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Semiclassical Estimates for Eigenvalue Means of Laplacians on Spheres

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Abstract

We compute three-term semiclassical asymptotic expansions of counting functions and Riesz-means of the eigenvalues of the Laplacian on spheres and hemispheres, for both Dirichlet and Neumann boundary conditions. Specifically for Riesz-means we prove upper and lower bounds involving asymptotically sharp shift terms, and we extend them to domains of $\mathbb{S}^d$. We also prove a Berezin–Li–Yau inequality for domains contained in the hemisphere $\mathbb{S}^2_+$.

Keywords Eigenvalues · Pólya's conjecture · Spheres and hemispheres · Riesz-means · Berezin–Li–Yau inequality · Kröger inequality · Averaged variational principle · Semiclassical expansions · Asymptotically sharp estimates.

Mathematics Subject Classification 58C40 · 35P15 · 35J25 · 35J40 · 53A05

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1 Introduction

In 1954, in the first edition of its monograph [38], Pólya stated his celebrated conjecture that the leading term in Weyl's law separates the spectrum of the Dirichlet Laplacian from that of the Neumann Laplacian. More precisely, given Ω an open bounded set in $\mathbb{R}^2$, and given $\lambda_k(\Omega)$, $\mu_k(\Omega)$ the k-th eigenvalue of the Laplace operator $-\Delta$ with respectively Dirichlet and Neumann boundary conditions, Pólya conjectured that

$$\mu_{k+1}(\Omega) \le \frac{4\pi k}{|\Omega|} \le \lambda_k(\Omega) \tag{1}$$

for any $k \in \mathbb{N}$ (the inequality for $\lambda_k(\Omega)$ is understood for $k \ge 1$). The same conjecture has been then formulated also in higher dimensions, and for $\Omega \subset \mathbb{R}^d$, $d \ge 2$ reads

$$\mu_{k+1}(\Omega) \le \frac{4\pi^2}{\omega_d^{2/d}} \left(\frac{k}{|\Omega|}\right)^{2/d} \le \lambda_k(\Omega) \tag{2}$$

for any $k \in \mathbb{N}$, where ω_d denotes the volume of the unit ball in $\mathbb{R}^d$. The quantity in the middle of (2) is the semiclassical approximation of the eigenvalues $\lambda_k(\Omega)$, $\mu_k(\Omega)$ since $\lambda_k(\Omega), \mu_k(\Omega) \sim \frac{4\pi^2}{\omega_d^{2/d}} \left(\frac{k}{|\Omega|}\right)^{2/d}$ for $k \to \infty$, as already proved by Weyl [42]. For this reason we say that these inequalities are asymptotically sharp (in leading order).

Pólya himself proved inequalities (1) when Ω is a tiling domain [37] and, while there have been some developments in two and in higher dimensions (see e.g., [10, 16, 18, 32]), the general case remains at the moment an open problem.

On the other hand, inequalities (2) in an averaged (weaker) version,

$$\frac{1}{k}\sum_{j=1}^{k} \mu_j(\Omega) \le \frac{d}{d+2}\frac{4\pi^2}{\omega_d^{2/d}} \left(\frac{k}{|\Omega|}\right)^{2/d} \le \frac{1}{k}\sum_{j=1}^{k} \lambda_j(\Omega), \tag{3}$$

were actually proven for any $\Omega \subseteq \mathbb{R}^d$ by Berezin [2] and Li and Yau [33] for the Dirichlet eigenvalues and by Kröger [31] for Neumann eigenvalues. For this reason, the first inequality in (3) is now known as Kröger inequality while the second one as Berezin–Li–Yau inequality. Note that inequalities (3) are asymptotically sharp, therefore Pólya's conjecture holds in an averaged sense. We remark that inequalities (2) on each eigenvalue can be rephrased as (reversed) bounds on the counting functions, i.e.,

$$N^D(z) = \#\{\lambda_k(\Omega) \le z\}, \quad N^N(z) = \#\{\mu_k(\Omega) \le z\} \qquad \forall z \ge 0,$$

while inequalities (3) on eigenvalue averages as (reversed) bounds on the first Riesz-means, i.e.

$$R_1^D(z) = \sum_j (z - \lambda_j(\Omega))_+, \quad R_1^N(z) = \sum_j (z - \mu_j(\Omega))_+ \qquad \forall z \ge 0.$$

Here $\sum_j$ denotes the summation over all $j \in \mathbb{N} \setminus \{0\}$. In other words, inequalities (2) are equivalent to

$$N^D(z) \leq L_{0,d}^{class} |\Omega| z^{\frac{d}{2}} \leq N^N(z), \tag{4}$$

while inequalities (3) are equivalent to

$$R_1^D(z) \leq L_{1,d}^{class} |\Omega| z^{1+\frac{d}{2}} \leq R_1^N(z). \tag{5}$$

Here $L_{0,d}^{class}$, $L_{0,d-1}^{class}$ are the semiclassical constants, which depend only on d (see (11) for the precise definition). Note that in (5) the term $L_{1,d}^{class}|\Omega|z^{1+\frac{d}{2}}$ is the leading term in the asymptotic expansion of $R_1^D(z)$, $R_1^N(z)$ as $z \to +\infty$.

Pólya's conjecture (2) is in some sense justified also by the semiclassical asymptotic expansions of the eigenvalues, which is equivalent to the expansion of the counting functions as $z \to +\infty$:

$$\begin{aligned} N^D(z) &\sim L_{0,d}^{class} |\Omega|\, z^{\frac{d}{2}} - \frac{1}{4} L_{0,d-1}^{class} |\partial\Omega|\, z^{\frac{d-1}{2}} \\ N^N(z) &\sim L_{0,d}^{class} |\Omega|\, z^{\frac{d}{2}} + \frac{1}{4} L_{0,d-1}^{class} |\partial\Omega|\, z^{\frac{d-1}{2}}. \end{aligned} \tag{6}$$

The first term in the expansions (6) was proved by Weyl [42], who later conjectured this two-term expansion (see [43]) that was proved only much later [30, 35] under suitable geometric conditions. More precisely, the set of periodic points of the geodesic billiard needs to have Lebesgue measure zero. We refer to [39] for a more exhaustive discussion on the history of semiclassical expansions, as well as two-term asymptotics for more general elliptic operators. Let us mention that the analogous expansions for the (more regular) first Riesz-mean hold under much weaker assumptions on the domain (see e.g., [13, 14]).

From this, the natural question arises whether the semiclassical expansions with the leading order term, or more terms as in (6), yield upper or lower bounds for all finite z. For the counting functions $N^D(z)$, $N^N(z)$ this amounts to Pólya's conjecture; for the Riesz-means $R_1^D(z)$, $R_1^N(z)$ this corresponds to the well-known bounds by Li-Yau [33] and Kröger [31].

In order to further understand the asymptotic behavior of the eigenvalues, several authors investigated Weyl-sharp inequalities for Riesz-means (or equivalently for eigenvalue averages) improving (3) with lower order terms and also reversed inequalities. In this regard, we mention the works [15, 20, 24, 26, 34, 41].

While the above discussion concerns inequalities and expansions in the Euclidean setting, a natural extension is to investigate similar questions for the Laplacian on different manifolds, for example when Ω is a domain of the sphere $\mathbb{S}^d = \{x \in \mathbb{R}^{d+1} : |x| = 1\}$. There has been a growing interest in the study of the properties of domains in manifolds for which Pólya's conjecture does not hold, the emblematic cases being the sphere $\mathbb{S}^d$ and the hemisphere $\mathbb{S}^d_+ = \mathbb{S}^d \cap \{x_{d+1} > 0\}$, $d \geq 2$. The hemisphere

$\mathbb{S}^2_+$ is an exception: Bérard and Besson [1] showed that Pólya's conjecture is true for the hemisphere, the quarter-sphere, and the eighth-sphere in dimension 2. Also, Freitas, Mao, and Salvessa [17] carry out a careful analysis of Pólya's conjecture on d-dimensional spheres and hemispheres, identifying precise subsequences of eigenvalues for which Pólya's conjecture fails, and others for which it is valid. Moreover, they prove a Pólya-type inequality with an asymptotically sharp correction term measuring how far the eigenvalues are from the leading term in Weyl's law. They also deduce Pólya-type inequalities for the eigenvalues on the whole sphere in the same spirit. These bounds suggest that a second term for the counting function $N(z)$ for $\mathbb{S}^d$ should be oscillating, but unbounded (which is expected due to the order of the remainder, which is known, see [7, 39]). Notice that, as the sphere violates the necessary geometrical conditions, an expansion like (6) cannot hold and is actually known to be false (cf. [39, Example 1.2.5]). Because of this, these bounds are fundamental to provide a better understanding of the remainder. The same conclusion can be deduced also for hemispheres, where again the expansion (6) cannot be inferred from classical arguments (cf. [39, Section 1.3.1]).

Considering sharp estimates of Riesz-means on domains of compact homogeneous manifolds (in particular spheres), Strichartz [40] proved a series of asymptotically sharp inequalities. We remark that the starting point is an observation due to Colin de Verdière and Gallot [19] relating the Riesz-mean in a domain with that in the manifold containing the domain. Improvements in the case of the sphere have been proved by Ilyin and Laptev [29]. We also mention El Soufi, Harrell, Ilias, Stubbe [9] where the authors present the so-called *averaged variational principle*, which is an efficient way to recover the result of Colin de Verdière and Gallot, and apply it to bound Riesz-means on general Riemannian manifolds, also for other types of operators [4, 6]. Related bounds for eigenvalue averages on domains of Riemannian manifolds can be found in [8]. However we remark that for domains in a general Riemannian manifold sharp upper or lower bounds for Riesz means are not available.

In this paper we consider specifically the Laplacian on the sphere and on the hemisphere and derive asymptotics and Weyl-sharp upper and lower bound for Riesz-means and counting functions that complement and improve those already present in the literature.

Our first aim is to investigate further terms in the asymptotic expansions for $N(z)$ in the case $\mathbb{S}^d$ and $N^D(z)$, $N^N(z)$ in the case of $\mathbb{S}^d_+$ in order to clarify the behavior highlighted in [17]. We will also consider subsequent terms in the expansion of the more regular Riesz-mean $R_1(z)$ for $\mathbb{S}^d$ and $R_1^D(z)$, $R_1^N(z)$ for $\mathbb{S}^d_+$. For example, for $R_1(z)$, the second term has a sign, but contains an oscillating part (see Theorem 4.1.1). We highlight the relation with the results in [40], where the lim-inf and lim-sup of the remainder term for $R_1(z)$ are computed. On the other hand, on $\mathbb{S}^d_+$, we derive three-term expansions both for the counting functions and the Riesz-means (see Theorems 4.2.1, 4.2.2, 4.2.3, 4.2.4). Note that $R_1^D(z)$, $R_1^N(z)$ have a second term which is coherent with the expansion (6) even though $N^D(z)$, $N^N(z)$ do not. This behavior is the expected one since Riesz-means present a higher regularity than counting functions. However, smoothing out the oscillations in the expansions of counting functions, we can see already from them what are the correct coefficients of lower order terms in the expansions of more regular quantities. We plan to make this argument rigorous and expand it to cover more general cases in a forthcoming paper [5].

Our results on asymptotic expansions for counting functions in some sense complete the study of [39, §1.7] where the authors consider non-classical two-terms expansions for the eigenvalues of the degree operator on spheres and hemispheres, namely the operator $-\Delta + \frac{(d-1)^2}{4}$. Note that the analysis for this operator is somehow easier since the energy levels are given by $\left(l + \frac{d-1}{2}\right)^2$. With some effort, it is possible to recover two-terms expansions for the eigenvalues of the Laplacian from the expansions in [39, §1.7]. However, up to our knowledge, the three-terms expansions established in our paper were not known.

Once precise spectral asymptotics are established, one naturally asks whether it is possible to obtain bounds, at least for the more regular Riesz-means. For example, in $\mathbb{S}^2$ we are able to improve the lower bounds for $R_1(z)$ present in [29, 40] and derive sharp bounds with lower order terms also for $\mathbb{S}^2_+$. As for the higher dimensional case, we prove upper and lower bounds for $R_1(z)$, containing an asymptotically sharp shift (see Theorems 4.1.3 and 4.1.4). Note that the upper bound with a shift in the form of Theorem 4.1.4 implies a Berezin–Li–Yau inequality for the shifted eigenvalues. The case of $\mathbb{S}^1$ is intrinsically different, since the leading term in Weyl's law is neither an upper bound nor a lower bound. Nevertheless, we provide an upper bound containing an asymptotically sharp shift (see Theorem 5.0.1).

The second aim of the present paper is to consider Berezin–Li–Yau bounds for Dirichlet eigenvalues on domains. It is well-known that, in general, it is not possible to bound from above the Riesz-mean $R_1^D(z)$ with the leading term in Weyl's law. In other words, the first inequality in (5) in general does not hold. The natural counterexample is a domain which is invading the whole sphere: its Dirichlet spectrum is converging to the spectrum on the whole sphere, for which Berezin–Li–Yau inequality in the form of the first inequality of (5) does not hold. However, bounds of Berezin–Li–Yau-type with a correction term can be obtained in the spirit of [19], as done in [29, 40]. In this paper we observe that if we restrict to domains on the hemisphere $\mathbb{S}^2_+$, then Berezin–Li–Yau bounds do hold (see Theorems 3.2.4 and 3.2.5). Moreover, they hold for $\mathbb{S}^d_+$ when $d = 3, 4, 5$ (see Theorem 4.2.5). For domains in $\mathbb{S}^d$ we complete the picture of [29, 40] by establishing Berezin–Li–Yau bounds with a shift term, which is asymptotically sharp when the domain is $\mathbb{S}^d$ (see Theorem 4.1.6).

For what concerns the techniques used, to obtain the results in $\mathbb{S}^d$ and $\mathbb{S}^d_+$ we mainly exploit the fact that in both cases the eigenvalues and their multiplicities are completely known and easily described, and this in turn allows for a somewhat explicit but rather complicated representation of any related quantity. Careful manipulations then permit to recover in a new manner known bounds and to derive new ones. The results for domains instead take advantage of what we proved for $\mathbb{S}^d$ together with the *averaged variational principle* of Harrell and Stubbe (see Theorem 2.0.1, see also [25]). We remark that there are other situations where eigenvalues and their multiplicities are explicitly known. This is the case, for example, of compact symmetric spaces of rank one (the sphere belongs to this family), see [27, 28]. Our techniques can be used to treat the case of these spaces and their domains. However we believe that is is more instructive to focus on the case of the sphere.

The paper is organized as follows. In Sect. 2 we introduce the notation and some preliminaries: we state the eigenvalue problems, the functional setting and the tools needed in our analysis. Section 3 contains our results in dimension 2, that is for the sphere $\mathbb{S}^2$, the hemisphere $\mathbb{S}^2_+$, and domains of the hemisphere. Then in Sect. 4 we consider the d-dimensional case of the sphere $\mathbb{S}^d$, its domains, and the hemisphere $\mathbb{S}^d_+$. In Sect. 5 we deal with the case of the circle $\mathbb{S}^1$. For the sake of clarity, we have postponed some technical results to Appendix A. In Appendix B we discuss a duality principle for Riesz-means.

2 Preliminaries and Notation

Let M^d be a d-dimensional, compact, Riemannian manifold, and let Ω be a domain in M^d (possibly $\overline{\Omega} = M^d$). We recall that Ω is called a domain if it is an open, bounded, connected set. By $L^2(\Omega)$ we denote the classical Lebesgue space of square integrable functions. By $H^m(\Omega)$ we denote the standard Sobolev space of functions in $L^2(\Omega)$ with all weak partial derivatives up to the order m in $L^2(\Omega)$. By $H_0^m(\Omega)$ we denote the closure of $C_c^\infty(\Omega)$ in $H^m(\Omega)$ with respect to its standard norm. Throughout the paper, by $\mathbb{N}$ we denote the set of natural numbers including zero.

On M^d we consider the (closed) eigenvalue problem for the Laplacian

$$-\Delta u = \lambda u, \tag{7}$$

and on domains $\Omega \subset M^d$ we consider the Dirichlet problem

$$\begin{cases} -\Delta u = \lambda u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{8}$$

and the Neumann problem

$$\begin{cases} -\Delta u = \lambda u, & \text{in } \Omega, \\ \partial_\nu u = 0, & \text{on } \partial\Omega. \end{cases} \tag{9}$$

We will understand problems (8) and (9) in their weak formulations. For problem (8) it amounts to finding a function $u \in H_0^1(\Omega)$ and a real number $\lambda \in \mathbb{R}$ such that

$$\int_\Omega \nabla u \cdot \nabla \phi = \lambda \int_\Omega u\phi, \quad \forall \phi \in H_0^1(\Omega). \tag{10}$$

For problem (9), the variational formulation is the same as (10) but with the energy space $H_0^1(\Omega)$ replaced by $H^1(\Omega)$.

We denote the eigenvalues of (7) as

$$0 = \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_j \leq \cdots \nearrow +\infty.$$

As for the Dirichlet and Neumann problems (8)-(9) on domains of M^d, we shall denote the eigenvalues by

$$0 < \lambda_1(\Omega) < \lambda_2(\Omega) \leq \cdots \leq \lambda_j(\Omega) \leq \cdots \nearrow +\infty$$

and

$$0 = \mu_1(\Omega) < \mu_2(\Omega) \leq \cdots \leq \mu_j(\Omega) \leq \cdots \nearrow +\infty,$$

respectively. In order to ease the notation, we will omit the explicit dependence on Ω when there is no possibility of confusion.

In many situations (e.g., $M^d = \mathbb{S}^d$, the round sphere), the eigenvalues of the Laplacian appear as *energy levels*, namely, the values assumed by the eigenvalues (without multiplicities) form a sequence which we shall denote by

$$0 = \lambda_{(0)} < \lambda_{(1)} < \lambda_{(2)} < \cdots < \lambda_{(l)} < \cdots \nearrow +\infty.$$

Each eigenvalue corresponding to an energy level $\lambda_{(l)}, l \in \mathbb{N}$, has a certain multiplicity, which depends on d and l, and which we shall denote by $m_{l,d}$. To clarify the situation, let us just consider the eigenvalues of the Laplacian on $\mathbb{S}^1$, which are given by the sequence

$$0, 1, 1, 4, 4, 9, 9, \cdots, l^2, l^2, \cdots$$

therefore $\lambda_{(l)} = l^2, l \in \mathbb{N}$, and $m_{0,1} = 1, m_{l,1} = 2$ for all $l \geq 1$. In general $m_{0,d} = 1$ for all d. If we want to enumerate the eigenvalues of $\mathbb{S}^1$ in increasing order, counting multiplicities, we will denote them as $\lambda_1 = 0, \lambda_2 = 1, \lambda_3 = 1, \lambda_4 = 4, \lambda_5 = 4$, etc. Energy levels and their multiplicities are explicitly known, for example, for all the compact symmetric spaces of rank 1 (the sphere belongs to this family), see e.g., [27, 28].

Concerning $\mathbb{S}^d$ and its domains, we will consider semiclassical estimates for *Riesz-means* of eigenvalues, namely for

$$R_\gamma(z) = \sum_j (z - \lambda_j)_+^\gamma \qquad \forall z \geq 0$$

where $\gamma \geq 0$, a_+ denotes the positive part of a real number a, and the sum is taken over $j \in \mathbb{N}$. As a convention, when the summation is over all $j \in \mathbb{N}$, we will just write the index j at the bottom of the summation symbol. If the sum is over some subset $J \subset \mathbb{N}$ we will write $\sum_{j \in J}$; if the sum starts from some $k_0 \in \mathbb{N}$ we will write $\sum_{j \geq k_0}$. When $\gamma = 0$, then $R_0(z)$ is just the counting function $N(z)$ which counts the number of eigenvalues λ_j below z.

We will denote by $R_1(z)$ and $N(z)$ the Riesz-mean and the counting function for the whole manifold M^d, namely, for problem (7). Moreover, we shall denote by $R_1^D(z), N^D(z)$ and by $R_1^N(z), N^N(z)$ the Riesz-means and the counting functions for the Dirichlet (8) and Neumann (9) problems on domains of M^d, respectively.

In this paper we will be interested in $\gamma = 1$, i.e., the first Riesz-mean, since semiclassical estimates can be deduced in a very efficient way by means of the *averaged variational principle*, introduced by Harrell and Stubbe in [9, 25], generalizing a work of Kröger [31] by averaging over test functions which form a complete frame of the underlying Hilbert space. We shall state it here for sake of completeness.

Theorem 2.0.1 *Let H be a self-adjoint operator in a Hilbert space $(\mathcal{H}, \langle\cdot,\cdot\rangle_{\mathcal{H}})$, the spectrum of which is discrete at least in its lower portion, and we denote it by*

$$\omega_1 \le \omega_2 \le \cdots \le \omega_j \le \cdots$$

with corresponding orthonormalized eigenvectors $\{g_j\}_{j\in\mathbb{N}\setminus\{0\}}$. The closed quadratic form corresponding to H is denoted $Q(\varphi,\varphi)$ for any φ in the quadratic form domain $\mathcal{Q}(H) \subset \mathcal{H}$. Let $f_p \in \mathcal{Q}(H)$ be a family of vectors indexed by a variable p ranging over a measure space $(\mathfrak{M}, \Sigma, \sigma)$. Suppose that $\mathfrak{M}_0$ is a measurable subset of $\mathfrak{M}$. Then for any $z \in \mathbb{R}$,

$$\sum_j (z-\omega_j)_+ \int_{\mathfrak{M}} \left|\langle g_j, f_p\rangle_{\mathcal{H}}\right|^2 d\sigma_p \ge \int_{\mathfrak{M}_0} \left(z\|f_p\|^2_{\mathcal{H}} - Q(f_p, f_p)\right) d\sigma_p,$$

provided that the integrals converge.

In particular, in the situation of $\mathbb{S}^d$ the *averaged variational principle* turns out to be equivalent to generalizations of the Berezin–Li–Yau method, which was first observed by Colin de Verdière and Gallot [19], and employed in various form in Ilyin and Laptev [29] and Strichartz [40]. This application of the averaged variational principle to recover in an efficient way the results of Strichartz [40] is contained in [9], and it is employed not only for homogeneous spaces but for more general Riemannian manifolds.

We shall denote by $L^{class}_{\gamma,d}$ the semiclassical constant for Laplacian eigenvalues in dimension d, which is given by

$$L^{class}_{\gamma,d} = (4\pi)^{-d/2}\frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1+d/2)}. \tag{11}$$

It is also convenient to recall that

$$L^{class}_{0,d}|\mathbb{S}^d| = \frac{2}{\Gamma(d+1)} = \frac{2}{d!}, \quad L^{class}_{1,d}|\mathbb{S}^d| = \frac{4}{(d+2)\Gamma(d+1)} = \frac{4}{(d+2)d!}.$$

Finally, we introduce the *fluctuation function* ψ defined by

$$\psi(\eta) = \eta - \lfloor\eta\rfloor - \frac{1}{2} \qquad \forall\eta \ge 0, \tag{12}$$

where $\lfloor\eta\rfloor$ denotes the integer part of η.

3 The Two-Dimensional Sphere $\mathbb{S}^2$ and the Hemisphere $\mathbb{S}^2_+$

In this section we will consider semiclassical estimates for Laplacian eigenvalues in the exceptional case of the two-dimensional sphere. In particular, we shall consider the closed problem on $\mathbb{S}^2$, the Dirichlet and Neumann problems for the hemisphere $\mathbb{S}^2_+$, and on domains of $\mathbb{S}^2_+$.

3.1 The Sphere $\mathbb{S}^2$

As is well-known, the energy levels of the Laplacian on $\mathbb{S}^2$ are given by $\lambda_{(l)} = l(l+1)$ with corresponding multiplicities $m_{l,2} = 2l+1$, $l \in \mathbb{N}$, see e.g., [3]. It is well-known [39, 42] that Weyl's law for the counting function of the Laplacian eigenvalues on $\mathbb{S}^2$ reads

$$N(z) = L^{class}_{0,2}|\mathbb{S}^2|z + o(z) = z + o(z) \quad \text{as } z \to +\infty,$$

and, accordingly, the semiclassical limit for the first Riesz-mean R_1 is

$$R_1(z) = L^{class}_{1,2}|\mathbb{S}^2|z^2 + o(z^2) = \frac{1}{2}z^2 + o(z^2) \quad \text{as } z \to +\infty.$$

Strichartz [40, (3.11)-(3.13) p. 166] proves a Weyl sharp lower bound with a correction term and a Weyl sharp upper bound for the eigenvalue means of the Laplacian eigenvalues on $\mathbb{S}^2$. These bounds are equivalent to a Weyl sharp shifted upper bound and a Weyl sharp lower bound the first Riesz-mean R_1 which we show in the following proposition. The upper bound was also shown by Ilyin and Laptev [29]. Our technique will allow a more careful analysis, improving these bounds in Theorem 3.1.1. For a discussion of $d > 2$ and our significant improvements based on the techniques introduced here, see Sect. 4.1, in particular Theorems 4.1.3 and 4.1.4 and the subsequent remarks.

Proposition 3.1.1 *For all $z \geq 0$, the following bounds hold for the first Riesz-mean R_1 of the Laplacian eigenvalues on $\mathbb{S}^2$:*

$$\frac{1}{2}z^2 \leq R_1(z) \leq \frac{1}{2}\left(z + \frac{1}{2}\right)^2.$$

Equality in the lower bound holds if and only if $z = \lambda_{(l)}$ for some $l \in \mathbb{N}$. For the upper bound, equality holds if and only if $z = (l+1)^2 - \frac{1}{2} \in [\lambda_{(l)}, \lambda_{(l+1)}]$ for some $l \in \mathbb{N}$.

Proof For the sake of simplicity, we prove the bounds for $z = w(w+1)$ where $w \geq 0$. We note that

$$\begin{aligned} R_1(w(w+1)) &= \sum_{l=0}^{\lfloor w \rfloor}(2l+1)(w(w+1) - l(l+1)) \\ &= \frac{1}{2}(w + \lfloor w \rfloor + 1)(w + \lfloor w \rfloor + 2)(w - \lfloor w \rfloor)(\lfloor w \rfloor + 1 - w) + \frac{1}{2}w^2(w+1)^2. \end{aligned} \tag{13}$$

Since $\lfloor w \rfloor \leq w < \lfloor w \rfloor + 1$ the first term in the right hand side of the above equation is non-negative. Moreover, it equals zero if and only if $w \in \mathbb{N}$, that is when $w(w+1)$ equals an energy level $\lambda_{(l)}$ for some $l \in \mathbb{N}$. For the upper bound we write R_1 as follows

$$R_1(w(w+1)) = -\frac{1}{8}\left(-2(w + \lfloor w \rfloor + 1)(w - \lfloor w \rfloor) + 2\lfloor w \rfloor + 1 \right)^2 + \frac{1}{2}\left(w(w+1) + \frac{1}{2} \right)^2. \tag{14}$$

Since

$$\begin{aligned} &-2(w + \lfloor w \rfloor + 1)(w - \lfloor w \rfloor) + 2\lfloor w \rfloor + 1 \\ &= 2\left(\lfloor w \rfloor + \frac{1}{2} \right)\left(\lfloor w \rfloor + \frac{3}{2} \right) - 2\left(w + \frac{1}{2} \right)^2, \end{aligned}$$

the first term in the right hand side of equation (14) is non-positive and equals zero if

$$w = -\frac{1}{2} + \sqrt{\left(\lfloor w \rfloor + \frac{1}{2} \right)\left(\lfloor w \rfloor + \frac{3}{2} \right)}$$

which has a solution $w = -\frac{1}{2} + \sqrt{\left(l + \frac{1}{2}\right)\left(l + \frac{3}{2}\right)}$ in each interval $[l, l+1]$. Hence, recalling the substitution $z = w(w+1)$ we have that the equality in the upper bound holds if and only if $z = (l+1)^2 - \frac{1}{2}$ which is in the interval $[\lambda_{(l)}, \lambda_{(l+1)}]$. □

As anticipated, a careful inspection of the proof of Proposition 3.1.1 allows to establish an improved two-sided bound for the first Riesz-mean with a sharp first term and a second term of order z with an oscillating, but positive, coefficient, thus improving the results of [29, 40].

Theorem 3.1.1 *For all $z \geq 0$, the following bounds hold for the first Riesz-mean R_1 of the Laplacian eigenvalues on $\mathbb{S}^2$:*

$$\begin{aligned} &\frac{1}{2}z^2 + 2\left(\frac{1}{4} - \psi(w)^2 \right)\left(z - \frac{\sqrt{z}}{2} \right) \leq R_1(z) \\ &\leq \frac{1}{2}z^2 + 2\left(\frac{1}{4} - \psi(w)^2 \right)\left(z + \frac{\sqrt{z}}{2} + \frac{1}{2} \right), \end{aligned}$$

where ψ is the fluctuation function (12) and w is defined by the relation $w(w+1) = z$. Consequently, for any $\epsilon > 0$:

$$\lim_{z \to +\infty} z^{-\epsilon - 1/2}\left(R_1(z) - \frac{1}{2}z^2 - 2\left(\frac{1}{4} - \psi(w)^2 \right) z \right) = 0.$$

Proof It is sufficient to consider the third line of (13), substitute $\lfloor w \rfloor$ with $w - \psi(w) - \frac{1}{2}$, and use the bounds $w \leq \sqrt{z} \leq w + 1$ and $|\psi| \leq \frac{1}{2}$. □

We remark that the lower bound is given by the first two terms of the asymptotic expansion of $R_1(z)$ which we prove in general for $d \geq 2$ (see Theorem 4.1.1), plus a term of negative sign of (lower) order $\sqrt{z}$, and the upper bound is given by the same two terms, plus a term of positive sign of (lower) order $\sqrt{z}$, as it is expected.

3.2 The Hemisphere $\mathbb{S}^2_+$

Now we pass to consider the case of the two dimensional hemisphere $\mathbb{S}^2_+$. Since the hemisphere $\mathbb{S}^2_+$ has a non-empty boundary, to consider problems on $\mathbb{S}^2_+$ it is necessary to impose boundary conditions. We will consider both the cases of Dirichlet and Neumann boundary conditions imposed on the equator, that is problems (8) and (9) with $M^d = \mathbb{S}^2$ and $\Omega = \mathbb{S}^2_+$.

3.2.1 Dirichlet Laplacian

We start with the case of Dirichlet boundary condition imposed on the equator. As is well-known, the energy levels of the Dirichlet Laplacian on $\mathbb{S}^2_+$ are the same of the Laplacian on $\mathbb{S}^2$, that is $\lambda_{(l)} = l(l+1)$, but with corresponding multiplicities l, where $l \in \mathbb{N} \setminus \{0\}$.

Since the work by Bérard and Besson [1] it is known that the eigenvalues of the Dirichlet Laplacian on $\mathbb{S}^2_+$ satisfy Pólya's conjecture. The same result, together with many more on Pólya's-type inequalities on spheres and hemisphere, is proved by Freitas, Mao and Salvessa [17]. First, we provide another elementary proof of Pólya's conjecture for $\mathbb{S}^2_+$.

Proposition 3.2.1 *For all $z \geq 0$, the counting function $N^D(z)$ for the Dirichlet Laplacian on $\mathbb{S}^2_+$ satisfies the following inequality:*

$$N^D(z) \leq \frac{1}{2}z. \tag{15}$$

Proof As already done in previous proofs, we set $z = w(w+1)$ with $w \geq 0$. Then we have

$$\begin{aligned} N^D(w(w+1)) - \frac{w(w+1)}{2} &= \sum_{l=1}^{\lfloor w \rfloor} l - \frac{w(w+1)}{2} \\ &= -\frac{(w - \lfloor w \rfloor)(w + 1 + \lfloor w \rfloor)}{2} \leq 0, \end{aligned}$$

that clearly proves the bound. □

Actually, we are able to prove a two-sided bound for the counting function, where the upper bound improves the result of Proposition 3.2.1.

Theorem 3.2.1 *For all $z \geq 0$, the counting function $N^D(z)$ for the Dirichlet Laplacian on $\mathbb{S}^2_+$ satisfies the following inequality:*

$$\frac{z}{2}\left(1-\left(\psi(w)+\frac{1}{2}\right)z^{-\frac{1}{2}}\right)^2-\frac{1}{8}\left(\psi(w)+\frac{1}{2}\right)z^{-\frac{1}{2}} \leq N^D(z)$$
$$\leq \frac{z}{2}\left(1-\left(\psi(w)+\frac{1}{2}\right)z^{-\frac{1}{2}}\right)^2,$$

where w is defined by the relation $w(w+1)=z$.

Proof We prove the inequalities for $z=w(w+1)$, $w \geq 0$. We have

$$N(w(w+1)) = \frac{\lfloor w\rfloor(\lfloor w\rfloor+1)}{2} = \frac{(w+\lfloor w\rfloor-w)(w+1+\lfloor w\rfloor-w)}{2}$$
$$= \frac{w(w+1)-\left(\psi(w)+\frac{1}{2}\right)(2w+1)+\left(\psi(w)+\frac{1}{2}\right)^2}{2}.$$

For the upper bound it suffices to note that $2w+1 \geq 2\sqrt{w(w+1)}$ and recall the substitution $z=w(w+1)$. In fact,

$$\frac{w(w+1)-\left(\psi(w)+\frac{1}{2}\right)(2w+1)+\left(\psi(w)+\frac{1}{2}\right)^2}{2}$$
$$\leq \frac{w(w+1)}{2}-\left(\psi(w)+\frac{1}{2}\right)\sqrt{w(w+1)}+\frac{1}{2}\left(\psi(w)+\frac{1}{2}\right)^2$$
$$= \frac{w(w+1)}{2}\left(1-\left(\psi(w)+\frac{1}{2}\right)(w(w+1))^{-\frac{1}{2}}\right)^2.$$

The lower bound can be proved in the same way by noting that $2w+1 \leq 2\sqrt{w(w+1)}+\frac{1}{4\sqrt{w(w+1)}}$. □

Remark 3.2.1 We remark that the upper bound coincides with the expression given by the leading term in Weyl's law plus the second and the third terms found in the expansion (44) for $N^D(z)$ in Theorem 4.2.1 for all $d \geq 2$. The lower bound coincides with the same three terms, and the further term which one can find going further in the asymptotic expansion of $N^D(z)$ (which is not difficult in the case $d=2$).

We pass now to consider Weyl sharp upper and lower bounds for the first Riesz-mean R_1^D. The semiclassical expansion of R_1^D reads

$$R_1^D(z) = L_{1,2}^{class}|\mathbb{S}^2_+|z^2-\frac{1}{4}L_{1,1}^{class}|\partial\mathbb{S}^2_+|z^{3/2}+O(z)$$
$$= \frac{1}{4}z^2-\frac{1}{3}z^{3/2}+O(z) \quad \text{as } z \to +\infty$$

where the $O(z)$ term is oscillatory and non-negative (see Theorem 4.2.2; see [12] for Euclidean domains). Note that R_1^D admits a two-term "standard" expansion as in (6) (the second term is a power-like function), contrarily to N^D (see (44)).

Note also that the leading term in Weyl's law is an upper bound for R_1^D and this follows immediately from the validity of Pólya's conjecture (Proposition 3.2.1).

In the following theorem we derive upper and lower bounds for R_1^D with also lower order terms.

Theorem 3.2.2 *For all $z \geq 0$ the following bounds hold for the first Riesz-mean R_1^D of the Dirichlet Laplacian eigenvalues on $\mathbb{S}_+^2$:*

$$\frac{1}{4}z^2 - \frac{1}{3}z\sqrt{z+\frac{1}{4}} \leq R_1^D(z) \leq \frac{1}{4}z^2 - \frac{1}{3}z\sqrt{z+\frac{1}{4}} + \frac{1}{4}z.$$

Moreover, equality in the lower bound occurs if and only if $z = \lambda_{(l)}$ for some $l \in \mathbb{N}\backslash\{0\}$.

Proof As in the previous proofs, we derive the bounds for $z = w(w+1)$, for all $w > 0$. We first note that

$$\begin{aligned}
R_1^D(w(w+1)) &= \sum_{l=1}^{\lfloor w \rfloor} l(w(w+1) - l(l+1)) \\
&= -\frac{1}{4}(w - \lfloor w \rfloor)^2 (w + \lfloor w \rfloor + 1)^2 \\
&\quad - \frac{1}{6}\lfloor w \rfloor(\lfloor w \rfloor + 1)(2\lfloor w \rfloor + 1) + \frac{1}{4}w^2(w+1)^2.
\end{aligned}$$

For the lower bound we write R_1^D as follows

$$\begin{aligned}
&R_1^D(w(w+1)) - \frac{1}{4}w^2(w+1)^2 + \frac{1}{6}w(w+1)(2w+1) \\
&= -\frac{1}{4}(w - \lfloor w \rfloor)^2 (w + \lfloor w \rfloor + 1)^2 \\
&\quad + \frac{1}{6}\Big(w(w+1)(2w+1) - \lfloor w \rfloor(\lfloor w \rfloor + 1)(2\lfloor w \rfloor + 1)\Big).
\end{aligned}$$

We add and subtract $\frac{1}{4}(w - \lfloor w \rfloor)(w + \lfloor w \rfloor + 1)^2$ to the right hand side of the previous equality. First we note that

$$\begin{aligned}
-\frac{1}{4}(w - \lfloor w \rfloor)^2 (w + \lfloor w \rfloor + 1)^2 &+ \frac{1}{4}(w - \lfloor w \rfloor)(w + \lfloor w \rfloor + 1)^2 \\
&= \frac{1}{4}(w - \lfloor w \rfloor)(1 + \lfloor w \rfloor - w)(w + \lfloor w \rfloor + 1)^2.
\end{aligned}$$

Moreover,

$$
\begin{aligned}
&\frac{1}{6}\left(w(w+1)(2w+1)-\lfloor w\rfloor(\lfloor w\rfloor+1)(2\lfloor w\rfloor+1)\right)\\
&=\frac{1}{6}(w-\lfloor w\rfloor)\left(2\lfloor w\rfloor^2+2w^2+2w\lfloor w\rfloor+3w+3\lfloor w\rfloor+1\right)\\
&=\frac{1}{4}(w-\lfloor w\rfloor)\left(\frac{1}{3}(w-\lfloor w\rfloor)^2+(w+\lfloor w\rfloor)^2+2w+2\lfloor w\rfloor+\frac{2}{3}\right),
\end{aligned}
$$

and therefore

$$
\begin{aligned}
&-\frac{1}{4}(w-\lfloor w\rfloor)(w+\lfloor w\rfloor+1)^2\\
&+\frac{1}{6}\left(w(w+1)(2w+1)-\lfloor w\rfloor(\lfloor w\rfloor+1)(2\lfloor w\rfloor+1)\right)\\
&=\frac{1}{12}(w-\lfloor w\rfloor)\left((w-\lfloor w\rfloor)^2-1\right).
\end{aligned}
$$

Combining both we get

$$
\begin{aligned}
&R_1^D(w(w+1))-\frac{1}{4}w^2(w+1)^2+\frac{1}{6}w(w+1)(2w+1)\\
&=\frac{1}{12}(w-\lfloor w\rfloor)\Big(3(1+\lfloor w\rfloor-w)(w+\lfloor w\rfloor+1)^2+(w-\lfloor w\rfloor)^2-1\Big)\\
&=\frac{1}{12}(w-\lfloor w\rfloor)(1+\lfloor w\rfloor-w)\left(3(w+\lfloor w\rfloor+1)^2-(1+w-\lfloor w\rfloor)\right),
\end{aligned}
$$

which is obviously non-negative. In particular, the right hand side of the previous equality vanishes if and only if w is a non-negative integer. Recalling the substitution $z=w(w+1)$ the statement for the lower bound is proved.

Next we pass to consider the upper bound. For the sake of simplicity from now up to the end of the proof we write $w=\lfloor w\rfloor+x$ where $x\in[0,1[$ denotes the fractional part of w. We rewrite R_1^D in the following way

$$
\begin{aligned}
&R_1(w(w+1))-\frac{1}{4}w^2(w+1)^2+\frac{1}{6}w(w+1)(2w+1)\\
&=\frac{1}{12}x(1-x)\left(12\lfloor w\rfloor^2+12\lfloor w\rfloor x+12\lfloor w\rfloor+3x^2+5x+2\right)\\
&=\frac{1}{4}(\lfloor w\rfloor+x)(\lfloor w\rfloor+1+x)-\lfloor w\rfloor(\lfloor w\rfloor+1)\left(x-\frac{1}{2}\right)^2\\
&-\frac{x}{2}(1-2x(1-x))\lfloor w\rfloor-\frac{x}{12}\left(3x^3+2x^2+1\right)\le\frac{1}{4}(\lfloor w\rfloor+x)(\lfloor w\rfloor+1+x)
\end{aligned}
$$

which is the claimed upper bound. □

Remark 3.2.2 In Theorem 3.2.2 the lower bound is negative for $0 \leq z \leq 2$ and since $R_1^D(z) = 0$ for $0 \leq z \leq 2$ one can clearly replace it by the trivial bound 0. Moreover, the upper bound in Theorem 3.2.2 also implies the Weyl-sharp upper bound $R_1^D(z) \leq \frac{z^2}{2}$ since $-\frac{1}{3}z\sqrt{z+\frac{1}{4}} + \frac{1}{4}z \leq 0$ for all $z \geq 5/16$.

3.2.2 Neumann Laplacian

Next we pass to consider the case of Neumann boundary conditions. The energy levels of the Neumann Laplacian on $\mathbb{S}^2_+$ are again the same of the Laplacian on $\mathbb{S}^2$, that is $\lambda_{(l)} = l(l+1)$, but with corresponding multiplicities $l+1$, where $l \in \mathbb{N}$.

As we have done for the Dirichlet Laplacian on $\mathbb{S}^2_+$, we show Weyl sharp upper and lower bound for the first Riesz-mean R_1^N of the Neumann eigenvalues. The semiclassical expansion of R_1^N is given by

$$R_1^N(z) = L^{class}_{1,2}|\mathbb{S}^2_+|z^2 + \frac{1}{4}L^{class}_{1,1}|\partial\mathbb{S}^2_+| + O(z) = \frac{1}{4}z^2 + \frac{1}{3}z^{3/2} + O(z) \qquad \text{as } z \to +\infty,$$

where the $O(z)$ term is oscillatory and non-negative (see Theorem 4.2.4). We have the following two-sided bound with two sharp terms.

Theorem 3.2.3 *For all $z \geq 0$ the following bounds hold for the first Riesz-mean R_1^N of the Neumann Laplacian eigenvalues on $\mathbb{S}^2_+$:*

$$\frac{1}{4}z^2 + \frac{1}{3}z\sqrt{z+\frac{1}{4}} \leq R_1^N(z) \leq \frac{1}{4}z^2 + \frac{1}{3}z\sqrt{z+\frac{1}{4}} + z.$$

Moreover, equality in the lower bound occurs if and only if $z = \lambda_{(l)}$ for some $l \in \mathbb{N}$.

Proof As in the previous proofs, we derive the bounds for $z = w(w+1)$, for all $w \geq 0$. By a direct computation one can verify that

$$\begin{aligned} &R_1^N(w(w+1)) - \frac{1}{4}w^2(w+1)^2 - \frac{1}{6}w(w+1)(2w+1) \\ &\quad = (1-4\psi^2(w))\left(w^2 + \frac{3-2\psi(w)}{8}w + \frac{(7-6\psi(w))(3-2\psi(w))}{192}\right). \end{aligned} \tag{16}$$

The right hand side of equation (16) is clearly non-negative, and thus the lower bound holds. Moreover, the right hand side vanishes for those $w > 0$ such that $\psi(w) = \pm 1/2$, that is when w is a natural number and hence $w(w+1)$ is an energy level.

For the upper bound we may assume $w \geq 1$ since for $w \leq 1$ we have $R_1(w(w+1)) = w(w+1)$ and the upper bound is trivially verified. Now we note that $\frac{3-2\psi(w)}{8} \leq \frac{1}{2}$ and $\frac{(7-6\psi)(3-2\psi)}{192} \leq \frac{5}{14} \leq \frac{w}{2}$. Hence

$$\begin{aligned} &(1-4\psi^2(w))\left(w^2 + \frac{3-2\psi(w)}{8}w + \frac{(7-6\psi(w))(3-2\psi(w))}{192}\right) \\ &\leq w^2 + \frac{w}{2} + \frac{w}{2} = w(w+1). \end{aligned}$$

That is the right hand side of equation (16) is bounded above by $w(w+1)$, which concludes the proof. $\square$

3.2.3 Domains in $\mathbb{S}^2_+$

Here we derive upper bounds in the spirit of Berezin–Li–Yau [2, 33] for the first Riesz-mean of the eigenvalues of the Dirichlet Laplacian on domains Ω contained in the hemisphere $\mathbb{S}^2_+$. Namely, we prove that for these domains the leading term in Weyl's law is an upper bound for R_1^D, i.e., the first inequality in (5) holds.

We recall that we denote by

$$0 < \lambda_1(\Omega) < \lambda_2(\Omega) \leq \ldots \leq \lambda_j(\Omega) \leq \ldots \nearrow +\infty$$

the eigenvalues of the Dirichlet Laplacian on Ω, each repeated in accordance with its multiplicity, and by $\{u_j\}_{j\geq 1}$ the corresponding $L^2(\Omega)$-orthonormal sequence of eigenfunctions.

We note that Strichartz [40] considered Berezin–Li–Yau-type inequalities for the eigenvalues of the Laplacian on domains of the sphere. For general domains of the sphere, a Berezin–Li–Yau inequality in the form of the first inequality of (5) cannot hold since the first eigenvalue on the whole sphere is zero, and actually, the opposite bound holds, see Proposition 3.1.1. In [40] the author proves a Berezin–Li–Yau-type inequality with a first sharp term and with a lower order correction. The basic estimate relies on an observation of Colin de Verdière and Gallot, already contained in [19]. We also refer to [9, 29] for equivalent approaches leading to analogous results.

The principal idea of this section in order to recover an analogue of the Berezin–Li–Yau inequality and improve the result of [29, 40] is to get rid of the eigenvalue 0 of the Laplacian on $\mathbb{S}^2$ by considering only domains $\Omega \subset \mathbb{S}^2_+$. Note that the Berezin–Li–Yau inequality in the form of the first inequality of (5) cannot hold in general as long as the domain is not contained in a hemisphere, even if it is close to it. In fact, for a spherical cap of radius $\pi/2 + \epsilon$ in $\mathbb{S}^2$, Pólya's conjecture already fails for λ_1.

We are now ready state our first result. Its proof is based on the *averaged variational principle* (i.e., Theorem 2.0.1) with the use of the eigenfunction u_j extended to zero outside Ω as test functions for the Dirichlet Laplacian eigenvalues on $\mathbb{S}^2_+$. For the sake of clarity we have postponed two technical lemmas used in the proof to the end of this subsection.

Theorem 3.2.4 *Let Ω be a domain in $\mathbb{S}^2_+$. Then for all $z \geq 0$ the following inequality for the first Riesz-mean R_1^D of the eigenvalues of the Dirichlet Laplacian on Ω holds:*

$$R_1^D(z) = \sum_{j\geq 1} \big(z - \lambda_j(\Omega)\big)_+ \leq \frac{1}{8\pi} |\Omega| z^2.$$

Proof The eigenfunctions of the Dirichlet Laplacian on $\mathbb{S}^2_+$ associated with the energy level $\lambda_{(l)}$ are the spherical harmonics $Y_l^{-l-1+2h}$, where $h = 1, \ldots, l$. We note one more time here that the index l is not the numbering of the eigenvalues counting multiplicities, as it is j for $\lambda_j(\Omega)$, but it is the numbering of the energy levels.

Let $z \geq 0$. We apply Theorem 2.0.1 with $\mathcal{H} = L^2(\mathbb{S}^2_+)$, $H = -\Delta$, $\mathcal{Q} = H^1_0(\mathbb{S}^2_+)$, $Q(u,u) = \int_{\mathbb{S}^2_+} |\nabla u|^2$, $\mathfrak{M} = \mathbb{N}\backslash\{0\}$, $\mathfrak{M}_0 = \{j \in \mathbb{N}\backslash\{0\} : z - \lambda_j(\Omega) \geq 0\}$, $f_p = u_j$. We get

$$\sum_{l\geq 1}\sum_{h=1}^{l} (z - l(l+1))_+ \sum_{j\geq 1}\left|\int_\Omega Y_l^{-l-1+2h} u_j \, dS\right|^2 \geq \sum_{j\geq 1}\left(z - \lambda_j(\Omega)\right)_+,$$

which, since $\{u_j\}_k$ form a complete set in $L^2(\Omega)$, implies that

$$\sum_{l\geq 1}\sum_{h=1}^{l} \left(z - l(l+1)\right)_+ \int_\Omega |Y_l^{-l-1+2h}|^2 \, dS \geq \sum_{j\geq 1}\left(z - \lambda_j(\Omega)\right)_+ .$$

Now we note that

$$\sum_{h=1}^{l} |Y_l^{-l-1+2h}|^2 \leq \sum_{m=-l}^{l} |Y_l^m|^2 = \frac{2l+1}{4\pi}$$

by the addition formula for spherical harmonics (see [11, Chapter 2,§H], see also [21]), and, accordingly we get

$$\frac{|\Omega|}{4\pi}\sum_{l\geq 1}(2l+1)\left(z - l(l+1)\right)_+ \geq \sum_{j\geq 1}\left(z - \lambda_j(\Omega)\right)_+ . \tag{17}$$

Then the statement follows by Lemma 3.2.1 below. □

Remark 3.2.3 Alternatively, in order to recover the above Berezin–Li–Yau bound, we could have followed a more physical idea. Let $\Omega \subset \mathbb{S}^2_+$. Let $\widetilde{\Omega}$ be the set obtained by reflecting Ω at the equator. Then we consider the Dirichlet eigenvalues of $\Omega \cup \widetilde{\Omega}$ but restricted to functions antisymmetric with respect to the equator (and the same for the entire sphere). In this space the eigenvalues on $\Omega \cup \widetilde{\Omega}$ are of course $\lambda_j(\Omega)$ with the same multiplicities. Finally we apply the averaged variational principle of Theorem 2.0.1 as above.

Remark 3.2.4 As already pointed out, Theorem 3.2.4 cannot hold for large domains Ω approaching the entire sphere for which the reversed inequality of the theorem holds (see Proposition 3.1.1).

We also prove another upper bound for R_1^D containing lower order terms which improves Theorem 3.2.4 when $z > 1$.

Theorem 3.2.5 *Let* Ω *be a domain in* $\mathbb{S}^2_+$. *Then for all* $z \geq 0$ *the following inequality for the first Riesz-mean* R_1^D *of the eigenvalues of the Dirichlet Laplacian on* Ω *holds:*

$$R_1^D(z) = \sum_{j\geq 1}\big(z - \lambda_j(\Omega)\big)_+ \leq \frac{1}{8\pi}|\Omega|\left(z - \frac{1}{2}\right)^2.$$

Proof The proof can be performed following the same lines of that of Theorem 3.2.4 together with the use of Lemma 3.2.2 instead of Lemma 3.2.1. □

We conclude with the two technical lemmas we used to prove the previous results.

Lemma 3.2.1 *For all* $z \geq 0$ *the following inequality holds:*

$$\sum_{l\geq 1}(2l+1)\big(z - l(l+1)\big)_+ \leq \frac{z^2}{2}.$$

Proof The proof can be performed by direct computations.

□

It is possible to improve the previous lemma adding lower order terms (see [29, Lemma 3.2]).

Lemma 3.2.2 *For all* $z \geq 0$ *the following inequality holds:*

$$\sum_{l\geq 1}(2l+1)\big(z - l(l+1)\big)_+ \leq \frac{1}{2}\left(z - \frac{1}{2}\right)^2.$$

Proof We prove the inequality for $z = w(w+1)$, $w \geq 1$. For the sake of simplicity we write $w = \lfloor w \rfloor + x$ where $x \in [0, 1[$ denotes the fractional part of w. Then

$$\begin{aligned}
&\sum_{l=1}^{\lfloor w \rfloor}(2l+1)((\lfloor w \rfloor + x)(\lfloor w \rfloor + x + 1) - l(l+1))_+ \\
&-\frac{1}{2}\left((\lfloor w \rfloor + x)(\lfloor w \rfloor + x + 1) - \frac{1}{2}\right)^2 \\
&= -\frac{1}{8}\left(\lfloor w \rfloor(4x-2) + 2x(x+1) - 1\right)^2 \leq 0.
\end{aligned}$$

□

4 The *d*-Dimensional Sphere $\mathbb{S}^d$ and the Hemisphere $\mathbb{S}^d_+$

The general case $d \geq 3$ presents a few peculiar features: for example, Pólya's conjecture does not hold for $\mathbb{S}^d_+$ as shown in [17]. Actually, it should be remarked that the two-dimensional case is the special case. In what follows we shall treat $d \geq 2$.

4.1 The Sphere $\mathbb{S}^d$

We recall that the eigenvalues of the Laplacian on $\mathbb{S}^d$ are given as energy levels by $\lambda_{(l)} = l(l+d-1)$ with multiplicities $m_{l,d} = H_{l,d} - H_{l-2,d}$ where

$$H_{l,d} = \binom{d+l}{l},$$

see e.g., [3].

The first result of this subsection is a two-term expansion for $R_1(z)$. Note that the second term has a sign, though it contains an oscillating part. We stress the fact that, since the sphere has no boundary, the classical second term in $z^{\frac{d}{2}}$ is not present in the expansion, and the term we obtain may be regarded as a "third term" in the semiclassical expansion. This asymptotic expansion improves the result in Strichartz [40, Theorem 3.3 p. 168] on eigenvalue means, where the lim inf and the lim sup of the second term was given. Moreover, in Theorems 4.1.3 and 4.1.4 below we shall prove lower and and upper bounds on R_1 corresponding to the lower and upper envelope of the asymptotic expansion (via the estimates $0 \le \frac{1}{4} - \psi^2 \le \frac{1}{4}$ for the fluctuation function, see also Remark 4.1.2 below).

Theorem 4.1.1 *As z tends to infinity we have the following asymptotic expansion for the first Riesz mean R_1 on $\mathbb{S}^d$:*

$$\frac{R_1(z)}{L^{class}_{1,d} |\mathbb{S}^d| \, z^{\frac{d}{2}+1}} = 1 + \frac{d(d+2)}{12}\left(d - 2 + 6\left(\frac{1}{4} - \psi^2(w)\right)\right) z^{-1} + o(z^{-1}), \tag{18}$$

where w is defined by the relation $w(w+d-1) = z$.

Proof We first prove that

$$\begin{aligned} R_1(z) &= \sum_{l=0}^{L} m_{l,d}\big(z - l(l+d-1)\big) \\ &= \frac{(2L+d)\Gamma(L+d)}{(d+2)\Gamma(L+1)\Gamma(d+1)}\,(-dL(L+d) + (d+2)z), \end{aligned} \tag{19}$$

where $L = \lfloor w \rfloor$. Note that (19) can be deduced by [29, Appendix A] (see also [40, Theorem 3.2]). We prove it here for the reader's convenience. We start by recalling the following well-known formula (see e.g., [22, 0.15 p.3])

$$\sum_{k=0}^{m} \binom{n+k}{n} = \binom{n+m+1}{n+1} \tag{20}$$

and we write

$$\sum_{l=0}^{L} m_{l,d} z = z\left(\sum_{l=0}^{L}\binom{d+l}{l} - \sum_{l=0}^{L}\binom{d+l-2}{l-2}\right). \tag{21}$$

Thus

$$\begin{aligned}\sum_{l=0}^{L} m_{l,d} l(l+d-1) &= \sum_{l=0}^{L}\binom{d+l}{l}(l(l-1)+ld)\\ &\quad -\sum_{l=0}^{L}\binom{d+l-2}{l-2}((l+d)(l+d-1)-d(l+d-1))\\ &= \sum_{l=0}^{L} d(d+1)\binom{d+l}{l-1}\\ &\quad +\sum_{l=0}^{L}(d+1)(d+2)\binom{d+l}{l-2} - \sum_{l=0}^{L}(d+1)(d+2)\binom{d+l}{l-2}\\ &\quad +\sum_{l=0}^{L} d(d+1)\binom{d+l-1}{l-2}.\end{aligned} \tag{22}$$

Using (20) to compute (21) and (22), we get (19).

Since

$$L_{1,d}^{class}|\mathbb{S}^d| = \frac{4}{(d+2)\Gamma(d+1)},$$

we get

$$\frac{R_1(z)}{L_{1,d}^{class}|\mathbb{S}^d|z^{1+d/2}} = \frac{(2L+d)\Gamma(L+d)}{4\Gamma(L+1)z^{1+d/2}}(-dL(L+d)+(d+2)z). \tag{23}$$

This proves (19). We apply the asymptotic expansions given in Appendix A for the Gamma function (Lemma A.0.3), and the Taylor expansions of the quadratic polynomials $P_{a,b}(x) = 1 + ax + bx^2$ (Lemma A.0.2) with $x = 1/L$. First we note that

$$\begin{aligned}\frac{(2L+d)\Gamma(L+d)}{4\Gamma(L+1)} &= L^d\left(\frac{(2+dx)e^{-d}(1+dx)^{1/x}(1+dx)^{d-1/2}P_{\frac{1}{12},\frac{1}{288}}(\frac{x}{1+dx})}{4\,P_{\frac{1}{12},\frac{1}{288}}(x)} + O(x^3)\right)\\ &= L^d\left(\frac{1}{2}(1+\frac{dx}{2})P_{-\frac{d^2}{2},\frac{8d^3+3d^4}{24}}(x)P_{\frac{(2d-1)d}{2},\frac{d^2(2d-1)(2d-3)}{8}}(x)\cdot\frac{P_{\frac{1}{12},\frac{1}{288}}(\frac{x}{1+dx})}{P_{\frac{1}{12},\frac{1}{288}}(x)} + O(x^3)\right).\end{aligned}$$

We rewrite the last term in (23) as $L^2(-d(d+x)+(d+2)\,\frac{z}{L^2})$. Since $z = w(w+d-1)$ and L is the integer part of w, we write z using the fluctuation function, which is then given by

$$\psi(w) = w - L - \frac{1}{2},$$

as $z = (L + \psi(w) + \frac{1}{2})(L + d + \psi(w) - \frac{1}{2})$, and therefore

$$\frac{z}{L^2} = \left(1 + \left(\psi(w) + \frac{1}{2}\right)x\right)\left(1 + \left(d + \psi(w) - \frac{1}{2}\right)x\right).$$

According to (53) and (54) of Lemma A.0.2 we have

$$P_{\frac{1}{12},\frac{1}{288}}\left(\frac{x}{1+dx}\right) = P_{\frac{1}{12},\frac{1}{288}-\frac{d}{12}}(x) + O(x^3)$$

and

$$\frac{1}{P_{\frac{1}{12},\frac{1}{288}}(x)} = P_{-\frac{1}{12},\frac{1}{288}}(x) + O(x^3).$$

We compute the coefficients A, B, C of the product in (23) according to (55) of Lemma A.0.2 as follows:

$$\begin{aligned}
A &= \frac{d}{2} - \frac{d^2}{2} + \frac{(2d-1)d}{2} + \frac{1}{12} - \frac{1}{12} = \frac{d^2}{2},\\
B &= \frac{8d^3 + 3d^4}{24} + \frac{d^2(2d-1)(2d-3)}{8} + \frac{1}{288} - \frac{d}{12} + \frac{1}{288}\\
&= \frac{d(5d-2)(3d^2-2d+1)}{24} + \frac{1}{144},
\end{aligned}$$

and

$$C = \frac{d^4}{8} - \frac{d^2}{8} - \frac{d^4}{8} - \frac{(2d-1)^2 d^2}{8} - \frac{1}{144} = -\frac{d^2(2d^2-2d+1)}{4} - \frac{1}{144}.$$

Hence the coefficient of x^2 is given by

$$B + C = \frac{d(d-1)(3d^2-d+2)}{24}.$$

Therefore we have

$$\begin{aligned}
&\frac{R_1(z)}{L_{1,d}^{class}|\mathbb{S}^d|z^{1+d/2}} = \frac{1}{2}\left(P_{A,B+C}(x) + O(x^3)\right)\\
&\cdot\left((d+2)\left(1 + \left(\psi(w) + \frac{1}{2}\right)x\right)\left(1 + \left(d + \psi(w) - \frac{1}{2}\right)x\right) - d(1+dx)\right)\cdot\left(\frac{L^2}{z}\right)^{1+d/2}
\end{aligned}$$

$$= \left(P_{A,B+C}(x) + O(x^3)\right)$$
$$\cdot \left(1 + ((d+2)\psi(w) + d)x + \left(\psi(w) + \frac{1}{2}\right)\left(1 + \frac{d}{2}\right)\left(\psi(w) - \frac{1}{2} + d\right)x^2\right) \cdot \left(\frac{L^2}{z}\right)^{1+d/2}.$$

Next we expand

$$\left(\frac{L^2}{z}\right)^{1+d/2} = \left(1 + \left(\psi(w) + \frac{1}{2}\right)x\right)^{-1-d/2}\left(1 + \left(d + \psi(w) - \frac{1}{2}\right)x\right)^{-1-d/2}.$$

Combining all terms as above we finally get

$$\frac{R_1(z)}{L_{1,d}^{class}|\mathbb{S}^d|z^{1+d/2}} = 1 + \frac{d(d+2)}{12}\left(d - 2 + 6\left(\frac{1}{4} - \psi^2(w)\right)\right)x^2 + O(x^3).$$

Since $x^2 = z^{-1} + O(z^{-3/2})$, the theorem is proven. □

In view of (18), we now derive a Weyl sharp lower bound for $R_1(z)$. To do so, we first prove the following

Lemma 4.1.1 *The ratio*

$$\frac{R_1(z)}{L_{1,d}^{class}|\mathbb{S}^d|z^{1+\frac{d}{2}}}$$

has a unique critical point which is a strict maximum in each interval $[\lambda_{(l)}, \lambda_{(l+1)}]$.

Proof As before we write $z = w(w+d-1)$ and put $L := \lfloor w \rfloor$. Since

$$R_1(w(w+d-1)) = \sum_{l=0}^{L} m_{l,d}\big(w(w+d-1) - l(l+d-1)\big)$$
$$= \frac{(2L+d)\Gamma(L+d)}{(d+2)\Gamma(L+1)\Gamma(d+1)}(-dL^2 - d^2L + (d+2)w(w+d-1))$$

we get

$$\frac{R_1(w(w+d-1))}{L_{1,d}^{class}|\mathbb{S}^d|w^{1+d/2}(w+d-1)^{1+d/2}}$$
$$= \frac{(2L+d)\Gamma(L+d)}{4\Gamma(L+1)w^{1+d/2}(w+d-1)^{1+d/2}}(-dL^2 - d^2L + (d+2)w(w+d-1)).$$

The aim is then to show that in each interval $[L, L+1]$, $L \geq 1$, the ratio of R_1 and the leading term in Weyl's law has a unique maximum w. When $L = 0$ the ratio is a strictly decreasing function and singular at $w = 0$. For this we fix L and put $w = w(x) = L+x$

with $x \in [0, 1[$ the fractional part of w. Note that $\lambda_{(L)} = w(0)(w(0) + d - 1)$, $\lambda_{(L+1)} = w(1)(w(1) + d - 1)$. Therefore

$$\begin{aligned}
&\frac{R_1(w(w+d-1))}{L_{1,d}^{class}|\mathbb{S}^d|w^{1+d/2}(w+d-1)^{1+d/2}} \\
&= \frac{(L+d/2)\Gamma(L+d)}{\Gamma(L+1)(L+x)^{1+d/2}(L+x+d-1)^{1+d/2}}\,A(x),
\end{aligned} \tag{24}$$

with

$$A(x) = \left(\frac{d}{2}+1\right)x^2 + (d+2)\left(\frac{d-1}{2}+L\right)x + \left(L-1+\frac{d}{2}\right)L. \tag{25}$$

We consider the logarithm of the quantities in equation (24) that is

$$\begin{aligned}
&Q(x) := \log\left(\frac{(L+d/2)\Gamma(L+d)}{\Gamma(L+1)}\right) \\
&-\left(\frac{d}{2}+1\right)\log\big((L+x)(L+x+d-1)\big) + \log A(x).
\end{aligned}$$

An easy computation shows that

$$Q'(x) = \frac{A'(x)}{A(x)} - \left(\frac{d}{2}+1\right)\frac{2L+d-1+2x}{(L+x)(L+d-1+x)}$$

and

$$\begin{aligned}
Q''(x) &= \frac{A''(x)}{A(x)} - \frac{A'(x)^2}{A(x)^2} - \frac{d+2}{(L+x)(L+d-1+x)} \\
&+\left(\frac{d}{2}+1\right)\frac{(2L+d-1+2x)^2}{(L+x)^2(L+d-1+x)^2}.
\end{aligned}$$

We compute the right derivatives of A and Q at $x = 0$ and the left derivatives at $x = 1$. By (25) we have $A(0) = (L - 1 + \frac{d}{2})L > 0$, $A'(0) = (\frac{d}{2} + 1)(2L + d - 1) > 0$ and therefore $Q'(0) = \dfrac{d(d+2)(2L+d-1)}{2L(2L+d-2)(L+d-1)} > 0$. Similarly, $A(1) = (L + 1 + \frac{d}{2})(L + d) > 0$, $A'(1) = (\frac{d}{2} + 1)(2L + d + 1) > 0$ and therefore $Q'(1) = -\dfrac{d(d+2)(2L+d+1)}{2(L+1)(2L+d+2)(L+d+2)} < 0$. Therefore $Q(x)$ has (at least) one critical point in $]L, L+1[$. We show that it is unique. Suppose $Q'(x_0) = 0$. Since $A'(x) = (\frac{d}{2} + 1)(2L + d - 1 + 2x) > 0$ the condition $Q'(x_0) = 0$ is also equivalent to

$$A(x_0) = (L + x_0)(L + d - 1 + x_0).$$

Then, since $A''(x) = d + 2$ and $Q'(x_0) = 0$ we get

$$Q''(x_0) = \frac{d+2}{A(x_0)} - \frac{d}{d+2}\frac{A'(x_0)^2}{A(x_0)^2} - \frac{d+2}{A(x_0)} = -\frac{d}{d+2}\frac{A'(x_0)^2}{A(x_0)^2} < 0.$$

Hence any critical point is a strict local maximum and therefore $Q(x)$ has exactly one critical point in each interval $]L, L+1[$. This concludes the proof. □

Now we are ready to prove a Weyl-sharp lower bound for $R_1(z)$. This result can be found in Ilyin and Laptev [29], however Lemma 4.1.1 gives a new insight on the typical behavior of Riesz-means R_1 and opens the door to the improvement which we present in Theorem 4.1.3 below, confirming thereby the study of the asymptotics for eigenvalue sums done by Strichartz [40] (see Remark 4.1.2 below).

Theorem 4.1.2 *For all $z \geq 0$ the following lower bound for the first Riesz-mean R_1 on $\mathbb{S}^d$ holds:*

$$R_1(z) \geq L^{class}_{1,d} |\mathbb{S}^d|\, z^{\frac{d}{2}+1}. \tag{26}$$

Proof From Lemma 4.1.1 we deduce that it is sufficient to prove the bound for each $z = \lambda_{(l)}$, $l \in \mathbb{N}$. Since the bound trivially holds for $\lambda_{(0)} = 0$ we consider $R_1(\lambda_{(l+1)})$. According to (23) we have

$$\frac{R_1(\lambda_{(l+1)})}{L^{class}_{1,d}|\mathbb{S}^d|\lambda_{(l+1)}^{1+d/2}} = \frac{(l+d/2)(l+1+d/2)\Gamma(l+d+1)}{\Gamma(l+1)(l+1)^{1+d/2}(l+d)^{1+d/2}}. \tag{27}$$

We rewrite $(l+d/2)(l+1+d/2) = (l+1)(l+d) + \frac{d(d-2)}{4}$. Since $\frac{d(d-2)}{4} \geq 0$, we therefore have the lower bound

$$\frac{R_1(\lambda_{(l+1)})}{L^{class}_{1,d}|\mathbb{S}^d|\lambda_{(l+1)}^{1+d/2}} \geq \frac{\Gamma(l+d+1)}{\Gamma(l+1)(l+1)^{d/2}(l+d)^{d/2}}.$$

Since

$$\frac{\Gamma(l+d+1)}{\Gamma(l+1)} = \prod_{j=1}^{d}(l+j) = \left(\prod_{j=1}^{d}(l+j)(l+d+1-j)\right)^{1/2}$$
$$= \left(\prod_{j=1}^{d}((l+1)(l+d) + (j-1)(d-j))\right)^{1/2},$$

we finally obtain

$$\frac{R_1(\lambda_{(l+1)})}{L^{class}_{1,d}|\mathbb{S}^d|\lambda_{(l+1)}^{1+d/2}} \geq \left(\prod_{j=1}^{d}\left(1 + \frac{(j-1)(d-j)}{(l+1)(l+d)}\right)\right)^{1/2} \geq 1.$$

□

We note that, taking into account the term $d(d-2)/4$ in the proof of the above theorem (which we have dropped at the beginning of the estimate) we get the following estimates for $R_1(z)$ when $z = \lambda_{(l+1)}$, improving the result of [29]:

Corollary 4.1.1 *For all $l \geq 0$ and $d \geq 2$:*

$$R_1(\lambda_{(l+1)}) \geq L_{1,d}^{class} |\mathbb{S}^d| \, \lambda_{(l+1)}^{\frac{d}{2}+1} \left(1 + \frac{d(d-2)(d+2)}{12\lambda_{(l+1)}}\right).$$

Proof Since for $d = 2$ the inequality has already been shown, we assume $d \geq 3$. We start from (27)

and we rewrite $(l + d/2)(l + 1 + d/2) = (l+1)(l+d) + \frac{d(d-2)}{4}$. Then

$$\frac{R_1(\lambda_{(l+1)})}{L_{1,d}^{class} |\mathbb{S}^d| \lambda_{(l+1)}^{1+d/2}} = \frac{\Gamma(l+d+1)}{\Gamma(l+1)(l+1)^{d/2}(l+d)^{d/2}} \left(1 + \frac{d(d-2)}{4\lambda_{(l+1)}}\right).$$

Writing as before

$$\frac{\Gamma(l+d+1)}{\Gamma(l+1)} = \left(\prod_{j=1}^{d} ((l+1)(l+d) + (j-1)(d-j))\right)^{1/2},$$

we finally obtain

$$\frac{R_1(\lambda_{(l+1)})}{L_{1,d}^{class} |\mathbb{S}^d| \lambda_{(l+1)}^{1+d/2}} = \left(\prod_{j=1}^{d} \left(1 + \frac{(j-1)(d-j)}{\lambda_{(l+1)}}\right)\right)^{1/2} \left(1 + \frac{d(d-2)}{4\lambda_{(l+1)}}\right).$$

We consider the function $f(x)$ defined for $x \geq 0$ by

$$f(x) = \left(1 + \frac{d(d-2)}{4} x\right)^2 \left(\prod_{j=1}^{d} (1 + (j-1)(d-j)x)\right) - \left(1 + \frac{d(d-2)(d+2)}{12} x\right)^2. \tag{28}$$

We have $f(0) = 0$. We will show $f'(0) = 0$, $f''(0) > 0$. Since obviously $f'''(x) \geq 0$ this implies $f(x) \geq 0$ for $x \geq 0$ and hence the claim. We note $f(x) = (1+Bx)^2 P(x) - (1 + Ax)^2$ where $P(x) = \prod_{j=1}^{d} (1 + a_j x)$ denotes the polynomial given by the product. The coefficients a_j, A, B are easily identified by (28). We have

$$P'(x) = P(x) \sum_{j=1}^{d} \frac{a_j}{1 + a_j x}, \quad P''(x) = P(x) \left(\sum_{j=1}^{d} \frac{a_j}{1 + a_j x}\right)^2 - P(x) \sum_{j=1}^{d} \frac{a_j^2}{(1 + a_j x)^2}.$$

Hence

$$
\begin{aligned}
f'(x) &= 2B(1+Bx)P(x) + (1+Bx)^2 P'(x) - 2A(1+Ax),\\
f''(x) &= 2B^2 P(x) + 4B(1+Bx)P'(x) + (1+Bx)^2 P''(x) - 2A^2.
\end{aligned}
$$

First of all, we see that

$$f'(0) = 2B + \sum_{j=1}^{d} a_j - 2A = 0.$$

The coefficient $A = \frac{d(d-2)(d+2)}{12}$ is indeed determined by this condition. Finally,

$$
\begin{aligned}
f''(0) &= 2B^2 + 4B\sum_{j=1}^{d} a_j + \left(\sum_{j=1}^{d} a_j\right)^2 - \sum_{j=1}^{d} a_j^2 - 2A^2\\
&= 2B^2 + 8B(A-B) + 4(A-B)^2 - 2A^2 - \sum_{j=1}^{d} a_j^2 = 2A^2 - 2B^2 - \sum_{j=1}^{d} a_j^2.
\end{aligned}
$$

Together with

$$\sum_{j=1}^{d} a_j^2 = \frac{d(d-1)(d-2)(d^2-2d+2)}{30}$$

we get

$$f''(0) = \frac{d(d-1)(d-2)(d+1)(d+2)(5d-12)}{360}$$

which is non-negative for positive integers d proving the assertion. □

From Corollary 4.1.1 and a careful inspection of the proof of Lemma 4.1.1 and Theorem 4.1.2 we deduce the following improvement of (26), which is optimal in a suitable sense, as we will explain in Remark 4.1.3 below.

Theorem 4.1.3 *For all $z \geq 0$ the following lower bound for the first Riesz-mean R_1 on $\mathbb{S}^d$ holds:*

$$R_1(z) \geq L_{1,d}^{class} |\mathbb{S}^d|\, z^{\frac{d}{2}+1} \left(1 + \frac{d(d-2)(d+2)}{12z}\right). \tag{29}$$

Proof The proof follows the same lines as the proof of Lemma 4.1.1, showing that the ratio of the right-hand side and left-hand side of (29) as a function of z has exactly one local maximum in each interval $[L(L+d-1), (L+1)(L+d)]$. Then we conclude by Corollary 4.1.1.

□

We turn our attention to upper bounds for $R_1(z)$. The upper bound contains a shift term, which is again optimal in a suitable sense (see Remark 4.1.1 below).

Theorem 4.1.4 *For all $z \geq 0$ the following upper bound for the first Riesz-mean R_1 on $\mathbb{S}^d$ holds:*

$$R_1(z) \leq L_{1,d}^{class} |\mathbb{S}^d| \, (z + z_d)^{\frac{d}{2}+1} \tag{30}$$

with

$$z_d = \frac{(2d-1)d}{12}. \tag{31}$$

Proof Again, let us set $z = w(w + d - 1)$ and $L = \lfloor w \rfloor$. Let $b \geq 0$. We analyze the quantity

$$\begin{aligned} &\frac{R_1(w(w+d-1))}{L_{1,d}^{class}|\mathbb{S}^d| \, (w(w+d-1)+b)^{1+d/2}} \\ &= \frac{(2L+d)\Gamma(L+d)}{4\Gamma(L+1)\,(w(w+d-1)+b)^{1+d/2}} \left(-dL(L+d) + (d+2)w(w+d-1)\right). \end{aligned} \tag{32}$$

We show that in each interval $[L, L+1]$, $L \geq 1$, the ratio in (32) has a unique maximum. When $L = 0$ the ratio is a strictly decreasing function and singular at $w = 0$ if $b = 0$. For this we fix L and put $w = L + x$ with $x \in [0, 1[$ the fractional part of w. Note that $\lambda_{(L)} = w(0)(w(0) + d - 1)$, $\lambda_{(L+1)} = w(1)(w(1) + d - 1)$. Therefore

$$\begin{aligned} &\frac{R_1(w(w+d-1))}{L_{1,d}^{class}|\mathbb{S}^d|\big(w(w+d-1)+b\big)^{1+d/2}} \\ &= \frac{(L+d/2)\Gamma(L+d)}{\Gamma(L+1)\big((L+x)(L+x+d-1)+b\big)^{1+d/2}} A(x) \end{aligned} \tag{33}$$

with

$$A(x) = \left(\frac{d}{2}+1\right)(x+L)(x+L+d-1) - \frac{Ld(L+d)}{2}.$$

We also define

$$\rho(x) := (L+x)(L+x+d-1) + b.$$

Then $A(x) = \left(\frac{d}{2}+1\right)\rho(x) - \frac{Ld(L+d)}{2} - \left(\frac{d+2}{2}\right)b$, and (33) reads as follows

$$\frac{R_1(w(w+d-1))}{L_{1,d}^{class}|\mathbb{S}^d|\big(w(w+d-1)+b\big)^{1+d/2}}$$

$$= \frac{(L+d/2)\Gamma(L+d)}{\Gamma(L+1)} \left(\frac{d+2}{2} \rho(x)^{-d/2} - \left(\frac{Ld(L+d)}{2} + \frac{d+2}{2} b \right) \rho^{-1-d/2} \right). \tag{34}$$

The right-hand side of (34) has a unique maximum at $\rho_b = L(L+d) + \frac{d+2}{d} b$. It is easy to check that $\lambda_{(L)} + b \leq \rho_b \leq \lambda_{(L+1)} + b$ when $b \leq d^2/2$. Therefore we get the inequality

$$\begin{aligned} &\frac{R_1(w(w+d-1))}{L_{1,d}^{class}|\mathbb{S}^d|\big(w(w+d-1)+b\big)^{1+d/2}} \\ &\leq \frac{(L+d/2)\Gamma(L+d)}{\Gamma(L+1)} \left(L(L+d) + \frac{d+2}{d} b \right)^{-d/2}, \end{aligned} \tag{35}$$

which holds for all $w \in [L, L+1]$. Now, we note that

$$\begin{aligned} \frac{\Gamma(L+d)}{\Gamma(L+1)} &= \prod_{j=1}^{d-1} (L+j) = \left(\prod_{j=1}^{d-1} (L+j)(L+d-j) \right)^{1/2} \\ &= \left(\prod_{j=1}^{d-1} \Big((L+d/2)^2 - (j-d/2)^2 \Big) \right)^{1/2}. \end{aligned}$$

Therefore we may rewrite (35) as follows:

$$\begin{aligned} &\frac{R_1(w(w+d-1))}{L_{1,d}^{class}|\mathbb{S}^d|\,(w(w+d-1)+b)^{1+d/2}} \\ &\leq \left(\prod_{j=1}^{d-1} 1 - \frac{(j-d/2)^2}{(L+d/2)^2} \right)^{1/2} \left(1 + \frac{\frac{d+2}{d} b - \frac{d^2}{4}}{(L+d/2)^2} \right)^{-d/2}. \end{aligned} \tag{36}$$

We see that the right-hand side of (36) is bounded above by 1 if $b \geq \frac{d^3}{4(d+2)}$. However, here we want to show a that a choice $b \leq \frac{d^3}{4(d+2)}$ also yields the upper bound 1 in (36). For this we apply the arithmetic–geometric mean inequality to the product:

$$\begin{aligned} &\left(\prod_{j=1}^{d-1} 1 - \frac{(j-d/2)^2}{(L+d/2)^2} \right)^{1/2} \\ &\leq \left(1 - \frac{1}{d-1} \sum_{j=1}^{d-1} \frac{(j-d/2)^2}{(L+d/2)^2} \right)^{(d-1)/2} \\ &= \left(1 - \frac{d(d-2)}{12(L+d/2)^2} \right)^{(d-1)/2}. \end{aligned}$$

It is now sufficient to show that the function $f(t)$ defined by

$$f(t)=\frac{d-1}{2}\log\left(1-\frac{d(d-2)}{12}t\right)-\frac{d}{2}\log\left(1+\left(\frac{d+2}{d}b-\frac{d^2}{4}\right)t\right)$$

is decreasing for $t>0$ for b suitably chosen (we will use this fact with $t=(L+d/2)^{-2}$). In particular, we want to show that this is the case for $b=z_d=\frac{(2d-1)d}{12}$ which will be the optimal choice. We easily compute

$$f'(t)=-\frac{\frac{d}{2}\left(\frac{d+2}{d}b-\frac{(d+2)(2d-1)}{12}-\frac{d-2}{12}\left(\frac{d+2}{d}b-\frac{d^2}{4}\right)t\right)}{(1-\frac{d(d-2)}{12}t)(1+(\frac{d+2}{d}b-\frac{d^2}{4})t)}.$$

The best choice is obviously $b=z_d=\frac{(2d-1)d}{12}$ eliminating the constant term. With this choice

$$f'(t)=-\frac{d(d-1)(d-2)^2t}{24(1-\frac{d(d-2)}{12}t)(1+(\frac{d+2}{d}b-\frac{d^2}{4})t)}\leq 0.$$

The proof is now completed. $\square$

Remark 4.1.1 We remark that the shift z_d in the upper bound (30) is, in a sense, optimal. We observe that for $d=2$ the upper bound coincides with the one found in Proposition 3.1.1 for $\mathbb{S}^2$, which we have already shown to be sharp. For $d\geq 3$, in general we cannot find $z\in[\lambda_{(l)},\lambda_{(l+1)}]$ such that the equality is attained in (30). When $z\in[\lambda_{(l)},\lambda_{(l+1)}]$ one uses the explicit form of $R_1(z)$ (as in (19)) and considers the function $f(z)=R_1(z)-L_{1,d}^{class}|\mathbb{S}^d|(z+b)^{\frac{d}{2}+1}$. Computing $f'(z)$, finding z_0 such that $f'(z_0)=0$, and substituting z_0 in $f(z)$, we find the minimum distance from $R_1(z)$ to $L_{1,d}^{class}|\mathbb{S}^d|(z+b)^{\frac{d}{2}+1}$, namely, $|f(z_0)|$. If we want this distance to be zero, then we must chose $b=b(l)$. If $d=2$, then $b=b(l)=1/2$ for all l and this corresponds to the optimal upper bound of Proposition 3.1.1 (see also [29, 40]). If $d\geq 3$, one has that in each interval $[\lambda_{(l)},\lambda_{(l+1)}]$ the optimal shift would be given by

$$b(l)=\frac{d}{d+2}(4^{-1/d}((d+2l)(d+l-1)!/l!)^{2/d}-l(l+d)).$$

We highlight that $b(l)\to z_d$ as $l\to+\infty$, so in this sense the shift z_d becomes sharp as $z\to\infty$.

Remark 4.1.2 In [40] the author estimates the liminf and limsup of the remainder of Weyl's law for R_1 on $\mathbb{S}^d$ (Theorem 3.3). These expressions agree with the two-term Weyl's law we have proved in Theorem 4.1.1 and with the corresponding upper and lower bounds. In fact, the bounds of Theorems 4.1.3 and 4.1.4 are optimal since they provide the precise envelopes for the second term of the asymptotic expansion (18). In particular, the upper bound is obtained when $\psi(w)=0$ (meaning $w=\lfloor w\rfloor$), and the lower bound when $\psi^2(w)=\frac{1}{4}$ (meaning $w=\lfloor w\rfloor\pm\frac{1}{2}$). Here w is defined by $w(w+d-1)=z$.

Remark 4.1.3 A consequence of the upper bound (30) is that the average of $\lambda_j + z_d$ satisfies a Berezin–Li–Yau lower bound. One may wonder whether the lower bound (26) holds with a shift, namely, with z replaced by $z + b_d$ where $b_d = \frac{d(d-2)}{6}$ which is the optimal choice (this is the liminf of Strichartz, like z_d is for the upper bound). Clearly this is true for $d = 2$ but it is already false for $d = 3$. It is enough to observe that the corresponding inequality $R_1(z) \geq L^{class}_{1,d} |\mathbb{S}^d|(z + b_d)^{\frac{d}{2}+1}$ fails for $z \leq d$.

For the reader's convenience, we restate the results of Theorems 4.1.2 and 4.1.4 in terms of inequalities for eigenvalues averages, i.e., in the form of (3). For the equivalence between inequalities on Riesz means and averages we refer e.g., to [23].

Corollary 4.1.2 *For all $k \geq 1$ the following inequalities hold:*

$$\frac{d}{d+2}\frac{4\pi^2}{\omega_d^{2/d}}\left(\frac{k}{|\mathbb{S}^d|}\right)^{2/d} - \frac{(2d-1)d}{12} \leq \frac{1}{k}\sum_{j=1}^{k}\lambda_j \leq \frac{d}{d+2}\frac{4\pi^2}{\omega_d^{2/d}}\left(\frac{k}{|\mathbb{S}^d|}\right)^{2/d} \tag{37}$$

where λ_j are the eigenvalues of the Laplacian on $\mathbb{S}^d$.

Remark 4.1.4 We note that for $d = 2$ the first inequality of (37) reads

$$\frac{1}{k}\sum_{j=1}^{k}\lambda_j + \frac{1}{2} \geq \frac{2\pi k}{|\mathbb{S}^2|}$$

which is a Berezin–Li–Yau inequality in the form (3) for the shifted eigenvalues, namely, for $\lambda_j + \frac{1}{2}$. One can rearrange this inequality and re-write it as

$$\frac{1}{k+1}\sum_{j=1}^{k+1}\lambda_j \geq \frac{2\pi k}{|\mathbb{S}^2|}$$

which is a Berezin–Li–Yau inequality with a shift in the index. One may ask whether also for $d > 2$ there is a Berezin–Li–Yau lower bound for averages with a shift in the index. From Theorem 4.1.1 however we see that the second term of $R_1(z)$ is of order $z^{d/2}$, which, after Legendre transforming yields the expression already found by Strichartz in [40, Theorem 3.3]:

$$\frac{1}{k}\sum_{j=1}^{k}\lambda_j = \frac{d}{d+2}C_d\left(\frac{k}{|\mathbb{S}^d|}\right)^{\frac{2}{d}} - R(k) \tag{38}$$

with $\limsup R(k) = \frac{d(2d-1)}{12}$. Assume that for some $N \in \mathbb{N}$ the following inequality holds for $k \geq N$:

$$\frac{1}{k}\sum_{j=1}^{k}\lambda_j \geq \frac{d}{d+2}C_d\left(\frac{k-N}{|\mathbb{S}^d|}\right)^{\frac{2}{d}}. \tag{39}$$

We get, combining (38) and (39),

$$\frac{d}{d+2}C_d\left(\frac{k}{|\mathbb{S}^d|}\right)^{\frac{2}{d}}\left(1-\left(1-\frac{N}{k}\right)^{\frac{2}{d}}\right)-R(k)\geq 0. \tag{40}$$

Taking a subsequence $k_n \to +\infty$ such that $R(k_n) \to \frac{d(2d-1)}{12}$ in (40) we get a contradiction when $d \geq 3$, while the inequality is possible for $d = 2$ when $N \geq 1$. This remark motivates the fact that in the non flat case, we should look at Berezin–Li–Yau inequalities for the shifted eigenvalues, where the shift is related to the curvature of the space.

We conclude this section with a three-term asymptotic expansion for the counting function $N(z)$.

Theorem 4.1.5 *As $z \to \infty$ we have the following asymptotic expansion for the counting function N on $\mathbb{S}^d$:*

$$\frac{N(z)}{L^{class}_{0,d}|\mathbb{S}^d|z^{\frac{d}{2}}}=1-\frac{L^{class}_{0,d-1}}{L^{class}_{0,d}}\frac{|\partial\mathbb{S}^d_+|}{|\mathbb{S}^d|}\psi(w)z^{-\frac{1}{2}}+\frac{d(d-1)(12\psi^2(w)+2d-1)}{24}z^{-1}$$
$$+O(z^{-\frac{3}{2}}). \tag{41}$$

Here w is defined by $w(w+d-1)=z$ and $|\partial\mathbb{S}^d_+|$ denotes the measure of the boundary of the hemisphere.

Proof The proof follows from the identity $N(z) = N^D(z) + N^N(z)$, where $N^D(z)$ and $N^N(z)$ are the counting functions for the Dirichlet and Neumann Laplacian on the hemisphere $\mathbb{S}^d_+$. We prove the corresponding three-term expansions in Theorem 4.2.1 and 4.2.3 in the next section. □

It is interesting to see that the second term is oscillatory, but it is not bounded: along suitable subsequences it behaves like $\pm z^{\frac{d}{2}-\frac{1}{2}}$. This is natural as this is the correct order of the remainder after the first term, see [7, 39]. This also provides an interpretation of the results of [17, Theorem F] for the eigenvalues on the whole sphere. Note that in [39], the authors present a quasi-Weyl formula in the case of manifolds or domains not satisfying the geometric conditions ensuring the existence of a second term of the form $c_1 z^{\frac{d-1}{2}}$. They present the explicit example of $-\Delta+\frac{(d-1)^2}{4}$ ([39, Examples 1.2.5, 1.7.1 and 1.7.11]). The eigenvalues are given as energy levels $\left(l+\frac{d-1}{2}\right)^2$ (they are $\lambda_{(l)}+\frac{(d-1)^2}{4}$, with multiplicities $m_{l,d}$). For such eigenvalues, a two-term quasi-Weyl formula in the sense of [39, Formula (1.7.5)] does hold, and the function Q which describes the behavior of the second term in [39, Formulas (1.7.4)-(1.7.5)] agrees with the second term of (41).

4.1.1 Domains in $\mathbb{S}^d$

Here we derive upper bounds in the spirit of Berezin–Li–Yau [2, 33] for the first Riesz-mean of the eigenvalues of the Dirichlet Laplacian on domains Ω of $\mathbb{S}^d$. We denote by

$$0 < \lambda_1(\Omega) < \lambda_2(\Omega) \leq \ldots \leq \lambda_j(\Omega) \leq \ldots \nearrow +\infty$$

the eigenvalues of the Dirichlet Laplacian on Ω, each repeated in accordance with its multiplicity, and by $\{u_j\}_{j\geq 1}$ the corresponding $L^2(\Omega)$-orthonormal sequence of eigenfunctions. In [40] the author establishes Berezin–Li–Yau-type inequalities for domains of $\mathbb{S}^2$ (see also [29]), and provides an expansion for R_1^D in the higher dimensional case, highlighting the sharp behavior of the remainder. Here we establish a Berezin–Li–Yau inequality with a shift term in any dimension, which coincides with that proved in [29, 40] when $d = 2$, and which contains a shift term which is asymptotically sharp when $\Omega = \mathbb{S}^d$, see Remark 4.1.1. In particular, the "generalized conjecture of Pólya" stated in [8, Formula (1.11)] holds for the sphere in a stronger form: the correct shift constant is z_d and not $d^2/4$ as conjectured in [8] (this was clear for $d = 2$ by [40]).

The proof is based on the *averaged variational principle* and is in the spirit of that of Theorem 3.2.4.

Theorem 4.1.6 *Let Ω be a domain in $\mathbb{S}^d$. Then for all $z \geq 0$ the following inequality for the first Riesz-mean R_1^D of the eigenvalues of the Dirichlet Laplacian on Ω holds:*

$$R_1^D(z) = \sum_{j\geq 1}(z - \lambda_j(\Omega))_+ \leq L_{1,d}^{class}|\Omega|(z + z_d)^{\frac{d}{2}+1}$$

with $z_d = \frac{(2d-1)d}{12}$. Equivalently, the following inequality holds for all $k \geq 1$:

$$\frac{1}{k}\sum_{j=1}^{k}\lambda_j(\Omega) \geq \frac{d}{d+2}\frac{4\pi^2}{\omega_d^{2/d}}\left(\frac{k}{|\Omega|}\right)^{2/d} - z_d.$$

Proof The proof follows the same lines as that of Theorem 3.2.4. The eigenfunctions of the Laplacian on $\mathbb{S}^d$ associated with the energy level $\lambda_{(l)} = l(l + d - 1)$ are the spherical harmonics Y_l^m, where $m = 1, \ldots, m_{l,d}$. Let $z \geq 0$. We apply Theorem 2.0.1 with $\mathcal{H} = L^2(\mathbb{S}^d)$, $H = -\Delta$, $\mathcal{Q} = H^1(\mathbb{S}^d)$, $Q(u,u) = \int_{\mathbb{S}^d}|\nabla u|^2$, $\mathfrak{M} = \mathbb{N}\setminus\{0\}$, $\mathfrak{M}_0 = \{j \in \mathbb{N}\setminus\{0\} : z - \lambda_j(\Omega) \geq 0\}$, $f_p = u_j$. Following the proof of Theorem 3.2.4 we obtain

$$\frac{|\Omega|}{|\mathbb{S}^d|}R_1(z) \geq R_1^D(z),$$

where $R_1(z) = \sum_l m_{l,d}(z - l(l + d - 1))_+$ is the first Riesz mean for the whole $\mathbb{S}^d$. The upper bound for R_1^D follows then from Theorem 4.1.4. By Legendre transforming the inequality for R_1^D we get the inequality on the average (see Corollary 4.1.2, see also [23]). □

Remark 4.1.5 Concerning the Neumann eigenvalues, it has been shown by Ilyin and Laptev [29] that any domain of $\mathbb{S}^d$ satisfies a Kröger-type bound, namely, the leading term in Weyl's law is a lower bound for $R_1^N = \sum_{j\geq 1}(z-\mu_j(\Omega))_+$. This is a consequence of the fact that

$$\sum_{j\geq 1}(z-\mu_j(\Omega))_+ \geq \frac{|\Omega|}{|\mathbb{S}^d|}\sum_{l\geq 0} m_{l,d}(z-l(l+d-1))_+$$

which is proved in [29] (or can be easily deduced as an application of the averaged variational principle as for (17)), and from the inequality (26). However, from our improved inequality (29), we can improve the result for domains in $\mathbb{S}^d$. Namely, for any domain Ω in $\mathbb{S}^d$ we have

$$R_1^N(z) = \sum_{j\geq 1}(z-\mu_j(\Omega))_+ \geq L_{1,d}^{class}|\Omega| z^{\frac{d}{2}+1}\left(1+\frac{d(d-2)(d+2)}{12z}\right).$$

4.2 The Hemisphere $\mathbb{S}^d_+$

In this subsection we shall consider the eigenvalues of the Dirichlet and Neumann Laplacian on the hemisphere $\mathbb{S}^d_+$. In particular, we will compute three-term expansions for N^D, R_1^D, N^N, R_1^N.

4.2.1 The Dirichlet Laplacian

The eigenvalues are of the form $\lambda_{(l)} = l(l+d-1)$, $l \in \mathbb{N}\setminus\{0\}$, with multiplicities $m_{l,d}^D$ given by

$$m_{l,d}^D = \binom{d+l-2}{d-1}. \tag{42}$$

The counting function $N^D(z)$ is easily computed. Again let w be defined by the relation $z = w(w+d-1)$ and $L = \lfloor w \rfloor$ be the integer part of w. Then

$$N^D(z) = \sum_{l=1}^{L} m_{l,d}^D = \frac{\Gamma(d+L)}{\Gamma(L)\Gamma(d+1)}. \tag{43}$$

Since the hemisphere does not satisfy the billiard condition, the counting function N^D does not admit an expansion with just a power-like surface term of order $z^{\frac{d-1}{2}}$ after the leading term in Weyl's law as in (6). This is explained in [39], and a major consequence is the failure of Pólya's conjecture, as pointed out in [17].

We prove here a three-term asymptotic expansion for N^D and we show that the second term contains oscillations, but, at any rate, it has a sign. In fact, it is non-positive. Moreover, the third term is oscillating but again, it has a sign and it is non-negative.

Moreover, it is strictly positive along the sequences where the second term vanishes. This explains the failure of Pólya's conjecture along certain sequences of eigenvalues, as pointed out in [17, Theorem A]. The second and third terms should be instead compared with the sharp corrections to the Pólya's inequality proved in [17, Theorems B, C, D].

Theorem 4.2.1 *As $z \to \infty$ we have the following asymptotic expansion for the counting function N^D of the Dirichlet Laplacian eigenvalues on $\mathbb{S}^d_+$:*

$$\begin{aligned}\frac{N^D(z)}{L^{class}_{0,d}|\mathbb{S}^d_+|\,z^{\frac{d}{2}}} &= 1 - \frac{1}{4}\frac{L^{class}_{0,d-1}}{L^{class}_{0,d}}\frac{|\partial\mathbb{S}^d_+|}{|\mathbb{S}^d_+|}(1+2\psi(w))\,z^{-1/2} \\ &\quad + \frac{d(d-1)}{2}\left(\left(\frac{1}{2}+\psi(w)\right)^2 + \frac{d-2}{6}\right)z^{-1} + O(z^{-3/2})\end{aligned} \tag{44}$$

or equivalently

$$\begin{aligned}\frac{N^D(z)}{L^{class}_{0,d}|\mathbb{S}^d_+|\,z^{\frac{d}{2}}} &= 1 - \frac{d(1+2\psi(w))}{2}\,z^{-1/2} \\ &\quad + \frac{d(d-1)}{2}\left(\left(\frac{1}{2}+\psi(w)\right)^2 + \frac{d-2}{6}\right)z^{-1} + O(z^{-3/2}).\end{aligned}$$

Here w is defined by the relation $w(w+d-1) = z$.

Proof As in the proof of Theorem 4.1.1 for $\mathbb{S}^d$, we set $L = \lfloor w \rfloor$ and we expand in $x = 1/L$. From (43) we have

$$\frac{N^D(z)}{L^{class}_{0,d}|\mathbb{S}^d_+|\,z^{\frac{d}{2}}} = \frac{\Gamma(L+d)}{\Gamma(L)}\,z^{-\frac{d}{2}}. \tag{45}$$

Moreover,

$$\frac{\Gamma(L+d)}{\Gamma(L)} = L^d\left(P_{-\frac{d^2}{2},\frac{8d^3+3d^4}{24}}(x)P_{\frac{(2d-1)d}{2},\frac{d^2(2d-1)(2d-3)}{8}}(x)\cdot\frac{P_{\frac{1}{12},\frac{1}{288}}(\frac{x}{1+dx})}{P_{\frac{1}{12},\frac{1}{288}}(x)} + O(x^3)\right).$$

According to (53) and (54) of Lemma A.0.1, we have

$$P_{\frac{1}{12},\frac{1}{288}}\left(\frac{x}{1+dx}\right) = P_{\frac{1}{12},\frac{1}{288}-\frac{d}{12}}(x) + O(x^3)$$

and

$$\frac{1}{P_{\frac{1}{12},\frac{1}{288}}(x)} = P_{-\frac{1}{12},\frac{1}{288}}(x) + O(x^3).$$

We compute the coefficients A, B, C of the product in (45) according to (55) of Lemma A.0.2 as follows:

$$\begin{aligned}
A &= -\frac{d^2}{2} + \frac{(2d-1)d}{2} + \frac{1}{12} - \frac{1}{12} = \frac{d(d-1)}{2}, \\
B &= \frac{8d^3 + 3d^4}{24} + \frac{d^2(2d-1)(2d-3)}{8} + \frac{1}{288} - \frac{d}{12} + \frac{1}{288} \\
&= \frac{d(5d-2)(3d^2-2d+1)}{24} + \frac{1}{144}
\end{aligned}$$

and

$$C = \frac{d^2(d-1)^2}{8} - \frac{d^4}{8} - \frac{(2d-1)^2 d^2}{8} - \frac{1}{144} = -\frac{d^3(2d-1)}{4} - \frac{1}{144}.$$

Hence the coefficient of x^2 is given by

$$B + C = \frac{d(d-1)(d-2)(3d-1)}{24}.$$

Therefore

$$\frac{N^D(z)}{L_{0,d}^{class} |\mathbb{S}_+^d| \, z^{\frac{d}{2}}} = \left(P_{A,B+C}(x) + O(x^3)\right) \left(\frac{z}{L^2}\right)^{-d/2}.$$

Since $z = w(w+d-1) = (\psi(w) + \frac{1}{2} + L)(\psi(w) - \frac{1}{2} + d + L)$ we have

$$\begin{aligned}
&\left(\frac{z}{L^2}\right)^{-d/2} = \left(1 + (d + 2\psi(w))x + \left(\psi(w) + \frac{1}{2}\right)\left(d + \frac{1}{2} - \psi(w)\right)x^2\right)^{-d/2} \\
&= 1 - \frac{d}{2}(d + 2\psi(w))x + \frac{d}{8}((d+1)(2\psi(w)+d)^2 + (d-1)^2)x^2 \\
&+ O(x^3) =: 1 + \alpha x + \beta x^2 + O(x^3).
\end{aligned}$$

Now

$$P_{A,B+C}(x) P_{\alpha,\beta}(x) = P_{A',B'+C'}(x)$$

with

$$A' = A + \alpha, \quad B' = B + C + \beta, \quad C' = A\alpha.$$

We compute

$$A' = -\frac{d(1 + 2\psi(w))}{2}, \quad C' = -\frac{d^2(d-1)(d+2\psi(w))}{4}$$

and

$$B' + C' = \frac{d}{12}\left(6(d+1)(\psi(w)+1/2)^2 + (d-1)(d+6\psi(w)+1)\right).$$

Finally, in order to reconvert $x = 1/L$ into the variable z we use

$$L = \sqrt{z + \left(\frac{d-1}{2}\right)^2} - \frac{d-1}{2} - \left(\psi(w) + \frac{1}{2}\right)$$

and therefore

$$\frac{1}{L} = z^{-\frac{1}{2}} + \frac{d + 2\psi(w)}{2} z^{-1} + O(z^{-\frac{1}{2}}).$$

Inserting the first two terms $P_{A',B'+C'}(x)$ we obtain that the coefficient of z^{-1} is given by

$$B' + C' + A'\frac{d+2\psi(w)}{2} = \frac{d(d-1)}{2}\left((\frac{1}{2} + \psi(w))^2 + \frac{d-2}{6}\right),$$

proving the theorem. □

On the other hand one may expect that, similarly to the Euclidean setting, the more regular Riesz-mean $R_1^D(z)$ admits an expansion with a surface term after the leading term in Weyl's law as in (6), that is,

$$R_1^D(z) \sim L_{1,d}^{class}|\mathbb{S}_+^d|\, z^{\frac{d}{2}+1} - \frac{1}{4} L_{1,d-1}^{class}|\partial\mathbb{S}_+^d|\, z^{\frac{d}{2}+\frac{1}{2}}$$

as z goes to infinity. Note that

$$L_{1,d}^{class}|\mathbb{S}_+^d| = \frac{2}{(d+2)\Gamma(d+1)}$$

and

$$L_{1,d-1}^{class}|\partial\mathbb{S}_+^d| = L_{1,d-1}^{class}|\mathbb{S}^{d-1}| = \frac{4}{(d+1)\Gamma(d)}.$$

We prove the following theorem stating that $R_1^D(z)$ has a second term of order $z^{\frac{d}{2}+\frac{1}{2}}$, and a third term, of negative sign, which includes an oscillatory part.

Theorem 4.2.2 *As $z \to \infty$ we have the following asymptotic expansion for the first Riesz-mean R_1^D of the Dirichlet Laplacian eigenvalues on $\mathbb{S}_+^d$:*

$$\frac{R_1^D(z)}{L_{1,d}^{class}|\mathbb{S}_+^d|\, z^{\frac{d}{2}+1}} = 1 - \frac{1}{4}\frac{L_{1,d-1}^{class}}{L_{1,d}^{class}}\frac{|\partial\mathbb{S}_+^d|}{|\mathbb{S}_+^d|}\, z^{-1/2}$$
$$+\frac{d(d+2)}{2}\left(\frac{1}{4} - \psi^2(w) + \frac{d-2}{6}\right) z^{-1} + O(z^{-3/2})$$

or, equivalently,

$$\frac{R_1^D(z)}{L_{1,d}^{class}|\mathbb{S}_+^d|\, z^{\frac{d}{2}+1}} = 1 - \frac{d(d+2)}{2(d+1)}\, z^{-1/2}$$
$$-\frac{d(d+2)}{2}\left(\frac{1}{4} - \psi^2(w) + \frac{d-2}{6}\right) z^{-1} + O(z^{-3/2}).$$

Here w is defined by the relation $w(w+d-1) = z$.

Proof As before, we set $L = \lfloor w \rfloor$ and $x = 1/L$. One easily computes the Riesz-mean as in the case of $\mathbb{S}^d$ (see Theorem 4.1.1)

$$\frac{R_1^D(z)}{L_{1,d}^{class}|\mathbb{S}_+^d|\, z^{\frac{d}{2}+1}} = \frac{d+2}{2}\,\frac{\Gamma(L+d)}{\Gamma(L)}\left(z - \frac{d(L+d)(L(d+1)+1)}{(d+1)(d+2)}\right) z^{-1-\frac{d}{2}}. \tag{46}$$

As in the proof of Theorem 4.1.1, we expand

$$\frac{\Gamma(L+d)}{\Gamma(L)} = L^d\left(P_{-\frac{d^2}{2},\,\frac{8d^3+3d^4}{24}}(x)\,P_{\frac{(2d-1)d}{2},\,\frac{d^2(2d-1)(2d-3)}{8}}(x)\cdot\frac{P_{\frac{1}{12},\frac{1}{288}}\left(\frac{x}{1+dx}\right)}{P_{\frac{1}{12},\frac{1}{288}}(x)} + O(x^3)\right)$$

as well as

$$z - \frac{d(L+d)(L(d+1)+1)}{(d+1)(d+2)} = L^2\left(\frac{z}{L^2} - \frac{d(1+dx)(1+\frac{x}{d+1})}{(d+2)}\right).$$

According to (53) and (54) of lemma A.0.1 we have

$$P_{\frac{1}{12},\frac{1}{288}}\left(\frac{x}{1+dx}\right) = P_{\frac{1}{12},\frac{1}{288}-\frac{d}{12}}(x) + O(x^3)$$

and

$$\frac{1}{P_{\frac{1}{12},\frac{1}{288}}(x)} = P_{-\frac{1}{12},\frac{1}{288}}(x) + O(x^3).$$

We compute the coefficients A, B, C of the product in (46) according to (55) of Lemma A.0.2 as follows:

$$A = -\frac{d^2}{2} + \frac{(2d-1)d}{2} + \frac{1}{12} - \frac{1}{12} = \frac{d(d-1)}{2},$$

$$B = \frac{8d^3 + 3d^4}{24} + \frac{d^2(2d-1)(2d-3)}{8} + \frac{1}{288} - \frac{d}{12} + \frac{1}{288} = \frac{d(5d-2)(3d^2-2d+1)}{24} + \frac{1}{144},$$

and

$$C = \frac{d^2(d-1)^2}{8} - \frac{d^4}{8} - \frac{(2d-1)^2 d^2}{8} - \frac{1}{144} = -\frac{d^3(2d-1)}{4} - \frac{1}{144}.$$

Hence the coefficient of x^2 is given by

$$B + C = \frac{d(d-1)(d-2)(3d-1)}{24}.$$

Therefore we have

$$\frac{R_1(z)}{L^{class}_{1,d}|\mathbb{S}^d_+|z^{1+d/2}} = \frac{d+2}{2}\left(P_{A,B+C}(x) + O(x^3)\right) \cdot \left(\frac{z}{L^2} - \frac{d(1+dx)(1+\frac{x}{d+1})}{(d+2)}\right) \cdot \left(\frac{L^2}{z}\right)^{1+d/2}.$$

□

Remark 4.2.1 We remark that this result suggests that the leading term in Weyl's law could be an upper bound for $R^D_1(z)$ for all $d \geq 2$. In Subsect. 4.2.4 below we show that it is false for $d \geq 6$, and prove the Weyl upper bound for $d = 3, 4, 5$ in Theorem 4.2.5.

4.2.2 The Neumann Laplacian

When we consider the Laplacian on $\mathbb{S}^d_+$ with Neumann boundary conditions, the eigenvalues are of the form $\lambda_{(l)} = l(l+d-1)$, $l \in \mathbb{N}$, with multiplicities $m^N_{l,d}$ given by

$$m^N_{l,d} = \binom{d+l-1}{d-1}. \tag{47}$$

The counting function $N^N(z)$ is easily computed. Again let w be defined by the relation $z = w(w+d-1)$ and L be the integer part of w. Then

$$N^N(z) = \sum_{l=1}^{L} m^N_{l,d} = \frac{\Gamma(d+L+1)}{\Gamma(L+1)\Gamma(d+1)} = \left(1 + \frac{d}{L}\right) N^D(z), \tag{48}$$

where $N^D(z)$ denotes the counting function for the Dirichlet Laplacian on the hemisphere. In view of this relation the asymptotic expansion of $N^N(z)$ is easily determined form the expansion for $N^D(z)$. we have the following result.

Theorem 4.2.3 *As $z \to \infty$ we have the following asymptotic expansion for the counting function N^N of the Neumann Laplacian eigenvalues on $\mathbb{S}^d_+$:*

$$\frac{N^N(z)}{L^{class}_{0,d}|\mathbb{S}^d_+|\, z^{\frac{d}{2}}} = 1 + \frac{1}{4}\frac{L^{class}_{0,d-1}}{L^{class}_{0,d}}\frac{|\partial\mathbb{S}^d_+|}{|\mathbb{S}^d_+|}(1-2\psi(w))\, z^{-1/2} + \frac{d(d-1)}{2}\left(\left(\frac{1}{2}-\psi(w)\right)^2 + \frac{d-2}{6}\right) z^{-1} + O(z^{-3/2})$$

or, equivalently,

$$\frac{N^N(z)}{L^{class}_{0,d}|\mathbb{S}^d|\, z^{\frac{d}{2}}} = 1 + \frac{d(1-2\psi(w))}{2}\, z^{-1/2} + \frac{d(d-1)}{2}\left(\left(\frac{1}{2}-\psi(w)\right)^2 + \frac{d-2}{6}\right) z^{-1} + O(z^{-3/2}).$$

Here w is defined by the relation $w(w+d-1) = z$.

Proof As usual, let w be defined by $w(w+d-1) = z$ and let $L = \lfloor w \rfloor$. From (48), expanding $1/L$ in terms of z we have

$$N^N(z) = \left(1 + dz^{-1/2} + \frac{d+2\psi(w)}{2}\, z^{-1} + O(z^{-3/2})\right) N^D(z).$$

Hence

$$\frac{N^N(z)}{L^{class}_{0,d}|\mathbb{S}^d|\, z^{\frac{d}{2}}} = (1 + dz^{-1/2} + \frac{d+2\psi(w)}{2}\, z^{-1})$$
$$\cdot\left(1 - \frac{d(1+2\psi(w))}{2}\, z^{-1/2} + \frac{d(d-1)}{2}\left(\left(\frac{1}{2}-\psi(w)\right)^2 + \frac{d-2}{6}\right) z^{-1})\right) + O(z^{-3/2})$$

as $z \to \infty$, from which we easily compute the coefficients of $z^{-1/2}$ and z^{-1}, respectively. □

For the more regular Riesz-mean $R^N_1(z)$ we prove the following three-term expansion

Theorem 4.2.4 *As $z \to \infty$ we have the following asymptotic expansion for the first Riesz-mean R^N_1 of the Neumann Laplacian eigenvalues on $\mathbb{S}^d_+$:*

$$\frac{R^N_1(z)}{L^{class}_{1,d}|\mathbb{S}^d_+|\, z^{\frac{d}{2}+1}} = 1 + \frac{1}{4}\frac{L^{class}_{1,d-1}}{L^{class}_{1,d}}\frac{|\partial\mathbb{S}^d_+|}{|\mathbb{S}^d_+|}\, z^{-1/2} + \frac{d(d+2)}{2}\left(\frac{1}{4} - \psi^2(w) + \frac{d-2}{6}\right) z^{-1} + O(z^{-3/2})$$

or, equivalently,

$$\frac{R_1^N(z)}{L_{1,d}^{class}|\mathbb{S}_+^d|\,z^{\frac{d}{2}+1}} = 1 + \frac{d(d+2)}{2(d+1)}\,z^{-1/2}$$
$$-\frac{d(d+2)}{2}\left(\frac{1}{4} - \psi^2(w) + \frac{d-2}{6}\right)z^{-1} + O(z^{-3/2}).$$

Here w is defined by the relation $w(w+d-1)=z$.

Proof From explicit but long computations one can get

$$R_1^N(z) = \left(1 + \frac{d(d+2)}{d+1}\,z^{-1/2} + \frac{1}{2}\,\frac{d^2(d+2)^2}{(d+1)^2}\,z^{-1} + O(z^{-3/2})\right) R_1^D(z) \quad (49)$$

from which the result easily follows. A simpler way of proving (49) is to directly link the Riesz-mean for the Neumann Laplacian to the Riesz-mean for the Dirichlet Laplacian via counting function $N^D(z)$ and to use of the explicit sum

$$\sum_{l=0}^{L}\left(\binom{d+l-2}{d-1} - \binom{d+l-1}{d-1}\right) l(l+d-1) = -\,\frac{d-1}{d+1}\cdot\frac{\Gamma(L+1+d)}{\Gamma(L)\Gamma(d)}.$$

This sum equals to the sum of the difference of Dirichlet and Neumann energy levels weighed by their multiplicities. This quantity is negative since there are more Neumann eigenvalues for each energy level. Therefore we obtain the following expression for the difference of the Riesz-means divided by the leading term in Weyl's law.

$$\frac{R_1^N(z) - R_1^D(z)}{L_{1,d}^{class}|\mathbb{S}_+^d|\,z^{\frac{d}{2}+1}} = \frac{N^N(z) - N^D(z)}{L_{1,d}^{class}|\mathbb{S}_+^d|\,z^{\frac{d}{2}}} - \frac{d-1}{d+1}\cdot\frac{\Gamma(L+1+d)}{\Gamma(L)\Gamma(d)}\cdot\frac{1}{L_{1,d}^{class}|\mathbb{S}_+^d|\,z^{\frac{d}{2}+1}}$$

where L is the integer part of w and $z = w(w+d-1)$. We have already shown that $N^N(z) = \left(1 + \frac{d}{L}\right) N^D(z)$ and

$$N^D(z) = \frac{\Gamma(L+d)}{\Gamma(L)\Gamma(d+1)}.$$

Since $L_{1,d}^{class} = \frac{2}{d+2}\,L_{0,d}^{class}$ we therefore have the relation

$$\frac{R_1^N(z) - R_1^D(z)}{L_{1,d}^{class}|\mathbb{S}_+^d|\,z^{\frac{d}{2}+1}} = \frac{d+2}{2}\left(\frac{d}{L} - \frac{d(d-1)(L+d)}{(d+1)z}\right)\frac{N^D(z)}{L_{0,d}^{class}|\mathbb{S}_+^d|\,z^{\frac{d}{2}}}.$$

We expand the term in parentheses using $L = \sqrt{z + \frac{(d-1)^2}{4}} - \frac{d}{2} - \psi(w)$ and therefore

$$L = z^{1/2} - \left(\frac{d}{2} + \psi(w)\right) + O(z^{-1/2}), \quad \frac{1}{L} = z^{-1/2} + \left(\frac{d}{2} + \psi(w)\right) z^{-1} O(z^{-3/2}).$$

For counting function $N^D(z)$ we have by the previous result

$$\frac{N^D(z)}{L_{0,d}^{class} |\mathbb{S}_+^d| \, z^{\frac{d}{2}}} = 1 - \frac{d}{2}\,(1 + 2\psi(w))\, z^{-1/2}.$$

Therefore we finally obtain

$$\frac{R_1^N(z) - R_1^D(z)}{L_{1,d}^{class} |\mathbb{S}_+^d| \, z^{\frac{d}{2}+1}} = \frac{d(d+2)}{d+1}\, z^{-1/2} + O(z^{-3/2})$$

which in particular implies (49), concluding the proof. □

Remark 4.2.2 We recall the following identities which, in fact, we have used in the proof of Theorems 4.2.3 and 4.2.4

$$N^N(w(w+d-1)) = \frac{\lfloor w \rfloor + d}{\lfloor w \rfloor} N^D(w(w+d-1)) \tag{50}$$

or

$$N^N(w(w+d-1)) = N^D((w+1)(w+d)). \tag{51}$$

Identity (50) corresponds to (48). Identity (51) says that the two counting functions N^D, N^N, are equal when the w variable is shifted by 1. This fact is equivalent to a statement about the multiplicities (and clearly seen from these) defined in (42) and (47).

4.2.3 Pólya's Conjecture

It is well-known that Pólya's conjecture in general fails for the Dirichlet eigenvalues of $\mathbb{S}_+^d$ when $d \geq 3$, while it is satisfied for $d = 2$. As proved in [17], for the Dirichlet eigenvalues of the hemisphere, one can find subsequences of eigenvalues (corresponding to the last eigenvalue in a chain of multiple eigenvalues) which don't satisfy Pólya's conjecture, as well as subsequences of eigenvalues which satisfy it (corresponding to the first eigenvalues in a chain of multiple eigenvalues, but starting from an energy level in general higher than 2). We have already discussed the relation of our results, especially three-term asymptotic expansions, with those presented in [17].

For the sake of completeness, we briefly show here that Pólya's conjecture does not hold in general for Dirichlet eigenvalues on $\mathbb{S}^d_+$ when $d \geq 3$.

By Pólya's conjecture we understand that the counting function N^D is bounded above by the leading term in Weyl's law, that is

$$N^D(z) \leq L^{class}_{0,d} |\mathbb{S}^d_+| \, z^{d/2} = \frac{1}{\Gamma(d+1)} \, z^{d/2}.$$

This inequality is equivalent to the eigenvalue bound

$$\lambda_j \geq \left(L^{class}_{0,d} |\mathbb{S}^d_+| \right)^{-2/d} j^{2/d} = \Gamma(d+1)^{2/d} \, j^{2/d}.$$

Here λ_j are the Dirichlet eigenvalues on $\mathbb{S}^d_+$. We have shown in Subsect. 3.2 that this bounds hold when $d = 2$. Let $d \geq 3$. We have $\lambda_1 = d$ and

$$\frac{\Gamma(d+1)^2}{\lambda_1^d} = \prod_{j=1}^{d} \frac{j(d+1-j)}{d} = \prod_{j=1}^{d} \left(1 + \frac{(j-1)(d-j)}{d} \right) > 1.$$

On the other hand, the counting function $N^N(z)$ for the Neumann eigenvalues on the hemisphere satisfies

$$N^N(z) \geq L^{class}_{0,d} |\mathbb{S}^d_+| \, z^{d/2} = \frac{1}{\Gamma(d+1)} \, z^{d/2}$$

as one easily sees from the identity

$$N^N(z) = L^{class}_{0,d} |\mathbb{S}^d_+| \, \frac{\Gamma(L+d+1)}{\Gamma(L+1)},$$

where, as usual, $L = \lfloor w \rfloor$ with $w(w+d-1) = z$. However, the stronger version of Pólya's inequality for the Neumann eigenvalues (i.e., taking into account the 0 eigenvalue, see [26, Corollary 1.4] where it is proved for certain Euclidean domains), which reads

$$N^N(z) \geq L^{class}_{0,d} |\mathbb{S}^d_+| \, z^{d/2} + 1 = \frac{1}{\Gamma(d+1)} \, z^{d/2} + 1.$$

does not hold.

4.2.4 Li–Yau Estimates

Usually, averaging the eigenvalues leads to a more regular behavior, as we have already seen in the previous sections.

The Weyl-sharp upper bound $R_1^D(z) \leq L_{1,d}^{class} |\mathbb{S}_+^d| z^{1+d/2}$ for all $z \geq 0$ for the first Riesz-mean of Dirichlet Laplacian eigenvalues on $\mathbb{S}_+^d$ is equivalent to the following estimate for averages of eigenvalues:

$$\frac{1}{k} \sum_{j=1}^{k} \lambda_j \geq \frac{d}{d+2} \Gamma(d+1)^{2/d} k^{2/d} \tag{52}$$

for all positive integers k. Note that, in our notation, λ_j denotes the $j-$th eigenvalue (and not the numbering of the energy level). Since $\lambda_1 = d$ the Li-Yau estimate (52) for $k = 1$ is equivalent to

$$(d+2)^d \geq \Gamma(d+1)^2$$

which only holds provided $d \leq 5$.

Therefore, an estimate on averages as (52) cannot hold if $d \geq 6$. Clearly it holds for $d = 2$ (as a consequence of the validity of Pólya's conjecture). We actually are able to prove that (52) holds for $d = 3, 4, 5$.

Theorem 4.2.5 *For all $z \geq 0$ the following inequality for the first Riesz-mean R_1^D of the eigenvalues of the Dirichlet Laplacian on $\mathbb{S}_+^d$, $d = 3, 4, 5$, holds:*

$$R_1^D(z) \leq L_{1,d}^{class} |\mathbb{S}_+^d| z^{1+\frac{d}{2}}.$$

Proof As usual, we write $z = w(w+d-1)$, $L = \lfloor w \rfloor$ and $w = L + x$ with $x \in [0, 1[$ the fractional part of w. As in the proof of Theorem 4.2.2, we write explicitly the quotient $\frac{R_1^D(z)}{L_{1,d}^{class} |\mathbb{S}_+^d| z^{1+\frac{d}{2}}}$ as a function of x. Here, however, we shall not expand in power series with respect to x.

When $L = 0$ the claimed bound is clearly satisfied in any dimension. Hence let us consider $L \geq 1$. For any fixed L, the function $f_L(x) = \frac{R_1^D(z(x))}{L_{1,d}^{class} |\mathbb{S}_+^d| z(x)^{1+\frac{d}{2}}}$, with $z(x) = (L+x)(L+x+d-1)$, is smooth in $x \in [0, 1]$. Computing its derivative, it vanishes in $(0, 1)$ only at the point

$$x = x_L = \frac{1-d-2L}{2} + \sqrt{\frac{(1-d-2L)}{4} + \frac{d+(d+2)L}{d+1}},$$

(it is easily proven that $x_L \in (0, 1)$ for any $L \geq 1$). Therefore it is sufficient to prove that $f_L(0) \leq 1$ and $f_L(x_L) \leq 1$ (since $f_L(1) = f_{L+1}(0)$). A standard computation shows that

$$f_L(0) = \frac{(L-1)(L+1)\cdots(L+d-2)\left(L + \frac{d^2}{2(d+1)}\right)}{(L(L+d-1))^{\frac{d}{2}}}$$

and

$$f_L(x_L) = \frac{L(L+1)\cdots(L+d-1)}{\left((L+d)\left(L+\frac{1}{d+1}\right)\right)^{\frac{d}{2}}}.$$

Let us prove that for $d = 3, 4, 5$, $f_L(x_L) \le 1$. The same proof allows to show that $f_L(0) \le 1$ for all $d \ge 0$ (actually, x_L is a local - and global - maximum of $f_L(x)$ for $x \in [0, 1]$ and all $L \ge 1$).

We write

$$L(L+1)\cdots(L+d-1) = \left(\prod_{j=1}^{d}(L+j-1)(L+d-j)\right)^{\frac{1}{2}}.$$

Applying the arithmetic–geometric inequality we get

$$L(L+1)\cdots(L+d-1) \le \left(\frac{1}{d}\sum_{j=1}^{d}(L+j-1)(L+d-j)\right)^{\frac{d}{2}}$$
$$= \left(L^2 + (d-1)L + \frac{(d-1)(d-2)}{6}\right)^{\frac{d}{2}}.$$

Therefore $f_L(x_L) \le 1$ if and only if

$$L^2 + (d-1)L + \frac{(d-1)(d-2)}{6} \le (L+d)\left(L + \frac{1}{d+1}\right).$$

An explicit computation shows that

$$L^2 + (d-1)L + \frac{(d-1)(d-2)}{6} - (L+d)\left(L + \frac{1}{d+1}\right)$$
$$= -\frac{(d+2)(L-1)}{d+1} + \frac{(d-2)(d-5)}{6}$$

and the right-hand side is negative for all $L \ge 1$ provided $d \le 5$. This concludes the proof.

□

Concerning the proof of Theorem 4.2.5, we remark that for any $d \ge 2$ there exists always $L_0 > 1$ such that $f_L(x_L) \le 1$ for all $L \ge L_0$, so that Berezin–Li–Yau holds for all $z \ge z_0$, where z_0 depends on d. It doesn't hold for all $z \ge 0$, as already mentioned. In fact, for $d \ge 6$, we always have $f_1(x_1) > 1$.

5 The Circle $\mathbb{S}^1$

For the sake of completeness, in this brief section we consider the case of the one dimensional sphere $\mathbb{S}^1$, that is the circle. We recall that the energy levels of the Laplacian on $\mathbb{S}^1$ are:

$$\lambda_{(l)} = l^2, \quad l \in \mathbb{N},$$

with corresponding multiplicities $m_{0,1} = 1$, $m_{l,1} = 2$ for all $l \in \mathbb{N} \setminus \{0\}$. We also recall that $L_{1,1}^{class} = \frac{2}{3}\pi$ and then $L_{1,1}^{class}|\mathbb{S}^1| = \frac{4}{3}$.

As a first observation, we show that the leading term in Weyl's law $\frac{4}{3}\, z^{3/2}$ cannot be a either lower or upper bound for the Riesz-mean $R_1(z)$. As already done several times in the previous sections, we use an auxiliary variable to simplify the computations. Namely we set $z = w^2$, $w \geq 0$. Clearly, for $0 \leq w \leq 1$ we have $R_1(w^2) = w^2$ and then $\frac{R_1(w^2)}{(L_{1,1}^{class}|S^1|w^3)} = \frac{3}{4}w$ which is strictly less than 1. For $w > 1$ we have

$$R_1(w^2) = w^2 + \sum_{l=1}^{\lfloor w \rfloor} 2(w^2 - l^2) = \frac{4w^3}{3} + \frac{w}{6} - 2w\psi^2(w) - \frac{\psi(w)}{6} + \frac{2\psi^3(w)}{3}.$$

Clearly, in any interval between two integers there exist two $w_\pm$ such that $\psi(w_\pm) = \pm\frac{\sqrt{3}}{6}$. Then

$$R_1(w_\pm^2) = \frac{4}{3}\, w_\pm^3 \pm \frac{\sqrt{3}}{54}$$

proving that $\frac{4}{3}\, w^3$ is neither a lower bound nor an upper bound for For $R_1(w^2)$. However, if we introduce a shift we are able to get the following Weyl sharp upper bound.

Proposition 5.0.1 *For all $z \geq 0$ the first Riesz-mean R_1 of the Laplacian eigenvalue on $\mathbb{S}^1$ satisfies the following inequality:*

$$R_1(z) \leq \frac{4}{3}\left(z + \frac{1}{12}\right)^{\frac{3}{2}}.$$

Moreover, in each interval $]l^2, (l+1)^2[$ with $l \in \mathbb{N}$ there exists a z_l such that equality holds.

Proof We prove the inequality for $z = w^2$, $w \geq 0$. We start considering the difference of the squares of both sides of the claimed inequality. We have

$$R_1(w^2)^2 - \frac{16}{9}\left(w^2 + \frac{1}{12}\right)^3 = \frac{(12\psi^2(w) - 24\psi(w)w - 1)^2(3\psi^2(w) - 6\psi(w)w - 9w^2 - 1)}{972}.$$

We note that for $w \geq 1$ the right hand side of the above inequality is always negative since $-\frac{1}{2} \leq \psi \leq \frac{1}{2}$. Instead, for $0 \leq w < 1$ we have

$$R_1(w^2)^2 - \frac{16}{9}\left(w^2 + \frac{1}{12}\right)^3 = w^4 - \frac{16}{9}\left(w^2 + \frac{1}{12}\right)^3$$
$$= -\frac{(48w^2+1)(6w^2-1)^2}{972} \leq 0.$$

Equality is attained when $\psi(w) = w - \sqrt{w^2 + \frac{1}{12}} = -\frac{1}{12\left(w+\sqrt{w+\frac{1}{12}}\right)}$ which has a solution in each interval $]l, l+1[$ with $l \in \mathbb{N}$. $\square$

Note that Proposition 5.0.1 has been already proved (Theorem 4.1.4). The new information of Proposition 5.0.1 is that the bound is saturated. Note also that the shift $\frac{1}{12}$ corresponds exactly to z_d with $d = 1$ for the general Theorem 4.1.4.

Remark 5.0.3 In Theorem 3.2.5 we have shown that Berezin–Li–Yau inequality holds for domains of $\mathbb{S}^2_+$, and that it cannot hold in general, in the form of the first inequality of (5), for domains invading the whole $\mathbb{S}^2$, since there is spectral convergence and on $\mathbb{S}^2$ Berezin–Li–Yau inequality in the sense specified above does not hold. On the other hand, Kröger inequality is proved for domains in $\mathbb{S}^d$ and this is a consequence of the fact that it holds on the whole sphere. In the case of $\mathbb{S}^1$ the leading term in Weyl's law is neither a lower nor an upper bound for R_1, but clearly, the Dirichlet and Neumann eigenvalues of any domain (i.e., each arc of length smaller than 2π) satisfy Berezin–Li–Yau and Kröger inequalities. This is not a contradiction, since in this case we do not have convergence of the spectrum of an arc with Dirichlet/Neumann conditions to the spectrum of $\mathbb{S}^1$ when the arc invades $\mathbb{S}^1$.

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Declarations

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A Taylor and Asymptotic Expansions

We collect in this appendix a few Taylor expansions for real-valued functions and also asymptotic expansions for Gamma functions, which are used in the proofs of Theorems 4.1.1, 4.2.1, and 4.2.2. We will also recall a formula for sums of binomial coefficients and an example of a related computation.

For real a, b, x let $P_{a,b}$ the quadratic polynomial in x defined by

$$P_{a,b}(x) = 1 + ax + bx^2$$

We have the following

Lemma A.0.1 *As $x \to 0$ we have*

$$\frac{1}{P_{a,b}(x)} = P_{-a,a^2-b}(x) + O(x^3). \tag{53}$$

For $c \in \mathbb{R}$, as $x \to 0$ we have

$$P_{a,b}\left(\frac{x}{1+cx}\right) = P_{a,b-ac}(x) + O(x^3). \tag{54}$$

Note that the above expansions remain valid if we add an $O(x^3)$-term to $P_{a,b}(x)$. Moreover, we have also the following

Lemma A.0.2 *For any positive integer n and for $a_j, b_j \in \mathbb{R}$, $j = 1, \ldots, n$, let $A = \sum_{j=1}^n a_j$, $B = \sum_{j=1}^n b_j$ and $C = \sum_{j=1}^n \sum_{i=1}^{j-1} a_i a_j = \frac{1}{2}\left(A^2 - \sum_{j=1}^n a_j^2\right)$. Then*

$$\prod_{j=1}^n P_{a_j,b_j}(x) = P_{A,B+C}(x) + O(x^3) \tag{55}$$

We shall need the following expansions of Gamma, power-type and exponential functions (see e.g., [36, Chapter 5]).

Lemma A.0.3 *The following asymptotic expansions hold:*

i)

$$\begin{aligned}\Gamma(x) &= \sqrt{2\pi}\, x^{x-1/2} e^{-x}\left(1 + \frac{1}{12x} + \frac{1}{288x^2} + O(x^{-3})\right)\\ &= \sqrt{2\pi}\, x^{x-1/2} e^{-x}\left(P_{\frac{1}{12},\frac{1}{288}}(x^{-1}) + O(x^{-3})\right)\end{aligned}$$

as $x \to \infty$.

ii)

$$e^{-a}(1+ax)^{1/x} = 1 - \frac{a^2}{2}x + a^3(\frac{1}{3} + \frac{a}{8})x^2 + O(x^3).$$

as $x \to 0$.

iii) For any $p > 0$

$$(\sqrt{x+a^2} - b)^p = x^{p/2}\left(1 - pbx^{-1/2} + p\,\frac{(p-1)b^2 + a^2}{2}x^{-1} + O(x^{-3/2})\right)$$

as $x \to \infty$.

B Duality in the Averaged Variational Principle for Estimating Averages of Increasing Sequences

In this appendix we discuss a duality aspect in the averaged variational principle, which reflects in a duality principle for Berezin–Li–Yau and Kröger bounds on sums.

Let $(a_j)_j, (b_j)_j$ be two sequences of non-negative increasing numbers. When applying the averaged variational principle (see Theorem 2.0.1) we show typically an inequality of the following form: for all $z \in [a_N, a_{N+1}]$

$$\sum_{k=1}^{N}(z - a_k) \geq p \sum_{j \in J}(z - b_j) \tag{56}$$

where $N \in \mathbb{N}\backslash\{0\}$, $J \subset \mathbb{N}\backslash\{0\}$ are arbitrary, and $p > 0$ is some positive constant. Let

$$R_1^{(a)}(z) = \sum_k (z - a_k)_+, \quad R_1^{(b)}(z) = \sum_k (z - b_k)_+$$

be the Riesz-means of the sequences $(a_j)_j, (b_j)_j$. Choosing J such that the sum on the right-hand side of (56) equals $R_1^{(b)}(z)$ one has, for all $z \geq 0$, the Riesz-mean inequality

$$R_1^{(a)}(z) \geq p\,R_1^{(b)}(z).$$

Moreover, for all positive integers N and $z \in [a_N, a_{N+1}]$ one also has

$$\sum_{k=1}^{N} a_k \leq Nz - p\,R_1^{(b)}(z)$$

which trivially implies

$$\sum_{k=1}^{N} a_k \leq \max_{z\geq 0} \left(Nz - p\, R_1^{(b)}(z) \right)$$

On the other hand, choosing $J = \{1, \dots, N\}$ and isolating $\sum_{j\in J} b_j$ in (56) we get for all $z \geq 0$ the inequality

$$\sum_{j=1}^{N} b_j \geq Nz - p^{-1}\, R_1^{(a)}(z).$$

In particular, the above inequality holds at the maximum of the r.h.s., that is

$$\sum_{j=1}^{N} b_j \geq \max_{z\geq 0} \left(Nz - p^{-1}\, R_1^{(a)}(z) \right).$$

In the applications, we typically have simple lower bounds on $R_1^{(b)}(z)$ and upper bounds on $R_1^{(a)}(z)$ so that the maxima can be computed explicitly.

References

1. Bérard, P., Besson, G.: Spectres et groupes cristallographiques. II. Domaines sphériques. Ann. Inst. Fourier (Grenoble) **30**(3), 237–248 (1980)
2. Berezin, F.A.: Covariant and contravariant symbols of operators. Izv. Akad. Nauk SSSR Ser. Mat. **36**, 1134–1167 (1972)
3. Berger, M., Gauduchon, P., Mazet, E.: Le spectre d'une variété riemannienne (The spectrum of a Riemannian manifold) Lect. Notes Math. Springer, Cham (1971)
4. Buoso, D., Luzzini, P., Provenzano, L., Stubbe, J.: On the spectral asymptotics for the buckling problem. J. Math. Phys. **62**(12), 18 (2021). (**Paper No. 121501**)
5. Buoso, D., Luzzini, P., Provenzano, L., Stubbe, J.: Semiclassical expansions and estimates for eigenvalue means of Laplacians on compact homogeneous spaces. In preparation (2023)
6. Buoso, D., Provenzano, L., Stubbe, J.: Semiclassical bounds for spectra of biharmonic operators. Rend. Mat. Appl. **7**(43(4)), 267–314 (2022)
7. Canzani, Y., Galkowski, J.: Weyl remainders: an application of geodesic beams. Invent. Math. (2023)
8. Cheng, Q.-M., Yang, H.: Estimates for eigenvalues on Riemannian manifolds. J. Differ. Equ. **247**(8), 2270–2281 (2009)
9. El Soufi, A., Harrell, E.M., II., Ilias, S., Stubbe, J.: On sums of eigenvalues of elliptic operators on manifolds. J. Spectr. Theory **7**(4), 985–1022 (2017)
10. Filonov, N., Levitin, M., Polterovich, I. et al.: Póya's conjecture for Euclidean balls. Invent. math. (2023). https://doi.org/10.1007/s00222-023-01198-1
11. Folland, G.B.: Introduction to Partial Differential Equations. Math Notes (Princeton), vol. 17. Princeton University Press, Princeton, NJ (1976)
12. Frank, R.L., Geisinger, L.: Two-term spectral asymptotics for the Dirichlet Laplacian on a bounded domain. In: Mathematical Results in Quantum Physics, pp. 138–147. World Scientific Publications, Hackensack, NJ (2011)
13. Frank, R.L., Geisinger, L.: Semi-classical analysis of the Laplace operator with Robin boundary conditions. Bull. Math. Sci. **2**(2), 281–319 (2012)

14. Frank, R.L., Larson, S.: Two-term spectral asymptotics for the Dirichlet Laplacian in a Lipschitz domain. J. Reine Angew. Math. **766**, 195–228 (2020)
15. Freitas, P.: Asymptotic behaviour of extremal averages of Laplacian eigenvalues. J. Stat. Phys. **167**(6), 1511–1518 (2017)
16. Freitas, P.: A remark on Pólya's conjecture at low frequencies. Arch. Math. (Basel) **112**(3), 305–311 (2019)
17. Freitas, P., Mao, J., Salavessa, I.: Pólya-type inequalities on spheres and hemispheres. arXiv:2204.07277, (2022)
18. Freitas, P., Salavessa, I.: Families of non-tiling domains satisfying pólya's conjecture. arXiv:2204.08902, (2022)
19. Gallot, S.: Inégalités isopérimétriques et analytiques sur les variétés riemanniennes. Astérisque, (163-164):5–6, 31–91, 281 (1989), 1988. On the geometry of differentiable manifolds (Rome, 1986)
20. Geisinger, L., Laptev, A., Weidl, T.: Geometrical versions of improved Berezin–Li–Yau inequalities. J. Spectr. Theory **1**(1), 87–109 (2011)
21. Giné, E., The, M.: addition formula for the eigenfunctions of the Laplacian. Adv. Math. **18**(1), 102–107 (1975)
22. Gradshteyn, I.S., Ryzhik I.M.: Table of integrals, series, and products. Elsevier/Academic Press, Amsterdam, eighth edition, (2015). Translated from the Russian, Translation edited and with a preface by Daniel Zwillinger and Victor Moll, Revised from the seventh edition [MR2360010]
23. Harrell, E.M., II., Hermi, L.: On Riesz means of eigenvalues. Commun. Partial Differ. Equ. **36**(9), 1521–1543 (2011)
24. Harrell, E.M., II., Provenzano, L., Stubbe, J.: Complementary asymptotically sharp estimates for eigenvalue means of Laplacians. Int. Math. Res. Not. IMRN **11**, 8405–8450 (2021)
25. Harrell, E.M., II., Stubbe, J.: On sums of graph eigenvalues. Linear Algebra Appl. **455**, 168–186 (2014)
26. Harrell, E.M., II., Stubbe, J.: Two-term, asymptotically sharp estimates for eigenvalue means of the Laplacian. J. Spectr. Theory **8**(4), 1529–1550 (2018)
27. Helgason, S.: The Radon transform on Euclidean spaces, compact two-point homogeneous spaces and Grassmann manifolds. Acta Math. **113**, 153–180 (1965)
28. Helgason, S.: Groups and geometric analysis, volume 83 of Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, 2000. Integral geometry, invariant differential operators, and spherical functions, Corrected reprint of the (1984) original
29. Ilyin, A., Laptev, A.: Berezin–Li–Yau inequalities on domains on the sphere. J. Math. Anal. Appl. **473**(2), 1253–1269 (2019)
30. Ivriĭ, V.J.: The second term of the spectral asymptotics for a Laplace–Beltrami operator on manifolds with boundary. Funktsional. Anal. Prilozhen. **14**(2), 25–34 (1980)
31. Kröger, P.: Upper bounds for the Neumann eigenvalues on a bounded domain in Euclidean space. J. Funct. Anal. **106**(2), 353–357 (1992)
32. Laptev, A.: Dirichlet and Neumann eigenvalue problems on domains in Euclidean spaces. J. Funct. Anal. **151**(2), 531–545 (1997)
33. Li, P., Yau, S.T.: On the Schrödinger equation and the eigenvalue problem. Commun. Math. Phys. **88**(3), 309–318 (1983)
34. Melas, A.D.: A lower bound for sums of eigenvalues of the Laplacian. Proc. Am. Math. Soc. **131**(2), 631–636 (2003)
35. Melrose, R.B.: Weyl's conjecture for manifolds with concave boundary. Geometry of the Laplace operator, Honolulu/Hawaii 1979. Proc. Symp. Pure Math. **36**, 257–274 (1980)
36. Olver, F.W.J., Lozier, D.W., Boisvert, R.F., Clark, C.W. (eds).: NIST handbook of mathematical functions. U.S. Department of Commerce, National Institute of Standards and Technology, Washington, DC; Cambridge University Press, Cambridge (2010). With 1 CD-ROM (Windows, Macintosh and UNIX)
37. Pólya, G.: On the eigenvalues of vibrating membranes. Proc. Lond. Math. Soc. **3**(11), 419–433 (1961)
38. Pólya, G.: Mathematics and plausible reasoning. Vol. II. Princeton University Press, Princeton, NJ, second edition (1990). Patterns of plausible inference
39. Safarov, Y., Vassiliev, D.: The asymptotic distribution of eigenvalues of partial differential operators, volume 155 of Translations of Mathematical Monographs. American Mathematical Society, Providence, RI (1997). Translated from the Russian manuscript by the authors
40. Strichartz, R.S.: Estimates for sums of eigenvalues for domains in homogeneous spaces. J. Funct. Anal. **137**(1), 152–190 (1996)

41. Weidl, T.: Improved Berezin–Li–Yau inequalities with a remainder term. In Spectral theory of differential operators, volume 225 of Amer. Math. Soc. Transl. Ser. 2, pp. 253–263. Amer. Math. Soc., Providence, RI (2008)
42. Weyl, H.: Das asymptotische Verteilungsgesetz der Eigenwerte linearer partieller Differentialgleichungen (mit einer Anwendung auf die Theorie der Hohlraumstrahlung). Math. Ann. **71**(4), 441–479 (1912)
43. Weyl, H.: Über die Randwertaufgabe der Strahlungstheorie und asymptotische Spektralgesetze. J. Reine Angew. Math. **143**, 177–202 (1913)