes. Moreover, the composition of the optimal input portfolio depends only on the ratio of the grade-attainment likelihoods and not the cost—as long as the ratio of the input prices does not move past a threshold (γ_0 or $\hat{\gamma}_0$ in the model). The optimal composition of the input portfolio can be gleaned, at least approximately, from the solution to the envelope optimization problem, leading to a useful rule of thumb: the ratio of the optimal input quantities (implied by Theorem 2) corresponds to the ratio of the grade-attainment likelihoods (i.e., either p/q for grade A, or $\hat{p}/\hat{q}$ for grade B). This formalizes that, quite unsurprisingly, the input that is "relatively cheaper" serves as the buffer component, whereby the assessment of what it means to be

relatively cheaper involves both the per-unit input costs as well as the grade-attainment likelihoods of the different component types; cf. Table 1.

Beyond providing structural insights, the solution to the envelope optimization problem (**) can be used as a high-quality seed for a numerical solution of the optimality conditions (9)–(10) which in turn pinpoint a solution to the original (nonconvex) minimum-cost matching problem (*), with the expected output in Eq. (7) that uses the (standard) normal approximation of the underlying binomial distribution. Finally, the results in Lemma 16 show that the explicit input portfolio obtained from the envelope optimization problem serves as an effective approximation of the minimum-cost input portfolio, with an arbitrarily small relative approximation error for a sufficiently large expected production volume.

Declaration of competing interest

The author declares that he has no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Appendix. Proofs

Proof of Lemma 1. We restrict our attention to establishing the monotonicity of $U(m,n)$ in m; the proof for the monotonicity of U in n is symmetric. Let $\pi(x|m) = \text{Prob}(X = x|m)$ denote the discrete probability distribution function in Eq. (1), which vanishes for $x \notin \{0,1,\ldots,m\}$. Increasing the number of input draws yields a first-order stochastically dominant distribution (i.e., $\pi(\cdot|m+1) \succ_{\text{FOSD}} \pi(\cdot|m)$; see, e.g., Klenke and Mattner, 2010), which implies for any realization $y \in \{1,\ldots,n\}$ that

$$\mathbb{E}\left[\lfloor X, y\rfloor | m+1\right] > \mathbb{E}[\lfloor X, y\rfloor | m].$$

Thus, $\mathbb{E}[Z|m,n]$ is increasing in m, for $n > 0$. On the other hand, the random variable $\hat{X}(m) = m - X$ is distributed with the discrete probability distribution

$$\begin{aligned}\text{Prob}(\hat{X} = x|m) &= \binom{m}{m-x} p^{m-x}(1-p)^x \\ &= \binom{m}{x}(1-\hat{p})^{m-x}\hat{p}^x, \quad x \in \{0,\ldots,m\},\end{aligned}$$

where $\hat{p} = 1 - p$, so that we obtain an equivalent setup as before. The preceding argument therefore implies that $\mathbb{E}[\hat{Z}|m,n]$ is increasing in m, for $n > 0$. This concludes our proof. □

Proof of Lemma 2. By Lemma 1, the output-achievement constraint $U(m,n) \geq u$ defines an upper contour set $\mathcal{C}(u)$ which is such that $(m,n) \in \mathcal{C}(u)$ implies that $(\hat{m},\hat{n}) \in \mathcal{C}(u)$ for all $(\hat{m},\hat{n}) \geq (m,n)$. Let $(m_0,n_0) \in \mathcal{C}(u)$; then the optimal cost $C(u,c)$ is bounded from above by $C_0 = c_1 m_0 + c_2 n_0$. This means that one can add the constraint $C_0 \geq c_1\, m + c_2 n$ without changing the feasibility of any solution to the problem. Moreover, the set $\mathcal{S}(u) = \{(m,n) \geq 0 : c_1\, m + c_2 n\} \cap \mathcal{C}(u)$ is finite for integer m,n, so that a solution exists.[16] □

[16] For (nonnegative) real-valued m,n, a solution exists by the Weierstrass theorem (see, e.g., Bertsekas, 1995, p. 540), since the objective function is continuous and the set $\mathcal{S}(u)$ is compact; as mentioned in footnote 11, Ilienko (2013) discusses a continuous version of the binomial distribution. In a real-world setting, the firm would need to round a real-valued input to a suitable integer number of pieces, e.g., depending on the batch size.

Proof of Lemma 3. Since φ is by assumption increasing, $U(m,n) \geq u$ if and only if $\hat{U}(m,n) \geq \hat{u} = \varphi(u) \geq \varphi(0)$, for all $u \geq 0$. Thus,

$$\hat{C}(\varphi(u), \alpha c) = \alpha \min_{m,n\geq 0}\{c_1\, m + c_2 n\}, \quad \text{s.t.} \quad U(m,n) \geq u, \tag{17}$$

which implies that $C(\varphi(u), \alpha c) = \alpha C(u,c)$, as claimed. □

Proof of Lemma 4. The claim follows immediately from the equivalence of the minimum-cost matching problem (*) and the scaled optimization problem in Eq. (17) for $\varphi(u) \equiv u$. □

Proof of Lemma 5. Differentiating the normal distribution $F(x|m)$ in Eq. (5) with respect to m yields

$$\partial_m F(x|m) = \frac{-p\sqrt{mp\hat{p}} - (x - mp)p\hat{p}/(2\sqrt{mp\hat{p}})}{mp\hat{p}}\phi\left(\frac{x - mp}{\sqrt{mp\hat{p}}}\right),$$

or equivalently,

$$\partial_m F(x|m) = -\frac{p + (x/m)}{2\sigma_x(m)}\phi\left(\frac{x - \mu_x(m)}{\sigma_x(m)}\right) < 0, \quad x \geq 0.$$

This establishes first-order stochastic dominance with respect to an increase in the number of type-1 parts ordered. Similarly,

$$\partial_n G(y|n) = -\frac{q + (y/n)}{2\sigma_y(n)}\phi\left(\frac{y - \mu_y(n)}{\sigma_y(n)}\right) < 0, \quad y \geq 0,$$

which shows that increasing the number of type-2 parts induces a first-order stochastically dominant shift in the random number of these components in the grade-A category. □

Proof of Lemma 6. Let $z \in [0, \lfloor m,n\rfloor]$. Then, taking into account the dependence of Z on X and Y,

$$\begin{aligned}H(z|m,n) &= \text{Prob}(Z \leq z|m,n) \\ &= \text{Prob}(X \leq z|m) + \text{Prob}(Y \leq z|n) \\ &\quad -\text{Prob}(X, Y \leq z|m,n),\end{aligned}$$

or equivalently,

$$H(z|m,n) = F(z|m) + G(z|n) - F(z|m)G(z|n),$$

so

$$H(z|m,n) = 1 - (1 - F(z|m))(1 - G(z|n)),$$

which establishes the result. □

Proof of Lemma 7. Note first that $Z = \lfloor X, Y\rfloor = X + \lfloor 0, Y - X\rfloor$. Thus, given m and n, the expected value of Z can be written in the form

$$\mathbb{E}[Z|m,n] = \mu_x(m) + \mathbb{E}\left[\lfloor 0, Y - X\rfloor | m,n\right].$$

Since X, Y are by assumption normally distributed (and independent), the random difference $\Delta = Y - X$ is also normally distributed—with mean

$$\mu(m,n) = \mu_y(n) - \mu_x(m) = nq - mp$$

and variance

$$\sigma^2(m,n) = \sigma_x^2(m) + \sigma_y^2(n) = mp\hat{p} + nq\hat{q}.$$

By virtue of the linearity of the expectation operator, we can introduce the standard normal random variable $(\Delta - \mu(m,n))/\sigma(m,n)$ through an affine transformation as follows:

$$\begin{aligned}\mathbb{E}\left[\lfloor 0, \Delta\rfloor | m,n\right] &= \mu(m,n) + \mathbb{E}\left[\lfloor -\mu(m,n), \Delta - \mu(m,n)\rfloor | m,n\right] \\ &= \mu(m,n) + \sigma(m,n) \\ &\quad \times \mathbb{E}\left[\left\lfloor -\tfrac{\mu(m,n)}{\sigma(m,n)}, \tfrac{\Delta - \mu(m,n)}{\sigma(m,n)}\right\rfloor \middle| m,n\right].\end{aligned}$$

As a result, the expected value in the preceding line can be evaluated directly,

$$E\left[\left\lfloor -\tfrac{\mu(m,n)}{\sigma(m,n)}, \tfrac{\Delta-\mu(m,n)}{\sigma(m,n)}\right\rfloor \middle| m,n\right] = -\tfrac{\mu(m,n)}{\sigma(m,n)}\left(1-\Phi\left(-\tfrac{\mu(m,n)}{\sigma(m,n)}\right)\right) -\phi\left(-\tfrac{\mu(m,n)}{\sigma(m,n)}\right) = -\tfrac{\mu(m,n)}{\sigma(m,n)}\Phi\left(\tfrac{\mu(m,n)}{\sigma(m,n)}\right) - \phi\left(\tfrac{\mu(m,n)}{\sigma(m,n)}\right),$$

where we have used the fact that for any $\xi \in \mathbb{R}$ by symmetry of the standard normal distribution it is $\Phi(\xi) = 1 - \Phi(-\xi)$ and $\phi(\xi) = \phi(-\xi)$. The last equation yields

$$\mathbb{E}[\lfloor 0, Y - X\rfloor | m,n] = \mu(m,n)\left(1-\Phi\left(\tfrac{\mu(m,n)}{\sigma(m,n)}\right)\right) -\sigma(m,n)\phi\left(\tfrac{\mu(m,n)}{\sigma(m,n)}\right),$$

so that

$$\begin{aligned}\mathbb{E}[Z|m,n] &= \mu_x(m)\Phi\left(\tfrac{\mu(m,n)}{\sigma(m,n)}\right) \\ &\quad +\mu_y(n)\left(1-\Phi\left(\tfrac{\mu(m,n)}{\sigma(m,n)}\right)\right) - \sigma(m,n)\phi\left(\tfrac{\mu(m,n)}{\sigma(m,n)}\right) \\ &= mp\,\Phi\left(\frac{nq-mp}{\sigma(m,n)}\right) + nq\,\Phi\left(\frac{mp-nq}{\sigma(m,n)}\right) \\ &\quad -\sigma(m,n)\phi\left(\frac{nq-mp}{\sigma(m,n)}\right),\end{aligned}$$

which completes our proof. □

Proof of Lemma 8. Let $\hat{z} \in [0, \lfloor m,n\rfloor]$. Then

$$\begin{aligned}\hat{H}(\hat{z}|m,n) &= \text{Prob}(\hat{Z} \le \hat{z}|m,n) \\ &= \text{Prob}(m - X \le \hat{z}|m) \\ &\quad +\text{Prob}(n - Y \le \hat{z}|n) - \text{Prob}(m - X, n - Y \le \hat{z}|m,n),\end{aligned}$$

or equivalently,

$$\begin{aligned}\hat{H}(\hat{z}|m,n) &= \big(1 - F(m-\hat{z}|m)\big) + \big(1 - G(n-\hat{z}|n)\big) \\ &\quad -\big(1 - F(m-\hat{z}|m)\big)\big(1 - G(n-\hat{z}|n)\big),\end{aligned}$$

which establishes the result. □

Proof of Lemma 9. The proof of this result is analogous to the proof of Lemma 7 when replacing the grade-attainment probabilities p, q by their respective complements $\hat{p}, \hat{q}$. □

Proof of Lemma 10. Note first that based on Eqs. (5) and (6) the probability densities for type-1 and type-2 parts are given by

$$f(x|m) = \frac{1}{\sigma_x(m)}\phi\left(\frac{x-\mu_x(m)}{\sigma_x(m)}\right) \quad \text{and}$$
$$g(y|n) = \frac{1}{\sigma_y(n)}\phi\left(\frac{y-\mu_y(n)}{\sigma_y(n)}\right),$$

respectively, for all $(x,y) \in \mathbb{R}^2_+$. The gradient of $U(m,n)$ is $(\partial_m U, \partial_n U)(m,n)$, where

$$\begin{aligned}\partial_m U &= \int_0^{\lfloor m,n\rfloor} \Big[\,\rho\,(\partial_m F(m-z|m) + f(m-z|m))\,G(n-z|n) \\ &\quad -\partial_m F(z|m)\,(1-G(z|n))\,\Big]\,dz\end{aligned}$$

and

$$\begin{aligned}\partial_n U &= \int_0^{\lfloor m,n\rfloor} \Big[\,\rho\,(\partial_n G(n-z|n) + g(n-z|n))\,F(m-z|m) \\ &\quad -\partial_n G(z|n)\,(1-F(z|m))\,\Big]\,dz.\end{aligned}$$

We consider first the sign of $\partial_m U$. The function $G(n-z|n)$ is positive on the interval $(0, \lfloor m,n\rfloor)$. Moreover,

$$\partial_m F(m-z|m) + f(m-z|m) = \left(\frac{1-p}{2} + \frac{z}{2m}\right) f(m-z|m) > 0,$$

$z \in [0, \lfloor m,n\rfloor]$.

Since $\partial_m F < 0$ and $(1 - G(z|n)) > 0$ on $(0, \lfloor m,n\rfloor)$, this implies $\partial_m U > 0$. The fact that $\partial_n U > 0$ can be shown in an analogous manner. □

Proof of Lemma 11. Let $(u,c) \in \mathbb{R}_+ \times \mathbb{R}^2_{++}$, and assume that the input tuple $(m^*(u,c), n^*(u,c))$ solves the minimum-cost matching problem (*) with objective function $U(m,n)$, as defined in Eq. (8) for all $(m,n) \in \mathbb{R}^2_+$. Clearly, it is not possible that $U(m^*(u,c), n^*(u,c)) < u$, as then the utility attainment constraint is violated and the solution is not feasible. On the other hand, it is also impossible that $U(m^*(u,c), n^*(u,c)) > u$, since then by Lemma 10 a better input tuple $(\hat{m}^*(u,c), \hat{n}^*(u,c))$ can be found with $\hat{m}^*(u,c) < m^*(u,c)$ and $\hat{n}^*(u,c) < n^*(u,c)$ (i.e., lower cost) which still satisfies the output-attainability constraint (i.e., $U(\hat{m}^*(u,c), \hat{n}^*(u,c)) \ge u$. Hence, at the optimum the output-efficiency condition (9) holds for any solution of the minimum-cost matching problem. □

Proof of Theorem 1. Let $(u,c) \in \mathbb{R}_+ \times \mathbb{R}^2_{++}$. We first introduce the Lagrangian $L(m,n;\lambda) = c_1\, m + c_2 n - \lambda(U(m,n) - u)$ for the minimum-cost matching problem (*), where $\lambda \ge 0$ is an adjoint variable, representing the shadow cost of relaxing the output-attainment condition by one unit. The first-order necessary optimality conditions become

$$\partial_m L(m,n;\lambda) = c_1 - \lambda\partial_m U(m,n) = 0, \tag{18}$$
$$\partial_n L(m,n;\lambda) = c_2 - \lambda\partial_n U(m,n) = 0. \tag{19}$$

Given that by Lemma 10 the gradient of U is positive at the frontier $U(m,n) = u$, and that by Lemma 11 the output-attainment constraint is binding at the optimum (resulting in Eq. (9)), the shadow cost must be positive (i.e., $\lambda > 0$). Thus, combining Eqs. (18) and (19) yields Eq. (10). The gradient of the expected grade-A output with respect to the input vector (m,n) can be obtained via direct differentiation of $\mu_z(m,n)$ as specified in Lemma 7:

$$\nabla_{(m,n)}\mu_z(m,n) = \left(p\Phi\left(\tfrac{nq-mp}{\sigma(m,n)}\right) - \tfrac{p\hat{p}}{2\sigma(m,n)}\varphi\left(\tfrac{nq-mp}{\sigma(m,n)}\right), q\Phi\left(\tfrac{mp-nq}{\sigma(m,n)}\right) - \tfrac{q\hat{q}}{2\sigma(m,n)}\varphi\left(\tfrac{nq-mp}{\sigma(m,n)}\right)\right);$$

similarly, for the gradient of the expected grade-B output with respect to the input vector,

$$\nabla_{(m,n)}\hat{\mu}_z(m,n) = \left(\hat{p}\Phi\left(\tfrac{n\hat{q}-m\hat{p}}{\sigma(m,n)}\right) - \tfrac{p\hat{p}}{2\sigma(m,n)}\varphi\left(\tfrac{n\hat{q}-m\hat{p}}{\sigma(m,n)}\right), \hat{q}\Phi\left(\tfrac{m\hat{p}-n\hat{q}}{\sigma(m,n)}\right) - \tfrac{q\hat{q}}{2\sigma(m,n)}\varphi\left(\tfrac{n\hat{q}-m\hat{p}}{\sigma(m,n)}\right)\right).$$

But this implies the gradient of the expected output,

$$\begin{aligned}\nabla_{(m,n)}U(m,n) &= \big(\partial_m U(m,n), \partial_n U(m,n)\big) \\ &= \nabla_{(m,n)}\mu_z(m,n) + \rho\left[\nabla_{(m,n)}\hat{\mu}_z(m,n)\right],\end{aligned}$$

resulting in the expressions given in the theorem. This concludes our proof. □

Proof of Lemma 12. Taking into account that the constraint $U(m,n) \ge u$ in (*) is independent of c, by the envelope theorem (Mas-Colell et al., 1995, Thm. M.L.1., pp. 965–966) it is for any $i \in \{1,2\}$:

$$\frac{\partial C(u,c)}{\partial c_i} = \frac{\partial}{\partial c_i}\bigg|_{(m^*(u,c),n^*(u,c))} L(m,n;\lambda) = \begin{cases} m^*(u,c), & \text{if } i = 1, \\ n^*(u,c), & \text{if } i = 2,\end{cases}$$

independent of λ, where $L(m,n;\lambda) = c_1\, m + c_2 n - \lambda(U(m,n) - u)$. Similarly, the envelope theorem also yields:

$$\frac{\partial C(u,c)}{\partial u} = \frac{\partial}{\partial u}\bigg|_{(m^*(u,c),n^*(u,c))} L(m,n;\lambda) = \lambda,$$

which, by virtue of Eqs. (18) and (19), implies the corresponding assertion in this result. □

Proof of Lemma 13. Since $C(u,c)$ is twice continuously differentiable in c, the claim follows directly from Young's theorem, as the Hessian of C is symmetric. □

Proof of Lemma 14. Let $(m,n) \in \mathbb{R}^2_+$. By Jensen's inequality,

$$\mathbb{E}[\lfloor X, Y\rfloor | m,n] \le \lfloor \mathbb{E}[X|m], \mathbb{E}[Y|n]\rfloor = \lfloor mp, nq\rfloor.$$

Similarly, Jensen's inequality also implies that

$$\mathbb{E}[\lfloor m - X, n - Y\rfloor | m,n] \le \lfloor m - \mathbb{E}[X|m], n - \mathbb{E}[Y|n]\rfloor = \lfloor m\hat{p}, n\hat{q}\rfloor.$$

Hence, taking into account the definitions of U, $\bar{U}$ in Eqs. (3) and (11), one obtains

$$U(m,n) = \mathbb{E}[\lfloor X, Y\rfloor + \rho\lfloor m - X, n - Y\rfloor | m,n] \le \bar{U}(m,n),$$

which concludes the proof. □

Proof of Lemma 15. Let $m,n > 0$. If we set $\delta(m,n) = nq - mp$, then by Lemma 7 for $\delta \ge 0$ it is

$$0 \le \frac{\lfloor mp, nq\rfloor - \mu_z(m,n)}{\sigma(m,n)} = \phi\left(\frac{\delta(m,n)}{\sigma(m,n)}\right) - \frac{\delta(m,n)}{\sigma(m,n)}\left(1 - \Phi\left(\frac{\delta(m,n)}{\sigma(m,n)}\right)\right).$$

On the other hand, since $(d/dx)\,(\phi(x) - x\,(1-\Phi(x))) = -(1-\Phi(x)) < 0$, for all $x \in \mathbb{R}$, the expression on the right-hand side of the preceding equality is downward-sloping in $\delta/\sigma > 0$, so

$$\delta(m,n) \ge 0 \quad\Rightarrow\quad 0 \le \frac{\lfloor mp, nq\rfloor - \mu_z(m,n)}{\sigma(m,n)} \le \frac{1}{\sqrt{2\pi}} = \phi(0). \tag{20}$$

For $\delta \le 0$, Lemma 7 yields

$$0 \le \frac{\lfloor mp, nq\rfloor - \mu_z(m,n)}{\sigma(m,n)} = \phi\left(\frac{\delta(m,n)}{\sigma(m,n)}\right) + \frac{\delta(m,n)}{\sigma(m,n)}\,\Phi\left(\frac{\delta(m,n)}{\sigma(m,n)}\right).$$

Thus, taking account of the fact that $(d/dx)\,(\phi(x) + x\Phi(x)) = \Phi(x) > 0$, for all $x \in \mathbb{R}$, we find that the expression on the right-hand side must be upward-sloping in $\delta/\sigma < 0$, whence

$$\delta(m,n) \le 0 \quad\Rightarrow\quad 0 \le \frac{\lfloor mp, nq\rfloor - \mu_z(m,n)}{\sigma(m,n)} \le \frac{1}{\sqrt{2\pi}} = \phi(0). \tag{21}$$

Combining Eqs. (20) and (21) yields

$$0 \le \frac{\lfloor mp, nq\rfloor - \mu_z(m,n)}{\sigma(m,n)} \le \frac{1}{\sqrt{2\pi}}, \tag{22}$$

for all $m,n > 0$. In a completely analogous manner, after introducing $\hat{\delta}(m,n) = n\hat{q} - m\hat{p}$, Lemma 9 yields, independent of the sign of $\hat{\delta}$, that

$$0 \le \frac{\lfloor m\hat{p}, n\hat{q}\rfloor - \hat{\mu}_z(m,n)}{\sigma(m,n)} \le \frac{1}{\sqrt{2\pi}}, \tag{23}$$

for all $m,n > 0$. Using Eqs. (22) and (23), together with the fact that $U = \mu_z + \rho\hat{\mu}_z$, we can conclude by the triangular inequality for the Euclidean norm on $\mathbb{R}$ that

$$0 \le \frac{\bar{U}(m,n) - U(m,n)}{\sigma(m,n)} \le \frac{1+\rho}{\sqrt{2\pi}}, \tag{24}$$

for all $m,n > 0$. The (maximum) approximation error is therefore at most linear in the joint standard deviation $\sigma(m,n)$, so

$$\frac{R(m,n)}{\sigma(m,n)} \le \frac{1+\rho}{\sqrt{2\pi}},$$

which concludes our proof. □

Proof of Theorem 2. Let $\bar{u} > 0$ and fix $p,q \in (0,1)$. We first consider the level set in (m,n)-space where $\bar{U}(m,n) = \bar{u}$. For relatively small quantities of type-1 components, when $m \le n\lfloor(q/p), \hat{q}/\hat{p}\rfloor$, one obtains that

$$m = \frac{\bar{u}}{p + \rho\hat{p}}.$$

For relatively small quantities of type-2 components, when $n \le m\lfloor p/q, \hat{p}/\hat{q}\rfloor$, it is

$$n = \frac{\bar{u}}{q + \rho\hat{q}}.$$

The situation in between depends on whether $p \le q$ or not. In the former case,

$$n = \frac{\bar{u} - mp}{\rho\hat{q}},$$

whereas in the latter case (when $p > q$),

$$n = \frac{\bar{u} - m\rho\hat{p}}{q}.$$

Thus, when $p \le q$, the points $(m_1,n_1) = (\frac{q}{p}, 1)\frac{\bar{u}}{q+\rho\hat{q}}$ and $(m_2,n_2) = (1, \frac{\hat{p}}{\hat{q}})\frac{\bar{u}}{p+\rho\hat{p}}$ define the iso-$\bar{U}$ contour. For $p = q$, the two points coincide and lie on the 45-degree line, so that (m^*,n^*) with $m^* = n^* = \bar{u}/(p+\rho\hat{p})$ solves the envelope optimization problem (**) as the only available solution candidate. For $p < q$, the points (m_1,n_1) and (m_2,n_2) are such that $m_1 > m_2$ and $n_1 < n_2$. The marginal rate of substitution (i.e., the slope of the iso-utility contour) in between the points is

$$\frac{n_2 - n_1}{m_2 - m_1} = -\frac{p}{\rho\hat{q}} = -\gamma_0.$$

Comparing this with the slope $-\gamma = -(c_1/c_2)$ of the iso-$\bar{C}$ contour leads to the optimal solution of (**) for $\gamma \ne \gamma_0$:

$$(\bar{m}^*, \bar{n}^*) = \begin{cases}(m_1,n_1), & \text{if } \gamma < \gamma_0,\\ (m_2,n_2), & \text{if } \gamma > \gamma_0.\end{cases} \tag{25}$$

When the marginal rate of substitution is equal to the slope of the iso-$\bar{C}$ contour, any intermediate point solves (**), so

$$(\bar{m}^*, \bar{n}^*) \in \{\lambda(m_1,n_1) + (1-\lambda)(m_2,n_2) : \lambda \in [0,1]\}, \quad \gamma = \gamma_0. \tag{26}$$

In the case where $p > q$, the points $(\hat{m}_1,\hat{n}_1) = (1, \frac{p}{q})\frac{\bar{u}}{p+\rho\hat{p}}$ and $(\hat{m}_2,\hat{n}_2) = (\frac{\hat{q}}{\hat{p}}, 1)\frac{\bar{u}}{q+\rho\hat{q}}$ define the iso-$\bar{U}$ contour. Switching the axes (compared to the earlier case where $p < q$), the marginal rate of substitution between the two points is

$$\frac{\hat{m}_2 - \hat{m}_1}{\hat{n}_2 - \hat{n}_1} = -\frac{q}{\rho\hat{p}} = -\hat{\gamma}_0.$$

A comparison with the slope $-\hat{\gamma} = -(c_2/c_1)$ then yields the solution to (**) for $\hat{\gamma} \ne \hat{\gamma}_0$:

$$(\bar{m}^*, \bar{n}^*) = \begin{cases}(\hat{m}_1,\hat{n}_1), & \text{if } \hat{\gamma} < \hat{\gamma}_0,\\ (\hat{m}_2,\hat{n}_2), & \text{if } \hat{\gamma} > \hat{\gamma}_0.\end{cases} \tag{27}$$

For $\hat{\gamma} = \hat{\gamma}_0$, the solution to (**) is again set-valued:

$$(\bar{m}^*, \bar{n}^*) \in \{\lambda(\hat{m}_1,\hat{n}_1) + (1-\lambda)(\hat{m}_2,\hat{n}_2) : \lambda \in [0,1]\}, \quad \hat{\gamma} = \hat{\gamma}_0. \tag{28}$$

Table 1 summarizes the results contained in Eqs. (25)–(28), which concludes our proof. □

Table 4
Minimum output $\underline{u}(\varepsilon)$ with the input bounds $\underline{m}(M(\varepsilon))$ and $\hat{\underline{m}}(M(\varepsilon))$, given $\varepsilon > 0$.

	$p \le q$		$p > q$	
	$\gamma \le \gamma_0$	$\gamma > \gamma_0$	$\hat{\gamma} \le \hat{\gamma}_0$	$\hat{\gamma} > \hat{\gamma}_0$
	$\delta = 0$	$\hat{\delta} = 0$	$\delta = 0$	$\hat{\delta} = 0$
$\underline{u}(\varepsilon)$	$\left(\frac{q+\rho\hat{q}}{q/p}\right)\underline{m}(M(\varepsilon))$	$(p+\rho\hat{p})\,\hat{\underline{m}}(M(\varepsilon))$	$(p+\rho\hat{p})\,\underline{m}(M(\varepsilon))$	$\left(\frac{q+\rho\hat{q}}{\hat{q}/\hat{p}}\right)\hat{\underline{m}}(M(\varepsilon))$

Proof of Corollary 1. Let $p, q \in (0, 1)$ and $\bar{u} \ge 0$. By definition $\bar{C}(\bar{u}, c) = c_1\bar{m}^* + c_2\bar{n}^*$, where $(\bar{m}^*, \bar{n}^*)$ is the solution to the envelope optimization problem (**) in Theorem 2. This directly implies all the entries in Table 2. □

Proof of Corollary 2. The claim follows immediately by direct differentiation of the various entries in Table 2 which were established by Corollary 1. □

Proof of Lemma 16. Let $M > 0$ be any given constant. To exclude trivialities, we restrict attention to positive $m, n > 0$. As in the proof of Lemma 15, we set $\delta(m, n) = nq - mp$ and $\hat{\delta}(m, n) = n\hat{q} - m\hat{p}$. The bound for the absolute error in that result was derived under the somewhat conservative (and generically infeasible) premise that the least favorable contingencies $\delta(m, n) = 0$ and $\hat{\delta}(m, n) = 0$ apply simultaneously. We now show that under either of these two contingencies the relative approximation error, $r(m, n) = R(m, n)/U(m, n)$ can be made arbitrarily small for large enough input quantities. For this, consider first the case where $\delta(m, n) = 0$, i.e., $mp = nq$, so

$$\frac{U(m,n)}{\sigma(m,n)} \ge \frac{mp\,\Phi(0) + nq\,\Phi(0)}{\sigma(m,n)} = \sqrt{\frac{mp}{\hat{p}+\hat{q}}} \ge M,$$

as long as $m \ge \underline{m}(M) \triangleq M^2(\hat{p}+\hat{q})/p$. Similarly, for $\rho \in (0, 1]$, in the case where $\hat{\delta}(m, n) = 0$, i.e., $m\hat{p} = n\hat{q}$, it is

$$\frac{U(m,n)}{\sigma(m,n)} \ge \rho\frac{m\hat{p}\,\Phi(0) + n\hat{q}\,\Phi(0)}{\sigma(m,n)} = \rho\sqrt{\frac{m\hat{p}}{p+q}} \ge M,$$

as long as $m \ge \hat{\underline{m}}(M) \triangleq (M/\rho)^2(p+q)/\hat{p}$. Thus, if under either contingency m is larger than the corresponding lower bound, it is

$$r(m,n) = \frac{R(m,n)}{U(m,n)} \le \frac{1}{M}\left(\frac{1+\rho}{\sqrt{2\pi}}\right).$$

Hence, given an $\varepsilon > 0$, one obtains

$$r(\bar{m}^*, \bar{n}^*) \le \varepsilon,$$

as long as

$$\bar{m}^* \ge \max\left\{\mathbf{1}_{\{\delta(\bar{m}^*,\bar{n}^*)=0\}}\underline{m}(M(\varepsilon)),\ \mathbf{1}_{\{\hat{\delta}(\bar{m}^*,\bar{n}^*)=0\}}\hat{\underline{m}}(M(\varepsilon))\right\},$$

where

$$M(\varepsilon) \triangleq \frac{1}{\varepsilon}\left(\frac{1+\rho}{\sqrt{2\pi}}\right).$$

By Theorem 2 the optimal solution $(\bar{m}^*, \bar{n}^*)$ to the envelope optimization problem (**) is proportional to $\bar{u}$, so that Table 1 implies a suitable lower bound for the expected envelope output $\bar{u}$. Indeed, if the type-1 input m (which implies the value of the type-2 input n) is such that

$$m \ge \begin{cases} \underline{m}(M(\varepsilon)), & \text{if } \delta(m,n) = 0, \\ \hat{\underline{m}}(M(\varepsilon)), & \text{if } \hat{\delta}(m,n) = 0, \end{cases}$$

then the relative error, by construction, cannot exceed ε. This yields the input bounds

$$\underline{m}(M(\varepsilon)) = \frac{\hat{p}+\hat{q}}{2\pi p}\left(\frac{1+\rho}{\varepsilon}\right)^2 \quad \text{and}$$

$$\hat{\underline{m}}(M(\varepsilon)) = \frac{p+q}{2\pi\hat{p}}\left(\frac{1+\rho}{\rho\varepsilon}\right)^2.$$

Table 4 summarizes the minimum output, which is proportional to the input bounds. Straightforward substitution of the inputs into the corresponding entries for the minimum output yields all the relevant entries in Table 3, which establishes the result. □

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