

NORM TORI OF ÉTALE ALGEBRAS AND UNRAMIFIED BRAUER GROUPS

BY

EVA BAYER-FLUCKIGER

*Institute of Mathematics, School of Basic Sciences, EPFL
Station 8, CH-1015 Lausanne, Switzerland
e-mail: eva.bayer@epfl.ch*

AND

TING-YU LEE

*Department of Mathematics, National Taiwan University
Taipei 10617, Taiwan (R.O.C.)
e-mail: tingyulee@ntu.edu.tw*

Dedicated to Moshe Jarden for his 80th birthday

ABSTRACT

Let k be a field, and let L be an étale k -algebra of finite rank. If $a \in k^\times$, let X_a be the affine variety defined by $N_{L/k}(x) = a$. Assuming that L has at least one factor that is a cyclic field extension of k , we give a combinatorial description of the unramified Brauer group of X_a .

Received October 12, 2021 and in revised form February 19, 2022

Introduction

Let k be a field, let L be an étale k -algebra of finite rank, and let $N_{L/k} : L \rightarrow k$ be the norm map. Let $a \in k^\times$, and let X_a be the affine k -variety determined by

$$N_{L/k}(t) = a.$$

Let X_a^c be a smooth compactification of X_a (see [CTHsk05]). The aim of this paper is to describe the group $\text{Br}(X_a^c)/\text{Im}(\text{Br}(k))$ under the hypothesis that L has at least one cyclic factor. We first give a combinatorial description of a group associated to the étale algebra L (see §3), and then give an explicit isomorphism between this group and $\text{Br}(X_a^c)/\text{Im}(\text{Br}(k))$ (see §4, in particular Theorem 4.3).

Let $\text{Br}_{\text{ur}}(k(X_a))$ be the subgroup of $\text{Br}(k(X_a))$ consisting of all elements which are unramified at all discrete valuations of $k(X_a)$ with residue fields containing k and with fields of fraction $k(X_a)$; recall that $\text{Br}_{\text{ur}}(k(X_a))$ is isomorphic to $\text{Br}(X_a^c)$ (see Česnavičius [C19], Theorem 1.2); this group is often called the **unramified Brauer group** of X_a (or of $k(X_a)$).

Let us illustrate our results by a special case. Let p be a prime number and let $n \geq 1$ be an integer; assume that $\text{char}(k) \neq p$ and let F be a Galois extension of k with Galois group $\mathbf{Z}/p^n\mathbf{Z} \times \mathbf{Z}/p^n\mathbf{Z}$. Suppose that L is a product of r linearly disjoint cyclic subfields of F of degree p^n . Then we have (see Theorem 4.9):

THEOREM:

$$\text{Br}(X_a^c)/\text{Im}(\text{Br}(k)) \simeq (\mathbf{Z}/p^n\mathbf{Z})^{r-2}.$$

We also give explicit generators of this group, as follows. With the above notation, let K be one of the cyclic subfields of degree p^n of F , and let χ be an injective morphism from $\text{Gal}(K/k)$ to \mathbf{Q}/\mathbf{Z} . Let us write $L = K \times K'$, with $K' = \prod_{i \in I} K_i$, where K_i is a cyclic subfield of F of degree p^n of F for all $i \in I$, and assume that K and the fields K_i are linearly disjoint in F . For all $i \in I$, set $N_i = N_{K_i/k}(y_i)$, considered as elements of $k(X_a)^\times$. Assume that the cardinality of I is $r - 1$, so that L is a product of the r linearly disjoint cyclic subfields K and K_i of F of degree p^n . Let I' be a subset of cardinality $r - 2$ of I .

Let (N_i, χ) denote the class of the cyclic algebra over $k(X_a)$ associated to χ and the element $N_i \in k(X_a)^\times$.

THEOREM: *The group $\mathrm{Br}(X_a^c)/\mathrm{Im}(\mathrm{Br}(k))$ is generated by the elements (N_i, χ) for $i \in I'$.*

This is also proved in Theorem 4.9. Note that the above results are generalizations of [BP20], Theorems 12.1 and 12.2. Similar questions are considered in [CT14], [DW14], [H84], [Lee22], [Ma19], [P14] and [PR13].

The methods combine algebraic and arithmetic arguments. The overall strategy is to give combinatorial constructions of groups, and show that in the case of global ground fields k , these are isomorphic to $\mathrm{Br}(T^c)/\mathrm{Br}(k)$ (see Theorem 3.10 and Corollary 3.11). The arithmetic results rely on [BLP19]; the new feature is the construction of the group in algebraic and combinatorial terms. The second step is to transfer these results to more general fields, using a strategy of [BP20]; note that other methods for proving algebraic results of similar nature using arithmetic ones are available in earlier papers; see, for instance, [CK98, §3] and [BDH13, §8].

The paper is organized as follows. Throughout the paper, K is a finite cyclic extension of k , and $L = K \times K'$, where K' is an étale k -algebra of finite rank. Sections 1 and 2 are preliminary: in particular, it is shown in §2 that we may assume K to be cyclic of prime power degree. Sections 3 and 4 contain the description of the unramified Brauer group. When k is a global field, we obtain additional results concerning the “locally trivial” Brauer group (cf. §5). Finally, in §6 we apply Theorem 4.3 to give an alternative proof of [BLP19] Theorem 7.1 for k a global field with $\mathrm{char}(k) \neq p$; we show that the Brauer–Manin map of [BLP19] is the Brauer–Manin pairing, and hence deduce the Hasse principle from results of [Sa81] and [DH22].

1. Definitions and notation

GENERALITIES. Let k be a field, let k_s be a separable closure of k and let $\mathcal{G}_k = \mathrm{Gal}(k_s/k)$ be the absolute Galois group of k . We fix once and for all this separable closure k_s , and all separable extensions of k that will appear in the paper will be contained in k_s . We use standard notation in Galois cohomology; in particular, if M is a discrete \mathcal{G}_k -module and i is an integer ≥ 0 , we set $H^i(k, M) = H^i(\mathcal{G}_k, M)$. A \mathcal{G}_k -**lattice** will be a torsion free

\mathbf{Z} -module of finite rank on which \mathcal{G}_k acts continuously. For a k -torus T , we denote by $\hat{T} = \text{Hom}(T, \mathbb{G}_m)$ its character group; it is a \mathcal{G}_k -lattice.

Let G be a finite group. A G -**lattice** is by definition a $\mathbf{Z}[G]$ -module of finite rank. If $g \in G$, we denote by $\langle g \rangle$ the cyclic subgroup of G generated by g . Let M be a G -lattice. Set

$$\text{III}_{\text{cycl}}^2(G, M) = \text{Ker} \left[H^2(G, M) \rightarrow \prod_{g \in G} H^2(\langle g \rangle, M) \right].$$

We recall a result of Colliot-Thélène and Sansuc (cf. [CTS87, Prop. 9.5]).

THEOREM 1.1: *Let G be a finite group, let T be a k -torus, and assume that the character group of T is a G -lattice via a surjection $\mathcal{G}_k \rightarrow G$. Let T^c be a smooth compactification of T . We have*

$$\text{Br}(T^c)/\text{Br}(k) \simeq \text{III}_{\text{cycl}}^2(G, \hat{T}).$$

Proof. See [BP20, Theorem 2.3]. ■

NORM EQUATIONS. Let L be an étale k -algebra of finite rank; in other words, a product of a finite number of separable extensions of k . Let $T_{L/k} = R_{L/k}^{(1)}(\mathbf{G}_m)$ be the k -torus defined by

$$1 \rightarrow T_{L/k} \rightarrow R_{L/k}(\mathbf{G}_m) \xrightarrow{N_{L/k}} \mathbf{G}_m \rightarrow 1.$$

Let $a \in k^\times$. Let X_a be the affine k -variety associated to the **norm equation**

$$N_{L/k}(t) = a.$$

The variety X_a is a torsor under $T_{L/k}$; let X_a^c be a smooth compactification of X_a . We have a natural map $\text{Br}(k) \rightarrow \text{Br}(X_a^c)$; if $a = 1$ then $X_1 = T_{L/k}$, and the map $\text{Br}(k) \rightarrow \text{Br}(T_{L/k}^c)$ is injective, and moreover we have an injection

$$\text{Br}(X^c)/\text{Im}(\text{Br}(k)) \rightarrow \text{Br}(T^c)/\text{Br}(k)$$

(see, for instance, [BP20, §6]). Recall a result from [BP20, Theorem 7.1]:

THEOREM 1.2: *Assume that $L = K \times K'$, where K/k is a cyclic extension and K' an étale k -algebra. Then the map $\text{Br}(X^c)/\text{Im}(\text{Br}(k)) \rightarrow \text{Br}(T^c)/\text{Br}(k)$ is an isomorphism.*

GLOBAL FIELDS. If k is a global field, we denote by V_k the set of all places of k ; if $v \in V_k$, we denote by k_v the completion of k at v .

For any k -torus T , set

$$\text{III}^i(k, T) = \text{Ker} \left(H^i(k, T) \rightarrow \prod_{v \in V_k} H^i(k_v, T) \right).$$

If M is a \mathcal{G}_k -module, set

$$\text{III}^i(k, M) = \text{Ker} \left(H^i(k, M) \rightarrow \prod_{v \in V_k} H^i(k_v, M) \right),$$

and let $\text{III}_\omega^i(k, M)$ be the set of $x \in H^i(k, M)$ that map to 0 in $H^i(k_v, M)$ for almost all $v \in V_k$.

2. Norm equations and étale algebras

In the sequel, we consider norm equations of étale algebras having at least one cyclic factor. The aim of this section is to introduce some notation and prove some results that will be used throughout the paper.

Let K be a cyclic extension of k , and let K' be an étale k -algebra of finite rank; set $L = K \times K'$. We first show that it is enough to consider the case when K/k is cyclic of prime power degree.

REDUCTION TO THE PRIME POWER DEGREE CASE. Let \mathcal{P} be the set of prime numbers dividing $[K : k]$. For all $p \in \mathcal{P}$, let $K[p]$ be the largest subfield of K such that $[K[p] : k]$ is a power of p , and set $L[p] = K[p] \times K'$. Recall from [BLP19] the following result.

PROPOSITION 2.1: *Assume that k is a global field. We have*

$$\text{III}^2(k, \hat{T}_{L/k}) \simeq \bigoplus_{p \in \mathcal{P}} \text{III}^2(k, \hat{T}_{L[p]/k}).$$

Proof. This follows from [BLP19], Lemma 3.1 and Proposition 5.16. ■

Let k'/k be a Galois extension of minimal degree splitting $T_{L/k}$, and let $G = \text{Gal}(k'/k)$.

PROPOSITION 2.2: *We have $\text{III}_{\text{cycl}}^2(G, \hat{T}_{L/k}) \simeq \bigoplus_{p \in \mathcal{P}} \text{III}_{\text{cycl}}^2(G, \hat{T}_{L[p]/k})$.*

Proof. Let us write $K' = \prod_{i \in I} K_i$, where the K_i are finite field extensions of k . Let H be the subgroup of G such that $K = (k')^H$, and for all $i \in I$, let H_i be the subgroup of G such that $K_i = (k')^{H_i}$. Set $M = \hat{T}_{L/k}$. We have the exact sequence of G -modules

$$0 \rightarrow \mathbf{Z} \rightarrow \mathbf{Z}[G/H] \oplus \bigoplus_{i \in I} \mathbf{Z}[G/H_i] \rightarrow M \rightarrow 0.$$

For all $p \in \mathcal{P}$, let $H[p]$ be the subgroup of G such that $K[p] = (k')^{H[p]}$. Set $M[p] = \hat{T}_{L[p]/k}$. We have the exact sequence of G -modules

$$0 \rightarrow \mathbf{Z} \rightarrow \mathbf{Z}[G/H[p]] \oplus \bigoplus_{i \in I} \mathbf{Z}[G/H_i] \rightarrow M[p] \rightarrow 0.$$

Let ℓ'/ℓ be an unramified extension of number fields with Galois group G (cf. [F62]). Set $L_0 = (\ell')^H$, $L_0[p] = (\ell')^{H[p]}$, and $L_i = (\ell')^{H_i}$. Let

$$E = L_0 \times \prod_{i \in I} L_i \quad \text{and} \quad E[p] = L_0[p] \times \prod_{i \in I} L_i.$$

We have

$$\hat{T}_{E/\ell} \simeq M \quad \text{and} \quad \hat{T}_{E[p]/\ell} \simeq M[p].$$

By Proposition 2.1 we have $\text{III}^2(\ell, M) \simeq \bigoplus_{p \in \mathcal{P}} \text{III}^2(\ell, M[p])$. Since ℓ'/ℓ is unramified, we have $\text{III}_{\text{cycl}}^2(G, M) \simeq \text{III}^2(\ell, M)$ and $\text{III}_{\text{cycl}}^2(G, M[p]) \simeq \text{III}^2(\ell, M[p])$ (see [BP20, Proposition 3.1]), hence $\text{III}_{\text{cycl}}^2(G, M) \simeq \bigoplus_{p \in \mathcal{P}} \text{III}_{\text{cycl}}^2(G, M[p])$. ■

PROPOSITION 2.3: *Assume that k is a global field. We have*

$$\text{III}_{\omega}^2(k, \hat{T}_{L/k}) \simeq \bigoplus_{p \in \mathcal{P}} \text{III}_{\omega}^2(k, \hat{T}_{L[p]/k}).$$

Proof. This follows from Proposition 2.2 and [BP20, Corollary 3.4]. ■

THE PRIME POWER DEGREE CASE. Let p be a prime number, and assume that K/k is cyclic of degree a power of p . Let us write $K' = \prod_{i \in I} K_i$, where the K_i are finite field extensions of k , and let $[K : k] = p^n$.

Notation 2.4: For all integers $1 \leq m \leq n$, let $K(m)$ be the unique subfield of K of degree p^m over k . The K_i -algebra $K(m) \otimes_k K_i$ is a product of cyclic extensions of K_i ; let $p^{e_i(m)}$ be the degree of these extensions, and set $\mathcal{E}(m) = \{e_i(m) \mid i \in I\}$. For all $i \in I$, let us choose one of the cyclic factors E_i/K_i of $K \otimes_k K_i$. For all m let $E_i(m)$ be the subfield of E_i which corresponds to a cyclic factor of $K(m) \otimes_k K_i$.

Let \mathcal{K} be a Galois extension of k containing K and all the fields K_i , and let $G = \text{Gal}(\mathcal{K}/k)$. If F is a subfield of \mathcal{K} , we denote by G_F the subgroup of G such that $F = \mathcal{K}^{G_F}$.

For all integers $0 \leq m \leq n$, let Γ_i^m be the set of conjugacy classes of elements $g \in G$ such that $\langle g \rangle \cap (G_{E_i(n-m)} \setminus G_{E_i(n-m+1)}) \neq \emptyset$.

Notation 2.5: Assume moreover that k is a global field. Let V_i^m be the set of places v of k such that there exists a place w of K_i above v having the property that $K \otimes_k (K_i)_w$ is a product of field extensions of degree at least p^m of $(K_i)_w$.

PROPOSITION 2.6: *Assume that k is a global field. Let V_{r_m} be the set of places of k which are ramified in \mathcal{K} . For all integers $0 \leq m \leq n$, sending a place $v \in V_i^m \setminus V_{r_m}$ to the conjugacy class of its Frobenius element $f_v \in G$ gives rise to a surjection from $V_i^m \setminus V_{r_m}$ onto Γ_i^m .*

In order to prove the Proposition, we need the following lemma.

LEMMA 2.7: *Let F be a field, and let E be a cyclic extension of F of prime degree. Let M be an extension of E , and assume that M is a Galois extension of F . Set $G_F = \text{Gal}(M/F)$ and $G_E = \text{Gal}(M/E)$. Let $v : M^\times \rightarrow \mathbf{Z}$ be a discrete valuation of M ; assume that the restriction v_F of v to F^\times is surjective, and that the residue field of v is perfect.*

Let $D_{M/F}$ be the decomposition group of v . Then v_F is inert in E if and only if

$$D_{M/F} \cap (G_F \setminus G_E) \neq \emptyset.$$

Proof. Let $G_{E/F}$ be the Galois group of the extension E/F , and let $D_{E/F}$ be the decomposition group of v_F ; note that v_F is inert in E if and only if $D_{E/F} = G_{E/F}$. Since E/F is cyclic of prime degree, this amounts to saying that $D_{E/F}$ is not trivial.

We have the exact sequence

$$1 \rightarrow G_E \rightarrow G_F \rightarrow G_{E/F} \rightarrow 1.$$

The image of $D_{M/F}$ by the homomorphism $G_F \rightarrow G_{E/F}$ is equal to $D_{E/F}$ (see for instance [Se79], Chap. I, Proposition 22). Hence $D_{E/F}$ is non trivial if and only if $D_{M/F} \cap (G_F \setminus G_E) \neq \emptyset$. ■

PROPOSITION 2.8: *Let F , E and M be as in Lemma 2.7. Assume moreover that k is a subfield of F , and that M is a Galois extension of k . Let*

$G = \text{Gal}(M/k)$. Let $v_k : k^\times \rightarrow \mathbf{Z}$ be a discrete valuation such that the extensions of v_k to M are unramified; let \mathcal{D} be the set of corresponding decomposition groups. The following are equivalent:

- (a) There exists an extension of v_k to F that is inert in E .
- (b) There exists $D \in \mathcal{D}$ such that $D \cap (G_F \setminus G_E) \neq \emptyset$.

Proof. Let us prove that (a) implies (b). Let v_F be an extension of v_k to F that is inert in E , let v be an extension of v_F to M , and let D be the decomposition group of v . By Lemma 2.7, we have (b). Conversely, assume that (b) holds. Let $D \in \mathcal{D}$ be as in (b), and let v be the corresponding valuation. Let v_F be the restriction of v to F . By Lemma 2.7, we see that v_F is inert in E , hence (a) holds. ■

Proof of Proposition 2.6. If $v \in V_i^m \setminus V_{rm}$, then by definition there exists a place of $E_i(n - m)$ that is inert in the extension $E_i(n)/E_i(n - m)$, and hence also in $E_i(n - m + 1)/E_i(n - m)$; therefore by Proposition 2.8 the conjugacy class of its Frobenius element f_v belongs to Γ_i^m .

Conversely, if the conjugacy class of $g \in G$ belongs to Γ_i^m , then by Chebotarev’s density theorem there exists an unramified place v such that its Frobenius element f_v is the conjugacy class of g . Since the conjugacy class of g belongs to Γ_i^m , there is a place w of $E_i(n - m)$ above v such that w is inert in $E_i(n - m + 1)/E_i(n - m)$. Therefore w is also inert in $E_i(n)/E_i(n - m)$ as $E_i(n)/E_i(n - m)$ is cyclic of p -power degree and is unramified at w . This implies that $v \in V_i^m \setminus V_{rm}$. ■

We get immediately the following corollary.

COROLLARY 2.9: *For $i, j \in I$, the map defined in Proposition 2.6 induces a surjection from $V_i^m \cap V_j^m \setminus V_{rm}$ to $\Gamma_i^m \cap \Gamma_j^m$.*

Remark 2.10: Keep the notation in Proposition 2.6. For each conjugacy class in Γ_i^m , by Chebotarev’s density theorem there are infinitely many unramified places $v \in V_i^m$ mapped to it.

3. Norm equations—unramified Brauer group

We keep the notation of the previous section. In particular, k is a field, K is a cyclic extension of k , and $L = K \times K'$ where K' is an étale k -algebra of finite

rank. Let $T_{L/k} = R_{L/k}^{(1)}(\mathbf{G}_m)$ be the k -torus defined by

$$1 \rightarrow T_{L/k} \rightarrow R_{L/k}(\mathbf{G}_m) \xrightarrow{N_{L/k}} \mathbf{G}_m \rightarrow 1.$$

Let $a \in k^\times$, and let X_a be the affine k -variety associated to the norm equation $N_{L/k}(t) = a$. The variety X_a is a torsor under $T_{L/k}$. Let $T_{L/k}^c$ be a smooth compactification of $T_{L/k}$, and let X_a^c be a smooth compactification of X_a .

The aim of this section is to describe the group $\text{Br}(X_a^c)/\text{Im}(\text{Br}(k))$. Using the results of §2, we can assume that K/k is of degree p^n , where p is a prime number.

We use the notation of §2 (see Notation 2.4). In addition, we need the following.

Notation 3.1: For all integers $n \geq 1$, we denote by $C(I, \mathbf{Z}/p^n\mathbf{Z})$ the set of maps $I \rightarrow \mathbf{Z}/p^n\mathbf{Z}$.

If $1 \leq m \leq n$, let $\pi_{n,m}$ be the projection $C(I, \mathbf{Z}/p^n\mathbf{Z}) \rightarrow C(I, \mathbf{Z}/p^m\mathbf{Z})$.

For $x \in \mathbf{Z}/p^m\mathbf{Z}$ and $y \in \mathbf{Z}/p^r\mathbf{Z}$, we denote by $\delta(x, y)$ the maximum integer $d \leq \min\{m, r\}$ such that $x = y \pmod{p^d\mathbf{Z}}$.

We start with some special cases, in which the results are especially simple.

K/k CYCLIC OF DEGREE p . Assume first that $[K : k] = p$, and that K is not contained in any of the fields K_i . Then for all $i \in I$, E_i is a cyclic field extension of degree p of K_i . Let $\Gamma_i = \Gamma_i^1$ be the set of conjugacy classes of elements $g \in G$ such that $\langle g \rangle \cap (G_{K_i} \setminus G_{E_i}) \neq \emptyset$ (cf. Notation 2.4).

Let $C(L)$ be the group

$$\{c \in C(I, \mathbf{Z}/p\mathbf{Z}) \mid c(i) = c(j) \text{ if } \Gamma_i \cap \Gamma_j \neq \emptyset\},$$

and D be the subgroup of constant maps $I \rightarrow \mathbf{Z}/p\mathbf{Z}$.

As a consequence of Theorem 3.8, we'll show the following.

PROPOSITION 3.2: *Assume that K/k is cyclic of degree p , and that K is not contained in any of the fields K_i . Then we have*

$$\text{III}_{\text{cycl}}^2(G, \hat{T}_{L/k}) \simeq C(L)/D.$$

By Theorem 1.1, this implies the following.

COROLLARY 3.3: *Assume that K/k is cyclic of degree p , and that K is not contained in any of the fields K_i . Then we have*

$$\text{Br}(T^c)/\text{Br}(k) \simeq C(L)/D.$$

K/k OF DEGREE p^n AND K LINEARLY DISJOINT OF ALL THE K_i . For all integers m with $1 \leq m \leq n$ set

$$C^m = \{c \in C(I, \mathbf{Z}/p^m\mathbf{Z}) \mid c(i) = c(j) \text{ if } \Gamma_i^m \cap \Gamma_j^m \neq \emptyset\}.$$

Set

$$C(L) = \{c \in C^n \mid \pi_{n,m}(c) \in C^m \text{ for all } m \leq n\},$$

and denote by D the subgroup of constant maps $I \rightarrow \mathbf{Z}/p^n\mathbf{Z}$.

PROPOSITION 3.4: Assume that K/k is cyclic of degree p^n , and that K is linearly disjoint of all the fields K_i . Then we have

$$\text{III}_{\text{cycl}}^2(G, \hat{T}_{L/k}) \simeq C(L)/D.$$

As in the case where K/k is of degree p , this follows from Theorem 3.8, and has the immediate corollary

COROLLARY 3.5: Assume that K/k is cyclic of degree p^n , and that K is linearly disjoint of all the fields K_i . Then we have

$$\text{Br}(T^c)/\text{Br}(k) \simeq C(L)/D.$$

THE GENERAL CASE. Recall that K/k is cyclic of degree p^n , and that we use Notation 2.4. Recall that $\mathcal{E} = \mathcal{E}(n)$.

Notation 3.6: For all $e \in \mathcal{E}$, set $I_e = \{i \in I \mid e_i(n) = e\}$. Denote by \hat{e} the maximum element in \mathcal{E} . Note that the index i belongs to I_e if and only if $K \cap K_i$ is an extension of degree p^{n-e} of k . As K is a cyclic extension, this means that given $0 \leq m \leq n$, the $e_i(m)$ are the same for all $i \in I_e$ and we denote it by $e(m)$.

For all integers m with $1 \leq m \leq n$ set

$$C^m = \left\{ c \in \bigoplus_{e \in \mathcal{E}} C(I_e, \mathbf{Z}/p^{e-e(n-m)}\mathbf{Z}) \mid c(i) = c(j) \text{ if } \Gamma_i^m \cap \Gamma_j^m \neq \emptyset \right\}.$$

We still denote by $\pi_{n,m}$ the map from $\bigoplus_{e \in \mathcal{E}} C(I_e, \mathbf{Z}/p^e\mathbf{Z})$ to

$$\bigoplus_{e \in \mathcal{E}} C(I_e, \mathbf{Z}/p^{e-e(n-m)}\mathbf{Z})$$

induced by the natural projection.

Set

$$C(L) = \{c \in C^n \mid \pi_{n,m}(c) \in C^m \text{ for all } m \leq n\},$$

and denote by D the image of constant maps $I \rightarrow \mathbf{Z}/p^n\mathbf{Z}$ in C^n under the natural projection .

Remark 3.7: If $E_i(n - m + 1) \supsetneq E_i(n - m)$, then $K(n - m) \supseteq K \cap K_i$. In this case $e_i(n) \geq m$ and $e_i(n) - e_i(n - m) = m$.

The main results of this section are:

THEOREM 3.8: *Assume that K/k is cyclic of degree p^n . Then we have*

$$\text{III}_{\text{cycl}}^2(G, \hat{T}_{L/k}) \simeq C(L)/D.$$

By Theorem 1.1, this implies the following:

COROLLARY 3.9: *Assume that K/k is cyclic of degree p^n . Then we have*

$$\text{Br}(T^c)/\text{Br}(k) \simeq C(L)/D.$$

The proof of Theorem 3.8 will be given below, using some arithmetic results of [BLP19]. We start by recalling and developing some results concerning global fields.

GLOBAL FIELDS. Assume that k is a global field. Recall that K/k is cyclic of degree p^n , and that we use Notations 2.4 as well as 3.6. In addition, for global fields, we also use Notation 2.5.

For all integers m with $1 \leq m \leq n$ set

$$C_{\text{arith}}^m = \left\{ c \in \bigoplus_{e \in \mathcal{E}} C(I_e, \mathbf{Z}/p^{e-e(n-m)}\mathbf{Z}) \mid c(i) = c(j) \text{ if } V_i^m \cap V_j^m \neq \emptyset \right\}$$

and

$$C_{\omega}^m = \left\{ c \in \bigoplus_{e \in \mathcal{E}} C(I_e, \mathbf{Z}/p^{e-e(n-m)}\mathbf{Z}) \mid c(i) = c(j) \text{ if } V_i^m \cap V_j^m \text{ is infinite} \right\}.$$

Set

$$C_{\text{arith}}(L) = \{ c \in C_{\text{arith}}^n \mid \pi_{n,m}(c) \in C_{\text{arith}}^m \text{ for all } m \leq n \}$$

and

$$C_{\omega}(L) = \{ c \in C_{\omega}^n \mid \pi_{n,m}(c) \in C_{\omega}^m \text{ for all } m \leq n \}.$$

THEOREM 3.10: *Assume that K/k is cyclic of degree p^n . Then we have*

- (1) $\text{III}^2(k, \hat{T}_{L/k}) \simeq C_{\text{arith}}(L)/D,$
- (2) $\text{III}_{\omega}^2(k, \hat{T}_{L/k}) \simeq C_{\omega}(L)/D.$

Proof. Recall some notation of [BLP19].

For $a = (a_i) \in \bigoplus_{e \in \mathcal{E}} \bigoplus_{i \in I_e} (\mathbf{Z}/p^e \mathbf{Z})$ and $r \in \mathbf{Z}/p^{\hat{e}} \mathbf{Z}$, set

$$I_r(a) = \{i \in I \mid a_i = r \pmod{p^{e_i(n)} \mathbf{Z}}\}.$$

Set

$$G(K, K') = \left\{ a = (a_i) \in \bigoplus_{e \in \mathcal{E}} \bigoplus_{i \in I_e} (\mathbf{Z}/p^e \mathbf{Z}) \mid \bigcap_{r \in \mathbf{Z}/p^{\hat{e}} \mathbf{Z}} \bigcup_{i \notin I_r(a)} V_i^{\delta(r, a_i)+1} = \emptyset \right\};$$

$$G_\omega(K, K') = \left\{ a = (a_i) \in \bigoplus_{e \in \mathcal{E}} \bigoplus_{i \in I_e} (\mathbf{Z}/p^e \mathbf{Z}) \mid \bigcap_{r \in \mathbf{Z}/p^{\hat{e}} \mathbf{Z}} \bigcup_{i \notin I_r(a)} V_i^{\delta(r, a_i)+1} \text{ is finite} \right\}.$$

With the notation of [BLP19], we have

$$\text{III}^2(k, \hat{T}_{L/k}) \simeq \text{III}(K, K') = G(K, K')/D',$$

where D' is the subgroup generated by $(1, 1, \dots, 1)$ (see [BLP19, Theorem 5.3 and Lemma 3.1]). Similarly, it is shown in [Lee22, Theorem 2.5] that

$$\text{III}_\omega^2(k, \hat{T}_{L/k}) \simeq G_\omega(K, K')/D'.$$

Hence it suffices to show that $G(K, K') \simeq C_{\text{arith}}(L)$ and that $G_\omega(K, K') \simeq C_\omega(L)$.

We show that $G(K, K') \simeq C_{\text{arith}}(L)$; the proof of $G_\omega(K, K') \simeq C_\omega(L)$ is the same.

Let

$$f : \bigoplus_{e \in \mathcal{E}} \bigoplus_{i \in I_e} (\mathbf{Z}/p^e \mathbf{Z}) \rightarrow \bigoplus_{e \in \mathcal{E}} C(I_e, \mathbf{Z}/p^e \mathbf{Z})$$

be the map sending $(a_i) \in \bigoplus_{i \in I_e} (\mathbf{Z}/p^e \mathbf{Z})$ to $c : I_e \rightarrow \mathbf{Z}/p^e \mathbf{Z}$ such that $c(i) = a_i$.

We claim that the isomorphism f gives rise to an isomorphism

$$G(K, K') \rightarrow C_{\text{arith}}(L).$$

For $c \in \bigoplus_{e \in \mathcal{E}} C(I_e, \mathbf{Z}/p^e \mathbf{Z})$, we denote $\pi_{n,m}(c)$ by c_m .

Let $a = (a_i) \in G(K, K')$ and $c = f(a)$. We show that $c_m \in C^m$ for $1 \leq m \leq n$.

Suppose that $V_i^m \cap V_j^m \neq \emptyset$. By Remark 3.7, we have

$$e_i - e_i(n - m) = e_j - e_j(n - m) = m.$$

Let $v \in V_i^m \cap V_j^m$. As $a \in G(K, K')$, there is $r \in \mathbf{Z}/p^{\hat{e}} \mathbf{Z}$ such that $v \notin \bigcup_{l \notin I_r(a)} V_l^{\delta(r, a_l)+1}$. If $i \notin I_r(a)$, then $\delta(r, a_i) + 1 > m$ since $v \notin V_i^{\delta(r, a_i)+1}$. Hence $c(i) = r \pmod{p^m \mathbf{Z}}$. If $i \in I_r(a)$, then $c(i) = r \pmod{p^{e_i(n)} \mathbf{Z}}$. In both cases we have $c_m(i) = r \pmod{p^m \mathbf{Z}}$. The same argument works for j . Therefore $c_m(i) = c_m(j)$ and $c_m \in C^m$.

Let $c \in C^m$ be such that $c_m \in C^m$ for $1 \leq m \leq n$. Let $a = f^{-1}(c)$. If $c \in D'$, then clearly $a \in D'$. Suppose that $c \notin D'$. We claim that

$$\bigcap_{r \in \mathbf{Z}/p^{\hat{e}}\mathbf{Z}} \bigcup_{i \notin I_r(a)} V_i^{\delta(r, a_i)+1} = \emptyset.$$

Suppose not. Let $v \in \bigcap_{r \in \mathbf{Z}/p^{\hat{e}}\mathbf{Z}} \bigcup_{i \notin I_r(a)} V_i^{\delta(r, a_i)+1}$. Choose $r_0 \in \mathbf{Z}/p^{\hat{e}}\mathbf{Z}$. Since $c \notin D'$, there exist $r_1 \in \mathbf{Z}/p^{\hat{e}}\mathbf{Z}$ and $i \in I_{r_1}(a) \setminus I_{r_0}(a)$ such that $v \in V_i^{\delta(r_0, a_i)+1}$. For the same reason, there is $r_2 \in \mathbf{Z}/p^{\hat{e}}\mathbf{Z}$ and $j \in I_{r_2}(a) \setminus I_{r_1}(a)$ such that $v \in V_j^{\delta(r_1, a_j)+1}$.

By the choice of i and j , we have $\delta(r_0, a_i) = \delta(r_0, r_1)$ and $\delta(r_1, a_j) = \delta(r_1, r_2)$. Suppose that $\delta(r_0, r_1) \geq \delta(r_1, r_2)$. Then $v \in V_i^m \cap V_j^m$, where $m = \delta(r_1, r_2) + 1$. Hence $c_m(i) = c_m(j)$ and $\delta(a_i, a_j) \geq m = \delta(r_1, r_2) + 1$, which contradicts that $\delta(r_1, r_2) \geq \delta(a_i, a_j)$. Therefore $\delta(r_0, r_1) < \delta(r_1, r_2)$.

We can continue the above process to get an infinite sequence of $r_l \in \mathbf{Z}/p^{\hat{e}}\mathbf{Z}$ such that $\delta(r_l, r_{l+1}) < \delta(r_{l+1}, r_{l+2})$. It is a contradiction as $\delta(r_l, r_{l+1})$ ranges from 0 to \hat{e} . Hence $\bigcap_{r \in \mathbf{Z}/p^{\hat{e}}\mathbf{Z}} \bigcup_{i \notin I_r(a)} V_i^{\delta(r, a_i)+1} = \emptyset$ and $a \in G(K, K')$. As a consequence f induces an isomorphism $G(K, K') \rightarrow C_{\text{arith}}(L)$. ■

COROLLARY 3.11: *Let k be a global field. Then $\text{III}_{\omega}^2(k, \hat{T}_{L/k}) \simeq C(L)/D$.*

Proof. By Corollary 2.9 and Remark 2.10, the two sets C^m and C_{ω}^m are the same. Our claim then follows from Theorem 3.10. ■

Proof of Theorem 3.8. Recall that \mathcal{K} is a Galois extension of k containing K and all the fields K_i , and that $G = \text{Gal}(\mathcal{K}/k)$; if F is a subfield of \mathcal{K} , we denote by G_F the subgroup of G such that $F = \mathcal{K}^{G_F}$.

Note that k is not necessarily a global field here. However, there is always an unramified extension ℓ'/ℓ with Galois group $\text{Gal}(\ell'/\ell) \simeq G$ ([F62]). Hence we can regard $\hat{T}_{L/K}$ as a $\text{Gal}(\ell'/\ell)$ -module.

To be precise, set $F = (\ell')^{G_K}$, $L_i = (\ell')^{G_{K_i}}$ and $E = F \times \prod_{i \in I} L_i$. By construction, the G -lattices $\hat{T}_{E/\ell}$ and $\hat{T}_{L/k}$ are isomorphic.

Since the extension ℓ'/ℓ is unramified, we have

$$\text{III}^2(\ell, \hat{T}_{E/\ell}) \simeq \text{III}_{\omega}^2(\ell, \hat{T}_{E/\ell}) \simeq \text{III}_{\text{cycl}}^2(G, \hat{T}_{E/\ell}).$$

By Corollary 3.11 the group $\text{III}_{\text{cycl}}^2(G, \hat{T}_{E/\ell})$ is isomorphic to $C(E)/D$. However, $C(L)$ only depends on the group G . Hence $C(L) \simeq C(E)$. Therefore $\text{III}_{\text{cycl}}^2(G, \hat{T}_{L/k}) \simeq \text{III}_{\text{cycl}}^2(G, \hat{T}_{E/\ell}) \simeq C(L)/D$. ■

4. Unramified Brauer groups and generators

We keep the notation of the previous sections. Recall that p is a prime number, K/k a cyclic field extension of degree p^n , and $L = K \times K'$, where K' is an étale k -algebra of finite rank. In the previous section, we introduced a group $C(L)$ and proved that

$$\text{Br}(X_a^c)/\text{Im}(\text{Br}(k)) \simeq C(L)/D.$$

The aim of this section is to give more precise information about the isomorphism $C(L)/D \rightarrow \text{Br}(X_a^c)/\text{Im}(\text{Br}(k))$ when $\text{char}(k) \neq p$.

Let $\text{Br}_{\text{ur}}(k(X_a))$ be the subgroup of $\text{Br}(k(X_a))$ consisting of all elements which are unramified at all discrete valuations of $k(X_a)$ with residue fields containing k and with fields of fraction $k(X_a)$; recall that $\text{Br}_{\text{ur}}(k(X_a))$ is isomorphic to $\text{Br}(X_a^c)$ when $\text{char}(k) \neq p$ (see [Po17, Theorem 6.8.3]).

As in the previous sections, let us write $K' = \prod_{i \in I} K_i$, where the K_i are finite separable field extensions of k .

Notation 4.1: We denote by \mathcal{G}_k the absolute Galois group of k , $\mathcal{G}_{k(X_a)}$ the absolute Galois group of $k(X_a)$. Let R be a discrete valuation ring of $k(X_a)$ with residue field κ_R containing k and with field of fractions $k(X_a)$. We denote by \mathcal{G}_R the absolute Galois group of κ_R .

Notation 4.2: For all $i \in I$, let $\{\beta_{ij}\}$ be a basis of K_i over k . Let

$$y_i = \sum_j \beta_{ij} x_{ij},$$

where x_{ij} are variables. Set

$$N_i = N_{K_i \otimes k(X_a)/k(X_a)}(y_i)$$

considered as an element of $k(X_a)^\times$. We define

$$N = N_{K \otimes k(X_a)/k(X_a)}(y)$$

in a similar way. Fix an isomorphism $\chi: \text{Gal}(K/k) \rightarrow \mathbf{Z}/p^n \mathbf{Z}$. Then χ gives rise to a homomorphism $\tilde{\chi}: \mathcal{G}_{k(X_a)} \rightarrow \mathbf{Z}/p^n \mathbf{Z}$ and a homomorphism $\chi_R: \mathcal{G}_R \rightarrow \mathbf{Z}/p^n \mathbf{Z}$. Let $(N_i, \tilde{\chi})$ denote the class of the cyclic algebra over $k(X_a)$ associated to $\tilde{\chi}$ and the element $N_i \in k(X_a)^\times$ ([GS06, Prop. 4.7.3]).

The main result of this section is

THEOREM 4.3: *Assume $\text{char}(k) \neq p$. Then the map*

$$u : C(L) \rightarrow \text{Br}(k(X_a))$$

given by

$$u(c) = \sum_{i \in I} c(i)(N_i, \tilde{\chi})$$

induces an isomorphism

$$C(L)/D \rightarrow \text{Br}(X_a^c)/\text{Im}(\text{Br}(k)).$$

Remark 4.4: Note that $(N_i, \tilde{\chi}) \in \text{Br}(k(X_a))$ has order at most $p^{e_i(n)}$, so $c(i)(N_i, \tilde{\chi})$ is well-defined for $c(i) \in \mathbf{Z}/p^{e_i(n)}\mathbf{Z}$ (ref. [BLP19, Lemma 6.1]).

We start with the following lemmas.

LEMMA 4.5: *The group $u(D)$ is contained in the image of $\text{Br}(k)$ in $\text{Br}(k(X^c))$.*

Proof. Since $N \cdot \prod_{i \in I} N_i = c$, we have $\sum_{i \in I} (N_i, \tilde{\chi}) = (c/N, \tilde{\chi}) = (c, \tilde{\chi})$, which is the image of (c, χ) in $\text{Br}(k(X_a))$. Hence $u(D) \subseteq \text{Im}(\text{Br}(k))$. ■

The following lemma can be found in [Lee22] §3. Here we use the notation $C(L)$ to simplify the proof.

LEMMA 4.6: (1) *Let $c \in C(L) \setminus D$. Pick $i_0 \in I_{\hat{e}}$. Let $\hat{c} \in D$ be the image of the constant map from I to $c(i_0)$. Set m to be the maximal integer such that $\pi_{n,m}(c) = \pi_{n,m}(\hat{c})$. Choose $r \in \mathbf{Z}/p^{\hat{e}}\mathbf{Z}$ such that $\delta(r, c(i_0)) = m$. Consider the element $c' \in \bigoplus_{e \in \mathcal{E}} C(I_e, \mathbf{Z}/p^e\mathbf{Z})$ defined as follows:*

$$(4.1) \quad c'(i) = \begin{cases} \pi_{\hat{e}, e_i(n)}(r), & \text{if } e_i(n) > m \text{ and } m = \delta(c(i), c(i_0)); \\ \pi_{\hat{e}, e_i(n)}(c(i_0)), & \text{otherwise.} \end{cases}$$

Then $c' \in C(L) \setminus D$.

(2) *Suppose that k is a global field and $c \in C_{\text{arith}}(L) \setminus D$. Then the element c' defined above is in $C_{\text{arith}}(L) \setminus D$.*

Proof. As c is not in D , by the choice of m there is some $i \in I$ such that $e_i(n) > m$ and $\delta(c(i), c(i_0)) = m$. Hence $c'(i) \neq c'(i_0) \pmod{p^{e_i(n)}\mathbf{Z}}$ by our construction and $c' \notin D$.

Now we show that $\pi_{n,l}(c') \in C^l(L)$ for $0 \leq l \leq n$. If $l \leq m$, then by the choice of r we have $\pi_{n,l}(c(i_0)) = \pi_{n,l}(r)$. Clearly $\pi_{n,l}(c') \in C^l(L)$.

Suppose $l > m$. If $\Gamma_i^l \cap \Gamma_j^l \neq \emptyset$, then by Remark 3.7 $e_i(n)$ and $e_j(n)$ are at least l and $c(i) = c(j) \pmod{p^l \mathbf{Z}}$. Hence $\delta(c(i_0), c(i)) = m$ if and only if $\delta(c(i_0), c(j)) = m$. By construction $c'(i) = c'(j) \pmod{p^l \mathbf{Z}}$ and $\pi_{n,l}(c') \in C^l(L)$.

The proof of statement (2) is similar. ■

LEMMA 4.7: *Assume that $\text{char}(k) \neq p$. Let R be a discrete valuation ring as in Notation 4.1. Denote by ∂_R the residue map from $\text{Br}(k(X_a))$ to $H^1(\kappa_R, \mathbf{Q}/\mathbf{Z})$. Suppose that the order of $\partial_R(N_i, \tilde{\chi})$ and the order of $\partial_R(N_j, \tilde{\chi})$ are both at least p^m . Then $\Gamma_i^m \cap \Gamma_j^m$ is nonempty.*

Proof. Let ν_R be the discrete valuation associated to R . Denote the completion of $k(X_a)$ with respect to ν_R by $k(X_a)_R$. Choose an extension ω_R of ν_R to a separable closure of $k(X_a)_R$. By the construction of $\tilde{\chi}$, $\partial_R(N_i, \tilde{\chi}) = \nu_R(N_i)\chi_R$. (See [GS06, 6.8.4 and 6.8.5.]) Write $\nu_R(N_i)$ as $p^{m_i}q_i$ where $p \nmid q_i$. Let p^{n_R} be the order of χ_R . As the order of $\partial_R(N_i, \tilde{\chi}) \geq p^m$, we have $n_R - m_i \geq m$.

Since $\nu_R(N_i) = p^{m_i}q_i$, there is some factor M_i of $K_i \otimes_k k(X_a)_R$ such that p^{m_i+1} does not divide $\nu_R(N_{M_i/k(X_a)_R}(y_{M_i}))$, where y_{M_i} is the projection of y_i in M_i . Let $\omega_{i,R}$ be the restriction of ω_R to M_i . Write the inertia degree of $\omega_{i,R}$ over ν_R as $p^{f_i}q'_i$ where $p \nmid q'_i$. As p^{m_i+1} does not divide $\nu_R(N_{M_i/k(X_a)_R}(y_{M_i}))$, we have $f_i \leq m_i$.

Choose a factor M of $K \otimes_k k(X_a)_R$ and let \overline{M} be its residue field. Let \overline{M}_i be the residue field of $\omega_{i,R}$. Both fields are considered as subfields of a separable closure κ_R^s of κ_R .

As $f_i \leq m_i$ and $n_R - m_i \geq m$, the cyclic extension $\overline{M}M_i/\overline{M}_i$ is of degree at least p^m . Choose $g_R \in \mathcal{G}_R$ such that $\chi_R(g_R)$ generates the image of χ_R in \mathbf{Q}/\mathbf{Z} . Let \mathcal{H}_i be the subgroup of \mathcal{G}_R which fixes \overline{M}_i . We claim that there are some $h_i \in \mathcal{G}_R$ and some $\sigma_R \in \langle h_i^{-1}g_R h_i \rangle \cap \mathcal{H}_i$ such that $\chi_R(\sigma_R)$ is of order at least p^m .

Consider the group action of $\langle g_R \rangle$ on the set of left cosets of \mathcal{H}_i in \mathcal{G}_R . As $|\mathcal{G}_R/\mathcal{H}_i| = p^{f_i}q'_i$ with $p \nmid q'_i$, there is some $h_i \in \mathcal{G}_R$ such that p^{f_i+1} does not divide the length of the orbit of $h_i\mathcal{H}_i$. Hence the stabilizer of $h_i\mathcal{H}_i$ is $\langle g_R^{p^{f'_i}r_i} \rangle$ for some $f'_i \leq f_i$ and some r_i coprime to p . Let $\sigma_R = h_i^{-1}g_R^{p^{f'_i}r_i}h_i$. Then

$$\chi_R(\sigma_R) = \chi_R(g_R^{p^{f'_i}r_i}),$$

which is of order $p^{n_R-f'_i}$. Since $f'_i \leq f_i \leq m_i$ and $n_R - m_i \geq m$, the order of $\chi_R(\sigma_R)$ is at least p^m .

Let g and σ be the image of g_R and σ_R in G . Then σ fixes K_i and σ is an element of order at least p^m in $\text{Gal}(K/k)$. Hence the conjugacy class of g belongs to Γ_i^m . The same argument proves that the conjugacy class of g belongs to Γ_j^m . Hence $\Gamma_i^m \cap \Gamma_j^m$ is nonempty. ■

Next we prove that for all $c \in C(L)$, the element $\sum_{i \in I} c(i)(N_i, \tilde{\chi})$ is unramified.

PROPOSITION 4.8: *Suppose that $\text{char}(k) \neq p$. The image of u is an unramified subgroup of $\text{Br}(k(X_a))$.*

Proof. By Lemma 4.5 we can assume that $c(i) = 0$ for some $i \in I_{\hat{e}}$.

Let R be a discrete valuation ring of k with residue field κ_R containing k and with field of fractions $k(X_a)$. Let ν_R be the discrete valuation associated to R . Denote by ∂_R the residue map from $\text{Br}(k(X_a))$ to $H^1(\kappa, \mathbf{Q}/\mathbf{Z})$. We claim that $u(c) = \sum_{i \in I} c(i)(N_i, \tilde{\chi})$ is unramified at R .

Let $J(c) = \{i \in I \mid c(i) \neq 0 \text{ in } \mathbf{Z}/p^{e_i(n)}\mathbf{Z}\}$. Let m be the maximum integer such that $\pi_{n,m}(c) = 0$ and set $J_m(c) = \{i \in J(c) \mid m = \delta(0, c(i))\}$. We prove by induction on $|J(c)|$. If $|J(c)| = 0$, then $c = 0$ and our claim is trivial. Suppose that our claim is true for $|J(c)| \leq h$.

Let $|J(c)| = h + 1$. Then $c \notin D$ and $J_m(c)$ is nonempty. Pick $j \in J_m(c)$ and choose $r \in \mathbf{Z}/p^{\hat{e}}\mathbf{Z}$ such that $c(j) = r \pmod{p^{e_j(n)}\mathbf{Z}}$. Let c' be defined as in Lemma 4.6. We first prove that $u(c')$ is unramified at R , i.e., $\partial_R(u(c')) = 0$.

Since $c(i) = 0$, by the definition of c' we have $u(c') = \sum_{s \in J_m(c)} r(N_s, \tilde{\chi})$. Hence

$$\partial_R(u(c')) = \sum_{s \in J_m(c)} r \cdot \nu_R(N_s)\chi_R.$$

Suppose that $\partial_R(u(c'))$ is not zero. Then there is some $s \in J_m(c)$ such that $r \cdot \nu_R(N_s)\chi_R \neq 0$. As $\delta(0, r) = m$, the order of $\nu_R(N_s)\chi_R$ is at least p^{m+1} .

By Lemma 4.5, there is some $t \in I \setminus J_m(c)$ such that $r \cdot \nu_R(N_t)\chi_R(g) \neq 0$ and the order of $\nu_R(N_t)\chi_R$ is at least p^{m+1} . By Lemma 4.7 the set $\Gamma_s^{m+1} \cap \Gamma_t^{m+1}$ is nonempty. As $\delta(0, r) = m$, we have $c'(s) \neq c'(t) \pmod{p^{m+1}\mathbf{Z}}$. This contradicts that $c' \in C(L)$. Therefore $\partial_R(u(c')) = 0$ and $u(c')$ is unramified.

Consider the element $c - c' \in C(L)$. By our construction of c' , the cardinality of $J(c - c')$ decreases by at least one. By induction hypothesis $u(c - c')$ is unramified. Hence $u(c)$ is unramified. ■

Proof of Theorem 4.3. By Lemma 4.5 and Proposition 4.8, the map

$$u : C(L)/D \rightarrow \text{Br}(X_a^c)/\text{Im}(\text{Br}(k))$$

is well-defined.

A similar argument as in [BP20, Thm. 12.2] proves the injectivity of u . Consider the projection from X_a to the affine space \mathbb{A}^d , where $d = \sum_{i \in I} [K_i : k]$ and the coordinates are given by x_{ij} defined in Notation 4.2.

Let M be the function field of \mathbb{A}^d . Denote by χ_M the image of χ in $H^1(M, \mathbf{Q}/\mathbf{Z})$. Suppose that $u(c) = \alpha \in \text{Im}(\text{Br}(k))$. Then $u(c) - \alpha$ is in the kernel of $\text{Br}(M) \rightarrow \text{Br}(k(X_a))$, which is generated by $(a \prod_{i \in I} N_i^{-1}, \chi)$ (see [BP20, Lemma 12.3]). Therefore

$$u(c) - \alpha = r \left(a \prod_{i \in I} N_i^{-1}, \chi \right).$$

Consider the discrete valuation v_{N_i} on M and let κ_{N_i} be its residue field. Denote by χ_{N_i} the image of χ in $H^1(\kappa_{N_i}, \mathbf{Q}/\mathbf{Z})$. We claim that χ_{N_i} is of order $p^{e_i(n)}$. Let M_i be the function field of the subvariety of $\mathbb{A}^{[K_i:k]}$ defined by N_i . Then $\kappa_{N_i} = M_i(x_{jl})$ where x_{jl} are defined as in Notation 4.2 with $j \neq i$. Hence κ_{N_i} is purely transcendental over M_i . Let χ_{M_i} be the image of χ in $H^1(M_i, \mathbf{Q}/\mathbf{Z})$. It suffices to prove the order of χ_{M_i} is $p^{e_i(n)}$. Note that $M_i \otimes_k K_i$ is isomorphic to a product of extensions $F_{i\ell}$ of K_i with one factor purely transcendental over K_i . Denote this factor by F_{i1} . Then

$$K \otimes_k M_i \otimes_k K_i \simeq K \otimes_k \left(\prod_{\ell} F_{i\ell} \right)$$

and $K \otimes_k F_{i1}$ is isomorphic to a product of extensions of F_{i1} with degree $p^{e_i(n)}$. On the other hand

$$K \otimes_k M_i \otimes_k K_i \simeq \left(\prod \tilde{M}_i \right) \otimes_k K_i,$$

where \tilde{M}_i is a factor of $K \otimes_k M_i$. If $[\tilde{M}_i : M_i] < p^{e_i(n)}$, then

$$\tilde{M}_i \otimes_k K_i = \tilde{M}_i \otimes_{M_i} M_i \otimes_k K_i$$

is isomorphic to a product of extensions of $F_{i\ell}$, and each factor is of degree less than $p^{e_i(n)}$. This contradicts that there are factors of $K \otimes_k M_i \otimes_k K_i$ isomorphic to extensions of F_{i1} with degree $p^{e_i(n)}$. Hence $[\tilde{M}_i : M_i] = p^{e_i(n)}$ and χ_{M_i} is of order $p^{e_i(n)}$.

As χ_{N_i} is of order $p^{e_i(n)}$, after taking residue of $u(c) - \alpha$ at v_{N_i} , we see that $c(i) = -r \pmod{p^{e_i(n)}\mathbf{Z}}$. Hence $c \in D$ and u is injective.

Since u is injective, $|C(L)/D| \leq |\text{Br}(X_a^c)/\text{Im}(\text{Br}(k))|$. By Theorem 3.8 and Theorem 1.1, the order of $C(L)/D$ is equal to the order of $\text{Br}(T_{L/k}^c)/\text{Br}(k)$. By Theorem 1.2, the map u is surjective and hence is an isomorphism. This concludes the proof of the theorem. ■

We now prove the results announced in the introduction:

THEOREM 4.9: *Let k be a field of $\text{char}(k) \neq p$. Let F be a bicyclic extension of k with Galois group $\mathbf{Z}/p^n\mathbf{Z} \times \mathbf{Z}/p^n\mathbf{Z}$. Let K and K_i be linearly disjoint cyclic subfields of F with degree p^n for $i = 1, \dots, m$. Then*

$$\text{Br}(X_a^c)/\text{Im}(\text{Br}(k)) \simeq (\mathbf{Z}/p^n\mathbf{Z})^{m-1},$$

and is generated by (N_i, χ) for $i = 1, \dots, m - 1$.

Proof. There exist a number field ℓ and an unramified Galois extension ℓ'/ℓ such that $\text{Gal}(\ell'/\ell) \simeq \text{Gal}(F/k)$ ([F62]). Set

$$F = (\ell')^{G_K}, \quad L_i = (\ell')^{G_{K_i}} \quad \text{and} \quad E = F \times \prod_{i \in I} L_i.$$

By construction, the G -lattices $\hat{T}_{E/\ell}$ and $\hat{T}_{L/k}$ are isomorphic.

By [Lee22, Proposition 7.3], the group $\text{III}_\omega^2(\ell, \hat{T}_{E/\ell}) \simeq (\mathbf{Z}/p^n\mathbf{Z})^{m-1}$. Since $\text{III}_\omega^2(\ell, \hat{T}_{E/\ell}) \simeq \text{III}_{\text{cycl}}^2(G, \hat{T}_{E/\ell}) \simeq \text{III}_{\text{cycl}}^2(G, \hat{T}_{L/k})$, we have

$$C(L)/D \simeq (\mathbf{Z}/p^n\mathbf{Z})^{m-1}$$

by Theorem 3.8. The assertion then follows from Theorem 4.3. ■

5. Global fields

We keep the notation of the previous section, and in addition we assume that k is a global field. Denote by $\mathbb{B}(X_a^c)$ the subgroup of $\text{Br}(X_a^c)/\text{Im}(\text{Br}(k))$ consisting of locally trivial elements, and by $\mathbb{B}_\omega(X_a^c)$ the subgroup consisting of elements which are trivial at almost all places of k .

THEOREM 5.1: *Suppose that k is a global field with $\text{char}(k) \neq p$. Then*

- (1) *u induces an isomorphism between $C(L)/D$ and $\mathbb{B}_\omega(X_a^c)$.*
- (2) *u induces an isomorphism between $C_{\text{arith}}(L)/D$ and $\mathbb{B}(X_a^c)$.*

Proof. First note that [Sa81, 6.1.4] remains true over global fields. Hence $B_\omega(X_a^c) \simeq B_\omega(X_a)$ and $B(X_a^c) \simeq B(X_a)$.

By [Sa81, 6.8 and 6.9 (ii)], we have $B(X_a) \simeq \text{III}^2(k, \hat{T}_{L/k})$ (respectively $B_\omega(X_a) \simeq \text{III}_\omega^2(k, \hat{T}_{L/k})$). By Theorem 3.10 and Corollary 3.11, we conclude that $C(L)/D \simeq B_\omega(X_a^c)$ and $C_{\text{arith}}(L)/D \simeq B(X_a^c)$.

As $B_\omega(X_a^c)$ is a subgroup of $\text{Br}(X_a^c)/\text{Im}(\text{Br}(k))$ with the same cardinality, the first statement follows from Theorem 4.3.

To see that u gives rise to the desired isomorphism in (2), it is sufficient to show that $u(c)_v \in \text{Br}(X_a^c \times_k k_v)$ is in the image of $\text{Br}(k_v)$ for $c \in C_{\text{arith}}(L)$ and for $v \in V_k$.

By Lemma 4.5 we can assume that $c(i_0) = 0$ for some $i_0 \in I_{\hat{e}}$. Let

$$J(c) = \{i \in I \mid c(i) \neq 0 \text{ in } \mathbf{Z}/p^{e_i(n)}\mathbf{Z}\}.$$

Set m to be the maximum integer such that $\pi_{n,m}(c) = 0$ and set

$$J_m(c) = \{i \in J(c) \mid m = \delta(0, c(i))\}.$$

We prove by induction on $|J(c)|$. If $|J(c)| = 0$, then $c = 0$ and our claim is trivial. Suppose that our claim is true for $|J(c)| \leq h$.

Let $|J(c)| = h + 1$. Then $c \notin D$ and $J_m(c)$ is nonempty. Pick $j \in J_m(c)$ and choose $r \in \mathbf{Z}/p^{\hat{e}}\mathbf{Z}$ such that $c(j) = r \pmod{p^{e_j(n)}\mathbf{Z}}$. Let c' be defined as in Lemma 4.6.

For $v \in V_k$, let χ_v be the image of χ in $H^1(k_v, \mathbf{Q}/\mathbf{Z})$, and $\chi_{i,w}$ be the image of χ in $H^1((K_i)_w, \mathbf{Q}/\mathbf{Z})$ where w is a place of K_i .

By the definition of $C_{\text{arith}}(L)$, the set $V_i^{m+1} \cap V_j^{m+1}$ is empty for any $i \in J_m(c)$ and for any $j \notin J_m(c)$.

Let $v \in V_k$. Suppose that $v \notin \bigcup_{i \in J_m(c)} V_i^{m+1}$. Then for all $i \in J_m(c)$, $\chi_{i,w}$ is of order at most m for all $w \mid v$. By the projection formula, $(N_i, \chi)_v$ has order at most p^m . Hence $c'(i)(N_i, \chi)_v = 0$ and $u(c')_v = 0$ in this case.

Suppose that $v \in V_i^{m+1}$ for some $i \in J_m(c)$. Let $d \in D$ be the image of the constant map from I to r . Set $\bar{c} = d - c'$. As the set $V_i^{m+1} \cap V_j^{m+1}$ is empty for any $j \notin J_m(c)$, $v \notin \bigcup_{j \notin J_m(c)} V_j^{m+1}$. The same argument shows that $u(\bar{c})_v = 0$. Hence $u(c')_v$ is in the image of $\text{Br}(k_v)$.

Since the cardinality of $J(c - c')$ decreases by at least one, by induction hypothesis $u(c - c') \in B(X_a^c)$. In combination with $u(c') \in B(X_a^c)$, we see that $u(c)$ is in $B(X_a^c)$. ■

Example 5.2: Let $k = \mathbf{Q}(i)$. Let

$$K = k(\sqrt[4]{17}), \quad K_1 = k(\sqrt[4]{17 \times 13}) \quad \text{and} \quad K_2 = k(\sqrt[4]{13}).$$

By [Lee22, Example 7.4],

$$\text{III}_\omega^2(k, \hat{T}_{L/k}) \simeq \mathbf{Z}/4\mathbf{Z} \quad \text{and} \quad \text{III}^2(k, \hat{T}_{L/k}) \simeq \mathbf{Z}/2\mathbf{Z}.$$

By Theorem 4.3 and Theorem 5.1, the element (N_2, χ) generates the group $\text{Br}(X_a^c)/\text{Im}(\text{Br}(k))$ and $2(N_2, \chi)$ generates $\text{B}(X_a^c)$.

More generally we have the following.

PROPOSITION 5.3: *Let k be a global field of $\text{char}(k) \neq p$. Let F be a bicyclic extension of k with Galois group $\mathbf{Z}/p^n\mathbf{Z} \times \mathbf{Z}/p^n\mathbf{Z}$. Let K and K_i be linearly disjoint cyclic subfields of F with degree p^n for $i = 1, \dots, m$. Assume moreover that $F \otimes_k k_v$ is a product of cyclic extensions for all $v \in V_k$. Then (N_i, χ) generates $\text{B}(X_a^c)$.*

Proof. By [Lee22, Prop. 7.3], we have $\text{III}^2(k, \hat{T}_{L/k}) \simeq (\mathbf{Z}/p^n\mathbf{Z})^{m-1}$. We apply Theorem 3.10 (1) and Theorem 5.1 (2) to conclude. \blacksquare

6. An application to Hasse principles

In this section we apply Theorem 4.3 to give an alternative proof of [BLP19] Theorem 7.1 for k a global field and K/k a cyclic extension of degree prime to $\text{char}(k)$. Moreover, we can assume that K/k is a cyclic extension of p -power degree where $p \neq \text{char}(k)$. (See §2 and [BLP19, 6.3].)

We use the notation of the previous sections. In particular, X_a is the affine variety defined in the introduction, K/k is a cyclic extension of p -power degree with $p \neq \text{char}(k)$, and K_i/k is a finite separable extension for all $i \in I$. Recall that χ is an injective homomorphism from $\text{Gal}(K/k)$ to \mathbf{Q}/\mathbf{Z} .

Let χ_v be the image of χ in $H^1(k_v, \mathbf{Q}/\mathbf{Z})$. Let inv be the Hasse invariant map $\text{inv} : \text{Br}(k_v) \rightarrow \mathbf{Q}/\mathbf{Z}$.

Denote by K_i^v the algebra $K_i \otimes_k k_v$. Suppose that there is a local point $(x_i^v) \in \prod_{v \in V_k} X_a(k_v)$, where $x_i^v \in K_i^v$ for $i \in I$ and $x_0^v \in K \otimes_k k_v$. Define $\alpha_a : C_{\text{arith}}(L)/D \rightarrow \mathbf{Q}/\mathbf{Z}$ as

$$\alpha_a(c) = \sum_{v \in V_k} \sum_{i \in I} c(i) \text{inv}(N_{K_i^v/k_v}(x_i^v), \chi_v).$$

THEOREM 6.1: *Suppose that there is a local point $(x_i^v) \in \prod_{v \in V_k} X_a(k_v)$. Then the map α_a is the Brauer–Manin pairing and X_a has a k -point if and only if $\alpha_a = 0$.*

Proof. First we consider the case where k is a number field. By Theorem 5.1 and [Sa81, Lemma 6.2], the map α_a is the Brauer–Manin pairing of X_a^c . Our claim then follows from Sansuc’s result [Sa81, Cor. 8.7].

For k a global function field, we apply Theorem 5.1 and [DH22, Theorem 2.5] to conclude. ■

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Open access funding provided by EPFL Lausanne.

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