Data-driven IQC-Based Uncertainty Modelling for Robust Control Design

Vaibhav Gupta * **Elias Klauser** *,** **Alireza Karimi** *

* *Laboratoire d'Automatique, École Polytechnique Fédérale de Lausanne, CH-1015 Lausanne, Switzerland (e-mail: vaibhav.gupta@epfl.ch, alireza.karimi@epfl.ch)*
** *CSEM SA, Rue Jaquet-Droz 1, CH-2002 Neuchâtel, Switzerland (e-mail: elias.klauser@csem.ch)*

Abstract: A new approach is presented in the paper for modelling uncertainty as an elliptical set for robust controller synthesis. The method involves finding the best linear nominal model and the corresponding elliptical uncertainty set that is consistent with a set of frequency response functions of linear time-invariant (LTI) single-input single-output (SISO) systems. The uncertainty set is then converted into an equivalent integral quadratic constraint (IQC) using a novel split representation of uncertainty. Finally, the IQC is integrated into a data-driven frequency-domain controller synthesis method through convex optimization. The simulation and experimental results demonstrate that the proposed method yields a "tighter" uncertainty set and improved stability margins compared to classical methods that use disk uncertainty.

Keywords: Data-based control, Robust control (linear case), Uncertainty descriptions, Convex optimization, Frequency-domain identification for control.

1. INTRODUCTION

With recent advancements in computational power and sensor technologies, data-driven approaches for control design are becoming more attractive for industrial applications compared to classical model-based approaches. These data-driven techniques allow a control criterion to be directly minimised based on measured input-output data and are particularly advantageous when a parametric plant model is not available. The frequency response data can effectively be used for analysis and synthesis of linear control systems. It can be easily computed from input-output data as presented by (Pintelon and Schoukens, 2012) and is therefore widely used in industry as the classical loop-shaping method. Since most of the control performance and robust stability conditions can be represented in the frequency domain, new data-driven methods using only frequency-domain data and optimization techniques to compute robust controllers have been proposed in literature.

The controller design using frequency-domain data leads to a non-convex optimization problem. This optimization problem is solved by a non-smooth optimisation technique in Apkarian and Noll (2018) to compute fixed structure $\mathcal{H}_\infty$ controllers for systems represented by their frequency-domain data. Several solutions using convex approximation are proposed as well. In Hast et al. (2013) and Saeki (2014), the use of frequency-domain data for computing SISO-PID controllers by convex optimisation is proposed using the same type of linearization of the constraints as in the work of Karimi and Galdos (2010). The design of MIMO-PID controllers is presented as a convex-concave optimization problem in Boyd et al. (2016) and solved by linearization of quadratic matrix inequalities. The same linearization method is used in Saeki et al. (2010) for designing linearly parametrized MIMO controllers. In Karimi et al. (2018), a frequency-based data-driven control design methodology with an $\mathcal{H}_\infty$ control objective based on coprime factorization of the controller is proposed and extended to systems with sector nonlinearity (Nicoletti and Karimi, 2019). This method is also employed for linear parameter varying controller design in Bloemers et al. (2022a). Finally, a fixed-structure data-driven controller design method for multivariable systems with mixed $\mathcal{H}_2/\mathcal{H}_\infty$ sensitivity performance is proposed in Karimi and Kammer (2017) and applied to the distributed control of microgrids (Madani et al., 2021) and passivity-based controller design (Madani and Karimi, 2020). Although the frequency-domain data-driven methods can take into account multimodel uncertainty and have been extended to some nonlinear systems, e.g. LPV systems, sector nonlinearity and passive nonlinearity, a general data-driven framework for uncertainty representation and controller design is missing.

Integral Quadratic Constraints (IQC), proposed by Megretski and Rantzer (1997), is a very generic and flexible formalism to represent and analyze various types of uncertainties and nonlinearities such as parametric uncertainties, rate-bounded uncertainties, time-delay uncertainties and norm- and sector-bounded nonlinearities. One of the key benefits of the IQC framework is the ability to analyze systems affected by multiple types of uncertainty through the use of a single composite IQC. Sufficient

⋆ This work is funded by Swiss National Science Foundation under grant no. 200021-204962 and ESA Contract No. 4000133258/20/NL/MH/hm.

conditions for closed-loop stability can be formulated in time- and frequency-domain in this framework. A variety of model-based robust control analysis and synthesis methods based on the IQC framework have also been developed (Veenman et al., 2016; Michalowsky et al., 2021) and there is even a MATLAB toolbox available called IQC-Lab (Veenman et al., 2021). In a data-driven setting, the available results are limited. However, Koch et al. (2021) developed a necessary and sufficient condition for a linear time-invariant (LTI) system to satisfy a given IQC using only one input–output trajectory of finite length. Data-driven methods combining robust stability with robust performance analysis integrated in an LMI-based IQC have been studied allowing for the direct design of MIMO LPV controllers from frequency-domain data (Bloemers et al., 2022b). An interesting problem in this context is to identify the best LTI model and a measure of additive uncertainty or nonlinearity from a set of input/output data (Martin and Allgöwer, 2019). The problem becomes more challenging when the best non-parametric LTI model and corresponding IQC set in the frequency domain need to be obtained for robust control synthesis.

In this paper, we consider that several sets of frequency-domain data of an LTI system are available. These sets may represent different noise realizations or operating points. A common approach is to use an average model or one of the models as the nominal model and find a weighting filter by computing the smallest disk covering all frequency responses for every frequency point. However, it has been demonstrated in Hindi et al. (2002) that the uncertainty in the nominal model can be significantly reduced by performing a combined optimization of the nominal model and the weighting filter. This optimization can be formulated as a convex optimization problem with linear matrix inequalities (LMIs). A weighting filter containing the radii of the disks at every frequency point can be incorporated in an LMI-based constraint which can be used for robust control design (Karimi and Kammer, 2017). However, note that this disk uncertainty in the complex plane will introduce some conservatism to the controller design. If the uncertainties can be represented as ellipses in the complex plane, this conservatism can be significantly reduced depending on the distribution of the data points. However, elliptical uncertainties in the frequency-domain cannot be represented by the classical additive weighting filters and a more general IQC uncertainty set is required.

The contributions of this paper are summarised as:

- Computation of the optimal elliptical uncertainty sets and the best non-parametric linear frequency-domain model based on multiple sets of frequency-domain data.
- Representation of the optimal elliptical uncertainty set in the form of a non-parametric IQC.
- Integration of the non-parametric IQC into a data-driven robust controller design, which reduces conservatism compared to existing methods.

The paper is organised as follows: Section 2 presents the notation and a brief introduction to IQC and data-driven controller design using frequency-domain data. A method for additive elliptical uncertainty modelling using a set of frequency functions of the plant and converting it to a set of IQC is proposed in Section 3. Then, the IQC is integrated into the data-driven controller design in the same section and a solution based on convex optimization using LMIs is proposed. A simulation example of an active suspension system subject to parametric uncertainties is given in Section 4. It is shown that the parametric uncertainty can be well approximated by ellipses in the frequency domain and leads to a significant reduction of the conservatism. Section 5 presents the experimental results of a laboratory setup with multimodel uncertainty. It is shown that, based only on the frequency-domain data at different operating points, the optimal LTI non-parametric model and IQC uncertainty set can be computed by convex optimization and used to compute a robust controller. Finally, the concluding remarks are given in Section 6.

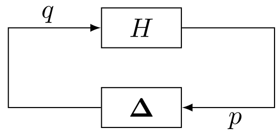

Fig. 1. Basic feedback configuration

2. PRELIMINARIES

Notation: $M \succ (\succeq) N$ indicates that $M - N$ is a positive (semi-) definite matrix and $M \prec (\preceq) N$ indicates $M - N$ is negative (semi-) definite. The zero and identity matrix of appropriate size are denoted $\mathbf{0}$ and I respectively. The transpose of a matrix M is denoted by M^T and its conjugate transpose by M^*. For continuous-time systems $\Omega := \mathbb{R}$ and for discrete-time systems $\Omega := [-\pi/T_s, \pi/T_s)$, where T_s is the sampling time. $G(j\omega)$ will be used to denote the frequency response of the system G in both cases.

2.1 Integral Quadratic Constraint

Two signals p and q are said to satisfy the IQC defined by a multiplier Π, if

$$\int_\Omega \begin{bmatrix} P(j\omega) \\ Q(j\omega) \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} P(j\omega) \\ Q(j\omega) \end{bmatrix} d\omega \geq 0 \tag{1}$$

where $P(j\omega)$ and $Q(j\omega)$ are the Fourier transform of p and q respectively. From Megretski and Rantzer (1997, Theorem 1), the feedback connection between H, a stable LTI system with bounded infinity norm, and a bounded causal operator $\mathbf{\Delta}$ (see Fig. 1) is stable if,

(1) Interconnection of H and $\tau\mathbf{\Delta}$ is well-posed for all $\tau \in [0, 1]$;
(2) $\tau\mathbf{\Delta}$ satisfies the IQC defined by Π for all $\tau \in [0, 1]$;
(3) $\exists \epsilon > 0$ such that

$$\begin{bmatrix} H(j\omega) \\ I \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} H(j\omega) \\ I \end{bmatrix} \preceq -\epsilon I \qquad \forall \omega \in \Omega. \tag{2}$$

Remark 1. If the upper left corner of Π is positive semi-definite and lower right corner is negative semi-definite, then using Megretski and Rantzer (1997, Remark 2), $\tau\mathbf{\Delta}$ satisfies the IQC defined by Π for all $\tau \in [0, 1]$ if and only if $\mathbf{\Delta}$ satisfies the IQC.

Remark 2. If $\mathbf{\Delta}$ is a linear operator such that $Q(j\omega) = \mathbf{\Delta}(j\omega)P(j\omega)$, then $\mathbf{\Delta}$ satisfies the IQC defined by Π, if

$$\begin{bmatrix} I \\ \mathbf{\Delta}(j\omega) \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} I \\ \mathbf{\Delta}(j\omega) \end{bmatrix} \succeq \mathbf{0} \qquad \forall \omega \in \Omega. \tag{3}$$

2.2 Data-driven frequency-domain controller synthesis

Given the frequency response of the plant G, the data-driven approach presented in Karimi and Kammer (2017) can be used to design a controller for a given control performance criterion. The approach uses iterative convex optimisation to compute a stabilising controller with the desired performance specified by a set of weighting filters.

For example, consider the $\mathcal{H}_\infty$ performance criterion,

$$\min_{K} \; \|W\mathcal{S}\|_\infty \tag{4}$$

where $\mathcal{S} = (1 + GK)^{-1}$ and W is an appropriate weighting filter.

Under the closed-loop stability condition, this problem can be converted to an optimisation problem on the spectral norm,

$$\min_{K,\,\gamma} \gamma \tag{5}$$
$$\text{s.t.} \quad [W(j\omega)\mathcal{S}(j\omega)]\,[W(j\omega)\mathcal{S}(j\omega)]^* \leq \gamma \qquad \forall \omega \in \Omega.$$

Note that the argument $j\omega$ will be omitted for $W(j\omega)$ and $\mathcal{S}(j\omega)$ to simplify the notation. Using a controller parametrization $K = XY^{-1}$, where X and Y are polynomials in z (z-transform variable) or s (Laplace transform variable), we obtain:

$$\mathcal{S} = Y(Y + GX)^{-1} = Y\Phi^{-1}.$$

with the evident definition of Φ. Therefore, the constraint in (5) can be rewritten as,

$$(WY)\,(\Phi^*\Phi)^{-1}\,(WY)^* \leq \gamma \qquad \forall \omega \in \Omega$$

Using the Schur complement lemma, an equivalent matrix inequality can be found,

$$\begin{bmatrix} \Phi^*\Phi & (WY)^* \\ (WY) & \gamma \end{bmatrix} (j\omega) \succeq \mathbf{0} \qquad \forall \omega \in \Omega. \tag{6}$$

This inequality can be convexified around a known stabilising initial controller $K_c = X_c Y_c^{-1}$, using a lower bound on $\Phi^*\Phi$:

$$\Phi^*\Phi \succeq \Phi^*\Phi_c + \Phi_c^*\Phi - \Phi_c^*\Phi_c$$

where, $\Phi_c = Y_c + GX_c$. This gives a convex optimisation problem with LMIs,

$$\min_{X,\,Y,\,\gamma} \gamma \tag{7}$$
$$\text{s.t.} \quad \begin{bmatrix} \Phi^*\Phi_c + \Phi_c^*\Phi - \Phi_c^*\Phi_c & (WY)^* \\ (WY) & \gamma \end{bmatrix} (j\omega) \succeq \mathbf{0} \qquad \forall \omega \in \Omega.$$

If the known initial controller K_c is stabilising, then using Theorem 2 in Karimi and Kammer (2017), it can be proven that the designed controller K is also stabilising. The optimisation problem can be solved by gridding over the frequencies and using the designed controller as the initial stabilising controller for the next optimisation. This sequence of convex optimisation problems will converge to a local optimal solution of the original nonlinear problem.

For small discrete sets of models, this approach can deal with the multimodel uncertainty by adding a set of constraints for each model with $\Phi_i = Y + G_i X$. However, if the set of models is continuous or has a large number of models, the optimisation problem might become intractable.

2.3 Basic problem statement

This paper focuses on the computation of uncertainty models that are not invalidated by the available data as defined in Newlin and Smith (1998), and their use for robust data-driven control design.

The frequency response function (FRF) for each measurement i is denoted using $G_i(j\omega)$. These measurements could be several independent measurements of the plant G, either at the same or different operating points, or of multiple plants. The aim is to find a linear nominal model $\hat{G}$ with the uncertainty set such that it is consistent with (not invalidated by) the data. This uncertainty set should be a "'tight" elliptical area around the nominal model to minimise the impact on the controller performance. In this paper, "tight" is defined in the sense that the area of uncertainty at all frequencies should be minimised.

For robust data-driven control design using the obtained model, a robustness constraint for stability is to be found. As traditional approaches are limited to disk uncertainties, an IQC-based approach is presented which allows for elliptical uncertainty. The final step is to integrate this constraint with the approach in Section 2.2 to find a robust controller that minimizes a performance objective.

3. MAIN RESULTS

First, a best linear nominal model $\hat{G}$ and its additive elliptical uncertainty set, which is consistent with the data, are found. Next, this uncertainty set is converted into an equivalent IQC formulation. Finally, the IQC formulation is converted into a robust stability constraint in frequency-domain and added to the data-driven approach in Section 2.2.

3.1 Optimal Additive Elliptical Uncertainty Set

In this section, an optimal nonparametric additive elliptical uncertainty set is computed using tools from convex optimisation. The system under consideration is a linear time-invariant single-input single-output (LTI-SISO) plant represented using FRF $\{G_i(j\omega)\}$ which can be obtained from a series of m experiments using the Fourier analysis on the sampled input-output data as presented in Pintelon and Schoukens (2012).

Recall that the set of points inside a rotated ellipse, in an xy-plane can be represented as,

$$\begin{bmatrix} x \\ y \end{bmatrix}^T A^T A \begin{bmatrix} x \\ y \end{bmatrix} \leq 1 \Leftrightarrow \left\| A \begin{bmatrix} x \\ y \end{bmatrix} \right\|_2^2 \leq 1$$

and the area of the ellipse is given by $\pi \det\{A^{-1}\}$.

Definition: A model with additive elliptical uncertainty set can be represented as,

$$\mathcal{M}(\hat{G}, A)(j\omega) \triangleq \left\{ \hat{G}(j\omega) + \Delta \;\middle|\; \left\| A(\omega) \begin{bmatrix} \mathrm{Re}\{\Delta\} \\ \mathrm{Im}\{\Delta\} \end{bmatrix} \right\|_2 \leq 1 \right\} \tag{8}$$

where, $\hat{G}(j\omega)$ is the nominal FRF model and $A(\omega) \in \mathbb{R}^{2\times 2}$ characterises the uncertainty elliptical set with an area of $\pi \det\{A^{-1}(\omega)\}$.

Given the dataset $\{G_i(j\omega)\}$, a model $\mathcal{M}(\hat{G}, A)(j\omega)$ needs to be found such that

$$G_i(j\omega) \in \mathcal{M}(\hat{G}, A)(j\omega) \qquad \forall i \; \forall \omega \in \Omega$$

and $\mathcal{M}(\hat{G}, A)(j\omega)$ should have minimal area at all frequencies. It can be easily shown that a measurement $G_i(j\omega)$ belongs to $\mathcal{M}(\hat{G}, A)(j\omega)$ iff

$$\left\| A(\omega) \begin{bmatrix} \mathrm{Re}\left\{ G_i(j\omega) - \hat{G}(j\omega) \right\} \\ \mathrm{Im}\left\{ G_i(j\omega) - \hat{G}(j\omega) \right\} \end{bmatrix} \right\|_2 \leq 1. \tag{9}$$

This can be equivalently defined as an optimisation problem at each frequency by,

$$\min_{\hat{G},\, A} \; \mathrm{area}\, \mathcal{M}(\hat{G}, A)(j\omega) \tag{10}$$

$$\text{s.t.} \quad \left\| A(\omega) \begin{bmatrix} \mathrm{Re}\left\{ G_i(j\omega) - \hat{G}(j\omega) \right\} \\ \mathrm{Im}\left\{ G_i(j\omega) - \hat{G}(j\omega) \right\} \end{bmatrix} \right\|_2 \leq 1 \qquad \forall i.$$

The objective function can be replaced by a convex function such that the optimisation remains equivalent,

$$\min_{\hat{G},\, A} \; \mathrm{area}\, \mathcal{M}(\hat{G}, A)(j\omega) \Leftrightarrow \min_{\hat{G},\, A} \; -\log\det\{A(\omega)\}.$$

To convert the constraint into a convex constraint, a change of variable can be performed,

$$b(\omega) = A(\omega) \begin{bmatrix} \mathrm{Re}\left\{ \hat{G}(j\omega) \right\} \\ \mathrm{Im}\left\{ \hat{G}(j\omega) \right\} \end{bmatrix}$$

such that

$$\left\| A(\omega) \begin{bmatrix} \mathrm{Re}\{G_i(j\omega)\} \\ \mathrm{Im}\{G_i(j\omega)\} \end{bmatrix} - b(\omega) \right\|_2 \leq 1 \qquad \forall i.$$

Then, (10) can be converted to a convex optimisation problem with a log-det objective and a conic constraint for each measurement in the dataset at all frequencies. In practice, this optimisation only needs to be solved at a finite number of frequency points.

$$\min_{A,b} \; -\log\det\{A(\omega_k)\} \tag{11}$$

$$\text{s.t.} \quad \left\| A(\omega_k) \begin{bmatrix} \mathrm{Re}\{G_i(j\omega_k)\} \\ \mathrm{Im}\{G_i(j\omega_k)\} \end{bmatrix} - b(\omega_k) \right\|_2 \leq 1 \qquad \forall i$$

The best linear nominal model $\hat{G}(j\omega)$ for the elliptical uncertainty set can be found from the optimal solution of the optimisation problem as

$$\begin{bmatrix} \mathrm{Re}\left\{ \hat{G}(j\omega_k) \right\} \\ \mathrm{Im}\left\{ \hat{G}(j\omega_k) \right\} \end{bmatrix} = A^{-1}(\omega_k) b(\omega_k).$$

Note that, in general, this nominal model might not be any of the models of the system or the average model.

Remark 3. In order for the matrix $A(\omega)$ to be finite, the area of the elliptical uncertainty must be non-zero. This implies that there should be at least three non-collinear points at each frequency. In the real world, the presence of noise will typically cause the points to shift, which ensures that this assumption is satisfied.

3.2 Uncertainty set as IQC

In this section, the IQC multiplier Π is found such that the uncertainty Δ of $\mathcal{M}(\hat{G}, A)$ satisfies the IQC defined by Π.

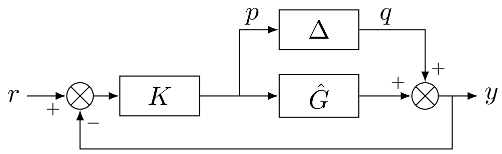

Fig. 2. Feedback system with additive uncertainty block

Consider a transformation matrix

$$J = \begin{bmatrix} 1 & 0 \\ 0 & j \end{bmatrix} \quad \text{with} \quad J^* J = I.$$

From the definition of $\mathcal{M}(\hat{G}, A)$, the uncertainty Δ satisfies

$$\left\| A(\omega) \begin{bmatrix} \mathrm{Re}\{\Delta\} \\ \mathrm{Im}\{\Delta\} \end{bmatrix} \right\|_2 \leq 1 \qquad \forall \omega$$

$$\Leftrightarrow \left\| A(\omega) J^* \begin{bmatrix} \mathrm{Re}\{\Delta\} \\ j\,\mathrm{Im}\{\Delta\} \end{bmatrix} \right\|_2 \leq 1 \qquad \forall \omega$$

$$\Leftrightarrow \begin{bmatrix} \mathrm{Re}\{\Delta\} \\ j\,\mathrm{Im}\{\Delta\} \end{bmatrix}^* J A^T(\omega) A(\omega) J^* \begin{bmatrix} \mathrm{Re}\{\Delta\} \\ j\,\mathrm{Im}\{\Delta\} \end{bmatrix} \leq 1 \qquad \forall \omega$$

which can be written as,

$$\begin{bmatrix} 1 \\ \mathrm{Re}\{\Delta\} \\ j\,\mathrm{Im}\{\Delta\} \end{bmatrix}^T \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & -J A^T(\omega) A(\omega) J^* \end{bmatrix} \begin{bmatrix} 1 \\ \mathrm{Re}\{\Delta\} \\ j\,\mathrm{Im}\{\Delta\} \end{bmatrix} \geq 0 \qquad \forall \omega.$$

Taking $\mathbf{\Delta}$ as $\begin{bmatrix} \mathrm{Re}\{\Delta\} \\ j\,\mathrm{Im}\{\Delta\} \end{bmatrix}$, the uncertainty of $\mathcal{M}(\hat{G}, A)$ can be shown to satisfy the IQC defined by

$$\Pi(j\omega) = \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & -\bar{A}^*(j\omega)\bar{A}(j\omega) \end{bmatrix} \tag{12}$$

where $\bar{A}(j\omega) = A(\omega) J^*$. Note that $\Pi(j\omega)$ is a dynamic multiplier for the elliptical uncertainty set, in contrast to the frequency-dependent static gain for the disk uncertainty set. Since (12) satisfies the condition of Remark 1, $\tau\mathbf{\Delta}$ also satisfies the IQC defined by Π for all $\tau \in [0, 1]$.

This split representation of $\mathbf{\Delta}$ as real and imaginary part of the true uncertainty Δ allows for an elliptical uncertainty set as a quadratic constraint. As shown in the next section, the stability condition arising from this representation is related to a nonparametric multiplier $\Pi(j\omega)$ and hence, non-trivial to deal with using traditional model-based approaches. Using the data-driven approach in Section 2.2, this stability constraint can be integrated straightforwardly in the LMIs of the convex optimisation problem.

3.3 Design for Robust Stability

A robust controller K needs to be synthesised for the system $\mathcal{M}(\hat{G}, A)$, which is graphically represented in Fig. 2. Using the data-driven approach described in Section 2.2, a stabilising controller for $\hat{G}$ can be designed while minimising the performance costs. To make the controller robust, an additional robustness constraint is added at all frequency points.

For the uncertainty models described by $\mathcal{M}(\hat{G}, A)$, the uncertainty block can be split into its real and imaginary components and regrouped as shown in Fig. 3, where $\mathcal{U}$ is the closed loop transfer function from the output to the input of the uncertainty block. Using the controller parametrization $K = Y^{-1}X$,

$$\mathcal{U}(j\omega) = (1 + K(j\omega)\hat{G}(j\omega))^{-1} K(j\omega) = \Phi^{-1}(j\omega) X(j\omega)$$

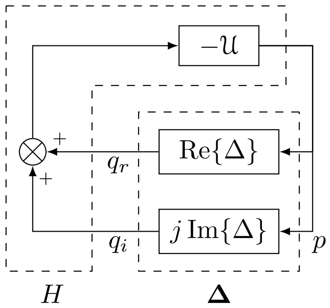

Fig. 3. Feedback system with uncertainty block split into its real and imaginary components

where, $\Phi(j\omega) = Y(j\omega) + X(j\omega)\hat{G}(j\omega)$.

Since $\mathfrak{U}$ is stable by design (refer Section 2.2),

$$H(j\omega) = [-\mathfrak{U}(j\omega) \;\; -\mathfrak{U}(j\omega)] = -\Phi^{-1}(j\omega)\bar{X}(j\omega)$$

with $\bar{X}(j\omega) = [X(j\omega) \;\; X(j\omega)]$, is also stable. Then, using Megretski and Rantzer (1997, Theorem 1), if

$$\left(\begin{bmatrix} H \\ I \end{bmatrix}^* \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & -\bar{A}^*\bar{A} \end{bmatrix} \begin{bmatrix} H \\ I \end{bmatrix}\right)(j\omega) \prec \mathbf{0} \qquad \forall \omega \in \Omega \tag{13}$$

then the feedback connection between system H and $\boldsymbol{\Delta}$ is stable.

After simplification, (13) can be written as,

$$\left(\bar{X}^*(\Phi\Phi^*)^{-1}\bar{X} - \bar{A}^*\bar{A}\right)(j\omega) \prec \mathbf{0} \qquad \forall \omega \in \Omega.$$

Using the Schur complement lemma, an equivalent matrix inequality can be found,

$$\begin{bmatrix} \Phi\Phi^* & \bar{X} \\ \bar{X}^* & \bar{A}^*\bar{A} \end{bmatrix}(j\omega) \succ \mathbf{0} \qquad \forall \omega \in \Omega. \tag{14}$$

This inequality can be convexified around a known controller $K_c = Y_c^{-1}X_c$ (as presented in section 2.2),

$$\Phi\Phi^* \succeq \Phi\Phi_c^* + \Phi_c\Phi^* - \Phi_c\Phi_c^*$$

where, $\Phi_c = Y_c + \hat{G}X_c$. This gives a sufficient condition for robust stability as an LMI,

$$\begin{bmatrix} \Phi\Phi_c^* + \Phi_c\Phi^* - \Phi_c\Phi_c^* & \bar{X} \\ \bar{X}^* & \bar{A}^*\bar{A} \end{bmatrix}(j\omega) \succ \mathbf{0} \qquad \forall \omega \in \Omega \tag{15}$$

which can be added as an additional constraint in the data-driven approach presented in Section 2.2. For example, for $\mathcal{H}_\infty$ performance, the new optimisation problem for robust controller synthesis would be,

$$\min_{X,Y,\gamma} \; \gamma \tag{16}$$

$$\text{s.t.} \begin{bmatrix} \Phi^*\Phi_c + \Phi_c^*\Phi - \Phi_c^*\Phi_c & (WY)^* \\ (WY) & \gamma \end{bmatrix}(j\omega) \succeq \mathbf{0} \qquad \forall \omega \in \Omega$$

$$\begin{bmatrix} \Phi\Phi_c^* + \Phi_c\Phi^* - \Phi_c\Phi_c^* & \bar{X} \\ \bar{X}^* & \bar{A}^*\bar{A} \end{bmatrix}(j\omega) \succ \mathbf{0} \qquad \forall \omega \in \Omega.$$

The optimisation problem can be solved by gridding over the frequencies and using the synthesised controller as the initial stabilizing controller for the next iteration (see section 2.2).

4. SIMULATION EXAMPLE

For this section, the effect of actuation force on the body acceleration is to be studied in an active suspension system. An example from the MATLAB Robust Control Toolbox is taken, with a 10% uncertainty in all system parameters, and converted to discrete time at a sampling frequency of 400 Hz.

$$G = \left[\begin{array}{cccc|c} 0 & 1 & 0 & 0 & 0 \\ \frac{-k_s}{m_b} & \frac{-b_s}{m_b} & \frac{k_s}{m_b} & \frac{b_s}{m_b} & \frac{1000}{m_b} \\ 0 & 0 & 0 & 1 & 0 \\ \frac{k_s}{m_w} & \frac{b_s}{m_w} & \frac{-k_s-k_t}{m_w} & \frac{-b_s}{m_w} & \frac{-1000}{m_w} \\ \hline \frac{-k_s}{m_b} & \frac{-b_s}{m_b} & \frac{k_s}{m_b} & \frac{b_s}{m_b} & \frac{1000}{m_b} \end{array}\right]$$

Table 1. Parameters and their uncertainty for the active suspension system example

	Nominal	Uncertainty
m_b	300 kg	±10 %
m_w	60 kg	±10 %
b_s	1000 N s m^{-1}	±10 %
k_s	16 000 N m^{-1}	±10 %
k_t	190 000 N m^{-1}	±10 %

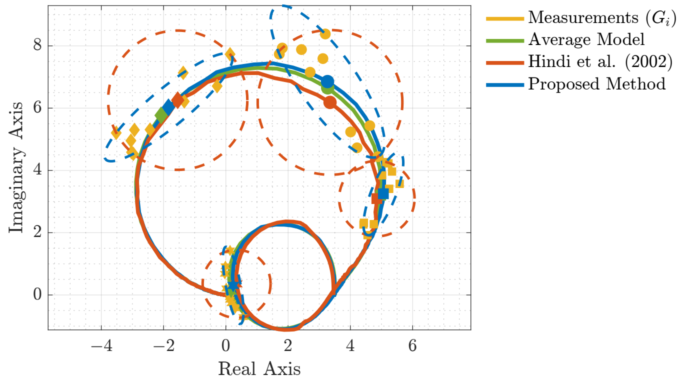

Fig. 4. Nyquist plot of the estimated model using averaging, the proposed method and the method by Hindi et al. (2002). Dashed lines shows the uncertainty boundary estimated by the respective methods.

Uncertainty modelling using the proposed method and the method by Hindi et al. (2002) is performed for 10 different measurements at 500 frequency points (see Fig. 4). It can be observed that the estimated $\hat{G}$ is much closer to the average model for the proposed method compared to the method by Hindi et al. (2002).

From Fig. 5, it can be seen that a significant reduction in the area of uncertainty can be achieved by using the proposed method compared to the method by Hindi et al. (2002). In frequencies with large uncertainties, this reduction can be up to a factor of 2. It is easy to see that the area of the elliptical uncertainty set will always be smaller than the area of the disk uncertainty set, and under the worst case scenario, it would be equal.

5. EXPERIMENTAL RESULTS

In this section, real data from a laboratory setup is used to design a discrete-time controller sampled at 2 ms, which is then implemented on the real system. It is shown how the proposed approach can deal with multimodel uncertainty and measurement noise for model reference control synthesis.

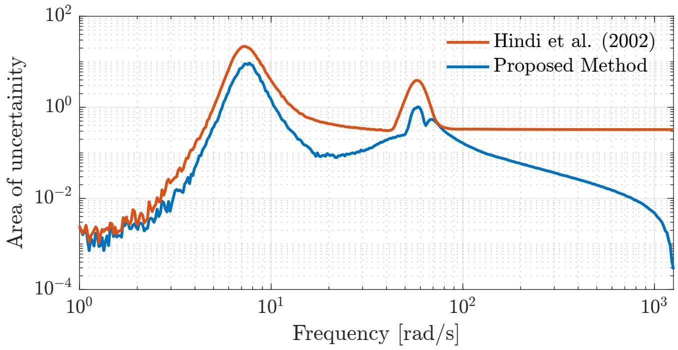

Fig. 5. Area of uncertainty model using the proposed method and the method by Hindi et al. (2002)

Fig. 6. Quanser Servo-Qube with weights attached on top

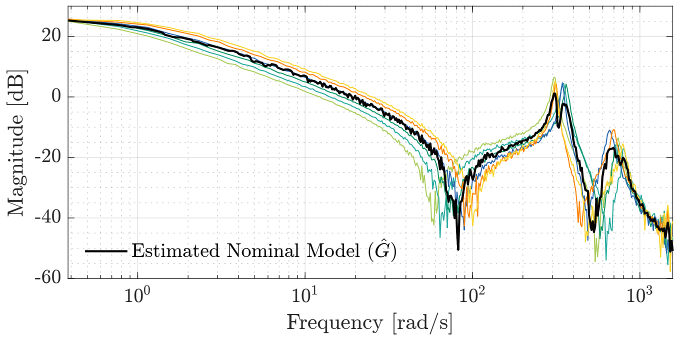

Fig. 7. Frequency responses of the measurements $\{G_i(j\omega)\}$ and the nominal model $\hat{G}(j\omega)$ obtained using the proposed method

This example focuses on the velocity control of a DC motor with a flexible element attached on top, resulting in large resonant modes at high frequencies. Different weights can be attached to the flexible element at different positions, resulting in different loadings and creating a multimodel uncertainty (Fig. 6).

The frequency function is obtained from input-output data at 6 positions of the weights, leading to different models $\{G_1(j\omega), \cdots, G_6(j\omega)\}$. The frequency responses of $\{G_i(j\omega)\}$ and the obtained nominal model $\hat{G}(j\omega)$ are shown in Fig. 7. Note that the nominal model has more modes than any single model, particularly between $300\,\mathrm{rad\,s^{-1}}$ to $400\,\mathrm{rad\,s^{-1}}$, there are 2 modes in the nominal model compared to a single mode in individual models.

Fig. 8 shows the nominal model and the respective uncertainty set using the proposed method and the method by Hindi et al. (2002). The proposed approach gives "'tighter" uncertainty set compared to Hindi et al. (2002).

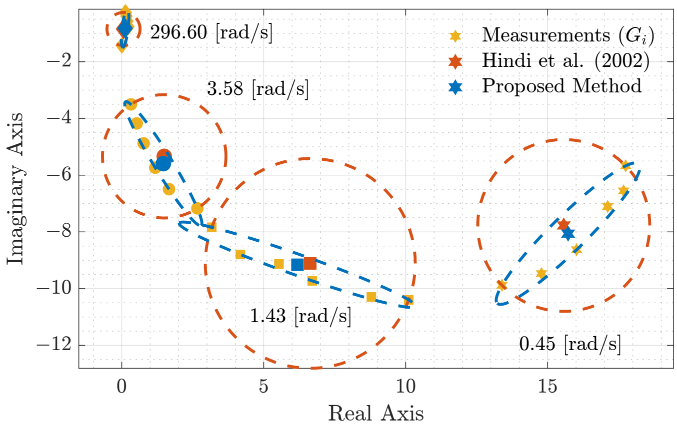

Fig. 8. Frequency responses of the measurements $\{G_i(j\omega)\}$ and the estimated model using the proposed method and the method by Hindi et al. (2002). Dashed lines shows the uncertainty boundary estimated by the respective methods.

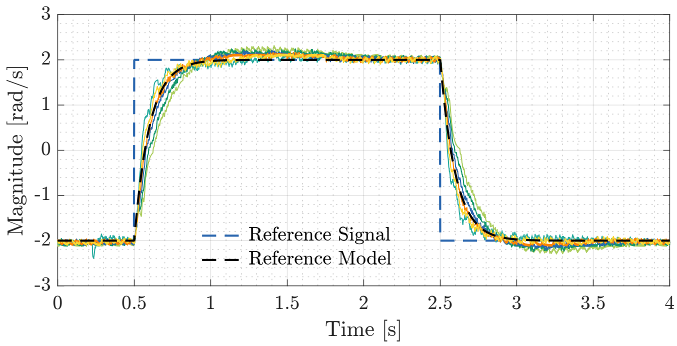

Fig. 9. Step closed-loop responses of the Servo-Qube with different loadings

The design objective is formulated as minimisation of the $\mathcal{H}_2$ error between the desired closed-loop performance $\mathcal{T}_r$ and the true closed-loop performance $\hat{\mathcal{T}} = K\hat{G}(1+K\hat{G})^{-1}$.

$$\min_K \left\| W\left(\mathcal{T}_r - \hat{\mathcal{T}}\right)\right\|_2$$

where $W(j\omega) = 1 + 1/j\omega$ is a weighting filter.

Using the approach in Section 2.2, this can be converted into a convex optimisation problem and the additional constraint (15) is added to each frequency for robustness. Additionally, an integrator is fixed in the controller. The problem is then sampled at 500 logarithmically spaced points and solved.

The closed-loop tracking of a square waveform reference is shown in Fig. 9, along with the desired output of the reference model $\mathcal{T}_r$. As it can be seen, the different closed-loop outputs for different loadings follow the output of the reference model.

The stability margin under the IQC description can be defined as the minimum scaling factor of the uncertainty set at which the feedback connection becomes unstable (Hai-rong et al., 2002). Using the proposed approach, a stability margin of 4.72 is obtained, while the approach by Hindi et al. (2002) yields a margin of 3.44.

6. CONCLUSION

A convex optimization technique is used to find the best linear nominal model and corresponding elliptical uncertainty set that are consistent with a given set of frequency response functions (FRFs) of LTI-SISO systems. Next, a novel split representation of uncertainty is employed to transform the uncertainty set into an equivalent IQC. The resulting IQC is integrated with a data-driven frequency-domain controller synthesis method by converting it into a set of LMI constraints for robust stability.

The proposed method results in a "tighter" uncertainty set compared to the disk uncertainty. Experimental results show that using the elliptical uncertainty set, the area of uncertainty can be reduced by up to a factor of 2 compared to the disk uncertainty. Additionally, the stability margin of the synthesized controller sees an improvement of up to 30% compared to the controller synthesized using the disk uncertainty set.

For LTI-MIMO systems, uncertainty can be represented as either elementwise additive uncertainty or matrix additive uncertainty. While the proposed method can be applied to the elementwise additive uncertainty representation with some effort, extending it to the matrix additive uncertainty case is of interest and is the focus of ongoing research.

REFERENCES

Apkarian, P. and Noll, D. (2018). Structured $\mathcal{H}_\infty$ control of infinite-dimensional systems. *International Journal of Robust and Nonlinear Control*, 28(9), 3212–3238.

Bloemers, T., Oomen, T., and Toth, R. (2022a). Frequency response data-based LPV controller synthesis applied to a control moment gyroscope. *IEEE Transactions on Control Systems Technology*, 30(6), 2734–2742.

Bloemers, T., Oomen, T., and Toth, R. (2022b). Frequency Response Data-Driven LPV Controller Synthesis for MIMO Systems. *IEEE Control Systems Letters*, 6, 2264–2269.

Boyd, S., Hast, M., and Åström, K.J. (2016). MIMO PID tuning via iterated LMI restriction. *Int. J. Robust Nonlinear Control*, 26(8), 1718–1731.

Hai-rong, D., Zhi-yong, G., Jin-zhi, W., and Lin, H. (2002). Stability margin of systems with mixed uncertainties under the IQC descriptions. *Applied Mathematics and Mechanics*, 23(11), 1274–1281.

Hast, M., Åström, K.J., Bernhardsson, B., and Boyd, S. (2013). PID design by convex-concave optimization. In *European Control Conference*, 4460–4465. Zurich, Switzerland.

Hindi, H., Seong, C.Y., and Boyd, S. (2002). Computing optimal uncertainty models from frequency domain data. In *Proceedings of the 41st IEEE Conference on Decision and Control*, volume 3, 2898–2905.

Karimi, A. and Galdos, G. (2010). Fixed-order $\mathcal{H}_\infty$ controller design for nonparametric models by convex optimization. *Automatica*, 46(8), 1388–1394.

Karimi, A. and Kammer, C. (2017). A data-driven approach to robust control of multivariable systems by convex optimization. *Automatica*, 85, 227–233.

Karimi, A., Nicoletti, A., and Zhu, Y. (2018). Robust $\mathcal{H}_\infty$ controller design using frequency-domain data via convex optimization. *Int. J. Robust Nonlinear Control*, 28(12), 3766–3783.

Koch, A., Berberich, J., Köhler, J., and Allgöwer, F. (2021). Determining optimal input–output properties: A data-driven approach. *Automatica*, 134, 109906.

Madani, S.S., Kammer, C., and Karimi, A. (2021). Data-driven distributed combined primary and secondary control in microgrids. *IEEE Transactions on Control Systems Technology*, 29(3), 1340–1347.

Madani, S.S. and Karimi, A. (2020). Data-driven passivity-based current controller design for power-electronic converters of traction systems. In *2020 59th IEEE Conference on Decision and Control (CDC)*, 842–847.

Martin, T. and Allgöwer, F. (2019). Nonlinearity measures for data-driven system analysis and control. In *2019 IEEE 58th Conference on Decision and Control (CDC)*, 3605–3610.

Megretski, A. and Rantzer, A. (1997). System analysis via integral quadratic constraints. *IEEE Transactions on Automatic Control*, 42(6), 819–830.

Michalowsky, S., Scherer, C., and Ebenbauer, C. (2021). Robust and structure exploiting optimisation algorithms: an integral quadratic constraint approach. *International Journal of Control*, 94(11), 2956–2979.

Newlin, M. and Smith, R. (1998). A generalization of the structured singular value and its application to model validation. *IEEE Transactions on Automatic Control*, 43(7), 901–907.

Nicoletti, A. and Karimi, A. (2019). Robust control of systems with sector nonlinearities via convex optimization: A data-driven approach. *International Journal of Robust and Nonlinear Control*, 29(5), 1361–1376.

Pintelon, R. and Schoukens, J. (2012). *System Identification: A Frequency Domain Approach*. John Wiley & Sons, Ltd, second edition.

Saeki, M. (2014). Data-driven loop-shaping design of PID controllers for stable plants. *International Journal of Adaptive Control and Signal Processing*, 28(12), 1325–1340.

Saeki, M., Ogawa, M., and Wada, N. (2010). Low-order $\mathcal{H}_\infty$ controller design on the frequency domain by partial optimization. *International Journal of Robust and Nonlinear Control*, 20(3), 323–333.

Veenman, J., Scherer, C.W., Ardura, C., Bennani, S., Preda, V., and Girouart, B. (2021). IQClab: a new IQC based toolbox for robustness analysis and control design. *IFAC-PapersOnLine*, 54(8), 69–74. 4th IFAC Workshop on Linear Parameter Varying Systems LPVS 2021.

Veenman, J., Scherer, C.W., and Köroglu, H. (2016). Robust stability and performance analysis based on integral quadratic constraints. *European Journal of Control*, 31, 1–32.