Discount models

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In memory of Tomas Björk

Abstract
Discount is the difference between the face value of a bond and its present value. We propose an arbitrage-free dynamic framework for discount models, which provides an alternative to the Heath–Jarrow–Morton framework for forward rates. We derive general consistency conditions for factor models, and discuss affine term structure models in particular. There are several open problems, and we outline possible directions for further research.

Keywords Discount · Factor models · Stochastic partial differential equation · Term structure models · Zero-coupon bonds

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1 Discount

Let $P(t, T)$ denote the time-t price of a zero-coupon bond with maturity T, or in short, a T-bond. Define the corresponding *discount*

$$H(t, T) := 1 - P(t, T).$$

The discount $H(t, T)$ is the difference between the face value of the bond and its present value. It is the interest earned on investing in a T-bond at t and holding it to maturity T. As such, it quantifies the time value of money. It also equals the time-t price of a long position in a floating rate note paying overnight short rates $r_t = -\partial_T P(t, T)|_{T=t}$ minus a short position in a T-bond. We call this long/short

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portfolio the *T-discount*. The infinitesimal net cash flow generated by the T-discount is $r_t \, dt$ at any $t \leq T$ and 0 after T. The payments of the principals of the floating rate note and the zero-coupon bond at T offset each other. A T-discount is therefore identical to the floating leg of an overnight indexed swap with maturity T.

The gains process, say $G(t, T)$, from holding a T-discount over $[0, t]$, where the cash flows are continuously invested in the money market account that earns interest at the short rate, is given by the sum

$$G(t, T) = \underbrace{\int_0^t e^{\int_s^t r_u \, du} r_s \, ds}_{\text{accumulated cash flow}} + \underbrace{H(t, T)}_{\text{spot value}} = e^{\int_0^t r_s \, ds} - P(t, T). \tag{1.1}$$

2 Discount framework

In analogy to the Heath–Jarrow–Morton (HJM) approach [8] for modelling the forward rates $f(t, T) = -\partial_T \log P(t, T)$, we now formulate an arbitrage-free dynamic framework for discount models. We start with a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{Q})$ satisfying the usual conditions and carrying an n-dimensional standard Brownian motion W. For simplicity of exposition, we skip technical details and assume throughout that all stochastic processes are adapted, and regular enough such that the differential and integral operations are well defined. For more background and technical details, also on arbitrage pricing, we refer to Björk [1, Chap. 19] and Filipović [4, Chap. 4].

We recall that $\mathbb{Q}$ is a risk-neutral or *(local) martingale measure* for the bond market if all discounted T-bond price processes $(e^{-\int_0^t r_s \, ds} P(t, T))_{0 \leq t \leq T}$ are (local) $\mathbb{Q}$-martingales. In view of (1.1), this holds if and only if all discounted T-discount gains processes $(e^{-\int_0^t r_s \, ds} G(t, T))_{0 \leq t \leq T}$ are (local) $\mathbb{Q}$-martingales. It is well known that the bond market is arbitrage-free if and (essentially) only if $\mathbb{Q}$ is a local martingale measure; see e.g. [1, Chap. 11] or [4, Sect. 4.3.4].

We represent the T-discount price at any $t \leq T$ in terms of its maturity derivative,

$$H(t, T) = \int_t^T h(t, s) \, ds,$$

where the *discount derivative* $h(t, T)$ is assumed to follow an Itô process with dynamics of the form

$$dh(t, T) = \alpha(t, T) \, dt + \sigma(t, T) \, dW_t \tag{2.1}$$

for some drift and volatility processes $(\alpha(t, T))$ and $(\sigma(t, T))$, respectively. We then specify the T-bond price by

$$P(t, T) = 1 - \int_t^T h(t, s) \, ds. \tag{2.2}$$

The T-bond specification (2.2) is the linearised counterpart to the familiar relation $P(t,T) = e^{-\int_t^T f(t,s)\,ds}$. Differentiating in T, the relationship between the discount derivative and the forward rate is obtained as

$$h(t,T) = P(t,T)f(t,T).$$

In particular,

$$h(t,t) = f(t,t) = r_t \qquad (2.3)$$

equals the short rate.

The challenge of our linear approach is that T-bond prices should be positive, $P(t,T) > 0$, which is equivalent to requiring that

$$\int_t^T h(t,s)\,ds < 1 \qquad \text{for all } t \leq T. \qquad (2.4)$$

Positivity of bond prices follows in particular if we can show that $\mathbb{Q}$ is a martingale measure for the bond market specified by (2.1) and (2.2). We now derive necessary and sufficient conditions for this to hold. We thus let $h(t,T)$ be given by (2.1) and define the T-bond prices by (2.2) and the short rates by (2.3). Here is our first result, which shows that the drift $\alpha(t,T)$ is fully determined by the requirement that $\mathbb{Q}$ is a local martingale measure.

Proposition 2.1 *Measure $\mathbb{Q}$ is a local martingale measure if and only if we have the discount drift condition*

$$\alpha(t,T) = h(t,T)h(t,t). \qquad (2.5)$$

Proof The implied T-bond price dynamics is

$$dP(t,T) = -d\left(\int_t^T h(t,s)\,ds\right)$$

$$= r_t\,dt - \int_t^T \alpha(t,s)\,ds\,dt - \int_t^T \sigma(t,s)\,ds\,dW_t. \qquad (2.6)$$

On the other hand, the discounted T-price process $(e^{-\int_0^t r_s\,ds}P(t,T))$ is a local $\mathbb{Q}$-martingale if and only if the drift of $P(t,T)$ equals $P(t,T)r_t\,dt$. Matching this with the drift in (2.6) gives $\int_t^T \alpha(t,s)\,ds = H(t,T)r_t$. Differentiating in T, we obtain (2.5), which proves the result. □

Given the drift condition (2.5) and using the existence of a local martingale measure as synonymous for the absence of arbitrage, we can paraphrase that the generic

dynamics (2.1) for an arbitrage-free discount derivative model is of the form

$$dh(t, T) = h(t, T)h(t, t)\, dt + \sigma(t, T)\, dW_t, \qquad 0 \le t \le T,$$

$$h(0, T) = h_0(T), \tag{2.7}$$

for an initial discount derivative curve h_0. Note that in contrast to the HJM drift condition on the forward rates, see e.g. [4, Theorem 6.1], the derivative drift condition (2.5) does not depend on the volatility $\sigma(t, T)$.

We next show that for a given volatility structure $\sigma(t, T)$, the system of stochastic differential equations (2.7) uniquely determines $h(t, T)$.

Lemma 2.2 *For a given volatility process $\sigma(t, T)$, $0 \le t \le T < \infty$, and initial discount derivative curve h_0, there exists at most one solution $h(t, T)$, $0 \le t \le T < \infty$, to (2.7).*

Proof Let h and $\tilde{h}$ be solutions to (2.7). Then their discounted difference

$$q(t, T) := e^{-\int_0^t \tilde{h}(s,s)\, ds} \big(\tilde{h}(t, T) - h(t, T) \big)$$

satisfies

$$dq(t, T) = h(t, T)q(t, t)\, dt, \qquad q(0, T) = 0. \tag{2.8}$$

Integrating gives, for $T = t$,

$$q(t, t) = \int_0^t h(s, t)q(s, s)\, ds \qquad \text{for all } t \ge 0.$$

We claim that $q(t, t) = 0$ for all $t \ge 0$, and by (2.8) thus $\tilde{h} = h$. Indeed, by contradiction, assume that there exist $0 \le t_0 < t_1$ such that $q(s, s) = 0$ for all $s \le t_0$, and $|q(t_1, t_1)| > 0$. We let $\beta(s)$ be a positive process such that $|h(s, t)| \le \beta(s)$ for all $t_0 \le s \le t \le t_1$. Then $|q(t, t)| \le \int_{t_0}^t \beta(s)|q(s, s)|\, ds$ for all $t \in [t_0, t_1]$, and Gronwall's inequality implies that $q(t, t) = 0$ for all $t \in [t_0, t_1]$. This contradicts the assumption, whence the claim is proved. $\qquad\square$

The problem remains that we still have no guarantee that bond prices in (2.2) are positive. Our main result is the following theorem, which provides sufficient conditions such that the discount framework (2.7) defines an arbitrage-free price system for T-bonds. It thus represents an alternative to the HJM framework of forward rates.

Theorem 2.3 *Let $h(t, T)$, $0 \le t \le T < \infty$, be any solution to (2.7) such that $r_t = h(t, t)$ is well defined and*

$$\int_0^t e^{-\int_0^s r_u\, du} \sigma(s, T)\, dW_s, 0 \le t \le T, \text{ is a } \mathbb{Q}\text{-martingale, for all } T. \tag{2.9}$$

Then $\mathbb{Q}$ is a martingale measure and the implied T-bond prices in (2.2) satisfy

$$P(t, T) = \mathbb{E}_{\mathbb{Q}} \big[e^{-\int_t^T r_s\, ds} \big| \mathcal{F}_t \big],$$

and are thus positive, that is, (2.4) holds, in particular.

Proof Let $h(t, T), 0 \le t \le T < \infty$, be a solution to (2.7). Denote $r_t = h(t, t)$ and set

$$M(t, T) := e^{-\int_0^t r_s \, ds} h(t, T).$$

Then $(M(t, T))$ has zero drift and dynamics

$$dM(t, T) = e^{-\int_0^t r_s \, ds} \sigma(t, T) \, dW_t,$$

and thus is a $\mathbb{Q}$-martingale by the assumption (2.9). We obtain

$$
\begin{aligned}
h(t, T) &= e^{\int_0^t r_s \, ds} M(t, T) \\
&= e^{\int_0^t r_s \, ds} \mathbb{E}_{\mathbb{Q}}[M(T, T)|\mathcal{F}_t] \\
&= \mathbb{E}_{\mathbb{Q}}\big[e^{-\int_t^T r_s \, ds} r_T \big| \mathcal{F}_t\big].
\end{aligned}
\tag{2.10}
$$

Integrating over T gives

$$
\begin{aligned}
H(t, T) &= \int_t^T h(t, u) \, du \\
&= \mathbb{E}_{\mathbb{Q}}\bigg[\int_t^T e^{-\int_t^u r_s \, ds} r_u \, du \bigg| \mathcal{F}_t\bigg] \\
&= 1 - \mathbb{E}_{\mathbb{Q}}\big[e^{-\int_t^T r_s \, ds} \big| \mathcal{F}_t\big],
\end{aligned}
\tag{2.11}
$$

as desired. $\square$

The following remarks provide further details and discussion on the above discount framework.

Remark 2.4 The identities (2.10) and (2.11) are of independent interest, and show the economic meaning of $h(t, T)$ and $H(t, T)$ as present values of future cash flows r_T and r_u for $u \in [t, T]$, respectively.

Remark 2.5 The expression (2.6) shows that the induced volatility $v(t, T)$ of the T-bond returns is given by $P(t, T) v(t, T) = -\int_t^T \sigma(t, s) \, ds$.

Remark 2.6 The equivalent physical measure $\mathbb{P} \approx \mathbb{Q}$ is related to $\mathbb{Q}$ by the market price of risk θ such that the Radon–Nikodým derivative satisfies

$$\mathbb{E}_{\mathbb{Q}}\bigg[\frac{d\mathbb{P}}{d\mathbb{Q}}\bigg|\mathcal{F}_T\bigg] = \exp\bigg(\int_0^T \theta_t dW_t - \frac{1}{2}\int_0^T |\theta_t|^2 dt\bigg)$$

for any time horizon $T > 0$. This yields the $\mathbb{P}$-Brownian motion $dW_t^{\mathbb{P}} = dW_t - \theta_t \, dt$. Hence the dynamics under $\mathbb{P}$ of $h(t, T)$ is

$$dh(t, T) = \big(h(t, T)h(t, t) + \sigma(t, T)\theta_t\big) dt + \sigma(t, T) \, dW_t^{\mathbb{P}}.$$

Remark 2.7 The proofs of Lemma 2.2 and Theorem 2.3 rely on technical properties, e.g. that $T \mapsto h(t, T)$ is locally bounded in $T \geq t$ a.s., or that the order of integration and expectation can be changed. Such properties can be asserted by imposing sufficient technical assumptions.

Remark 2.8 Existence of a solution for the system (2.7) of SDEs is an open problem. Existence could be an issue in view of the quadratic drift (2.5), which may cause explosion in finite time. Example 2.9 and Sect. 3.3 below give non-exploding specifications. A natural approach for a systematic study is to state (2.7) as a stochastic partial differential equation for the discount derivative curve $\psi_t(x) := h(t, t + x)$ in an appropriate function space $\mathcal{H}$. Such an SPDE is of the form

$$d\psi_t(x) = \big(\partial_x \psi_t(x) + \psi_t(x)\psi_t(0)\big)dt + B(\psi_t)\,dW_t, \qquad (2.12)$$

for some appropriate volatility operator $B : \mathcal{H} \to \mathcal{H}^n$ such that $\sigma(t, t + \cdot) = B(\psi_t)$. An example of an appropriate function space is the weighted Sobolev space $\mathcal{H}_w$ consisting of weakly differentiable functions $\psi : [0, \infty) \to \mathbb{R}$ with

$$\|\psi\|_w^2 := \int_0^\infty \big(\psi'(x)\big)^2 w(x)\,dx < \infty$$

and $\psi(\infty) = 0$, for some increasing and continuously differentiable weight function w such that $\int_0^\infty w^{-1/3}(x)\,dx < \infty$. An example could be $w(x) = e^{\alpha x}$ for some $\alpha > 0$. This is similar to the space H_w introduced in Filipović [3, Chap. 5]. It can be shown as in [3, Equation (5.7)] that the L^1-norm of any $\psi \in \mathcal{H}_w$ is bounded by $\int_0^\infty |\psi(x)|\,dx \leq C_w \|\psi\|_w$ for some finite constant C_w. It can further be shown as in [3, Theorem 5.1.1] that the differential operator ∂_x generates a strongly continuous semigroup on $\mathcal{H}_w$. Hence one can study existence and uniqueness of (local) mild and weak solutions to (2.12) in the spirit of Da Prato and Zabczyk [2, Chap. 7]. In fact, uniqueness follows as soon as the volatility operator $B(\psi_t)$ is Lipschitz-continuous in ψ_t; see [3, Corollary 2.4.1]. This is a direct improvement of Lemma 2.2, since there we assumed that the volatility process $(\sigma(t, T))$ is given as exogenous and does not depend on ψ_t. Similarly, global existence would follow if one could show that any local weak solution of (2.12) with $\|\psi_0\|_w < C_w^{-1}$ remains bounded in the sense that $\|\psi_t\|_w < C_w^{-1}$ for all $t \geq 0$. Combined with the above L^1-bound, this would imply that $h(t, s) = \psi_t(s - t)$ satisfies (2.4), so that the induced bond prices (2.2) are positive.

For the special deterministic case where $B \equiv 0$, the unique local solution to (2.12) is given by

$$\psi_t(x) = \frac{\psi_0(t + x)}{1 - \int_0^t \psi_0(s)\,ds}.$$

This solution does not explode in finite time if and only if the initial curve satisfies (2.4), that is, $\int_0^t \psi_0(s)\,ds < 1$ for all finite t. In this case, it follows easily that also ψ_t satisfies (2.4), as $\int_0^{T-t} \psi_t(x)\,dx = (\int_0^T \psi_0(s)\,ds - \int_0^t \psi_0(s)\,ds)(1 - \int_0^t \psi_0(s)\,ds)^{-1} < 1$ for all finite $T \geq t$.

The following example illustrates that the discount framework (2.7) admits global solutions.

Example 2.9 A simple toy model specification is $h(t, T) = \phi(T - t)r_t$ for some deterministic function ϕ with $\phi(0) = 1$, and where the short rate r_t follows an Itô process of the form $dr_t = \mu_t\, dt + \nu_t\, dW_t$. The induced dynamics of $h(t, T)$ is

$$dh(t, T) = \big(\phi(T - t)\mu_t - \phi'(T - t)r_t\big)\, dt + \phi(T - t)\nu_t\, dW_t.$$

The drift condition (2.5) now reads as the consistency condition

$$\phi(T - t)\mu_t - \phi'(T - t)r_t = \phi(T - t)r_t^2, \tag{2.13}$$

which does not depend on the volatility process (ν_t), whereas the volatility in (2.7) is simply induced as $\sigma(t, T) = \phi(T - t)\nu_t$. This property holds more generally in affine discount term structure models; see Remark 3.1 below.

For $T = t$, as $\phi(0) = 1$, we obtain the short rate drift $\mu_t = (r_t + \phi'(0))r_t$ and volatility $\nu_t = \sigma(t, t)$. Assume that $\phi'(0) = -\theta$ is negative for some parameter $\theta > 0$, and assume that $\sigma(t, t) \to 0$ fast enough as $r_t \to 0$ and $r_t \to \theta$, respectively. Then the risk-neutral dynamics

$$dr_t = -(\theta - r_t)r_t\, dt + \sigma(t, t)\, dW_t$$

is well behaved with values in $[0, \theta]$, for any initial value $r_0 \in [0, \theta]$. This short rate dynamics is reminiscent of the linear–rational framework; see Filipovic et al. [7, Sect. IV.E].

Plugging $\mu_t = -(\theta - r_t)r_t$ back into (2.13), we obtain

$$-\phi(T - t)\theta r_t - \phi'(T - t)r_t = 0,$$

which is equivalent to $\phi(s) = e^{-\theta s}$. This results in a term structure of T-discounts of the form

$$H(t, T) = r_t \int_0^{T-t} \phi(s)\, ds = (1 - e^{-(T-t)\theta})\frac{r_t}{\theta},$$

and thus a term structure of T-bonds of the form

$$P(t, T) = 1 - (1 - e^{-(T-t)\theta})\frac{r_t}{\theta}.$$

As a technical note, we have from the above that r_t takes values in $[0, \theta]$, which implies that the discount curve $T \mapsto P(t, T)$ is decreasing and $P(t, T) > 0$ for all finite maturities $T > t$.

3 Discount factor models

We now elaborate on discount factor models, extending Example 2.9. We first derive the consistency conditions for a general factor model. We then study in more detail the affine discount factor models.

3.1 General factor models

We study consistent discount factor models of the form

$$h(t, T) = \phi(T - t, Z_t)$$

for some function $\phi : [0, \infty) \times \mathcal{Z} \to \mathbb{R}$, where $\mathcal{Z} \subseteq \mathbb{R}^d$ is some state space with non-empty interior, and the factor (Z_t) is some $\mathcal{Z}$-valued diffusion process with dynamics

$$dZ_t = \mu(Z_t)\, dt + \nu(Z_t)\, dW_t$$

for a drift function $\mu : \mathcal{Z} \to \mathbb{R}^d$ and volatility function $\nu : \mathcal{Z} \to \mathbb{R}^{d \times n}$. We denote the corresponding diffusion function by $c(\cdot) = \nu(\cdot)^\top \nu(\cdot)$. The induced dynamics of $h(t, T)$ is

$$dh(t, T) = \left(-\partial_1 \phi(T - t, Z_t) + \mu(Z_t)^\top \nabla_z \phi(T - t, Z_t) \right.$$

$$\left. + \frac{1}{2} \operatorname{trace}\left(c(Z_t)\nabla_z^2 \phi(T - t, Z_t) \right) \right) dt + \nabla_z \phi(T - t, Z_t)^\top \nu(Z_t)\, dW_t.$$

Matching the drift term with the arbitrage-free dynamics (2.7) pointwise gives the consistency equation

$$-\partial_x \phi(x, z) + \mu(z)^\top \nabla_z \phi(x, z) + \frac{1}{2} \operatorname{trace}\left(c(z)\nabla_z^2 \phi(x, z) \right) = \phi(x, z)\phi(0, z), \quad (3.1)$$

whereas the induced volatility is

$$\sigma(t, T) = \nabla_z \phi(T - t, Z_t)^\top \nu(Z_t). \tag{3.2}$$

3.2 Affine discount term structure models

We now assume an affine term structure

$$\phi(x, z) = \phi_0(x) + \sum_{i=1}^d \phi_i(x) z_i \tag{3.3}$$

for some functions $\phi_j : [0, \infty) \to \mathbb{R}$, $j = 0, 1, \ldots, d$. Plugging in the affine term structure (3.3) in the consistency equation (3.1) gives

$$-\phi_0'(x) - \sum_{i=1}^d \phi_i'(x) z_i + \sum_{i=1}^d \phi_i(x)\mu_i(z)$$

$$= \phi_0(x)\gamma_0 + \sum_{j=1}^d \left(\phi_0(x)\gamma_j + \phi_j(x)\gamma_0 \right) z_j + \sum_{j,k=1}^d \phi_j(x)\gamma_k z_j z_k, \tag{3.4}$$

where we denote

$$\gamma_j := \phi_j(0), \qquad j = 0, 1, \ldots, d.$$

Remark 3.1 We note that the consistency condition (3.4) does not depend on the diffusion matrix $c(z)$ of the factor process (Z_t). This is in contrast to the nonlinear case as seen in (3.1), where $c(z)$ shows up. In other words, an affine discount term structure leaves the underlying volatility unspanned. This is in contrast to affine models of the forward rates, see Filipović et al. [6], and reminiscent of the linear–rational framework, see Filipović et al. [7, Sect. I]. In fact, it follows directly from (3.3) that T-bond prices become affine expressions in Z_t. We derive the explicit expressions in (3.17) below.

We henceforth assume that the functions

$$\phi_1, \ldots, \phi_d \text{ are linearly independent.} \tag{3.5}$$

Equation (3.4) then implies that every drift function $\mu_i(z)$ is a quadratic polynomial in z,

$$\mu_i(z) = b_i + \sum_{j=1}^{d} \beta_{ij} z_j + \sum_{j,k=1}^{d} B_{i,jk} z_j z_k \tag{3.6}$$

for some coefficients b_i, β_{ij}, $B_{i,jk}$. Plugging (3.6) into (3.4) and matching coefficients of the same order in z gives

$$-\phi_0'(x) + \sum_{i=1}^{d} b_i \phi_i(x) = \phi_0(x) \gamma_0, \tag{3.7}$$

$$-\phi_j'(x) + \sum_{i=1}^{d} \beta_{ij} \phi_i(x) = \phi_0(x) \gamma_j + \phi_j(x) \gamma_0, \qquad j = 1, \ldots, d, \tag{3.8}$$

$$\sum_{i=1}^{d} (B_{i,jk} + B_{i,kj}) \phi_i(x) = \phi_j(x) \gamma_k + \phi_k(x) \gamma_j, \qquad j, k = 1, \ldots, d. \tag{3.9}$$

Writing $\bar{Z}_t = (1, Z_{1,t}, \ldots, Z_{d,t})^\top$ for the extended factor process including the constant 1 and setting $e_0 = (1, 0, \ldots, 0)^\top$, $\bar{\phi}(x) := (\phi_0(x), \ldots, \phi_d(x))^\top$ and $\bar{\gamma} := (\gamma_0, \ldots, \gamma_d)^\top = \bar{\phi}(0)$, we arrive at the following result.

Proposition 3.2 *Assume (3.5). The factor process (Z_t) in an arbitrage-free affine discount term structure model of the form (3.3) has a quadratic drift of the form*

$$\mu_i(z) = b_i + \sum_{j=1}^{d} \beta_{ij} z_j + z_i \sum_{j=1}^{d} \gamma_j z_j, \qquad i = 1, \ldots, d, \tag{3.10}$$

for some coefficients b_i, β_{ij} and γ_j. The functions ϕ_i in turn are given by

$$\bar{\phi}(x) = e^{Ax} \bar{\gamma}, \tag{3.11}$$

where A is defined as the $(d+1) \times (d+1)$ matrix on the right-hand side of (3.14) below. The T-discounts and short rates are linear in $\bar{Z}_t$,

$$H(t,T) = 1 - e_0^\top e^{A^\top(T-t)} \bar{Z}_t, \tag{3.12}$$

$$r_t = \bar{\gamma}^\top \bar{Z}_t. \tag{3.13}$$

Proof In view of (3.5), condition (3.9) is equivalent to

$$B_{i,jk} = \begin{cases} \gamma_k, & \text{if } i = j, \\ \gamma_j, & \text{if } i = k, \\ 0, & \text{otherwise.} \end{cases}$$

Hence the drift (3.6) takes the form (3.10).

Equations (3.7) and (3.8) are equivalent to the linear system of ordinary differential equations

$$\begin{pmatrix} \phi_0'(x) \\ \phi_1'(x) \\ \vdots \\ \phi_i'(x) \\ \vdots \\ \phi_d'(x) \end{pmatrix} = \underbrace{\begin{pmatrix} -\gamma_0 & b_1 & b_2 & \cdots & b_{d-1} & b_d \\ -\gamma_1 & \beta_{11} - \gamma_0 & \beta_{21} & \cdots & \beta_{d-1,1} & \beta_{d1} \\ \vdots & \vdots & \ddots & & & \vdots \\ -\gamma_i & \beta_{1i} & & \beta_{ii} - \gamma_0 & & \beta_{di} \\ \vdots & \vdots & & & \ddots & \vdots \\ -\gamma_d & \beta_{1d} & \beta_{2d} & \cdots & \beta_{d-1,d} & \beta_{dd} - \gamma_0 \end{pmatrix}}_{=:A} \begin{pmatrix} \phi_0(x) \\ \phi_1(x) \\ \vdots \\ \phi_i(x) \\ \vdots \\ \phi_d(x) \end{pmatrix}. \tag{3.14}$$

The system (3.14) of ODEs can be written in compact form as

$$\bar{\phi}'(x) = A\bar{\phi}(x), \qquad \bar{\phi}(0) = \bar{\gamma}. \tag{3.15}$$

The solution of (3.15) is (3.11).

Note that

$$\bar{\gamma} = -A e_0. \tag{3.16}$$

Hence $\bar{\phi}(x) = -e^{Ax} A e_0 = -\partial_x e^{Ax} e_0$, and the primitive functions

$$\Phi_j(x) = \int_0^x \phi_j(u)\,du, \qquad j = 0, \ldots, d,$$

are given by

$$\bar{\Phi}(x) = \big(\Phi_0(x), \ldots, \Phi_d(x)\big)^\top = (I_{d+1} - e^{Ax})e_0.$$

The expressions (3.12) and (3.13) for the T-discount $H(t,T) = \bar{\Phi}(T-t)^\top \bar{Z}_t$ and short rates r_t now follow by elementary calculus. $\qquad \square$

As announced in Remark 3.1, we conclude from (3.12) that the induced T-bond prices are linear in $\bar{Z}_t$,

$$P(t, T) = e_0^\top e^{A^\top (T-t)} \bar{Z}_t. \tag{3.17}$$

Differentiation using (3.16) shows that forward rates are linear–rational in $\bar{Z}_t$,

$$f(t, T) = \frac{\bar{\gamma}^\top e^{A^\top (T-t)} \bar{Z}_t}{e_0^\top e^{A^\top (T-t)} \bar{Z}_t}.$$

3.3 Well-behaved factor processes

As noted in Remark 2.8, the existence of global factor processes may be an issue in view of their quadratic drift (3.10), which may cause explosion in finite time. We discuss here some well-behaved specifications.

More specifically, we show that there exist factor processes (Z_t) with quadratic drift of the form (3.10) and taking values in the half-open solid simplex

$$\mathcal{Z} := \left\{ z \in [0, 1] : \sum_{i=1}^d z_i < 1 \right\}.$$

This can always be achieved, as we show now. To that end, we first specify a diffusion (U_t) with values in $[0, \infty)^d$ and then apply the diffeomorphism $G : [0, \infty)^d \to \mathcal{Z}$ defined by

$$G_i(u) := u_i \left(1 + \sum_{j=1}^d u_j \right)^{-1}, \qquad i = 1, \ldots, d,$$

with inverse $G_i^{-1}(z) = z_i (1 - \sum_{j=1}^d z_j)^{-1}$.

The dynamics of (U_t) could be of the form

$$dU_{t,i} = \left(\kappa_i U_{t,i} + \theta_i \left(1 + \sum_{j=1}^d U_{t,j} \right) \right) dt + q_i \sqrt{U_{t,i} \left(1 + \sum_{j=1}^d U_{t,j} \right)} \, dW_{t,i} \tag{3.18}$$

for some parameters $\kappa_i \in \mathbb{R}$, $\theta_i, q_i \geq 0$.

Lemma 3.3 *There exists a $[0, \infty)^d$-valued weak solution (U_t) to (3.18). The drift of the transformed process $(Z_t) := (G(U_t))$ is quadratic in Z_t of the form (3.10), and assumption (2.9) holds so that Theorem 2.3 applies.*

Proof The existence of a $[0, \infty)^d$-valued weak solution (U_t) to (3.18) follows from Filipović and Larsson [5, Theorem 5.3 and Proposition 6.4].

Now define $V_t := 1 + \sum_{j=1}^d U_{t,j}$ so that we obtain $Z_{t,i} = U_{t,i}(V_t)^{-1}$, and denote $\theta_V := \sum_{j=1}^d \theta_j$. Using Itô calculus, we get

$$dU_{t,i} = (\kappa_i U_{t,i} + \theta_i V_t) \, dt + q_i \sqrt{U_{t,i}} \sqrt{V_t} \, dW_{t,i},$$

and thus

$$dV_t = \sum_{j=1}^{d} dU_{t,j} = \left(\sum_{j=1}^{d} \kappa_j U_{t,j} + \theta_V V_t \right) dt + \sum_{j=1}^{d} q_j \sqrt{U_{t,j}} \sqrt{V_t} \, dW_{t,j}$$

and

$$
\begin{aligned}
d(V_t^{-1}) &= -V_t^{-2} dV_t + V_t^{-3} d\langle V, V \rangle_t \\
&= -V_t^{-1} \left(\sum_{j=1}^{d} \kappa_j Z_{t,j} + \theta_V \right) dt - V_t^{-1} \sum_{j=1}^{d} q_j \sqrt{Z_{t,j}} \, dW_{t,j} \\
&\quad + V_t^{-1} \sum_{j=1}^{d} q_j^2 Z_{t,j} \, dt.
\end{aligned}
$$

Integration by parts thus gives

$$
\begin{aligned}
dZ_{t,i} &= U_{t,i} \, d(V_t^{-1}) + V_t^{-1} \, dU_{t,i} + d\langle U_i, V^{-1} \rangle_t \\
&= -Z_{t,i} \left(\sum_{j=1}^{d} \kappa_j Z_{t,j} + \theta_V \right) dt - Z_{t,i} \sum_{j=1}^{d} q_j \sqrt{Z_{t,j}} \, dW_{t,j} + Z_{t,i} \sum_{j=1}^{d} q_j^2 Z_{t,j} \, dt \\
&\quad + (\kappa_i Z_{t,i} + \theta_i) \, dt + q_i \sqrt{Z_{t,i}} \, dW_{t,i} + q_i^2 Z_{t,i} \, dt \\
&= \left(\theta_i + (-\theta_V + \kappa_i + q_i^2) Z_{t,i} + Z_{t,i} \sum_{j=1}^{d} (-\kappa_j + q_j^2) Z_{t,j} \right) dt \\
&\quad + q_i (1 - Z_{t,i}) \sqrt{Z_{t,i}} \, dW_{t,i} - Z_{t,i} \sum_{j \neq i} q_j \sqrt{Z_{t,j}} \, dW_{t,j}. \tag{3.19}
\end{aligned}
$$

This shows that the drift function of (Z_t) is of the form (3.10).

The induced short rate process (r_t) is linear in Z_t as in (3.13) and thus uniformly bounded. The volatility function of (Z_t) is also uniformly bounded, and hence the induced discount derivative volatility is given by (3.2). Hence assumption (2.9) holds, which completes the proof. □

The existence result in Lemma 3.3 is only partially an answer to the question about the global existence of a solution to the SPDE (2.12) in Remark 2.8. In fact, the affine discount term structure model (3.3) generates only discount derivative curves of the form $\psi_t(x) = \phi_0(x) + \sum_{i=1}^{d} \phi_i(x) Z_{t,i}$, which lie in a d-dimensional affine subspace, say $\mathcal{A}$, of $\mathcal{H}$. In view of (3.5), there exist points $0 \leq x_1 < x_2 < \cdots < x_d$ such that the factor Z_t is an affine function of the d discount derivative values $\psi_t(x_1), \ldots, \psi_t(x_d)$. Combining this with (3.2) and (3.19), we easily see that the volatility operator $B(\psi_t)$ in (2.12) is a (non-Lipschitz-continuous) function of the values $\psi_t(x_1), \ldots, \psi_t(x_d)$ when restricted to $\psi_t \in \mathcal{A}$. However, it is not clear how to extend B beyond $\mathcal{A}$ in

such a way that a solution to (2.12) still exists outside $\mathcal{A}$. Moreover, uniqueness remains a problem anyway because of the lack of Lipschitz continuity at $Z_{t,i} = 0$ of the volatility of Z_t given in (3.19), as the following remark shows.

Remark 3.4 While pathwise uniqueness for (3.18) is proved in Filipović and Larsson [5, Theorem 4.3] in dimension $d = 1$, it remains an open problem in higher dimensions $d > 1$. We conjecture that pathwise uniqueness for (3.18) can be proved along similar arguments as used in Yamada and Watanabe [9]. However, as discussed in [5, Sect. 4], the problem is far from trivial. For instance, the results in [9] do not directly apply here as they assume that the ith element of the diagonal diffusion matrix only depends on the ith coordinate of the process.

4 Conclusion

An arbitrage-free dynamic discount model defines an arbitrage-free price system for bonds. Modelling discount derivatives thus provides a valuable alternative to modelling forward rates. Of particular interest are affine discount term structure models, for which we provide some concrete specifications.

The paper identifies various open problems outlined in Remarks 2.7, 2.8 and 3.4, which point to directions for further research. Other research directions include the implementation of discount models for pricing and hedging interest rate derivatives.

Declarations

Competing Interests The author declares no competing interests.

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