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# Hyperbolic representations of PU(1,n)

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A mi familia y amigos a ambos lados del mar.

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# Abstract

In this thesis we study the isometric representations of the groups $\mathrm{PU}(1,n)$ in the infinite-dimensional hyperbolic spaces. These spaces and their isometry groups are described.

Invariants for hyperbolic representations of such groups are introduced and in terms of them partial results about the classification of such representations are obtained.

In the case of $\mathrm{PU}(1,1)$ a method to obtain new representations from the known ones is developed. With it, a family of representations is described which has not been described before.

# Résumé

Dans cette thèse, nous étudions les représentations isométriques des groupes $\mathrm{PU}(1,n)$ dans les espaces hyperboliques de dimension infinie. Ces espaces et leurs groupes d'isométrie sont décrits.

Des invariants pour les représentations hyperboliques de ces groupes sont introduits et des résultats partiels sur la classification de ces représentations sont obtenus en fonction de ces invariants.

Dans le cas de $\mathrm{PU}(1,1)$, une méthode permettant d'obtenir de nouvelles représentations à partir des représentations connues est développée. Cette méthode permet de décrire une famille de représentations qui n'a jamais été décrite auparavant.

# Contents

# Introduction

This work investigates the problem of hyperbolic representations of isometry groups of finite-dimensional complex hyperbolic spaces. Representations of one semi-simple Lie group into another have no mysteries: by the Karpelevich–Mostow theorem (see [43], or for a proof in the hyperbolic case, see [5]), they are all "standard" in the sense that they correspond to totally geodesic or trivial embeddings of the corresponding symmetric spaces of the simple factors.

The situation changes radically for representations on the infinite-dimensional spaces. There are "exotic" representations that do not correspond to totally geodesic embeddings (see for example [41]).

The infinite-dimensional hyperbolic spaces can be understood as limits of nested finite-dimensional ones. Indeed, if $\mathbf{F} = \mathbf{R}, \mathbf{C}$ and $H$ is a Hilbert space over $\mathbf{F}$, define the Hermitian form $B$ on $\mathbf{F} \oplus V$, given by

$$B(a \oplus u, b \oplus v) = a\overline{b} - \langle u, v \rangle.$$

If $\kappa$ is the dimension of $H$, the set of positive $\mathbf{F}$-lines (with respect to $B$) in $\mathbf{F} \oplus H$ is defined as the $\kappa$-dimensional $\mathbf{F}$-hyperbolic space $\mathbf{H}^{\kappa}_{\mathbf{F}}$. If the Hilbert space $H$ is considered finite-dimensional, this definition recovers the classical definition of the finite-dimensional hyperbolic spaces.

The group $\mathrm{O}_{\mathbf{F}}(B)$ of invertible $\mathbf{F}$-linear transformations of $\mathbf{F} \oplus H$ preserving the form $B$ acts through isometries on $\mathbf{H}^{\kappa}_{\mathbf{F}}$. In the case $F = \mathbf{R}$, the group $\mathrm{O}_{\mathbf{F}}(B)$ induces through projectivization the entire isometry group of $\mathbf{H}^{\kappa}_{\mathbf{F}}$. For $\mathbf{F} = \mathbf{C}$, the group induced has index 2. If $\mathbf{F} = \mathbf{R}$, the group of isometries induced by $\mathrm{O}_{\mathbf{F}}(B)$ is denoted $\mathrm{PO}(1, \kappa)$ and if $\mathbf{F} = \mathbf{C}$, the group is denoted $\mathrm{PU}(1, \kappa)$.

The object of study of this work, therefore, is homomorphisms $\mathrm{PU}(1, n) \to \mathrm{PO}(B)^{\kappa}_{\mathbf{F}}$, for every $n \geqslant 1$. As mentioned above, the cases that are of interest or that have not already been classified are the cases when in the target the groups considered are $\mathrm{PO}(1, \infty)$ or $\mathrm{PU}(1, \infty)$.

The study of hyperbolic representations in general is interesting because it lies at an intersection between the theory of Lie groups, the theory of hyperbolic lattices, geometric group theory, theory of CAT(-1) metric spaces as well as the classical theories of unitary, affine and projective representations of groups.

It could be said, in particular with respect to the latter two theories, that hyperbolic representations emerged alongside unitary representations in the context of Pontryagin spaces. Moreover, through horospherical representations, hyperbolic representations can be considered a generalization of affine representations, and therefore also of unitary representations.

Even if this subject has been studied from many perspectives for some time, a systematic study for a general group is very recent and it is still to be developed. However, important progress has been made and classification results have been obtained for the cases of irreducible representations $\mathrm{PO}(1,n) \to \mathrm{PO}(1,\infty)$ (Monod & Py [41]), $\mathrm{PO}(1,\infty) \to \mathrm{PO}(1,\infty)$ (Monod & Py [42]) and $\mathrm{Isom}(T_n) \to \mathrm{PO}(1,\infty)$, where $T_n$ is a homogeneous tree of degree $n \geqslant 3$ (Burger, Iozzi & Monod [9]).

The three classification results mentioned above have a common characteristic, the classification is given by the *displacement*. Given a metric space $X$ and an isometry $g$, the *displacement* of $g$ is defined as

$$\ell(g) = \inf_{x \in X}\{d(gx, x)\}.$$

If $G$ is $\mathrm{PO}(1,n)$, $\mathrm{PO}(1,\infty)$ or $\mathrm{Isom}(T_n)$, for any (irreducible) representation $G \xrightarrow{\rho} \mathrm{PO}(1,\infty)$, there exists $t > 0$ such that for every $g \in G$, $\ell(\rho(g)) = t\ell(g)$. Through this text, the parameter $t$ is called the *displacement* of $\rho$ and is denoted $\ell(\rho)$.

Specifically, in all three cases, the respective authors showed that two irreducible representations are equivalent if, and only if, they have the same displacement. They also showed that for a representation $\rho$ of $G$, $\ell(\rho) \in (0,1)$, if $G$ is equal to $\mathrm{PO}(1,n)$ or $\mathrm{PO}(1,\infty)$, and $\ell(\rho) \in (0,\infty)$, if $G = \mathrm{Isom}(T_n)$. Moreover, every $t$ in the aforementioned intervals is realized as the displacement of a representation.

The classification of representations $\mathrm{PO}(1,n) \to \mathrm{PO}(1,\infty)$ will be of relevance for this work. Due to the fact that the continuous cohomology group in degree 2 of $\mathrm{PO}(1,n)$ is trivial, for every $n > 3$ the same classification is valid If $\mathrm{PU}(1,\infty)$ is considered instead of $\mathrm{PO}(1,\infty)$. This classification, the statement of which is the same when considering the group $\mathrm{PO}(1,n)_o$ instead of $\mathrm{PO}(1,n)$, will be used when studying representations of the group $\mathrm{PU}(1,1)$ because $\mathrm{PU}(1,1)$ and $\mathrm{PO}(1,2)_o$ are isomorphic.

One of the questions that this thesis addresses is whether this same behavior occurs for representations $\mathrm{PU}(1,n) \to \mathrm{PU}(1,\infty)$. The answer for $n=1$ is negative. There are non-equivalent irreducible representations with the same displacement (see Theorem 3.3.1).

This leads to mention another invariant for representations $\mathrm{PU}(1,n) \to \mathrm{PU}(1,\infty)$. Given three points $x,y,z \in \mathbf{H}^{\kappa}_{\mathbf{C}}$, with $\kappa$ finite or infinite, the *Cartan argument* of the triple $(x,y,z)$ is defined as follows. Recall that $\mathbf{H}^{\kappa}_{\mathbf{C}}$ is the set of positive complex lines in $\mathbf{C} \oplus H$ with respect to the form $B$. The Cartan argument of $(x,y,z)$ is defined as

$$\mathrm{Arg}\big(B(\tilde{x},\tilde{y})B(\tilde{y},\tilde{z})B(\tilde{z},\tilde{x})\big) \in \left(-\tfrac{\pi}{2},\tfrac{\pi}{2}\right),$$

for any representatives $\tilde{x},\tilde{y},\tilde{z}$, and is denoted $\mathrm{Cart}(x,y,z)$.

If $\mathrm{PU}(1,n) \xrightarrow{\rho} \mathrm{PU}(1,\infty)$ is an irreducible representation and $x \in \mathbf{H}^n_{\mathbf{C}}$, there exists a unique $\mathrm{PU}(1,n)$-equivariant map $\mathbf{H}^n_{\mathbf{C}} \xrightarrow{f} \mathbf{H}^{\infty}_{\mathbf{C}}$ (see Proposition 5.8 in [40]). If $x \in \mathbf{H}^n_{\mathbf{C}}$, there exists $s \in [0,1]$ such that for every $g_1,g_2,g_3 \in \mathrm{PU}(1,n)$,

$$\mathrm{Cart}\big(f(g_1x), f(g_2x), f(g_3x)\big) = s\mathrm{Cart}\big(g_1x, g_2x, g_3x\big).$$

The scalar $s$ does not depend on the choice of $x$ and the *argument* of $\rho$ is defined as $\mathrm{Arg}(\rho) = \frac{s\pi}{2}$ (see Remarks 1.3.15 and 3.1.6).

Constructing irreducible representations has proven to be a difficult task. In [40] Monod developed a fruitful method for generating representations from existing ones. In the spirit of the relationship between functions of positive type and cyclic unitary representations of a given group, in the aforementioned paper the author defined the *functions of complex hyperbolic type*.

A pair $(\beta,\alpha)$ is a *function of complex hyperbolic type* defined on a group $G$ if, and only if, there exists a representation $G \xrightarrow{\rho} \mathrm{Isom}(\mathbf{H}^{\kappa}_{\mathbf{C}})_o$ and $x \in \mathbf{H}^{\kappa}_{\mathbf{C}}$ with a total orbit such that for $g,g_1,g_2,g_3 \in G$, $\beta(g) = \cosh(d(\rho(g)x,x)$ and

$$\alpha(g_1,g_2,g_3) = \mathrm{Cart}\big(\rho(g_1)x, \rho(g_2)x, \rho(g_3)x\big).$$

Monod also showed that if $t \in (0,1)$ and $(\beta,\alpha)$ is a function of complex hyperbolic type, then $(\beta^t, t\alpha)$ is a function of complex hyperbolic type. This is so far the only general method for constructing new representations. In fact, with this method it is possible to construct all irreducible representations of $\mathrm{PO}(1,n)$.

The group $\mathrm{PU}(1,n)$ acts on $\mathbf{H}^n_{\mathbf{C}}$, hence there exists $(\beta,\alpha)$ a "tautological" function of complex hyperbolic type coming from that action. Using the "exponentiation" of functions of complex hyperbolic type, Monod showed the existence, for every $t \in (0,1)$, of irreducible representation $\mathrm{PU}(1,n) \xrightarrow{\rho_t} \mathrm{PU}(1,\infty)$ such that

$$(\ell(\rho_t),\mathrm{Arg}(\rho_t)) = (t, \tfrac{t\pi}{2}).$$

This indicates that the behaviour of the representations of $\mathrm{PO}(1,n)$ and of $\mathrm{PU}(1,n)$ are different. In fact, since for $n > 2$ the continuous cohomology in degree 2 of $\mathrm{PO}(1,n)$ is zero, for every irreducible representation $\mathrm{PO}(1,n) \xrightarrow{\rho} \mathrm{PU}(1,\infty)$, $\mathrm{Arg}(\rho) = 0$. The following is Theorem 4.1.8.

**Theorem** *For $n > 1$, every irreducible representation $\mathrm{PU}(1,n) \xrightarrow{\rho} \mathrm{PU}(1,\infty)$ is such that* $\mathrm{Arg}(\rho) \neq 0$.

Things change if $n = 1$ is consider. As the group $\mathrm{PU}(1,1)$ is isomorphic to $\mathrm{PO}(1,2)_o$, there are two families of irreducible representations of different nature. On the one hand, the representations $\mathrm{PU}(1,1) \xrightarrow{\rho_t} \mathrm{PU}(1,\infty)$ coming from the "exponentiation" described before, and on the other hand, the representations $\mathrm{PU}(1,1) \xrightarrow{\tau_t} \mathrm{PU}(1,\infty)$ coming from the classification of irreducible representations $\mathrm{PO}(1,2) \to \mathrm{PO}(1,\infty)$. These two families are such that

$$(\ell(\rho_t),\mathrm{Arg}(\rho_t)) = (t, \tfrac{t\pi}{2})$$

and

$$(\ell(\tau_t),\mathrm{Arg}(\tau_t)) = (t, 0).$$

It is clear that $\rho_t$ and $\tau_t$ are non-equivalent because they have different argument.

In that direction, using the theory of functions of complex hyperbolic type, Theorem 3.1.11 is proved.

**Theorem** *Two irreducible representations* $\mathrm{PU}(1,1) \xrightarrow{\rho_i} \mathrm{PU}(1,\infty)$ *are equivalent if, and only if,*

$$(\ell(\rho_1),\mathrm{Arg}(\rho_1)) = (\ell(\rho_2),\mathrm{Arg}(\rho_2)).$$

The results described so far are enough to show that for the groups $\mathrm{PU}(1,n)$, with $n > 1$, $\mathrm{PO}(1,n)$, with $n > 2$, and $\mathrm{PU}(1,1)$ the theories of complex hyperbolic representations are completely different.

Furthermore, for $\mathrm{PU}(1,1)$ a phenomenon is observed that had not been described for the other groups mentioned. It had already been commented that the "exponentiation" was the only general method available to obtain new representations from existing ones. For $\mathrm{PU}(1,1)$ a new method, called the *horospherical combination*, is available. The following is Theorem 3.2.12.

**Theorem** *If $\varphi$ and $\psi$ are two irreducible representations of $\mathrm{PU}(1,1)$ in $\mathrm{PU}(1,\infty)$ with displacement $t$, for every $u \in [0,1]$, there exists an irreducible representation $\varphi \underset{u}{\wedge} \psi$ such that $\ell(\varphi \underset{u}{\wedge} \psi) = t$ and*

$$\mathrm{Arg}(\varphi \underset{u}{\wedge} \psi) = (1-u)\mathrm{Arg}(\varphi) + u\mathrm{Arg}(\psi).$$

With the horospherical combination it is possible to produce from the representations $\rho_t$ and $\tau_t$ described before, a continuum of non-equivalent representations all of them with the same displacement.

Although a classification has not been achieved for the representations of $\mathrm{PU}(1,n)$, it is clear that the complex representations of these groups present a behaviour that differs radically from the representations of $\mathrm{PO}(1,n)$.

The text is organized as follows. The Chapter 1 consists of preliminaries. Many of the arguments used through the text work in a CAT(-1) generality, for this reason this spaces are briefly reviewed. In this chapter the functions of complex hyperbolic type are addressed and some results about them are presented.

In Chapter 2 the restriction of a representation to the stabilizer of a point at infinity is analyzed. This technique, that was used in [41] and can be tracked back to [9], is fundamental for the study of the invariants defined before and in the process of constructing the horospherical combination in the case $n = 1$.

Chapter 3 focuses on the representations of PU(1,1). It is shown that the argument and displacement of a representation is a complete invariant. The horospherical combination is introduced and with it a new family of representations is described.

Chapter 4 deals with non-elementary representations of $\mathrm{PU}(1,n)$, with $n > 1$. It is shown that every representation of $\mathrm{PU}(1,n)$ has non-zero argument.

# Chapter 1

# Preliminaries

This chapter will present the preliminaries necessary for the rest of this work. None of the results presented in this chapter is original and if any proof is given it is due to the fact that the references in the literature are either unknown to the author or because in the specific context of this work, such proofs can be simplified. Nevertheless, this chapter is intended to be presented in a coherent way and not strictly as a list of results to be used.

In Section 1.1 some general results about CAT(-1) spaces are described placing special emphasis on its visual boundary. In Section 1.2 groups of isometries of CAT(-1) spaces are considered, as well as isometric representations of groups in CAT(-1) spaces. In this section, particular attention is paid to the elementary and non-elementary cases.

In Section 1.3 hyperbolic spaces of any dimension are introduced with emphasis in the non-locally compact setting. Isometry groups of the hyperbolic spaces and especially their description with linear transformations are addressed.

Section 1.4 will briefly outline the functions of complex hyperbolic type introduced by Monod in [40]. In Section 1.5 some applications of the theory of functions of complex hyperbolic type that will be used in the following chapters are described. As well as its implications with respect to the existence of “non-trivial" representations for PU(1,n).

## 1.1 CAT(-1) spaces

Let $X$ be a metric space. Given an interval $I \subset \mathbf{R}$, a curve $I \xrightarrow{\gamma} X$ is called a *geodesic* if for every $x, y \in I$, $d(x, y) = d(\gamma(x), \gamma(y))$. The geodesic $\gamma$ is called *complete* if $I = \mathbf{R}$. A metric space is called *geodesic* if for every two points there exists a geodesic (segment) connecting them.

Let $X$ be a geodesic metric space. For three points $x, y, z \in X$, a *geodesic triangle* with vertex $x, y, z$, denoted $\Delta(x, y, z)$, is the union of the images of three geodesic (segments) connecting the points $x, y, z$.

Let $\mathbf{H}^2_{\mathbf{R}}$ be the real hyperbolic plane (see Section 1.3). For any geodesic metric space $X$, three points $x_1, x_2, x_3 \in X$ and a geodesic triangle $\Delta(x_1, x_2, x_3)$, a *comparison triangle* in $\mathbf{H}^2_{\mathbf{R}}$ is a geodesic triangle $\Delta(\overline{x}_1, \overline{x}_2, \overline{x}_3) \subset \mathbf{H}^2_{\mathbf{R}}$ such that such that $d(x_i, x_j) = d(p(x_i), p(x_j))$. Observe that there exists a map

$$\triangle(x_1, x_2, x_3) \xrightarrow{p} \triangle(\overline{x}_1, \overline{x}_2, \overline{x}_3),$$

called *comparison map*, that restricted to the sides of the triangles is an isometry. Comparison triangles in $\mathbf{H}^2_{\mathbf{R}}$ always exist for any triple of points in any metric space (see Lemma I.2.4 [7]).

A geodesic metric space $X$ is called a CAT(-1) space if for every three points $x_1, x_2, x_3 \in X$, every comparison map

$$\triangle(x_1, x_2, x_3) \xrightarrow{p} \triangle(\overline{x}_1, \overline{x}_2, \overline{x}_3)$$

is such that for every $x, y \in \triangle(x_1, x_2, x_3)$, $d(x, y) \leqslant d(p(x), p(y))$.

Analogously, a geodesic space is called CAT(0) if the same property holds, but the comparison triangles are considered in $\mathbf{R}^2$. In this work CAT(-1) spaces will always be considered complete.

For a CAT(-1) space $X$, there exists a constant $\delta > 0$ such that every triangle in $X$ is $\delta$-*slim*. That is to say that given three points $x_1, x_2, x_3 \in X$, if $[x_i, x_j]$ is the geodesic segment connecting $x_i$ and $x_j$, then every $p \in [x_1, x_2]$ is at distance less than $\delta$ to some point in $[x_1, x_3] \cup [x_2, x_3]$ (see Proposition III.H.1.2 in [7]).

Let $X$ be a metric space. Given three points $x, y, z \in X$ define the *Gromov product* of $y$

and $z$ with respect to $x$ as,

$$(y,z)_x = \frac{1}{2}\left(d(y,x)+d(z,x)-d(y,z)\right).$$

A metric space $X$ is called *Gromov hyperbolic* if there exists $\delta > 0$ such that for every $w, x, y, z$,

$$(x,y)_z \geqslant \min\{(x,w)_z, (w,y)_z\} - \delta.$$

A sequence $(x_i)$ in $X$ is called a *Gromov sequence* if for $z_0$ a (any) base point,

$$\lim_{n,m\to\infty}(x_n, x_m)_{z_0} = \infty.$$

Two Gromov sequences, $(x_i)$ and $(y_i)$, are called *equivalent* if for $z_0$ a (any) base point,

$$\lim_{n,m\to\infty}(x_n, y_m)_{z_0} = \infty.$$

The relation defined above in the set of Gromov sequences is an equivalence relation. Denote $\partial_g X$ the set of equivalence classes of Gromov sequences in $X$. The set $\partial_g X$ will be called the *boundary at infinity* of X.

Every CAT(-1) space is Gromov hyperbolic (Proposition 3.3.4 in [20]), therefore there are two natural ways to define and topologize a boundary at infinity for a complete CAT(-1) space. The first one is considering $X$ as a Gromov hyperbolic space and taking $\partial_g X$. The second is considering $X$ as a CAT(0) space and defining the boundary at infinity as the set of equivalence classes of asymptotic geodesic rays. Two geodesic rays $\sigma, \tau$ are called *asymptotic* if the map $t \mapsto d(\sigma(t), \tau(t))$ is bounded.

It is a classical result that for a CAT(-1) space these two notions are equivalent. A sketch of a proof will be given later due to the author's lack of knowledge of a reference in the literature.

**Remark 1.1.1.** There is a unique topology on $X \cup \partial_g X$ such that for $S \subset X \cup \partial_g X$, $S$ is open if, and only if, $S \cap X$ is open for the metric topology and for every $\xi \in S \cap \partial_g X$, there exists $t \geqslant 0$ such that $N_t(\xi) \subset S$, where

$$N_t(\xi) = \{y \in X \cup \partial_g X \mid (y,\xi)_{x_0} > t\}.$$

Let $X$ be a complete CAT(0) space and $x_0 \in X$ a base point. Given two geodesic rays $\sigma$ and $\tau$ that issue from $x_0$, the map $t \mapsto d(\sigma(t), \tau(t))$ is a convex non negative function that vanishes at 0, therefore if it is bounded it has to be constant. This observation allows to make sense of the following definitions.

For $s > r$ there is a projection

$$\overline{B(x_0, s)} \xrightarrow{p_{r,s}} \overline{B(x_0, s)}.$$

This defines and inverse system of topological spaces indexed by the positive numbers. Let

$$\overline{X} = \{[0,\infty) \xrightarrow{\sigma} X \mid \sigma(0) = x_0 \text{ and } \sigma \text{ is a generalized geodesic ray}\}$$

be the inverse limit associated to this inverse system. Here a generalized geodesic ray is either a geodesic ray rooted at $x_0$ or a geodesic segment rooted at $x_0$ defined in an interval $[0, r]$ and considered constant in $[r,\infty)$.

**Remark 1.1.2.** The topology of inverse limit in $\overline{X}$ (the subspace topology of the product $X^{\mathbf{R}_{\geqslant 0}}$) is the same as the topology of uniform convergence on compact sets. This topology on $\overline{X}$, often called the *cone topology*, restricts to the metric topology on $X$ and it does not depend on the choice of the base point $x_0$ (see II.8.8 in [7]). Denote $\partial_c X$ the set of geodesic (infinite) rays with base point in $x_0$ provided with subspace topology of the cone topology.

For every $r > 0$, let

$$\overline{X} \xrightarrow{p_r} \overline{B(x_0, r)}$$

be the function that is the identity in $\overline{B(x_0, r)}$ and $p_r(\sigma) = \sigma(r)$, for any $\sigma$ generalized geodesic ray such is not constant on $[r,\infty)$.

Given a geodesic ray $\xi$, denote $U(\xi, R,\epsilon)$ the set of generalized rays $\tau$ such that $\tau|_{[R,\infty)}$ is not constant and $d(p_R(\tau), p_R(\xi)) < \epsilon$. Observe that given a geodesic ray $\xi$, the sets $U(\xi, R,\epsilon)$ are a neighborhood basis for the cone topology.

The following is Lemma 3.4.10 in [20].

**Lemma 1.1.3.** *Let $X$ be a CAT(-1) space and suppose $\xi,\eta \in \partial_g X$ and $z, w \in X$. If $(x_i) \in \xi$ and $(y_i) \in \eta$, the limits*

$$(\xi,\eta)_z = \lim_{n,m\to\infty} (x_n, y_m)_z$$

*and*

$$(\xi, w)_z = \lim_{n\to\infty} (x_n, w)_z$$

*exist and do not depend on the choice of representatives.*

The next proposition is Lemma 3.4.22 in [20].

**Proposition 1.1.4.** *Let $X$ be a CAT(-1) space. Suppose $(z_n)$ is a sequence in $X$ and suppose $(x_n)$ and $(y_n)$ are sequences in $X \cup \partial_g X$ converging with the topology $\mathcal{T}_g$ (see Remark 1.1.1) to $z \in X$ and $x, y \in X \cup \partial_g X$, respectively. Therefore*

$$\lim_{n\to\infty} (x_n, y_n)_{z_n} = (x, y)_z.$$

The next result is often called the finite approximation Lemma, see for example Theorem 1 in Chapter 8 of [17].

**Lemma 1.1.5.** *Suppose $(X, x_0)$ is a Gromov hyperbolic (with constant $\delta$) geodesic space and consider*

$$\{x_1, \ldots, x_n\} \subset X \cup \partial X.$$

*Here a point at infinity is understood as the limit of a geodesic ray. Define $Y$ as the union of the geodesic segments or geodesic rays $[x_0, x_i]$. If $2n \leqslant 2^k + 1$, there exists a simplicial tree $Tr(Y)$ and a map $Y \xrightarrow{f} Tr(Y)$ with the following properties:*

1. *For every $i$, the restriction of $f$ to $[x_0, x_i]$ is an isometry.*
2. *For every $x, y \in Y$,*
$$d(x, y) - 2k\delta \leqslant d(f(x), f(y)) \leqslant d(x, y).$$

When $n = 2$ the tree of the finite approximation Lemma is a tripod where the extremes are $f(x_i)$, with $i = 0, 1, 2$ (see Proposition 3.1 of Chapter 1 in [17]).

**Lemma 1.1.6.** *If $X$ is a CAT(-1) space, then there exists a constant $C > 0$ such that for every $x, y, z \in X$,*

$$|d(x, [y, z]) - (x, y)_z| < C.$$

**Proof.** This is just an easy application of the Lemma 1.1.5. Given $x, y, z \in X$, consider the point $w \in [y, z]$ that minimizes the distance between $x$ and the geodesic segment connecting $y$ and $z$. Consider $w$ as the root, and $x, y, z$ as the other three points for the finite approximation Lemma for four points. So it is just a matter of analyzing the possible combinatorics of the tree that approximates the possible configurations for four points. □

**Lemma 1.1.7.** *Let $X$ be a CAT(-1) space, suppose $(x_n)$ is a Gromov sequence and fix $x_0$ a base point. If $\sigma_n$ is the geodesic connecting $x_0$ with $x_n$, then for every $s > 0$, the sequence $(\sigma_n(s))_{n\in\mathbf{N}}$ is Cauchy.*

**Proof.** There is an abuse of notation because for a given $s>0$, the geodesics $\sigma_n$ are defined on $s$ just for $n$ big enough. Fix $s>0$ and consider $x_n$ and $x_m$ with $n$ and $m$ big enough such that

$$s<(x_n,x_m)_{x_0}=t.$$

Consider now the geodesic triangle $\triangle(x_0,\sigma_n(t),\sigma_m(t))$ and its comparison triangle in $\mathbf{R}^2$, $\triangle(\overline{x_0},\overline{\sigma_n(t)},\overline{\sigma_m(t)})$. From the finite approximation Lemma, there is a constant $C>0$ that does not depend on $n$ or $m$, such that

$$\frac{C}{t}\geqslant\frac{d(\sigma_n(t),\sigma_m(t))}{t}=\frac{d(\overline{\sigma_n(t)},\overline{\sigma_m(t)})}{t}=\frac{d(\overline{\sigma_n(s)},\overline{\sigma_m(s)})}{s}\geqslant\frac{d(\sigma_n(s),\sigma_m(s))}{s}.$$

This proofs the claim because $C$ and $S$ are fixed and $t$ goes to infinity when $n$ and $m$ go to infinity. □

**Lemma 1.1.8.** *Every $CAT(-1)$ space is a visibility space, in other words, every two points at infinity are connected by a geodesic, moreover this geodesic is unique.*

**Proof.** See Proposition 10.1 in [11] for the existence. Observe that if $\tau$ and $\sigma$ are two different geodesics, the function $t\mapsto d(\sigma(t),\tau(t))$ is a convex function. If $\tau$ and $\sigma$ share their extreme points this function is bounded, but this is a contradiction (see Theorem II.2.13 in [7]). □

As it was mentioned before, the following theorem is a classic result for which the author could not find a reference for non-proper spaces in the literature.

**Theorem 1.1.9.** *Let $X$ be a CAT(-1) space. There is a natural homeomorphism*

$$((X,\partial_c X),\mathscr{T}_c)\xrightarrow{\Psi}((X,\partial_g X),\mathscr{T}_g).$$

**Proof.** Fix a base point $z_0\in X$. Observe that for every geodesic ray $\tau$ with $\tau(0)=z_0$, the sequence $(\tau(t_n))$ is a Gromov sequence for any sequence $(t_n)\to\infty$ and the class of equivalence of this Gromov sequence does not depend on the choice of the sequence $(t_n)$. Hence for every geodesic ray $\tau$ with starting point at $z_0$ there is a well defined Gromov sequence $[\tau]$.

Let $\Psi$ be the map such that $\partial_c X\xrightarrow{\Psi|_{\partial_c X}}\partial_g X$ is defined by $\Psi(\sigma)=[\sigma]$ and such that $\Psi|_X$ is the identity. In Proposition 4 of Chapter 7 in [29] the authors showed, for proper CAT(-1) spaces, that $\Psi|_{\partial_c X}$ is a bijection. The same proof can be applied in this context if convergence arguments of Arzelà-Ascoli type are exchanged by properties of convergence of Gromov sequences and applications of the finite approximation Lemma.

Indeed, if $\sigma$ and $\tau$ are two different geodesic rays issuing from $x_0$, consider the geodesic $\gamma$ that connects the limits at infinity of $\sigma$ and $\tau$. Let $z \in \gamma$ be the minimizer of the distance of points in $\gamma$ to $x_0$. Without lost of generality, suppose $\gamma(0) = z$. By Proposition 1.1.4,

$$([\sigma],[\tau])_{x_0} = \lim_{n\to\infty} (\gamma(n),\gamma(-n))_{x_0},$$

and by Lemma 1.1.6, for every $n$,

$$d(x_0,\gamma) = d(x_0,[\gamma(n),\gamma(-n)])$$

is at distance at most $C$ from $(\gamma(n),\gamma(-n))_{x_0}$. Therefore $([\sigma],[\tau])_{x_0} < \infty$, or in other words, $[\sigma] \neq [\tau]$. This shows that the map $\Psi$ is injective.

Let $(x_n)$ be a Gromov sequence and let $\sigma_n$ the geodesic ray issuing from $x_0$ that contains $x_n$. Lemma 1.1.7 shows that $\gamma(s) = \lim_{n\to\infty} \sigma_n(s)$ exists, the claim is that $\gamma$ is a geodesic ray and that $(x_n) \in [\gamma]$. Indeed, if $s, t > 0$,

$$d(\gamma(s),\gamma(t)) = d\left(\lim_{n\to\infty} \sigma_n(s), \lim_{n\to\infty} \sigma_n(t)\right) = \lim_{n\to\infty} d(\sigma_n(s),\sigma_n(t)).$$

For every $n$, if $t_n > 0$ is such that $x_n = \sigma_n(t_n)$, then $\lim_{n\to\infty} t_n = \infty$. Without lost of generality, suppose $(t_n)_n$ is increasing and observe that for fixed $n$,

$$(x_n,\gamma(t_n))_{x_0} = \lim_{r\to\infty} (x_n,\sigma_{n+r}(t_n))_{x_0}.$$

It is a consequence of Lemma 1.1.5 in its version for three points that there exists an independent constant $C > 0$ such that

$$\left|(x_n,x_{n+r})_{x_0} - (x_n,\sigma_{n+r}(t_n))_{x_0}\right| < C,$$

in other words,

$$\left|(x_n,\gamma(t_n))_{x_0} - \lim_{r\to\infty} (x_n,x_{n+r})_{x_0}\right| \leqslant C.$$

The sequence $(x_n)$ is Gromov, therefore

$$\lim_{n\to\infty} (x_n,\gamma(t_n))_{x_0} = \infty.$$

This shows that $(x_n)_n \in [\gamma]$ and that $\Psi$ is a bijection.

The will be shown now that $\Psi$ is a homeomorphism. Fix $N_t([\sigma])$ for $t>0$ and a geodesic ray $\sigma$ issuing from $z_0$. Call $C$ the general constant error coming from the tree approximation for 3 points. Fix $R,\epsilon>0$ such that $R-\epsilon-C>t+1$. Let $\tau\neq\sigma$ be a geodesic ray from $x_0$ such that $d(\tau(R),\sigma(R))<\epsilon$ and consider any $s>R$. The claim now is that if

$$(\sigma(s),\tau(s))_{z_0}>t+1,$$

then $\sigma(s)\in N_t([\sigma])$. If this is not the case, then $R>(\sigma(s),\tau(s))_{z_0}$ and from the tripod approximation for the points $\{z_0,\sigma(s),\tau(s)\}$,

$$|(\sigma(s),\tau(s))_{z_0}-(\sigma(R),\tau(R))_{z_0}|<C.$$

But $(\sigma(R),\tau(R))_{z_0}>R-\frac{\epsilon}{2}$, and therefore, $(\sigma(s),\tau(s))_{x_0}>t+1$, which is a contradiction. This shows that $(\sigma(s),\tau(s))_{x_0}>t+1$ and that

$$([\tau],[\sigma])_{x_0}=\lim_{s\to\infty}(\sigma(s),\tau(s))_{x_0}\geqslant t+1,$$

or in other words, that $[\tau]\in N_t([\sigma])$.

Observe now that for every $r>0$,

$$\begin{aligned}(\sigma(s+r),\tau(s))_{x_0} &= \tfrac{1}{2}\big(2s+r-d(\sigma(s+r),\tau(s))\big)\\ &\geqslant \tfrac{1}{2}\big(2s-d(\sigma(s),\tau(s))\big)\\ &= (\sigma(s),\tau(s))_{z_0}.\end{aligned}$$

This implies that

$$\lim_{r\to\infty}(\sigma(s+r),\tau(s))_{x_0}=([\sigma],\tau(s))_{x_0}\geqslant t+1,$$

showing that $\Psi(U(\sigma,R,\epsilon))\subset N_t([\sigma])$.

Fix $R,\epsilon>0$ and consider $U(\sigma,R,\epsilon)$. Suppose that for every $t>0$

$$N_t([\sigma])\not\subset U(\sigma,R,\epsilon).$$

Thus, for every $n\in\mathbf{N}$ there exists $x_n\in N_n([\sigma])\backslash U(\sigma,R,\epsilon)$. This means that for every $n$, $(x_n,[\sigma])_{z_0}\geqslant n$. Choose $s_n$ such that for every $r\geqslant s_n$,

$$(x_n,\sigma(r))_{z_0}\geqslant n.$$

Without lost of generality, suppose that $(s_n)_n$ and $(d(x_n,x_0))_n$ are increasing sequences.

Using the finite approximation lemma for

$$\{z_0, x_n, x_{n+r}, \sigma(s_{n+r})\},$$

it is possible to show that $(x_n)$ is a Gromov sequence. If $\sigma_n$ is the geodesic segment that connects $z_0$ to $x_n$, then

$$\gamma(t) = \lim_{n\to\infty} \sigma_n(t)$$

is a geodesic, in fact $\gamma$ is such that $\Psi(\gamma) = [(x_n)]$. Here an abuse of notation is made because only for $n$ bigger than $t$ it is possible to assume that $\sigma_n(t)$ is defined. By construction $d(\gamma(R), \sigma(R)) \geqslant \epsilon$, therefore $\gamma \neq \sigma$, but this is a contradiction because $(x_n)$ belongs to $[\gamma]$ and $[\sigma]$. Therefore there exists $t > 0$ such that

$$N_t([\sigma]) \subset U(\sigma, R, \epsilon). \qquad \square$$

**Proposition 1.1.10.** *If $X$ is a CAT(-1) space and $T \in \mathrm{Isom}(X)$, then $T$ induces a homeomorphism of $\partial X$.*

**Proof.** Consider $x_0 \in X$ and two geodesic rays $\sigma$ and $\tau$ issuing from $x_0$. Observe that for $R > 0$, $d(\sigma(R), \tau(R)) = d(T(\sigma(R)), T(\tau(R)))$. This shows that $T$ is an open map because $\mathbf{R} \circ \sigma$ and $R \circ \tau$ are two geodesic rays issuing from $T(x_0)$ and the cone topology does not depend on the base point. As $T^{-1}$ has the same property, $T$ induces a homeomorphism $\partial X \to \partial X$. $\square$

Suppose $X$ is a geodesic metric space. A function $X \xrightarrow{f} \mathbf{R}$ is called *convex* if for every geodesic $I \xrightarrow{\gamma} X$, the function $t \mapsto f(\gamma(t))$ is convex.

If $X$ is a CAT$(0)$ space, $x_0$ is a base of point of $X$ and $\xi \in \partial X$, then $b_{\xi,x_0}$, the *Busemann function* based on $\xi$ and normalized in $x_0$, is defined as follows. If $\sigma$ is the geodesic ray that starts at $x_0$ and points towards $\xi$, then

$$b_{\xi,x_0}(y) = \lim_{t\to\infty} d(y, \sigma(t)) - t.$$

This limit exists because the function $t \mapsto d(\sigma(t), y) - t$ is decreasing and and for every $t$, $|d(\sigma(t), y) - t| \leqslant d(x_0, y)$. Observe that the map $b_{\xi,x_0}$ is the pointwise limit of the maps $b_t(y) = d(y, \sigma(t)) - t$. For every $t \geqslant 0$, the maps $b_t$ are convex, therefore $b_{\xi,x_0}$ is convex too.

Given two points $y_1, y_2 \in X$,

$$\begin{aligned} |b_{\xi,x_0}(y_1) - b_{\xi,x_0}(y_2)| &= \lim_{t\to\infty} |d(y_1,\sigma(t)) - d(y_2,\sigma(t))| \\ &= \leqslant d(y_1, y_2). \end{aligned}$$

Hence the following lemma.

**Lemma 1.1.11.** *If $\xi \in \partial X$ and $x_0 \in X$, then $b_{\xi,x_0}$ is a convex 1-Lipschitz function.*

Observe that if $X$ is a CAT(-1) space, $x_0, y \in X$ and $\xi \in \partial X$, then

$$\begin{aligned} b_{\xi,x_0}(y) + 2(y,\xi)_{x_0} &= \lim_{t\to\infty} (d(y,\sigma(t)) - t) + \lim_{t\to\infty} \big(d(y,x_0) + t - d(y,\sigma(t))\big) \\ &= d(y,x_0). \end{aligned}$$

Given two asymptotic rays $\sigma$ and $\tau$, there are associated Busemann functions $b_{\xi,\sigma(0)}$ and $b_{\xi,\tau(0)}$. In a CAT(0) generality, two Busemann functions associated to asymptotic geodesic rays differ by a constant (see Corollary II.8.20 in [7]). In the CAT(-1) context, this can be deduced from Proposition 1.1.4. Observe that for every $y \in X$,

$$\begin{aligned} b_{\xi,\sigma(0)}(y) &= d(y,x_0) - 2(y,\xi)_{x_0} \\ &= \lim_{n\to\infty} (-d(\tau(n),x_0) + d(y,\tau(n))) \\ &= -b_{\xi,\tau(0)}(x_0) + b_{\xi,\tau(0)}(y). \end{aligned}$$

The level sets of any Busemann function $b_{\xi,\sigma(0)}$ are called *horospheres* (centered at $\xi$).

In Theorem 1.1 of [12] the authors proved the main statement of 1) in the following lemma in a more general setting. Also in Proposition 2.1 of [1] there is a similar result for locally compact CAT(0) spaces. Using the idea of that proof, here an elementary argument for 1) is given. Part 2) of the next lemma is Proposition 1.2 of [35]).

**Lemma 1.1.12.** *If $X$ is a CAT(-1) space, $x_0 \in X$ and $\mathscr{C} = \{C_i\}_{i\in\mathbf{N}}$ is a family of non-empty, closed and convex subsets of $X$ such that for every $n$, $C_{n+1} \subset C_n$, then the following hold:*

1. *If $\lim_{n\to\infty} d(x_0, C_n) = \infty$, then there exists $\xi \in \partial X$ such that,*

$$\{\xi\} = \bigcap_n \partial C_n.$$

   *In particular if there is a group $G$ acting by isometries on $X$ and permuting the elements of $\mathscr{C}$, then $\xi$ is a $G$-fixed point.*

2. $\lim_{n\to\infty} d(x_0, C_n) = \infty$ *if, and only if,* $\bigcap_n C_n = \emptyset$.

**Proof.** For every $n$ there is $x_n \in C_n$ such that $d(z_0, x_n) = d(z_0, C_n)$. There is a constant $C > 0$ coming from the finite approximation lemma such that for every $n, m \in \mathbf{N}$,

$$|d(z_0, [x_n, x_m]) - (x_n, x_m)_{z_0}| < C.$$

If $m$ is bigger than $n$,

$$d(z_0, [x_n, x_m]) \geqslant d(z_0, x_n),$$

therefore $(x_n)$ is a Gromov sequence. If $\xi$ is its equivalence class, then $\xi \in \bigcap_n \partial C_n$.

Suppose there is $\eta \neq \xi$ such that $\eta \in \bigcap_n \partial C_n$. If $\tau$ is the unique geodesic connecting $\eta$ and $\xi$ (see Proposition 4.4.4 of [20]), then the image of $\tau$ is contained in every $C_n$. This is a contradiction because $\bigcap_n C_n = \varnothing$.

The last claim of 1) follows from the fact that $G$ also permutes the elements of $\{\partial C_n\}_n$.

For 2) observe that if $d(x_0, C_n)$ is bounded, without lost of generality the $C_n$ can be considered bounded. Denote $r = \sup\{d(x_0, C_n)\}_{n \in \mathbf{N}}$ and define for every $n$, the convex and closed set

$$D_n = \{y \in C_n \mid d(x_0, y) \leqslant 2r - d(x_0, C_n)\}.$$

For every $n$, $D_n \subset C_n$, because if $x_n \in C_n$ is such that $d(x_0, C_n) = d(x_0, x_n)$, then $x_n \in D_n$. Observe that if $n < m$, $C_m \subset C_n$.

In a triangle $\Delta(a, b, c) \in \mathbf{R}^2$, if $m$ is the midpoint of $[a, b]$, then

$$d(c, m)^2 = \frac{1}{2} d(c, a)^2 + \frac{1}{2} d(c, b)^2 - \frac{1}{4} d(a, b)^2,$$

thus for every $x, y, z \in X$, if $m$ is the midpoint of $[x, y]$, then

$$d(z, m)^2 \leqslant \frac{1}{2} d(z, x)^2 + \frac{1}{2} d(z, y)^2 - \frac{1}{4} d(x, y)^2.$$

Therefore if $y, w \in D_n$ and $z \in D_n$ is the midpoint of $[y, w]$, then

$$\begin{array}{rcl} d(x_0, C_n)^2 & \leqslant & d(x_0, z)^2 \\ & \leqslant & \frac{1}{2} d(x_0, y)^2 + \frac{1}{2} d(x_0, w)^2 - \frac{1}{4} d(y, w)^2 \\ & \leqslant & 2r - d(x_0, C_n) - \frac{1}{4} d(y, w)^2. \end{array}$$

This implies that

$$d(y, w)^2 \leqslant 8(r - d(x_0, C_n)).$$

This shows that every sequence $(y_n)$, with $y_n \in D_n$, is a Cauchy sequence, therefore it has a limit and because every $C_n$ is closed, this limit is in $\bigcap_n C_n$. □

**Remark 1.1.13.** Observe that the previous lemma is valid for nested families of convex sets even if the families considered are not countable. This can shown by considering nets instead of sequences in the proofs of the Lemmas 1.1.6, 1.1.7 and 1.1.12.

## 1.2 Groups of isometries of CAT(-1) spaces

If $X$ is CAT(-1) and $g \in \mathrm{Isom}(X)$, define the *displacement* of g as $\ell(g) = \inf\{d(gx, x)\}_{x\in X}$. There are two possibilities, either $\ell(g)$ is achieved or not. If it is achieved and $\ell(g) = 0$, $g$ is called *elliptic*. If $\ell(g)$ is achieved and positive, then $g$ is called *hyperbolic*. And last, if $\ell(g)$ is not achieved, $g$ is called *parabolic*. The type of an isometry of $X$ (hyperbolic, parabolic or elliptic) is invariant under conjugations.

A proof for the following well known fact can be found in Proposition 3.1 of [6].

**Lemma 1.2.1.** *If $X$ is a CAT(-1) space and $g \in \mathrm{Isom}(X)$, then for every $x \in X$,*

$$\ell(g) = \lim_{n\to\infty} \frac{d(g^n x, x)}{n}$$

*and for every $n \in \mathbf{N}$, $\ell(g^n) = n\ell(g)$.*

**Lemma 1.2.2.** *If $X$ is a CAT(-1) space and $g \in \mathrm{Isom}(X)$, the following hold:*

1. *If $g$ is hyperbolic, then $g$ preserves a unique geodesic line and $\ell(g)$ is achieved in it.*
2. *If $g$ is parabolic, then $\ell(g) = 0$, $g$ fixes a unique point in $\partial X$ and $g$ preserves all the horospheres centered at it.*

**Proof.** 1) The existence of the geodesic can be found for example in Theorem II.6.8 in [7]. If a hyperbolic isometry preserves two geodesics, then these are asymptotic, thus, up to a reparametrization, they are equal (see Lemma 1.1.8).

For 2), consider the convex function $x \stackrel{d_g}{\longmapsto} d(gx, x)$. The convex and closed sets

$$C_n = d_g^{-1}[\ell(g), \ell(g) + 1/n]$$

are non-empty. Due to the fact that $\ell(g)$ is not achieved, $\bigcap_n C_n = \emptyset$. By Lemma 1.1.12, there exists $\xi \in \partial X$ fixed by $g$. The claim is that this fixed point is unique. Indeed, suppose that $g$ fixes another point at infinity, then $g$ preserves the geodesic line connecting the two fixed points. The projection onto this $g$-invariant subspace is a contraction and commutes with $g$ (see Proposition II.2.4 in [7]). This implies that $g$ is elliptic or hyperbolic because $\ell(g)$ is achieved in the preserved geodesic line, which is a contradiction.

The fact that $\ell(g)=0$ appears in [11]. Here a proof is given for completeness. Let $\delta$ be a constant such that all the triangles in $X$ are $\delta$-slim. Suppose that $g$ is parabolic and $\ell(g)\neq 0$. By Lemma 1.2.1, without lost of generality, it is possible to suppose that $\ell(g)>2\delta$. Fix $x_0\in X$ and let $x_n=P_{C_n}(x_0)$, where $P_{C_n}$ is the projection onto $C_n$. As $\bigcap_n \mathbf{C}_n=\emptyset$, by Lemma 1.1.12, $d(x_0,x_n)\to\infty$. Observe that the sets $C_n$ are preserved by $g$, thus the projection onto $C_n$ commutes with $g$. Consider $n$ large enough such that $x_0$ and $gx_0$ do not belong to $C_n$. Consider now the points $x_0$, $gx_0$, $x_n$ and $gx_n$. Define the sets

$$A=\{y\in[gx_0,x_n]\mid d(y,[gx_0,x_0])<\delta\}$$

and

$$B=\{y\in[gx_0,x_n]\mid d(y,[gx_n,x_n])<\delta\}.$$

If $A\cup B=[x_0,gx_0]$, then there exists $y\in A\cap B$, therefore $d(y,[gx_0,gx_n])\leqslant\delta$ and $d(y,[x_0,x_n])\leqslant\delta$. This implies that there exist $z_1\in[gx_0,gx_n]$ and $z_2\in[x_0,x_n]$ such that $d(z_1,z_2)<2\delta$. This is a contradiction because

$$2\delta<d(x_n,gx_n)\leqslant d(z_1,z_2).$$

The last inequality is a consequence of the fact that $gx_n=P_{C_n}(z_1)$ and $x_n=P_{C_n}(z_2)$.

Therefore, $A\cup B\neq[x_0,gx_0]$, which means that there exists $y\in[gx_0,x_n]$ such that $d(y,[x_0,gx_0])<\delta$ and $d(y,[x_n,gx_n])<\delta$. This implies that there exist $z_1\in[x_0,gx_0]$ and $z_2\in[gx_n,x_n]$. such that $d(z_1,z_2)<2\delta$.

For $n$ big enough, define $w_n\in[x_0,gx_0]$ and $z_n\in[x_n,gx_n]$ the points minimizing the distance between $[x_0,gx_0]$ and $[x_n,gx_n]$. Observe that $d(z_n,w_n)\leqslant 2\delta$ and notice that

$$\begin{array}{rcl} d(x_0,gx_0) & = & d(gx_0,w_n)+d(w_n,x_0)\\ & \geqslant & d(gx_0,z_n)+d(x_0,z_n)-2d(w_n,z_n)\\ & \geqslant & d(gx_0,gx_n)+d(x_0,x_n)-2\delta. \end{array}$$

But this a contradiction because $\{x_n\}_{n\in\mathbf{N}}$ is unbounded, therefore $\ell(g)=0$.

Let $\xi\in\partial X$ be the point fixed by $g$. For $x,y\in X$, $b_{\xi,\sigma(0)}(x)-b_{\xi,\sigma(0)}(y)$ does not depend on the choice of the geodesic ray $\sigma$ with limit $\xi$. Observe that

$$b_{\xi,\sigma(0)}(gx) = b_{\xi,g^{-1}\sigma(0)}(x)$$

and

$$|b_{\xi,g^{-1}\sigma(0)}(x) - b_{\xi,\sigma(0)}(x)| \leqslant d(g\sigma(0),\sigma(0)).$$

Thus

$$|b_{\xi,\sigma(0)}(gx) - b_{\xi,\sigma(0)}(x)| \leqslant \ell(g) = 0.$$

Hence $g$ preserves the horospheres centered at $\xi$. □

**Lemma 1.2.3.** *If $G$ acts by isometries on a CAT(0) space $X$, then $G$ fixes a point in $X$ if, and only if, $G$ has bounded orbits.*

**Proof.** By Corollary II.2.7 of [7], if $B \subset X$ is bounded there exist unique $x_0 \in X$ and $r_0 > 0$ such that $B \subset B(x_0, r_0)$ and such that

$$r_0 = \inf\{s > 0 \mid B \subset B(y, s), \text{ for some } y \in X\}.$$

Thus, if $B$ is a bounded orbit of $G$, then $gx_0 = x_0$. □

The following proposition can be found in Theorem 6.2.3 in [20] in a Gromov hyperbolic generality. The arguments presented there can be simplified in the CAT(-1) context, doing so, a much simpler proof for the CAT(-1) case will be given.

**Proposition 1.2.4.** *If $X$ is a CAT(-1) space and $G < \mathrm{Isom}(X)$, then one, and only one, of the following cases occurs.*

1. *$G$ fixes a point in $X$.*
2. *$G$ has unbounded orbits, fixes a point $\xi \in \partial X$ and leaves invariant all the horospheres centered at $\xi$.*
3. *$G$ contains an hyperbolic element.*

**Lemma 1.2.5.** *If $X$ is a CAT(-1) space and $s > 0$, there exists $r = r(s)$ such that for every $g$ non-hyperbolic isometry of $X$, if $d(gx, x) > r$ then $(gx, g^{-1}x)_x > s$, for every $x \in X$.*

**Proof.** Let $s > 0$. Suppose that $g$ is elliptic and non-trivial. Let $y$ be a fixed point and $x$ any other point. As $X$ is Gromov hyperbolic for some constant $\delta$,

$$(gx, g^{-1}x)_x \geqslant \min\{(gx, y)_x, (g^{-1}x, y)_x\} - \delta.$$

Observe that

$$2(gx,y)_x = d(gx,x) = d(g^{-1}x,x) = 2(g^{-1}x,y).$$

Thus, if $d(gx,x) \geqslant s+\delta+1$, then $2(gx,g^{-1}x)_x \geqslant s$.

If $g$ is parabolic, then let $\sigma$ be the geodesic segment issuing from $x$ and with limit the fixed point $\xi \in \partial X$ fixed by $g$. Observe that for every $x \in X$,

$$\begin{aligned} 2(gx,\xi)_x &= \lim_{t\to\infty}\big(d(gx,x)+t-d(gx,\sigma(t))\big) \\ &= d(gx,x) - b_{\xi,\sigma(0)}(gx) \\ &= d(gx,x) - b_{\xi,\sigma(0)}(x) \\ &= d(gx,x). \end{aligned}$$

Observe that

$$(gx,\xi)_x = (g^{-1}x,\xi)_x = d(gx,x),$$

hence there exists $T>0$ such that

$$2\min\{(gx,\sigma(T))_x,(g^{-1}x,\sigma(T))_x\} \geqslant d(gx,x)-1.$$

Therefore

$$\begin{aligned} 2(gx,g^{-1}x)_x &\geqslant 2\min\{(gx,\sigma(T))_x,(g^{-1}x,\sigma(T))_x\}-\delta \\ &\geqslant d(gx,x)-\delta-1. \end{aligned}$$

Thus if $d(gx,x) \geqslant s+\delta+1$, then $2(gx,g^{-1}x)_x \geqslant s$. □

**Lemma 1.2.6.** *If $X$ is a CAT(-1) space and $(g_n)$ is a sequence in $Isom(X)$ of non-hyperbolic isometries such that $d(g_nx,x)\to\infty$, then $(g_nx)$ is a Gromov sequence.*

**Proof.** Fix $s>0$ such that $s>4\delta$ and let $r(s)$ be like in Lemma 1.2.5. Let $r$ be such that $r-s/2>r(s)$. Suppose that

$$\min\{d(g_nx,x),d(g_mx,x)\} > r(s),$$

and therefore that,

$$\min\{(g_nx,g_n^{-1}x)_x,(g_mx,g_m^{-1}x)_x\} \geqslant s.$$

Suppose that $(g_nx,g_mx)_x < s/4$. Hence

$$s/2 > (g_nx,g_mx)_x+\delta \geqslant (g_n^{-1}x,g_mx)_x,$$

and therefore,

$$\begin{aligned} s &\geqslant 2(g_n^{-1}x, g_m x)_x. \\ &= d(g_n x, x) + d(g_m x, x) - d(g_n g_m x, x) \\ &\geqslant 2r - d(g_n g_m x, x). \end{aligned}$$

Therefore,

$$d(g_n g_m x, x) \geqslant 2r - s > r(s),$$

and analogously,

$$d(g_m g_n x, x) > r(s).$$

The last two inequalities imply that

$$\min\left\{(g_n g_m x, g_m^{-1} g_n^{-1} x)_x, (g_m g_n x, g_n^{-1} g_m^{-1} x)_x\right\} > s.$$

Notice that also either $(g_n x, g_m x)_x \geqslant s/2$ or the two inequalities

$$(g_n x, g_n g_m x)_x = d(g_n x, x) - (g_n^{-1} x, g_m x)_x \geqslant r - s/2 > r(s) > s$$

and analogously,

$$(g_m x, g_m g_n x)_x > s.$$

Observe that with the same argument,

$$\min\left\{(g_m^{-1} x, g_m^{-1} g_n^{-1} x)_x, (g_n^{-1} x, g_n^{-1} g_m^{-1} x)_x\right\} > s.$$

Observe that

$$\begin{aligned} (g_n g_m x, g_m x)_x &\geqslant \min\{(g_n g_m x, g_m^{-1} g_n^{-1} x)_x, (g_m^{-1} g_n^{-1} x, g_m x)_x\} - \delta \\ &\geqslant \min\{s, (g_m^{-1} g_n^{-1} x, g_m x)_x\} - \delta. \end{aligned}$$

But also,

$$\begin{aligned} (g_m^{-1} g_n^{-1} x, g_m x)_x &\geqslant \min\{(g_m^{-1} g_n^{-1} x, g_m^{-1} x)_x, (g_m^{-1} x, g_m x)_x\} - \delta \\ &\geqslant s - \delta. \end{aligned}$$

Therefore, as

$$(g_n x, g_m x)_x + \delta \geqslant \min\{(g_n . g_m, g_n)_x, (g_n . g_m x, g_m)_x\},$$

it is clear that $(g_n x, g_m x)_x \geqslant s - C$, for some $C > 0$ not depending on $n, m$ or $s$. □

**Proof of Proposition 1.2.4.** It is clear that the three cases are exclusive. Suppose that $G$ does not contain any hyperbolic element and that $G$ has unbounded orbits. Let $(g_n)$ be a sequence in $G$ such that for $x \in X$, $d(g_n x, x) \to \infty$. By Lemma 1.2.6, $(g_n x)$ is a representative of some $\xi \in \partial X$. The claim is that $\xi$ is fixed by $G$. Observe that for any other sequence $(h_n)$ in $G$, if $d(h_n x, x) \to \infty$, then $(h_n x)$ is a Gromov sequence. Observe that if $k_{2n} = g_n$ and $k_{2n+1} = h_n$, then, by Lemma 1.2.6, $(k_n x)$ is a Gromov sequence. This shows that $(g_n x)$ and $(h_n x)$ are equivalent. For every $g \in G$ and every $n$,

$$d(gg_n x, x) \geqslant d(g_n x, x) - d(gx, x).$$

Hence $\xi$ is $G$- fixed.

If $g \in G$ is parabolic, by Lemma 1.2.2, $g$ preserves the horospheres centered at $\xi$. If $g$ is elliptic, then $g$ fixes a point $x_0 \in X$. Consider $\sigma$ the geodesic ray issuing from $x_0$ and with limit $\xi$. Thus for every $y \in X$, $b_{\xi,\sigma(0)}(gy) = b_{\xi,\sigma(0)}(y)$. □

Let $G$ be a group acting on a space $X$. A function $X \xrightarrow{f} \mathbf{R}$ is called *quasi-invariant* if for every $g$ there exists a constant $c(g)$ such that for every $x \in X$,

$$f(gx) - f(x) = c(g).$$

Observe that the map $c$ in the previous definition has to be a homomorphism. The statement of the next lemma, but in the context of proper CAT(0) spaces, appears in Section 2 of [1]. Using Lemma 1.1.12, the arguments in the aforementioned article also work for CAT(-1) spaces.

**Lemma 1.2.7.** *Let a group $G$ act by isometries on a CAT(-1) space $X$. If the action does not have fixed points in $X \cup \partial X$, then every continuous quasi-invariant convex function defined on $X$ is $G$-invariant, has a lower bound and the non-empty sublevel sets of it are $G$-invariant and unbounded.*

**Proof.** Suppose $X \xrightarrow{F} \mathbf{R}$ is a convex quasi-invariant function. If $F$ is not invariant, there exists $c(g) \neq 0$ for some $g \in G$ and therefore $F$ is not bounded below or above. Without lost of generality suppose $c(g) < 0$ and define, for every $n \in \mathbf{Z}$,

$$C_n = F^{-1}(-\infty, c(g^n)].$$

Observe the sets $C_n$ are convex, closed and nested and $\bigcap_{n \in \mathbf{Z}} C_n = \emptyset$. By Lemma 1.1.12, there exists a $G$-fixed point in $\partial X$, which is a contradiction. Observe that the same arguments show that $F$ has to be bounded from below. As $G$ does not fix points in $X$, every non-empty sublevel set of $F$ has to be unbounded (see Lemma 1.2.3).

□

Let $G$ be a topological group and let $X$ be a topological space. An action of $G$ on $X$ is called *orbitally continuous* if for every $x \in X$, the map $g \mapsto g \cdot x$ is continuous. From now on through all the text the representations will be considered orbitally continuous.

**Proposition 1.2.8.** *If $X$ is a CAT(-1) space such that $\partial X \neq \emptyset$ and $G \xrightarrow{\rho} \mathrm{Isom}(X)$ is a representation, then for every $\xi \in \partial X$, the map $G \to \partial X$, given by $g \mapsto \rho(g)\xi$, is continuous.*

**Proof.** It is enough to show that the map $t \mapsto \rho(g)\xi$ is continuous at $e \in G$. Fix $R > 0$ and $x_0 \in X$. For $g \in G$, let $\sigma$ and $\tau$ be geodesic rays issuing from $x_0$ representing $\xi$ and $g\xi$ respectively. Observe that $g\sigma$ and $\tau$ are asymptotic rays, thus the map $t \mapsto d(\tau(t), g\sigma(t))$ is bounded and convex, therefore it has to be decreasing. Notice that

$$\begin{aligned} d(\sigma(R), \tau(R)) &\leqslant d(\sigma(R), g\sigma(R)) + d(\tau(R), g\sigma(R)) \\ &\leqslant d(\sigma(R), g\sigma(R)) + d(x_0, gx_0). \end{aligned}$$

Hence using the fact that the action of $G$ on $X$ is orbitally continuous it is possible to show the continuity at $e$ with respect to the cone topology in $\partial X$. □

If $X$ is a $CAT(-1)$ space, an orbitally continuous representation $G \xrightarrow{\rho} \mathrm{Isom}(X)$ is called *non-elementary* if it does not have finite orbits in $X \cup \partial X$. Observe that if $X$ is a CAT(-1) space and $G \xrightarrow{\rho} \mathrm{Isom}(X)$ is a non-elementary representation, then there exists $g \in G$ such that $\rho(g)$ is a hyperbolic isometry (see Proposition 1.2.4).

The following lemma is well known, but due to the author's lack of knowledge of a reference in the literature, a proof will be given.

**Lemma 1.2.9.** *Let $X$ be a CAT(-1) space. A representation $G \xrightarrow{\rho} Isom(X)$ is non-elementary if, and only if, it does not fix a point in $X \cup \partial X$ and it does not preserve a geodesic.*

**Proof.** Suppose that $\rho$ does not have fixed points in $X \cup \partial X$ and that it does not preserve a geodesic. If $\rho$ has a finite orbit in $X$, then it has a fixed point in $X$ (see Lemma 1.2.3). Suppose that there is $\{\xi_1, \ldots, \xi_l\}$ a $G$-invariant set in $\partial X$ with $n \geqslant 3$. Fix a base point $x_0 \in X$ and consider the function $f = \sum_{i=1}^{n} b_{\xi_i, x_0}$.

Observe that $b_{\xi,x_0}(gy) = b_{g^{-1}\xi,g^{-1}x_0}(y)$. As the set $\{\xi_1,\ldots,\xi_l\}$ is $G$-invariant, there is a permutation of $\{1,\ldots,l\}$ defined by $g^{-1}\xi_i = \xi_{\varphi(i)}$. Therefore

$$\begin{aligned} b_{\xi,x_0}(gy) &= b_{g^{-1}\xi_i,g^{-1}x_0}(y) \\ &= b_{\xi_{\varphi(i)},g^{-1}x_0}(y) \\ &= b_{\xi_{\varphi(i)},x_0}(y) - b_{\xi_{\varphi(i)},x_0}(g^{-1}x_0). \end{aligned}$$

As a consequence, the convex function $f$ is quasi-invariant because

$$f(gy) = \sum_{i=1}^{n} b_{\xi_i,x_0}(gy) = \sum_{i=1}^{n} b_{\xi_i,x_0}(y) - \sum_{i=1}^{n} b_{\xi_i,x_0}(g^{-1}x_0).$$

By Lemma 1.2.7, any non-empty sublevel set of $f$ is unbounded. Fix one non-empty sublevel set $C_r$ and let $(y_n)$ be an unbounded sequence in $C_r$. Up to taking a subsequence, suppose that $(y_n)$ converges to at most one point at infinity $\eta$. Observe that for every $\xi_i$,

$$b_{\xi_i,x_0}(y_n) = d(y_n,x_0) - 2(y_n,\xi_i)_{x_0}.$$

Thus if $\eta \neq \xi_1,\ldots,\xi_l$, there exists $C > 0$ such that for every $n$,

$$|f(y_n) - l d(y_n,x_0)| < C.$$

This is a contradiction because $\min(f) \leqslant f(y_n) \leqslant r$ (see Lemma 1.2.7) and

$$\lim_{n\to\infty} d(y_n,x_0) = \infty.$$

Now suppose that $(y_n)$ converges to $\eta = \xi_1$. Observe that $b_{\xi_1,x_0}(y) \geqslant -d(y,x_0)$ and, because of the same arguments used in the previous case, there exists $C' > 0$ such that for every $y_n$,

$$\begin{aligned} f(y_n) &= b_{\xi_1,x_0}(y_n) + b_{\xi_2,x_0}(y_n) + \cdots + b_{\xi_l,x_0}(y_n) \\ &\geqslant -d(y_n,x_0) + (l-1)d(y_n,x_0) - C' \\ &\geqslant (l-2)d(y_n,x_0) - C'. \end{aligned}$$

Therefore $\{d(y_n,x_0)\}_n$ is bounded, which is a contradiction. □

The next lemma appears in Proposition 2.1 of [13] in the context of proper CAT(0) spaces. The ideas in that article can be used with slight modifications for the case of CAT(-1) spaces.

**Lemma 1.2.10.** *Let $G$ be a group and let $X$ be a CAT(-1) space such that $\partial X \neq \emptyset$. If*

$$G \xrightarrow{\rho} \mathrm{Isom}(X)$$

*is a representation and $G$ preserves a Borel probability measure in $\partial X$, then $\rho$ fixes a point in $X \cup \partial X$.*

**Proof.** Suppose that $\rho$ does not have fixed points in $X \cup \partial X$. Fix a point $x_0 \in X$ and consider the function

$$F(y) = \int_{\partial X} b_{\xi,x_0}(y)d\mu(\xi),$$

where $\mu$ is the $G$-invariant probability measure in $\partial X$. The function $\xi \mapsto b_{\xi,x_0}(y)$ is continuous (see Lemma 3.4.22 in [20]) and for every $\xi \in \partial X$, $|b_{\xi,x_0}(y)| \leqslant d(y,x_0)$. This shows that the integral makes sense.

Observe that for every $\xi$, $b_{\xi,x_0}$ is 1-Lipschitz and convex, hence $F$ has the same properties. Moreover, for every $g \in G$,

$$\begin{aligned} F(g^{-1}y) &= \int_{\partial X} b_{\xi,x_0}(g^{-1}y)d\mu(\xi) \\ &= \int_{\partial X} b_{g\xi,gx_0}(y)d\mu(\xi) \\ &= \int_{\partial X} \Big(b_{g\xi,x_0}(y) + b_{g\xi,gx_0}(x_0)\Big)d\mu(\xi) \\ &= \int_{\partial X} \Big(b_{g\xi,x_0}(y) + b_{\xi,x_0}(g^{-1}x_0)\Big)d\mu(\xi) \\ &= F(y) + F(g^{-1}x_0). \end{aligned}$$

The last equality holds because $\mu$ is $G$-invariant. Therefore $F$ is quasi-invariant, and by Lemma 1.2.7, it is a $G$-invariant function.

Notice that $x_0 \in C_0$, the sublevel set of $F$ associated to 0. Observe that for every $n \in \mathbf{N}$ there exists $x_n \in C_0$ such that $d(x_0,x_n) > n$ (see Lemma 1.2.7). Up to taking a subsequence, it is possible to suppose that $(x_n)$ converges at most to $\xi_0 \in \partial X$. The claim is that $F(x_n) \to \infty$, which would be a contradiction. The proof for this statement will follow the ideas of Lemma 2.4 in [10].

By Lemma 1.2.9, the orbit of every $\eta \in \partial X$ is infinite, hence $\mu$ is a non-atomic measure, therefore

$$F(y) = \int_{\partial X \setminus \xi_0} b_{\xi,x_0}(y)d\mu(\xi).$$

For every $y, z \in X$,

$$(y,z)_{x_0} \leqslant \min\{d(y,x_0), d(z,x_0)\},$$

thus, for every $\eta \in \partial X$, $(y,\eta)_{x_0} \leqslant d(y,x_0)$. Therefore, for every $y \in X$ and $\eta \in \partial X$,

$$b_{\eta,x_0}(y) = d(y,x_0) - 2(y,\eta)_{x_0} \geqslant -d(y,x_0).$$

Define for every $n \in \mathbf{N}$ the measurable set

$$V(n) = \left\{\eta \in \partial X \mid \sup_{m\in\mathbf{N}}\{2(x_m,\eta)_{x_0}\} \leqslant n\right\}.$$

The sequence $(x_n)$ belongs to at most $\xi_0$, therefore

$$\partial X\backslash\xi_0 \subset \bigcup_n V(n).$$

For every $n$, $V(n) \subset V(n+1)$, thus there exists some $n_0$ such that $\mu(V(n_0)) > \frac{1}{2}$. Therefore for every $x_m$,

$$\begin{aligned} F(x_m) &= \int_{V(n_0)\backslash\xi_0} b_{\xi,x_0}(x_m)d\mu(\xi) + \int_{(\partial X\backslash\xi_0)\backslash V(n_0)} b_{\xi,x_0}(x_m)d\mu(\xi) \\ &\geqslant \big(d(x_m,x_0) - n_0\big)\mu(V(n_0)) - \big(1-\mu(V(n_0))\big)d(x_m,x_0) \\ &= \big(2\mu(V(n_0)) - 1\big)d(x_m,x_0) - n_0\mu(V(n_0)). \end{aligned}$$

Thus $F(x_m) \to \infty$, which is a contradiction. □

Let $G$ be a (Hausdorff) locally compact group. A discrete subgroup $\Gamma$ is called a *lattice* if the space $G/\Gamma$ admits a non-zero finite $G$-invariant Radon measure.

**Corollary 1.2.11.** *Suppose that $G$ is a locally compact and $\sigma$-compact group, $\Gamma \leqslant G$ is a lattice and $X$ is a CAT(-1) space. If $G \xrightarrow{\rho} \mathrm{Isom}(X)$ is a non-elementary representation, then $\rho|_\Gamma$ is non-elementary.*

**Proof.** If there exists $\eta \in \partial X$ fixed by the action of $\Gamma$, using the continuous map $G/\Gamma \to \partial X$, induced by the orbit map $g \mapsto g\eta$, it is possible to define a $G$-invariant probability measure $\mu$ in $\partial X$. This is a contradiction (see Lemma 1.2.10), thus $\rho|_\Gamma$ does not fix any point in $\partial X$.

If $\rho|_\Gamma$ permutes two points at infinity there is an index two subgroup of $\Gamma$ that preserves a point at infinity. A finite index subgroup of a lattice is a lattice (see for example Lemma 1.6 in [46]), thus this assumption leads to a contradiction.

If $\Gamma$ has a fixed point $x \in X$, the orbit map $g \mapsto g \cdot x$ induces in $X$ a $G$-invariant probability measure $\mu$. Consider a nested family of compact sets $\{K_i\}_{i \in \mathbf{N}}$ such that $\bigcup_i K_i = G$. There exists $i$ such that $\mu(K_i x) > 1/2$, therefore for every $g \in G$,

$$g \cdot K_i x \cap K_i x \neq \emptyset,$$

or in other words, there are $k_1, k_2 \in K_i$ such that $g k_1 x = k_2 x$. Observe that

$$d(gx, x) \leqslant d(gx, gk_1 x) + d(k_2 x, x).$$

This shows that $x$ has a bounded orbit, but this is a contradiction because $G$ does not fix any point in $X$. □

An action of a group $G$ on a CAT(-1) space $X$ is called *minimal* if there is no non-empty, closed, convex and $G$-invariant proper subset of $X$ (see [14]).

A group $G$ is called *amenable* if whenever $G$ has a jointly continuous action by affine maps on $V$, a locally convex Hausdorff topological vector space, such that there exists $K \subset V$, a non-empty, convex and compact $G$-invariant set, then $G$ fixes a point in $K$.

The following theorem can be found in a higher generality in Theorem 1.6 of [12]. An easier proof for the purposes of this text will be given later adapting the arguments used in the aforementioned paper to the context of CAT(-1) spaces.

**Theorem 1.2.12.** *If $G$ is an amenable group and $X$ is a CAT(-1) space, then every representation $G \to Isom(X)$ is elementary.*

**Lemma 1.2.13.** *If $X$ is a CAT(-1) space and $G \xrightarrow{\rho} \mathrm{Isom}(X)$ is a representation without fixed points in $X \cup \partial X$, then there exists $\emptyset \neq Y \subset X$ convex, closed $G$-invariant such that the action of $G$ on $Y$ is minimal.*

**Proof.** If the action of $G$ on $X$ is not minimal, then there exists a non-trivial $G$-invariant convex and closed subsets of $X$. By Lemma 1.2.3, all of these sets have to be unbounded. Consider the non-empty set

$$A = \{\emptyset \neq C \subset X \mid C \text{ is } G\text{-invariant convex and closed}\}$$

ordered with the inclusion. Consider a (descending) chain $D$ in $C$. If $\bigcap D = \emptyset$, then by Lemma 1.1.12 and Remark 1.1.13, there exists $\xi \in \partial X$ such that

$$\{\xi\} = \bigcap \{\partial C_\alpha \mid C_\alpha \in D\}.$$

This is a contradiction because $\xi$ would be a $G$-fixed point.

Then by Zorn's Lemma, there exists $D_0 \in A$ that does not contain any proper convex, closed and $G$-invariant subset. $\square$

If $X$ is a CAT(-1) space, a function $X \xrightarrow{f} \mathbf{R}$ is called *affine* if for every geodesic $I \xrightarrow{\gamma} X$, the function $I \xrightarrow{f\circ\gamma} \mathbf{R}$ is affine.

Let $X$ be a CAT(-1) space and fix a point $x_0 \in X$. Denote $B$ the set of 1-Lipschitz functions $X \xrightarrow{f} \mathbf{R}$ such that $f(x_0) = 0$. The vector space $B$ endowed with the pointwise convergence topology is a locally convex Hausdorff topological vector space. Denote $K \subset B$ the set of convex and 1-Lipschitz functions defined on $X$. Observe that with the subspace topology, $K$ is a convex and compact subset of $B$.

For every $x \in X$, define $\iota(x) \in K$, given by $\iota(x)(y) = d(x, y) - d(x, x_0)$. Denote $C_{x_0} \subset K$ the closure, with respect to the pointwise convergence topology, of $\{\iota(x)\}_{x\in X}$.

**Lemma 1.2.14.** *If $X$ is a CAT(-1) space such that $|\partial X| \geqslant 3$ and $x_0 \in X$, then $C_{x_0}$ does not contain any affine function.*

**Proof.** Suppose that $F \in C_{x_0}$ is affine and consider $\sigma$ and $\tau$ two distinct geodesic rays issuing from $x_0$. Denote $Y \subset X$ the union of the images of $\sigma$ and $\tau$. The space $Y$ is separable, and therefore, there exists a sequence $(x_n)$ in $Y$ such that $\iota(x_n)|_Y \to F|_Y$. If $(x_n)$ is bounded then, up to taking a subsequence, $(x_n) \to y_0$, for some $y_0 \in Y$. In that case $F|_Y = \iota(y_0)|_Y$, but this is a contradiction because for every $x \in X$, $\iota(x)$ is strictly convex.

Without lost of generality, suppose that for every $n \in N$, $d(x_n, x_0) \geqslant n$. If $F$ is constant in the image of $\sigma$, then for a fixed $T > 0$ and for every $n \in N$, there exists $M(n) \in \mathbf{N}$, such that for every $m > M(n)$,

$$|d(\sigma(n), x_m) - d(x_m, x_0)| < T.$$

Observe that for every $m, m' > M(n)$

$$\begin{aligned} 2(x_m, x'_m)_{x_0} &\geqslant \min\{2(x_m, \sigma(n))_{x_0}, 2(x'_m, \sigma(n))_{x_0}\} - 2\delta \\ &\geqslant n - T - 2\delta. \end{aligned}$$

The previous computation shows that $(x_n)$ is a Gromov sequence and that is equivalent to $(\sigma(n))$. Observe that as $\tau$ and $\sigma$ are not asymptotic, $F$ is not constant on the image of $\tau$.

Let $\gamma$ be the geodesic connecting the limits of $\tau$ and $\sigma$. Suppose that the positive part of

$\gamma$ points towards the limit of $\tau$. Therefore, there exists $T_0 > 0$ such that for every $t > 0$,

$$d(F(\tau(t)), F(\gamma(t)))) \leqslant d(\tau(t), \gamma(t)) \leqslant T_0.$$

This shows that $\lim_{t\to\infty} F(\gamma(t)) = \pm\infty$. With the same arguments it is possible to show that $\lim_{t\to\infty} F(\gamma(-t)) = 0$. But this is a contradiction. Therefore $F$ is not constant on any geodesic ray in $X$.

Consider $\xi_1, \xi_2, \xi_3$ three distinct points in $\partial X$ and let $\gamma_1, \gamma_2, \gamma_3$ be the three geodesics connecting $\xi_1$ with $\xi_2$, $\xi_2$ with $\xi_3$ and $\xi_3$ with $\xi_1$, respectively. Without lost of generality suppose that $\lim_{t\to\infty} \gamma_1(F(t)) = \infty$. Thus

$$\lim_{t\to\infty} \gamma_1(F(-t)) = \lim_{t\to\infty} \gamma_3(F(t)) = -\infty,$$

and therefore,

$$\lim_{t\to\infty} \gamma_3(F(-t)) = \lim_{t\to\infty} \gamma_2(F(t)) = \infty.$$

This is a contradiction because

$$-\infty = \lim_{t\to\infty} \gamma_2(F(-t)) = \lim_{t\to\infty} \gamma_1(F(t)) = \infty.$$

□

If $G$ is a group acting by isometries on $X$, then $G$ acts on $B$ in the following way. If $f \in B$ and $g \in G$, then

$$g \cdot f(x) = f(g^{-1}x) - f(g^{-1}x_0).$$

It is clear that $K$ is invariant under this action. The claim is that this action is jointly continuous on $K$. Indeed, if $(g_\alpha, f_\alpha) \to (g, f) \in G \times K$ and $x \in X$,

$$\begin{aligned}
d(g_\alpha f_\alpha(x), g f(x)) &= \\
d\big(f_\alpha(g_\alpha^{-1}x) - f_\alpha(g_\alpha^{-1}x_0), f(g^{-1}x) - f(g^{-1}x_0)\big) &\leqslant \\
d\big(f_\alpha(g_\alpha^{-1}x) - f_\alpha(g_\alpha^{-1}x_0), f_\alpha(g^{-1}x) - f_\alpha(g^{-1}x_0)\big) + \\
d\big(f_\alpha(g^{-1}x) - f_\alpha(g^{-1}x_0), f(g^{-1}x) - f(g^{-1}x_0)\big) &\leqslant \\
d(g_\alpha^{-1}x, g^{-1}x) + d(g_\alpha^{-1}x_0, g^{-1}x_0) + \\
d\big(f_\alpha(g^{-1}x) - f_\alpha(g^{-1}x_0), f(g^{-1}x) - f(g^{-1}x_0)\big).
\end{aligned}$$

**Proof of Theorem 1.2.12.** Suppose $G \xrightarrow{\rho} \mathrm{Isom}(X)$ is a non-elementary representation. Let $Y \subset X$ be a non-empty closed convex and $G$-invariant set such that the action of $G$ restricted to it is minimal (see Lemma 1.2.13), in particular non-elementary. By Proposition 1.2.4, $\partial Y \neq \varnothing$, because there are no $G$-fixed points in $Y$, and by Lemma 1.2.9, $\partial Y$ is infinite.

Without lost of generality suppose $Y = X$. If $x_0 \in X$, by Lemma 1.2.14, $C_{x_0}$ does not contain any affine function. Let $conv(C)$ be the set of convex combinations of functions in $C$. It is immediate that $conv(C)$ does not contain any affine function. The set $K$ is compact, therefore $\overline{conv(C)}$ is compact. By Lemma 4.10 in [12], $\overline{conv(C)}$ does not contain any affine function, in particular it does not contain any constant function.

Observe that $\overline{conv(C)}$ is $G$-invariant convex and compact, thus there exists $F \in \overline{conv(C)}$ fixed by $G$. The function $F$ is convex, non-constant and quasi-invariant. By Lemma 1.2.7, $F$ is $G$-invariant, but this is a contradiction because the action of $G$ is supposed to be minimal and the sublevel sets of $F$ are closed and convex invariant subsets of $X$. □

## 1.3 The hyperbolic spaces

Following Burger, Iozzi & Monod [9], let $H$ be a vector space over $\mathbf{F} = \mathbf{R}, \mathbf{C}$ such that $\dim_{\mathbf{F}}(H) \in \mathbf{N}_{\geqslant 2} \cup \{\infty\}$. Suppose $H$ is endowed with a non-degenerate form $B$, linear in the first argument and antilinear in the second.

Define

$$\iota(B) = \sup\{\dim_{\mathbf{F}}(W) \mid W \leqslant H \text{ and } B|_{W\times W} = 0\},$$

the *index* of $B$, and

$$\iota_{\pm}(B) = \sup\{\dim_{\mathbf{F}}(W) \mid W \leqslant H \text{ and } B|_{W\times W} \text{ is positive (resp. negative) definite}\}.$$

From now on $H$ will be a $\mathbf{F}$-Hilbert space endowed with $B$, a non-degenerate form such that $\iota(B) = 1$ and $\iota_+(B) \leqslant \iota_-(B)$.

**Lemma 1.3.1.** *If $B$ restricted to a finite-dimensional $\mathbf{F}$-subspace $W$ is non-degenerate, then $H = W \oplus W^{\perp}$ and $B$ restricted to $W^{\perp}$ is non-degenerate.*

**Proof.** Define the map $H \xrightarrow{\varphi} W^*$, given by $\varphi(h)(w) = B(w,h)$. Observe that $\dim_{\mathbf{F}}(H/\ker(\varphi)) \leqslant \dim_{\mathbf{F}}(W^*) = \dim_{\mathbf{F}}(W)$. On the other hand, $W \cap \ker(\varphi) = 0$ because $B$ restricted to $W$ is non-degenerate. Therefore $W \oplus \ker\varphi = H$ and $B$ restricted to $\ker(\varphi) = W^{\perp}$ is non-degenerate. □

**Lemma 1.3.2.** *For the form $B$,*

$$\iota(B) = \iota_+(B).$$

**Proof.** Suppose $B$ restricted to some finite-dimensional subspace $W \leqslant H$ is positive definite. It is supposed that $\iota_-(B) \geqslant \iota_+(B)$, thus there exists $V \leqslant H$ such that $\dim_{\mathbf{F}}(W) = \dim_{\mathbf{F}}(V)$ and such that $B$ restricted to $V$ is negative definite.

For every $0 \neq v \in V$, $v = w + u$, for some $w \in W$ and $u \in W^{\perp}$. Since $B(v, v) < 0$, then $u \neq 0$ and $B(u, u) < 0$. By the previous observation, if $\pi_{W^{\perp}}$ is the projection on $W^{\perp}$, then $B$ restricted to $\pi_{W^{\perp}}(V) = U$ is negative definite and $\dim_{\mathbf{F}}(U) = \dim_{\mathbf{F}}(V)$. Observe that $B$ restricted to $W \oplus U$ is non-degenerate. Using the characterization of non-degenerate forms defined on finite-dimensional spaces,

$$\iota(B|_{W\oplus U}) = \iota_+(W \oplus U) = \iota_-(W \oplus U) = \dim_{\mathbf{F}}(W).$$

This shows that $\iota(B) \geqslant \iota_+(B)$. As $\iota(B) = 1$ and $B$ is not negative definite, it is possible to conclude that $\iota(B) = \iota_+(B)$. □

A form with these properties will be called a *form of signature* $(1, m)$, where $m = \dim_{\mathbf{F}}(H) - 1$. Throughout this section, and for the rest of this work, $m$ will be used to refer to dimensions either finite or infinite and $n$ will be used for only finite ones.

**Lemma 1.3.3.** *Let $v$ and $w$ be two distinct non-zero elements of $H$. If $B(v, v) \geqslant 0$ and $B(w, w) \geqslant 0$, then $B(v, w) \neq 0$.*

**Proof.** As $\iota_+(B) = 1$, it is enough to assume that $B(v, v) = 0$ and $B(w, w) \geqslant 0$. Suppose that $B(v, w) = 0$. Fix $y \in H$ such that $B(y, y) = 1$. The space $H$ admits a decomposition $H = \mathbf{F}y \oplus y^{\perp}$. Without lost of generality, suppose that in that decomposition, $v = y + a$. Notice that $B(a, a) = -1$. Observe that $y^{\perp}$ admits a decomposition $\mathbf{F}a \oplus (a^{\perp} \cap y^{\perp})$, hence $w = \lambda y + \gamma a + u$. The fact that $B(v, w) = 0$ implies that $\lambda = \gamma$. Thus $u \neq 0$, but this is a contradiction because

$$B(w, w) = |\lambda|^2 - |\lambda|^2 + B(u, u) < 0.$$ □

Consider $W \leqslant H$ such that for some $w \in W$, $B(w, w) > 0$. Observe that by Lemma 1.3.1,

$$W = \mathbf{F}w \oplus (w^{\perp} \cap W),$$

and because $\iota_+(B) = 1$, $B$ restricted to $w^{\perp} \cap W$ is negative semi-definite. By Lemma 1.3.2, if $0 \neq v \in w^{\perp} \cap W$, then $B(v, v) < 0$.

**Remark 1.3.4.** This shows that for every $\mathbf{F}$-vector subspace $W$ of $H$ with $dim_{\mathbf{F}}(W) \geqslant 2$, if there exists $w \in W$ such that $B(w,w) > 0$, then $B$ restricted to $W$ is a non-degenerate form of signature $(1, \dim_{\mathbf{F}}(W) - 1)$.

For $v \in H$, denote $[v] = \mathbf{F}v$. Suppose $dim_{\mathbf{F}}(H) = m+1$, where $m \geqslant 2$ if $\mathbf{F} = \mathbf{R}$, and $m \geqslant 1$ if $\mathbf{F} = \mathbf{C}$. Define

$$\mathbf{H}^m_{\mathbf{F}} = \{[v] \mid B(v,v) > 0\}.$$

The space $\mathbf{H}^m_{\mathbf{F}}$ is equipped with a metric given by the formula

$$\cosh(d([v],[w])) = \frac{|B(v,w)|}{B(v,v)^{\frac{1}{2}} B(w,w)^{\frac{1}{2}}}.$$

It can be shown that $d$ is a metric using the fact that $B$ restricted to any finite-dimensional subspace of $H$ containing positive vectors is a form of signature $(1,n)$ with $n < \infty$, and that the formula above is the usual metric defined on the finite-dimensional hyperbolic spaces.

A $\pm$-*orthogonal decomposition* of $H$ is a $B$-orthogonal decomposition

$$H = W_+ \oplus W_-,$$

with $B|_{W_\pm \times W_\pm}$ positive/negative definite. Given a $\pm$-orthogonal decomposition of $H$, define the sesquilinear form $B_\pm$ as $B_\pm|_{W_+ \times W_+} = B$, $B_\pm|_{W_- \times W_-} = -B$ and $B(W_+, W_-) = 0$. A form of signature $(1,\infty)$ on $H$ is called *strongly non-degenerate* if for every (any) $\pm$-orthogonal decomposition, the space $(H, B_\pm)$ is a Hilbert space (see Lemma 2.4 of [9]). The metric space $(\mathbf{H}^m_{\mathbf{F}}, d)$ is complete if, and only if, $B$ is a strongly non-degenerate form (see Proposition 3.3 in [9]).

From now on the space $\mathbf{H}^m_{\mathbf{F}}$ will be always considered associated to a strongly non-degenerate sesquilinear form and it will be called the *m-dimensional* $\mathbf{F}$-*hyperbolic space* (see Proposition 3.7 of [9]). For further reading on these spaces see [9, 20], for the infinite-dimensional case, and see [31], for the finite-dimensional complex case. From now on $H$ will denote a separable Hilbert space over $\mathbf{F}$ provided with $B$, a strongly non-degenerate sesquilinear from of signature $(1,m)$.

If $\mathbf{F} = \mathbf{C}$, let $\mathbf{K} = \mathbf{R}, \mathbf{C}$ and if $\mathbf{F} = \mathbf{R}$ define $\mathbf{K} = \mathbf{R}$. Denote $\pi$ the projectivization map $H \backslash \{0\} \to \mathbf{P}(H)$. A $\mathbf{K}-$*hyperbolic subspace* of $\mathbf{H}^m_{\mathbf{F}}$ is the image under $\pi$ of a closed $\mathbf{K}$-vector subspace $L$ of $H$ such that $B|_{L\times L}$ is non-degenerate of signature $(1,m')$. The restriction of $B$ to $L$ is strongly non-degenerate (see Proposition 2.8 of [9]), therefore $\pi(L)$ is a (complete) hyperbolic space. In the finite-dimensional case, this is a characterization for totally geodesic subspaces (see 3.1.11 of [31]).

For every finite set of points $X$ of $\mathbf{H}_{\mathbf{F}}^m$ there is $W \subset H$, a finite-dimensional space over $\mathbf{F}$, that contains representatives of each of the elements of $X$. The restriction of $B$ to $W$ is a non-degenerate form of signature $(1,n)$, therefore $\pi(W)$ is isometric to a finite-dimensional $\mathbf{F}$-hyperbolic space. This shows that many statements about finite sets of points in $\mathbf{H}_{\mathbf{F}}^\infty$ can be reduced to a finite-dimensional question. For example, the space $\mathbf{H}_{\mathbf{F}}^m$ is a geodesically complete CAT(-1) space because this is true for every finite dimensional $\mathbf{H}_{\mathbf{F}}^n$ (see Proposition II.10.10 of [7]).

Every geodesic ray in $\mathbf{H}_{\mathbf{F}}^m$ lies inside a finite dimensional $\mathbf{F}$-hyperbolic space. It is not surprising that $\partial\mathbf{H}_{\mathbf{F}}^m$, the visual boundary of $\mathbf{H}_{\mathbf{F}}^m$, is in a natural bijection with the set of isotropic lines of $H$, because this is true at a finite-dimensional level (see Proposition 3.5.3 in [20]).

The vector space $H$ has a well defined topology. Indeed, for any $\pm$-orthogonal decomposition of $H$, the space $H$ can be provided with a positive definite Hermitian form $B_\pm$ such that $(H, B_\pm)$ is a Hilbert space. The Hilbert topology on $H$ does not depend on the $\pm$-orthogonal decomposition (see Lemma 2.4 in [9]).

Observe that the space

$$\{[v] \in \mathbf{P}(H) \mid B(v,v) \geqslant 0\}$$

can be provided with the subspace topology of the projective space (with the quotient topology) associated to $H$. The hyperbolic spaces are Gromov hyperbolic, hence $\mathbf{H}_{\mathbf{F}}^m \cup \partial\mathbf{H}_{\mathbf{F}}^m$ has a natural topology (see Remark 1.1.1). In this case both topologies are the same and coincide in $\mathbf{H}_{\mathbf{F}}^m$ with the metric topology (see Proposition 3.5.3 of [20]).

For $\mathbf{H}_{\mathbf{F}}^m$ there is an explicit description of the Busemann functions. If $x \in \mathbf{H}_{\mathbf{F}}^m$, it can be shown at a finite-dimensional level that every geodesic ray $\sigma$ issuing from $x$ admits a lift to $H$ of the shape $t \mapsto \cosh(t)\tilde{x} + \sinh(t)u$, where $\tilde{x}$ is a lift of $x$, $B(\tilde{x},\tilde{x}) = 1 = -B(u,u)$ and $B(\tilde{x},u) = 0$. If $y \in \mathbf{H}_{\mathbf{F}}^m$ and $\tilde{y}$ is a lift of $y$ such that $B(\tilde{y},\tilde{y}) = 1$, then

$$\begin{aligned} b_{\xi,\sigma(0)}(y) &= \lim_{t\to\infty} d(y,\sigma(t)) - t \\ &= \lim_{t\to\infty} \operatorname{arccosh}\big(|B(\tilde{y}, \cosh(t)\tilde{x} + \sinh(t)u)|\big) - t \\ &= \ln(|B(\tilde{y}, \tilde{x} + u)|). \end{aligned}$$

Observe that $\xi$ is represented by the isotropic vector $\tilde{x} + u$ (see Proposition 3.5.3 of [20]).

Denote $\mathrm{O}_{\mathbf{F}}(B)$ the set of $\mathbf{F}$-isomorphisms of $H$ that preserve $B$. Every $A \in \mathrm{O}_{\mathbf{F}}(B)$ is a bounded operator (for the Hilbert norm) of $H$ (see Lemma 1.4.8). It is clear from the definition of the metric that every element of $O_{\mathbf{F}}(B)$ induces an isometry of $\mathbf{H}^m_{\mathbf{F}}$.

**Proposition 1.3.5.** *If $x \in \mathbf{H}^m_{\mathbf{F}}$, $y_1, y_2 \in \partial\mathbf{H}^m_{\mathbf{F}}$ and if $s \in O_{\mathbf{F}}(B)$ is such that $sy_1 = y_2$ and $sy_2 = y_1$, then the following hold:*

1. *The action of $O_{\mathbf{F}}(B)$ on $\mathbf{H}^m_{\mathbf{F}}$ is transitive.*
2. *The action of $O_{\mathbf{F}}(B)_x$ is transitive on metric spheres centered at $x$.*
3. *The action of $O_{\mathbf{F}}(B)_{y_1}$ is transitive on $\partial\mathbf{H}^m_{\mathbf{F}} \setminus \{y_1\}$.*
4. *The action of $O_{\mathbf{F}}(B)$ is double transitive on $\partial\mathbf{H}^m_{\mathbf{F}}$.*
5. *If $m < \infty$ and $\mathbf{F} = \mathbf{R}$, 1.,2., and 3. hold for $SO(1,m)$.*
6. $O_{\mathbf{F}}(B) = O_{\mathbf{F}}(B)_{y_1} \sqcup \left(O_{\mathbf{F}}(B)_{y_1} \cdot s \cdot O_{\mathbf{F}}(B)_{y_1}\right)$.
7. *If $< \infty$ and $\mathbf{F} = \mathbf{R}$, then $SO(1,m) = SO(1,m)_{y_1} \sqcup SO(1,m)_{y_1} \cdot s \cdot SO(1,m)_{y_1}$.*

**Proof.** From the discussion at the beginning of this section, it is clear that 1. holds. For 2., let $x \in \mathbf{H}^m_{\mathbf{F}}$ and fix $\tilde{x}$ a lift of $x$ such that $B(\tilde{x}, \tilde{x}) = 1$. Observe that the action of $O_{\mathbf{F}}(B)_x$ on

$$\{v \in H \mid B(\tilde{x}, v) = 0 \text{ and } B(v, v) = -1\}$$

is transitive, therefore 2. holds.

For 3., let $\xi_1$ and $\xi_2$ be two elements of $\partial\mathbf{H}^m_{\mathbf{F}}$ different than $y_1$. Chose $\tilde{y}_1, \tilde{\xi}_2$ and $\tilde{\xi}_2$ respective lifts of $y_1, \xi_1$ and $\xi_2$ such that $B(\tilde{y}_1, \tilde{\xi}_i) = 1$. The claim is that

$$H = \mathbf{F}\tilde{y}_1 \oplus \mathbf{F}\tilde{\xi}_i \oplus (\tilde{y}_1^{\perp} \cap \tilde{\xi}_i^{\perp}).$$

As

$$H = \mathbf{F}(\tilde{y}_1 + \tilde{\xi}_i) + (\tilde{y}_1 + \tilde{\xi}_i)^{\perp},$$

then for every $h \in H$, $h = a\tilde{y}_1 + a\tilde{\xi}_i + u$. Define

$$v = B(u, \tilde{y}_1)\tilde{y}_1 + B(u, \tilde{\xi}_i)\tilde{\xi}_i + u$$

and observe that $v \in \tilde{y}_1^{\perp} \cap \tilde{\xi}_i^{\perp}$. Therefore

$$h = (a - B(u, \tilde{y}_1))\tilde{y}_1 + (a - B(u, \tilde{\xi}_i))\tilde{\xi}_i + v.$$

To conclude observe that $\tilde{y}_1^{\perp} \cap \tilde{\xi}_1^{\perp}$ and $\tilde{y}_1^{\perp} \cap \tilde{\xi}_2^{\perp}$ have the same dimension and $B$ restricted to both spaces is negative definite.

For 4. observe that 2. implies that the action of $\mathrm{O}_{\mathbf{F}}(B)_x$ is transitive in $\partial\mathbf{H}_{\mathbf{F}}^m$. Thus, it follows form 2. and 3. The claim of 5. is clear from the arguments given for 1. to 4.

For 6. observe that the intersection is empty because non of the elements of

$$\mathrm{O}_{\mathbf{F}}(B)_{y_1} \cdot s \cdot \mathrm{O}_{\mathbf{F}}(B)_{y_1}$$

fixes $y_1$. By 3., the elements of $\mathrm{O}_{\mathbf{F}}(B)_{y_1} \cdot s \cdot \mathrm{O}_{\mathbf{F}}(B)_{y_1}$ can send $y_1$ to any element of $\partial\mathbf{H}_{\mathbf{F}}^m \setminus \{y_1\}$. The point 7. follows from the same arguments. □

**Corollary 1.3.6.** *For every $y \in \partial\mathbf{H}_{\mathbf{F}}^m$, the group $\mathrm{O}_{\mathbf{F}}(B)_y$ is not contained in any proper subgroup of $\mathrm{O}_{\mathbf{F}}(B)$.*

**Proof.** Suppose $L$ is a proper subgroup of $\mathrm{O}_{\mathbf{F}}(B)$ containing $\mathrm{O}_{\mathbf{F}}(B)_y$ and suppose there exist $l \in L$ and $g \in \mathrm{O}_{\mathbf{F}}(B)$ such that $ly \neq y$ and $g \notin L$. Observe that by 3. in Proposition 1.3.5, there exists $k \in \mathrm{O}_{\mathbf{F}}(B)_y$ such that $kly = gy$. This implies that $g \in L$, which is a contradiction. □

The group $\mathrm{O}_{\mathbf{F}}(B)$ is denoted by $\mathrm{U}(1,m)$ (resp. $\mathrm{O}(1,m)$) if $\mathbf{F} = \mathbf{C}$ (resp. $\mathbf{F} = \mathbf{R}$). For every $G \leqslant \mathrm{O}_{\mathbf{F}}(B)$, denote $\mathrm{P}G$ the natural image under projectivization. The group $\mathrm{PO}(1,n)$ is equal to $\mathrm{Isom}(\mathbf{H}_{\mathbf{R}}^m)$ and $\mathrm{PU}(1,m)$ is an index 2 subgroup of $\mathrm{Isom}(\mathbf{H}_{\mathbf{C}}^m)$. In fact, every isometry of $\mathbf{H}_{\mathbf{C}}^m$ is induced by either a $\mathbf{C}$-linear map or by an antilinear one (see Theorem 2.2.3 of [20]). Denote

$$\mathrm{Isom}(\mathbf{H}_{\mathbf{C}}^m)_o = \mathrm{PU}(1,m)$$

and for $m < \infty$,

$$\mathrm{Isom}(\mathbf{H}_{\mathbf{R}}^m)_o = \mathrm{PSO}(1,m).$$

Observe that the diagonal matrix act trivially on $\mathbf{H}_{\mathbf{F}}^m$, therefore if $m < \infty$, then $\mathrm{PSU}(1,m) = \mathrm{PU}(1,m)$. For $m < \infty$, the topology of these groups will be the quotient topology of the projectivization map.

Suppose $\xi \in \partial\mathbf{H}_{\mathbf{F}}^m$ and $G < \mathrm{Isom}_o(\mathbf{H}_{\mathbf{F}}^m)_\xi$. Let $b_{\xi,\sigma(0)}$ be a Busemann function centered at $\xi$ and normalized in $\sigma(0)$, for some geodesic ray $\sigma$. The geodesic ray $\sigma$ admits a lift

$$\tilde{\sigma}(t) = \cosh(t)\tilde{x} + \sinh(t)u,$$

with $u, \tilde{x} \in H$ such that $B(\tilde{x}, \tilde{x}) = 1 = -B(u, u)$ and $B(u, \tilde{x}) = 0$. For every $g \in G$, there exists $c(g) \in \mathbf{R}$ such that for every $y \in \mathbf{H}^m_{\mathbf{F}}$,

$$b_{\xi,\sigma(0)}(y) = b_{\xi,\sigma(0)}(gy) + c(g).$$

The map $c : G \to \mathbf{R}$, called the *Busemann character* associated to $\xi$, is a continuous homomorphism and does not depend on the choice of $\sigma$.

**Remark 1.3.7.** Observe that if $\tilde{y}$ is a normalized lift of $y$, then

$$c(g) = \ln\left(\frac{|B(\tilde{y}, \tilde{x} + u|}{|B(\tilde{g}\tilde{y}, \tilde{x} + u)|}\right),$$

where $\tilde{g}$ is any linear representative of the isometry $g$. Thus, if

$$\tilde{g}(\tilde{x} + u) = \theta(\tilde{g})(\tilde{x} + u),$$

with $\theta(\tilde{g}) \in \mathbf{C}\backslash\{0\}$, then $c(g) = \ln(|\theta(\tilde{g})|)$. Therefore the map $g \mapsto |\theta(\tilde{g})| \in \mathbf{R}_{>0}$ is a continuous homomorphism.

**Proposition 1.3.8.** *If $G < \mathrm{PO}_{\mathbf{F}}(B)_\xi$ and $c : G \to \mathbf{R}$ is the Busemann character associated to $\xi$, then*

1. $\ker(c) = \{T \in G \mid T$ *is elliptic or parabolic*$\}$.
2. *For every $T \in G$, $\ell(T) = |c(T)|$.*

**Proof.** 1. Suppose $\xi$ is represented by the isotropic element $y_1$. Let $T \in G$ and let $\tilde{T}$ be a linear representative of $G$. If $T$ is hyperbolic, $\tilde{T}$ leaves invariant two isotropic lines with respective representatives $y_1$ and $y_2$. Suppose that $B(y_1, y_2) = 1$. Thus, if $\tilde{T}(y_i) = \theta_i y_i$, then $\theta_1\overline{\theta_2} = 1$.

The point $x$ represented by $\frac{1}{\sqrt{2}}(y_1 + y_2)$ belongs to the geodesic connecting $y_1$ and $y_2$ because $2d(T(x), x) = d(T^2(x), x)$. Observe that $d(T(x), x) = |\ln(|\theta_1|)|$. This implies that $|\theta_1| \neq 1$, and as it was noticed before, $c(g) = \ln(|\theta_1|)$. Therefore $T \notin \ker(c)$.

If $T$ is parabolic, by Lemma 1.2.2, $c(T) = 0$. If $T$ is elliptic then $T$ fixes pointwise every geodesic ray representing $\xi$ that starts on a $T$-fixed point in $\mathbf{H}^m_{\mathbf{F}}$. Hence $c(T) = 0$.

The point 2. follows from the arguments of 1 and Lemma 1.2.2. □

**Remark 1.3.9.** Consider an $\mathbf{F}$-vector space $H$ equipped with a strongly non-degenerate form $B$ of signature $(1,m)$, where $\mathbf{F}=\mathbf{R},\mathbf{C}$ and $m$ either finite or infinite, and consider $\mathbf{H}^m_{\mathbf{F}}$ the hyperbolic space associated. Fix $\eta_1,\eta_2\in H$ two isotropic vectors such that $B(\eta_1,\eta_2)=1$.

Let $[\eta_1]\in\partial\mathbf{H}^m_{\mathbf{F}}$ be the point represented by $\eta_1$. If $T\in \mathrm{O}_{\mathbf{F}}(B)_{[\eta_1]}$ is such that $B(T(\eta_1),\eta_2)>0$, then with respect to the decomposition

$$H=\mathbf{F}\eta_1\oplus\mathbf{F}\eta_2\oplus(\eta_1^\perp\cap\eta_2^\perp),$$

$T$ (with a small abuse of notation when $\mathbf{F}=\mathbf{R}$) admits the matrix representation by blocks,

$$T=\begin{pmatrix}\lambda & -\frac{\lambda B(v,v)}{2}+ib & -\lambda B(A(\cdot),v)\\ 0 & \lambda^{-1} & 0\\ 0 & v & A\end{pmatrix},$$

where $\lambda>0$, $b\in\mathbf{R}$, $v\in\eta_1^\perp\cap\eta_2^\perp$ and $A$ is an $\mathbf{F}$-isomorphism of $\eta_1^\perp\cap\eta_2^\perp$ preserving $B$. If $\mathbf{F}=\mathbf{R}$, $b$ is always supposed to be 0. Denote $T=g(\lambda,v,A,b)$.

The set

$$P=\{A\in \mathrm{O}_{\mathbf{F}}(B)_{[\eta_1]}\mid B(A(\eta_1),\eta_2)>0\}$$

is a subgroup of $\mathrm{O}_{\mathbf{F}}(B)$ where the formula for the product and the inverse are

$$g(\lambda,v,A,b)g(\gamma,w,D,d)=g\big(\lambda\gamma,\gamma^{-1}v+A(w),AD,\lambda d+\gamma^{-1}b-\lambda\mathrm{Im}(B(A(w),v))\big)$$

and

$$g(\lambda,v,A,b)^{-1}=g(\lambda^{-1},-\lambda A^{-1}(v),A^{-1},-b).$$

When $\mathbf{F}=\mathbf{R}$ the previous formulas apply if $b,d$ and $\mathrm{Im}(B(A(w),v))$ are identified with 0.

The map $P\to\mathrm{Isom}_o(\mathbf{H}^m_{\mathbf{F}})_{[\eta_1]}$ is surjective because for every $A\in\mathrm{O}_{\mathbf{F}}(B)$ there exists $z\in\mathbf{S}^1\subset\mathbf{C}$ such that $B(zA(\eta_1),\eta_2)>0$. It is clearly also injective.

If $m<\infty$, the group $P$ is closed because it is the intersection of $\mathrm{O}_{\mathbf{F}}(B)_{[\eta_1]}$ and the closed set

$$\{T\in\mathrm{O}_{\mathbf{F}}(B)\mid B(T(\eta_1),\eta_2)\geqslant 0\}.$$

The map $\mathrm{O}^m_{\mathbf{F}}\to\mathrm{PO}^m_{\mathbf{F}}$ is closed because it has a compact kernel. Therefore the map

$$P\to\mathrm{Isom}_o(\mathbf{H}^m_{\mathbf{F}})_{[\eta_1]}$$

is bijective, continuous and closed, hence it is an isomorphism of topological groups. From now on the group $P$ will be identified with $\text{Isom}_o(\mathbf{H}^m_{\mathbf{F}})_{[\eta_1]}$. If $\mathbf{F} = \mathbf{C}$ and $n = 1$, then every $g \in P$ is of the shape $g(\lambda, b)$, for some $\lambda > 0$ and $b \in \mathbf{R}$. With a small abuse of notation, $g(\lambda, b)$ will be sometimes identified with a transformation $g(\lambda, 0, Id, b)$.

Observe that if $P \xrightarrow{c} \mathbf{R}$ is the Busemann character associated to $[\eta_1]$, then

$$c(g(\lambda, v, A, b)) = |\ln(\lambda)|.$$

Therefore, by Proposition 1.3.8, the following proposition holds.

**Proposition 1.3.10.** *An isometry $g(\lambda, v, A, b) \in P$ is hyperbolic if, and only if $\lambda \neq 1$.*

**Proposition 1.3.11.** *An isometry $g(\lambda, v, A, b) \in P$ is parabolic if, and only if, $\lambda = 1$ and one of the following properties holds.*

1. *The vector $v$ is not contained in Im(A-Id).*

2. *If $u$ is a vector such that $(A - Id)u = -v$, then Im$(B(A(u), u)) \neq b$.*

**Proof.** Observe that if an element $g \in P$ is elliptic, then it fixes pointwise an entire geodesic. Therefore $g$ fixes at least two points in $\partial \mathbf{H}^m_{\mathbf{F}}$. Thus, the question if a non-hyperbolic element $g \in P$ is parabolic or elliptic, is actually about if $g$ fixes only $\eta_1$ in $\partial \mathbf{H}^m_{\mathbf{F}}$ or if it fixes another point too. By Proposition 1.3.10, if $g(\lambda, v, A, b)$ is parabolic, then $\lambda = 1$.

If $x \neq \eta_1$ is an isotropic vector, then without lost of generality, $x = a\eta_1 + \eta_2 + w$. Observe that

$$g(1, v, A, b)(a\eta_1 + \eta_2 + w) = \left(a - \tfrac{B(v,v)}{2} + ib - B(A(w), v)\right)\eta_1 + \eta_2 + A(w) + v.$$

Thus, $g(1, v, A, b)(x) = x$ if, and only if, $(A - Id)(w) = -v$ and $b = \text{Im}(B(A(w), w))$. Indeed, observe that $(A - Id)(w) = -v$ implies that

$$B(v, v) = 2B(w, w) - 2\text{Re}(B(A(w), w))$$

and that

$$B(A(w), v) = -B(w, w) + B(A(w), w).$$

Therefore

$$\begin{aligned} a - \tfrac{B(v,v)}{2} + ib - B(A(w), v) &= \\ a + -B(w,w) + \mathrm{Re}(B(A(w),w)) + B(w,w) - B(A(w),w) + ib &= \\ a + i\big(-\mathrm{Im}(B(A(w),w)) + b\big). & \end{aligned}$$

□

Observe that in the case $\mathbf{F} = \mathbf{C}$ and $n = 1$, the previous proposition states that the only parabolic elements of $P$ are the ones of the shape $g(1,b)$, with $b \neq 0$. For the case $\mathbf{F} = \mathbf{R}$ what the previous proposition states is that $g(\lambda, v, A, b)$ is parabolic if, and only if, $\lambda = 1$ and $v \notin \mathrm{Im}(A - Id)$.

The focus will be now on the groups $\mathrm{SU}(1,1)$ and $\mathrm{SO}(1,3)$ and their relationship with $\mathrm{SL}_2(\mathbf{R})$.

Let $\{e_1, e_2\}$ be the canonical base of $\mathbf{C}^2$. Fix the basis $\{\xi_1, \xi_2\}$, where $\xi_1 = \frac{1}{\sqrt{2}}(e_1 + e_2)$ and $\xi_2 = \frac{1}{\sqrt{2}}(e_1 - e_2)$. Observe that

$$B(\xi_1, \xi_1) = 0 = B(\xi_2, \xi_2)$$

and $B(\xi_1, \xi_2) = 1$. With respect to the basis $\{\xi_1, \xi_2\}$, every

$$g(\lambda, b) = \begin{pmatrix} \lambda & ib \\ 0 & \lambda^{-1} \end{pmatrix} \in \mathrm{SU}(1,1)_{[\eta_1]}.$$

Let $s \in \mathrm{SU}(1,1)$ be defined by $s(\xi_1) = i\xi_2$ and $s(\xi_2) = i\xi_1$. If

$$\mathrm{U}(1,1) \xrightarrow{\pi} \mathrm{PU}(1,1)$$

is the projectivization map, by the arguments used in Proposition 1.3.5,

$$\mathrm{Isom}(\mathbf{H}^1_{\mathbf{C}})_o = \pi(P) \sqcup \pi(P)\pi(s)\pi(P).$$

Every element of $\mathrm{SU}(1,1)$ has the form, with respect to the canonical basis,

$$M(\alpha, \beta) = \begin{pmatrix} \alpha & \beta \\ \overline{\beta} & \overline{\alpha} \end{pmatrix},$$

where $|\alpha|^2 - |\beta|^2 = 1$. The map $\mathrm{SU}(1,1) \xrightarrow{\psi} \mathrm{SL}_2(\mathbf{R})$ given by

$$\psi(M(\alpha,\beta)) = \begin{pmatrix} \mathrm{Re}(\alpha)+\mathrm{Im}(\beta) & \mathrm{Re}(\beta)+\mathrm{Im}(\alpha) \\ \mathrm{Re}(\beta)-\mathrm{Im}(\alpha) & \mathrm{Re}(\alpha)-\mathrm{Im}(\beta) \end{pmatrix}$$

is an isomorphism. Let

$$T = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

and define the map $\mathrm{SU}(1,1) \xrightarrow{\Psi} \mathrm{SL}_2(R)$ as $\Psi(A) = T^{-1}\psi(A)T$. The map $\Psi$ is such that

$$\Psi(g(\lambda,b)) = \begin{pmatrix} \lambda & b \\ 0 & \lambda^{-1} \end{pmatrix}$$

and

$$\Psi(s) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The group $\mathrm{SU}(1,1)$ admits a simple description in terms of generators and the relations between them. The following theorem is a well known fact and a proof for it can be found in p. 209 of [37].

**Theorem 1.3.12.** *Let $F$ be the free group generated by the family $\{u(r)\}_{r\in\mathbf{R}\setminus 0}$ and an element $w$. For $r \neq 0$, denote*

$$s(r) = wu(r^{-1})wu(r)wu(r^{-1}).$$

*Consider the relations*

1. *$u$ is an additive homomorphism.*
2. *$s$ is a multiplicative homomorphism.*
3. *$w^2 = s(-1)$*
4. *$s(a)u(b)s(a^{-1}) = u(ba^2)$, for every $a, b \neq 0$.*

*If $G$ is the quotient of $F$ under these relations then $G$ is isomorphic to* $\mathrm{SU}(1,1)$.

Let $SU(1,1) \xrightarrow{\phi} GL_3(\mathbf{R})$ be the map defined by

$$\phi(M(\alpha,\beta)) = \begin{pmatrix} -\frac{1}{2}(\beta^2+\overline{\beta}^2-\alpha^2-\overline{\alpha}^2) & \frac{i}{2}(-\beta^2+\overline{\beta}^2-\alpha^2+\overline{\alpha}^2) & i(\overline{\alpha\beta}-\alpha\beta) \\ -\frac{i}{2}(\beta^2-\overline{\beta}^2-\alpha^2+\overline{\alpha}^2) & \frac{1}{2}(\beta^2+\overline{\beta}^2+\alpha^2+\overline{\alpha}^2) & \overline{\alpha\beta}+\alpha\beta \\ i(\overline{\alpha}\beta-\alpha\overline{\beta}) & \overline{\alpha}\beta+\alpha\overline{\beta} & |\alpha|^2+|\beta|^2 \end{pmatrix}$$

and let

$$T' = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & \sqrt{2} \\ 1 & -1 & 0 \\ 1 & 1 & 0 \end{pmatrix}.$$

Define the map $SU(1,1) \xrightarrow{\Phi} SO(1,2)$ given by

$$\Phi(M(\alpha,\beta)) = T'^{-1}\phi(M(\alpha,\beta))T'.$$

The map $\Phi$ is a homomorphism and $\ker(\Phi) = \{Id, -Id\}$. With an appropriate choice of a basis $\{\xi'_1, \xi'_2, u\}$ of $\mathbf{R}^3$, where

$$B(\xi'_i, \xi'_i) = 0 = B(\xi'_i, u)$$

and

$$B(\xi'_1, \xi'_2) = 1 = -B(u,u),$$

the map $\Phi$ is such that,

$$\Phi(s) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

$$\Phi(g(1,b)) = \begin{pmatrix} 1 & b^2 & -\sqrt{b} \\ 0 & 1 & 0 \\ 0 & -\sqrt{b} & 1 \end{pmatrix}$$

and

$$\Phi(g(\lambda,0)) = \begin{pmatrix} \lambda^2 & 0 & 0 \\ 0 & \lambda^{-2} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Every elliptic transformation is contained in a compact subgroup, therefore its image under $\Phi$ is elliptic too (see Proposition II.2.7 of [7]). Up to conjugation, every parabolic isometry is of the shape $g(1,b)$ and every hyperbolic isometry is of the shape $g(\lambda,0)$ (see Proposition 1.3.5). Thus, by Propositions 1.3.10 and 1.3.11, $\Phi$ preserves the type (hyperbolic, parabolic or elliptic).

Observe that if $g(1,v,A) \in O(1,2)_{[\xi_1']}$, then $A = \pm 1$, $v \in R$ and $\det(g(1,v,A)) = A$. This shows that $SO(1,2)_{[\xi_1']} < \mathrm{Im}(\Phi)$. Thus, by Proposition 1.3.5, $\Phi$ is surjective.

Consider the following commutative diagram,

$$\begin{array}{ccc} SU(1,1) & \xrightarrow{\Phi} & SO(1,2) \\ \downarrow & & \downarrow \\ \mathrm{Isom}(\mathbf{H}^1_{\mathbf{C}})_o & \xrightarrow{\overline{\Phi}} & \mathrm{Isom}(\mathbf{H}^2_{\mathbf{R}})_o, \end{array}$$

where the vertical arrows are the projectivization maps and $\overline{\Phi}$ is the induced isomorphism.

**Lemma 1.3.13.** *For every $g \in Isom_o(\mathbf{H}^1_{\mathbf{C}})$,*

$$2\ell(g) = \ell(\overline{\Phi}(g)).$$

**Proof.** The map $\overline{\Phi}$ preserves the type (elliptic, parabolic and hyperbolic), therefore it is enough to prove the claim for hyperbolic elements. Up to conjugation, every hyperbolic element in $\mathrm{Isom}(\mathbf{H}^1_{\mathbf{C}})_o$ has a representative $g(\lambda, 0)$ for some $\lambda > 0$. For these elements the claim follows from the arguments in the proof of Proposition 1.3.8 and Remark 1.3.7. □

**Proposition 1.3.14.** *The group* $PU(1,1)$ *is simple.*

**Proof.** The elements of $U(1,1)$ will be expressed with respect to a basis $\{\xi_1, \xi_2\}$ of isotropic vectors such that $B(\xi_1, \xi_2) = 1$. Consider the subgroups $U = \{g(1,b)\}_{b \in \mathbf{R}}$ and

$$P = \{g(\lambda, b)\}_{\lambda > 0, b \in \mathbf{R}}$$

(see Remark 1.3.9). Let $y \in \partial \mathbf{H}^1_{\mathbf{C}}$ be the point such that $[\xi_1] = y$.

Denote, with a small abuse of notation, $U$ and $P$ the isomorphic images of $U$ and $P$ in $PU(1,1)$. As it was noted before, $P = PU(1,1)_y$ and by Corollary 1.3.6, $P$ is not contained properly in any proper subgroup of $PU(1,1)$.

Let $L$ be a normal subgroup of $PU(1,1)$. Observe that $L$ is not contained in $P$. Indeed, if that is the case, for every $g \in PU(1,1)$ and $l \in L$, $lg(y) = g(y)$, which is a contradiction. Thus $PL = PU(1,1)$.

The group $B$ is normal in $P$, therefore the group $BL$ is normal in $PU(1,1)$. This means that $BL$ contains all the groups conjugated to $B$. In particular it contains the subgroup of

PU$(1,1)$ represented by U$(1,1)_{\xi_2}$. Observe that U$(1,1)_{\xi_2}$. contains all the matrix

$$\begin{pmatrix} 1 & 0 \\ ib & 1 \end{pmatrix},$$

with $b \in \mathbf{R}$ and notice that for $a > 0$,

$$\begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ i(1-a) & 1 \end{pmatrix} \begin{pmatrix} 1 & -ia^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ i(a^2-a) & 1 \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}.$$

This shows that $P$ is contained in $BL$ which means that $BL = \mathrm{PU}(1,1)$.

The group $B$ is abelian, and $\mathrm{PU}(1,1)/L = BL/L$. Therefore $L$ contains $\mathrm{PU}(1,1)'$, the commutator subgroup of PU$(1,1)$. Observe that for $\lambda > 0$ and $b \in \mathbf{R}$

$$g(\lambda,0)g(1,b)g(\lambda^{-1},0)g(1,-d) = g(1,d(\lambda^2-1)),$$

which means that $B$ is contained in $\mathrm{PU}(1,1)'$ and therefore, $L = \mathrm{PU}(1,1)$. □

Given a topological group $G$, define $C^n(G)$ the set of continuous functions $G^{n+1} \to \mathbf{R}$ that are invariant for the diagonal action on the left of $G$ on $G^{n+1}$. Define $C^n(G) \xrightarrow{\partial_n} C^{n+1}(G)$ given by

$$\partial_n(f)(g_0,\ldots,g_{n+1}) = \sum_{i=0}^{n+1}(-1)^i f(g_0,\ldots,\hat{g}_i,\ldots,g_{n+1}).$$

For $n \geqslant 1$, the n-*continuous cohomology group* of $G$ is defined as

$$\mathrm{H}^n_c(G) = \ker(\partial_n)/\mathrm{im}(\partial_{n-1}).$$

**Remark 1.3.15.** For every $n \geqslant 1$, $\mathrm{H}^2_c(\mathrm{PU}(1,n) \cong \mathbf{R}$ (see [33]). If $K$ is a maximal compact subgroup of PU$(1,n)$, define

$$C^m_K(\mathrm{PU}(1,n)) \subset C^m(\mathrm{PU}(1,n))$$

as the set of $f \in C^m(\mathrm{PU}(1,n))$ that are alternating and such that for every $g_i \in \mathrm{PU}(1,n)$ and $k_i \in K$,

$$f(g_0k_0,\ldots,g_mk_m) = f(g_0,\ldots,g_m).$$

If

$$C^m_K(\mathrm{PU}(1,n)) \xrightarrow{\partial'_m} C^{m+1}_K(\mathrm{PU}(1,n))$$

is just the restriction of $\partial_m$ then $\ker(\partial'_m)/\mathrm{im}(\partial'_{m-1})$ is isomorphic to $\mathrm{H}^m_c(\mathrm{PU}(1,n))$ (see Theorem 7.4.5 in [39] and the comments after it). Observe that by Proposition 1.3.5, $C^1_K(\mathrm{PU}(1,n))=0$, thus $\mathrm{H}^2_c(\mathrm{PU}(1,n))=\ker(\partial'_2)$, which is a 1-dimensional real vector space.

## 1.4 Functions of complex hyperbolic type

In [40], Monod developed a Gelfand-Naimark-Segal type of construction for actions by isometries on complex hyperbolic spaces. Here a brief discussion about the results of the aforementioned paper is presented.

For $m$ finite or infinite, denote $\mathscr{C}^3\partial\mathbf{H}^m_{\mathbf{C}}$ and $\mathscr{C}^4\partial\mathbf{H}^m_{\mathbf{C}}$ the set of pairwise distinct 3-tuples and 4-tuples of $\partial\mathbf{H}^m_{\mathbf{C}}$, respectively.

**Lemma 1.4.1.** *Given any $(x,y,z)\in(\mathbf{H}^m_{\mathbf{C}})^3$ or any pairwise distinct $(x,y,z)\in\mathscr{C}^3\partial\mathbf{H}^m_{\mathbf{C}}$ , with $m$ either finite or infinite, it is claimed that for any lifts $\tilde{x},\tilde{y},\tilde{z}$ of $x,y,z$,*

$$Re\Big(B(\tilde{x},\tilde{y})B(\tilde{y},\tilde{z})B(\tilde{z},\tilde{x})\Big)\geqslant 0.$$

*If $(x,y,z) in (\mathbf{H}^m_{\mathbf{C}})^3$, the inequality is strict. For $(x,y,z)\in\mathscr{C}^3\partial\mathbf{H}^m_{\mathbf{C}}$, there is an equality if, and only if, for any $\tilde{x},\tilde{y},\tilde{z}$ lifts, the complex vector space generated by them has complex dimension 2.*

**Proof.** The product

$$B(\tilde{x},\tilde{y})B(\tilde{y},\tilde{z})B(\tilde{z},\tilde{x})$$

rescales by a positive real number when changing lifts of $x$, $y$ or $z$. Thus, the statement of this lemma does not depend on the choice of the lifts.

Suppose that $x,y,z\in\mathbf{H}^m_{\mathbf{C}}$ and choose $\tilde{x},\tilde{y},\tilde{z}$ respective lifts such that $B(\tilde{x},\tilde{x})=1$ and

$$B(\tilde{x},\tilde{y})=1=B(\tilde{x},\tilde{z}).$$

With respect to the decomposition $H=\mathbf{C}\tilde{x}\oplus\tilde{x}^{\perp}$, $\tilde{y}=1\oplus u_y$ and $\tilde{z}=1\oplus u_z$.

Now observe that

$$\begin{aligned} B(\tilde{x},\tilde{y})B(\tilde{y},\tilde{z})B(\tilde{z},\tilde{x}) &= B(\tilde{y},\tilde{z}) \\ &= 1+B(u_y,u_z). \end{aligned}$$

Because of $B(\tilde{y},\tilde{y})$ and $B(\tilde{z},\tilde{z})$ are positive, $|B(u_y,u_y)|$ and $|B(u_z,u_z)|$ are smaller than 1. This implies that

$$\mathrm{Re}(B(u_y,u_z))^2 \leqslant B(u_y,u_y)B(u_z,u_z) < 1,$$

which concludes the proof in this case.

Suppose now that $x,y,z \in \partial\mathbf{H}^m_{\mathbf{C}}$ are pairwise distinct and that $\tilde{x},\tilde{y},\tilde{z}$ are respective lifts such that $B(\tilde{x},\tilde{y}) = 1$ and that $B(\tilde{x},\tilde{z}) = 1$. With respect to a decomposition

$$H = \mathbf{C}\tilde{x} \oplus \mathbf{C}\tilde{y} \oplus (\tilde{x}^{\perp} \cap \tilde{y}^{\perp}),$$

$z = 1 \oplus a \oplus u$, with $a \neq 0$.

Observe that

$$\begin{aligned} B(\tilde{x},\tilde{y})B(\tilde{y},\tilde{z})B(\tilde{z},\tilde{x}) &= B(\tilde{y},\tilde{z}) \\ &= \overline{a}. \end{aligned}$$

Observe that

$$B(\tilde{z},\tilde{z}) = 2\mathrm{Re}(a) + B(u,u) = 0,$$

hence $\mathrm{Re}(a) \geqslant 0$ and $\mathrm{Re}(a) = 0$ if, and only if, $\tilde{z}$ belongs to the space generated by $\tilde{x}$ and $\tilde{y}$. □

For $(x,y,z) \in (\mathbf{H}^m_{\mathbf{C}})^3$ or $(x,y,z) \in \mathscr{C}^3\partial\mathbf{H}^m_{\mathbf{C}}$, with $m$ either finite or infinite, the *Cartan argument* of $(x,y,z)$ is defined as

$$\mathrm{Cart}(x,y,z) = \mathrm{Arg}\big(B(\tilde{x},\tilde{y})B(\tilde{y},\tilde{z})B(\tilde{z},\tilde{x})\big),$$

for any $\tilde{x},\tilde{y},\tilde{z}$ lifts of $x,y,z$. Here Arg denotes the principal value of the argument.

It is clear that for any $g \in \mathrm{PU}(1,m)$ and $(x,y,z) \in (\mathbf{H}^m_{\mathbf{C}})^3$ or $(x,y,z) \in \mathscr{C}^3\mathbf{H}^m_{\mathbf{C}}$,

$$\mathrm{Cart}(gx,gy,gz) = \mathrm{Cart}(x,y,z).$$

**Proposition 1.4.2.** *Let* $(x_1,x_2,x_3),(y_1,y_2,y_3) \in \mathscr{C}^3\partial\mathbf{H}^m_{\mathbf{C}}$. *The triples are such that*

$$\mathrm{Cart}(x_1,x_2,x_3) = \mathrm{Cart}(y_1,y_2,y_3)$$

*if, and only if, there exists* $g \in \mathrm{PU}(1,m)$ *such that* $g(x_i) = y_i$.

**Proof.** Suppose that $\tilde{x}_i$ and $\tilde{y}_i$ are respective lifts such that

$$B(\tilde{x}_1, \tilde{x}_2) = B(\tilde{x}_2, \tilde{x}_3) = 1 = B(\tilde{y}_1, \tilde{y}_2) = B(\tilde{y}_2, \tilde{y}_3).$$

Observe that for $\lambda > 0$,

$$B(\lambda \tilde{x}_1, \lambda^{-1}\tilde{x}_2) = B(\lambda^{-1}\tilde{x}_2, \lambda \tilde{x}_3) = 1$$

and

$$B(\lambda \tilde{x}_1, \lambda \tilde{x}_3) = \lambda^2 B(\tilde{x}_1, \tilde{x}_3).$$

This shows that it is possible to choose lifts $\tilde{x}_1, \tilde{x}_2, \tilde{x}_3$. such that

$$B(\tilde{x}_1, \tilde{x}_2) = B(\tilde{x}_2, \tilde{x}_3) = 1$$

and $|B(\tilde{x}_1, \tilde{x}_3)| = 1$. The same can be done when choosing the lifts $\tilde{y}_i$. The fact that

$$\mathrm{Arg}(B(\tilde{x}_3, \tilde{x}_1)) = \mathrm{Cart}(x_1, x_2, x_3) = \mathrm{Cart}(y_1, y_2, y_3) = \mathrm{Arg}(B(\tilde{y}_3, \tilde{x}_1)),$$

implies that

$$B(\tilde{x}_3, \tilde{x}_1) = B(\tilde{y}_3, \tilde{y}_1).$$

Observe that by Lemma 1.4.1, the vectors $\tilde{x}_i$ are linear independent if, and only if, the vectors $\tilde{y}_i$ are linear independent.

If the vectors $\tilde{x}_i$ are linear independent, with the choice made taking representatives and by Remark 1.3.4, it is possible to conclude that there exists $g \in \mathrm{U}(1, m)$ such that $g(\tilde{x}_i) = \tilde{y}_i$.

If the vectors $\tilde{x}_i$ are not linearly independent, then $\tilde{x}_3 = \tilde{x}_1 + a\tilde{x}_2$ and $\tilde{y}_3 = \tilde{y}_1 + b\tilde{y}_2$, with $a = b$. Thus, again by the same arguments used in the previous case, there exists $g \in \mathrm{U}(1, m)$ such that $g(\tilde{x}_1) = \tilde{y}_1$ and $g(\tilde{x}_2) = \tilde{y}_2$, and by the previous observation, $g(\tilde{x}_3) = \tilde{y}_3$. $\square$

**Proposition 1.4.3.** *Suppose* $(x_1, x_2, x_3), (y_1, y_2, y_3) \in (\mathbf{H}^m_{\mathbf{C}})^3$ *are such that* $d(x_i, x_j) = d(y_i, y_j)$. *Then*

$$\mathrm{Cart}(x_1, x_2, x_3) = \mathrm{Cart}(y_1, y_2, y_3)$$

*if, and only if, there exists* $g \in \mathrm{PU}(1, m)$ *such that* $g(x_i) = y_i$.

**Proof.** Choose $\tilde{x}_i$ and $\tilde{y}_i$ lifts such that

$$B(\tilde{x}_i, \tilde{x}_i) = 1 = B(\tilde{y}_i, \tilde{y}_i).$$

Observe that $d(x_i, x_j) = d(y_i, y_j)$, implies that $|B(\tilde{x}_i, \tilde{x}_j)| = |B(\tilde{y}_i, \tilde{y}_j)|$. It is possible to choose lifts such that

$$\{B(\tilde{x}_1,\tilde{x}_2),B(\tilde{x}_2,\tilde{x}_3),B(\tilde{y}_1,\tilde{y}_2),B(\tilde{y}_2,\tilde{y}_3)\}\subset \mathbf{R}_{>0}.$$

Hence $B(\tilde{x}_1,\tilde{x}_2)=B(\tilde{y}_1,\tilde{y}_2)$ and $B(\tilde{x}_2,\tilde{x}_3)=B(\tilde{y}_2,\tilde{y}_3)$. Observe that

$$\mathrm{Arg}(B(\tilde{x}_3,\tilde{x}_1))=\mathrm{Cart}(x_1,x_2,x_3)=\mathrm{Cart}(y_1,y_2,y_3)=\mathrm{Arg}(B(\tilde{y}_3,\tilde{y}_1)).$$

Therefore $B(\tilde{x}_1,\tilde{x}_3)=B(\tilde{y}_1,\tilde{y}_3)$.

By Remark 1.3.4, if $z$ is a vector contained in $W$, the complex vector subspace generated by $\{\tilde{x}_1,\tilde{x}_2,\tilde{x}_3\}$, then $z=0$ if, and only if $B(z,\tilde{x}_i)=0$, for $i=1,2,3$.

If $z=\lambda_1\tilde{x}_1+\lambda_2\tilde{x}_2+\lambda_3\tilde{x}_3$, then $z=0$ if, and only if,

$$\lambda_2B(\tilde{x}_2,\tilde{x}_1)+\lambda_3B(\tilde{x}_3,\tilde{x}_1)=0,$$

$$\lambda_1B(\tilde{x}_1,\tilde{x}_2)+\lambda_3B(\tilde{x}_3,\tilde{x}_2)=0$$

and

$$\lambda_1B(\tilde{x}_1,\tilde{x}_3)+\lambda_2B(\tilde{x}_2,\tilde{x}_3)=0.$$

Hence, there exists $g$ a complex linear map between the complex vector subspaces generated by $\{\tilde{x}_i\}$ and $\{\tilde{y}_i\}$, respectively, such that $g(\tilde{x}_i)=\tilde{y}_i$. Reversing the arguments, it is clear that $g$ is an isomorphism preserving the restriction of $B$ to these subspaces. By Lemma 1.3.1 and because $\iota(B)=1$, the map $g$ can be extended to a map in $\mathrm{U}(1,m)$. □

**Proposition 1.4.4.** *The map*

$$\mathbf{H}^m_{\mathbf{C}}\times\mathbf{H}^m_{\mathbf{C}}\times\mathbf{H}^m_{\mathbf{C}}\xrightarrow{\mathrm{Cart}}\mathbf{R}$$

*is continuous.*

**Proof.** The first remark is that the map $H\times H\xrightarrow{B}\mathbf{C}$ is continuous. Observe that this is clear because the topology of $H$ is the one coming from considering $H$ with a (any) Hermitian product $B_\pm$ with respect to a (any) $\pm$-orthogonal decomposition of $H$ (see Section 1.3). Denote $C_{>0}$ the set of positive vectors in $H$. By Lemma 1.4.1, the function $C^3_{>0}\xrightarrow{S}\mathbf{R}$ given by

$$S(x,y,z)=\mathrm{Arg}\big(B(x,y)B(y,z)B(z,x)\big)$$

is continuous.

The restriction of the projectivization map $C_{>0}\xrightarrow{\pi}\mathbf{H}^m_{\mathbf{C}}$ is a quotient map. Thus, if $m$

is finite, the product $C^3_{>0} \xrightarrow{\pi^3} (\mathbf{H}^m_{\mathbf{C}})^3$ is a quotient map (see for example Theorem 3.3.17 of [26]). Therefore the map

$$(\mathbf{H}^m_{\mathbf{C}})^3 \xrightarrow{\text{Cart}} \mathbf{R}$$

is continuous.

For the case $m = \infty$, fix $x, y, z \in \mathbf{H}^m_{\mathbf{C}}$. Denote $V$ the complex subspace of $H$ generated by the lifts of $x, y$ and $z$ and let $W$ be a 3-dimensional complex subspace of $H$ orthogonal to $V$. By Lemma 1.3.3, the form $B$ restricted to $W$ is negative definite. Denote $\mathbf{H}$ the complex hyperbolic space induced by $V \oplus W$.

Given any other three points $p, q, r \in \mathbf{H}^m_{\mathbf{C}}$ denote $U$ the complex subspace generated by the lifts of $x, y, z, p, q$ and $r$. Observe that $U$ admits a decomposition $V \oplus U'$, where $U'$ is orthogonal to $V$. Again by Lemma 1.3.3, there exists a complex linear map $U \xrightarrow{A} V \oplus W$ that preserves $B$ and such that $A|_V = Id$. The form $B$ is strongly non-degenerate, therefore $A$ can be extended to a map in $\mathrm{U}(1, m)$. If $T$ is the isometry induced by the extension of $A$, then $T|_{\mathbf{H}} = Id$ and $T(p), T(q), T(r) \in \mathbf{H}$. This shows that Cart is continuous at $(p, q, r)$. □

**Lemma 1.4.5.** *If $(w, x, y, z) \in (\mathbf{H}^m_{\mathbf{C}})^4$, then*

$$\text{Cart}(x, y, z) - \text{Cart}(w, y, z) + \text{Cart}(w, x, z) - \text{Cart}(w, x, y) = 0.$$

**Proof.** Choose respective lifts $\tilde{w}, \tilde{x}, \tilde{y}, \tilde{z}$ such that

$$B(\tilde{w}, \tilde{x}) = B(\tilde{x}, \tilde{y}) = B(\tilde{y}, \tilde{z}) = 1.$$

Then

$$\begin{aligned} \text{Cart}(x, y, z) - \text{Cart}(w, y, z) + \text{Cart}(w, x, z) - \text{Cart}(w, x, y) &= \\ \text{Arg}(B(\tilde{z}, \tilde{x})) - \text{Arg}\Big(B(\tilde{w}, \tilde{y})B(\tilde{z}, \tilde{w})\Big) + & \\ \text{Arg}\Big(B(\tilde{x}, \tilde{z})B(\tilde{z}, \tilde{w})\Big) - \text{Arg}(B(\tilde{y}, \tilde{w})) &= \\ \text{Arg}(B(\tilde{z}, \tilde{w})) - \text{Arg}(B(\tilde{z}, \tilde{w})) &= 0. \end{aligned}$$

The last equality is true because the four arguments involved are smaller in absolute value than $\frac{\pi}{2}$. □

The map

$$\mathbf{H}^m_{\mathbf{C}} \times \mathbf{H}^m_{\mathbf{C}} \times \mathbf{H}^m_{\mathbf{C}} \xrightarrow{\text{Cart}} \mathbf{R}$$

is an alternating 2-cocycle and its image is contained in $(-\frac{\pi}{2}, \frac{\pi}{2})$. For further reading about the Cartan argument see [8, 31, 40].

A set $X \subset \mathbf{H}^m_{\mathbf{C}}$ is contained in a real hyperbolic subspace if, and only if, for every $x, y, z \in X$, $\mathrm{Cart}(x, y, z) = 0$ (see Lemma 2.1 in [8]).

A set $X \subset \mathbf{H}^m_{\mathbf{F}}$ is called *total* if there is not a proper and closed $\mathbf{F}$-vector space that contains the lifts of $X$.

The concept of function of complex hyperbolic type, due to Monod [40], is the analogous for hyperbolic representations of the functions of positive type. It is a fundamental tool for the study of hyperbolic representations, particularly for the results of Chapter 3.

Given a topological space $X$, a continuous function $X \times X \xrightarrow{\varphi} \mathbf{C}$ is called a *complex kernel of positive type*, if for every $\lambda_1, \dots \lambda_n \in \mathbf{C}$ and every $x_1, \dots, x_n \in X$,

$$\sum_{i,j} \lambda_i \overline{\lambda}_j \varphi(x_i, x_j) \geqslant 0.$$

See Chapter II.C of [3] and [4] for further reading on kernels of positive type.

Following [40], a pair $(\alpha, \beta)$ is called a *function of hyperbolic type* defined on a topological group $G$, if

$$\alpha : G^3 \to \left(-\tfrac{\pi}{2}, \tfrac{\pi}{2}\right)$$

is a continuous $G$-invariant (with respect to the diagonal action) alternating 2-cocycle,

$$\beta : G \to \mathbf{R}_{>0}$$

is a continuous function, symmetric with respect to the inversion of the group, such that $\beta(e) = 1$ and such that the map

$$(g, k) \mapsto \beta(g)\beta(k) - e^{-i\alpha(g,k,e)}\beta(g^{-1}k)$$

is a complex kernel of positive type.

Given a representation $G \xrightarrow{\rho} \mathrm{Isom}(\mathbf{H}^m_{\mathbf{C}})_o$ and $x \in \mathbf{H}^m_{\mathbf{C}}$, if

$$\beta(g) = \cosh(d(\rho(g)x, x))$$

and

$$\alpha(g, k, l) = \mathrm{Cart}\big(\rho(g)x, \rho(k)x, \rho(l)x\big),$$

then $(\beta, \alpha)$ is a function of complex hyperbolic type (see Proposition 1.9 in [40]). Later on it will be clear that this example is prototypical (see Theorem 1.4.9).

The following is Theorem 1.12 in [40]. It will be crucial in the rest of this work. It constitutes a fundamental tool in the search for new representations for a given group.

**Theorem 1.4.6.** *Let $(\beta,\alpha)$ be a function of complex hyperbolic type defined on a group $G$. If $0 \leqslant t \leqslant 1$, then $(\beta^t, t\alpha)$ is a function of complex hyperbolic type.*

**Lemma 1.4.7.** *Fix $x \in H$ such that $B(x,x)=1$ and consider the Hilbert norm associated to the $\pm$-decomposition $H = \mathbf{C}x \oplus x^{\perp}$. Thus, for every $v, w \in H$ the following hold:*

1. $\|v\|^2 = 2|B(v,x)|^2 - B(v,v)$.
2. $|B(v,w)| \leqslant \|v\|\|w\|$.

**Proof.** For 1., observe that $v = B(v,x)x + u$, for some $u$ orthogonal to $x$. Therefore,

$$\begin{aligned} \|v\|^2 &= |B(v,x)|^2 - B(u,u) \\ &= 2|B(v,x)|^2 - B(v,v). \end{aligned}$$

For 2., notice that if $w = B(w,x)x + u'$, for some $v$ orthogonal to $x$, then if $B_{\pm}$ is the positive definite Hermitian product associated to the $\pm$-decomposition induced by $x$, then

$$\begin{aligned} |B(v,w)|^2 &= |B_{\pm}\big(B(v,x)x + u, B(w,x)x - u'\big)|^2 \\ &\leqslant \|v\|^2 \|B(w,x)x - u'\|^2 \\ &= \|v\|^2\|w\|^2. \end{aligned}$$

□

**Lemma 1.4.8.** *Every $A \in U(1,\infty)$ is a bounded operator with respect to the Hilbert norm induced by $B_{\pm}$.*

**Proof.** Observe that, if the same conventions of the previous lemma are used, for $v \in H$ such that $\|v\| = 1$,

$$\begin{aligned} \|A(v)\|^2 &= 2|B(A(v),x)|^2 - B(A(v),A(v)) \\ &= 2|B(v,A^{-1}(x))|^2 - \|v\|^2 - 2|B(v,x)|^2 \\ &\leqslant 2(\|A^{-1}x\|^2 + \|x\|^2) + 1. \end{aligned}$$

□

The next result is Theorem 1.11 of [40]. A sketch of the proof will be given in order to outline arguments that will often be used in the rest of this text.

**Theorem 1.4.9.** *The pair $(\alpha,\beta)$ is a function of hyperbolic type if, and only if, up to a conjugation by a holomorphic isometry of $\mathbf{H}^m_{\mathbf{C}}$, there exist a unique representation $G \xrightarrow{\rho} \mathrm{Isom}(\mathbf{H}^m_{\mathbf{C}})_o$ and $p \in \mathbf{H}^m_{\mathbf{C}}$ such that the orbit of $p$ is total and*

$$\beta(g) = \cosh d(\rho(g)p, p)$$

*and*

$$\alpha(g_1, g_2, g_3) = \mathrm{Cart}\big(\rho(g_1)p, \rho(g_2)p, \rho(g_3)p\big).$$

*Moreover $\beta$ and $\alpha$ are continuous if, and only if, $\rho$ is orbitally continuous.*

**Proof.** The unicity will be clear from the arguments used in the proof. By definition

$$(g,k) \mapsto \beta(g)\beta(k) - e^{-i\alpha(g,k,e)}\beta(g^{-1}k)$$

is a complex kernel of positive type. By Theorem C.1.4 in [3], there exist a complex Hilbert space $L$ and a continuous function $G \xrightarrow{h} L$ such that the image of $h$ generates a dense subspace of $L$ and such that for every $g,k \in G$,

$$\langle h(g), h(k)\rangle = \beta(g)\beta(k) - e^{i\alpha(e,g,k)}\beta(g^{-1}k).$$

Define $H = \mathbf{C} \oplus L$ and provide it with the Hermitian form $B$ given by

$$B(a \oplus v, b \oplus w) = a\bar{b} - \langle v, w\rangle.$$

By construction $B$ is a strongly non-degenerate form of signature $(1,\infty)$ defined on $H$.

Denote $G \xrightarrow{f} H$ the map given by $f(g) = \beta(g) \oplus h(g)$. Observe that by construction, for every $g \in G$, $B(f(g), f(g)) = 1$ and $|B(f(e), f(g))| = \beta(g)$. Let $\mathbf{H}$ be the complex hyperbolic space induced by $H$ and $B$. Thus, if $\bar{f}(g)$ is the point in $\mathbf{H}$ represented by $f(g)$, then for every $g,k \in G$,

$$\cosh\big(d(\bar{f}(g)\bar{f}(e))\big) = \beta(g)$$

and

$$\begin{aligned} \mathrm{Cart}\big(\bar{f}(g), \bar{f}(k), \bar{f}(e)\big) &= \mathrm{Arg}\big(B(\beta(g) \oplus h(g), \beta(k) \oplus h(k))\big) \\ &= \alpha(g,k,e). \end{aligned}$$

The image of $f$ generates a closed vector space of $H$ because $h(e) = 0$ and because the image of $h$ generates a closed vector space of $L$. Define, for every $g \in G$,

$$T_g(f(k)) = e^{i\alpha(e,g,gk)} f(gk).$$

The claim is that $T_g$ can be extended to a map in $\mathrm{U}(1,\infty)$. Thus, for every $g, k, l \in G$,

$$B\big(T_g(f(k)), T_g(f(k))\big) = e^{i\alpha(e,g,gk)} e^{-i\alpha(e,g,gl)} e^{\alpha(e,gk,gl)} \beta(k^{-1}l).$$

As $\alpha$ is a 2-cocycle,

$$\begin{array}{rcl} \partial\alpha(e,g,gk,gl) & = & \\ \alpha(e,k,l) - \alpha(e,gk,gl) + \alpha(e,g,gl) - \alpha(e,g,gk) & = & 0. \end{array}$$

Therefore,

$$\begin{array}{rcl} B\big(T_g(f(k)), T_g(f(k))\big) & = & e^{i\alpha(e,k,l)} \beta(k^{-1}l) \\ & = & B(f(k), f(l)). \end{array}$$

In particular, this shows, together with the previous observation, that for every $g, k, l \in G$,

$$\mathrm{Cart}\big(\bar{f}(g), \bar{f}(k), \bar{f}(l)\big) = \alpha(g,k,l).$$

If $\sum_{i=1}^{n} \lambda_i f(k_i) = 0$, for some $\lambda_i \in \mathbf{C}$, then

$$B\Big(\sum_{i=1}^{n} \lambda_i T_g(f(k_i)), T_g(f(l))\Big) = 0,$$

for every $l \in G$. The complex vector space generated by $\mathrm{Im}(f)$ is the one generated by $\{T_g(f(l))\}_{l\in G}$. Hence

$$\sum_{i=1}^{n} \lambda_i T_g(f(k_i)) = 0.$$

Denote $V$ the complex vector space generated by $\mathrm{Im}(f)$. The previous observation shows that for every $g \in G$, $T_g$ can be extended to a complex linear map $V \xrightarrow{T_g} V$ that preserves $B$. Using the arguments of Lemma 1.4.8, it is possible to show that for every $g \in G$, $V \xrightarrow{T_g} H$ is uniformly continuous, thus it can be extended to a complex linear map $H \xrightarrow{T_g} H$ that preserves $B$.

For every $g, k, l \in G$, on the one hand

$$\begin{array}{rcl} T_g \circ T_k(f(l)) & = & T_g\big(e^{i\alpha(e,k,kl)} f(kl)\big) \\ & = & e^{i\alpha(e,g,gkl)} e^{i\alpha(e,k,kl)} f(gkl), \end{array}$$

and on the other hand,

$$T_{gk}(f(l)) = e^{i\alpha(e,gk,gkl)} f(gkl).$$

Again, as $\alpha$ is a 2-cocycle,

$$-\alpha(g,gk,gkl) + \alpha(e,gk,gkl) - \alpha(e,g,gkl) + \alpha(e,g,gk) = 0.$$

This shows that

$$T_g \circ T_k = e^{i\alpha(e,g,gk)} T_{gk}.$$

Observe that this identity shows, in particular, that for every $g \in G$, the map $T_g$ is invertible.

Even if $g \mapsto T_g$ is not an homomorphism, there is a well defined homomorphism $G \xrightarrow{\rho} \text{Isom}(\mathbf{H})$, where $\rho(g)$ is the isometry represented by $T_g$.

The last claim is that for every $w \in H$, the map $g \mapsto T_g(w)$ is continuous. Observe that for every $v \in V$ this is true because $\beta$ and $\alpha$ are continuous. Let $U \subset G$ be a symmetric neighborhood of $e$ such that there exists $M' > 0$ such that for every $g \in U$, $\|T_g(f(e))\| < M'$. Hence, by Lemma 1.4.7, for every $w \in H$, $\{\|T_g(w)\|\}_{g \in U}$ is bounded. By the uniform boundedness principle, there exists $M > 0$ such that for every $g \in U$, $\|T_g\| \leqslant M$ (see for example Section 3.3 in [49]).

This implies that for every $w \in H$, the map $g \mapsto T_g(w)$ is continuous at $e$. Indeed, if $v \in V$ and $g \in U$, then

$$\begin{aligned} \|T_g(w) - w\| &\leqslant \|T_g(w) - T_g(v)\| + \|T_g(v) - v\| + \|v - w\| \\ &\leqslant (M'+1)\|v - w\| + \|T_g(v) - u\|. \end{aligned}$$

To conclude the proof for the continuity in $e$, just observe that $V$ is dense and the map $g \mapsto T_g(v)$ is continuous at $e$. This implies that $g \mapsto T_g(v)$ is continuous at every $g_0 \in G$ because the map $g \mapsto T_g T_{g_0}$ is continuous at $e$. To finish the proof just consider $p = \bar{f}(e)$. □

## 1.5 Hyperbolic representations

In this section, following the ideas of Monod [40], some results about representations using the language of functions of positive type are described.

**Proposition 1.5.1.** *Let $G \xrightarrow{\rho} \text{Isom}(\mathbf{H}^m_{\mathbf{C}})_o$ be a representation and suppose $x \in \mathbf{H}^m_{\mathbf{C}}$ is a point with total orbit. If there exists $\omega$, an alternating $G$-invariant 1-cochain, such that $\partial\omega = \alpha$,* *where $\alpha$ is the 2-cocycle associated to $x$, then $\rho$ admits a lift to a representation $G \stackrel{\tilde{\rho}}{\longrightarrow} U(1,m)$.*

**Proof.** Let $T_g$ the map defined in the proof of Theorem 1.4.9. Define

$$T'_g = e^{-i\omega(e,g)} T_g.$$

Observe that on one side for every $g,k \in G$,

$$\begin{array}{rcl} T'_g T'_k & = & e^{-i\omega(e,g)} e^{-i\omega(e,k)} T_g T_k \\ & = & e^{-i\omega(e,g)} e^{-i\omega(e,k)} e^{i\alpha(e,g,gk,)} T_{gh}, \end{array}$$

and that on the other side,

$$\alpha(e,g,gk) = \omega(e,k) - \omega(e,gk) + \omega(e,g).$$

Therefore the map $g \mapsto T'_g$ is a homomorphism. □

Given $x,y \in \mathbf{H}^m_{\mathbf{C}}$ and $\xi \in \partial \mathbf{H}^m_{\mathbf{C}}$, the Cartan argument of the triple $(x,y,\xi)$ can be defined in an analogous way that it has been done for triples of points in $\mathbf{H}^m_{\mathbf{C}}$ or triples of pairwise distinct points in $\partial \mathbf{H}^m_{\mathbf{C}}$. Indeed, for any lifts respective lifts $\tilde{x}, \tilde{y}, \tilde{\xi}$,

$$\mathrm{Re}\Big( B(\tilde{x},\tilde{y}) B(\tilde{y},\tilde{\xi}) B(\tilde{\xi},\tilde{x}) \Big) > 0.$$

This can be shown using similar arguments as the ones used in Lemma 1.4.1. Denote

$$\mathrm{Cart}(x,y,\xi) = \mathrm{Arg}\Big( B(\tilde{x},\tilde{y}) B(\tilde{y},\tilde{\xi}) B(\tilde{\xi},\tilde{x}) \Big).$$

Observe that for every $\xi \in \partial \mathbf{H}^m_{\mathbf{C}}$, the map $(x,y) \mapsto \mathrm{Cart}(x,y,\xi)$ is alternating.

**Lemma 1.5.2.** *For every $\xi \in \partial \mathbf{H}^m_{\mathbf{C}}$ the map $\mathbf{H}^m_{\mathbf{C}} \times \mathbf{H}^m_{\mathbf{C}} \stackrel{\omega_\xi}{\longrightarrow} \mathbf{R}$, given by*

$$\omega_\xi(x,y) = \mathrm{Cart}(x,y,\xi),$$

*is continuous.*

**Proof.** The proof follows from the arguments used in Proposition 1.4.4 and the fact that $\omega_\xi(T(x),T(y)) = \omega_\xi(x,y)$, for every $T \in \mathrm{Isom}(\mathbf{H}^m_{\mathbf{C}})_o$ that fixes $\xi$. □

**Proposition 1.5.3.** *Let $G \xrightarrow{\rho} \mathrm{Isom}(\mathbf{H}^m_{\mathbf{C}})_o$ be a representation and let $x \in \mathbf{H}^m_{\mathbf{C}}$. If $\rho$ fixes a point $\xi \in \partial \mathbf{H}^m_{\mathbf{C}}$, then there exists $\omega$, a continuous and alternating $G$-invariant 1-cochain, such that $\partial\omega = \alpha$, where $\alpha$ is the 2-cocycle associated to $x$.*

**Proof.** Choose respective lifts $\tilde{x}, \tilde{\xi}$ such that $B(\tilde{x}, \tilde{\xi}) > 0$. Let $\tilde{\rho}$ be a linear lift (not necessarily a homomorphism) of $\rho$ such that for every $g \in G$, $\tilde{\rho}(g)(\tilde{\xi}) = \theta_g \tilde{\xi}$, with $\theta_g > 0$. Define, for every $l, k \in G$,

$$\omega(l,k) = \omega_\xi(\rho(l)x, \rho(k)x).$$

The cochain $\omega$ is continuous, $G$-invariant and alternating. Observe that

$$\begin{aligned} \omega(l,k) &= \\ \mathrm{Cart}\big(\rho(l)x, \rho(k)x, \xi\big) &= \\ \mathrm{Arg}\Big(B\big(\tilde{\rho}(l)\tilde{x}, \tilde{\rho}(k)\tilde{x}\big)B\big(\tilde{x}, \tilde{\rho}(k^{-1})\tilde{\xi}\big)B\big(\tilde{\rho}(l^{-1})\tilde{\xi}, \tilde{x}\big)\Big) &= \\ \mathrm{Arg}(B(\tilde{\rho}(l)\tilde{x}, \tilde{\rho}(k)\tilde{x})). & \end{aligned}$$

For every $x_1, x_2, x_3 \in \mathbf{H}^m_{\mathbf{C}}$,

$$|\mathrm{Cart}(x_1, x_2, x_3)| < \pi/2$$

and for $y \in \partial \mathbf{H}^m_{\mathbf{C}}$,

$$|\mathrm{Cart}(x_1, x_2, y)| \leqslant \pi/2.$$

Therefore,

$$\begin{aligned} \mathrm{Arg}(B(\tilde{\rho}(g)\tilde{x}, \tilde{\rho}(l)\tilde{x})) + \mathrm{Arg}(B(\tilde{\rho}(l)\tilde{x}, \tilde{\rho}(k)\tilde{x})) - \mathrm{Arg}(B(\tilde{\rho}(g)\tilde{x}, \tilde{\rho}(k)\tilde{x})) &= \\ \mathrm{Arg}\Big(B\big(\tilde{\rho}(g)\tilde{x}, \tilde{\rho}(l)\tilde{x}\big)B\big(\tilde{\rho}(l)\tilde{x}, \tilde{\rho}(k)\tilde{x}\big)B\big(\tilde{\rho}(k)\tilde{x}, \tilde{\rho}(g)\tilde{x}\big)\Big). & \end{aligned}$$

In other words, $\partial\omega = \alpha$. □

The following corollary is a consequence of Propositions 1.5.1 and 1.5.3.

**Corollary 1.5.4.** *Let $G \xrightarrow{\rho} \mathrm{Isom}(\mathbf{H}^m_{\mathbf{C}})_o$ be a representation and suppose $x \in \mathbf{H}^m_{\mathbf{C}}$ has a total orbit. If $\rho$ fixes a point at infinity, then $\rho$ admits an orbitally continuous lift $\tilde{\rho}$ to $U(1,m)$.*

A group topological $G$ is called *topologically perfect* if the closed group generated by $\{xyx^{-1}y^{-1}\}_{x,y\in G}$ is equal to $G$.

The following proposition is an immediate consequence of Lemma 1.2.3 and Lemma 2.2 and 2.6 in [40].

**Proposition 1.5.5.** *Let* $G \xrightarrow{\rho} \mathrm{Isom}(\mathbf{H}^m_{\mathbf{C}})$ *be a representation and let* $x \in \mathbf{H}^m_{\mathbf{C}}$*. If* $(\beta, \alpha)$ *is the function of complex hyperbolic type associated to* $\rho$ *and* $x$*, then the following hold:*

1. *The function* $\beta$ *is bounded if, and only if,* $\rho$ *fixes a point in* $\mathbf{H}^m_{\mathbf{C}}$*.*
2. *The representation* $\rho$ *preserves a real hyperbolic subspace of* $\mathbf{H}^m_{\mathbf{C}}$ *if, and only if,* $\alpha = 0$*.*
3. *If* $G$ *is perfect and for some* $g \in G$*,* $\lim_{n\to\infty} \beta(g^n)^{\frac{1}{n}} > 1$*, then* $\rho$ *is non-elementary.*

Observe that the asymptotic condition in 3. of the previous proposition, just says that for some $g$, $\rho(g)$ is hyperbolic.

A non-elementary representation $G \xrightarrow{\rho} \mathrm{Isom}(\mathbf{H}^m_{\mathbf{F}})$, with $m$ finite or infinite and $\mathbf{F} = \mathbf{R}, \mathbf{C}$ is called *irreducible* if there is no proper $\mathbf{F}$-hyperbolic subspace of $\mathbf{H}^m_{\mathbf{F}}$ invariant under $\rho$. Observe that for such representation any point in $\mathbf{H}^m_{\mathbf{F}}$ has a total orbit.

The following theorem is Proposition 4.3 in [9]. The statement there is about real hyperbolic spaces, however the proof for the complex case works exactly in the same way.

**Theorem 1.5.6.** *Let* $\mathbf{F} = \mathbf{R}, \mathbf{C}$ *and let* $G \xrightarrow{\rho} \mathrm{Isom}(\mathbf{H}^m_{\mathbf{F}})$ *be a representation. If* $\rho$ *is non-elementary, then there exists* $\mathbf{H}$*, an* $\mathbf{F}$*-hyperbolic subspace of* $\mathbf{H}^m_{\mathbf{F}}$*, invariant under* $\rho$ *and such that if* $\mathbf{H}'$ *is a* $\rho$*-invariant* $\mathbf{F}$*-hyperbolic subspace of* $\mathbf{H}^m_{\mathbf{F}}$*, then* $\mathbf{H} \subset \mathbf{H}'$*.*

The space $\mathbf{H}$ in the previous representation will be called the *irreducible part* of the representation $\rho$.

# Chapter 2

# Representations of PU(1,n)

In this chapter the ideas of [41], that can be track to [9] are adapted for the complex case. In Section 2.1 the restrictions of hyperbolic representations of $\mathrm{PU}(1,n)$ to the stabilizers, either of a point in $\mathbf{H}^n_{\mathbf{C}}$ or of a point in $\partial\mathbf{H}^n_{\mathbf{C}}$, are studied. The latter play a fundamental role: every irreducible representation of $\mathrm{PU}(1,n)$ is determined by its restriction to a stabiliser of a point in $\partial\mathbf{H}^n_{\mathbf{C}}$. This result was proved by Monod & Py [41] for irreducible representations of $\mathrm{PO}(1,n)$. With the ideas of that proof it is possible to derive analogous results for the complex case.

In Section 2.2 the concept of displacement of a representation is studied. This, by way of a common thread, allows to put in the same perspective the results of this work with the classifications made for irreducible representations of the isometry groups of a regular tree (Burger, Iozzi & Monod [9]), the infinite-dimensional real hyperbolic space (Monod & Py [42]) and the finite-dimensional real hyperbolic spaces (Monod & Py [41]).

## 2.1 Non-elementary representations and stabilizers

Let $\xi_1,\xi_2\in\partial\mathbf{H}^m_{\mathbf{F}}$ and let $p\in\mathbf{H}^m_{\mathbf{F}}$ be the point represented by $\frac{1}{\sqrt{2}}(\tilde{\xi}_1+\tilde{\xi}_2)$, where $\tilde{\xi}_i$ are lifts of $\xi_i$ such that $B(\xi_1,\xi_2)=1$. Denote (with a small abuse of notation if $\mathbf{F}=\mathbf{R}$)

$$A=\{g(\lambda,0,Id,0)\}_{\lambda>0}$$

and $K=\mathrm{PO}_{\mathbf{F}}(B)_x$. Observe that by Proposition 1.3.5, $\mathrm{PO}_{\mathbf{F}}(B)=KAK$.

If $G$ is a topological group acting on a metric space $X$, define for $x \in X$ the function $G \xrightarrow{d_x} \mathbf{R}_{\geqslant 0}$ given by $d_x(g) = d(gx, x)$. The following proposition is an observation done in [18] in a much higher generality. A simple proof adapted to the context of this work is shown below.

**Proposition 2.1.1.** *Let $X$ be a metric space and let $H$ a finite-dimensional $\mathbf{F}$-vector space provided with a form $B$ of signature $(1, n)$. If $\mathrm{PO}_{\mathbf{F}}(B) \xrightarrow{\rho} \mathrm{Isom}(X)$ is an orbitally continuous representation, then for every $x \in X$, the map $d_x$, given by*

$$d_x(g) = d(\rho(g)x, x),$$

*is either bounded or proper.*

**Lemma 2.1.2.** *If $G$ acts on a metric space $X$ orbitally continuously and there exist $K_1, K_2$ and $A$ closed subsets of $G$ such that $K_i$ is compact and $G = K_1 A K_2$, then for $x \in X$ the following hold:*

1. *The map $d_x$ is bounded if, and only if, $d_x|_A$ is bounded.*
2. *The map $d_x$ is proper if, and only if, $d_x|_A$ is proper.*

**Proof.** Denote $Q_i = \sup\{d(kx, x)\}_{k \in K_i}$. Observe that for $k_i \in K_i$ and $a \in A$,

$$d_x(k_1 a k_2) - Q_1 - Q_2 \leqslant d_x(a) \leqslant d_x(k_1 a k_2) + Q_1 + Q_2,$$

which proves 1. In order to show that 2. holds, observe that for every $M > 0$,

$$d_x|_A^{-1}[0, M] = A \cap d_x^{-1}[0, M]$$

and that

$$d_x^{-1}[0, M] \subset K_1(d_x|_A^{-1}[0, M + Q_1 + Q_2])K_2. \qquad \square$$

**Proof of Proposition 2.1.1.** Suppose for now that $\mathbf{F} = \mathbf{C}$. If for some $x \in X$, $d_x$ is proper (resp. bounded), then for every $y \in X$, $d_y$ is proper (resp. bounded). Suppose $x \in X$ is such that $d_x$ is non-proper, then by Lemma 2.1.2, $d_x|_A$ is non-proper. Denote $g(\lambda, 0, Id, b) = g(\lambda, b)$ and suppose that $(\lambda_n)_n$ is a sequence escaping compacts such that there exists $M > 0$ such that for every $n$, $d_x(g(\lambda_n, 0)) < M$. Notice that

$$d_x(\lambda, 0) = d_x(\lambda^{-1}, 0)$$

and

$$g(\lambda,0)g(1,b)g(\lambda^{-1},0) = g(1,\lambda^2 b).$$

Denote the subgroup $L = \{g(\lambda,b)\}_{\lambda>0,b\in\mathbf{R}}$. Observe that there exists a compact neighborhood $e \in W \subset L$ such that there exists $M' > 0$ such that for every $w \in W$, $d_x(w) < M'$. Define $\gamma_n = \min\{\lambda_n, \lambda_n^{-1}\}$ and notice that for every $l \in L$, there exists $n$ large enough such that $d_x(g(\gamma_n,0)lg(\gamma_n^{-1},0)) = u \in W$. Therefore

$$d_x(l) \leqslant 2d_x(g(\lambda_n,0)) + d_x(u) \leqslant 2M + M',$$

and by Proposition 2.1.1, $d_x$ is bounded in $\mathrm{PU}(1,m)$. The same proof works for the real case (see the comments before Lemma 1.3.13). □

Recall that the notation $\mathrm{PU}(1,n)$ or $\mathbf{H}^n_{\mathbf{C}}$ is used for the case $n < \infty$. If $\rho$ is a representation of $\mathrm{PU}(1,n)$ and if $g = g(\lambda, v, A, b)$, from now on, $\rho(g)$ will be denoted $\rho(\lambda, v, A, b)$.

The following theorem, but stated for $\mathbf{R}$ and a non-elementary representation of $\mathrm{PO}(1,n)$, is Proposition 2.1 in [41]. The proof given in the aforementioned paper works, just with few changes, for the complex case.

**Theorem 2.1.3.** *If* $\mathrm{PU}(1,n) \xrightarrow{\rho} \mathrm{Isom}(\mathbf{H}^m_{\mathbf{C}})$ *is a non-elementary representation, then* $\rho$ *preserves the type (hyperbolic, elliptic or parabolic). If* $\xi \in \partial\mathbf{H}^n_{\mathbf{C}}$*, then* $\rho(\mathrm{PU}(1,n)_\xi)$ *fixes a unique point in* $\partial\mathbf{H}^m_{\mathbf{C}}$.

**Proof.** The representation is supposed non-elementary, thus by Proposition 2.1.1, for every $x \in \mathbf{H}^m_{\mathbf{C}}$, $d_x$ is proper.

The first claim is that such a representation preserves the elliptic and the non-elliptic types. Observe that if $g \in \mathrm{PU}(1,n)$ is elliptic, then $g$ is contained in a compact subgroup, thus for every $x \in \mathbf{H}^m_{\mathbf{C}}$, the orbit $\{\rho(g^z)x\}_{z\in\mathbf{Z}}$ is bounded. Hence, $\rho(g)$ is elliptic.

Suppose $\rho(g)$ is elliptic and that $x \in \mathbf{H}^m_{\mathbf{C}}$ is fixed by it. The fact that $d_x$ is proper implies that $Stab_x \leqslant \mathrm{PU}(1,n)$ is a compact group, or in other words, is the stabilizer of some point in $\mathbf{H}^n_{\mathbf{C}}$. This shows that $g$ has to be elliptic.

The claim now is that the parabolic type is preserved. Let $g \in \mathrm{PU}(1,n)$ be parabolic and, up to conjugation, it is possible to suppose that $\xi \in \partial\mathbf{H}^n_{\mathbf{C}}$ is the $g$-fixed point. Fix another $\eta \in \partial\mathbf{H}^n_{\mathbf{C}}$ and respective lifts $\tilde{\xi}$ and $\tilde{\eta}$ such that $B(\tilde{\xi},\tilde{\eta}) = 1$. With respect to the decomposition that this choice induces, suppose that $g = g(1, v, A, b)$ (see Proposition 1.3.11).

The first possibility (only if $n > 1$) is that $v \notin \mathrm{Im}(A - Id)$. Consider the non-trivial decomposition of $\mathbf{C}^{n-1}$

$$\ker(A - Id) \oplus \ker(A - Id)^{\perp}.$$

In this decompositon the vector $v$ has the form $v' + v''$, for some $v'$ and some $v'' \neq 0$. The restriction of $A$ to $\ker(A - Id)^{\perp}$ is an automorphism of this subspace, therefore there exists $w \in \ker(A - Id)^{\perp}$ such that $A(w) - w = v''$. Let $D \in \mathrm{U}(n-1)$ be such that if $s = \dim(\ker(A - Id))$, then $D(\mathbf{C}^s \oplus 0) = \ker(A - Id)$. Observe that

$$D^{-1}AD = \begin{pmatrix} Id & 0 \\ 0 & A' \end{pmatrix}.$$

Now notice that

$$\begin{array}{rcl} g(1,w,D,0)^{-1}g(1,v,A,b)g(1,w,D,0) & = & \\ g\big(1,-D^{-1}(w),D^{-1},-e\big)g\big(a,v+A(w),AD,*\big) & = & \\ g\big(1,B^{-1}(A(w)-w+v),D^{-1}AD,*\big) & = & \\ g\big(1,D^{-1}(v''-v),D^{-1}AD,*\big) & = & \\ g(1,D^{-1}(v'),D^{-1}AD,d), & & \end{array}$$

for some $d \in \mathbf{R}$. From the definition of $D$, the vector $u = D^{-1}(v')$ belongs to the space generated $\mathbf{C}^s \oplus 0$. Therefore

$$\begin{array}{rcl} g(1,u,Id,0)g(1,u,Id \oplus A',d) & = & g(1,2u,Id \oplus A',d) \\ & = & g(1,u,Id \oplus A',d)g(1,u,Id,0). \end{array}$$

Observe that

$$\begin{array}{rcl} g(2,0,Id,0)g(1,u,Id,0)g(1/2,0,Id,0) & = & g(2,0,Id,0)g(1/2,2u,Id,0) \\ & = & g(1,2u,Id,0) \\ & = & g(1,u,Id,0)^2. \end{array}$$

The isometry $h = \rho(1,u,Id,0)$ cannot be hyperbolic because it is conjugated to $h^2$, and therefore,

$$\ell(h) = \ell(h^2) = 2\ell(h).$$

As the non-elliptic type is preserved, $\rho(1,u,Id,0)$ is parabolic.

If $\rho(1,u,Id \oplus A',d)$ is hyperbolic then on the one hand, because the two isometries commute, $\rho(h)$ preserves the set of points in $\partial\mathbf{H}^m_{\mathbf{C}}$ fixed by $\rho(1,u,Id\oplus A',d)$ in $\partial\mathbf{H}^m_{\mathbf{C}}$. On the other hand, the isometry $\rho(h)$ is parabolic, therefore $\rho(1,u,Id\oplus A',d)$ has to preserve the $h$-fixed point in $\partial\mathbf{H}^m_{\mathbf{C}}$, which is a contradiction. Indeed, $h$ just has one fixed point in $\partial\mathbf{H}^m_{\mathbf{C}}$. This shows that $\rho(1,u,Id\oplus A',d)$, and therefore $\rho(1,v,A,b)$, is parabolic.

For the other case, suppose $g(1,v,A,b)$ is parabolic and such that there exists $u$ such that $A(u)-u=v$ and $\mathrm{Im}(B(A(u),v))\neq b$.

Observe that,

$$\begin{array}{rcl} g(1,-u,Id,0)^{-1}g(1,v,A,b)g(1,-u,Id,0) & = \\ g\big(1,v+u,A,b-\mathrm{Im}(B(v,u))\big)g(1,-u,Id,0) & = \\ g\Big(1,v+u-A(u),A,b-\mathrm{Im}(B(v,u))-\mathrm{Im}(B(-A(u),v+u))\Big) & = \\ g\Big(1,0,A,b-\mathrm{Im}(B(v,u))+\mathrm{Im}(B(A(u),v))+\mathrm{Im}(B(A(u),u))\Big) & = \\ g\big(1,0,A,b+\mathrm{Im}(B(A(u),v))\big) & = \\ g\big(1,0,A,b-\mathrm{Im}(B(A(u),u))\big) & = \\ g(1,0,A,d), & \end{array},$$

for some $d\neq 0$.

The transformations $g(1,0,A,d)$ and $g(1,0,Id,d)$ commute and

$$\begin{array}{rcl} g\big(\tfrac{1}{\sqrt{2}},0,Id,0\big)^{-1}g(1,0,Id,d)g\big(\tfrac{1}{\sqrt{2}},0,Id,0\big) & = \\ g(\sqrt{2},0,Id,0)g\big(\tfrac{1}{\sqrt{2}},0,Id,\sqrt{2}d\big) & = \\ g(1,0,Id,2d) & = \\ g(1,0,Id,d)^2. & \end{array}.$$

From this point the proof for the claim is exactly the same as in the first case. Thus the parabolic type is preserved.

Define $H=\{g(1,0,Id,b)\}_{b\in\mathbf{R}}$. For each $h\in H\backslash\{e\}$, the transformation $h$ is parabolic. Since $H$ is abelian, $\rho(H)$ has a unique common fixed point $\omega\in\partial\mathbf{H}^m_{\mathbf{C}}$. Notice that $H$ is a normal subgroup of $P$, therefore for every $g\in P$, the transformation $\rho(g)$ fixes $\omega$.

Consider now the hyperbolic isometry $g=g(\gamma,0,Id,0)$, for some $\gamma>0$, and suppose $\rho(g)$ is parabolic. The group $\{\rho(\lambda,0,Id,0)\}_{\lambda>\mathbf{R}}$ is abelian, hence for every $\lambda>0$, $\rho(\lambda,0,Id,0)$ is parabolic. Let $s\in\mathrm{Isom}(\mathbf{H}^n_{\mathbf{C}})$ be the isometry represented, in the decomposition

$$\mathbf{C}\tilde{\xi}\oplus\mathbf{C}\tilde{\eta}\oplus(\tilde{\xi}^{\perp}\cap\tilde{\eta}^{\perp}),$$

by the matrix

$$\sigma = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & Id \end{pmatrix}.$$

Observe that

$$sg(\lambda, 0, Id, 0)s^{-1} = g(\lambda^{-1}, 0, Id, 0).$$

Thus, as $\rho(\lambda, 0, Id, 0)$ is parabolic, then $\rho(s)$ fixes $\omega$. This is a contradiction because by Proposition 1.3.5, $\mathrm{PU}(1,n)$ is generated by $P$ and $s$ and the representation $\rho$ is non-elementary.

To conclude, observe that given the double transitivity of $\mathrm{PU}(1,n)$ in $\partial\mathbf{H}^n_{\mathbf{C}}$, up to a conjugation, every $g \in \mathrm{PU}(1,n)$ hyperbolic can be supposed of the shape $g(\lambda, 0, A, 0)$, for some $\lambda > 0$ and $A \in \mathrm{U}(n-1)$.

Observe that for every $T \in \mathrm{U}(n-1)$ and $\gamma > 0$, $g(\gamma, 0, Id, 0)$ and $g(1, 0, T, 0)$ commute, thus if $\omega' \neq \omega$ is the other fixed point at infinity by $\{\rho(\lambda, 0, Id, 0)\}_{\lambda>0}$, then for every $T$, $\rho(1, 0, T, 0)(\omega') = \omega'$. Therefore,

$$\rho(\lambda, 0, A, 0)(\omega') = \rho(\lambda, 0, Id, 0)\rho(1, 0, A, 0)(\omega') = \omega'.$$

As $g(\lambda, 0, A, 0)$ is hyperbolic and $\rho(\lambda, 0, A, 0)$ is neither parabolic nor elliptic, then it is possible to conclude that the hyperbolic type is preserved. □

**Remark 2.1.4.** The previous theorem shows that given $\mathrm{PU}(1,n) \xrightarrow{\rho} \mathrm{Isom}(\mathbf{H}^m_{\mathbf{C}})_o$, a non-elementary representation, there exists a map $\partial\mathbf{H}^n_{\mathbf{C}} \xrightarrow{\Gamma} \partial\mathbf{H}^m_{\mathbf{C}}$ defined in the following way. If $\xi \in \partial\mathbf{H}^n_{\mathbf{C}}$ and $P < \mathrm{PU}(1,n)$ is its stabilizer, let $\Gamma(\xi)$ be the unique $\rho(P)$-fixed point in $\partial\mathbf{H}^m_{\mathbf{C}}$. Observe that $\Gamma$ is injective because given two distinct $\xi_1, \xi_2 \in \partial\mathbf{H}^n_{\mathbf{C}}$, the group $\mathrm{PU}(1,n)$ is generated by $P_{\xi_1} \cup P_{\xi_2}$, the respective stabilizers in $\mathrm{PU}(1,n)$ of $\xi_1$ and $\xi_2$ (see Corollary 1.3.6).

**Proposition 2.1.5.** *Let* $\mathrm{PU}(1,n) \xrightarrow{\rho} \mathrm{Isom}(\mathbf{H}^m_{\mathbf{C}})_o$ *be non-elementary. For a group* $G < \mathrm{PU}(1,n)$ *the following hold:*

1. *The group* $G$ *fixes a point* $x \in \mathbf{H}^n_{\mathbf{C}}$ *if, and only if, it fixes a point* $y \in \mathbf{H}^m_{\mathbf{C}}$.
2. *If* $G$ *fixes a point* $\eta \in \partial\mathbf{H}^m_{\mathbf{C}}$, *then* $G$ *fixes a point in* $\mathbf{H}^n_{\mathbf{C}} \cup \partial\mathbf{H}^n_{\mathbf{C}}$.

**Proof.** The point 1. is an immediate consequence of Propositions 1.2.4 and 2.1.1.

For 2., observe that by Proposition 1.2.4, if $G$ does not contain hyperbolic isometries then the claim of the proposition holds. Suppose then that $G$ contains hyperbolic elements.

Let $\eta \in \partial\mathbf{H}^m_{\mathbf{C}}$ be a $\rho(G)$-fixed point. Suppose $g, h \in G$ are two distinct hyperbolic isometries and let $\xi^g_1, \xi^g_2$ and $\xi^h_1, \xi^h_2$ be the respective fixed points of $g$ and $h$ in $\partial\mathbf{H}^n_{\mathbf{C}}$. The claim is that there exist $i, j \in \{1,2\}$ such that $\xi^g_i = \xi^h_j$. If that is not the case and $\Gamma$ is the map defined in Remark 2.1.4, observe that

$$\{\Gamma(\xi^g_1), \Gamma(\xi^g_2)\} \cap \{\Gamma(\xi^h_1), \Gamma(\xi^h_2)\} = \varnothing.$$

This is a contradiction because $\rho(g)$ and $\rho(h)$ fix $\eta \in \partial\mathbf{H}^m_{\mathbf{C}}$.

This argument shows that there exists $v \in \partial\mathbf{H}^n_{\mathbf{C}}$ fixed by $g$ and $h$ and such that $\Gamma(v) = \eta$. Observe that with the same argument, it is clear that every hyperbolic isometry of $G$ fixes $v$.

Let $P_\eta < \mathrm{Isom}(\mathbf{H}^m_{\mathbf{C}})_o$ be the stabilizer of $\eta$ and let $P_\eta \xrightarrow{\kappa} G$ be the Busemann character associated to $\eta$ (see Remark 1.3.7). The map $G \xrightarrow{\kappa\circ\rho} \mathbf{R}$ is a homomorphism and denote $L$ its kernel.

Due to the fact that $\rho$ preserves the type and $\kappa(g) \neq 0$, if and only if $g$ is hyperbolic, then $L$ does not contain any hyperbolic isometry. By Proposition 1.2.4, either $L$ fixes a point in $\mathbf{H}^n_{\mathbf{C}}$ or it fixes a point in $\partial\mathbf{H}^n_{\mathbf{C}}$.

Suppose that $L$ fixes a point in $\mathbf{H}^n_{\mathbf{C}}$. Denote $Y \subset \mathbf{H}^n_{\mathbf{C}}$ the set of $L$-fixed points. The space $Y$ is closed and convex, therefore there exists a projection $\mathbf{H}^n_{\mathbf{C}} \xrightarrow{p_Y} Y$. Since $L$ is normal in $G$, the space $Y$ is $G$-invariant and the map $p_Y$ is $G$-equivariant. Hence for every $g \in G$ hyperbolic, the space $Y$ contains the axis preserved by $g$. This shows that $v$ is $L$-fixed, and therefore, a $G$-fixed point.

Now suppose that $L$ does not fix a point in $\mathbf{H}^n_{\mathbf{C}}$. There exists, by Proposition 1.2.4, $\xi \in \partial\mathbf{H}^n_{\mathbf{C}}$ that is fixed by $L$. If $\xi \neq v$, then $\rho(L)$ preserves, and therefore fixes pointwise, the axis connecting $v$ and $\xi$. This is a contradiction because $L$ does not fix a point in $\mathbf{H}^n_{\mathbf{C}}$, hence $\xi = v$. □

**Remark 2.1.6.** If the representation $\mathrm{PU}(1,n) \xrightarrow{\rho} \mathrm{Isom}(\mathbf{H}^m_{\mathbf{C}})_o$ considered is irreducible the point 1 of the previous proposition admits a much stronger statement. Indeed, if $K < \mathrm{PU}(1,n)$ is a maximal compact subgroup, by Proposition 5.8 and Remark 5.9 in [40], there exists a unique $\rho(K)$ fixed point in $\mathbf{H}^m_{\mathbf{C}}$. This fact and Theorem 2.1.3 show that if $\mathrm{PU}(1,n) \to \mathrm{Isom}(\mathbf{H}^m_{\mathbf{C}})_o$ is an irreducible representation there exist unique $\mathrm{PU}(1,n)$-equivariant maps $\mathbf{H}^n_{\mathbf{C}} \to \mathbf{H}^m_{\mathbf{C}}$ and $\partial\mathbf{H}^n_{\mathbf{C}} \to \partial\mathbf{H}^m_{\mathbf{C}}$.

Let $\mathrm{PU}(1,n) \xrightarrow{\rho} \mathrm{Isom}(\mathbf{H}^m_{\mathbf{C}})_o$ be a non-elementary representation and let $\xi_1, \xi_2$ in $\partial\mathbf{H}^n_{\mathbf{C}}$ be two distinct points. By Theorem 2.1.3, after fixing $\tilde{\xi}_1, \tilde{\xi}_2$, respective lifts of $\xi_1, \xi_2$ such that $B(\tilde{\xi}_1, \tilde{\xi}_2) = 1$, for every $g \in P = Stab(\xi_1)$, $g = g(\lambda, v, A, b)$. Moreover there are two distinguished points $\eta_1, \eta_2 \in \partial \mathbf{H}^m_{\mathbf{C}}$ such that $\rho(P)(\eta_1) = \eta_1$ and such that for every $\lambda > 0$, $\rho(\lambda, 0, Id, 0)(\eta_2) = \eta_2$.

Observe that If $\mathrm{PU}(1,n) \xrightarrow{\rho'} \mathrm{Isom}(\mathbf{H})_o$, with $\mathbf{H} \subset \mathbf{H}^m_{\mathbf{C}}$, is the irreducible part of $\rho$ (see Theorem 1.5.6), then $\eta_i \in \partial \mathbf{H}$. Indeed, the axis fixed by $\rho(\lambda, 0, Id, 0)$ has to be contained in $\mathbf{H}$ because the projection $\mathbf{H}^m_{\mathbf{C}} \to \mathbf{H}$ is $\rho(\mathrm{PU}(1,n))$-equivariant and contracting.

**Remark 2.1.7.** Let $\tilde{\eta}_i$ be lifts of $\eta_i$ such that $B(\tilde{\eta}_1, \tilde{\eta}_2) = 1$. Observe that with respect to the decomposition $\mathbf{C}\tilde{\eta}_1 \oplus \mathbf{C}\tilde{\eta}_2 \oplus \tilde{\eta}_1^\perp \cap \tilde{\eta}_2^\perp$, each transformation $\rho(g_{\lambda,v,A,b})$ has a unique linear representation such that the image of $\eta_1$ under this linear transformation is a positive multiple of $\eta_1$. That is to say, if $\mathbf{H}^m_{\mathbf{C}}$ is induced from a Hilbert space $H$ and a form $B$, for every $g \in P$, $\rho(g)$ has a representation with the shape,

$$\begin{pmatrix} \chi(g) & -\frac{\chi(g)B(c(g),c(g))}{2} + i\Delta(g) & -\chi(g)B\big(\pi(g)(\cdot), c(g)\big) \\ 0 & \chi(g)^{-1} & 0 \\ 0 & c(g) & \pi(g) \end{pmatrix},$$

where, by Corollary 1.5.4, $P \xrightarrow{c} \tilde{\eta}_1^\perp \cap \tilde{\eta}_1^\perp$ and $P \xrightarrow{\Delta} \mathbf{R}$ are continuous functions, $P \xrightarrow{\chi} \mathbf{R}_{>0}$ is a continuous homomorphism, and $P \xrightarrow{\pi} \mathrm{U}$, where U is the group of unitary transformations of $\tilde{\eta}_1^\perp \cap \tilde{\eta}_1^\perp$, is a strongly continuous unitary representation of $P$. For $g = g(\lambda, v, A, b)$, denote $c(g) = c(\lambda, v, A, b)$ and $\pi(g) = \pi(\lambda, v, A, b)$. These conventions and notation will be used through all this work.

Given a group $G$ and a unitary representation $\pi$ into a Hilbert space $H$, a map $G \xrightarrow{c} H$ is called an *affine cocycle* (of $\pi$) if for every $g, k \in G$,

$$c(gh) = c(g) + \pi(g)c(h)$$

(see Chapter 2 of [3]).

The following is an immediate observation using the matrix representations of the elements in $\rho(P)$.

**Lemma 2.1.8.** *For $g, k \in P$, the following hold:*

1. $c(gk) = \chi(k)^{-1}c(g) + \pi(g)c(k)$.
2. $\Delta(gk) = \chi(g)\Delta(k) + \chi(k)^{-1}\Delta(g) - Im\big(B(\pi(g)c(k), c(g))\big)$.

Observe that if $L = \ker(\chi)$, then $c|_L$ is an affine cocycle of $\pi_L$.

**Lemma 2.1.9.** *For every $g = g(\lambda, v, A, b) \in P$ the following hold:*

1. $\chi(g) = \lambda^t$, *for some* $t \in \mathbf{R}\backslash\{0\}$.

2. $c(\lambda, 0, A, 0) = 0$.

3. $\Delta(\lambda, 0, A, 0) = 0$.

4. $c(1, \lambda v, Id, \lambda^2 b) = \lambda^t \pi(\lambda, 0, Id, 0) c(1, v, Id, b)$.

5. $\Delta(1, \lambda v, Id, \lambda^2 b) = \lambda^{2t} \Delta(1, v, Id, b)$

6. *If* $g = g(1, 0, Id, b)$, *then*

$$\Delta(1, 0, Id, 2b) = 2\Delta(g) - Im\big(B(\pi(g)c(g), c(g))\big).$$

**Proof.** For 1., observe that, by Proposition 1.3.8 and Remark 1.3.7, $g(1, v, A, b)$ is not hyperbolic. Thus, by Theorem 2.1.3, $\rho(1, v, A, b)$ is not hyperbolic and $\chi(g(1, v, A, b)) = 1$. As

$$g = g(\lambda, 0, Id, 0) g(1, v, A, \lambda^{-1} b),$$

then $\chi(g(\lambda, v, A, b)) = \chi(g(\lambda, 0, Id, 0))$.

Observe that for $\lambda \neq 1$, $\chi(g(\lambda, 0, Id, 0)) \neq 1$ (see Proposition 1.3.8 and Remark 1.3.7), therefore the map $\lambda \mapsto \chi(g(\lambda, 0, Id, 0))$ is a continuous isomorphism of $\mathbf{R}_{>0}$.

The points 2. and 3. are a consequence of the fact that the isometries $\rho(\lambda, 0, Id, 0)$ and $\rho(1, 0, A, 0)$ commute, and therefore $\rho(\lambda, 0, A, 0)(\eta_i) = \eta_i$.

For the 4. and 5., observe that

$$g(\lambda, 0, Id, 0) g(1, v, Id, b) g(\lambda^{-1}, 0, Id, 0) = g(1, \lambda v, Id, \lambda^2 b).$$

Thus, points 2. and 3. and Lemma 2.1.8 imply that,

$$c(1, \lambda v, Id, \lambda^2 b) = \lambda^t \pi(\lambda, 0, Id, 0)\big(c(1, v, Id, b)\big)$$

and

$$\Delta(1, \lambda v, Id, \lambda^2 b) = \lambda^{2t} \Delta(1, v, Id, b).$$

The point 6. is an immediate consequence of the identity

$$g(1,0,Id,2b) = g(1,0,Id,b)^2. \qquad \square$$

Let

$$\sigma = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & Id \end{pmatrix}$$

and denote $s \in \mathrm{PU}(1,n)$ the isometry induced by $\sigma$.

**Lemma 2.1.10.** *For every $g(1,v,Id,b) \in \mathrm{PU}(1,n)$, if $\alpha = \frac{B(v,v)}{2} + ib$, then*

$$sg(1,v,Id,b)sg\big(1,\alpha^{-1}v,Id,b|\alpha|^{-2}\big)s = g\big(|\alpha|^{-1}, -\tfrac{\alpha}{|\alpha|}v, T, -b|\alpha|^{-1}\big),$$

*for some $T \in U(1,n)$.*

**Proof.** It will be shown first that there exists $\theta \in \mathbf{C}$ such that

$$\sigma g(1,v,Id,b)\sigma g\big(1,\alpha^{-1}v,Id,b|\alpha|^{-2}\big)\sigma(\tilde{\xi}_1) = \theta\tilde{\xi}_1.$$

Indeed,

$$\begin{aligned} \sigma g\big(1,\alpha^{-1}v,Id,b|\alpha|^{-2}\big)\sigma(\tilde{\xi}_1) &= \\ \sigma g\big(1,\alpha^{-1}v,Id,b|\alpha|^{-2}\big)(\tilde{\xi}_2) &= \\ \sigma\Big(-\tfrac{\overline{\alpha}}{|\alpha|^2}\tilde{\xi}_1 + \tilde{\xi}_2 + \alpha^{-1}v\Big) &= \\ \tilde{\xi}_1 - \alpha^{-1}\tilde{\xi}_2 + \alpha^{-1}v & \end{aligned}$$

and

$$\begin{aligned} g(1,-v,Id,-b)\sigma(\tilde{\xi}_1) &= g(1,-v,Id,-b)(\xi_2) \\ &= -\alpha\tilde{\xi}_1 + \tilde{\xi}_2 - v. \end{aligned}$$

Therefore

$$\sigma g(1,v,Id,b)\sigma g\big(1,\alpha^{-1}v,Id,b|\alpha|^{-2}\big)\sigma(\tilde{\xi}_1) = -\alpha^{-1}\tilde{\xi}_1.$$

Observe that

$$\begin{aligned} \sigma g(1,v,Id,b)\sigma g\big(1,\alpha^{-1}v,Id,b|\alpha|^{-2}\big)\sigma(\tilde{\xi}_2) &= \\ \sigma g(1,v,Id,b)(\tilde{\xi}_2) &= \\ \tilde{\xi}_1 + \Big(-\tfrac{B(v,v)}{2} + ib\Big)\tilde{\xi}_2 + v & \end{aligned}$$

and that for $u \in \tilde{\xi}_1^{\perp} \cap \tilde{\xi}_2^{\perp}$,

$$\begin{array}{rcl} \sigma g(1,v,Id,b)\sigma g\big(1,\alpha^{-1}v,Id,b|\alpha|^{-2}\big)\sigma(u) & = \\ \sigma g(1,v,Id,b)\sigma\left(-\frac{B(u,v)}{\overline{\alpha}}\tilde{\xi}_1+u\right) & = \\ \sigma g(1,v,Id,b)\left(-\frac{B(u,v)}{\overline{\alpha}}\tilde{\xi}_2+u\right) & = \\ \sigma\left(-\frac{B(u,v)}{\overline{\alpha}}\tilde{\xi}_2-\frac{B(u,v)}{\overline{\alpha}}v+u\right) & = \\ -\frac{B(u,v)}{\overline{\alpha}}\tilde{\xi}_1-\frac{B(u,v)}{\overline{\alpha}}v+u. \end{array}$$

Therefore, if

$$R=\sigma g(1,v,Id,b)\sigma g(1,\alpha^{-1}v,Id,b|\alpha|^{-2})\sigma,$$

it has been shown that

1. $R(\tilde{\xi}_1)=-\alpha^{-1}\tilde{\xi}_1$.
2. $R(\tilde{\xi}_2)=\tilde{\xi}_1-\overline{\alpha}\tilde{\xi}_2+v$.
3. $R(u)=-\frac{B(u,v)}{\overline{\alpha}}\tilde{\xi}_1-\frac{B(u,v)}{\overline{\alpha}}v+u$, for every $u\in\tilde{\xi}_1^{\perp}\cap\tilde{\xi}_2^{\perp}$.

Observe that $-\frac{\alpha}{|\alpha|}R(\tilde{\xi}_1)=|\alpha^{-1}|\tilde{\xi}_1$. Thus, after this normalization,

$$-\frac{\alpha}{|\alpha|}R(\xi_2)=-\frac{\alpha}{|\alpha|}\tilde{\xi}_1+|\alpha|\tilde{\xi}_2-\frac{\alpha}{|\alpha|}v.$$

This shows that

$$sg(1,v,Id,b)sg\big(1,\alpha^{-1}v,Id,b|\alpha|^{-2}\big)s=g\big(|\alpha|^{-1},-\frac{\alpha}{|\alpha|}v,T,-b|\alpha|^{-1}\big),$$

for some $T\in \mathrm{U}(1,n)$.

□

The following proposition is the complex version of Lemma 2.2 and Proposition 2.4 of [41]. The proof for the complex case require minor modifications.

**Proposition 2.1.11.** *Let* $\mathrm{PU}(1,n)\xrightarrow{\rho}\mathrm{Isom}(\mathbf{H}^m_{\mathbf{C}})_0$ *be irreducible. If* $V$ *is the closed complex vector space generated by*

$$\{c(1,v,Id,b)\mid v\in\mathbf{C}^{n-1}\ \textit{and}\ b\in\mathbf{R}\},$$

*then $V = \eta_1^\perp \cap \tilde{\eta}_2^\perp$ and the restriction of ρ to P determines the representation.*

**Proof.** The first claim is that the space $\mathbf{C}\tilde{\eta}_1 \oplus \mathbf{C}\tilde{\eta}_2 \oplus V$ is $\rho(P)$-invariant. Observe that for every for every $g(\lambda, v, A, b)$,

$$g(\lambda, v, A, b) = g(1, \lambda v, Id, \lambda b) g(\lambda, 0, A, 0).$$

Therefore, by Lemma 2.1.9,

$$c(\lambda, v, A, b) = \lambda^{-t} c(1, \lambda v, Id, \lambda b).$$

From this observation and by Lemma 2.1.8, it is possible to conclude that $\mathbf{C}\tilde{\eta}_1 \oplus \mathbf{C}\tilde{\eta}_2 \oplus V$ is $\rho(P)$-invariant.

Again, let $s \in \mathrm{PU}(1, n)$ be the isometry represented, with respect to the decomposition induced by $\tilde{\xi}_i$, by the matrix

$$\sigma = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & Id \end{pmatrix}.$$

Due to the identity

$$sg(\lambda, 0, Id, 0)s = g(\lambda^{-1}, 0, Id, 0),$$

the map $\rho(s)$ leaves invariant the set of fixed points at infinity of the family

$$\{\rho(\lambda, 0, Id, 0)\}_{\lambda > 0}.$$

The representation $\rho$ is non-elementary, therefore $\rho(s)$ exchanges $\eta_1$ with $\eta_2$ (see Proposition 1.3.5).

Observe that the map $\rho(s)$ admits a linear lift $\tau$ such that, with respect to the decomposition induced by $\tilde{\eta}_i$,

$$\tau = \begin{pmatrix} 0 & \nu^{-1} & 0 \\ \nu & 0 & 0 \\ 0 & 0 & A \end{pmatrix},$$

with $\nu > 0$ and $A$ a unitary transformation.

The claim now is that $A$ leaves invariant the space $V$. Define

$$K(v, b) = -\frac{B\big(c(1, v, Id, b), c(1, v, Id, b)\big)}{2} + i\Delta(1, v, Id, b)$$

and identify the isometries $\rho(1,v,Id,b)$ with their linear lifts such that

$$\tilde{\rho}(1,v,Id,b)(\tilde{\eta}_1)=\tilde{\eta}_1.$$

Using the notation of Lemma 2.1.10, observe that on the one hand,

$$\begin{array}{rl} \tau\rho(1,v,Id,b)\tau\rho\big(1,\alpha^{-1}v,Id,b|\alpha|^{-2}\big)\tau(\tilde{\eta}_2) & = \\ \nu^{-1}\tau\rho(1,v,Id,b)\tau\rho\big(1,\alpha^{-1}v,Id,b|\alpha|^{-2}\big)(\tilde{\eta}_1) & = \\ \nu^{-1}\tau\rho(1,v,Id,b)\tau(\tilde{\eta}_1) & = \\ \tau\rho(1,v,Id,b)(\tilde{\eta}_2) & = \\ \tau\big(K(v,b)\tilde{\eta}_1+\tilde{\eta}_2+c(1,v,Id,b)\big) & = \\ \nu^{-1}\tilde{\eta}_1+\nu K(v,b)\tilde{\eta}_2+Ac(1,v,Id,b) & \end{array}.$$

On the other hand,

$$\begin{array}{rl} \rho\big(|\alpha|^{-1},-\frac{\alpha}{|\alpha|}v,T,-b|\alpha|^{-1}\big)(\tilde{\eta}_2) & = \\ \rho\big(1,-\frac{\alpha}{|\alpha|^2}v,Id,-b|\alpha|^{-2}\big)\rho(|\alpha|^{-1},0,T,0)(\tilde{\eta}_2) & = \\ |\alpha|^t\Big(K(-\frac{\alpha}{|\alpha|^2}v,-b|\alpha|^{-2})\tilde{\eta}_1+\eta_2+c\big(1,-\frac{\alpha}{|\alpha|^2}v,Id,-b|\alpha|^{-2}\big)\Big) & = \\ |\alpha|^{-t}K(-\alpha v,-b)\tilde{\eta}_1+|\alpha|^t\eta_2+|\alpha|^t c\big(1,-\frac{\alpha}{|\alpha|^2}v,Id,-b|\alpha|^{-2}\big). & \end{array}$$

Therefore

$$\begin{array}{rl} \rho\big(|\alpha|^{-1},-\frac{\alpha}{|\alpha|}v,T,-b|\alpha|^{-1}\big)(\tilde{\eta}_2) & = \\ \frac{\overline{K(v,b)}}{|K(v,b)|}\left(\nu^{-1}\eta_1+\nu K(v,b)\eta_2+Ac(1,v,Id,b)\right). & \end{array}$$

The last identity implies the following:

1. $\frac{\overline{K(v,b)}}{|K(v,b)|}Ac(1,v,Id,b)=|\alpha|^t c\big(1,-\frac{\alpha}{|\alpha|^2}v,Id,-\frac{b}{|\alpha|^2}\big).$
2. $\nu|K(v,b)|=|\alpha|^t.$

This computations show that $V$ is $A$-invariant and that $A$ and $\nu$ are determined by the restriction of $\rho$ to $P$. Observe that they also show that $\mathbf{C}\tilde{\eta}_1+\mathbf{C}\tilde{\eta}_2+V$ is $\rho$-invariant, so as $\rho$ is supposed irreducible, this finishes the proof. □

**Remark 2.1.12.** The following identities will be used extensively in Chapter 3.

1. $\frac{\overline{K(v,b)}}{|K(v,b)|}Ac(1,v,Id,b)=|\alpha|^t c\big(1,-\frac{\alpha}{|\alpha|^2}v,Id,-\frac{b}{|\alpha|^2}\big).$

2. $\nu|K(v,b)| = |\alpha|^t$.

Observe that, up to conjugating $\rho$ with an isometry $\rho(\lambda, 0, Id, 0)$, it is possible to suppose that $\nu$ takes any positive value.

## 2.2 Representations and displacement

Recall that given any non-elementary representation $\mathrm{PU}(1,n) \xrightarrow{\rho} \mathrm{Isom}(\mathbf{H}^m_{\mathbf{C}})_o$, there are fixed $\xi_i \in \partial \mathbf{H}^n_{\mathbf{C}}$, $\eta_i \in \partial \mathbf{H}^m_{\mathbf{C}}$ and respective lifts of them chosen like in Theorem 2.1.3. For $u \in \tilde{\eta}_1^{\perp} \cap \tilde{\eta}_2^{\perp}$, denote $\|u\| = (-B(u,u))^{\frac{1}{2}}$. The following theorem and its proof, in their real version, are contained in Proposition 2.3 in [41].

**Proposition 2.2.1.** *If* $\mathrm{PU}(1,n) \xrightarrow{\rho} \mathrm{Isom}(\mathbf{H}^m_{\mathbf{C}})_o$ *is a non-elementary representation with a total orbit and* $P \xrightarrow{\chi} \mathbf{R}_{>0}$ *is such that* $\rho(g)\tilde{\eta}_1 = \chi(g)\tilde{\eta}_1$*, then for every* $g = g(\lambda, v, A, b) \in P$ *(see Remark 1.3.9) there exists* $t > 0$ *such that* $\chi(g) = \lambda^t$*. Moreover:*

1. *If* $n > 1$*, then* $t \leqslant 1$.
2. *If* $n = 1$*, then* $t \leqslant 2$.

**Proof.** Observe that for $b \neq 0$, the isometry $\rho(1,0,Id,b)$ is parabolic. Therefore either $\Delta(1,0,Id,b) \neq 0$ or $c(1,0,Id,b) \neq 0$. In both cases, due to the continuity of $\Delta$ and $c$,

$$\lim_{\lambda \to 0} \Delta(1,0,Id,\lambda b) = 0$$

and

$$\lim_{\lambda \to 0} c(1,0,Id,\lambda b) = 0.$$

By Lemma 2.1.9,

$$\Delta(1,0,Id,\lambda b) = \lambda^t \Delta(1,0,Id,b)$$

and

$$\|c(1,0,Id,\lambda b)\| = \lambda^t \|c(1,0,Id,b)\|,$$

therefore $t > 0$.

For the case $n > 1$, observe that for every $v \in \mathbf{C}^{n-1}$ and $A \in \mathrm{U}(n-1)$,

$$g(1,v,A,0) = g(1,v,Id,0)g(1,0,A,0).$$

Thus, by Lemmas 2.1.8 and 2.1.9,

$$\Delta(1,v,A,0)=\Delta(1,v,Id,0).$$

But $g(1,v,Id,0)=g(1,0,A,0)g(1,A^{-1}(v),A^{-1},0)$, therefore

$$\Delta(1,v,Id,0)=\Delta(1,A^{-1}(v),A^{-1},0)=\Delta(1,A^{-1}(v),Id,0).$$

Recall that $g(1,v,Id,0)^{-1}=g(1,-v,Id,0)$, therefore

$$\begin{array}{rcl} \Delta(1,v,Id,0)+\Delta(1,-v,Id,0)- & & \\ \operatorname{Im}\Big(B\big(\pi(1,v,Id,0)c(1,-v,Id,0),c(1,v,Id,0)\big)\Big) & = & . \\ \Delta(1,v,Id,0)+\Delta(1,-v,Id,0) & = & 0 \end{array}$$

This equality shows that for every $v\in\mathbf{C}^{n-1}\backslash\{0\}$, $\Delta(1,v,Id,0)=0$, and therefore, as $\rho(1,v,Id,0)$ is parabolic, $c(1,v,Id,0)\neq 0$. Observe that

$$c(1,2v,Id,0)=c(1,v,Id,0)+\pi(1,v,Id,0)c(1,v,Id,0).$$

Therefore for every $v\neq 0$,

$$\begin{array}{rcl} 2^t\|c(1,v,Id,0)\| & = & \|c(1,2v,Id,0)\| \\ & \leqslant & 2\|c(1,v,Id,0)\|, \end{array}$$

which shows that if $n>1$,then $t\leqslant 1$.

For the case $n=1$, observe that

$$c(1,2b)=c(1,b)+\pi(1,b)c(1,b),$$

therefore $2^{\frac{t}{2}}|c(1,b)|\leqslant 2|c(1,b)|$. Thus, if $c(1,b)\neq 0$ for some (every) $b$, then $t\leqslant 2$. If this is not the case, by Lemma 2.1.8, the map $b\mapsto\Delta(1,b)$ is a (non-trivial) homomorphism. Therefore

$$2^t\Delta(1,b)=\Delta(1,2b)=2\Delta(1,b),$$

showing that $t=1$. □

Given an irreducible representation $\mathrm{PU}(1,n)\xrightarrow{\rho}\mathrm{Isom}(\mathbf{H}^m_{\mathbf{C}})_o$, define the *displacement* of $\rho$ as $\ell(\rho)=t$, where $t$ is such that for every $g=g(\lambda,v,A,b)\in P$, $\chi(g)=\lambda^t$. The definition of the displacement of a representation makes sense if $\rho$ is just supposed non-elementary. Indeed, If $\mathrm{PU}(1,n)\xrightarrow{\rho}\mathbf{H}$ is the irreducible part of $\rho$, then for every hyperbolic $g\in P$, the hyperbolic space **H** contains the axis preserved by $\rho(g)$ (see Theorem 2.1.3).

Observe that by Proposition 1.3.8 and Remark 1.3.7, if $g = g(\lambda, v, A, b)$, then $\ell(g) = |\ln(\lambda)|$. Every hyperbolic isometry $g \in \mathrm{PU}(1,n)$ can be conjugated to a hyperbolic isometry in $P$. As the displacement of an isometry is invariant under conjugations and for every $g$ non-hyperbolic, $\ell(g) = 0$, the following proposition holds.

**Proposition 2.2.2.** *If* $\mathrm{PU}(1,n) \xrightarrow{\rho} \mathrm{Isom}(\mathbf{H}^m_{\mathbf{C}})_o$ *is a non-elementary representation with a total orbit, then for every* $g \in \mathrm{PU}(1,n)$,

$$\ell(\rho(g)) = \ell(\rho)\ell(g).$$

Let $G$ be a locally compact group and $f, g :\to \mathbf{R}_+$ two function. Denote $f \preceq g$ if there exists $M > 0$ and a compact subset $K \subset G$ such that $f \leqslant Mg$ outside $K$. Denote $f \prec g$ if, for every $M > 0$, there exists a compact subset $K \subset G$ such that $f \leqslant Mg$ outside $K$.

Given a topological group $G$ and a strongly continuous unitary representation $\pi$ in a Hilbert space $H$, for every $v \in H$, the continuous map $G \xrightarrow{b_v} H$, given by $b_v(g) = \pi(g)v - v$ is a cocycle. The cocycles $b_v$ are called *coboundaries.*

Denote $Z^1(G,\pi)$ the set of continuous cocycles of $\pi$ and $B^1(G,\pi)$ the set of coboundaries.

Let $G$ be a locally compact, compactly generated group and let $S \subset G$ be an open, relatively compact generating set. For $g \in G$, denote $|g|_S$ the word length of $g$ with respect to $S$. That is to say for $g \in G$,

$$|g|_S = \inf\{k \in \mathbf{N} \mid g = s_1 \cdot \dots \cdot s_k, \text{ with } s_i \in S\}.$$

Observe that if $\pi$ a unitary representation of $G$ and $b \in Z^1(G,\pi)$, then $\|b(\cdot)\| \preceq |\cdot|_s$.

The following definitions and observations are in [19]. Let $G$ a locally compact, compactly generated group, with $S$ an open, relatively closed generating subset and let $\pi$ be a unitary representation of $G$. Define

$$\mathrm{lin}(G,\pi) = \{b \in Z^1(G,\pi) \mid \|b\| \succeq |\cdot|_S\}$$

and

$$\mathrm{sublin}(G,\pi) = \{b \in Z^1(G,\pi) \mid \|b\| \prec |\cdot|_S\},$$

namely the set of cocycles with linear (respectively sublinear) growth.

Let $\overline{B^1}(G,\pi)$ be the closure of $B^1(G,\pi)$ inside $Z^1(G,\pi)$ with respect of the topology of convergence on compact subsets. In [19], the authors showed that $\overline{B^1}(G,\pi) \subset \mathrm{sublin}(G,\pi)$. Denote $\overline{H^1}(G,\pi) = Z^1(G,\pi)/\overline{B^1}(G,\pi)$, the *reduced 1-cohomology* of the representation $\pi$.

**Theorem 2.2.3.** *If* $\mathrm{PU}(1,n) \xrightarrow{\rho} \mathrm{Isom}(\mathbf{H}^m_{\mathbf{C}})_o$ *is an irreducible representation with* $n > 1$*, then the following are equivalent:*

1. $\ell(\rho) = 1$.
2. *There exists* $b \in \mathbf{R}$ *such that* $c(1,0,Id,b) = 0$.
3. *For every* $b \in \mathbf{R}$, $c(1,0,Id,b) = 0$.
4. *There exists* $b \in \mathbf{R}$ *such that* $\pi(1,0,Id,b) = Id$.
5. *For every* $b \in \mathbf{R}$, $\pi(1,0,Id,b) = Id$.
6. *For every* $v, w \in \mathbf{C}^{n-1}$, $\pi(1,v,Id,0)$ *and* $\pi(1,w,Id,0)$ *commute.*
7. *The map* $b \mapsto \Delta(1,0,Id,b)$ *is a homomorphism and for every* $b \in \mathbf{R}$, $c(1,0,Id,b) = 0$.
8. *Up to conjugating* $\rho$ *with an isometry, The map* $v \mapsto c(1,v,Id,0)$ *is injective and* $\mathbf{C}$*-linear or antilinear.*
9. *The map* $v \mapsto c(1,v,Id,0)$ *is* $\mathbf{R}$*-linear and injective.*
10. *The group* $\{\pi(1,v,Id,b)\}_{v,b}$ *has a non-zero fixed point.*

**Proof.** It will be shown first that the properties from 1. to 7. are equivalent. It is trivial that 5. implies 4., 8. implies 9 and it is clear, by Lemma 2.1.9, that 2. and 3. are equivalent.

To prove that 4. implies 5., observe that if $b \in \mathbf{R}$ is such that $\pi(1,0,Id,b) = Id$, then for every $\lambda > 0$,

$$\begin{aligned}\pi(1,0,Id,b) &= \pi(\lambda,0,Id,0)\pi(1,0,Id,b)\pi(\lambda^{-1},0,Id,0) \\ &= \pi(1,0,Id,\lambda^2 b).\end{aligned}$$

This, together with the fact that $\pi(1,0,Id,-b) = Id$, shows the implication.

To show that 5. and 6. are equivalent, observe that for every $v, w \in \mathbf{C}^{n-1}$,

$$\begin{aligned}\pi(1,v,Id,0)\pi(1,w,Id,0)\pi(1,-v,Id,0)\pi(1,-w,Id,0) &= \\ \pi\big(1,0,Id,-2\mathrm{Im}(B(v,w))\big).\end{aligned}$$

To deduce 7. from 6., first notice that for every $b,d\in\mathbf{R}$,

$$\begin{aligned}-\mathrm{Im}\Big(B\big(\pi(1,0,Id,d)c(1,0,Id,b),c(1,0,Id,d)\big)\Big) &= \\ \Delta(1,0,Id,b+d)-\Delta(1,0,Id,b)-(1,0,Id,d) &= \\ -\mathrm{Im}\Big(B\big(\pi(1,0,Id,b)c(1,0,Id,d),c(1,0,Id,b)\big)\Big).\end{aligned}$$

Hence if for every $b\in R$, $\pi(1,0,Id,b)=Id$, then

$$-\mathrm{Im}\Big(B\big(c(1,0,Id,b),c(1,0,Id,d)\big)\Big)=-\mathrm{Im}\Big(B\big(c(1,0,Id,d),c(1,0,Id,b)\big)\Big).$$

The only way this can happen is if $b\mapsto\Delta(1,0,Id,b)$ is a homomorphism. Observe now that

$$c(1,0,Id,2b)=c(1,0,Id,b)+\pi(1,0,Id,b)c(1,0,Id,b)=2c(1,0,Id,b).$$

It is also true that $\|c(1,0,Id,2b)\|=2^{\frac{t}{2}}\|c(1,0,Id,b)\|$.This means that for every $b\in\mathbf{R}$, $c(1,0,Id,,b)=0$ and for every $b\neq 0$, $\Delta(1,0,Id,b)\neq 0$.

To show that 7. implies 1., notice that in this case, for every $b\in\mathbf{R}$,

$$2\Delta(1,0,Id,b)=\Delta(1,0,Id,2b)=2^t\Delta(1,0,Id,b).$$

The claim now is that 1. implies 2. The restrictions of $\pi$ and $c$ to the subgroup

$$L=\{g(1,v,Id,b)\}_{v\in\mathbf{C}^{n-1},b\in\mathbf{R}}$$

define an affine isometric action on the Hilbert space $\tilde{\eta}_1^{\perp}\cap\tilde{\eta}_2^{\perp}$. The claim is that the cocycle $c$ does not have sublinear growth, in other words, it is claimed that $\overline{H^1}(G,\pi)\neq 0$.

Recall from the proof of Proposition 2.2.1 that for every $v\in\mathbf{C}^{n-1}$,

$$\Delta(1,v,Id,0)=0,$$

and therefore,

$$c(1,v,Id,0)\neq 0.$$

Define

$$S=\{(v,t)\in\mathbf{C}^{n-1}\times\mathbf{R}\mid\|v\|<1\text{ and }|t|<1\}.$$

Observe that for $v \in \mathbf{C}^{n-1}$, we have that

$$\|v\| \leqslant |(v,0)|_S < \|v\| + 1.$$

As $\ell(\rho) = 1$, for every $v$ and every $\lambda > 0$,

$$\|c(1, \lambda v, Id, 0)\| = \lambda \|c(1, v, Id, 0)\|.$$

Fix $v_0 \in \mathbf{C}^{n-1}$ and let $M > 0$ be such that $\|c(1, v_0, Id, 0)\| = M\|v_0\|$.

If $c|_L \in \text{sublin}(L, \pi)$, then for given $0 < \epsilon_0$ there exists a compact subset $K \subset \mathbf{C}^{n+1} \times \mathbf{R}$ such that if $(v, b) \notin K$, then

$$\|c(1, v, Id, b)\| \leqslant \epsilon |(1, v, b)|_S.$$

Therefore for every $\lambda > 0$ large enough,

$$\lambda M \|v_0\| = \|c(1, \lambda v_0, Id, 0)\| \leqslant \epsilon |(\lambda v_0, 0)|_S < \epsilon(\|\lambda v_0\| + 1).$$

This is a contradiction because $M$ is fixed and the previous inequality implies that for every $\lambda > 0$ large enough,

$$\frac{\lambda M \|v_0\|}{\lambda \|v_0\| + 1} < \epsilon,$$

or in other words, $M \leqslant \epsilon$, which is a contradiction. Therefore $\overline{H^1}(L, \pi|_L) \neq 0$.

Now the claim is that for every $b \in \mathbf{R}$, $c(1, 0, Id, b) = 0$. If $\pi$ is a unitary representation of locally compact group in a separable Hilbert space, then there exists $(X, \mu)$ a standard measure space such that $\tau$ is equivalent to a direct integral of $\mu$-almost everywhere irreducible representation $\int^{\oplus} \tau_\alpha d\mu(\alpha)$. For an introductory read on this topic and references, see Chapter 7 of [27].

In Proposition 2.6 of Chapter III of [32], Guichardet showed that if $G$ is a locally compact group and $\tau = \int^{\oplus} \tau_\alpha d\mu(\alpha)$ is a unitary representation, then if for $\mu$-almost every $\alpha$, $\overline{H^1}(G, \tau_\alpha) = 0$, then $\overline{H^1}(G, \tau) = 0$.

In Theorem V.6 of [21], Delorme showed that if $G$ is a connected and solvable Lie group and $(K, \tau)$ is an irreducible unitary representation such that $\dim_{\mathbf{C}}(K) \geqslant 2$, then $\overline{H^1}(G, \tau) = 0$.

Putting together all these results if $(X, \mu)$ is the standard measure space from the integral decomposition of the representation $\pi|_L$, then there exists $Y \subset X$, a measurable subset such that $\mu(Y) > 0$ and such that for every $\alpha \in Y$, $\pi_\alpha$ is a one-dimensional representation.

Observe that for every $\alpha \in Y$, $\pi_\alpha|_{[L:L]}$ is trivial. This shows that

$$W = \{w \in \tilde{\eta}_1^\perp \cap \tilde{\eta}_2^\perp \mid \pi(l)w = w, \text{ for every } l \in [L:L]\}$$

is a non-zero closed vector space. The group $\{g(1,0,Id,b)\}_{b\in\mathbf{R}}$ is normal in $P$, therefore $W$ is $P$-invariant. Hence, the representation $\pi$ preserves the decomposition $W \oplus W^\perp = \tilde{\eta}_1^\perp \cap \tilde{\eta}_2^\perp$. Denote $c = c_1 \oplus c_2$ and $\pi = \pi_1 \oplus \pi_2$.

The cocycle $c_1|_{[L:L]}$ is a homomorphism, but for every $b \in \mathbf{R}$

$$\|c(1,0,Id,2b)\| = 2^{\frac{1}{2}}\|c(1,0,Id,b)\|.$$

Therefore $c_1$ vanishes on $[L:L]$.

Observe that for every $v \in \mathbf{C}^{n-1}$ and $\lambda > 0$,

$$\|c_2(1,\lambda v, Id, 0)\| = \lambda\|c_2(1,v,Id,0)\|).$$

Thus, if for some $v \in \mathbf{C}^{n-1}$, $c_2(1,v,Id,0) \neq 0$, then $c_2|_L$ does not have sublinear growth and therefore, $\overline{H^1}(L,\pi_2|_L) \neq 0$. Thus repeating the arguments already used, $W^\perp$ contains a non-zero $\pi_2([L:L])$-invariant vector, but this is a contradiction. All the $\pi([L:L])$-invariant vectors are contained in $W$.

This shows that for every $v \in \mathbf{C}^{n-1}$, $c_2(1,v,Id,0) = 0$. Observe that this implies that $c_2|_L = 0$, and therefore, for every $b \in \mathbf{R}$, $c(1,0,Id,b) = 0$. With this, the equivalence for the first seven properties is complete.

To show that 1.-7. imply 8., observe that for every $v, w \in \mathbf{C}^{n-1}$,

$$\pi(1, v+w, Id, 0) = \pi(1,v,Id,0)\pi(1,w,Id,0)$$

and that

$$c(1,v+w,Id,0) = c(1,v,Id,0) + \pi(1,v,Id,0)c(1,w,Id,0).$$

In this point the argument is the same as the one in the proof of Proposition 2.3 of [41]. For completeness it is reproduced here. For now change the notation and write $c(1,v,Id,0) = c(v)$ and $\pi(1,v,Id,0) = \pi(v)$. Notice that for every $v \in \mathbf{C}^{n-1}$,

$$\|c(2v)\| = \|c(v) + \pi(v)c(v)\| = 2\|c(v)\|,$$

and therefore, $\pi(v)c(v) = c(v)$. Moreover, for every $v, w \in \mathbf{C}^{n-1}$,

$$\begin{aligned}\pi(v)\pi(w)\big(c(v)+\pi(v)c(w)\big) &= \pi(v)\pi(w)\big(c(v)+c(v+w)-c(v)\big)\\ &= \pi(w)c(v)+c(v+w)-\pi(w)c(v)\\ &= c(w)+\pi(w)c(v).\end{aligned}$$

Hence,

$$\begin{aligned}c(w) &= \pi(v)\pi(w)\big(c(v)+\pi(v)c(w)\big)-\pi(w)c(v)\\ &= \pi(2v)c(w).\end{aligned},$$

This shows that $\pi(w)c(v) = c(v)$ and $c(v+w) = c(v)+c(w)$. The cocycle $c$ is a continuous homomorphism, so it is a $\mathbf{R}$-linear map.

From the proof of Proposition 2.1.11 and the fact that $\ell(\rho) = 1$, for every $v \neq 0$,

$$\nu = \frac{\|v\|^2}{\|c(1, v, Id, 0)\|^2}$$

and because for every $b \in \mathbf{R}\backslash 0$, $c(1,0,Id,b) = 0$, then

$$\nu = \frac{|\Delta(1,0,Id,b)|}{|b|}.$$

The map $b \mapsto \Delta(1,0,Id,b)$ is a nontrivial continuous homomorphism, hence

$$\Delta(1,0,Id,b) = \theta b,$$

for some $\theta \neq 0$. Up to conjugation it is possible to suppose, without lost of generality, that $\nu = 1$.

For every $v \in \mathbf{C}^{n-1}$,

$$g(1,0,iId,0)g(1,v,Id,0) = g(1,iv,iId,0),$$

and therefore,

$$c(1,iv,Id,0) = c(1,iv,iId,0) = \pi(1,0,iId,0)c(1,v,Id,0).$$

Observe also that

$$g(1,v,Id,0)g(1,iv,Id,0) = g\big(1,v+iv,Id,-B(v,v)\big)$$

and that

$$g\big(1, v+iv, Id, -B(v,v)\big) = g(1,0,Id,-B(v,v))g(1,v+iv,Id,0).$$

Hence,

$$\begin{aligned}\theta B(v,v) &= -\Delta(1,0,Id,-B(v,v))\\ &= -\Delta(1,v+iv,Id,-B(v,v))\\ &= \operatorname{Im}\Big(B\big(c(1,iv,Id,0),c(1,v,Id,0)\big)\Big)\\ &= \operatorname{Im}\Big(B\big(\pi(1,0,iId,0)c(1,v,Id,0),c(1,v,Id,0)\big)\Big).\end{aligned}$$

If we consider $\pi(1,0,iId,0)$ as a transformation of the finite-dimensional complex vector space generated by $\{c(1,v,Id,0)\}_{v\in\mathbf{C}^{n-1}}$, it is possible to decompose this space as the orthogonal sum of the (may be trivial) vector subspaces $W_1$, $W_{-1}$, $W_i$ and $W_{-i}$, where $\pi(1,0,iId,0)$ acts by multiplication by $1,-1,i$ and $-i$ respectively. For every $v$, if

$$c(1,v,Id,0) = v_1+v_{-1}+v_i+v_{-i},$$

where $v_i\in W_i$, observe that

$$\operatorname{Im}\Big(B\big(\pi(1,0,iId,0)c(1,v,Id,0),c(1,v;Id,0)\big)\Big) = B(v_i,v_i)-B(v_{-i},v_{-i})\neq 0$$

Therefore

$$\theta B(v,v) = B(v_i,v_i)-B(v_{-i},v_{-i}).$$

It will be shown later that the case $\theta=1$ or $\theta=-1$ are somehow equivalent (see Remark 3.1.4). So if $\theta=1$, $v_i=c(1,v,Id,0)$ and if $\theta=-1$, then $v_{-i}=c(1,v,Id,0)$, which shows that 8. holds.

It is clear that 9. implies 1. because

$$\|c(1,2v,Id,0)\| = 2^t\|c(1,v,Id,0)\| = 2\|c(1,v,Id,0)\|.$$

The claim now is that 1. implies 10. Indeed, for every $b\in\mathbf{R}$, $c(1,0,Id,b)=0$ and the map $v\mapsto c(1,v,Id,0)$ is linear, therefore

$$\begin{aligned}c(1,v,Id,0)+c(1,w,Id,0) &= c(1,v+w,Id,0)\\ &= c(1,v+w,Id,-\operatorname{Im}(B(w,v))\\ &= c(1,v,Id,0)+\pi(1,v,Id,0)c(1,w,Id,0).\end{aligned}$$

Hence for every $v, w \in \mathbf{C}^{n-1}$,

$$c(1,w,Id,0) = \pi(1,v,Id,0)c(1,w,Id,0).$$

Observe that for every $b \in \mathbf{R}$,

$$\begin{aligned}\pi(1,0,Id,b)c(1,v,Id,0) &= c(1,v,Id,b) \\ &= c(1,v,Id,0).\end{aligned}$$

This finishes the proof for the claim.

The claim now is that 10. implies 1. Let $W$ be the closed subspace of fixed vectors in $\tilde{\eta}_1 \perp \cap \tilde{\eta}_2^{\perp}$ and let $p$ be the orthogonal projection on this space.

Write $c(1,v,Id,b) = u_1 + u_1$ and $c(1,v',Id,b') = u_1' + u_2'$ in the decomposition $V = W \oplus W^{\perp}$. Then

$$\begin{aligned}p \circ c\big(1, v+v', Id, b+b' - \mathrm{Im}(B(v',v))\big) &= \\ p\big(c(1,v,Id,b) + \pi(1,v,Id,b)c(1,v',Id,b')\big) &= \\ p\big(u_1 + u_2 + \pi(1,v,Id,b)(u_1' + u_2')\big) &= \\ u_1 + u_1' + p(\pi(1,v,Id,b)u_2') &= \\ u_1 + u_1'.\end{aligned}$$

In other words, the map

$$g(1,v,Id,b) \mapsto p \circ c(1,v,Id,b)$$

is an homomorphism.

It is claimed that for every $b > 0$, $p \circ c(1,0,Id,b) = 0$. Denote

$$c(1,0,Id,1) = u_1 + u_2,$$

with respect to the decomposition $W \oplus W^{\perp}$. Observe that in the same decomposition,

$$c(1,0,Id,b) = bu_1 + w(b),$$

where $w$ is a function of $b$. But also

$$c(1,0,Id,b) = b^{\frac{t}{2}}\pi(b^{\frac{1}{2}},0,Id,0)c(1,0,Id,1) = b^{\frac{t}{2}}(a_1 + a_2),$$

where $\|a_1\|^2+\|a_2\|^2=\|u_1\|^2+\|u_2\|^2$. So for every $b\geqslant 0$ the following equality holds

$$b^2\|u_1\|^2+\|w(b)\|^2=b^t(\|u_1\|^2+\|u_2\|^2).$$

As $0<t\leqslant 1$ and $\|w(b)\|^2\geqslant 0$, then $u_1=0$.

If for every $v$, $p\circ c(1,v,Id,0)=0$, then $p\circ c(1,v,Id,b)=0$, but this is a contradiction since the vectors of the form $c(1,v,Id,b)$ generate a dense subspace. So let $v$ such that $p\circ c(1,v,Id,0)\neq 0$. With the same argument, in the decomposition $W\oplus W^{\perp}$, $c(1,v,Id,0)=u_1+u_2$ and for every $\lambda\geqslant 0$,

$$p\circ c(1,\lambda v,Id,0)=\lambda p\circ c(1,v,Id,0).$$

Therefore, $c(1,\lambda v,Id,0)=\lambda u_1+w(\lambda)$, with $w$ a function of $\lambda$. Again,

$$c(1,\lambda v,Id,0)=\lambda^t\pi(\lambda,0,Id,0)c(1,v,Id,0)=\lambda^t(a_1+a_2),$$

with $\|a_1\|^2+\|a_2\|^2=\|u_1\|^2+\|u_2\|^2$. This implies that for every $\lambda\geqslant 0$,

$$\lambda^2\|u_1\|^2+\|w(\lambda)\|^2=\lambda^{2t}(\|u_1\|^2+\|u_2\|^2).$$

Since $u_1\neq 0$, then $t=1$. □

**Proposition 2.2.4.** *If* $\mathrm{PU}(1,1)\xrightarrow{\rho}\mathrm{Isom}(\mathbf{H}^m_{\mathbf{C}})_o$ *is an irreducible representation then such that* $\ell(\rho)=2$, *then* $b\mapsto c(1,b)$ *is a non-trivial linear map and for every* $b\in\mathbf{R}$, $\Delta(1,b)=0$.

**Proof.** Observe that

$$\begin{aligned}2\|c(1,b)\| &= \|c(1,2b)\|\\ &= \|c(1,b)+\pi(1,b)c(1,b)\|.\end{aligned}$$

Therefore $\pi(1,b)c(1,b)=c(1,b)$. Observe that this implies that for every $b,d\in\mathbf{R}$

$$\begin{aligned}\pi(1,b)c(1,d)+c(1,b) &= c(b+d)\\ &= \pi(1,b+d)c(1,b+d)\\ &= \pi(1,b)\pi(1,d)\big(c(1,d)+\pi(1,d)c(1,b)\big)\\ &= \pi(1,b)c(1,d)+\pi(1,d)c(1,b).\end{aligned}$$

Thus, for every $b,d\in\mathbf{R}$, $\pi(d)c(b)=c(b)$, or in other words, the map $b\mapsto c(1,b)$ is linear.

This implies that for every $b \in \mathbf{R}$,

$$\operatorname{Im}\big(B(\pi(1,b)c(1,b),c(1,b))\big) = 0.$$

Hence for every $b \in \mathbf{R}$,

$$4\Delta(1,b) = \Delta(1,2b) = 2\Delta(1,b).$$

□

Compare this result to the construction before Lemma 1.3.13.

It has been shown in this section that for every irreducible representation $\mathrm{PU}(1,n) \xrightarrow{\rho} \mathrm{Isom}(\mathbf{H}^m_{\mathbf{C}})_o$, with $m$ infinite,

1. If $n = 1$, then $\ell(\rho) \in (0,2)$.

2. If $n > 1$, then $\ell(\rho) \in (0,1)$.

For irreducible representations $\mathrm{PO}(1,n) \xrightarrow{\rho} \mathrm{Isom}(\mathbf{H}^\infty_{\mathbf{R}})$, in [41], Monod & Py showed, among many other things, that $\ell(\rho) \in (0,1)$ and that the displacement is a complete invariant. That is to say, if two such representations $\rho$, $\rho'$ are such that $\ell(\rho) = \ell(\rho')$, then $\rho$ and $\rho'$ are equivalent. Moreover, for every $t \in (0,1)$ there exists $\rho$ and irreducible representation such that $\ell(\rho) = t$.

The theory of functions of complex hyperbolic type developed by Monod in [40] allowed him to show the existence part of the classification in a much easier way. However this has the setback of not being a constructive proof. The argument for the complex case is shown below, the real case is analogous.

Indeed for every $x \in \mathbf{H}^n_{\mathbf{R}}$ there is a function of complex hyperbolic type $(\beta, \alpha)$ defined on $\mathrm{PU}(1,n)$ given by $\beta(g) = d(gx, x)$ and

$$\alpha(g_1, g_2, g_3) = \mathrm{Cart}\big(g_1x, g_2x, g_3x\big).$$

The functions of complex hyperbolic type defined in this way for $\mathrm{PU}(1,n)$ will be called *tautological* functions of complex hyperbolic type. For every $t \in (0,1)$, $(\beta^t, t\alpha)$ is a function of complex hyperbolic type. Thus, there exists a representation $\mathrm{PU}(1,n) \xrightarrow{\rho'_t} \mathrm{Isom}(\mathbf{H}^m_{\mathbf{C}})$, for some $m$ such that there exists $x \in \mathbf{H}^m_{\mathbf{C}}$ with total orbit and such that

$\beta^t(g) = \cosh\big(d(\rho'_t(g)x, x)\big)$ and

$$t\alpha(g_1, g_2, g_3) = \mathrm{Cart}\big(\rho'_t(g_1)x, \rho'_t(g_2)x, \rho'_t(g_3)x\big).$$

The representation $\rho'_t$ does not need to be irreducible.

Observe that for every $n$, $\mathrm{PU}(1,n)$ admit copies of $\mathrm{PU}(1,1)$ acting naturally on respective copies of $\mathbf{H}^1_{\mathbf{C}}$. By Lemma 1.2.1 for every hyperbolic isometry $g$ contained in any of those copies of $\mathrm{PU}(1,1)$, $\rho'_t(g)$ is hyperbolic. By Propositions 1.3.14 and 1.5.5, the restriction of $\rho'_t$ to any of the copies of $\mathrm{PU}(1,1)$ is non-elementary. Hence, $\rho'_t$ itself is non-elementary.

By Theorem 1.5.6, there exists $\rho_t$, the irreducible part of $\rho'_t$. Observe that by Lemma 1.2.1, for every hyperbolic isometry $g$,

$$t\ell(g) = \ell(\rho_t(g)).$$

This shows that for every $t \in (0,1)$, with a small abuse of notation, there exist an irreducible representation $\mathrm{PU}(1,n) \xrightarrow{\rho_t} \mathrm{Isom}(\mathbf{H}^m_{\mathbf{C}})$ such that $\ell(\rho_t) = t$.

The representations $\rho_t$ are of the shape $\mathrm{PU}(1,n) \xrightarrow{\rho_t} \mathrm{Isom}(\mathbf{H}^m_{\mathbf{C}})_o$. The fact that $m = \infty$ can be shown with general principles (Mostow-Karpelevich), but in Lemma 3.1.9 an elementary proof for it will be given.

# Chapter 3

# Representations of PU(1,1)

The peculiarity of $\mathrm{PU}(1,1)$ among the groups $\mathrm{PU}(1,n)$ is that it lies at the intersection of the real and the complex worlds. With the classification of irreducible representations of $\mathrm{PO}(1,2)_o$ done by Monod & Py [41], it is known a family of irreducible representations with displacements between 0 and 2 and which preserve a real hyperbolic space in the target. After Monod's work [40], another family of representations is known which has a completely different behaviour. It will be shown that it is possible to "interpolate" these two families to obtain a third one which that, to the best author's knowledge, had not been described before.

In Section 3.3. a complete invariant for irreducible representations of $\mathrm{PU}(1,1)$ will be introduced. This is done by studying the functions of complex hyperbolic type associated with these representations.

It was shown in the previous chapter (see Proposition 2.1.11) that the restriction of an irreducible representation of $\mathrm{PU}(1,1)$ to the stabilizer of a point in $\partial\mathbf{H}^1_{\mathbf{C}}$ determines the representation. In Section 3.2 arguments in the reverse direction will be given. That is, for certain representations of a stabilizer of a point at infinity, an extension to the whole group $\mathrm{PU}(1,1)$ will be constructed.

With these arguments in hand, an operation between irreducible representations will be defined. With this operation, given two non-equivalent irreducible representations of $\mathrm{PU}(1,1)$ with the same displacement, it is possible to construct a third one that is not equivalent to either of the first two.

Hence, the family irreducible representations, up to conjugation, is closed under two operations, the "exponentiation" introduced by Monod in [40] and this new operation that resembles a convex combination.

Using this operation, in Section 3.3 a new family of irreducible representations of $\mathrm{PU}(1,1)$ is described.

## 3.1 Invariants of representations

Given an irreducible representation $\mathrm{PU}(1,1) \xrightarrow{\rho} \mathrm{Isom}(\mathbf{H}^m_{\mathbf{C}})_o$, the notations $\Delta(1,b) = \Delta(b)$, $c(1,b) = c(b)$ and $\pi(1,b) = \pi(b)$ will be used in this chapter (see Remark 2.1.7).

**Remark 3.1.1.** Recall from the proof of Proposition 2.1.11, the definition

$$K(b) = \tfrac{-B(c(b),c(b))}{2} + i\Delta(b)$$

and that, up to a conjugation, $|K(1)| = 1$. Observe that

$$K(b) = \frac{-B(\tilde{\eta}_2, \rho(1,b)\tilde{\eta}_1)}{|B(\tilde{\eta}_2, \rho(1,b)\tilde{\eta}_1)|} B(\rho(1,b)\tilde{\eta}_2, \tilde{\eta}_2).$$

Therefore $\mathrm{Arg}(K(b))$ does not depend on the representatives of $\eta_1$ and $\eta_2$ if the normalization condition $B(\tilde{\eta}_1, \tilde{\eta}_2) = 1$ is imposed.

**Lemma 3.1.2.** *If* $\mathrm{PU}(1,1) \xrightarrow{\rho} \mathrm{Isom}(\mathbf{H}^m_{\mathbf{C}})_o$ *is an irreducible representation, for every* $\lambda > 0$ *and every* $b \in \mathbf{R}$*, the following hold.*

1. $K(\lambda b) = \lambda^t K(b)$.
2. $K(-b) = \overline{K(b)}$.
3. $K(b+d) = K(b) + K(d) + B(c(d), c(-b))$.

**Proof.** Points 1. are 2. were proved in Lemma 2.1.9. For 3. observe that

$$\begin{aligned} B\big(c(b+d), c(b+d)\big) &= \\ B\big(c(b),c(b)\big) + B\big(c(d),c(d)\big) + 2\mathrm{Re}\Big(B\big(\pi(b)c(d), c(b)\big)\Big) &= \\ B\big(c(b),c(b)\big) + B\big(c(d),c(d)\big) - 2\mathrm{Re}\Big(B\big(c(d), c(-b)\big)\Big) & \end{aligned}$$

and that

$$\begin{aligned} \Delta(d+b) &= \Delta(b) + \Delta(d) - \mathrm{Im}\Big(B\big(\pi(b)c(d), c(b)\big)\Big) \\ &= \Delta(b) + \Delta(d) + \mathrm{Im}\Big(B\big(c(d), c(-b)\big)\Big). \end{aligned}$$

□

**Lemma 3.1.3.** *For every $y \in \mathbf{H}^1_{\mathbf{C}}$,*

$$\lim_{b\to\infty} \mathrm{Cart}\big(g(1,b)y, g(1,-b)y, y\big) = -\frac{\pi}{2}.$$

**Proof.** If $y$ is represented by $w = a\tilde{\xi}_1 + \tilde{\xi}_2$, then $\mathrm{Re}(a) > 0$ and

$$\begin{aligned} \mathrm{Cart}\big(g(1,b)y, g(1,-b)y, y\big) &= \\ \mathrm{Arg}\Big(B(g(1,2b)w,w)B(g(1,-b)w,w)^2\Big) &= \\ \mathrm{Arg}\Big(8\mathrm{Re}(a)^3 + 6\mathrm{Re}(a)b^2 - i2b^3\Big). \end{aligned}$$

Therefore,

$$\lim_{b\to\infty} \mathrm{Cart}\big(g(1,b)y, g(1,-b)y, y\big) = -\frac{\pi}{2}.$$

□

Let $\mathrm{PU}(1,n) \xrightarrow{\rho} \mathrm{Isom}(\mathbf{H}^m_{\mathbf{C}})_o$ be an irreducible representation. Suppose $y \in \mathbf{H}^1_{\mathbf{C}}$ and let $K = \mathrm{PU}(1,n)_y$. Denote $x \in \mathbf{H}^m_{\mathbf{C}}$ the unique point fixed by $\rho(K)$ (see Remark 2.1.6). By Remark 1.3.15, there exists $s \in \mathbf{R}$ such that for every $g_1, g_2 \in \mathrm{PU}(1,1)$,

$$s\mathrm{Cart}(g_1 y, g_2 y, y) = \mathrm{Cart}(\rho(g_1)x, \rho(g_2)x, x).$$

Observe that $|s| \leqslant 1$ because there exist $g_1, g_2 \in \mathrm{SU}(1,1)$ such that

$$|\mathrm{Cart}(g_1 y, g_2 y, y)|$$

is as close as desired to $\frac{\pi}{2}$ and

$$|\mathrm{Cart}(\rho(g_1)x, \rho(g_2)x, x)| < \frac{\pi}{2}.$$

**Remark 3.1.4.** The definition of the Cartan argument comes with an choice. In fact, suppose that $H$ is a complex Hilbert space provided with $B$, a strongly non-degenerated Hermitian form. Observe that if $B'$ is defined on $H$ as $B'(v,w) = B(w,v)$, for every $v, w \in H$, then

$$\mathbf{H}^{B'}_{\mathbf{C}} = \{[v] \mid v \in H \text{ and } B'(v,v) > 0\}$$

provided with the metric $d'$ given by

$$\cosh(d'([v],[w])) = |B'(v,w)|$$

is isometric to $\mathbf{H_C}$, the hyperbolic space induced from $(H,B)$. Regarding this, all the representations can be considered to be such that the scalar $s$ in the previous comment is positive.

**Lemma 3.1.5.** *If* $\mathrm{PU}(1,1) \xrightarrow{\rho} \mathrm{Isom}(\mathbf{H}^m_{\mathbf{C}})_o$ *is a non-elementary representation and* $x \in \mathbf{H}^m_{\mathbf{C}}$ *has a total orbit, then*

$$\lim_{b\to\infty} \mathrm{Cart}(\rho(b)x, \rho(-b)x, x) = \mathrm{Arg}(K(-1)).$$

*Moreover, if* $K \leqslant PU(1,1)$ *the stabilizer of* $x$ *is a maximal compact subgroup,* $y \in \mathbf{H}^1_{\mathbf{C}}$ *is the point fixed by* $K$ *and* $0 < s \leqslant 1$ *is such that*

$$s\mathrm{Cart}\big(g(1,b)y, g(1,-b)y, y\big) = \mathrm{Cart}(\rho(b)x, \rho(-b)x, x),$$

*then* $\frac{s\pi}{2} = \mathrm{Arg}(K(1))$.

**Proof.** After a conjugation if necessary, suppose $|K(1)| = 1$. If $\tilde{x} = a\tilde{\eta}_1 + \tilde{\eta}_2 + u$ is a representative of $x$, then

$$\begin{aligned}
\mathrm{Cart}(\rho(1,b)x, \rho(1,-b)x, x) &= \\
\mathrm{Arg}\Big(B\big(\rho(1,2b)\tilde{x},\tilde{x}\big)B\big(\rho(1,-b)\tilde{x},\tilde{x}\big)^2\Big) &= \\
\mathrm{Arg}\Big(\Big(2\mathrm{Re}(a) + K(2b) - B(\pi(2b)u, c(2b)) + B(c(2b) + \pi(2b)u, u)\times & \\
\Big(2\mathrm{Re}(a) + K(-b) - B(\pi(-b)u, c(-b)) + B\big(c(-b) + \pi(-b)u, u\big)\Big)^2\Big). &
\end{aligned}$$

Denote, for any $b \in \mathbf{R}$,

$$T(b) = -B(\pi(b)u, c(b)) + B\big(c(b) + \pi(b)u, u\big) + 2\mathrm{Re}(a).$$

Observe that

$$\begin{aligned}
\big(K(2b) + T(2b)\big)\big(K(-b) + T(-b)\big)^2 &= \\
\big(K(2b) + T(2b)\big)\Big(K(-b)^2 + T(-b)^2 + 2K(-b)T(-b)\Big) &= \\
2^t b^{3t} K(-1) + 2^{t+1} b^{2t} T(-b) + b^{2t} K(-1) T(2b) + & \\
2b^t K(-1) T(2b) T(-b) + 2^t b^t T(-b)^2 + T(2b) T(-b)^2 &
\end{aligned}$$

There exist constants $C_1, C_2 > 0$ such that for every $b > 0$,

$$\max\{|T(2b)|, |T(-b)|\} \leqslant C_1 b^{\frac{t}{2}} + C_2.$$

Therefore,

$$\begin{aligned}\lim_{b\to\infty} \mathrm{Cart}(\rho(b)x, \rho(-b)x, x) &= \lim_{b\to\infty} \mathrm{Arg}\big(K(2b)K(-b)^2\big) \\ &= \mathrm{Arg}(K(-1)).\end{aligned}$$

The second claim is immediate from Lemma 3.1.3. □

**Remark 3.1.6.** Observe that the previous lemma, Lemma 3.1.3 and the fact that the Cartan argument is left-invariant imply that neither $\mathrm{Arg}(K(1))$ nor $s$ depend on the choice of the point $x \in \mathbf{H}^{\infty}_{\mathbf{C}}$ fixed by a maximal compact subgroup of $\mathrm{PU}(1,1)$. The previous lemma shows also that $\Delta(1) \geqslant 0$.

In view of the previous remark define for an irreducible representation $\mathrm{PU}(1,1) \xrightarrow{\rho} \mathrm{Isom}(\mathbf{H}^m_{\mathbf{C}})_o$, $\mathrm{Arg}(\rho)$, the *angular invariant* of $\rho$, as $\mathrm{Arg}(K(1))$. With this normalization, for every $\rho$, $0 \leqslant \mathrm{Arg}(\rho) \leqslant \frac{\pi}{2}$.

**Proposition 3.1.7.** *If $\rho$ is non-elementary and $\mathrm{Arg}(\rho) = \pi/2$, then $\rho$ preserves a copy of $\mathbf{H}^1_{\mathbf{C}}$.*

**Proof.** Observe that if $\mathrm{Re}(K(1)) = 0$, then for every $b \in \mathbf{R}$, $c(b) = 0$. □

The previous proposition, trivial in this context, is contained in the much more general Theorem 1.1 of [23].

**Proposition 3.1.8.** *Let $x \in \mathbf{H}^1_{\mathbf{C}}$ and $0 < t < 1$. If $\rho$ is the irreducible part of the non-elementary representation associated to the function of hyperbolic type $(\beta^t, t\alpha)$, where $(\beta, \alpha)$ is the function of hyperbolic type associated to $x$ and the tautological action of $\mathrm{PU}(1,n)$ on $\mathbf{H}^1_{\mathbf{C}}$, then*

$$\mathrm{Arg}(\rho) = \frac{t\pi}{2}.$$

**Proof.** Let $\mathrm{PU}(1,1) \xrightarrow{\tau} \mathrm{Isom}(\mathbf{H}^m_{\mathbf{C}})_o$ be the representation associated to $(\beta^t, t\alpha)$ in Theorem 1.4.9 and consider its irreducible part

$$\mathrm{PU}(1,1) \xrightarrow{\rho} \mathrm{Isom}(\mathbf{H}_{\mathbf{C}})_o.$$

Denote $p$ the projection of $\mathbf{H}^m_{\mathbf{C}}$ onto $\mathbf{H}_{\mathbf{C}}$. The map $p$ is $\mathrm{PU}(1,1)$-equivariant, therefore if $y \in \mathbf{H}^m_{\mathbf{C}}$ is the distinguished point (see Theorem 1.4.9) and $K$ is the stabilizer of $x$, then $y$ and $p(y)$ are stabilized by $K$. By Remark 1.3.15, there exists $s \in \mathbf{R}$ such that for every $g_0, g_1, g_2 \in \mathrm{PU}(1,1)$, there exists $s \in \mathbf{R}$ such that

$$\mathrm{Cart}\big(\rho(g_0)p(y), \rho(g_1)p(y), \rho(g_2)p(y)\big) = s\mathrm{Cart}(g_0x, g_1x, g_3x).$$

By Theorem 1.4.9,

$$\mathrm{Cart}\big(\tau(g_0)y, \tau(g_1)y, \tau(g_2)y\big) = t\mathrm{Cart}(g_0, g_1, g_2).$$

Hence, by Lemma 3.1.5, $\frac{t\pi}{2} = \mathrm{Arg}(K_\tau(-1))$ and $\frac{s\pi}{2} = \mathrm{Arg}(K_\rho(-1)$, but

$$\mathrm{Arg}(K_\tau(-1)) = \mathrm{Arg}(K_\rho(-1).$$

The last statement is true because the irreducible part contains all the axis of the images of the hyperbolic isometries (see Remark 3.1.1). □

As it was mentioned in Section 2.2, in [41], among other things, the authors classified the irreducible representations $\mathrm{Isom}(\mathbf{H}^n_{\mathbf{R}}) \xrightarrow{\rho} \mathrm{Isom}(\mathbf{H}^\infty_{\mathbf{R}})$. They showed that for every $0 < t < 1$ there exists a unique, up to a conjugation, irreducible representation $\rho_t$ such that $\ell(\rho) = t$.

Every representation $\mathrm{Isom}(\mathbf{H}^2_{\mathbf{R}}) \xrightarrow{\rho} \mathrm{Isom}(\mathbf{H}^\infty_{\mathbf{R}})$ can be lifted to a representation into $\mathrm{O}(1,\infty)$. By Propositions 1.3.14 and 1.5.5, $\rho_t$ restricted to $\mathrm{Isom}(\mathbf{H}^2_{\mathbf{R}})_o$ remains non-elementary, thus it has an irreducible part. With a small abuse of notation, denote $\rho_t$ this irreducible representation. There is a natural embedding $\mathrm{O}(1,\infty) < \mathrm{U}(1,\infty)$ through complexification. In Proposition 5.10 of [40], the author showed that the complexification of any irreducible representation of $\mathrm{Isom}(\mathbf{H}^n_{\mathbf{R}})$ into $O(1,\infty)$ remains irreducible. The proof there works also for $\mathrm{Isom}(\mathbf{H}^2_{\mathbf{R}})_o$. Thus, the complexification of $\rho_t$,

$$\mathrm{Isom}(\mathbf{H}^2_{\mathbf{R}})_o \xrightarrow{\rho_t^{\mathbf{C}}} \mathrm{U}(1,\infty)$$

is irreducible and $\ell(\rho_t^{\mathbf{C}}) = t$. This last claim is true because the formulas for the displacement of the image of a hyperbolic isometry do not change.

Let

$$\mathrm{Isom}(\mathbf{H}^1_{\mathbf{C}})_o \xrightarrow{\overline{\Phi}} \mathrm{Isom}(\mathbf{H}^2_{\mathbf{R}})_o$$

be the homomorphism of Lemma 1.3.13 and recall that if $g \in \mathrm{Isom}(\mathbf{H}^1_{\mathbf{C}})_o$, then $\ell(\overline{\Phi}(g)) =$

$2\ell(g)$. Therefore for every $t \in (0,1)$ and every $g \in \mathrm{Isom}(\mathbf{H}^1_{\mathbf{C}})_o$,

$$\ell(\rho^{\mathbf{C}}_t \circ \overline{\Phi}(g)) = t\ell(\overline{\Phi}(g)) = 2t\ell(g).$$

The previous observation shows that for every $0 < t < 2$, there exists an irreducible representation

$$\mathrm{Isom}(\mathbf{H}^1_{\mathbf{C}})_o \xrightarrow{\rho_t} \mathrm{Isom}(\mathbf{H}^{\infty}_{\mathbf{C}})_o$$

such that $\ell(\rho_t) = t$ and $\mathrm{Arg}(\rho_t) = 0$.

By Proposition 3.1.8, for every $0 < t < 1$, there exist irreducible representations

$$\mathrm{Isom}(\mathbf{H}^1_{\mathbf{C}})_o \xrightarrow{\tau_t} \mathrm{Isom}(\mathbf{H}^{m}_{\mathbf{C}})_o$$

such that $\ell(\tau_t) = t$ and $\mathrm{Arg}(\tau_t) = \frac{t\pi}{2}$.

Due to general principles (Mostow-Karpelevich) $m = \infty$, however the next lemma provides an elementary proof for this fact.

**Lemma 3.1.9.** *If* $\mathrm{Isom}(\mathbf{H}^1_{\mathbf{C}})_o \xrightarrow{\rho} \mathrm{Isom}(\mathbf{H}^{m}_{\mathbf{C}})_o$ *is an irreducible representation such that* $\ell(\rho) \neq 1$, *then the family* $\{c(b)\}_{b \in \mathbf{R}\setminus 0}$ *is* $\mathbf{C}$*-linearly independent.*

**Proof.** Suppose $\sum^n a_i c(b_i) = 0$ with $b_i \neq 0$. Without lost of generality, suppose that $b_1 > b_i$ for every $i \neq 1$. For every $d \in \mathbf{R}$,

$$\begin{aligned} 0 &= \mathrm{Re}\big(\textstyle\sum^n a_i B(c(b_i), c(d))\big) \\ &= \textstyle\sum^n \mathrm{Re}(a_i)\mathrm{Re}(B(c(b_i), c(d))) - \mathrm{Im}(a_i)\mathrm{Im}(B(c(b_i), c(d))) \end{aligned}$$

and

$$\begin{aligned} 0 &= \mathrm{Im}\big(\textstyle\sum^n a_i B(c(b_i), c(d))\big) \\ &= \textstyle\sum^n \mathrm{Re}(a_i)\mathrm{Im}(B(c(b_i), c(d))) + \mathrm{Im}(a_i)\mathrm{Re}(B(c(b_i), c(d))). \end{aligned}$$

Consider an interval $(b_1, b_1 + r)$ such that $0 \notin (b_1, b_1 + r)$ and consider $d \in (b_1, b_1 + r)$. By Lemma 3.1.2, there are constants $C_0, C_1, D_0, D_1$ such that

$$C_0 d^t + \sum^n Re(\overline{K(1)} a_i)(d - b_i)^t = C_1$$

and

$$D_0 d^t + \sum^n Im(\overline{K(1)} a_i)(d - b_i)^t = D_1.$$

Thus, there exist constants $E_0, E_1$ such that for every $d \in (b_1, b_1 + r)$,

$$E_0 d^t + \sum^n \overline{K(1)} a_i (d - b_i)^t = E_1.$$

After differentiating twice the previous equality with respect to $d$ in the interval $(b_1, b_1 + r)$, it follows that

$$t(t-1)E_0 d^{t-2} + t(t-1) \sum^n \overline{K(1)} a_i ((d - b_i)^{t-2} = 0.$$

If $d \to b_1^+$, then $(d - b_1)^{t-2}$ is unbounded, but for every $i \neq 1$, $(d - b_i)^{t-2}$ is bounded. Therefore $a_1 = 0$. Repeating the same argument, it is possible to show that for every $i$, $a_i = 0$. □

Observe that the previous lemma is not valid for irreducible representations with displacement 1. Indeed, the identity $\text{Isom}(\mathbf{H}^1_{\mathbf{C}})_o \xrightarrow{Id} \text{Isom}(\mathbf{H}^1_{\mathbf{C}})_o$ is a representation with displacement 1.

**Lemma 3.1.10.** *Let* $\text{PU}(1,1) \xrightarrow{\rho} \text{Isom}(\mathbf{H}^m_{\mathbf{C}})_o$ *be an irreducible representation. If* $x \in \mathbf{H}^\infty_{\mathbf{C}}$ *is represented by* $\frac{1}{\sqrt{2}}(\tilde{\eta}_1 + \tilde{\eta}_2)$, *then the function of hyperbolic type associated to* $x$ *can be reconstructed from* $K(1)$ *and* $\ell(\rho)$.

**Proof.** The representation $\rho$ is determined by its restriction to $P$ (see Proposition 2.1.11). The claim is that the restriction of $\rho$ to $P$ is entirely determined by $K(1)$ and the parameter $t$.

For every $b \in \mathbf{R}$ and $\lambda, \gamma > 0$, by Lemmas 2.1.9 and 3.1.2, $B(\rho(\lambda, b)\frac{1}{\sqrt{2}}(\tilde{\eta}_1 + \tilde{\eta}_2))$ can be recovered from $K(1)$ and $\ell(t)$. Hence, the claim follows from Theorem 1.4.9 and the fact that the $P$-orbit of $\frac{1}{\sqrt{2}}(\tilde{\eta}_1 + \tilde{\eta}_2)$ is total (see Proposition 2.1.11). □

**Theorem 3.1.11.** *Let* $\rho_1$ *and* $\rho_2$ *be two irreducible representations of Isom*$(\mathbf{H}^1_{\mathbf{C}})_o$ *into* $\text{Isom}(\mathbf{H}^m_{\mathbf{C}})_o$ *such that* $\ell(\rho_1) = \ell(\rho_2)$. *Then* $\rho_1$ *and* $\rho_2$ *are equivalent if, and only if,* $\text{Arg}(\rho_1) = \text{Arg}(\rho_2)$.

**Proof.** Suppose that $\rho_1(P)$ and $\rho_2(P)$ share the same fixed point in $\eta_1 \partial \mathbf{H}^m_{\mathbf{C}}$ and that the families $\{\rho_1(\lambda, 0)\}_{\lambda > 0}$ and $\{\rho_2(\lambda, 0)\}_{\lambda > 0}$ preserve the axis connecting the points at infinity $\eta_1$ and $\eta_2$.

If $\rho_1$ and $\rho_2$ are equivalent, their restrictions to the group $P$ are equivalent. Therefore there exists $T$ an isometry of $\mathbf{H}^m_{\mathbf{C}}$ such that $T\rho_1|_P T^{-1} = \rho_2|_P$, and therefore $T(\eta_i) = \eta_i$.

If

$$\begin{aligned} K_i(1) &= \frac{-B(c_i(1), c_i(1))}{2} + i\Delta_i(1) \\ &= \frac{Q(\tilde{\eta}_2, \rho_i(1,1)\tilde{\eta}_1)}{|Q(\tilde{\eta}_2, \rho_i(1,1)\tilde{\eta}_1)|^2} B(\rho_i(1,1)\tilde{\eta}_2, \tilde{\eta}_2), \end{aligned}$$

with $\tilde{\eta}_i$ respective lifts such that $B(\tilde{\eta}_1, \tilde{\eta}_2) = 1$, then it is clear that

$$\mathrm{Arg}(K_1(1)) = \mathrm{Arg}(K_2(1)).$$

Indeed, $\mathrm{Arg}(K_i(1))$ does not depend on the choice of the representatives of $\eta_i$, as long as the condition $B(\tilde{\eta}_1, \tilde{\eta}_2) = 1$ is fulfilled.

Let $\eta_1^1, \eta_2^1 \in \partial\mathbf{H}^m_{\mathbf{C}}$ be the fixed points by $\rho_1(\lambda, 0)$ and let $\eta_1^2, \eta_2^2 \in \partial\mathbf{H}^m_{\mathbf{C}}$ be the points fixed by $\rho_2(\lambda, 0)$. Suppose $\mathrm{Arg}(K_1(1)) = \mathrm{Arg}(K_2(1))$. After conjugating $\rho_1$ by an isometry $\rho_1(\gamma, 0)$ if needed, it is possible to suppose that $K_1(1) = K_2(1)$. Observe that this conjugation preserves the points $\eta_i^1$.

Let $x_i \in \mathbf{H}^\infty_{\mathbf{C}}$ be the point represented by

$$\frac{1}{\sqrt{2}}(\tilde{\eta}_1^i + \tilde{\eta}_2^i).$$

Consider the functions of hyperbolic type of $\rho_i$ associated to $x_i$. By Lemma 3.1.10, the representations $\rho_1|_P$ and $\rho_2|_P$ can be supposed identical, therefore by Proposition 2.1.11, $\rho_1$ and $\rho_2$ are equivalent. □

**Proof.** Observe that $\mathrm{Arg}(K(b))$ does not depend on the choice of representatives $\eta_1$, $\eta_2$ as long as $B(\tilde{\eta}_1, \tilde{\eta}_2) = 1$ (see the definition before Lemma 3.1.2). Therefore, if $\tilde{\eta}_1$ and $\tilde{\eta}_2$ are chosen in the totally real subspace that contains the representatives of the real hyperbolic subspace of $\mathbf{H}^\infty_{\mathbf{C}}$ preserved by $\rho$, it is clear that $K(b) \in \mathbf{R}$. □

## 3.2 Extending certain parabolic representations

In this section a binary combination will be defined for representations of $\mathrm{PU}(1,1)$. The main tool used to define this combination is the Steinberg relations for $\mathrm{SL}_2(\mathbf{R})$. They will be used to determine if a hyperbolic representation defined on a the stabilizer of a point at infinity can be extended to the whole group.

Fix $\rho_1$ and $\rho_2$ two irreducible representations of $\mathrm{Isom}(\mathbf{H}^1_{\mathbf{C}})_o$ into $\mathrm{Isom}(\mathbf{H}^\infty_{\mathbf{C}})_o$ such that $\ell(\rho_1)=\ell(\rho_2)=t$. With the conventions of the previous section, suppose without lost of generality, that $\rho_i$ share the two distinguished points $\eta_i \in \partial\mathbf{H}^\infty_{\mathbf{C}}$. That is, for every $\lambda>0$ and $b\in\mathbf{R}$, $\rho_i(\lambda,b)(\eta_1)=\eta_1$ and $\rho_i(\lambda,0)(\eta_2)=\eta_2$.

In a matrix representation with respect to the decomposition

$$\mathbf{C}\eta_1\oplus\mathbf{C}\eta_2\oplus(\tilde{\eta}_1^\perp\cap\tilde{\eta}_2^\perp),$$

$\rho_i(\lambda,b)$ has the shape,

$$\begin{pmatrix} \lambda^t & \frac{-\lambda^t B(c_i(\lambda,b),c_i(\lambda,b))}{2}+i\Delta_i(\lambda,b) & -\lambda^t B\big(\pi_i(\lambda,b)(\cdot),c_i(\lambda,b)\big) \\ 0 & \lambda^{-t} & 0 \\ 0 & c_i(\lambda,b) & \pi_i(\lambda,d) \end{pmatrix},$$

and the isometry $\rho_i(\sigma)$ has the representation

$$\begin{pmatrix} 0 & \nu_i^{-1} & 0 \\ \nu_i & 0 & 0 \\ 0 & 0 & A_i \end{pmatrix},$$

where $\nu_i>0$ and, by Proposition 2.1.11,

$$A_i c_i(b)=\nu_i K_i(b)c_i(1,-1/b).$$

Define a model for the hyperbolic space in the following way. Consider $\mathbf{C}^2$ as $\mathbf{C}\tilde{\eta}_1\oplus\mathbf{C}\tilde{\eta}_2$ and consider the Hilbert space $L=H_1\oplus H_2$, where $H_i=\tilde{\eta}_1^\perp\cap\tilde{\eta}_2^\perp$. Define the form $Q$ in $\mathbf{C}\eta_1\oplus\mathbf{C}\eta_2\oplus L$ which is $\mathbf{C}$-linear in the first entry, antilinear in the second and that is given by

1. $Q|_{H_i}=B|_{H_i}$.
2. $Q(H_1,H_2)=0$.
3. $Q(\eta_i,H_j)=0$, for $i,j=1,2$.
4. $Q(\eta_i,\eta_i)=0$, for $i=1,2$.
5. $Q(\eta_1,\eta_2)=1$.

This defines a strongly non-degenerate form of signature $(1,\infty)$ in

$$\mathbf{C}\eta_1 \oplus \mathbf{C}\eta_2 \oplus L.$$

Define, for every $b \in \mathbf{R}$ and $\lambda > 0$, $c(b) = c_1(b) \oplus c_2(b)$ and $\pi(\lambda, b) = \pi_1(\lambda, b) \oplus \pi_2(\lambda, b)$. Observe that $\pi$ is a unitary representation of the group $P$ on $L$. Define $\rho(\lambda, 0)$ as the isometry represented by

$$\begin{pmatrix} \lambda^t & 0 & 0 \\ 0 & \lambda^{-t} & 0 \\ 0 & 0 & \pi(\lambda,0) \end{pmatrix}$$

and $\rho(1, b)$ as the isometry represented by

$$\begin{pmatrix} 1 & \frac{-B(c(b),c(b)}{2} + i\Delta(b) & -B\big(\pi(b)(\cdot), c(b)\big) \\ 0 & 1 & 0 \\ 0 & c(b) & \pi(b) \end{pmatrix},$$

where $\Delta(b) = \Delta_1(b) + \Delta_2(b)$. Denote

$$K(b) = \frac{-B(c(b),c(b))}{2} + i\Delta(b)$$

and

$$K_i(b) = \frac{-B(c_i(b),c_i(b))}{2} + i\Delta_i(b).$$

**Lemma 3.2.1.** *If $\rho$ is the complexification of an irreducible representation*

$$Isom(\mathbf{H}^1_{\mathbf{C}})_o \to \mathrm{Isom}(\mathbf{H}^\infty_{\mathbf{R}})_o,$$

*then for every $b \in \mathbf{R}$, $\Delta(b) = 0$.*

The next proposition is a consequence of Lemmas 2.1.8 2.1.9 and 3.1.2.

**Proposition 3.2.2.** *If $c$, $\pi$, $K$ and $K_i$ are defined as above, then for every $\lambda > 0$ and $b, d \in \mathbf{R}$, the following properties hold.*

1. $Im\big(B(c(b), c(d))\big) = \Delta(b-d) - \Delta(b) + \Delta(d)$.
2. $Re\big(B(c(b), c(d))\big) = -\frac{B(c(b-d),c(b-d))}{2} + \frac{B(c(b),c(b))}{2} + \frac{B(c(d),c(d))}{2}$.
3. $K(b) = K_1(b) + K_2(b)$.

4. $K(\lambda b) = \lambda^t K(b)$.

5. $K(-b) = \overline{K(b)}$.

6. $K(b+d) = K(b) + K(d) + B(c(b), c(-d))$.

7. $\pi(\lambda, 0)c(b) = \lambda^{-t} c(\lambda^2 b)$.

8. $c(b+d) = c(b) + \pi(b)c(d)$.

**Lemma 3.2.3.** *For every $b \neq 0$, $K(b) \neq 0$.*

**Proof.** Suppose $K(1) = 0$. By Lemma 3.1.5, $\Delta_i(1) \geqslant 0$, therefore $\Delta_i(1) = 0$. The isometries $\rho_i(1,1)$ are parabolic, thus $c_i(1) \neq 0$, which is a contradiction. □

Observe that

$$g(\lambda, 0)g(1, b) = g(\gamma, 0)g(1, d)$$

if, and only if, $\lambda = \gamma$ and $b = d$, and that

$$g(\lambda, 0)g(1, b) = g(1, \lambda^2 b)g(\lambda, 0).$$

It will be shown that with the formulas for $\rho(\lambda, 0)$ and $\rho(1, b)$ it is possible to define an homomorphism on $P$.

**Lemma 3.2.4.** *For every $\lambda, \gamma > 0$ and $b, d \in \mathbf{R}$, the following identities hold.*

1. $K(\lambda^{-1}\gamma^{-1}b + d) = K(d) + \gamma^{-t}K(\lambda^{-1}b) + B\big(\pi(\gamma, 0)c(\gamma^{-1}d), c(-\lambda^{-1}b)\big)$.

2. $\pi(\lambda^{-1}b)\pi(\gamma, 0)c(\gamma^{-1}d) + \gamma^{-t}c(\lambda^{-1}b) = \pi(\gamma, 0)c(\lambda^{-1}\gamma^{-2}b + \gamma^{-1}d)$.

**Proof.** By Proposition 3.2.2,

$$\begin{array}{rl} K(\lambda^{-1}\gamma^{-1}b + d) & = \\ K(d) + K(\lambda^{-1}\gamma^{-1}b) + B\big(c(d), c(-\lambda^{-1}\gamma^{-1}b)\big) & = \\ K(d) + K(\gamma^{-1}\lambda^{-1}b) + \Big(\gamma^{-\frac{t}{2}}c(d), \gamma^{\frac{t}{2}}c(-\gamma^{-1}\lambda^{-1}b)\Big) & = \\ K(d) + K(\gamma^{-1}\lambda^{-1}b) + B\Big(\pi(\gamma^{\frac{1}{2}}, 0)c(\gamma^{-1}d), \pi(\gamma^{-\frac{1}{2}}, 0)c(-\lambda^{-1}b)\Big) & = \\ K(d) + \gamma^{-t}K(\lambda^{-1}b) + B\big(\pi(\gamma, 0)c(\gamma^{-1}d), c(-\lambda^{-1}b)\big) & \end{array}$$

and

$$
\begin{aligned}
\pi(\lambda^{-1}b)\pi(\gamma,0)c(\gamma^{-1}d)+\gamma^{-t}c(\lambda^{-1}b) &= \\
\gamma^{-t}\pi(\lambda^{-1}b)c(\gamma d)+\gamma^{-t}c(\lambda^{-1}b) &= \\
\gamma^{-t}c(\lambda^{-1}b+\gamma d) &= \\
\pi(\gamma,0)c(\lambda^{-1}\gamma^{-2}b+\gamma^{-1}d).
\end{aligned}
$$

□

**Lemma 3.2.5.** *For every $\gamma,\lambda>0$, $b,d\in\mathbf{R}$ and $u\in\eta_1^{\perp}\cap\eta_2^{\perp}$,*

$$
\begin{aligned}
\gamma^t B(u,c(-\gamma^{-1}d))+B\Big(u,\pi(-\gamma^{-1}d)\pi(\gamma^{-1},0)c(-\lambda^{-1}b)\Big) &= \\
\gamma^t B\big(u,c(1,-\lambda^{-1}\gamma^{-2}b-\gamma^{-1}d)\big).
\end{aligned}
$$

**Proof.** By Proposition 3.2.2,

$$
\begin{aligned}
\gamma^t B(u,c(-\gamma^{-1}d))+B\Big(u,\pi(-\gamma^{-1}d)\pi(\gamma^{-1},0)c(-\lambda^{-1}b)\Big) &= \\
\gamma^t B(u,c(-\gamma^{-1}d))+\gamma^t B\big(u,\pi(-\gamma^{-1}d)c(-\lambda^{-1}\gamma^{-2}b)\big) &= \\
\gamma^t B\big(u,c(1,-\lambda^{-1}\gamma^{-2}b-\gamma^{-1}d)\big).
\end{aligned}
$$

□

Regarding Theorem 1.3.12, the representations will be supposed to be defined on $\mathrm{SU}(1,1)$.

**Proposition 3.2.6.** *If $\rho(\lambda,0)$ and $\rho(1,b)$ are the isometries defined at the beginning of this section, then the map $g(\lambda,b)\mapsto\rho(\lambda,0)\rho(1,\lambda^{-1}b)$ is a homomorphism and for every $x\in\mathbf{H}_{\mathbf{C}}^{\infty}$, the map $g(\lambda,b)\mapsto\rho(\lambda,0)\rho(1,\lambda^{-1}b)x$ is continuous.*

**Proof.** Observe that

$$
\begin{aligned}
g(\lambda,b)g(\gamma,d) &= g(\lambda,0)g(1,\lambda^{-1}b)g(\gamma,0)g(1,\gamma^{-1}d) \\
&= g(\lambda,0)g(\gamma,0)g(1,\lambda^{-1}\gamma^{-2}b)g(1,\gamma^{-1}d) \\
&= g(\lambda\gamma,0)g(1,\lambda^{-1}\gamma^{-2}b+\gamma^{-1}d).
\end{aligned}
$$

Therefore the first claim of the proposition is that for every $\lambda,\gamma>0$ and $b,d\in\mathbf{R}$,

$$
\rho(\lambda,0)\rho(1,\lambda^{-1}b)\rho(\gamma,0)\rho(1,\gamma^{-1}d)=\rho(\lambda\gamma,0)\rho(1,\lambda^{-1}\gamma^{-2}b+\gamma^{-1}d).
$$

It is clear that $\lambda\mapsto\rho(\lambda,0)$ is a homomorphism, thus to show the claim is equivalent to show that

$$
\rho(1,\lambda^{-1}b)\rho(\gamma,0)\rho(1,\gamma^{-1}d)=\rho(\gamma,0)\rho(1,\lambda^{-1}\gamma^{-2}b+\gamma^{-1}d).
$$

This will be done by comparing the columns of the matrix representations of both sides of the identities with respect to the decomposition $\mathbf{C}\tilde{\eta}_1 \oplus \mathbf{C}\tilde{\eta}_2 \oplus L$.

It is clear from the definition of $\rho$ that for $\tilde{\eta}_1$,

$$\begin{aligned} \rho(1,\lambda^{-1}b)\rho(\gamma,0)\rho(1,\gamma^{-1}d)\tilde{\eta}_1 &= \gamma^t\tilde{\eta}_1 \\ &= \rho(\gamma,0)\rho(1,\gamma^{-2}b+\gamma^{-1}d)\tilde{\eta}_1. \end{aligned}$$

By Proposition 3.2.2 and Lemma 3.2.4, for $\tilde{\eta}_2$ observe that,

$$\begin{aligned} \rho(1,\lambda^{-1}b)\rho(\gamma,0)\rho(1,\gamma^{-1}d)\tilde{\eta}_2 &= \\ \rho(1,\lambda^{-1}b)\rho(\gamma,0)\Big(K(\gamma^{-1}d)\tilde{\eta}_1+\tilde{\eta}_2+c(\gamma^{-1}d)\Big) &= \\ \rho(1,\lambda^{-1}b)\Big(\gamma^t K(\gamma^{-1}d)\tilde{\eta}_1+\gamma^{-t}\tilde{\eta}_2+\pi(\gamma,0)c(\gamma^{-1}d)\Big) &= \\ \Big(K(d)+\gamma^{-t}K(\lambda^{-1}b)+B\big(\pi(\gamma,0)c(\gamma^{-1}d),c(-\lambda^{-1}b)\big)\Big)\tilde{\eta}_1+ & \\ \gamma^{-t}\tilde{\eta}_2+\pi(\lambda^{-1}b)\pi(\gamma,0)c(\gamma^{-1}d)+\gamma^{-t}c(\lambda^{-1}b) &= \\ K(\lambda^{-1}\gamma^{-1}b+d)\tilde{\eta}_1+\gamma^{-t}\tilde{\eta}_2+\pi(\gamma,0)c(\lambda^{-1}\gamma^{-2}b+\gamma^{-1}d) &= \\ \rho(\gamma,0)\Big(K(\lambda^{-1}\gamma^{-2}b+\gamma^{-1}d)\tilde{\eta}_1+\tilde{\eta}_2+c(\lambda^{-1}\gamma^{-2}b+\gamma^{-1}d)\Big) &= \\ \rho(\gamma,0)\rho(1,\lambda^{-1}\gamma^{-2}b+\gamma^{-1}d)\tilde{\eta}_2. & \end{aligned}$$

And last, by Proposition 3.2.2 and Lemma 3.2.5, for $u\in\tilde{\eta}_1^{\perp}\cap\tilde{\eta}_2^{\perp}$,

$$\begin{aligned} \rho(1,\lambda^{-1}b)\rho(\gamma,0)\rho(1,\gamma^{-1}d)u &= \\ \rho(1,\lambda^{-1}b)\rho(\gamma,0)\Big(B\big(u,c(-\gamma^{-1}d)\big)\tilde{\eta}_1+\pi(\gamma^{1}d)u\Big) &= \\ \rho(1,\lambda^{-1}b)\Big(\gamma^t B\big(u,c(-\gamma^{-1}d)\big)\tilde{\eta}_1+\pi(\gamma,0)\pi(\gamma^{-1}d)u\Big) &= \\ \Big(\gamma^t B\big(u,c(-\gamma^{-1}d)\big)+B\big(\pi(\gamma,0)\pi(\gamma^{-1}d)u,c(-\lambda^{-1}b)\big)\Big)\tilde{\eta}_1+ & \\ \pi(\lambda^{-1}b)\pi(\gamma,0)\pi(\gamma^{-1}d)u &= \\ \Big(\gamma^t B\big(u,c(-\gamma^{-1}d)\big)+B\big(u,\pi(-\gamma^{-1}d)\pi(\gamma^{-1},0)c(-\lambda^{-1}b)\big)\Big)\tilde{\eta}_1+ & \\ \pi\big(\gamma,\gamma^{-1}\lambda^{-1}b+d\big)u &= \\ \gamma^t B\big(u,c\big(-\lambda^{-1}\gamma^{-2}b-\gamma^{-1}d\big)\big)\tilde{\eta}_1+\pi(\gamma,0)\pi\big(\lambda^{-1}\gamma^{-2}b+\gamma^{-1}d\big)u &= \\ \rho(\gamma,0)\Big(B\big(u,c(-\lambda^{-1}\gamma^{-2}b-\gamma^{-1}d)\big)\tilde{\eta}_1+\pi\big(\lambda^{-1}\gamma^{-2}b+\gamma^{-1}d\big)u\Big) &= \\ \rho(\gamma,0)\rho(1,\lambda^{-1}\gamma^{-2}b+\gamma^{-1}d)u. & \end{aligned}$$

Therefore the map

$$g(\lambda,b)\mapsto\rho(\lambda,0)\rho(1,\lambda^{-1}b)\in\operatorname{Isom}(\mathbf{H}^{\infty}_{\mathbf{C}})_o$$

is a homomorphism.

For the second claim of the proposition it is enough to show that for every $x \in \mathbf{H}^\infty_\mathbf{C}$, the map $g(\lambda, b) \mapsto \rho(\lambda,0)\rho(1,\lambda^{-1}b)x$ is continuous around the identity in $P$. Suppose $g_i = g(\lambda_i, b_i) \to Id$ in $P$, then

$$\frac{1}{4}\left|B\big(g(\lambda_i,b_i)(\tilde{\xi}_1+\tilde{\xi}_2),\tilde{\xi}_1+\tilde{\xi}_2\big)\right|^2 = \frac{1}{4}\big((\lambda_i+\lambda_i^{-1})^2+b_i^2\big) \to 1,$$

or equivalently,

$$(\lambda_i - \lambda_i^{-1})^2 + b_i^2 \to 0.$$

Therefore $\lambda_i \to 1$ and $b_i \to 0$. If $\tilde{x} = \alpha\tilde{\eta}_1 + \beta\tilde{\eta}_2 + u$ is such that $Q(\tilde{x},\tilde{x}) = 1$, then

$$\begin{aligned}
\rho(\lambda_i,0)\rho(1,\lambda_i^{-1}b_i)\tilde{x} &= \\
\rho(\lambda_i,0)\Big(\alpha+\beta K(\lambda_i^{-1}b_i)+B(u,c(-\lambda_i^{-1}b_i))\Big)\tilde{\eta}_1 + \\
\rho(\lambda_i,0)\Big(\beta\tilde{\eta}_2+\beta c(\lambda_i^{-1}b_i)+\pi(\lambda_i^{-1}b_i)u\Big) &= \\
\lambda_i^t\Big(\alpha+\beta K(\lambda_i^{-1}b_i)+B(u,c(-\lambda_i^{-1}b_i))\Big)\tilde{\eta}_1+\lambda_i^{-t}\beta\tilde{\eta}_2 + \\
\pi(\lambda_i,0)\big(\beta c(\lambda_i^{-1}b_i)+\pi(\lambda_i^{-1}b_i)u\big) &= \\
\Big(\lambda_i^t\alpha+\beta K(b_i)+B\big(u,\pi(\lambda_i^{-1},0)c(-\lambda_i b_i)\big)\Big)\tilde{\eta}_1+\lambda_i^{-t}\beta\tilde{\eta}_2 + \\
\lambda_i^{-t}\beta c(\lambda_i b_i)+\pi(\lambda_i,b_i)u.
\end{aligned}$$

Therefore, since

$$g(\lambda,b) \mapsto \pi(\lambda,b) = \pi_1(\lambda,b) \oplus \pi_2(\lambda,b)$$

is orbitally (jointly) continuous,

$$\begin{aligned}
\lim_{i\to\infty}\left|B\big(\rho(\lambda_i,0)\rho(1,\lambda_i^{-1}b_i)x,x\big)\right| &= \\
\lim_{i\to\infty}\Big|\overline{\beta}\Big(\lambda_i^t\alpha+\beta K(b_i)+B\big(u,\pi(\lambda_i^{-1},0)c(-\lambda_i b_i)\big)\Big)+\overline{\alpha}\lambda_i^{-t}\beta + \\
B\big(\lambda_i^{-t}\beta c(\lambda_i b_i)+\pi(\lambda_i,0)\pi(\lambda_i,b_i)u,u\big)\Big| &= \\
\left|\overline{\beta}\alpha+\overline{\alpha}\beta+B(u,u)\right| &= 1.
\end{aligned}$$

□

Now it is possible to define the representation $P \xrightarrow{\rho} \mathrm{Isom}(\mathbf{H}^\infty_\mathbf{C})_o$ given by

$$\rho(\lambda,b) = \rho(\lambda,0)\rho(1,\lambda^{-1}b).$$

The next results are devoted to prove that $\rho$ can be extended to a homomorphism defined on $\mathrm{SU}(1,1)$.

**Lemma 3.2.7.** *The only point fixed in $\mathbf{H}^\infty_\mathbf{C} \cup \partial\mathbf{H}^\infty_\mathbf{C}$ by $\rho$ is $\eta_1$.*

**Proof.** The isometries $\rho(\lambda,0)$ are hyperbolic by construction, therefore the only other candidate to be fixed by $\rho$ is $\eta_2$, but $\rho(1,b)$ does not fix it because $K(b)\neq 0$ (see Lemma 3.2.3). □

Observe that

$$K(b)=\frac{Q(\tilde{\eta}_2,\rho(1,b)\tilde{\eta}_1)}{|Q(\tilde{\eta}_2,\rho(1,b)\tilde{\eta}_1)|^2}Q(\rho(1,b)\tilde{\eta}_2,\tilde{\eta}_2)$$

is also true for the representation $\rho$. After a conjugation by an isometry $\rho(\gamma,0)$ if necessary, assume that $|K(1)|=1$. Notice that this conjugation does not change the argument of $K(1)$.

The following is the uniqueness part of the GNS construction (see Theorem C.1.4 of [3]).

**Lemma 3.2.8.** *Let $X$ be a set and let $H$ be a Hilbert space. Suppose $f$ and $g$ are two functions $X\to H$ such that their images are total in $H$. If for every $x,y\in X$, $\langle f(x),f(y)\rangle=\langle g(x),g(y)\rangle$, then there exists $A$, a unitary map of $H$, such that $Af(x)=g(x)$.*

**Proposition 3.2.9.** *The map*

$$Ac(b)=K(b)c(-1/b)$$

*defines a unitary map in $L'$, the closed subspace generated by $\{c(b)\}_{b\in\mathbf{R}}$, such that $A^2=Id$.*

**Proof.** Due to Lemma 3.2.8, it is enough to show that

$$B(Ac(b),Ac(d))=B(c(b),c(d)).$$

Denote $B(c(b),c(b))=|c(b)|^2$ and recall that $|K(1)|=1$. Suppose $b\neq d$. By Proposition 3.2.2, on one side,

$$\begin{aligned}
B(Ac(b),Ac(d)) &=\\
K(b)\overline{K(d)}B\big(c(-1/b),c(-1/d)\big) &=\\
K(b)\overline{K(d)}\Big(-\tfrac{|c(-1/b+1/d)|^2}{2}+\tfrac{|c(1/b)|^2}{2}+\tfrac{|c(1/d)|^2}{2}\Big)+&\\
K(b)\overline{K(d)}i\Big(\Delta(-1/b+1/d)-\Delta(-1/b)+\Delta(-1/d)\Big) &=\\
|b|^t|d|^tK\left(\tfrac{b}{|b|}\right)K\left(-\tfrac{d}{|d|}\right)\left(-\tfrac{|d-b|^t}{|bd|^t}+\tfrac{1}{|b|^t}+\tfrac{1}{|d|^t}\right)\tfrac{|c(1)|^2}{2}+&\\
|b|^t|d|^tK\left(\tfrac{b}{|b|}\right)K\left(-\tfrac{d}{|d|}\right)i\left(\tfrac{|b-d|^t(b-d)|bd|}{|bd|^tbd|b-d|}+\tfrac{b}{|b|^t|b|}-\tfrac{d}{|d|^t|d|}\right)\Delta(1) &=\\
K\left(\tfrac{b}{|b|}\right)K\left(-\tfrac{d}{|d|}\right)\left(-|d-b|^t+|d|^t+|b|^t\right)\tfrac{|c(1)|^2}{2}+&\\
K\left(\tfrac{b}{|b|}\right)K\left(-\tfrac{d}{|d|}\right)i\left(\tfrac{|b-d|^t(b-d)|bd|}{bd|b-d|}+\tfrac{|d|^tb}{|b|}-\tfrac{|b|^td}{|d|}\right)\Delta(1).&
\end{aligned}$$

On the other side,

$$\begin{aligned} B(c(b),c(d)) &= \\ -\tfrac{|c(b-d)|^2}{2}+\tfrac{|c(b)|^2}{2}+\tfrac{|c(d)|^2}{2}+i\big(\Delta(b-d)-\Delta(b)+\Delta(d)\big) &= \\ \big(-|b-d|^t+|b|^t+|d|^t\big)\tfrac{|c(1)|^2}{2}+i\big(\tfrac{|b-d|^t(b-d)}{|b-d|}-\tfrac{|b|^t b}{|b|}+\tfrac{|d|^t d}{|d|}\big)\Delta(1). \end{aligned}$$

There are three cases to analyse:

1. $b>0>d$.
2. $b>d>0$.
3. $b<d<0$.

1. If $b>0>d$,

$$K\left(\tfrac{b}{|b|}\right)K\left(-\tfrac{d}{|d|}\right) = K(1)^2$$

and

$$\tfrac{|b-d|^t(b-d)|bd|}{bd|b-d|}+\tfrac{|d|^t b}{|b|}-\tfrac{|b|^t d}{|d|} = -|b-d|^t+|b|^t+|d|^t.$$

Therefore

$$\begin{aligned} B(Ac(b),Ac(d)) &= \\ K(1)^2\big(-|b-d|^t+|b|^t+|d|^t\big)\left(\tfrac{|c(1)|^2}{2}+i\Delta(1)\right) &= \\ \big(-|b-d|^t+|b|^t+|d|^t\big)K(1)^2(-\overline{K(1)}) &= \\ -\big(-|b-d|^t+|b|^t+|d|^t\big)K(1) \end{aligned}$$

and

$$\begin{aligned} B(c(b),c(d)) &= \\ \big(-|b-d|^t+|b|^t+|d|^t\big)\left(\tfrac{|c(1)|^2}{2}-i\Delta(1)\right) &= \\ -\big(-|b-d|^t+|b|^t+|d|^t\big)K(1). \end{aligned}$$

2) If $b>d>0$,

$$K\left(\tfrac{b}{|b|}\right)K\left(-\tfrac{d}{|d|}\right) = 1$$

and

$$\tfrac{|b-d|^t(b-d)|bd|}{bd|b-d|}-\tfrac{|b|^t d}{|d|}+\tfrac{|d|^t b}{|b|} = |b-d|^t-|b|^t+|d|^t.$$

Thus,

$$\begin{aligned} B(Ac(b),Ac(d)) &= \\ \big(-|d-b|^t+|d|^t+|b|^t\big)\tfrac{|c(1)|^2}{2}+\big(|b-d|^t+|d|^t-|b|^t\big)\Delta(1) \end{aligned}$$

and

$$B(c(b),c(d)) = \left(-|d-b|^t+|d|^t+|b|^t\right)\frac{|c(1)|^2}{2}+i\left(|b-d|^t-|b|^t+|d|^t\right)\Delta(1).$$

3) If $b<d<0$,

$$K\left(\frac{b}{|b|}\right)K\left(-\frac{d}{|d|}\right) = 1$$

and

$$\frac{|b-d|^t(b-d)|bd|}{bd|b-d|}+\frac{|d|^t b}{|b|}-\frac{|b|^t d}{|d|} = -|b-d|^t-|d|^t+|b|^t.$$

Therefore

$$B(Ac(b),Ac(d)) = \left(-|d-b|^t+|d|^t+|b|^t\right)\frac{|c(1)|^2}{2}+i\left(-|b-d|^t-|d|^t+|b|^t\right)\Delta(1)$$

and

$$B(c(b),c(d)) = \left(-|b-d|^t+|b|^t+|d|^t\right)\frac{|c(1)|^2}{2}+i\left(-|b-d|^t+|b|^t-|d|^t\right)\Delta(1).$$

The case $b=d$ is an immediate consequence of Proposition 3.2.2.

By Lemma 3.2.8, $A$ induces a unitary map on $L'$. Observe that

$$\begin{aligned} A^2(c(b)) &= K(b)K(-1/b)c(b)\big) \\ &= |b|^t K\left(\tfrac{b}{|b|}\right)\tfrac{1}{|b|^t}K\left(-\tfrac{b}{|b|}\right)c(b) \\ &= c(b). \end{aligned}$$

□

Consider now $\mathbf{H}^\infty_\mathbf{C}$ as the hyperbolic space associated to $\mathbf{C}\eta_1\oplus\mathbf{C}\eta_2\oplus L'$ and the restriction of the form $Q$ defined in the beginning of this section. Denote by $\tilde\sigma\in \mathrm{Isom}(\mathbf{H}^\infty_\mathbf{C})_o$ the order two isometry represented by

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & A \end{pmatrix}.$$

The claim is that the representation $\rho$ can be extended to a representation of $\mathrm{SU}(1,1)$ using $\tilde\sigma$. That is to say, if $g(1,b)$, with $b\in\mathbf{R}$, $g(\lambda,0)$, with $\lambda>0$, and

$$s=\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

are understood as elements of $\mathrm{SU}(1,1)$, then the map defined by :

1. $T(g(\lambda,0)) = \rho(\lambda,0)$,

2. $T(-g(\lambda,0)) = \rho(\lambda,0)$,

3. $T(g(1,b)) = \rho(1,b)$,

4. $T(s) = \tilde{\sigma}$,

is a homomorphism. Here $\rho(\lambda,0)$, $\rho(1,b)$ and $\tilde{\sigma}$ are interpreted as elements of $\mathrm{Isom}(\mathbf{H}^{\infty}_{\mathbf{C}})_o$,

In order to prove that $T$ is a homomorphism it is enough to show that $T$ is coherent with the relations of Theorem 1.3.12, that is to show that, for $\lambda > 0$ and $b \in \mathbf{R}$,

1. $\rho(\lambda,0) = \tilde{\sigma}\rho(1,\lambda^{-1})\tilde{\sigma}\rho(1,\lambda)\tilde{\sigma}\rho(1,\lambda^{-1})$.

2. $\rho(\lambda,0) = \tilde{\sigma}\rho(1,-\lambda^{-1})\tilde{\sigma}\rho(1,-\lambda)\tilde{\sigma}\rho(1,-\lambda^{-1})$.

3. $\lambda \mapsto \rho(\lambda,0)$ is a homomorphism.

4. $b \mapsto \rho(1,b)$ is a homomorphism.

5. $\rho(\lambda,0)\rho(1,b)\rho(\lambda^{-1},0) = \rho(1,\lambda^2 b)$.

6. $\tilde{\sigma}^2 = Id$.

Observe that $T$ is coherent with the points from 3., 4. and 5. because $\rho$ is a homomorphism defined on $P$ (see Proposition 3.2.6). By Proposition 3.2.9, point 6. holds, therefore the only two families of relations left to be verified are that for every $b > 0$,

$$\begin{aligned}\rho(b,0) &= \sigma\rho(1,b^{-1})\sigma\rho(1,b)\sigma\rho(1,b^{-1}) \\ &= \sigma\rho(1,-b^{-1})\sigma\rho(1,-b)\sigma\rho(1,-b^{-1}).\end{aligned}$$

**Lemma 3.2.10.** *For $\epsilon = \pm 1$ and for every $b > 0$,*

*1.* $1 + K(\epsilon b)K(\epsilon b^{-1}) + B(Ac(\epsilon b), c(-\epsilon b^{-1})) = 0$.

*2.* $K(\epsilon b)c(\epsilon b^{-1}) + \pi(\epsilon b^{-1})Ac(\epsilon b) = 0$.

**Proof.** Indeed, if $B(c(-1),c(-1)) = |c(-1)|^2$,

$$\begin{aligned}
1+K(\epsilon b)K(\epsilon b^{-1})+B(Ac(\epsilon b),c(-\epsilon b^{-1})) &= \\
1+K(\epsilon)^2+K(\epsilon b)B(c(-\epsilon b^{-1}),c(-\epsilon b^{-1})) &= \\
1+K(\epsilon)^2+b^{-t}K(\epsilon)|c(-1)|^2 &= \\
K(\epsilon)\Big(K(-\epsilon)+K(\epsilon)+|c(-1)|^2\Big) &= 0.
\end{aligned}$$

and

$$\begin{aligned}
K(\epsilon b)c(\epsilon b^{-1})+\pi(\epsilon b^{-1})Ac(\epsilon b) &= \\
K(\epsilon b)\big(c(\epsilon b^{-1})+\pi(\epsilon b^{-1})c(-\epsilon b^{-1})\big) &= 0.
\end{aligned}$$

□

**Lemma 3.2.11.** *If $b > 0$ and $\epsilon = \pm 1$, then*

$$\rho(b,0) = \tilde{\sigma}\rho(1,\epsilon b^{-1})\tilde{\sigma}\rho(1,\epsilon b)\tilde{\sigma}\rho(1,\epsilon b^{-1}).$$

**Proof.** The procedure will be to compare the columns of the canonical matrix representation of both side of the identities with respect to the decomposition $\mathbf{C}\tilde{\eta}_1 \oplus \mathbf{C}\tilde{\eta}_2 \oplus L'$. In fact, it will be shown that

$$\tilde{\sigma}\rho(b,0)\rho(1,-\epsilon b^{-1})\tilde{\sigma} = \rho(1,\epsilon b^{-1})\tilde{\sigma}\rho(1,\epsilon b).$$

With a small abuse of notation, keep the notation above for the canonical linear representatives of each of the isometries. Indeed, on one side,

$$\begin{aligned}
\tilde{\sigma}\rho(b,0)\rho(1,-\epsilon b^{-1})\tilde{\sigma}(\tilde{\eta}_1) &= \\
\tilde{\sigma}\rho(b,0)\Big(K(-\epsilon b^{-1})\eta_1+\eta_2+c(-\epsilon b^{-1})\Big) &= \\
\tilde{\sigma}\Big(b^tK(-\epsilon b^{-1})\eta_1+b^{-t}\eta_2+b^{-t}c(-\epsilon b)\Big) &= \\
b^{-t}\eta_1+b^tK(-\epsilon b^{-1})\eta_2+b^{-t}K(-\epsilon b)c(\epsilon b^{-1}) &= \\
b^{-t}\eta_1+K(-\epsilon)\eta_2+K(-\epsilon)c(\epsilon b^{-1}),
\end{aligned}$$

and on the other side,

$$\rho(1,\epsilon b^{-1})\tilde{\sigma}\rho(1,\epsilon b)(\tilde{\eta}_1) = K(\epsilon b^{-1})\tilde{\eta}_1+\tilde{\eta}_2+c(\epsilon b^{-1}).$$

Observe that

$$K(-\epsilon)\big(K(\epsilon b^{-1})\tilde{\eta}_1+\tilde{\eta}_2+c(\epsilon b^{-1})\big) = b^{-t}\tilde{\eta}_1+K(-\epsilon)\tilde{\eta}_2+K(-\epsilon)c(\epsilon b^{-1}).$$

Therefore, as linear transformations, what has to be shown is that

$$\tilde{\sigma}\rho(b,0)\rho(1,-\epsilon b^{-1})\tilde{\sigma} = K(-\epsilon)\rho(1,\epsilon b^{-1})\tilde{\sigma}\rho(1,\epsilon b).$$

For $\tilde{\eta}_2$, observe that,

$$\tilde{\sigma}\rho(b,0)\rho(1,-\epsilon b^{-1})\tilde{\sigma}(\tilde{\eta}_2) = b^t\tilde{\eta}_2$$

and that

$$\begin{aligned}
\rho(1,\epsilon b^{-1})\tilde{\sigma}\rho(1,\epsilon b)(\tilde{\eta}_2) &= \\
\rho(1,\epsilon b^{-1})\tilde{\sigma}\Big(K(\epsilon b)\tilde{\eta}_1 + \tilde{\eta}_2 + c(\epsilon b)\Big) &= \\
\rho(1,\epsilon b^{-1})\Big(\tilde{\eta}_1 + K(\epsilon b)\tilde{\eta}_2 + Ac(\epsilon b)\Big) &= \\
\Big(1 + K(\epsilon b)K(\epsilon b^{-1}) + B\big(Ac(\epsilon b), c(-\epsilon b^{-1})\big)\Big)\tilde{\eta}_1 + K(\epsilon b)\tilde{\eta}_2 + \\
K(\epsilon b)c(\epsilon b^{-1}) + \pi(\epsilon b^{-1})Ac(\epsilon b).
\end{aligned}$$

Therefore, by Lemma 3.2.10,

$$\tilde{\sigma}\rho(b,0)\rho(1,-\epsilon b^{-1})\tilde{\sigma}(\tilde{\eta}_2) = K(-\epsilon)\rho(1,\epsilon b^{-1})\tilde{\sigma}\rho(1,\epsilon b)(\tilde{\eta}_2).$$

And last, for every $d \in \mathbf{R}\backslash\{0\}$,

$$\begin{aligned}
\tilde{\sigma}\rho(b,0)\rho(1,-\epsilon b^{-1})\tilde{\sigma}(c(d)) &= \\
K(d)\tilde{\sigma}\rho(b,0)\rho(1,-\epsilon b^{-1})c(-d^{-1}) &= \\
K(d)\tilde{\sigma}\rho(b,0)\Big(B\big(c(-d^{-1}), c(\epsilon b^{-1})\big)\tilde{\eta}_1 + \pi(-\epsilon b^{-1})c(-d^{-1})\Big) &= \\
K(d)\tilde{\sigma}\Big(b^t B\big(c(-d^{-1}), c(\epsilon b^{-1})\big)\tilde{\eta}_1 + \pi(b,0)\pi(-\epsilon b^{-1})c(-d^{-1})\Big) &= \\
K(d)\Big(b^t\big(c(-d^{-1}), c(\epsilon b^{-1})\big)\tilde{\eta}_2 + A\pi(b,0)\pi(-\epsilon b^{-1})c(-d^{-1})\Big).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\rho(1,\epsilon b^{-1})\tilde{\sigma}\rho(1,\epsilon b)(c(d)) &= \\
\rho(1,\epsilon b^{-1})\tilde{\sigma}\Big(B(c(d), c(-\epsilon b))\tilde{\eta}_1 + \pi(\epsilon b)c(d)\Big) &= \\
\rho(1,\epsilon b^{-1})\Big(B(c(d), c(-\epsilon b))\tilde{\eta}_2 + A\pi(\epsilon b)c(d)\Big) &= \\
\Big(B(c(d), c(-\epsilon b))K(\epsilon b^{-1}) + B\big(A\pi(\epsilon b)c(d), c(-\epsilon b^{-1})\big)\Big)\tilde{\eta}_1 + \\
B(c(d), c(-\epsilon b))\tilde{\eta}_2 + B(c(d), c(-\epsilon b))c(\epsilon b^{-1}) + \pi(\epsilon b^{-1})A\pi(\epsilon b)c(d).
\end{aligned}$$

Therefore the claim is that

$$\begin{aligned} K(-\epsilon)\Big(B(c(d),c(-\epsilon b))\tilde{\eta}_2 + B(c(d),c(-\epsilon b))c(\epsilon b^{-1}) + \pi(\epsilon b^{-1})A\pi(\epsilon b)c(d)\Big) &= \\ K(d)\Big(b^t B(c(-d^{-1}),c(\epsilon b^{-1}))\tilde{\eta}_2 + A\pi(b,0)\pi(-\epsilon b^{-1})c(-d^{-1})\Big). \end{aligned}$$

Observe that

$$\begin{aligned} K(\epsilon)K(d)b^t B(c(-d^{-1}),c(\epsilon b^{-1})) &= \\ B\big(K(d)c(-d^{-1}),K(-\epsilon b)c(b^{-1})\big) &= \\ B(Ac(d),Ac(-\epsilon b)) &= B(c(d),c(-\epsilon b)). \end{aligned}$$

Therefore

$$K(-\epsilon)B(c(d),c(-\epsilon b)) = K(d)b^t B(c(-d^{-1}),c(\epsilon b^{-1})).$$

The only identity remaining to show is that

$$\begin{aligned} K(d)A\pi(b,0)\pi(-\epsilon b^{-1})c(-d^{-1}) &= \\ K(-\epsilon)\Big(B(c(d),c(-\epsilon b))c(\epsilon b^{-1}) + \pi(\epsilon b^{-1})A\pi(\epsilon b)c(d)\Big). \end{aligned}$$

Suppose $0 \neq d \neq \epsilon b$. Notice that

$$\begin{aligned} \pi(\epsilon b^{-1})A\pi(\epsilon b)c(d) &= \\ \pi(\epsilon b^{-1})A\big(c(\epsilon b + d) - c(\epsilon b)\big) &= \\ \pi(\epsilon b^{-1})\Big(K(\epsilon b + d)c(-(\epsilon b + d)^{-1}) - K(\epsilon b)c(-\epsilon b^{-1}))\Big) &= \\ K(\epsilon b + d)\big(c(\tfrac{d}{\epsilon b(\epsilon b + d)}) - c(\epsilon b^{-1})\big) + K(\epsilon b)c(\epsilon b^{-1}) &= \\ \big(K(\epsilon b) + K(d) + B(c(d),c(-\epsilon b))\big)\big(c(\tfrac{d}{\epsilon b(\epsilon b + d)}) - c(\epsilon b^{-1})\big) + K(\epsilon b)c(\epsilon b^{-1}) &= \\ K(\epsilon b + d)c(\tfrac{d}{\epsilon b(\epsilon b + d)}) - \big(K(d) + B(c(d),c(-\epsilon b))\big)c(\epsilon b^{-1}). \end{aligned}$$

Therefore if $R = K(-\epsilon)\pi(\epsilon b^{-1})A\pi(\epsilon b)c(d)$,

$$\begin{aligned} K(-\epsilon)\Big(B(c(d),c(-\epsilon b))c(\epsilon b^{-1}) + \pi(\epsilon b^{-1})A\pi(\epsilon b)c(d)\Big) &= \\ K(-\epsilon)B(c(d),c(-\epsilon b))c(\epsilon b^{-1}) + R &= \\ K(-\epsilon)\Big(K(\epsilon b + d)c(\tfrac{d}{\epsilon b(\epsilon b + d)}) - K(d)c(\epsilon b^{-1})\Big). \end{aligned}$$

On the other hand

$$\begin{array}{rcl} A\pi(b,0)\pi(-\epsilon b^{-1})c(-d^{-1}) &=& \\ A\pi(b,0)\big(c(-\frac{d\epsilon b}{\epsilon bd})-c(-\epsilon b^{-1})\big) &=& \\ b^{-t}A\big(c(-\frac{\epsilon b(d\epsilon b)}{d})-c(-\epsilon b)\big) &=& \\ b^{-t}\Big(K(-\frac{\epsilon b(d\epsilon b)}{d})c(\frac{d}{\epsilon b(d\epsilon b)})-K(-\epsilon b)c(\epsilon b^{-1})\Big) &=& \\ K(-\frac{\epsilon(d\epsilon b)}{d})c(\frac{d}{\epsilon b(d\epsilon b)})-K(-\epsilon)c(\epsilon b^{-1}). && \end{array}$$

Therefore, what is left is to show that

$$\begin{array}{rcl} K(d)\Big(K(-\frac{\epsilon(d\epsilon b)}{d})c(\frac{d}{\epsilon b(d\epsilon b)})-K(-\epsilon)c(\epsilon b^{-1})\Big) &=& \\ K(-\epsilon)\Big(K(\epsilon b+d)c(\frac{d}{\epsilon b(\epsilon b+d)})-K(d)c(\epsilon b^{-1})\Big), && \end{array}$$

which is equivalent to show that

$$K(d)K(\frac{-\epsilon d-b}{d})=K(-\epsilon)K(\epsilon b+d).$$

This is can be easily proved considering all the different cases.

Define

$$f_{\epsilon b}(d)=K(d)A\pi(b,0)\pi(-\epsilon b^{-1})c(-d^{-1})$$

and

$$g_{\epsilon b}(d)=K(-\epsilon)\Big(B(c(d),c(-\epsilon b))c(\epsilon b^{-1})+\pi(\epsilon b^{-1})A\pi(\epsilon b)c(d)\Big).$$

Observe that for a given value $b_0\in\mathbf{R}\backslash\{0\}$, the functions $f_{b_0}(d)$ and $g_{b_0}(d)$ are continuous on $d$ and such that $f_{b_0}(0)=0=g_{b_0}(0)$. It has been shown that for every $0\neq d\neq\epsilon b_0$, $f_{b_0}(d)=g_{b_0}(d)$, therefore by continuity $f_{b_0}=g_{b_0}$.

This concludes the proof for the equalities,

$$\begin{array}{rcl} \rho(b,0) &=& \tilde\sigma\rho(1,b^{-1})\tilde\sigma\rho(1,b)\tilde\sigma\rho(1,b^{-1}) \\ &=& \tilde\sigma\rho(1,-b^{-1})\tilde\sigma\rho(1,-b)\tilde\sigma\rho(1,-b^{-1}). \end{array}$$

□

The previous lemma completes the argument that shows that

$$\mathrm{SU}(1,1)\xrightarrow{T}\mathrm{Isom}(\mathbf{H}^\infty_{\mathbf{C}})_o$$

is a homomorphism. Observe that by construction, $T(-Id) = Id$, therefore $T$ induces a representation $\mathrm{PU}(1,1) \to \mathrm{Isom}(\mathbf{H}^\infty_{\mathbf{C}})_o$.

**Theorem 3.2.12.** *The map $T$ induces an irreducible (orbitally continuous) representation $\mathrm{PU}(1,1) \xrightarrow{\rho} \mathrm{Isom}(\mathbf{H}^\infty_{\mathbf{C}})_o$ with $\ell(\rho) = t$ and*

$$\mathrm{Arg}(\rho) = \mathrm{Arg}(K_1(1) + K_2(1)).$$

**Proof.** Observe that $T$ does not have fixed points in $\mathbf{H}^\infty_{\mathbf{C}} \cup \partial \mathbf{H}^\infty_C$ because $\tilde{\sigma}$ does not fix $\eta_1$ (see Lemma 3.2.7). If $T$ preserves a geodesic, then it permutes the two limits of it, but this is a contradiction because every homomorphism $\mathrm{PU}(1,1) \to \mathbf{Z}_2$ is constant (see Proposition 1.3.14).

Let $\mathrm{SU}(1,1) \xrightarrow{\pi} \mathrm{Isom}(\mathbf{H}^1_{\mathbf{C}})_o$ be the projectivization map. The group $\pi(P)$ is closed in $\mathrm{Isom}(\mathbf{H}^1_{\mathbf{C}})_o$ and, by Proposition 1.3.5, there is a decomposition

$$\mathrm{Isom}(\mathbf{H}^1_{\mathbf{C}})_o = \pi(PsP) \sqcup \pi(P).$$

Therefore $\pi(PsPs)$ is an open neighborhood of $Id \in \mathrm{Isom}(\mathbf{H}^1_{\mathbf{C}})_o$. Thus, it is enough to show that, if $(g_j)$ is a sequence in $\pi(PsP)$ such that $(g_j) \to \pi(s)$, then for every $x \in \mathbf{H}^\infty_C$,

$$\rho(g_j)x \to T(s)x = \tilde{\sigma}x.$$

Observe that every element of $PsP$ can be written as

$$g(\lambda, b)sg(1,d) = \begin{pmatrix} -b & i(\lambda - bd) \\ i\lambda^{-1} & -\lambda^{-1}d \end{pmatrix}.$$

If

$$g_j = \pi\big((g(\lambda_j, b_j)sg(1,d_j)\big),$$

then $b_j \to 0$, $\lambda_j \to 1$ and $d_j \to 0$. Therefore, for every $x \in \mathbf{H}^\infty_C$, $\rho(\lambda_j, b_j)x \to x$ and $\rho(1,d_j)x \to x$, hence with a triangle inequality argument it is possible to conclude that $g_j x \to \tilde{\sigma}x$.

The irreducible part of $\rho$ contains the axis, and its limits, preserved by the maps $\rho(\lambda, 0)$, therefore $\rho$ is irreducible by construction. □

## 3.3 A new family of representations

With the results of the previous section a continuum of non-equivalent representations will be constructed.

Given an irreducible representation $\rho$ denote $K(1) = K(1)_\rho$. If $p, q \in \mathbf{R} > 0$ and $\rho, \tau : \mathrm{PU}(1,1) \to \mathrm{Isom}(\mathbf{H}^\infty_\mathbf{C})$ are two irreducible representations such that $\ell(\rho) = \ell(\tau) = t$, let $\rho_p$ and $\tau_q$ be two irreducible representations, equivalent to $\rho$ and $\tau$ respectively, such that $|K(1)_{\rho_p}| = p$ and $|K(1)_{\tau_q}| = q$ (see Proposition 2.1.11 and Remark 2.1.12). Observe that with the procedure describe in Theorem 3.2.12 it is possible to obtain an irreducible representation $\omega$ such that $\ell(\varphi) = t$ and

$$\mathrm{Arg}(\varphi) = \mathrm{Arg}\left(\frac{pK_\rho(1)}{|K_\rho(1)|} + \frac{qK_\tau(1)}{|K_\tau(1)|}\right).$$

Therefore for every

$$s \in [\min\{\mathrm{Arg}(\rho), \mathrm{Arg}(\tau)\}, \max\{\mathrm{Arg}(\rho), \mathrm{Arg}(\tau)\}]$$

there is an irreducible representation $\phi$ such that $\ell(\phi) = t$ and $\mathrm{Arg}(\phi) = s$.

Given $u \in [0,1]$, denote $\rho \underset{u}{\wedge} \tau$ the irreducible representation such that

1. $\ell(\rho \underset{u}{\wedge} \tau) = t$.
2. $\mathrm{Arg}(\rho \underset{u}{\wedge} \tau) = (1-u)\mathrm{Arg}(\rho) + u\mathrm{Arg}(\tau)$.

This representation will be called a *horospherical combination* of $\rho$ and $\tau$.

The representation $\rho \underset{u}{\wedge} \tau$ is well defined in the following sense. If $\ell(\rho) = \ell(\tau)$ and $\rho'$ and $\tau'$ are equivalent to $\rho$ and $\tau$ respectively, then $\rho \underset{u}{\wedge} \tau$ is equivalent to $\rho' \underset{u}{\wedge} \tau'$ (see Theorem 3.1.11).

Although in the definition of the horospherical combination, for simplicity, the representations were supposed acting on the same hyperbolic space, nothing prevents to define the horospherical combination of two irreducible representations with one possibly having finite-dimensional target. This could be the case, by Mostow-Karpelevich theorem or in particular by Lemma 3.1.9, only if $t = 1$.

Using the families constructed in [40, 41] and described in Section 3.1 and the horospherical combination, a new family of non-equivalent representations is built.

Recall that for every $0 < t < 2$ on the one hand, up to a conjugation, there exists a unique irreducible representation

$$\mathrm{Isom}(\mathbf{H}^1_{\mathbf{C}})_o \xrightarrow{\rho_t} \mathrm{Isom}(\mathbf{H}^\infty_{\mathbf{C}})_o$$

such that $\ell(\rho_t) = t$ and that preserves a real hyperbolic space. These representations are such that $\mathrm{Arg}(\rho_t) = 0$ (see Remark 3.1.1, Theorem 3.1.11 and Lemma 3.2.1). On the other hand, for every $0 < t < 1$ there exists an irreducible representation

$$\mathrm{Isom}(\mathbf{H}^1_{\mathbf{C}})_o \xrightarrow{\tau_t} \mathrm{Isom}(\mathbf{H}^\infty_{\mathbf{C}})_o$$

such that $\mathrm{Arg}(\tau_t) = \frac{t\pi}{2}$ and $\ell(\tau_t) = t$ (see Lemma 1.2.1 and Proposition 3.1.8).

**Theorem 3.3.1.** *If $0 < t < 1$ and $r \in [0, t\pi/2]$ or if $t = 1$ and $r \in [0, \pi/2)$, there exists a unique, up to a conjugation, irreducible representation $\rho_{t,r}$ such that*

1. $\mathrm{Arg}(\rho_{t,r}) = r$.
2. $\ell(\rho_{t,r}) = t$.

**Proof.** For $t < 1$, consider the family of irreducible representations $\rho_t \underset{u}{\wedge} \tau_t$. For $t = 1$, let $Id$ be the identity map $\mathrm{Isom}(\mathbf{H}^1_{\mathbf{C}})_o \to \mathrm{Isom}(\mathbf{H}^1_{\mathbf{C}})_o$. Observe that for every $u \in [0, 1)$, by construction the representation $\rho_t \underset{u}{\wedge} id$ is irreducible and the target is an infinite-dimensional complex hyperbolic space. This is true because the representation $\rho_t$ has as a target an infinite-dimensional hyperbolic space. The unicity is a consequence of Theorem 3.1.11. □

By Lemma 3.1.5, the representations listed in the previous theorem are representatives of all the irreducible representations of $\mathrm{PU}(1,1)$ into $\mathrm{Isom}(\mathbf{H}^\infty_{\mathbf{C}})_o$ with displacement 1.

# Chapter 4

# Non-elementary representations of PU(1,n), with n>1

In this chapter it will be shown that for $n > 1$, there are no non-elementary representations $\mathrm{PU}(1,n) \to \mathrm{Isom}(\mathbf{H}^\infty_\mathbf{R})$. In terms of the argument of a representation, contrary to the case when $n = 1$, this is equivalent to say that for every complex hyperbolic representation of $\mathrm{PU}(1,n)$, with $n > 1$, has non-zero argument.

The proof of this fact relies strongly on ideas of [15] and [22]: the existence of smooth harmonic maps $\mathbf{H}^n_\mathbf{C} \to \mathbf{H}^\infty_\mathbf{R}$ associated to a non-elementary representation of $\Gamma \to \mathrm{Isom}(\mathbf{H}^\infty_\mathbf{R})$, where $\Gamma < \mathrm{PU}(1,n)$ is a uniform lattice, together with the strong restrictions on the rank of such maps (see [47]).

## 4.1 Harmonic functions, lattices and representations

At the finite dimensional level, the main theorem of this chapter follows from rather elementary arguments.

**Proposition 4.1.1.** *If $n, n' < \infty$, there are no non-elementary representations*

$$\mathrm{PU}(n,1) \to \mathrm{PO}(1,n').$$

**Proof.** Suppose $\xi \in \partial \mathbf{H}^n_{\mathbf{C}}$ and consider the group

$$L = \{g(1, v, Id, b) \mid v \in \mathbf{C}^{n-1} \text{ and } b \in \mathbf{R}\}$$

contained in the stabilizer of $\xi$. By the arguments of Proposition 2.1.11, it can be shown that $\rho$ preserves the type and $\rho(L)$ preserves a point $\eta \in \partial \mathbf{H}^{n'}_{\mathbf{R}}$. Every element in $L$ is parabolic, therefore, $\rho(L) < B$, where

$$B = \{g(1, w, A) \mid w \in \mathbf{R}^{n'-1} \text{ and } A \in \mathrm{O}(n'-1)\}.$$

The group $B$ is equivalent to the group of affine isometries of $\mathbf{R}^{n'-1}$. Every connected nilpotent subgroup of the group of affine isometries of $\mathbf{R}^{n'-1}$ is abelian (see the proof of Corollary 4.1.13 in [48]). Hence, the group $\rho(L)$ is abelian. This is a contradiction because $L$ is a non-abelian solvable group consisting of parabolic isometries, and because the type is preserved by $\rho$, $\rho|_L$ is injective. □

In the infinite-dimensional case the arguments of the previous proposition do not hold. The group $L$ is solvable, hence amenable. Every amenable group satisfies Haagerup's property (see 1.2.6 in [16]). That is to say, $L$ admits a metrically proper representation in the group of affine isometries of real Hilbert space, which naturally induces a representation $L \xrightarrow{\tau} \mathrm{Isom}(\mathbf{H}^\infty_{\mathbf{R}})$ such that $\rho(L)$ fixes a point in $\partial \mathbf{H}^\infty_{\mathbf{R}}$ and all the horospheres centered at it (see Section 1.3).

**Lemma 4.1.2.** *Let $\Gamma_1$ and $\Gamma_2$ be two uniform lattices of a locally compact group $G$ and let $X \xrightarrow{f_i} Y$, $i = 1, 2$ be two continuous functions between $X$ a topological space and $Y$ a metric space. Suppose $G$ acts transitively on $X$ with compact stabilizers, by isometries on $Y$ and orbitally continuously on both. If $f_i$ is $\Gamma_i$-equivariant, then there exists $C > 0$ such that for every $x \in X$, $d(f_1(x), f_2(x)) < C$.*

**Proof.** There exist compact sets $K_i \subset G$ such that $\Gamma_i K_i = G$ (see for example Lemma 2.46 in [27]). Fix $x_0 \in X$ and take $y \in X$. There exist $\gamma_i \in \Gamma_i$ and $k_i \in K_i$, such that $\gamma_i k_i x_0 = y$. Therefore,

$$\begin{aligned} d(f_1(y), f_2(y)) &= d\big(\gamma_1 f_1(k_1 x_0), \gamma_2 f_2(k_2 x_0)\big) \\ &= d\big(\gamma_2^{-1}\gamma_1 f_1(k_1 x_0), f_2(k_2 x_0)\big) \\ &\leqslant \sup\{d\big(z f_1(l_1 x_0), f_2(l_2 x_0)\big) \mid z \in K_2 \mathrm{Stab}(x_0) K_1^{-1}, l_i \in K_i\}. \end{aligned}$$

□

**Lemma 4.1.3.** *Let $\{H_n\}_{n\in\mathbf{N}_{\geqslant 1}}$ be a sequence of finite-dimensional hyperbolic spaces embedded in $\mathbf{H}^\infty_\mathbf{R}$, where for $n \geqslant 2$, $H_n$ is isometric to $\mathbf{H}^n_\mathbf{R}$ and $H_1$ is a geodesic. Suppose that for every $n \geqslant 1$, $H_n \subset H_{n+1}$ and*

$$\overline{\bigcup_{n\geqslant 1} H_n} = \mathbf{H}^\infty_\mathbf{R},$$

*then for every $n \geqslant 2$ and $y_1, y_2 \in \mathbf{H}^\infty_\mathbf{R}$, there exists $\varphi \in \mathrm{Isom}(\mathbf{H}^\infty_\mathbf{R})$ such that, $\varphi|_{H_n} = Id$ and $\varphi(\{y_1, y_2\}) \subset H_{n+2}$.*

**Proof.** Given $H_n \subset H_{n+2}$ and $y_1, y_2 \in \mathbf{H}^\infty_\mathbf{R}$, there exists $m \geqslant n+2$ and $\mathbf{H}$ isometric to $\mathbf{H}^m_\mathbf{R}$ such that $y_i \in \mathbf{H}$ and $H_{n+2} \subset \mathbf{H}$. Observe that every isometry of $\mathbf{H}$ can be extended to an isometry of $\mathbf{H}^\infty_\mathbf{R}$. Therefore the problem can be reduced to a statement about $\mathbf{H}^m_\mathbf{R}$, where the claim is clear (see the proof of Proposition 1.4.4). □

Let $M$ be a Riemannian manifold and let $U \subset M$ be an open set contained in a chart $(V, \phi)$. Suppose that $\overline{U} \subset V$ and $\phi^{-1}(U) = B(p, r) \subset \mathbf{R}^m$. For every $\varphi \in \mathscr{C}^0(\partial U)$, there exists a unique $h_\varphi \in \mathscr{C}^0(\overline{U}) \cap \mathscr{C}^2(U)$ which solves the Dirichlet problem, in other words, $h_\varphi$ is harmonic in $U$ ($\Delta h_\varphi|_U = 0$) and $h_\varphi|_{\partial U} = \varphi|_{\partial U}$ (see Lemma 6.10 in [30]). For references about harmonic maps in the Riemannian setting see [44] and for harmonic maps with a CAT(0) codomain see [35, 36].

For every $x \in U$, the claim is that the map

$$\begin{array}{rcl} \mathscr{C}^0(\partial U) & \to & \mathbf{R} \\ \varphi & \mapsto & h_\varphi(x) \end{array}$$

is a positive linear functional, in other words, it defines a probability measure $p^U_x$ in $\partial U$. Indeed, in every $U$ as above, a harmonic map defined on $\overline{U}$ achieves its maximum (minimum) in $\partial U$ and if there exists $u \in U$ such that the maximum (minimum) of $h$ is achieved in $u$, then $h$ is constant in $\overline{U}$ (see Theorem 3.1 in [30]). Thus

$$h_\varphi \leqslant \max_{y\in\partial U} h_\varphi(y) = \max_{y\in\partial U} \varphi(y),$$

therefore the linear map $\varphi \mapsto h_\varphi(x)$ is positive and continuous for every $x \in U$.

A continuous function $M \xrightarrow{\varphi} \mathbf{R}$ is called *subharmonic* if for every $U$ as above and every $x \in U$,

$$\varphi(x) \leqslant \int_{\partial U} \varphi \, dp^U_x.$$

If $\varphi \in \mathscr{C}^2(M)$, then $\varphi$ is subharmonic if, and only if, $\Delta f \geqslant 0$ (see page 103 of [30]).

Observe that every non-constant subharmonic function $\varphi$ defined on $\overline{U}$ satisfies a maximum principle: the maximum of $\varphi$ is achieved only in the boundary.

**Lemma 4.1.4.** *Let $M$ be a Riemannian manifold and let $(\varphi_n)$ be a sequence of subharmonic functions defined in $M$. If $(\varphi_n) \to \varphi$ uniformly on compact sets, then $\varphi$ is subharmonic.*

The proof of the next lemma follows some of the ideas in Theorem 2.3 in [38].

**Lemma 4.1.5.** *Let $X$ be a homogeneous and complete Riemannian manifold and let*

$$u, v : X \to \mathbf{H}^\infty_\mathbf{R}$$

*be two harmonic and Lipschitz continuous functions of class $\mathscr{C}^2$. If there exists $C > 0$ such that for every $x \in X$, $d(u(x), v(x)) < C$, then either $u = v$ or the images of $u$ and $v$ are contained in one geodesic.*

**Proof.** Suppose that $K > 0$ is a Lipschitz constant for $u$ and $v$. Let $\{y_i\}_{i\in\mathbf{N}} \subset \mathbf{H}^\infty_\mathbf{R}$ be such that if for every $n \geqslant 1$, $H_n$ is the smallest hyperbolic space that contains $\{y_0, \ldots, y_n\}$, then the family $\{H_n\}_{n\geqslant 1}$ satisfies the hypothesis of Lemma 4.1.3.

Let $(x_n)_{n\geqslant 1}$ be a sequence in $X$ such that

$$d(u,v) = \sup_{x\in X}\{d(u(x), v(x))\} = \lim_{n\to\infty} d(u(x_n), v(x_n)).$$

Fix $x_0 \in X$ and for every $i$ choose $\varphi_i \in \mathrm{Isom}(\mathbf{H}^\infty_\mathbf{R})$ such that $\varphi_i(x_0) = x_i$. Define $u_i = u\circ\varphi_i$ and $v_i = v \circ \varphi_i$. For every $i$ there exist an isometry $T_i^1$ such that $T_i^1 \circ u_i(x_0) = y_0$ and $T_i^1 \circ v_i(x_0) \in H_1$. Observe that for every $i$,

$$d(T_i^1 \circ u_i(x_0), T_i^1 \circ v_i(x_0)) \leqslant d(u,v).$$

$H_1$ is locally compact, therefore there exists a subsequence $\big(T_{1,i}^1 \circ v_{1,i}(x_0)\big)_{i\in\mathbf{N}}$ of $\big(T_i^1 \circ v_i(x_0)\big)_{i\in\mathbf{N}}$ which is convergent.

Let $\{z_i\}_{i\in\mathbf{N}_{\geqslant 1}}$ be a dense subset of $X$. Observe that for every $i$, there exists an isometry $T_i^2$ such that $T_i^2|_{H_1} = Id$ and

$$\{T_i^2 \circ T_{1,i}^1 \circ u_{1,i}(z_1), T_i^2 \circ T_{1,i}^1 \circ v_{1,i}(z_1)\} \subset H_3.$$

Notice that for every $i$,

$$d\Big(T_i^2 \circ T_{1,i}^1 \circ u_{1,i}(z_1), T_i^2 \circ T_{1,i}^1 \circ u_{1,i}(x_0)\Big) \leqslant Kd(z_1, x_0),$$

but

$$T_i^2 \circ T_{1,i}^1 \circ u_{1,i}(x_0) = T_{1,i}^1 \circ u_{1,i}(x_0) = y_0.$$

Therefore

$$\Big(T_i^2 \circ T_{1,i}^1 \circ u_{1,i}(z_1)\Big)_{i \in \mathbf{N}_{\geqslant 1}}$$

is a bounded sequence in $H_3$. Also, for every $i$,

$$d\Big(T_i^2 \circ T_{1,i}^1 \circ u_{1,i}(z_1), T_i^2 \circ T_{1,i}^1 \circ v_{1,i}(z_1)\Big) \leqslant d(u, v).$$

Thus,

$$\Big(T_i^2 \circ T_{1,i}^1 \circ v_{1,i}(z_1)\Big)_{i \geqslant 1}$$

is again a bounded sequence in $H_3$. So it is possible to chose respective subsequences,

$$\Big(T_{2,i}^2 \circ T_{2,i}^1 \circ u_{2,i}(z_1)\Big)_{i \geqslant 1}$$

and

$$\Big(T_{2,i}^2 \circ T_{2,i}^1 \circ v_{2,i}(z_1)\Big)_{i \geqslant 1}$$

that are convergent.

By induction on $n$, suppose that for every for every $2 \leqslant m \leqslant n$ and for every $i \geqslant 1$ there are isometries $T_{m,i}^m$, and $T_{n,i}^1$ such that

1. $T_{n,i}^1 \circ u_i(x_0) = y_0$ and $\big(T_{n,i}^1 \circ v_i(x_0)\big)_{i \geqslant 1}$ is a convergent sequence in $H_1$.
2. $T_{n,i}^m|_{H_{1+2(m-2)}} = Id$.
3. $\Big(T_{n,i}^m \circ \cdots \circ T_{n,i}^1 \circ u_{n,i}(z_{m-1})\Big)_{i \geqslant 1}$ and $\Big(T_{n,i}^m \circ \cdots \circ T_{n,i}^1 \circ v_{n,i}(z_{m-1})\Big)_{i \geqslant 1}$ are converging sequences in $H_{1+2(m-1)}$.

For every $i \geqslant 1$, let $T_i^{n+1}$ be an isometry with the following properties,

1. $T_i^{n+1}|_{H_{1+2(n+1-2)}} = Id$.

2. $T_i^{n+1} \circ \cdots \circ T_{n,i}^1 \circ u_{n,i}(z_n)$ and $T_i^{n+1} \circ \cdots \circ T_{n,i}^1 \circ v_{n,i}(z_n)$ are elements of $H_{1+2(n+1-1)}$.

Observe that

$$\left(T_i^{n+1} \circ \cdots \circ T_{n,i}^1 \circ u_{n,i}(z_n)\right)_{i \geqslant 1}$$

is a bounded sequence in $H_{1+2(n+1-1)}$, indeed

$$d\left(T_i^{n+1} \circ \cdots \circ T_{n,i}^1 \circ u_{n,i}(z_n), T_i^{n+1} \circ \cdots \circ T_{n,i}^1 \circ u_{n,i}(x_0)\right) \leqslant K d(z_n, x_0),$$

but

$$T_i^{n+1} \circ \cdots \circ T_{n,i}^1 \circ u_{n,i}(x_0) = y_0.$$

Moreover, for every $i$,

$$d\left(T_i^{n+1} \circ \cdots \circ T_{n,i}^1 \circ u_{n,i}(z_n), T_i^{n+1} \circ \cdots \circ T_{n,i}^1 \circ v_{n,i}(z_n)\right) \leqslant d(u, v).$$

Therefore

$$\left(T_i^{n+1} \circ \cdots \circ T_{n,i}^1 \circ u_{n,i}(z_n)\right)_{i \geqslant 1}$$

and

$$\left(T_i^{n+1} \circ \cdots \circ T_{n,i}^1 \circ v_{n,i}(z_n)\right)_{i \geqslant 1}$$

are bounded sequences in $H_{1+2(n+1-1)}$. Hence it is possible to choose convergent subsequences

$$\left(T_{n+1,i}^{n+1} \circ \cdots \circ T_{n+1,i}^1 \circ u_{n+1,i}(z_n)\right)_{i \geqslant 1}$$

and

$$\left(T_{n+1,i}^{n+1} \circ \cdots \circ T_{n+1,i}^1 \circ v_{n+1,i}(z_n)\right)_{i \geqslant 1}.$$

Define now,

$$U(z_n) = \lim_{i \to \infty} T_{i,i}^i \circ \cdots \circ T_{i,i}^1 \circ u_{i,i}(z_n)$$

and

$$V(z_n) = \lim_{i \to \infty} T_{i,i}^i \circ \cdots \circ T_{i,i}^1 \circ v_{i,i}(z_n).$$

Observe that there exists $M > 0$ such that,

$$\begin{aligned} U(z_n) &= \lim_{i \to \infty} T_{i,i}^i \circ \cdots \circ T_{i,i}^1 \circ u_{i,i}(z_n) \\ &= \lim_{i \to \infty} T_{i,i}^M \circ \cdots \circ T_{i,i}^1 \circ u_{i,i}(z_n) \\ &= \lim_{i \to \infty} T_{M,i}^M \circ \cdots \circ T_{M,i}^1 \circ u_{M,i}(z_n) \end{aligned}$$

and

$$V(z_n) = \lim_{i\to\infty} T_{M,i}^{M} \circ \cdots \circ T_{M,i}^{1} \circ v_{M,i}(z_n).$$

Given $z_n$ and $z_m$, there exists $M' > 0$ such that

$$\begin{aligned} d(U(z_n), U(z_m)) &= \lim_{i\to\infty} d(u \circ \varphi_{M',i}(z_n), u \circ \varphi_{M',i}(z_m)) \\ &\leqslant Kd(z_n, z_m), \end{aligned}$$

and with the same reasoning,

$$d(V(z_n), V(z_m)) \leqslant K(d(z_n, z_m)).$$

Therefore $U$ and $V$ can be extended to $X$.

For every $m \geqslant 1$, define

$$R_m = T_{m,m}^{m} \circ \cdots \circ T_{m,m}^{1} \circ u_{m,m}$$

and

$$S_m = T_{m,m}^{m} \circ \cdots \circ T_{m,m}^{1} \circ v_{m,m}.$$

Observe that for every $m$, $R_m$ and $S_m$ are Lipschitz continuous functions with Lipschitz constant smaller or equal than $K$. Therefore $\{R_n\}_n$ and $\{S_n\}_n$ are equicontinuous families. If the function $L_n$ is defined as $L_n(z) = d(R_n(z), S_n(z))$, then the family $\{L_n\}_n$ is equicontinuous and pointwise convergent to $z \mapsto d(U(z), V(z))$, thus by Arzelà-Ascoli Theorem, the convergence is uniform on compact sets.

The functions $u$ and $v$ are $\mathscr{C}^2$, and for every $i$, $\varphi_i$ is an isometry, therefore $u_i$ and $v_i$ are harmonic functions (see for example Proposition 2.2 in [34]). Moreover, for every $i, j$, the map $T_{i,i}^{j}$ is an isometry, therefore for every $m$, the functions $R_m$ and $S_m$ defined above are harmonic. For one reference for the last statement see the corollary at the end of page 131 of [25].

The distance function $\mathbf{H}_{\mathbf{R}}^{\infty} \times \mathbf{H}_{\mathbf{R}}^{\infty} \xrightarrow{d} \mathbf{R}$ is a (geodesically) convex function and for every $m$, the map $x \mapsto d(R_m(x), S_m(x))$ is harmonic (see the second example in page 133 of [25]). Therefore, for every $m$ the function $L_m$ is subharmonic (see Theorem 3.4 in [34]) and by Lemma 4.1.4, the map $z \mapsto d(U(z), V(z))$ is subharmonic.

Notice that for every $z \in X$ $d(u,v) \geqslant d(U(z),V(z))$, also

$$\begin{aligned} d(U(x_0),V(x_0)) &= \lim_m d(T_m(x_0),S_m(x_0)) \\ &= \lim_m d(u_{m,m}(x_0),v_{m,m}(x_0)) \\ &= \lim_m d(u(x_{m,m}),v(x_{m,m})) = d(u,v). \end{aligned}$$

Therefore $d(U(z),V(z))$ is constant as a consequence of the maximum principle for subharmonic maps. By construction, for every $z$,

$$d(U(z),V(z)) = d(u(z),v(z)),$$

hence, by Lemma 2.2 in [38], either $u = v$ or the images of $u$ and $v$ are contained in a geodesic. □

**Lemma 4.1.6.** *If* $\Gamma$ *is a torsion-free uniform lattice of* $\mathrm{SU}(1,n)$*, then the following hold:*

1. *All the non-trivial elements act as hyperbolic isometries of* $\mathbf{H}^n_{\mathbf{C}}$.
2. *If* $\ell(g)$ *is the translation length of* $g$ *acting as an isometry of* $\mathbf{H}^n_{\mathbf{C}}$*, then*

$$\inf\{\ell(\gamma) \mid \gamma \in \Gamma\setminus\{e\}\} > 0.$$

3. *There exists* $g \in \mathrm{SU}(1,n)$ *such that* $g\Gamma g^{-1}$ *and* $\Gamma$ *are non-commensurable.*

**Proof.** For 1) and 2) see Proposition II.6.10 in [7] and observe that if $g \in \Gamma\setminus\{e\}$ acts as an elliptic isometry, then it is contained in a compact (finite) subgroup of $\Gamma$ and this cannot be the case.

For 3) observe that every $\gamma \in \Gamma\setminus\{e\}$ preserves a unique axis in $\mathbf{H}^n_{\mathbf{C}}$ and that $\Gamma$ is finitely generated (see Theorem 6.15 and Remark 6.18 in [46]). Define

$$X = \{\xi \in \partial\mathbf{H}^n_{\mathbf{C}} \mid \gamma\xi = \xi \text{ for some } \gamma \in \Gamma\}.$$

Let $x \in X$ and $g \in \mathrm{SU}(1,n)$ be such that $gx \notin X$. This is possible because $X$ is countable. The claim is that $g\Gamma g^{-1}$ and $\Gamma$ are not commensurable. Indeed, $gx$ is fixed by some $\theta \in g\Gamma g^{-1}$, but for every $n$, $\theta$ and $\theta^n$ share the axis, therefore the two lattices cannot be commensurable. □

The existence of uniform lattices in connected, non compact and semisimple groups is due to Borel, for one reference see Chapter XIV in [46]. Any of these lattices is finitely generated and as a consequence of Selberg's Lemma (see [2]) they are also virtually torsion-free. This two facts together with the previous observations show that there exist $\Gamma_1$ and $\Gamma_2$, non-commensurable uniform lattices in $\mathrm{SU}(1,n)$.

Following [28], a pair $(G,H)$ is called a *Borel pair* if $G$ does not admit non-trivial homomorphisms to a compact group, $H$ is a closed subgroup and $G/H$ admits a finite $G$-invariant measure. In this article the author showed that if $(G,H)$ is a Borel pair, where $G$ is a connected real algebraic group, then $H$ is Zariski-dense in $G$ (see Corollary 4 in [28]).

The claim is that $\mathrm{SU}(1,n)$ does not have homomorphisms to compact Lie groups. If $G \xrightarrow{\varphi} H$ is an homomorphism of Lie groups, where $G$ is connected, semisimple and with finite center, then the image of $\varphi$ is closed (see Corollary 1.2 and the proof of Corollary 1.3 in [45]). Therefore if there exists a non-trivial Lie group homomorphism $G \xrightarrow{\psi} K$, where $K$ is a compact Lie group, then the image is a compact and semisimple Lie group. This produces a decomposition in $\mathfrak{g}$, the Lie algebra of $G$, $\mathfrak{g} = \mathfrak{a} + \mathfrak{a}^{\perp}$, where $\mathfrak{a}$ is the kernel of $d\psi$ and $\mathfrak{a}^{\perp}$ is the complement with respect to the Killing form. The map $d\psi$ restricted to $\mathfrak{a}^{\perp}$ is an isomorphism, therefore the Killing form of $\mathfrak{a}^{\perp}$ is negative definite. The normal subgroup associated to this ideal is compact (see Corollary 3.6.3 in [24]). This shows that $\mathrm{SU}(1,n)$ does not admit non-trivial homomorphisms to compact Lie groups because $\mathrm{SU}(1,n)$ does not admit compact normal subgroups.

**Lemma 4.1.7.** *Given two non-commensurable lattices* $\Gamma_1$ *and* $\Gamma_2$ *of* $\mathrm{SU}(1,n)$ *(or any connected real semisimple linear algebraic group without compact factors), the group H generated by* $\Gamma_1 \cup \Gamma_2$ *is dense in* $\mathrm{SU}(1,n)$.

**Proof.** Observe that $\overline{H}$, the closure of $H$ for the usual topology, is Zariski-dense in $\mathrm{SU}(1,n)$. Consider $\mathfrak{h}$ the Lie subalgebra of $\overline{H}$. This space is invariant under the action of $H$, therefore it is $\mathrm{SU}(1,n)$-invariant because the action is Zariski-continuous. This means that $\overline{H}_0$ is a normal subgroup of $\mathrm{SU}(1,n)$, but $\mathrm{SU}(1,n)$ is simple. Suppose $\overline{H}_o$ is the trivial group. Observe that $\overline{H}/\Gamma_i$ carries a finite invariant measure (see Lemma 1.6 in [46]), therefore $\Gamma_1$ and $\Gamma_2$ have finite index in $\overline{H}$. This implies that $\Gamma_1$ and $\Gamma_2$ are commensurable, which is a contradiction. □

Let $\mathrm{SU}(1,n) \xrightarrow{\phi} \mathrm{PU}(1,n)$ be the projectivization map. This is a surjective homomorphism. The map $\phi$ has finite kernel, therefore if $\Gamma_1$ and $\Gamma_2$ are as above, $\phi(\Gamma_1)$ and $\phi(\Gamma_2)$ are two uniform non-commensurable lattices of $\mathrm{PU}(1,n)$. Indeed, observe that $\Gamma_i \ker(\phi)$ is closed and countable (discrete), therefore there is $U$ an open subset of $\mathrm{SU}(1,n)$ such that $U \cap (\Gamma_i \ker(\phi)) = \{e\}$. This shows that $\phi(\Gamma_i)$ is a discrete subgroup of $\mathrm{PU}(1,n)$. For the existence of a finite $\phi(\mathrm{SU}(1,n))$-invariant measure observe that there is a natural continuous

$G$-equivariant bijection

$$\mathrm{SU}(1,n)/\Gamma_i \to \phi(\mathrm{SU}(1,n))/\phi(\Gamma_i)$$

where the domain is compact. The lattices $\phi(\Gamma_1)$ and $\phi(\Gamma_2)$ are not commensurable because $\ker(\phi)$ is finite. The group generated by $\phi(\Gamma_1)$ and $\phi(\Gamma_2)$ is dense because $\Gamma_1$ and $\Gamma_2$ generate a dense subgroup of $\mathrm{SU}(1,n)$.

**Theorem 4.1.8.** *For $n \geqslant 2$, $\mathrm{PU}(1,n)$ does not admit non-elementary representations into $Isom(\mathbf{H}^{\infty}_{\mathbf{R}})$.*

**Proof.** Let $\rho$ be a non-elementary representation, given a torsion-free uniform lattice $\Gamma$ of $PU(1,n)$, the restriction of $\rho$ to $\Gamma$ is non-elementary. Therefore there exists a $\Gamma$-equivariant, harmonic and Lipschitz continuous map $\mathbf{H}^n_{\mathbf{C}} \xrightarrow{u} \mathbf{H}^{\infty}_{\mathbf{R}}$ (see Theorem 2.3.1 of [35]). In Section 3.2 of [22], the authors showed that this map is $\mathscr{C}^{\infty}$.

Given $\Gamma_1$ and $\Gamma_2$ two non-commensurable and uniform lattices of $\mathrm{PU}(1,n)$, there are $\mathscr{C}^2$, harmonic, Lipschitz and $\Gamma_i$-equivariant functions, $\mathbf{H}^n_{\mathbf{C}} \xrightarrow{u_i} \mathbf{H}^{\infty}_{\mathbf{R}}$. Therefore, it follows from Lemmas 4.1.2 and 4.1.5 that $u_1 = u_2$. This implies that the function $u = u_i$ is $\mathrm{PU}(1,n)$-equivariant. In Proposition 8 of [22], the authors showed that the real rank of $u$ is at most 2. The arguments used there go back to the work of Sampson (see [47]). If $x \in \mathbf{H}^n_{\mathbf{C}}$, the kernel of $df_x$ is nontrivial. The group $Stab(x)$ acts transitively in spheres of the tangent space of $x$ and $u$ is $\mathrm{PU}(1,n)$-equivariant, therefore $u$ is constant, but this is a contradiction. □

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