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Spectra of non-regular elements in irreducible representations of simple algebraic groups

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Abstract

We study the spectra of non-regular semisimple elements in irreducible representations of simple algebraic groups. More precisely, we prove that if G is a simply connected simple linear algebraic group and $\phi : G \to \mathrm{GL}(V)$ is a non-trivial irreducible representation for which there exists a non-regular non-central semisimple element $s \in G$ such that $\phi(s)$ has almost simple spectrum, then, with few exceptions, G is of classical type and $\dim V$ is minimal possible. Here the spectrum of a diagonalizable matrix is called *simple* if all eigenvalues are of multiplicity 1, and *almost simple* if at most one eigenvalue is of multiplicity greater than 1. This yields a kind of characterization of the natural representation (up to their Frobenius twists) of classical algebraic groups in terms of the behavior of semisimple elements.

Keywords: semisimple elements, irreducible representations, eigenvalue multiplicities, simple linear algebraic groups.

MSC: 20G05, 20G07, 20E28.

Dedicated to the memory of Ernest Vinberg

Introduction

A rather general problem which has received attention in the literature can be stated as that of classifying irreducible group representations whose image contains a matrix with a certain specified property. In this paper we concentrate on a property of the eigenvalue multiplicities of a semisimple element of simple linear algebraic groups in their irreducible representations. (Henceforth we will use "algebraic group" to mean "linear algebraic group".) Although problems on eigenvalues in group representations are important for many applications, little can be said

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in full generality. In fact, the behavior of individual elements in the image of a representation is quite unpredictable. For a discussion of this and related questions, we refer the reader to A. E. Zalesski (2009).

Here, we consider matrices with almost simple spectrum, that is, matrices having at most one eigenvalue of multiplicity greater than 1. More precisely, we will address the following:

Problem 1 – Let G be a simple algebraic group defined over an algebraically closed field. Determine the irreducible representations ϕ of G such that $\phi(G)$ contains a non-scalar diagonalizable matrix with almost simple spectrum.

Note that the notion of matrices with almost simple spectrum is a natural generalization of the similar notion of pseudo-reflections, the latter being diagonalizable matrices with two eigenvalues, one of which has multiplicity 1. The classification of irreducible matrix groups generated by pseudo-reflections was an important project enjoying numerous applications. (See Wagner (1978), Wagner (1981), and Zalesskii and Serežkin (1977, 1980).) We note as well that the consideration of Problem 1 is an extension of the analogous question for finite quasi-simple groups of Lie type and their representations in defining characteristic (see Suprunenko and Zalesskii (2000) and Suprunenko and Zalesskii (1998)), as well as the classification (in Seitz (1987) and Zalesskii and Suprunenko (1987)) of irreducible representations of simple algebraic groups for which a maximal torus acts with 1-dimensional weight spaces. A similar problem for irreducible representations of finite simple groups occurring as subgroups of $\mathrm{GL}_n(\mathbb{C})$ has been studied in Katz and Tiep (2021).

While Problem 1 is a question about semisimple elements, there is a natural generalization of the notions of simple and almost simple spectra to matrices that are not diagonalizable. Let V be a finite-dimensional vector space over a field F and $M \in \mathrm{GL}(V)$. Then M is called *cyclic* if, for some $v \in V$, the space V is spanned by the vectors $v, Mv, M^2v, \ldots$, and *almost cyclic* if, for some $\lambda \in F$, M is conjugate to a matrix $\mathrm{diag}(\lambda \cdot \mathrm{Id}, M_1)$, where M_1 is a cyclic matrix. Almost cyclic matrices in the images of irreducible representations of finite simple groups are studied in Di Martino, Pellegrini, and A. E. Zalesski (2014), Di Martino, Pellegrini, and A. E. Zalesski (2020), and Di Martino and A. E. Zalesski (2018) (in certain special cases). Now let G be as in Problem 1 above, $g \in G$, and let ϕ be an irreducible representation such that $\phi(g)$ is almost cyclic. If g is not semisimple, then $g = su = us$ with $u \neq 1$ unipotent and s semisimple, and one sees that $\phi(u)$ has a single non-trivial Jordan block. Such representations have been determined in Suprunenko (2013) and D. M. Testerman and A. E. Zalesski (2018). On the other hand, if g is semisimple, and $\phi(g)$ is almost cyclic, then $\phi(g)$ has almost simple spectrum; indeed $\phi(g)$ has at most two eigenvalues, one of which has multiplicity 1.

Let us now return to our considerations of semisimple elements of G whose spectrum in some irreducible representation of G is almost simple. As every semisimple element $s \in G$ lies in a maximal torus, the condition for $\phi(s)$ to have simple spectrum implies that all weight multiplicities of ϕ are equal to 1. The irreducible representations whose set of weights satifies this property are determined in Seitz (1987) for tensor-indecomposable representations and completed in Zalesskii and Suprunenko (1987). By analogy, one could expect ϕ in Problem 1 to have all but one weight multiplicity equal to 1. And indeed this is the case, as the following result, which will be etablished in §3, shows.

Theorem 1 – *Let G be a simple algebraic group defined over an algebraically closed field and ϕ an irreducible representation of G. Then the following statements are equivalent:*

(1) The matrix $\phi(s)$ has almost simple spectrum for some non-central semisimple element $s \in G$.

(2) All non-zero weights of ϕ are of multiplicity 1.

Theorem 1 will be relevant to our consideration of Problem 1, especially as the irreducible representations of simple algebraic groups satisfying 1 have been determined in D. M. Testerman and A. E. Zalesski (2015). The above theorem is best possible in the sense that in order to obtain a more precise result one has to specify the nature of the semisimple element s in question. We recall that an element $g \in G$ is said to be *regular* if $\dim(C_G(g))$ is equal to the rank of G; for g semisimple this is equivalent to $C_G(g)^\circ$ being abelian, see Springer and Steinberg (1970, Chapter III, Corollary 1.7). Our investigations show that, with very few exceptions, a non-central semisimple element s having an almost simple spectrum in an irreducible representation ϕ must be regular.

Theorem 2 – *Let G be a simply connected simple algebraic group defined over an algebraically closed field F of characteristic $p \geq 0$ and let $s \in G$ be a non-regular non-central semisimple element. Let V be a non-trivial irreducible G-module. If the spectrum of s on V is almost simple, then one of the following holds:*

(1) G is of Lie type A_n, B_n ($p \neq 2$), C_n or D_n and $\dim V = n+1, 2n+1, 2n, 2n$, respectively;

(2) G is of Lie type B_n, $p = 2$ and $\dim V = 2n$;

(3) $G = A_3$ and $\dim V = 6$;

(4) $G = C_2$, $p \neq 2$ and $\dim V = 5$.

The irreducible representations of G of the dimensions given in Theorem 2 are well known; a description of elements s which have almost simple spectrum on V is provided in Section 3.

Notation We fix an algebraically closed field F of characteristic $p \geq 0$.

Throughout the paper G is a simple simply connected linear algebraic group defined over F. All G-modules considered are rational finite-dimensional FG-modules. For a G-module V (or a representation ρ of G), we write $V \in \mathrm{Irr}(G)$ (or $\rho \in \mathrm{Irr}(G)$) to mean that V (or ρ) is rational and irreducible. If H is a subgroup of G then we write $V|_H$ for the restriction of a G-module V to H.

We fix a maximal torus T in G, which in turn defines the roots of G as well as the weights of G-modules and representations. The T-weights of a G-module V are the irreducible constituents of the restriction of V to T. As T is fixed, we will omit the reference to T and write "weights" in place of "T-weights". The set of weights of V is denoted by $\Omega(V)$. For $\mu \in \Omega(V)$, the dimension of the μ-weight space $\{v \in V : tv = \mu(t)v$ for all $t \in T\}$ is called the *multiplicity of μ in V*. The Weyl group of G is denoted by W; as $W = N_G(T)/T$, the conjugation action of $N_G(T)$ on T yields an action of W on T and consequently on the set of T-weights. The W-orbit of $\mu \in \Omega$ is denoted by $W\mu$. The set $\Omega = \mathrm{Hom}(T, F^{\times})$ (the rational homomorphisms of T to the multiplicative group of F) is called the *weight lattice*, which is a free $\mathbf{Z}$-module of finite rank called *the rank of* G.

With an algebraic group H is associated the Lie algebra of H denoted here by $\mathrm{Lie}(H)$. For the simple group G, we denote the set of roots (that is, the non-zero weights of the G-module $\mathrm{Lie}(G)$) by Φ or $\Phi(G)$. For the notions of closed subsystems of Φ and subsystem subgroups see Malle and D. Testerman (2011, §13.1). The $\mathbf{Z}$-span of Φ is called the *root lattice* and is denoted here by R or $R(G)$. In $\Phi(G)$, we fix a base $\Pi = \{\alpha_1, \ldots, \alpha_n\}$ and order the simple roots according to the Dynkin diagrams as in Bourbaki (1968). The associated set of positive roots will be denoted R^+ or $R^+(G)$. The weights in R are called *radical*. For each root $\alpha \in \Phi(G)$, we choose a non-zero element X_α in the α-weight space of T on $\mathrm{Lie}(G)$. Thus, FX_α is the Lie algebra of a T-invariant one-dimensional unipotent subgroup U_α of G; see Malle and D. Testerman (2011, Theorem 8.16) for details.

One defines a non-degenerate, W-invariant, symmetric bilinear form on $\Omega \otimes_{\mathbf{Z}} \mathbb{R}$, which we express as (μ, ν). For $\alpha \in \Phi$, let $w_\alpha \in W$ denote the corresponding reflection. The elements ω_i satisfying $2(\omega_j, \alpha_i) = (\alpha_i, \alpha_i)\delta_{ij}$ for $1 \leq i, j \leq n$ belong to Ω and are called *fundamental dominant weights*, see Bourbaki (1968, Ch. VI, §1, no.10). These form a $\mathbf{Z}$-basis of Ω, so every $\nu \in \Omega$ can be expressed in the form $\sum a_i\omega_i$, for $a_i \in \mathbb{Z}$; the set of ν with $a_1, \ldots, a_n \geq 0$ is denoted by Ω^+, the set of dominant weights. We set $\Omega^+(V) = \Omega^+ \cap \Omega(V)$, so $\Omega^+(V)$ is the set of dominant weights of V. In what follows, we will regularly use so-called "Bourbaki weights", when $R(G)$ is of type A_{r-1}, B_r, C_r or D_r, which are elements of a $\mathbf{Z}$-lattice containing Ω with basis $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_r$; the explicit expressions of the fundamental weights and the simple roots of G in terms of ε_i's are given in Bourbaki (1968, Planches I – IV).

There is a standard partial ordering of elements of Ω: for $\mu, \mu' \in \Omega$ we write $\mu \prec \mu'$ and $\mu' \succ \mu$ if and only if $\mu \neq \mu'$ and $\mu' - \mu \in R^+$. (We write $\mu \preceq \mu'$ and $\mu' \succeq \mu$ to allow $\mu = \mu'$.) If μ and μ' are dominant weights such that $\mu' \preceq \mu$, we say μ' *is* *subdominant to μ*. For the notion of a minuscule weight see Bourbaki (1975, Ch. VIII, §7.3), where they are tabulated. Every irreducible G-module has a unique weight ω such that $\mu \prec \omega$ for every $\mu \in \Omega(V)$ with $\mu \neq \omega$. This is called the *highest weight of V*. There is a bijection between Ω^+ and $\mathrm{Irr}(G)$, so for $\omega \in \Omega^+$ we denote by V_ω the irreducible G-module with highest weight ω. Suppose that $p > 0$; a dominant weight $\sum a_i\omega_i$ is called *p-restricted* if $0 \le a_i < p$ for all $i = 1,\ldots,n$. For uniformity, we often do not separate the cases with $p = 0$ and $p > 0$; by convention, when $p = 0$, a p-restricted weight is simply a dominant weight. An irreducible G-module is called *p-restricted* if its highest weight is p-restricted. For classical groups G, that is, those with root system one of A_n, B_n, C_n or D_n, the module with highest weight ω_1 is called the *natural module* and the associated representation the *natural representation*. (There is an exceptional case, when $G = B_n$ and $p = 2$, where the natural module is the Weyl module of highest weight ω_1.)

The maximal height root of $\Phi(G)$ is denoted by ω_a; this is the highest weight of $\mathrm{Lie}(G)$ and affords a non-trivial composition factor of the adjoint module $\mathrm{Lie}(G)$. The short root module for G of type B_n, C_n, F_4, and G_2 is the irreducible G-module all of whose non-zero weights are short roots. This is unique, and the highest weight of the short root module is maximal among short roots (with respect to $\prec$). An irreducible G-module is called *tensor-decomposable* if it is a tensor product of two or more non-trivial irreducible modules, similarly for representations.

If $h : G \to G$ is a surjective algebraic group homomorphism and ϕ is a representation of G then the h-twist ϕ^h of ϕ is defined as the mapping $g \mapsto \phi(h(g))$ for $g \in G$. Of fundamental importance is the Frobenius mapping $Fr : G \to G$ arising from the mapping $x \mapsto x^p$ $(x \in F)$ when $p > 0$. If V is a G-module and k a nonnegative integer, then the modules V^{Fr^k} are called *Frobenius twists of V*; if V is irreducible with highest weight ω then the highest weight of V^{Fr^k} (for $k \ge 0$) is $p^k\omega$.

If $p = 2$, then for every n there is a surjective algebraic group homomorphism $B_n \to C_n$ with trivial kernel (so this is an abstract group isomorphism); for our purposes, the choice between these two groups is irrelevant, so we choose to work with C_n when $p = 2$.

For the natural $2n$-dimensional module M of the group C_n, $n \ge 2$, a basis $\{e_i, f_i \mid 1 \le i \le n\}$ is called *symplectic* if $\{e_i, f_i\}$ is a hyperbolic pair for all i and M is the orthogonal direct sum of the spaces $\langle e_i, f_i\rangle$, $1 \le i \le n$.

Finally, we will assume $n \ge 1$ for A_n, $n > 1$ for C_n, $n > 2$ for $G = B_n$, and $n > 3$ for D_n. For brevity we write $G = A_n$ to say that G is a simple simply connected algebraic group of type A_n, and similarly for the other types.

Preliminaries

Lemma 1 – *Let $M = M_1 \otimes M_2$ be a Kronecker product of diagonal non-scalar matrices M_1, M_2 of sizes $m \le n$, respectively. Suppose that M has almost simple spectrum. Then*

(1) M_1 and M_2 have simple spectrum, and

(2) if M_i is similar to M_i^{-1} for $i = 1, 2$, then the eigenvalue multiplicities of M do not exceed 2.

Proof. (1) Suppose that M_1 has an eigenvalue e, say, of multiplicity $r > 1$. Let b_1, b_2 be distinct eigenvalues of M_2. Then eb_1, eb_2 are distinct eigenvalues of M, each of multiplicity greater than 1. This implies the claim.

(2) Suppose the contrary, and let e be an eigenvalue of M of multiplicity at least 3. By (1), M_1 and M_2 have simple spectra so $e = a_i b_i$ for $i = 1, 2, 3$ and some (distinct) eigenvalues a_i of M_1 and b_i of M_2. Then $e^{-1} = a_i^{-1} b_i^{-1}$ is an eigenvalue of M, of the same multiplicity as that of e. As M has almost simple spectrum and is similar to M^{-1} by hypothesis, we have $e = e^{-1}$, so $a_1 b_2 = a_2^{-1} b_1^{-1}$. If $(a_2^{-1}, b_2) \neq (a_1, b_1^{-1})$, then $a_1 b_2$ is an eigenvalue of M of multiplicity at least 2 and so is equal to e. But this then implies $a_1 b_2 = a_1 b_1$, contradicting that the b_i are distinct. Hence $a_2 = a_1^{-1}$ and $b_2 = b_1^{-1}$. Similarly, $a_1 b_3 = a_3^{-1} b_1^{-1}$ implies that $a_3 = a_1^{-1}$ and $b_3 = b_1^{-1}$. But now $a_2 = a_3$ contradicting that the a_i are distinct. □

Definition 1 – Let V be a G-module and $\mu, \nu \in \Omega(V)$, $\mu \neq \nu$. We say that $s \in T$ *separates the weights μ and ν* if $\mu(s) \neq \nu(s)$. If this holds for every pair of distinct weights μ, ν of V, we say that s *separates the weights of V*.

If s separates the weights of V then the eigenvalue multiplicities of s acting on V are simply the weight multiplicities of V.

Lemma 2 – *Let V be a non-trivial G-module. Let $S \subset T$ be the set of all $t \in T$ that separate the weights of V. Then*

(1) S is a nonempty Zariski open subset of T.

(2) Suppose that at most one weight of V has multiplicity greater than 1. Then, for all $s \in S$, the spectrum of s is almost simple.

Proof. (1) Let μ, ν be weights of V, $\mu \neq \nu$. Then $T_{\mu,\nu} := \{x \in T \mid \mu(x) = \nu(x)\}$ is a Zariski closed subset $T_{\mu,\nu}$ of T. The set of elements of T that do not separate some pair of weights of V, being the finite union of all $T_{\mu,\nu}$, is a proper closed subset of T. Moreover, $S = T \setminus (\cup T_{\mu,\nu})$, and so (1) follows.

(2) Let $s \in S$, so that $\mu(s) \neq \nu(s)$ whenever $\mu \neq \nu$ are weights of V. Then the eigenvalues of s on V are exactly $\mu(s)$, where μ runs over the weights of V, and the multiplicity of $\mu(s)$ equals that of μ, giving (2). □

We will require the following characterization of regular semisimple elements.

Proposition 1 – *Springer and Steinberg (1970, Ch. III, §1, Corollary 1.7) Let G, T be as usual, and let $s \in T$. Then the following conditions are equivalent:*

(1) s is regular;

(2) $C_G(s)$ consists of semisimple elements;

(3) for all $\alpha \in \Phi(G)$, $\alpha(s) \neq 1$;

(4) $C_G(s)^\circ$ is a torus.

Lemma 3 – *Let V, V_1, V_2 be non-trivial G-modules. Let $s \in T \setminus Z(G)$ have almost simple spectrum on V.*

(1) Suppose that $V = V_1 \otimes V_2$. Then all weights of V_1 and V_2 are of multiplicity 1, and s is regular.

(2) Suppose that $\Omega(V_1) + \Omega(V_2) = \Omega(V)$. Then s separates the weights of V_1 and V_2.

Proof. The first claim of (1) follows from Lemma 1. For the second assertion, suppose that s is not regular. Then by Proposition 1, $C_G(s)$ contains a unipotent element $u \neq 1$. As u stabilizes every eigenspace of s on V_1, at least one of them is of dimension greater than 1, contradicting Lemma 1(1).

(2) Suppose the contrary, that the weights of V_1, say, are not separated by s, so there exist distinct weights $\mu_1, \mu_2 \in \Omega(V_1)$ such that $\mu_1(s) = \mu_2(s)$. Then for every $\lambda, \mu \in \Omega(V_2)$, $\mu_i + \lambda, \mu_i + \mu \in \Omega(V)$ for $i = 1, 2$ and $(\mu_1 + \lambda)(s) = (\mu_2 + \lambda)(s)$ and $(\mu_1 + \mu)(s) = (\mu_2 + \mu)(s)$. As $s \notin Z(G)$, the spectrum of s on V is not almost simple, a contradiction.

With regards to applying Lemma 3(2), we note that $\Omega(V) = \Omega(V_1) + \Omega(V_2)$ if $V = V_1 \otimes V_2$. For certain choices of V, V_1, V_2, and under certain conditions on p, we may deduce that $\Omega(V) = \Omega(V_1) + \Omega(V_2)$, for V different from $V_1 \otimes V_2$. See Lemma 4(2) below.

We recall here some basic facts about the set of weights of irreducible representations of a simple algebraic group defined over a field of characteristic 0 (which are derived from analogous statements about the weights of irreducible representations of simple Lie algebras defined over $\mathbb{C}$). Fixing a maximal torus T_H of a simple algebraic group H defined over $\mathbb{C}$, and adopting the notation fixed earlier, so in particular, writing $W(H)$ for the Weyl group of H relative to T_H, let λ be a dominant T_H-weight. Then the set of weights of the irreducible $\mathbb{C}H$-module with highest weight λ is precisely the set

$$\{w(\mu) \mid \mu \in \Omega^+, \mu \preceq \lambda, w \in W(H)\},$$

that is, the $W(H)$-conjugates of all weights which are subdominant to the highest weight λ. From this one directly deduces the following facts:

(1) Let $\lambda, \mu \in \Omega^+$ and $\mu \prec \lambda$. Let V_λ, respectively V_μ, be the associated irreducible $\mathbb{C}H$-modules; then $\Omega(V_\mu) \subset \Omega(V_\lambda)$.

(2) Bourbaki (1975, Ch. VIII, §7, Proposition 10) Let $\lambda, \mu \in \Omega^+$, with associated irreducible $\mathbb{C}H$-modules V_λ, V_μ; then $\Omega(V_{\lambda+\mu}) = \Omega(V_\lambda \otimes V_\mu)$.

(3) Bourbaki (1975, Ch. VIII, §7, Propositions 4 and 6) Let $\lambda \in \Omega^+$, $\lambda \neq 0$. If λ is a radical weight, then some root is a weight of V_λ; otherwise $\Omega(V_\lambda)$ contains some minuscule weight.

We now return to the situation where the field F is of arbitrary characteristic. We will use a fundamental result of Premet, which relies on the following definition and notation.

Definition 2 – We set $e(G) = 1$ for G of type A_n, D_n, or E_n, $e(G) = 2$ for G of type B_n, C_n, or F_4, and $e(G) = 3$ for G of type G_2.

Theorem 3 – *Premet (1987, Theorem 1) Assume $p = 0$ or $p > e(G)$. Let λ be a p-restricted dominant weight. Then $\Omega(V_\lambda) = \{w(\mu) \mid \mu \in \Omega^+, \mu \preceq \lambda, w \in W\}$.*

An application of Theorem 3 and the preceding remarks now gives:

Lemma 4 – *Assume $p = 0$ or $p > e(G)$. Let $\lambda, \mu \in \Omega^+$, where λ is p-restricted, and let V_λ, respectively, V_μ be the associated irreducible G-modules. Then the following hold.*

(1) If $\mu \prec \lambda$ then $\Omega(V_\mu) \subseteq \Omega(V_\lambda)$.

(2) If $\lambda + \mu$ is p-restricted then $\Omega(V_{\lambda+\mu}) = \Omega(V_\lambda \otimes V_\mu) = \Omega(V_\lambda) + \Omega(V_\mu)$.

(3) If λ is a radical weight, then some root is a weight of V_λ; otherwise $\Omega(V_\lambda)$ contains some minuscule weight.

For the following result we introduce an additional notation. Let $\Psi \subset \Phi$ be a closed subsystem. Then we set $G(\Psi)$ to be the subgroup generated by the T-root subgroups corresponding to roots in Ψ.

Theorem 4 – *Suprunenko and A. E. Zalesski (2005, Theorem 1) Let G be a simple algebraic group with root system Φ. If Φ is of type B_n, assume* $\mathrm{char}(F) \neq 2$. *Let $R_1, R_2 \subset \Phi$ be closed subsystems such that the subgroups $G_1 := G(R_1)$ and $G_2 := G(R_2)$ are simple and $[G_1, G_2] = 1$. Let ϕ be an irreducible representation of G. Then one of the following holds:*

(1) $\phi|_{G_1 G_2}$ contains a composition factor which is non-trivial for both G_1 and G_2;

(2) G is classical and ϕ is a Frobenius twist of either the natural representation or the dual of the natural representation of G;

(3) $G = C_n$ with $p = 2$, $G = B_n$ with $n > 2$, or $G = D_n$ with $n \geq 4$, and ϕ is a Frobenius twist of the irreducible representation of highest weight ω_n, or one of ω_n and ω_{n-1} if $G = D_n$.

The following lemma will allow us in some cases to reduce our analysis of elements with almost simple spectrum to representations all of whose weights occur with multiplicity one.

Lemma 5 – *Let G be a simple algebraic group of rank greater than 1 and $s \in T \setminus Z(G)$. Assume that $p = 0$ or $p > e(G)$. Let $\mu \neq 0$ be a p-restricted dominant weight.*

(1) Let μ_m be the minimal non-zero weight subdominant to μ. Assume that the spectrum of s on V_{μ_m} is not almost simple. Then the following hold:

(i) if μ is not radical, then the spectrum of s on V_μ is not almost simple;

(ii) if μ is radical and the multiplicity of the weight 0 in V_{μ_m} is at most 1, then the spectrum of s on V_μ is not almost simple;

(iii) if μ is radical and the multiplicity of the weight 0 on both V_μ and V_{μ_m} is greater than 1, then the spectrum of s on V_μ is not almost simple;

(iv) if $0 \prec \mu_m \preceq \mu$ and s is non-regular, then the spectrum of s on V_μ is not almost simple.

(2) Suppose that $\omega_a \prec \mu$, the multiplicity of the weight 0 in V_μ is greater than 1, and the spectrum of s on V_{ω_a} is not almost simple. Then the spectrum of s on V_μ is not almost simple.

(3) Suppose that $\omega_a \prec \mu$, s is non-regular, and the spectrum of s on V_{ω_a} is not almost simple. Then the spectrum of s on V_μ is not almost simple.

Proof. By assumption, Theorem 3 applies, and we may apply Lemma 4. If μ_m is non-radical, then all weight multiplicities of V_{μ_m} are well known to be equal to 1; (i) follows. Together with the hypothesis in (ii) about the multiplicity of the zero weight, we observe that if the spectrum of s on V_{μ_m} is not almost simple then there are 4 distinct weights $\lambda_1, \lambda_2, \mu_1, \mu_2$ of V_{μ_m} such that $\lambda_1(s) = \lambda_2(s) \neq \mu_1(s) = \mu_2(s)$. Then Lemma 4(1) implies that these weights are weights of V_μ, and the result follows.

In case (iii), μ_m is the maximal height short root and the multiplicity of any non-zero weight in V_{μ_m} is equal to 1. Saying that the spectrum of s on V_{μ_m} is not almost simple means that there exist weights λ_1, λ_2 of V_{μ_m} such that $\lambda_1(s) = \lambda_2(s) \neq 1$. As these weights are weights of V_μ (again by Lemma 4(1)) and, by hypothesis, the weight 0 occurs in V_μ with multiplicity greater than 1, the result follows.

(iv) As s is non-regular, there exists $\alpha \in \Phi(G)$ such that $\pm\alpha(s) = 1$ (Proposition 1). Since the spectrum of s on V_{μ_m} is not almost simple, there are distinct short roots β, γ such that $\beta(s) = \gamma(s) \neq 1$. Then Lemma 4 implies that $\pm\alpha, \beta, \gamma$ are weights of V_μ, and the result follows.

For (2), first note that the multiplicity of the weight 0 in V_{ω_a} is greater than 1 unless $(G, p) = (A_2, 3)$ (here we again rely on the prime restrictions in the hypotheses). This case is considered in (ii). In all other cases, saying that the spectrum of s on

V_{ω_a} is not almost simple means that there are two roots α, β such that $\alpha(s) = \beta(s) \neq 1$. As the weights of V_{ω_a} occur as weights of V_μ and the weight 0 occurs in V_μ with multiplicity greater than 1, the result follows.

Finally, the case (3) follows as (iv) above, where one has to replace V_{μ_m} by V_{ω_a} and "short roots" by "roots". □

We complete this section with a straightforward observation about the natural modules for classical groups.

Lemma 6 – *Let G be a classical type group and assume $p \neq 2$ when G is of type B_n. Let $V = V_{\omega_1}$ and $s \in G$ be a non-central semisimple element.*

(1) For $G = A_n$ or C_n, if s is regular, then s has simple spectrum on V.

(2) Let $G = B_n$. Then s is regular if and only if the multiplicity of the eigenvalue -1 on V is at most 2 and the other eigenvalue multiplicities are equal to 1.

(3) Let $G = D_n$. Then s is regular if and only if the multiplicities of the eigenvalues 1 and -1 on V are at most 2 and the other eigenvalue multiplicities are equal to 1. In addition, if the spectrum of s on V is not almost simple then that of s on V_{ω_2} is not almost simple.

(4) If s is regular then the spectrum of s on V is almost simple unless $G = D_n$, $p \neq 2$ and $1, -1$ are eigenvalues of s on V, each of multiplicity 2.

Proof. (1) This is straightforward and well known.

For the remainder of the proof, we take T to be the maximal torus consisting of the diagonal matrices in the image of the natural representation of G. We now turn to (2) and the first statement of (3). Observe that $\Omega(V)$ consists of the weights $\pm\varepsilon_i$, $1 \leq i \leq n$, together with the weight 0 in case $G = B_n$. In addition, s is regular if and only if $\alpha(s) \neq 1$ for every root α. Set $a_i = \varepsilon_i(s)$ and recall that $\Phi(D_n) = \{\pm\varepsilon_i \pm \varepsilon_j \mid 1 \leq i < j \leq n\}$ and $\Phi(B_n) = \{\pm\varepsilon_i \pm \varepsilon_j, \pm\varepsilon_r \mid 1 \leq i < j \leq n, 1 \leq r \leq n\}$. So s is regular if and only if $a_i \neq a_j$ and $a_i \neq a_j^{-1}$ for every $i \neq j$, and if in addition, for $G = B_n$, $a_i \neq 1$ for all $1 \leq i \leq n$. So if $G = B_n$, we see that s is regular if and only if either all of the eigenvalues $a_1^{\pm1}, a_2^{\pm1}, \dots, a_n^{\pm1}$ are distinct and distinct from 1, or there exists a unique i with $a_i = a_i^{-1}$. If $a_i = a_i^{-1} = -1$, then s is regular if and only if all eigenvalues of s on V different from -1 occur with multiplicity 1, and -1 occurs with multiplicity at most 2. Now if $G = D_n$, then s is regular only if $a_1^{\pm1}, \dots, a_n^{\pm1}$ are distinct or there exists $1 \leq i \leq n$ such that $a_i = a_i^{-1}$, so $a_i \in \{1, -1\}$. In the latter case, s is regular if and only if all eigenvalues distinct from 1 and -1 occur with multiplicity 1 and each eigenvalue $a_i \in \{1, -1\}$ occurs with multiplicity at most 2, as claimed.

For the final statement of (3), let $G = D_n$ and suppose that the spectrum of s on V is not almost simple. Then, without loss of generality, we may assume $a_i = a_j$ for some $1 \leq i \neq j \leq n$. Then $(\varepsilon_i - \varepsilon_k)(s) = (\varepsilon_j - \varepsilon_k)(s)$ and $(-\varepsilon_i - \varepsilon_k)(s) = (-\varepsilon_j - \varepsilon_k)(s)$ for every $k \neq i, j$. Recall that the non-zero weights of V_{ω_2} are the roots in $\Phi(G)$, and the zero weight occurs with multiplicity at least 2. Assume for a contradiction that the spectrum of s on V_{ω_2} is almost simple. Then $(\varepsilon_i - \varepsilon_k)(s) = (\varepsilon_j - \varepsilon_k)(s) = (-\varepsilon_i - \varepsilon_k)(s) = (-\varepsilon_j - \varepsilon_k)(s) = 1$, whence $-\varepsilon_i(s) = \varepsilon_i(s) = \varepsilon_k(s)$ for all $1 \leq k \leq n$. As $s \notin Z(G)$, we get a contradiction.

(4) This follows from (1), (2) and (3). $\square$

1 Reduction theorem, and proof of Theorem 1

For an abelian group S, let $\mathrm{Irr}(S)$ denote the set of irreducible F-linear representations of S and write 1_S for the trivial representation. For V a finite-dimensional F-vector space, and $S \subset \mathrm{GL}(V)$ an abelian subgroup, and $\eta \in \mathrm{Irr}(S)$, set $V_S(\eta) = \{v \in V : sv = \eta(s)v \text{ for all } s \in S\}$. If $V_S(\eta) \neq \{0\}$, we say η is an S-weight of V and we call $V_S(\eta)$ the *η-weight space for S*. As throughout G is a simple algebraic group defined over F and $T \subset G$ is a maximal torus of G. If V is a rational G-module then V is a direct sum of T-weight spaces and for any subgroup $S \subseteq T$, these weight spaces are S-invariant. Thus for $\eta \in \mathrm{Irr}(S)$, $V_S(\eta)$ is a sum of T-weight spaces of V. We establish here a result about such subgroups S of T, and later will apply this to the case where S is the subgroup generated by an element $s \in T$.

Recall (see for instance Malle and D. Testerman (2011, §7)) that for any rational representation $\rho : G \to \mathrm{GL}(V)$, we have a corresponding representation of $\mathrm{Lie}(G)$, namely $d\rho : \mathrm{Lie}(G) \to \mathrm{Lie}(\mathrm{GL}(V))$. For $g \in G$, let $t_g : G \to G$ denote the automorphism induced by conjugation by g. Then using the basic definitions and properties of the differential, we have that $t_{\rho(g)} \circ \rho = \rho \circ t_g$ and so

$$\mathrm{Ad}(\rho(g)) \circ d\rho = d\rho \circ \mathrm{Ad}(g).$$

Theorem 5 (Reduction theorem) – *Let G be a simple algebraic group, T a maximal torus of G, and $S \subseteq T$ a subgroup such that $C_G(S) \neq G$. Let V be an irreducible G-module with p-restricted highest weight. Let $V_S(\eta)$ be an S-weight space of V, for some $\eta \in \mathrm{Irr}(S)$. Suppose that* $\dim V_S(\eta) = k > 1$ *and that all other S-weight spaces on V are of dimension* 1. *Then all non-zero T-weights of V are of multiplicity* 1.

Proof. Set $E = V_S(\eta)$. For $\mu \in \Omega(V)$, write M_μ for the T-weight space of V associated to μ. Suppose that $\dim M_\mu \geq 2$, for some $\mu \in \Omega(V)$. Then $M_\mu \subset E$. As $\dim M_\mu = \dim M_{w(\mu)}$ for any $w \in W$, we necessarily have $M_{w(\mu)} \subset E$. Now let $\rho : G \to \mathrm{GL}(V)$ be the corresponding rational representation of G. For a root $\alpha \in \Phi$, α induces a 1-dimensional representation λ_α of the group S.

Consider first the case where $\lambda_\alpha \neq 1_S$, for all $\alpha \in \Phi$. Recall the notation $X_\alpha \in \mathrm{Lie}(G)$, a root vector associated to the root α, a fixed element which spans the Lie algebra of the associated root group. Then $d\rho(X_\alpha)E \subset V_S(\eta\lambda_\alpha)$. Since $\eta\lambda_\alpha \neq \eta$, this latter S-weight space is of dimension at most 1. Hence $K_\alpha := \ker((d\rho(X_\alpha)) |_E)$ is of dimension at least $k-1$. Setting $K_1 = \cap_{\alpha\in\pm\Pi} K_\alpha$, we see that $K_1 \subset V$ is a proper Lie(G)-submodule on which Lie(G) acts trivially. But by Curtis (1960), V is an irreducible Lie(G)-module, and so $K_1 = \{0\}$. Therefore, $k = \dim E \leq 2n$, where n is the rank of G. We can now show that $\mu = 0$; for otherwise the W-orbit of μ is of length at least $n+1$ (the exact values are in A. E. Zalesski (2009, Table 1)). Therefore, $\dim \sum_{w\in W} M_{w(\mu)} \geq 2(n+1)$, which is a contradiction.

Consider now the case where there exists $\alpha \in \Phi$ such that $\lambda_\alpha = 1_S$. Set $M' := \sum_{w\in W} M_{w(\mu)}$, so that $M' \subseteq E$. Let $R_0 = \{\alpha \in \Phi : \lambda_\alpha = 1_S\}$, $R_2 = \Phi \setminus R_0$. Since S is non-central, $R_0 \neq \Phi$ and $R_2 \neq \varnothing$. Let R_1 be the set of roots α such that $\dim(d\rho(X_\alpha)M') \leq 1$. By the considerations of the first case above, $R_2 \subseteq R_1$. Moreover, we claim that R_1 is W-stable. Indeed for $w \in W$, choose $\dot{w} \in N_G(T)$ such that $w = \dot{w}T$. Then

$$\rho(\dot{w})d\rho(X_\alpha)M' = \rho(\dot{w})d\rho(X_\alpha)\rho(\dot{w})^{-1}\rho(\dot{w})M' = \mathrm{Ad}(\rho(\dot{w}))(d\rho(X_\alpha))M'.$$

By the remarks preceding the statement of the result, this latter is equal to

$$d\rho(\mathrm{Ad}(\dot{w})X_\alpha)M' = d\rho(X_{w(\alpha)})M'$$

and since $\dim(\rho(\dot{w})(d\rho(X_\alpha)M')) = \dim(d\rho(X_\alpha)M')$, we have the claim. Now, if all roots of Φ are of the same length then $R_1 = \Phi$, and we conclude as in the first case.

Hence we may assume that Φ has two root lengths and that the roots of R_1 are of a single length. Note that $R_0 = -R_0$ and $\beta, \gamma \in R_0$ implies $\beta + \gamma \in R_0$ provided $\beta + \gamma$ is a root. This implies (see for example Malle and D. Testerman (2011, B.14)) that R_0 is a root system, that is, R_0 is a closed subsystem of Φ. Moreover, R_0 is of maximal rank (equal to the rank of Φ) as otherwise, by Malle and D. Testerman (2011, B.18), R_0 lies in some subsystem corresponding to a proper subset of Π, in which case R_2, and so R_1 has roots of both lengths. So R_0 is a subsystem of maximal rank, and by the classification of such, Malle and D. Testerman (2011, B.18), one checks that in every case $\Phi \setminus R_0 = R_2$ again contains roots of both lengths and we conclude as above. □

Remark 1 – If $\omega = p^k\omega'$, with ω' p-restricted, then the weights of V_ω are $p^k\mu$ for μ a weight of $V_{\omega'}$. Then $p^k\mu(s) = \mu(s^{p^k})$. As the mapping $x \mapsto x^p$ for $x \in F$ is bijective on F, the spectrum of s on V_ω is almost simple if and only if the spectrum of s on $V_{\omega'}$ is almost simple.

We now take S to be generated by a single element $s \in T$ and consider the case of tensor-decomposable irreducible representations.

Lemma 7 – *Let $s \in T$ be a non-central element. Let ω be a dominant weight which is not p-restricted and not of the form $p^k\mu$ for μ a p-restricted weight. Suppose that the spectrum of s on V_ω is almost simple. Then all weights of V_ω are of multiplicity* 1.

Proof. By Steinberg's tensor product theorem, $V_\omega = V_{p^{k_1}\mu_1} \otimes V_{p^{k_2}\mu_2} \otimes \cdots \otimes V_{p^{k_t}\mu_t}$, where $t > 1$ and $\mu_1, \ldots, \mu_k$ are non-zero p-restricted weights and $(k_1, \ldots, k_t)$ are distinct non-negative integers. Then Lemma 3 implies that the spectrum of s on each tensor factor is simple so the weights of each tensor factor have multiplicity 1. Furthermore, Zalesskii and Suprunenko (1987, Proposition 2) implies that the weights of V_ω are of multiplicity 1 unless there exists $1 \le j < t$ such that $k_{j+1} = k_j + 1$ and one of the following holds:

(i) $G = C_n$, $p = 2$, $\mu_j = \omega_n$, $\mu_{j+1} = \omega_1$;

(ii) $G = G_2$, $p = 2$, $\mu_j = \omega_1$, $\mu_{j+1} = \omega_1$;

(iii) $G = G_2$, $p = 3$, $\mu_j = \omega_2$, $\mu_{j+1} = \omega_1$.

Moreover, in each of the cases (i), (ii) and (iii), the module $V_{\mu_j} \otimes V_{p\mu_{j+1}}$ has a weight of multiplicity greater than 1. Hence if one of the three cases occurs, we deduce that $t = 2$ and so we can also assume that $j = 1$ and $k_1 = 0$, that is, $V_\omega = V_{\mu_1} \otimes V_{p\mu_2}$. We consider the above cases in detail.

Case (i): Take T to be the set of diagonal matrices in the image of the natural representation of G. Here $\Omega(V_{\omega_n}) = \{\pm\varepsilon_1 \pm \cdots \pm \varepsilon_n\}$ and $\Omega(V_{2\omega_1}) = \{\pm 2\varepsilon_1, \ldots, \pm 2\varepsilon_n\}$. (As usual, we have adopted the notation of Bourbaki (1968, Planche III).) Let ν be a weight of V_{ω_n} with positive signs of both ε_i and ε_j, for some $1 \le i, j \le n$, $i \ne j$. As $\nu - 2\varepsilon_i$ and $\nu - 2\varepsilon_j$ are weights of V_{ω_n}, it follows that ν is also a weight of V_ω with multiplicity at least 2. This remains true for weights where both ε_i and ε_j have coefficient -1 or have opposite coefficients. It follows that the restriction of $V_{\omega_n + 2\omega_1}$ to T contains a direct sum of at least two copies of $V_{\omega_n}|_T$. Therefore, every eigenvalue of s on V_{ω_n} is also an eigenvalue of s on $V_{\omega_n+2\omega_1}$, and occurs with multiplicity at least 2. So this case is ruled out as the spectrum of s on V_{ω_n} is simple.

Case (ii): Here the weights of V_{ω_1} are the short roots of Φ, and the following weights occur with multiplicity 2 in V_ω: $3\alpha_1 + \alpha_2$, $3\alpha_1 + 2\alpha_2$. Since the spectrum of s on V_ω is almost simple, these roots must all take equal value on s. In particular, $\alpha_2(s) = 1$. But now the eigenvalue $5\alpha_1(s)$ occurs with multiplicity 2 as well as $3\alpha_1(s)$, implying that $\alpha_1(s) = 1$ as well, contradicting the fact that s is non-central.

Case (iii): This case is similar. Here the weights of V_{ω_2} are the long roots of Φ and the zero weight, and the weights of V_{ω_1} are the short roots and the zero weight. We find that each of the weights $3\alpha_1 + \alpha_2$ and α_2 occur with multiplicity 2, and deduce that $\alpha_1(s) = 1$. But now the eigenvalue $\alpha_2(s)$ occurs with multiplicity greater than 1, as well as the eigenvalue 1, and so $\alpha_2(s) = 1$ as well, again contradicting s non-central. □

Proof (of Theorem 1*).* Using Lemma 2, we see that assertion (1) follows from assertion (2). We apply Theorem 5, Remark 1 and Lemma 7 to obtain the reverse implication. □

2 Commuting subgroups and a partial proof of Theorem 2

An essential element of our proof of Theorem 2 is an application of Theorem 4, which allows us to treat many of the groups and representations in a uniform way. (See Proposition 2 below.) Let $s \in G$ be a non-regular semisimple element. In order to apply Theorem 4, we need to find a pair of subsystem subgroups K, Y such that $[K,Y] = 1$, $[K,s] = 1$ and $[s,Y] \neq 1$. For technical reasons, it will suffice to do this for groups other than B_n, D_n, and G_2.

Lemma 8 – *Let $G = \mathrm{SL}_n(F)$, $n > 3$, and let $s \in T \setminus Z(G)$ be a non-regular element. Then there are simple subsystem subgroups K, Y, normalized by T, such that $[K,Y] = 1$, $[K,s] = 1$ and $[s,Y] \neq 1$, unless $n = 4$ and, up to conjugacy in G, $s = \mathrm{diag}(a,a,a^{-1},a^{-1})$ or $s = \mathrm{diag}(a,a,-a^{-1},-a^{-1})$, for some $a \in F^{\times}$.*

Proof. We take T to be the torus of diagonal matrices in G. As s is non-regular and non-central, we may assume that $s = \mathrm{diag}(b,b,a_3,\ldots,a_n)$, where $a_3 \neq b$. Suppose first that $a_3 \neq a_i$ for some $i > 3$. Set $K = \mathrm{diag}(\mathrm{SL}_2(F), \mathrm{Id}_{n-2})$, $Y = \mathrm{diag}(\mathrm{Id}_2, \mathrm{SL}_{n-2}(F))$. Next, suppose $a_3 = \cdots = a_n$. If $n > 4$ then we can take $Y = \mathrm{diag}(1, \mathrm{SL}_2(F), \mathrm{Id}_{n-3})$ and $K = \mathrm{diag}(\mathrm{Id}_{n-2}, \mathrm{SL}_2(F))$. If $n = 4$, then $s = \mathrm{diag}(b,b,a,a)$ and $b^2a^2 = 1$, whence $b = \pm a^{-1}$.

Remark 2 – If $G = \mathrm{SL}_4(F)$, and $s = \mathrm{diag}(\lambda,\lambda,\lambda^{-1},\lambda^{-1})$ or $s = \mathrm{diag}(\lambda,\lambda,-\lambda^{-1},-\lambda^{-1})$, for $\lambda \in F$, $\lambda^4 \neq 1$, then s is non-regular, non-central, and it is impossible to find a pair of commuting subsystem subgroups K, Y such that $[s,K] = 1$ and $[s,Y] \neq 1$. Moreover, the Jordan form of s on the exterior square of the natural 4-dimensional module is $\mathrm{diag}(\lambda^2, \lambda^{-2}, 1,1,1,1)$, which is non-central with almost simple spectrum.

Lemma 9 – *Let $G = C_n$, $n > 1$, and let $s \in T \setminus Z(G)$ be a non-regular element. Then there are simple subsystem subgroups K, Y of G, normalized by T, such that $[K,Y] = 1$, $[K,s] = 1$ and $[s,Y] \neq 1$, unless $n = 2$ and with respect to an ordered symplectic basis (e_1, f_1, e_2, f_2) of V_{ω_1}, the Jordan form of s on the natural G-module is either $\mathrm{diag}(a,a^{-1},a,a^{-1})$, for $\pm 1 \neq a \in F$, or $s = \pm\mathrm{diag}(1,1,-1,-1)$, for $p \neq 2$.*

Proof. The group $G = C_n = \mathrm{Sp}_{2n}(F)$ contains a maximal rank subsystem subgroup H isomorphic to $\mathrm{Sp}_2(F) \times \cdots \times \mathrm{Sp}_2(F)$, so every semisimple element is conjugate to an element of H. Therefore, we can write the matrix of s with respect to a suitable basis of the natural G-module V_{ω_1} as $\mathrm{diag}(a_1, a_1^{-1}, \ldots, a_n, a_n^{-1})$ for some $a_1, \ldots, a_n \in F$. By Lemma 6, the diagonal entries of s are not distinct. Hence either $a_i = \pm 1$ for some $i \in \{1, \ldots, n\}$, or, replacing some a_i by a_i^{-1}, we can assume that $a_i = a_j$ for some $1 \leq i < j \leq n$.

Suppose first that $a_i = \pm 1$ for some $i \in \{1, \ldots, n\}$ and assume without loss of generality that $i = 1$. If there exists j such that $a_j \neq \pm 1$, we can assume $j = n$ and then take $K = \mathrm{diag}(\mathrm{Sp}_2(F), \mathrm{Id}_{2n-2})$, $Y = \mathrm{diag}(\mathrm{Id}_{2n-2}, \mathrm{Sp}_2(F))$. Otherwise, $s^2 = 1$ and $p \neq 2$. We can reorder $a_1, \dots, a_n$ so that $a_1 \neq a_2$, and if $n > 2$ we take $Y = \mathrm{diag}(\mathrm{Sp}_4(F), \mathrm{Id}_{2n-4})$, $K = \mathrm{diag}(\mathrm{Id}_{2n-2}, \mathrm{Sp}_2(F))$. If $n = 2$, $s^2 = 1$ and $p \neq 2$, such a choice is not possible and we have s as in the final statement.

Now suppose that $a_i \neq \pm 1$ for all $i \in \{1, \dots, n\}$, so there exists $1 \leq i < j \leq n$ such that $a_i = a_j$. In this case, there exists a 2-dimensional totally isotropic subspace of the underlying $2n$-dimensional symplectic space on which s acts as scalar multiplication. If $n > 2$, then s is contained in a Levi subgroup $L = L_1 \times L_2$ of G, where $L_1 \cong \mathrm{GL}_2(F)$ and $L_2 \cong \mathrm{Sp}_{2n-4}(F)$. Moreover $[s, L_1] = 1$, so we can take $K = L_1$, $Y = L_2$. If $n = 2$ then $s = \mathrm{diag}(a, a^{-1}, a, a^{-1})$ as in the statement of the result. □

Lemma 10 – *Let $G \in \{E_6, E_7, E_8, F_4\}$. Let $s \in T \setminus Z(G)$ be a non-regular element. Then there exist simple subsystem subgroups K, Y, normalized by T, such that K is of type A_1, $[K, Y] = 1$, $[K, s] = 1$, $[s, Y] \neq 1$.*

Proof. As s is not regular, $C_G(s)$ contains root subgroups $U_{\pm\alpha}$ for some root $\alpha \in \Phi$. Clearly, we can assume α to be a simple root. Moreover, we can assume that $\alpha = \alpha_1$ if $G \neq F_4$, otherwise, that $\alpha = \alpha_1$ or α_4.

Denote by R_α the set of roots orthogonal to α, and observe that R_α is not empty. Set $Y = \langle U_{\pm\beta} : \beta \in R_\alpha \rangle$ and $K = \langle U_{\pm\alpha} \rangle$. Then $[Y, K] = 1$ and $[K, s] = 1$. If $[Y, s] \neq 1$, replacing Y by a suitable simple subgroup of Y, we are done.

We now assume $[s, U_\beta] = 1$ for all $\beta \in R_\alpha$. In this situation, as s is non-central, $[s, U_\gamma] \neq 1$ for some simple root adjacent to α in the Dynkin diagram. Moreover, the Dynkin diagram of the above groups contains a node β, not adjacent to each of α, γ. In particular, $\beta \in R_\alpha$ and so $[s, U_\beta] = 1$, while $[s, U_\gamma] \neq 1$. So now we can take $K = \langle U_{\pm\beta} \rangle$ and $Y = \langle U_{\pm\gamma} \rangle$.

This completes the proof.

We now apply the previous three lemmas and Theorem 4 to establish Theorem 2 for certain groups.

Proposition 2 – *Let G be of type A_n for $n > 3$, C_n for $n > 2$, or of type F_4, E_6, E_7, or E_8. Let V be a non-trivial irreducible G-module and $s \in T \setminus Z(G)$. Suppose that the spectrum of s on V is almost simple. Then one of the following holds:*

(1) s is regular,

(2) $G = C_n$ with $p = 2$ and the highest weight of V is $2^m\omega_n$, or

(3) G is classical and V is a Frobenius twist of the natural or the dual of the natural module for G.

Proof. Suppose that s is not regular. By Lemma 8 for A_n, Lemma 9 for C_n, and Lemma 10 for the other groups in the statement, there are simple subsystem subgroups K, Y, normalized by T, such that $[K, Y] = 1$, $[K, s] = 1$ and $[Y, s] \neq 1$. Then we apply Theorem 4 to K, Y in place of $G(R_1), G(R_2)$ to conclude that either (2) or (3) holds or there is a KY-composition factor M of V afforded by an irreducible representation τ of KY, such that τ is non-trivial on both K and Y. So we assume neither (2) nor (3) holds, so we are in the latter situation, and aim for a contradiction.

We first note that $TY = Y \cdot Z(TY)$, as Y is simple. Therefore, as $s \in T$, $s = s_1 s_Y$ for some $s_1 \in Z(TY) \subset T$ and $s_Y \in (T \cap Y)$. As $[s, K] = 1$ and $[Y, K] = 1$, we have $[s_1, K] = 1$ and $[s_1, YK] = 1$. Also, as $[s, Y] \neq 1$, we have $[s_Y, Y] \neq 1$.

Now M is a direct sum of eigenspaces for s_1. It follows that τ is realized in one of the s_1-eigenspaces M_1, say, and hence the spectrum of s on M_1 is almost simple if and only if that of s_Y on M_1 is almost simple. Therefore, it suffices to show that the spectrum of $\tau(s_Y)$ is not almost simple.

Now $\tau = \tau_K \otimes \tau_Y$, where τ_K, τ_Y are non-trivial irreducible representations of K, Y, respectively. As $[s_Y, Y] \neq 1$, there are at least two distinct s_Y-eigenspaces on the representation space corresponding to τ_Y, each of them is of dimension at least 2 as $\tau_K(K)$ acts on each eigenspace and all $\tau_K(K)$ composition factors of M are of dimension strictly greater than 1. Hence, the spectrum of s_Y on M is not almost simple, giving the desired contradiction. □

Remark 3 – (1) Let $G = C_2$, p odd. If s is not as described in the exceptional cases of Lemma 9, then the conclusion of Proposition 2 remains valid.

(2) Note that the irreducible representation of $G = C_2$ with highest weight ω_2 induces an isomorphism between $\mathrm{PSp}_4(F)$ and $\mathrm{SO}_5(F)$, and the element $s = \pm\mathrm{diag}(1, 1, -1, -1)$ in Lemma 9 acts as $\mathrm{diag}(1, -1, -1, -1, -1)$, hence has almost simple spectrum. Similarly, the element $s = \mathrm{diag}(a, a^{-1}, a, a^{-1})$ acts as $\mathrm{diag}(a^2, 1, 1, 1, a^{-2})$, which has almost simple spectrum provided $a^2 \neq \pm 1$.

(3) In view of Lemma 3 and Proposition 2, to complete the proof of Theorem 2, it remains to consider p-restricted representations (of highest weight λ) of the groups B_n for $n > 2$, D_n for $n > 3$, C_n for $p = 2$ and $\lambda = \omega_n$, and the small rank groups A_2, A_3, C_2, and G_2. We will handle the small rank groups in Section 4.1 and complete the proof in Section 4.2 by dealing with the remaining groups.

3 Weight levels

Recall we have $\Omega = \sum_{i=1}^{n} \mathbb{Z}\omega_i$, the weight lattice associated with Φ, and Ω^+ the set of dominant weights in Ω. In this section we establish some results on Ω in view of applying the results in Section 2. Recall that a weight is radical if it is an integral linear combination of roots. The irreducible G-module whose highest weight is the maximal height short root is called the short root module. If all weights are of the same length then any root is regarded as short, and the short root module is V_{ω_a}.

Definition 3 – Let

$$\Lambda_1 = \{\mu \in \Omega^+ \mid \text{ if } \nu \preceq \mu \text{ for some } \nu \in \Omega^+ \text{ then } \mu = \nu\}.$$

For $i > 1$, let

$\Lambda_i = \{\mu \in \Omega^+, \mu \notin \Lambda_1 \cup \cdots \cup \Lambda_{i-1} \mid$ if $\nu \prec \mu$ for some $\nu \in \Omega^+$ then $\nu \in \Lambda_1 \cup \cdots \cup \Lambda_{i-1}\}$.

The elements of Λ_i are called *weights of level i*.

Lemma 11 – *Assume $p = 0$ or $p > e(G)$. Let $\omega \neq 0$ be a p-restricted dominant weight for G. If $\omega \notin \Lambda_1 \cup \cdots \cup \Lambda_i$ for some $i > 0$, then there are weights $\nu_1, \ldots, \nu_i$ of V_ω such that $\nu_j \in \Lambda_j$ for $j = 1, \ldots, i$. In addition, the weights of V_{ν_j} occur as weights of V_ω, for $1 \leq j \leq i$.*

Proof. This follows from the definition of Λ_j and Lemma 4. □

We conclude this section with some precise information about weights of level 1 or 2, and radical weights of level 3, for certain root systems.

Lemma 12 – *The sets Λ_1 and Λ_2 for the root systems of types A_n, B_n, C_n and D_n are as in Table 1. In addition, we have*

(1) for $\Phi = B_n$, $n > 2$, ω_2 is the only radical weight in Λ_3;

(2) for $\Phi = C_n$, $n > 3$, $2\omega_1, \omega_4$ are the only radical weights in Λ_3;

(3) for $\Phi = C_2$ or C_3, $2\omega_1$ is the only radical weight in Λ_3.

Φ	Λ_1	Λ_2
$A_n, n \geq 1$	$0, \omega_1, \ldots, \omega_n$	$2\omega_1, 2\omega_n, \omega_1 + \omega_n, \omega_1 + \omega_i, \omega_i + \omega_n, i = 2, \ldots, n-1$
$B_n, n \geq 3$	$0, \omega_n$	$\omega_1, \omega_1 + \omega_n$
$C_n, n > 2$	$0, \omega_1$	ω_2, ω_3
C_2	$0, \omega_1$	$\omega_2, \omega_1 + \omega_2$
$D_n, n > 4$	$0, \omega_1, \omega_{n-1}, \omega_n$	$\omega_2, \omega_3, \omega_1 + \omega_{n-1}, \omega_1 + \omega_n$
D_4	$0, \omega_1, \omega_3, \omega_4$	$\omega_2, \omega_1 + \omega_3, \omega_1 + \omega_4, \omega_3 + \omega_4$

Table 1 – Weights in Λ_1, Λ_2

Proof. By Lemma 4(3), Λ_1 consists of minuscule weights and the weight 0, justifying the entries in the column headed Λ_1 of the table. Furthermore, Λ_2 contains a unique radical weight, which is the maximal short root (see for instance Suprunenko and A. E. Zalesski (2007, Proposition 10)).

Let now $\omega = \sum a_i\omega_i \in \Lambda_2$ be a non-radical weight. Suppose that $a_i \geq 2$ for some i. Then $\omega' = \omega - \alpha_i \in \Omega^+$, so $\omega' \in \Lambda_1$. Inspecting Λ_1 and the expressions of simple roots in terms of fundamental dominant weights, we observe that $\omega' + \alpha_i$ (for $\omega' \in \Lambda_1$) is dominant only if Φ is of type A_n and $\omega \in \{2\omega_1, 2\omega_n\}$; furthermore, it is straightforward to see that in this latter case, we have $2\omega_1, 2\omega_n \in \Lambda_2$. So we can assume that $a_i \leq 1$ for all i. Next we proceed case-by-case, still assuming $\omega \in \Lambda_2$ a non-radical weight.

Consider first the case where $\Phi = A_n$. If $n = 1, 2$ then the result is clear, so assume now $n > 2$. Note that $\omega_i + \omega_j \succ \omega_{i-1} + \omega_{j+1}$ for $1 \leq i < j \leq n$ as $\omega_i + \omega_j - \omega_{i-1} - \omega_{j+1} = \alpha_i + \cdots + \alpha_j$. (Here ω_0 and ω_{n+1} are understood to be zero.) So if $a_i, a_j \neq 0$ for some $i \neq j$, then $\omega = \omega' + \omega_{i-1} + \omega_{j+1}$ with $\omega' \in \Lambda_1$. Using the same reasoning for different pairs of non-zero coefficients, we see that either $i = 1$ and $\omega' = \omega_1$ or $j = n$ and $\omega' = \omega_n$. Finally, one observes that no weight obtained is subdominant to another one. So Λ_2 is as in the table. This completes the consideration of $\Phi = A_n$.

For $\Phi \neq A_n$, the argument differs, as some fundamental dominant weights are radical. Recall that $\omega = \sum a_i \omega_i \in \Lambda_2$ is a non-radical weight and we have seen that $a_i \leq 1$ for all i. If ω_i is a radical weight and $a_i > 0$, then $\omega - \omega_i$ is subdominant to the weight ω, and hence $0 \neq \omega - \omega_i \in \Lambda_1$. So $\omega = \nu + \omega_i$, for some $\nu \in \Lambda_1$, $\nu \neq 0$. Moreover, $\omega_i = \mu$, where μ is the maximal height short root, as otherwise $\nu + \mu$ is subdominant to ω and $\omega \notin \Lambda_2$. So either $\omega = \nu + \mu$, for some $\nu \in \Lambda_1$, or $a_i = 0$ for all i such that ω_i is radical. For each root system, we determine when $\nu + \mu$ lies in Λ_2.

Consider the case $\Phi = B_n$, $n \geq 3$. Following the notation of the previous paragraph, we have $\nu = \omega_n$, $\mu = \omega_1$. Moreover, ω_i is radical for every $i < n$. So $\omega \in \Lambda_2$ non-radical implies that $\omega = \omega_1 + \omega_n$. It is straightforward to verify that $\omega_1 + \omega_n \in \Lambda_2$. We deduce that $\Lambda_2 = \{\omega_1, \omega_1 + \omega_n\}$. For the claim of (1), let $\omega \in \Lambda_3$ be a radical weight. If $a_i \geq 2$ for some i, then $\omega - \alpha_i$ is a radical dominant weight which must lie in Λ_2. We deduce that $\omega - \alpha_i = \omega_1$ and we find that $n = 2$, contradicting our hypothesis; so we may now assume $a_i \leq 1$ for all i. In particular, as ω is radical, $a_n = 0$. In addition, $\omega_i = \omega_{i-1} + \alpha_i + \cdots + \alpha_n$, see Bourbaki (1968, Planche II), i.e. $\omega_{i-1} \prec \omega_i$. So $\omega \in \Lambda_3$ then implies that $\omega = \omega_2$.

Consider now the case $\Phi = C_n$, for $n \geq 2$. If $a_i \neq 0$ or some i such that ω_i is radical (as above), we find that $\nu = \omega_1$, $\mu = \omega_2$. In this case $\mu + \nu = \omega_1 + \omega_2$. But $\omega_1 + \omega_2 - \alpha_1 - \alpha_2$ is subdominant to ω and lies in Λ_1 only if $n = 2$. We may now assume $a_i = 0$ if ω_i is radical, so $a_i = 0$ for i even. Also by the preliminary remarks, $a_i \leq 1$ for all i. It is easy to observe that $\omega_i \succ \omega_{i-2}$ for $i > 1$, which implies the result on Λ_2. We now turn to the claims of (2) and (3), so let $\omega \in \Lambda_3$ be a radical weight. If $a_i \geq 2$ for some i, then $\omega - \alpha_i \in \Lambda_2$ if only if $\omega = 2\omega_1$. So we now assume $a_i \leq 1$ for all i. Let $1 \leq i \leq n$ be maximal such that $a_i = 1$. Since the dominant weight $\omega - \omega_i + \omega_{i-2} \prec \omega$ must lie in $\Lambda_1 \cup \Lambda_2$ and is a radical weight, we find that $n \geq 4$ and $\omega = \omega_4$. Finally, one checks that ω_4 lies in Λ_3.

Finally consider the case $\Phi = D_n$, $n \geq 4$. Here, in the case where $a_i \neq 0$ for some i with ω_i radical, we have (in the previously defined notation) $\nu \in \{\omega_1, \omega_{n-1}, \omega_n\}$ and $\mu = \omega_2$, so $\mu + \nu \in \{\omega_1 + \omega_2, \omega_2 + \omega_{n-1}, \omega_2 + \omega_n\}$. Now $\omega_2 + \omega_n \succ \omega_1 + \omega_{n-1} \notin \Lambda_1$ and $\omega_2 + \omega_{n-1} \succ \omega_1 + \omega_n \notin \Lambda_1$ so $\omega_2 + \omega_n, \omega_2 + \omega_{n-1} \notin \Lambda_2$. Furthermore, as $\omega_1 + \omega_2 - \alpha_1 - \alpha_2 = \omega_3 + \delta_{n,4}\omega_4$, it follows that $\omega_1 + \omega_2 \notin \Lambda_2$. So we now assume that $a_i = 0$ for all i such that ω_i is radical, that is, $a_i = 0$ if $i < n-1$ is even and as established earlier $a_j \leq 1$ for all j. Moreover, there are at most two a_j which are non-zero, as otherwise there exists $\beta \in \Phi$ with $\omega - \beta$ dominant and not lying in Λ_1. Suppose $a_i = 1$ for some (odd) $i < n-1$. Then $\omega-(\omega_i-\omega_{i-2}) \prec \omega$ must lie in Λ_1 and so $i = 3$. So finally, recalling that ω is non-radical we have $\omega \in \{\omega_3 (n > 4), \omega_3 + \omega_{n-1} (n > 4), \omega_3 + \omega_n (n > 4), \omega_1 + \omega_n, \omega_1 + \omega_{n-1}, \omega_{n-1} + \omega_n\}$. It is straightforward to see that $\omega_3 (n > 4), \omega_1 + \omega_{n-1}$ and $\omega_1 + \omega_n$ all lie in Λ_2. In addition, $\omega_{n-1} + \omega_n \succ \omega_{n-3}$, and the latter lies in Λ_1 if and only if $n = 4$. So it remains to show that $\omega_3 + \omega_n, \omega_3 + \omega_{n-1} \notin \Lambda_2$ for $n > 4$. This is clear since $\omega_2 + \omega_{n-1}$, respectively $\omega_2 + \omega_n$, is subdominant to the given weight and does not lie in Λ_1. □

4 Proof of Theorem 2

In this section, we prove Theorem 2, so in particular we are concerned with the action of non-central non-regular semisimple elements on certain specific representations (as shown by Theorem 1). As noted earlier, in remark 3(3), we must handle some small rank groups as well as the groups B_n, D_n, and C_n when $p = 2$ and for certain highest weights; we do this in two separate subsections.

4.1 Groups of small rank

Lemma 13 – *Let $G = A_2$ and let $s \in T \setminus Z(G)$ be a non-regular element. Let $V = V_\omega$ be the irreducible G-module of p-restricted highest weight $\omega \neq 0$. Then the spectrum of s on V_ω is almost simple if and only if $\omega = \omega_1$ or ω_2.*

Proof. We take T to be the torus of diagonal matrices in $\mathrm{SL}_3(F)$. Since s is non-regular non-central, with respect to an appropriate choice of basis of V_{ω_1}, we may assume $s = \mathrm{diag}(a, a, a^{-2})$, for some $a \in F^\times$ with $a^3 \neq 1$. Clearly the spectrum of s on V_{ω_1} and V_{ω_2} is indeed almost simple. So we now assume $\omega \notin \{0, \omega_1, \omega_2\}$. In particular, Lemma 12 implies $\omega \notin \Lambda_1$ and by Lemmas 11 and 12, $\Omega(V_\omega)$ has some weight from $\Lambda_2 = \{\omega_1 + \omega_2, 2\omega_1, 2\omega_2\}$.

Suppose first that $\omega = 2\omega_1$, and so $p \neq 2$. The weights of V_{ω_1} are $\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$, so the weights of $V_{\omega_1} \otimes V_{\omega_1}$ are $2\varepsilon_1, 2\varepsilon_2, 2\varepsilon_3, \varepsilon_1+\varepsilon_2, \varepsilon_1+\varepsilon_3, \varepsilon_2+\varepsilon_3$, which by Lemma 4(2) coincide with the weights of V_ω. Now, $2\varepsilon_1(s) = 2\varepsilon_2(s) = a^2$, and $(\varepsilon_1 + \varepsilon_3)(s) = (\varepsilon_2 + \varepsilon_3)(s) = a^{-1}$. As $a^3 \neq 1$, the eigenvalues a^2, a^{-1} are distinct, so the spectrum of s on $V_{2\omega_1}$ is not almost simple, as claimed. Since $V_{2\omega_2}$ is dual to $V_{2\omega_1}$, the spectrum of s on $V_{2\omega_2}$ is not almost simple as well.

Suppose now that $\omega = \omega_1 + \omega_2$. Then the weights of V_ω are the roots and the zero weight. Then $(\alpha_1+\alpha_2)(s) = \alpha_2(s) = (\varepsilon_2-\varepsilon_3)(s) = a^3 \neq 1$ and $-(\alpha_1+\alpha_2)(s) = -\alpha_2(s) = a^{-3}$. If $p \neq 3$, the eigenvalue 1 is also of multiplicity 2, and we are done. If $p = 3$ and $a^3 \neq a^{-3}$ then we are done as well. So suppose $p = 3$ and $a^6 = 1$ and hence $a^3 = -1$, that is $a = -1$. Note that $\pm\alpha_2(s) = -1$ and $\pm\alpha_1(s) = 1$, so the result also follows in this case.

We now appeal to Lemma 11 to conclude. □

Lemma 14 – *Let $G = A_3$ and let $s \in T \setminus Z(G)$ be a non-regular element. Let $V = V_\omega$ be the irreducible G-module of p-restricted highest weight $\omega \neq 0$. If the spectrum of s on V is almost simple, then either $\omega = \omega_1$ or ω_3, or $\omega = \omega_2$ and there exists $a \in F^\times$, $a^4 \neq 1$ such that with respect to a suitably chosen basis, $s = \mathrm{diag}(a, a, \pm a^{-1}, \pm a^{-1})$.*

Proof. Without loss of generality, we take T to be the set of diagonal matrices in $\mathrm{SL}_4(F)$. We may assume $s = \mathrm{diag}(a, a, b, c)$ for some $a, b, c \in F$ such that $a^2bc = 1$. Fix the base of Φ such that $\alpha_i(\mathrm{diag}(a_1, a_2, a_3, a_4)) = a_i a_{i+1}^{-1}$ for $1 \leq i \leq 3$; in particular $\alpha_1(s) = 1$. It is clear that if $a^2b^2 \neq 1$ then s has almost simple spectrum on V_{ω_1} and on V_{ω_3}. If $\omega = \omega_2$, then the matrix of s on V is conjugate to $s_1 = \mathrm{diag}(a^2, a^{-2}, ab, ab, (ab)^{-1}, (ab)^{-1})$, so the spectrum of s is almost simple only if $b = \pm a^{-1}$ and $a^4 \neq 1$, and the result easily follows.

Now consider the general case, where $\omega \notin \{\omega_1, \omega_2, \omega_3\}$. Assume s has almost simple spectrum on V_ω. Factor s as

$$s = \mathrm{diag}(a\gamma, a\gamma, a^{-2}\gamma^{-2}, 1) \cdot \mathrm{diag}(\gamma^{-1}, \gamma^{-1}, \gamma^{-1}, c), \qquad \square$$

where $\gamma, c \in F^\times$ with $\gamma^3 = c$. Then viewing s as lying in the maximal parabolic $P = LQ$, $Q = R_u(P)$, corresponding to the root α_3, we see that the second factor acts as a scalar on the fixed point space V_ω^Q. Hence the eigenvalue multiplicities of s on this fixed point space are determined by those of the first factor. We now apply Lemma 13 to the element $h = \mathrm{diag}(\gamma a, \gamma a, (\gamma a)^{-2})$ and the weight $\omega \downarrow L'$, which is the highest weight of the irreducible L'-module V_ω^Q. In addition, we apply Lemma 13 to $(V_\omega^*)^Q$. By Lemma 13, the only p-restricted irreducible representations of $\mathrm{SL}_3(F)$ on which h has an almost simple spectrum are the natural representation and its dual. Writing $\omega = m_1\omega_1 + m_2\omega_2 + m_3\omega_3$, we deduce that $(m_1, m_2), (m_2, m_3) \in \{(0,0), (1,0), (0,1)\}$. We are therefore reduced to considering the case $\omega = \omega_1 + \omega_3$, (a quotient of) the adjoint representation. The multiplicity of the weight 0 is at least 2 and $\alpha_1(s) = 1$. Therefore, $(\alpha_1 + \alpha_2)(s) = \alpha_2(s)$, so $\alpha_2(s) = 1$ as well. But then $(\alpha_2 + \alpha_3)(s) = \alpha_3(s) \neq 1$, as s is non-central; hence s is not almost cyclic on $V_{\omega_1+\omega_3}$.

Lemma 15 – *Let $G = C_2$, $p = 2$, and let ω be a non-zero 2-restricted dominant weight. Let $s \in T$ be a non-regular element. Suppose that the spectrum of s on V_ω is almost simple. Then $\omega \in \{\omega_1, \omega_2\}$ and the spectrum of s is almost simple on precisely one of the modules V_{ω_1} and V_{ω_2}. Assume moreover that T is the torus of diagonal matrices in the group $\mathrm{Sp}_4(F)$, written with respect to a fixed symplectic basis (e_1, e_2, f_2, f_1) of the natural module V_{ω_1}. Let $g \in T$ be non-regular. If the spectrum of g on V_{ω_1} is almost simple then, up to conjugacy, $\varepsilon_1(g) = a$, $\varepsilon_2(g) = 1$ for $1 \neq a \in F^\times$; if the spectrum of g on V_{ω_2} is almost simple then, up to conjugacy $\varepsilon_1(g) = \varepsilon_2(g) = a$ for $1 \neq a \in F^\times$.*

Proof. As ω is 2-restricted, if $\omega \notin \{\omega_1, \omega_2\}$ then $\omega = \omega_1 + \omega_2$, and Steinberg (2016, §12, Corollary of Theorem 41) implies that $V_\omega = V_{\omega_1} \otimes V_{\omega_2}$. By Lemma 3, the spectrum of s is simple on V_{ω_1}, and hence s is regular, contradicting our hypothesis. One easily verifies the validity of the additional assertions. $\square$

Lemma 16 – *Let $G = C_2$, $p \neq 2$, and fix an ordered symplectic basis (e_1, e_2, f_2, f_1) of the natural module of G and let T be the torus of diagonal matrices of G in the natural representation. Let $s \in T \setminus Z(G)$ be a non-regular element and let $V_\omega \in \mathrm{Irr}(G)$ be a non-trivial p-restricted G-module. Then s has almost simple spectrum on V_ω if and only if one of the following holds:*

(i) $\omega = \omega_1$, and up to conjugacy, $\varepsilon_1(s) = 1$, $\varepsilon_2(s) = a$ or $\varepsilon_1(s) = -1$, $\varepsilon_2(s) = a$, where $a \in F^\times$, $a^2 \neq 1$;

(ii) $\omega = \omega_2$, and up to conjugacy, $\varepsilon_1(s) = 1$, $\varepsilon_2(s) = -1$ or $\varepsilon_1(s) = \varepsilon_2(s) = a$, where $a \in F^\times$, $a^2 \neq \pm 1$.

Proof. Let $\varepsilon_1(s) = b$, $\varepsilon_2(s) = a$, that is $s = \mathrm{diag}(b, a, a^{-1}, b^{-1})$.

We first consider $\omega = \omega_1$, so $\Omega(V_\omega) = \{\pm\varepsilon_1, \pm\varepsilon_2\}$. Since s is non-regular, we may assume that either $a = b$ or $b^2 = 1$. In the first case, s does not have almost simple spectrum on V_ω, while in the second case s has almost simple spectrum on V_ω if and only if $a^2 \neq 1$.

We now turn to the cases $\omega \neq \omega_1$. By Remark 3(1), we are left with the exceptional cases described in Lemma 9, $s_1 = \mathrm{diag}(a, a, a^{-1}, a^{-1})$ with $a^2 \neq 1$, or $s_2 = \pm\mathrm{diag}(1, -1, -1, 1)$. Note that $\alpha_1(s_1) = 1$ and $\alpha_2(s_2) = 1$. By Lemma 12, $\Lambda_1 = \{0, \omega_1\}$, and $\Lambda_2 = \{\omega_1 + \omega_2, \omega_2\}$ and $2\omega_1$ is the only radical weight in Λ_3. We consider these weights in turn, before turning to the general case.

The weights of V_{ω_2} are $0, \pm\varepsilon_1 \pm \varepsilon_2$. The remarks of the preceding paragraph imply that the cases in the statement are the only possible ones, and they yield the matrices of s_1, s_2 on V_{ω_2} (with respect to a suitable basis) $\mathrm{diag}(a^2, 1, 1, 1, a^{-2})$ and $\mathrm{diag}(-1, -1, 1, -1, -1)$, respectively.

Suppose $\omega = \omega_1 + \omega_2$. Then $\Omega(V_\omega) = \Omega(V_{\omega_1} \otimes V_{\omega_2})$, by Lemma 4. In terms of Bourbaki weights, the weights in $\Omega(V_\omega)$ are $\pm\varepsilon_1 + (\pm\varepsilon_1 \pm \varepsilon_2)$, $\pm\varepsilon_2 + (\pm\varepsilon_1 \pm \varepsilon_2)$, $\pm\varepsilon_1$, and $\pm\varepsilon_2$. Then $(\pm\varepsilon_1 + (\pm\varepsilon_1 \pm \varepsilon_2))(s_2) = -1$, $(\pm\varepsilon_2 + (\pm\varepsilon_1 \pm \varepsilon_2)(s_2) = 1$, so the spectrum of s_2 on V_ω is not almost simple. Furthermore, $(\varepsilon_1 + (-\varepsilon_1 + \varepsilon_2))(s_1) = a = (\varepsilon_2 + (\varepsilon_1 - \varepsilon_2))(s_1)$ and $(-\varepsilon_1 + (\varepsilon_1 - \varepsilon_2))(s_1) = a^{-1} = (-\varepsilon_2 + (-\varepsilon_1 + \varepsilon_2))(s_1)$. So the spectrum of s_1 on V_ω is not almost simple.

Finally, suppose $\omega = 2\omega_1$. Then by Lemma 4, the weights of V_ω are the same as those of $V_{\omega_1} \otimes V_{\omega_1}$. These are $\pm\varepsilon_i \pm \varepsilon_j$, for $i, j \in \{1, 2\}$. But now it is easy to see that neither s_1 nor s_2 has almost simple spectrum on V_ω.

We now turn to the general case and suppose that ω differs from the weights examined above. Then $\omega \notin \Lambda_1 \cup \Lambda_2$ and $\omega \neq 2\omega_1$. Recall that if $\mu \in \Lambda_i$ for some i then V_μ has a weight from Λ_j for every $j = 1, \ldots, i-1$ (Lemma 11). Then Lemma 12 implies that either $2\omega_1$ or $\omega_1 + \omega_2$ is a weight of V_ω and by Lemma 4, the weights of $V_{2\omega_1}$ or $V_{\omega_1+\omega_2}$ are weights of V_ω. The above considerations of $V_{\omega_1+\omega_2}$ and $V_{2\omega_1}$ show then that, given $s = s_1$ or s_2, there are 4 distinct weights $\lambda_1, \lambda_2, \nu_1, \nu_2$ in $\Omega(V_\omega)$ such that $\lambda_1(s) = \lambda_2(s) \neq \nu_1(s) = \nu_2(s)$. So s is not almost cyclic on V_ω, which completes the proof of the result. $\square$

Lemma 17 – *Theorem 2 is true for G of type G_2.*

Proof. Let G be of type G_2, and let V be a non-trivial G-module and $1 \neq s \in T$ a non-regular element. We have to show that the spectrum of s on V is not almost simple. Let ω be the highest weight of V. Suppose first that $\omega = \omega_1$ or $p = 3, \omega = \omega_2$, so $\dim V = 7$, or 6 for $p = 2$. The group G contains a maximal rank closed subgroup H isomorphic to A_2 such that the restriction of V_{ω_1} to H has composition factors the natural module for $\mathrm{SL}_3(F)$, and its dual and, if $p \neq 2$, an additional trivial summand. So the matrix of s on V_{ω_1} can be written as $\mathrm{diag}(a, b, c, 1, a^{-1}, b^{-1}, c^{-1})$ if $p \neq 2$, otherwise $\mathrm{diag}(a, b, c, a^{-1}, b^{-1}, c^{-1})$, where $abc = 1$ in both cases. This is also true if $p = 3$ and $V = V_{\omega_2}$. If all the entries are distinct, this matrix is a regular element in $\mathrm{SL}(V)$, and hence in G, contrary to the assumption.

Suppose that the entries are not distinct. As any permutation of a, b, c can be realized by an inner automorphism of G, we may assume that a equals some other diagonal entry and by the same reasoning, we may ignore the possibilities $a = c$ and $a = c^{-1}$. So we examine the cases $a = b$, $a = a^{-1}$, and $a = b^{-1}$.

Let $a = b$. Then s has almost simple spectrum on V_ω only if $a = a^{-1}$. But then $c = 1$ and s is not almost cyclic on V_ω.

Let $a = a^{-1} \neq b$, so $a = \pm 1$, $c = \pm b^{-1}$. If $a = 1$, then $b \neq 1$, s acts on V_ω as $\hat{s} = \mathrm{diag}(1, b, b^{-1}, 1, 1, b^{-1}, b)$ (where we drop the 1 in the middle if $p = 2$) which does not have almost simple spectrum. If $a = -1$ then $p \neq 2$ and s acts on V_ω as $\hat{s} = \mathrm{diag}(-1, b, -b^{-1}, 1, -1, b^{-1}, -b)$. If $b = \pm 1$ then the spectrum of $\hat{s}$ is not almost simple. Let $b \neq \pm 1$. As V is an orthogonal space, s is a regular element of $\mathrm{SO}(V)$ (Lemma 6), and hence in G, contrary to the assumption.

Let $a = b^{-1}$. Then $c = 1$. By reordering a, c, we arrive at the case $a = 1$, considered above. This completes the analysis of the cases $\omega = \omega_1$, and $(\omega, p) = (\omega_2, 3)$.

Suppose now that ω is an arbitrary p-restricted weight. If $p \neq 2, 3$ then the weights of V_{ω_1} occur as weights of V (Lemma 4), so the result follows from that for V_{ω_1}. Let $p = 2$; now $0, \omega_1, \omega_2, \omega_1 + \omega_2$ are the only 2-restricted dominant weights of G. By A. E. Zalesski (2009, Theorem 15), the weights of V_ω are the same as in characteristic 0, in particular all weights of V_{ω_1} are weights of V_ω, and we conclude as above.

Now turn to the case $p = 3$ and ω still p-restricted. By A. E. Zalesski (2009, Theorem 15), if $\omega \neq 2\omega_2$ then the weights of V_ω are the same as in characteristic 0, and in particular all weights of V_{ω_1} are weights of V_ω. So the result follows as above. For $p = 3$ and $\omega = 2\omega_2$, we use the tables of Lübeck (2018) to see that the weights of V_{ω_2} are weights of $V_{2\omega_2}$, and then conclude as before.

Finally, suppose that ω is not p-restricted. By Remark 1, we may assume that V is tensor-decomposable, say, $V = V_1 \otimes V_2$, where the highest weight of V_1 is of the form $p^k\omega'$ for some k. Then the result follows by Lemma 3.

4.2 Groups B_n with $n > 2$, D_n with $n > 3$, and C_n with $p = 2$ and $n > 2$

In this section, we consider the groups as indicated in the heading of the section. Recall that when $G = B_n$, we may assume $p \neq 2$. Note that for groups G of type B_n and of type D_n, the multiplicity of the 0 weight in the adjoint representation V_{ω_2} is greater than 1. Therefore, if ω is a dominant weight such that $\omega_2 \prec \omega$ then, by Lemma 5(2), it suffices to observe that a non-central non-regular semisimple element $s \in G$ is not of almost simple spectrum on V_{ω_2}. This is done in Lemma 18 below. The condition $\omega_2 \prec \omega$ holds provided ω is a radical weight and $\omega \neq 0, \omega_1, \omega_2$ for G of type B_n, and $\omega \neq 0, \omega_2$ for G of type D_n.

Lemma 18 – *Let $G = B_n$, $n > 2$, $p \neq 2$, $\omega \in \{\omega_2, \omega_n\}$ or $G = D_n$, $n > 3$, $\omega \in \{\omega_2, \omega_{n-1}, \omega_n\}$. Let $s \in T \setminus Z(G)$ be a non-regular element. Then the spectrum of s on V_ω is not almost simple, unless $G = D_4$, $\omega \in \{\omega_3, \omega_4\}$.*

Proof. Here we take T to be the preimage in G of the set of diagonal matrices in the image of G under the natural representation. We take $s \in T$ and assume the spectrum of s on V_ω is almost simple. Since s is not regular, there exists a root α with respect to T such that $\alpha(s) = 1$. We will assume without loss of generality that either $\alpha = \alpha_1$, or $G = B_n$ and $\alpha = \alpha_n$.

Suppose first that $\omega = \omega_2$. Set $R_0 = \{\alpha \in \Phi \mid \alpha(s) = 1\}$. Since s is non-central, there exists $\beta \in \Phi \setminus R_0$. Moreover, since Φ is an irreducible root system, there exists $\beta \in \Phi \setminus R_0$ which is not orthogonal to R_0. So for some $\alpha \in R_0$, $w_\alpha(\beta) \neq \beta$. Then $\beta(s) = w_\alpha(\beta)(s) \neq 1$, while $\alpha(s) = -\alpha(s) = 1$. So s is not almost cyclic on ω_2.

Let $\omega \in \{\omega_{n-1}, \omega_n\}$, for $G = D_n$ and $n > 4$, or $\omega = \omega_n$ for $G = B_n$. Then $\mu = \frac{1}{2}(\alpha_1 + \nu)$ is a weight of V_ω, for $\nu \in \{\pm\varepsilon_3 \pm \cdots \pm \varepsilon_n\}$, with certain conditions on the parity of the number of minus signs in the D_n-case. Suppose that $\alpha = \alpha_1$. Then $\mu - \alpha_1$ is a weight of V_ω for any admissible choice of the signs. As the spectrum of s on V_ω is almost simple, we deduce that $\mu(s)$ does not depend on the choice of ν and so $\varepsilon_3(s) = \cdots = \varepsilon_n(s) = 1$. Similarly, this then implies that $(\frac{1}{2}(\varepsilon_1 + \varepsilon_2 + \nu))(s)$ does not depend on the choice of ν, so again this value must be equal to $(\frac{1}{2}(\pm(\varepsilon_1 - \varepsilon_2) + \nu))(s)$, whence $\varepsilon_1(s) = 1 = \varepsilon_2(s)$ as well. This implies $s \in Z(G)$, a contradiction.

Finally, suppose that $G = B_n$, $\omega = \omega_n$ and $\alpha = \alpha_n$. Then for all $1 \leq i \leq n-1$, we have the two distinct weights of V_ω, $\omega - \alpha_i - \alpha_{i+1} - \cdots - \alpha_{n-1} - \alpha_n$ and $\omega - \alpha_i - \alpha_{i+1} - \cdots - \alpha_{n-1} - 2\alpha_n$, taking the same value on s, and therefore deduce that $\alpha_i(s) = 1$ for all i, again contradicting the fact that s is non-central.

Remark 4 – If $G = D_4$ then there exist non-central non-regular semisimple elements s with almost simple spectrum on V_{ω_3} or V_{ω_4}. Indeed, one easily observes that there are non-regular elements $s \in T \setminus Z(G)$ whose spectrum is almost simple on V_{ω_1}. Let σ be the triality automorphism of G. Then $\sigma(s)$ has almost simple spectrum on $V_{\omega_1}^\sigma$, whence the claim.

Lemma 19 – *Let $G = B_n$, $n > 2$, $p \neq 2$, or $G = D_n$, $n \geq 4$, and let $V_\omega \in \mathrm{Irr}(G)$, where $\omega \neq 0$ is p-restricted. Let $s \in T \setminus Z(G)$ be a non-regular element with almost simple spectrum on V_ω. Then either $\omega = \omega_1$ or $G = D_4$ and $\omega \in \{\omega_1, \omega_3, \omega_4\}$.*

Proof. If ω is radical, this follows from Lemmas 18, 5(2) and 4, both for B_n and D_n.

Suppose that ω is not radical. If $G = B_n$ then $\omega_n \preceq \omega$ by Lemma 4(2), so again the result follows from Lemmas 18 and 5(1)(i). Let $G = D_n$. By Theorem 5, all non-zero weights of V_ω are of multiplicity 1. Then, by D. M. Testerman and A. E. Zalesski (2015, Tables 1, 2), $\omega \in \{\omega_1, 2\omega_1, \omega_2, \omega_{n-1}, \omega_n\}$, where the radical weights $2\omega_1, \omega_2$ are to be dropped. Whence the result for $n = 4$. If $n > 4$ then the spectrum of s on V_ω is not almost simple by Lemma 18. □

We now handle the case $G = C_n$, for $n > 2$ and $p = 2$, which is excluded in Proposition 2. Moreover, we only need to consider V_{ω_n} (see Proposition 2).

Lemma 20 – *Let $G = C_n$, $n > 2$, $p = 2$. Let $1 \neq s \in T$ be a non-regular element. Then the spectrum of s on V_{ω_n} is not almost simple.*

Proof. We argue as in the proof of Lemma 18. We can assume that $\alpha(s) = 1$ for $\alpha = \alpha_1$ or $\alpha = 2\varepsilon_1$. The weights of V_{ω_n} are $\pm\varepsilon_1 \pm \cdots \pm \varepsilon_n$. Then $\mu := \varepsilon_1 \pm \varepsilon_2 + \nu$ are weights of V_{ω_n} for any $\nu = \pm\varepsilon_3 \pm \cdots \pm \varepsilon_n$. If $\alpha = 2\varepsilon_1$ then $\mu - \alpha$ is a weight of V_{ω_n}, and we conclude (as in the proof of Lemma 18) that $(2\varepsilon_1)(s) = \cdots = (2\varepsilon_n)(s)$. As $p = 2$, we have $\varepsilon_1(s) = \cdots = \varepsilon_1(s)$, whence $s \in Z(G) = 1$, a contradiction.

If $\alpha = \alpha_1$ then for $\mu = \varepsilon_1 - \varepsilon_2 + \nu$ we have $\mu - \alpha_1 \in \Omega(V_{\omega_n})$, whence $(2\varepsilon_3)(s) = \cdots = (2\varepsilon_n)(s) = 1$. This implies that $(\varepsilon_1 + \varepsilon_2 + \nu)(s)$ does not depend on ν, nor does $(\epsilon_1 - \epsilon_2 + \nu)(s)$, whence $(2\varepsilon_1)(s) = 1$, and again we conclude that $s = 1$.

4.3 Completion of the Proof of Theorem 2

Proof. Let G, s be as in the statement of Theorem 2. Note that $\mathrm{rank}(G) \geq 2$.

Suppose first that λ is p-restricted. The groups of rank 2 have been examined in Lemmas 13, 15, 16 and 17, and the group of type A_3 in Lemma 14. In Proposition 2, we handled the groups $A_n, n > 3$, F_4, E_6, E_7, E_8, and all p-restricted weights for the group C_n, $n > 2$, except the weight $\omega = \omega_n$ when $p = 2$. The latter is handled in Lemma 20.

Groups of type $B_n, n > 2$, and $p \neq 2$, and groups of type D_n are dealt with in Lemma 19.

By Remark 1, we may now assume that V is tensor-decomposable. Let $V = V_1 \otimes V_2$ be a non-trivial tensor decomposition of V. By Lemma 3, the spectra of s on V_1 and V_2 are simple. This contradicts Lemma 3. □

Finally, we conclude with a straightforward corollary of Theorem 2.

Corollary 1 – *Let $s \in T \setminus Z(G)$ be a non-regular element and V an irreducible G-module. Suppose that the spectrum of s on V is almost simple. Then the eigenvalue multiplicities of s on V do not exceed $m = m_V(s)$, where either $m \leq \mathrm{rank}(G)$ or one of the following holds:*

(1) $G = A_3$, $\dim V = 6$, $m = 4$;

(2) $G = B_n$, $n > 2$, $p \neq 2$, $\dim V = 2n+1$, $m = 2n$;

(3) $G = C_n$, *and either* $\dim V = 2n$ *and* $m = 2n-2$ *or* $n = 2$, $p \neq 2$, $\dim V = 5$ *and* $m = 4$;

(4) $G = D_n$, $n > 3$, $\dim V = 2n$, $m = 2n-2$.

Proof. This will follow from Theorem 2; we discuss each of the cases of the theorem. To get (1) above, we additionally use Lemma 14. For $G = C_2$, $p \neq 2$, we use Lemma 16. The modules of dimensions indicated in Theorem 2(1) are obtained by Frobenius twisting of V_{ω_1} (where the statement is clear); $m_V(s)$ remains unchanged under such a twist. This leaves us with $G = D_4$ and $\dim V = 8$. The modules V_{ω_1}, V_{ω_3}, V_{ω_4}, are obtained from each other by a graph automorphism of G, and the other modules of dimension 8 as in Theorem 2(4) are Frobenius twists of these. The result follows.

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References

Bourbaki, N. (1968). *Éléments de mathématique. Fasc. XXXIV. Groupes et algèbres de Lie. Chapitre IV: Groupes de Coxeter et Systèmes de Tits. Chapitre V: Groupes engendrés par des réflexions. Chapitre VI: systèmes de racines*. Actualités Scientifiques et Industrielles, No. 1337. Paris: Hermann, 288 pp. (loose errata) (cit. on pp. 4, 13, 18).

Bourbaki, N. (1975). *Éléments de mathématique. Fasc. XXXVIII: Groupes et algèbres de Lie. Chapitre VII: Sous-algèbres de Cartan, éléments réguliers. Chapitre VIII: Algèbres de Lie semi-simples déployées*. Actualités Sci. Indust., No. 1364. Hermann, Paris, p. 271 (cit. on pp. 4, 7).

Curtis, C. W. (1960). "Representations of Lie algebras of classical type with applications to linear groups". *J. Math. Mech.* **9**, pp. 307–326. DOI: 10.1512/iumj.1960.9.59018. URL: https://doi.org/10.1512/iumj.1960.9.59018 (cit. on p. 11).

Di Martino, L., M. A. Pellegrini, and A. E. Zalesski (2014). "On generators and representations of the sporadic simple groups". *Comm. Algebra* **42** (2), pp. 880–908. ISSN: 0092-7872. DOI: 10.1080/00927872.2012.729629. URL: https://doi.org/10.1080/00927872.2012.729629 (cit. on p. 2).

Di Martino, L., M. A. Pellegrini, and A. E. Zalesski (2020). "Almost cyclic elements in cross-characteristic representations of finite groups of Lie type". *J. Group Theory* **23** (2), pp. 235–285. ISSN: 1433-5883. DOI: 10.1515/jgth-2018-0162. URL: https://doi.org/10.1515/jgth-2018-0162 (cit. on p. 2).

Di Martino, L. and A. E. Zalesski (2018). "Almost cyclic elements in Weil representations of finite classical groups". *Comm. Algebra* **46** (7), pp. 2767–2810. ISSN: 0092-7872. DOI: 10.1080/00927872.2018.1435787. URL: https://doi.org/10.1080/00927872.2018.1435787 (cit. on p. 2).

Katz, N. M. and P. H. Tiep (2021). "Monodromy groups of Kloosterman and hypergeometric sheaves". *Geom. Funct. Anal.* **31** (3), pp. 562–662. ISSN: 1016-443X. DOI: 10.1007/s00039-021-00578-0. URL: https://doi.org/10.1007/s00039-021-00578-0 (cit. on p. 2).

Lübeck, F. (2018). *Online data for finite groups of Lie type and related algebraic groups*. URL: http://www.math.rwth-aachen.de/~Frank.Luebeck (visited on 08/2018) (cit. on p. 22).

Malle, G. and D. Testerman (2011). *Linear algebraic groups and finite groups of Lie type*. **133**. Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, pp. xiv+309. ISBN: 978-1-107-00854-0. DOI: 10.1017/CBO9780511994777. URL: https://doi.org/10.1017/CBO9780511994777 (cit. on pp. 4, 11, 12).

Premet, A. A. (1987). "Weights of infinitesimally irreducible representations of Chevalley groups over a field of prime characteristic". *Mat. Sb. (N.S.)* **133(175)** (2), pp. 167–183, 271. ISSN: 0368-8666 (cit. on p. 8).

Seitz, G. M. (1987). "The maximal subgroups of classical algebraic groups". *Mem. Amer. Math. Soc.* **67** (365), pp. iv+286. ISSN: 0065-9266 (cit. on p. 2).

Springer, T. A. and R. Steinberg (1970). "Conjugacy classes". In: *Seminar on Algebraic Groups and Related Finite Groups (The Institute for Advanced Study, Princeton, N.J., 1968/69)*. Lecture Notes in Mathematics, Vol. 131. Berlin: Springer, pp. 167–266 (cit. on pp. 3, 6).

Steinberg, R. (2016). *Lectures on Chevalley groups*. **66**. University Lecture Series. Notes prepared by John Faulkner and Robert Wilson, Revised and corrected edition of the 1968 original [MR0466335], With a foreword by Robert R. Snapp. American Mathematical Society, Providence, RI, pp. xi+160. ISBN: 978-1-4704-3105-1. DOI: 10.1090/ulect/066. URL: https://doi.org/10.1090/ulect/066 (cit. on p. 20).

Suprunenko, I. D. (2013). “Unipotent elements of nonprime order in representations of the classical algebraic groups: two big Jordan blocks”. *Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI)* **414** (Voprosy Teorii Predstavleniı Algebr i Grupp. 25), pp. 193–241. ISSN: 0373-2703. DOI: 10.1007/s10958-014-1863-6. URL: https://doi.org/10.1007/s10958-014-1863-6 (cit. on p. 2).

Suprunenko, I. D. and A. E. Zalesskii (2000). “Irreducible representations of finite exceptional groups of Lie type containing matrices with simple spectra”. *Comm. Algebra* **28** (4), pp. 1789–1833. ISSN: 0092-7872. DOI: 10.1080/00927870008826927. URL: https://doi.org/10.1080/00927870008826927 (cit. on p. 2).

Suprunenko, I. D. and A. E. Zalesski (2005). “On restricting representations of simple algebraic groups to semisimple subgroups with two simple components”. *Trudy Instituta Matematiki* **13** (2), pp. 109–115 (cit. on p. 8).

Suprunenko, I. D. and A. E. Zalesski (2007). “Fixed vectors for elements in modules for algebraic groups”. *Internat. J. Algebra Comput.* **17** (5-6), pp. 1249–1261. ISSN: 0218-1967. DOI: 10.1142/S0218196707004001. URL: https://doi.org/10.1142/S0218196707004001 (cit. on p. 17).

Suprunenko, I. D. and A. E. Zalesskii (1998). “Irreducible representations of finite classical groups containing matrices with simple spectra”. *Comm. Algebra* **26** (3), pp. 863–888. ISSN: 0092-7872. DOI: 10.1080/00927879808826170. URL: https://doi.org/10.1080/00927879808826170 (cit. on p. 2).

Testerman, D. M. and A. E. Zalesski (2015). “Subgroups of simple algebraic groups containing regular tori, and irreducible representations with multiplicity 1 non-zero weights”. *Transform. Groups* **20** (3), pp. 831–861. ISSN: 1083-4362. DOI: 10.1007/s00031-015-9321-1. URL: http://doi.org/10.1007/s00031-015-9321-1 (cit. on pp. 3, 24).

Testerman, D. M. and A. E. Zalesski (2018). “Irreducible representations of simple algebraic groups in which a unipotent element is represented by a matrix with a single non-trivial Jordan block”. *J. Group Theory* **21** (1), pp. 1–20. ISSN: 1433-5883. DOI: 10.1515/jgth-2017-0019. URL: https://doi.org/10.1515/jgth-2017-0019 (cit. on p. 2).

Wagner, A. (1978). “Collineation groups generated by homologies of order greater than 2”. *Geom. Dedicata* **7** (4), pp. 387–398. ISSN: 0046-5755. DOI: 10.1007/BF00152059. URL: https://doi.org/10.1007/BF00152059 (cit. on p. 2).

Wagner, A. (1981). “Determination of the finite primitive reflection groups over an arbitrary field of characteristic not two. II, III”. *Geom. Dedicata* **10** (1-4), pp. 191–203, 475–523. ISSN: 0046-5755. DOI: 10.1007/BF01447423. URL: https://doi.org/10.1007/BF01447423 (cit. on p. 2).

Zalesski, A. E. (2009). “On eigenvalues of group elements in representations of simple algebraic groups and finite Chevalley groups”. *Acta Appl. Math.* **108** (1), pp. 175–195. ISSN: 0167-8019. DOI: 10.1007/s10440-008-9373-5. URL: https://doi.org/10.1007/s10440-008-9373-5 (cit. on pp. 1, 11, 22).

Zalesskii, A. E. and V. N. Serežkin (1977). "Linear groups that are generated by pseudoreflections". *Vestsı Akad. Navuk BSSR Ser. Fız.-Mat. Navuk* (5), pp. 9–16, 137. ISSN: 0002-3574 (cit. on p. 2).

Zalesskii, A. E. and V. N. Serežkin (1980). "Finite linear groups generated by reflections". *Izv. Akad. Nauk SSSR Ser. Mat.* **44** (6), pp. 1279–1307, 38. ISSN: 0373-2436 (cit. on p. 2).

Zalesskii, A. E. and I. D. Suprunenko (1987). "Representations of dimension $(p^n \pm 1)/2$ of the symplectic group of degree $2n$ over a field of characteristic p". *Vestsı Akad. Navuk BSSR Ser. Fız.-Mat. Navuk* (6), pp. 9–15, 123. ISSN: 0002-3574 (cit. on pp. 2, 12).

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