Koopman-based Data-driven Robust Control of Nonlinear Systems Using Integral Quadratic Constraints

Mert Eyüboğlu and Alireza Karimi

Abstract— **This paper introduces a novel method for data-driven robust control of nonlinear systems based on the Koopman operator, utilizing Integral Quadratic Constraints (IQCs). The Koopman operator theory facilitates the linear representation of nonlinear system dynamics in a higher-dimensional space. Data-driven Koopman-based representations inherently yield only approximate models due to various factors. In addressing this, we focus on effective characterization of the modeling error, which is crucial for ensuring closed-loop guarantees. We identify non-parametric IQC multipliers to characterize the modeling error in a data-driven fashion through the solution of frequency domain (FD) linear matrix inequalities (LMIs), treating it as additive uncertainty for robust control design. These multipliers provide a convex set representation of stabilising robust controllers. We then obtain the optimal controller within this set by solving a different set of FD LMIs. Lastly, we propose an iterative approach alternating between IQC multiplier identification and robust controller synthesis, ensuring monotonic convergence of a robust performance index.**

I. INTRODUCTION

Due to the inherent challenges that data-driven modeling and control of nonlinear systems pose, the Koopman operator theory [1] has gained notable popularity within the past decade for its capacity to offer a global linear representation of nonlinear systems [2], [3]. Koopman operator theory focuses on the evolution of the so-called observable functions, expressing the nonlinear system dynamics linearly in a higher-dimensional space of observables. However, achieving a global linearization often requires lifting the system to an infinite-dimensional space. Therefore, in practice, a finite-dimensional truncation of the Koopman operator is considered, providing a linear but approximate representation of the nonlinear dynamics. The prevalent Extended Dynamic Mode Decomposition (EDMD) algorithm [4], facilitates the computation of such approximations from data.

One concern for Koopman theory in the context of non-autonomous systems is the fact that the linearity of the lifted dynamics in observables does not imply linearity in inputs. Many works, such as [5], impose linearity in inputs by only allowing the use of observable functions that are linear in inputs. Despite introducing an additional source of modeling error by this restriction, this approach has seen many successful applications. Several works consider bilinear lifted models, which are claimed to strike a compromise between modeling accuracy and ease of control design [6].

The authors are with Laboratoire d'Automatique, École Polytechnique Fédérale de Lausanne (EPFL), Switzerland (e-mail: mert.eyuboglu@epfl.ch; alireza.karimi@epfl.ch).

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While Koopman-based lifting of input-affine systems accepts an infinite-dimensional bilinear representation, in practice, the resulting models are still never exact due to finite-dimensional truncation and data-driven approximation. Thus, characterization of modeling error for data-driven lifted models is paramount for providing closed-loop guarantees.

Probabilistic error bounds for EDMD-based bilinear approximate models of input-affine systems are derived in [7]. In a complementary effort, [8] reformulates these error bounds and designs state feedback controller in the lifted space with closed-loop guarantees. These works focus on continuous-time systems, which necessitates state derivative measurements, posing a fundamental limitation for real-world applications. To alter this, the discrete-time counterpart of the error bounds, along with robust controller synthesis, is worked out more recently in [9]. Considering general nonlinear systems, a data-driven characterization of the model error in terms of the worst-case ℓ_2-gain is proposed in [10], where probabilistic guarantees are given by the scenario approach [11]. Utilizing these error bounds, [10] further presents robust controller synthesis tailored for LTI and open-loop stable LPV models. It is noteworthy that these works consider disc-shaped static error bounds, which are likely to yield conservative control design.

Introduced by [12], IQC approach provides an effective tool for analysis and control of uncertain dynamical systems by its flexibility of representing general nonlinearities. While most of the existing literature on IQCs focuses on the analysis of uncertain systems, an iterative algorithm alternating between nominal $\mathcal{H}_\infty$ controller synthesis and robustness analysis was presented in [13]. More recently, IQC synthesis methods based on non-smooth optimization for $\mathcal{H}_\infty$ and $\mathcal{H}_2$ performance are developed, respectively, in [14] and [15]. These methods alter the need for an iterative process; however, they require the use of structured IQC multipliers. Lately, a controller synthesis method robust against uncertainties characterized by non-parametric IQCs is proposed in [16]. This method performs controller synthesis by solving FD LMIs and enables less conservative designs through compatibility with non-parametric IQC multipliers.

In this work, we propose a novel robust controller synthesis approach for nonlinear systems based on Koopman operator theory and IQCs. We focus on LTI lifted models of nonlinear systems obtained via EDMD using only data collected from the system. To address the inherent errors of such models, we propose characterization of the model error using IQCs. Through solution of FD LMIs, we identify non-parametric IQC multipliers characterizing the modeling error. Next, we employ the control design method of [16] to design controllers with robust performance guarantees. Since the set of robustly stabilizing controllers during synthesis is dependent on the identified IQC multipliers, we propose an iterative algorithm alternating between IQC multiplier identification and controller synthesis, yielding monotonic convergence of a chosen performance index.

The paper is organized as follows: In Section II, a brief overview of the Koopman operator theory, EDMD, and IQCs are provided along with the main problem description. In Section III, the proposed method is presented. First, the IQC-based characterization of the modeling error and synthesis of robust controllers are presented separately. Next, the frequency sampling approach for implementation of the optimization problems is discussed, followed by the iterative algorithm combining the first two steps. The application of the proposed algorithm on a simulation example is presented in Section IV. A brief conclusion is offered in Section V.

II. PRELIMINARIES

Notations: $\mathbb{R}$ and $\mathbb{C}$ are used to denote the sets of real and complex numbers respectively. ℓ_2^p denotes the space of p dimensional square integrable signals. Identity matrix of an appropriate size is represented by I. $S \succ (\succeq) 0$ and $S \prec (\preceq) 0$ indicate that the matrix S is positive (-semi) definite and negative (-semi) definite respectively. The conjugate transpose of a complex matrix S is denoted by S^* and the pseudo-inverse of S is denoted by $S^\dagger$. If $S \in \mathbb{C}$ is full row rank, the right inverse is denoted as $S^{\mathsf{R}} = S^*(SS^*)^{-1}$. If $S \in \mathbb{C}$ is full column rank, the left inverse is denoted as $S^{\mathsf{L}} = (S^*S)^{-1}S^*$. The frequency response of a discrete-time system G is denoted by $G(e^{j\omega})$.

A. Koopman operator

Consider the discrete-time nonlinear system,

$$H : \left\{ x_{k+1} = f(x_k, u_k), \right. \tag{1}$$

where $x \in \mathbb{X} \subseteq \mathbb{R}^{n_x}$ is the state variable, $u \in \mathbb{U} \subseteq \mathbb{R}^{n_u}$ is the input and $f : \mathbb{X} \times \mathbb{U} \to \mathbb{X}$ is the nonlinear state transition map. The Koopman operator $\mathcal{K} : \mathcal{F} \to \mathcal{F}$ is a linear operator that advances an observable function $\xi(x_k, u_k)$ one-step ahead in time,

$$\xi(x_{k+1}, u_{k+1}) = \mathcal{K}\xi(x_k, u_k) = \xi(f(x_k, u_k), u_{k+1}), \tag{2}$$

where $\mathcal{F}$ is a Banach space of observable functions that is invariant under the action of the Koopman operator. Therefore, the Koopman operator $\mathcal{K}$ globally maps the nonlinear dynamics in the state space to linear dynamics in the lifted space of observables. In general, the Koopman operator is defined on an infinite-dimensional space. In practice, however, a finite-dimensional approximation of the Koopman operator denoted by $\mathbb{K}$, is used where a finite set of observable functions $\mathcal{D} = \{\xi_j\}_{j=1}^d$ called a dictionary is considered.

Due to the availability of well established tools for LTI systems, identifying such a model is often desirable. In order to obtain a lifted LTI representation we consider a dictionary structured as $\mathcal{D} = \begin{bmatrix} \boldsymbol{\xi}(x_k) & u_k \end{bmatrix}^T$ with $\boldsymbol{\xi}(x_k) = \begin{bmatrix} \xi_1(x_k) & \xi_2(x_k) & \dots & \xi_{d-1}(x_k) \end{bmatrix}^T$ yielding,

$$\begin{bmatrix} \boldsymbol{\xi}(x_{k+1}) \\ u_{k+1} \end{bmatrix} \approx \begin{bmatrix} \mathbb{K}_{11} & \mathbb{K}_{12} \\ \mathbb{K}_{21} & \mathbb{K}_{22} \end{bmatrix} \begin{bmatrix} \boldsymbol{\xi}(x_k) \\ u_k \end{bmatrix}. \tag{3}$$

Since predicting the future values of the input is not of interest we discard the last n_u rows of $\mathbb{K}$ resulting in,

$$\boldsymbol{\xi}(x_{k+1}) = A\boldsymbol{\xi}(x_k) + Bu_k + \varepsilon_k, \tag{4}$$

where $A = \mathbb{K}_{11}$, $B = \mathbb{K}_{12}$ and ε_k denotes the one step ahead prediction error. Note that the prediction error ε_k is introduced by the restriction of the Koopman operator to a finite dimensional space as well as the structure imposed on the dictionary. To alter the contribution of the latter, bilinear or LPV representations based on Koopman theory are also studied while this paper only considers LTI representations.

B. Data-driven approximation of the Koopman operator

EDMD [4] enables the computation of the matrices A and B in (4) by solving a least-squares problem as follows. Based on a set of data trajectories with N samples $\{x_k, u_k\}_{k=0}^{N-1}$ and a selected dictionary of observable functions $\boldsymbol{\xi}$, the matrices

$$Z := \begin{bmatrix} \boldsymbol{\xi}(x_0) & \dots & \boldsymbol{\xi}(x_{N-2}) \end{bmatrix},$$

$$Z^+ := \begin{bmatrix} \boldsymbol{\xi}(x_1) & \dots & \boldsymbol{\xi}(x_{N-1}) \end{bmatrix},$$

$$U := \begin{bmatrix} u_0 & \dots & u_{N-2} \end{bmatrix},$$

are constructed. A and B in (4) can be obtained by solving,

$$\min_{A,B} \quad \left\| Z^+ - \begin{bmatrix} A & B \end{bmatrix} \begin{bmatrix} Z \\ U \end{bmatrix} \right\| \tag{5}$$

which has the ℓ_2-optimal solution $\begin{bmatrix} A & B \end{bmatrix} = Z^+ \begin{bmatrix} Z \\ U \end{bmatrix}^\dagger$.

C. Integral Quadratic Constraints

Two discrete-time signals $p(k) \in \ell_2^{n_p}[0, \infty]$ and $q(k) \in \ell_2^{n_q}[0, \infty]$ with sampling time T_s are said to satisfy the IQC defined by Π if,

$$\int_{\omega \in \Omega} \begin{bmatrix} P(e^{j\omega}) \\ Q(e^{j\omega}) \end{bmatrix}^* \Pi(e^{j\omega}) \begin{bmatrix} P(e^{j\omega}) \\ Q(e^{j\omega}) \end{bmatrix} d\omega \geq 0, \tag{6}$$

where $P(e^{j\omega})$ and $Q(e^{j\omega})$ represent the discrete-time Fourier transforms of $p(k)$ and $q(k)$ respectively and $\Omega = (-\pi/T_s, \pi/T_s]$. Let a performance metric on the channel $w \to z$ with respect to the multiplier $\Pi_p(\gamma)$ be defined such that, performance with index γ is achieved if the signals w and z satisfy the IQC defined by $\Pi_p(\gamma)$. Considering the IQC theorem [17, Corollary 3]:

Theorem 1. *The feedback interconnection of a discrete-time stable LTI system T and a bounded causal operator Δ as depicted in Fig. 1, is robustly stable against Δ and has robust performance on the channel $w \to z$ with respect to Π_p if,*

1) *interconnection of T and $\tau\Delta$ is well-posed, $\forall \tau \in [0, 1]$;*
2) *the IQC defined by Π is satisfied by $\tau\Delta$, $\forall \tau \in [0, 1]$;*

3) for all $\omega \in \Omega$,

$$\begin{bmatrix} T(e^{j\omega}) \\ I \end{bmatrix}^* \Pi_{rp}(e^{j\omega}) \begin{bmatrix} T(e^{j\omega}) \\ I \end{bmatrix} \prec 0; \tag{7}$$

where,

$$\Pi_{rp} = \left[\begin{array}{cc|cc} \Pi_{11} & 0 & \Pi_{12} & 0 \\ 0 & \Pi_{p,11} & 0 & \Pi_{p,12} \\ \hline \Pi_{12}^* & 0 & \Pi_{22} & 0 \\ 0 & \Pi_{p,12}^* & 0 & \Pi_{p,22} \end{array}\right]. \tag{8}$$

By [17, Remark 3] if Π is partitioned as

$$\Pi = \begin{bmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{12}^* & \Pi_{22} \end{bmatrix},$$

with $\Pi_{11} \succeq 0$ and $\Pi_{22} \preceq 0$, then $\tau\Delta$ satisfies the IQC defined by Π for all $\tau \in [0, 1]$ if and only if Δ satisfies the IQC.

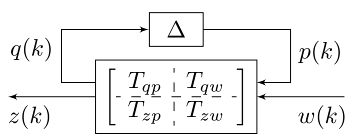

Fig. 1. General feedback interconnection.

D. Problem Formulation

Consider data $\{\{x_k^m, u_k^m\}_{k=0}^{N-1}\}_{m=1}^M$ collected from a general discrete-time nonlinear system (1) with sampling time T_s, as M trajectories of N samples. The data is collected from bounded sets such that $x \in \mathbb{X}$, $u \in \mathbb{U}$ and assumed to be rich enough to fully represent the systems behaviour within these sets. Using the data and a predetermined set of observable functions $\boldsymbol{\xi}(x_k)$, the discrete-time nonlinear dynamics can be approximated in the lifted space as,

$$H_0 : \left\{ \hat{\boldsymbol{\xi}}_{k+1} = A\hat{\boldsymbol{\xi}}_k + Bu_k, \right. \tag{9}$$

where $\hat{\boldsymbol{\xi}}_k \approx \boldsymbol{\xi}(x_k)$ and the matrices A, B are calculated by EDMD. Due to the prediction error term in (4), the LTI system H_0 is only an approximation of the true system H such that $H = H_0 + \Delta$, where Δ represents the error model to be treated as additive uncertainty for controller design. Thus, the interconnection of the nonlinear system H with a controller K can be represented as in Fig. 2.

Based on this, we formulate the problem of designing a data-driven controller providing closed-loop guarantees for the nonlinear system H, as the following two subproblems,

1) Characterization of the error system Δ using non-parametric dynamic IQC multipliers.
2) Synthesis of a fixed-structure controller $K = XY^{-1}$ for H_0 with guarantees of robust stability against Δ and robust performance with respect to Π_p on $w \to z$.

Remark 1. *For simplicity we consider the scenario where the controller is acting on the full lifted state. However, it is possible to pass only a subset of the observable functions to the controller. This would allow the synthesis of low dimensional controllers for high dimensional lifted LTI models.*

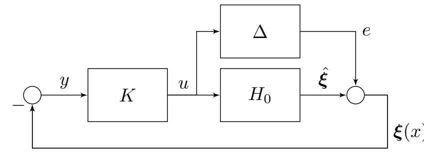

Fig. 2. Block diagram of the closed-loop system.

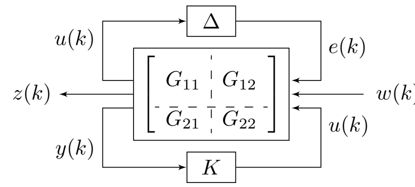

Fig. 3. Generalized plant structure of the feedback interconnection.

III. Data-driven Robust Control Design

For any arbitrary channel $w \to z$ on which the performance objective is defined, it is fairly standart to transform the block diagram in Fig. 2 to a generalized plant structure as in Fig. 3 where $G_{22} = -H_0$. Then, by applying a lower linear fractional transformation to the generalized plant G and controller K, the closed-loop system can be represented as in Fig. 1 with $T = G_{11} + G_{12}K(I - G_{22}K)^{-1}G_{21}$. Using A and B obtained by EDMD, the frequency response function (FRF) of the LTI model $H_0(e^{j\omega}) = (e^{j\omega}I - A)^{-1}B$ can be computed for any $\omega \in \Omega$. Based on $H_0(e^{j\omega})$ and following the corresponding generalized plant formulation G, the FRF $T(e^{j\omega})$ can be obtained similarly. For the generalized plant model G we assume that:

(A.1) $G_{21}(e^{j\omega})$ has full rank, $\forall\omega \in \Omega$.
(A.2) $G(e^{j\omega})$ is bounded, $\forall\omega \in \Omega$.

Next, we first discuss the problem of robust controller synthesis against uncertainty Δ characterized by an IQC multiplier Π. Since solving this problem requires a certain decomposition of Π, we discuss the error characterization problem afterwards so that we can already formulate this problem in terms of the decomposed elements of Π. Thus, we can readily use the solution of the IQC-based error characterization problem for robust controller synthesis.

A. Robust Controller Synthesis

The objective of the controller synthesis is to obtain a controller structured as $K = XY^{-1}$ which guarantees robust stability against Δ and robust performance on the channel $w \to z$ with respect to $\Pi_p(\gamma)$. For some Π such that the error system Δ satisfies the IQC defined by Π, this objective can be formulated as an optimization problem,

$$\begin{aligned} \min_{K} \quad & \gamma \\ \text{s.t.} \quad & \begin{bmatrix} T \\ I \end{bmatrix}^* \Pi_{rp} \begin{bmatrix} T \\ I \end{bmatrix} (e^{j\omega}) \prec 0, \ \forall\omega \in \Omega, \\ & T \text{ is stable.} \end{aligned} \tag{10}$$

For $\Phi = G_{21}^{\mathsf{R}}(Y - G_{22}X)$ and $\Psi = I - \Phi\Phi^{\mathsf{L}} = G_{21}^{\mathsf{R}}G_{21}$ the closed-loop transfer function T in (10) can be written as,

$$\begin{aligned} T &= G_{11} + G_{12}X\Phi^{\mathsf{L}} = G_{11}(\Phi\Phi^{\mathsf{L}} + \Psi) + G_{12}X\Phi^{\mathsf{L}} \\ &= (G_{11}\Phi + G_{12}X)\Phi^{\mathsf{L}} + G_{11}\Psi. \end{aligned} \tag{11}$$

Since Ψ is a hermitian idempotent matrix such that,

$$\Psi\Phi = \Phi - \Phi\Phi^{\mathsf{L}}\Phi = 0 \tag{12}$$

$$\Phi^{\mathsf{L}}\Psi = \Phi^{\mathsf{L}} - \Phi^{\mathsf{L}}\Phi\Phi^{\mathsf{L}} = 0 \tag{13}$$

we get,

$$\begin{bmatrix} T \\ I \end{bmatrix} = \begin{bmatrix} G_{11}\Phi + G_{12}X & G_{11}\Phi \\ \Phi & \Psi \end{bmatrix} \begin{bmatrix} \Phi^{\mathsf{L}} \\ \Psi \end{bmatrix} = L \begin{bmatrix} \Phi^{\mathsf{L}} \\ \Psi \end{bmatrix}. \tag{14}$$

Then, by [18, Proposition 8.1.2] the first constraint in (10) can be replaced by $L^*\Pi_{rp}L \prec 0$. Using the fact that any square matrix accepts a factorisation $\Pi_{rp} = \Pi_{rp}^+ + \Pi_{rp}^-$ with $\Pi_{rp}^+ \succ 0$ and $\Pi_{rp}^- \preceq 0$, $L^*\Pi_{rp}L \prec 0$ can be written as $L^*\Pi_{rp}^+L - (-L^*\Pi_{rp}^-L) \prec 0$. By the Schur complement lemma, this yields the constraint,

$$\begin{bmatrix} (\Pi_{rp}^+)^{-1} & L \\ L^* & -L^*\Pi_{rp}^-L \end{bmatrix} \succ 0. \tag{15}$$

The quadratic component $-L^*\Pi_{rp}^-L$ in (15) can be convexified around a known controller $K_c = X_cY_c^{-1}$ such that,

$$L^*\Pi_{rp}^-L \preceq L^*\Pi_{rp}^-L_c + L_c^*\Pi_{rp}^-L - L_c^*\Pi_{rp}^-L_c \prec 0, \tag{16}$$

where

$$\begin{aligned} L_c &= \begin{bmatrix} G_{11}\Phi_c + G_{12}X_c & G_{11}\Psi \\ \Phi_c & \Psi \end{bmatrix}, \\ \Phi_c &= G_{21}^{\mathsf{R}}(Y_c - G_{22}X_c). \end{aligned}$$

By expanding, it can be seen that (16) implies,

$$\Phi^*\Pi_{rp,22}^-\Phi_c + \Phi_c^*\Pi_{rp,22}^-\Phi - \Phi_c^*\Pi_{rp,22}^-\Phi_c \prec 0.$$

Therefore, by [16, Lemma 1], satisfying (16) also guarantees that T is stable if $\Pi_{rp,22}^- \prec 0$ and K_c is nominally stabilising.

Thus, by [16, Theorem 2], for a known robustly stabilising initial controller $K_c = X_cY_c^{-1}$ a solution of the convex problem,

$$\begin{aligned} \min_{\gamma, X, Y} \quad & \gamma \\ \text{s.t.} \quad & \begin{bmatrix} (\Pi_{rp}^+(\gamma))^{-1} & L \\ L^* & \mathcal{L} \end{bmatrix} (e^{j\omega}) \succ 0, \ \forall \omega \in \Omega, \end{aligned} \tag{17}$$

where $\mathcal{L} = L^*\Pi_{rp}^-L_c + L_c^*\Pi_{rp}^-L - L_c^*\Pi_{rp}^-L_c$, is also a solution to (10) if $\Pi_{rp,22}^- \prec 0$, for the full proof we refer to [16]. Thus, by solving (17) for any $\Pi_{rp}^+ \succ 0$ and $\Pi_{rp}^- \preceq 0$ such that $\Pi_{rp,22}^- \prec 0$, we can obtain the controller $K = XY^{-1}$ guaranteeing robust performance with index γ.

A known Π_p defining a performance objective can be easily decomposed in to $\Pi_p^+ \succ 0$ and $\Pi_p^- \preceq 0$ with $\Pi_{p,22}^- \prec 0$ such that $\Pi_p = \Pi_p^+ + \Pi_p^-$ for many conventional IQC multipliers. For some examples we refer to [16, Section III.A]. Following that, decomposition of Π_{rp} in to $\Pi_{rp}^+ \succ 0$ and $\Pi_{rp}^- \preceq 0$ can be obtained by applying the structure in (8) to Π^+, Π_p^+ and Π^-, Π_p^- respectively. Obtaining Π^+ and Π^- with $\Pi_{22}^- \prec 0$ from data is adressed in subsection III-B.

Remark 2. *Both constraints in (10) are convexified around the initial controller K_c arriving at (17), resulting in an over approximation of a convex-concave constraint. The conservatism due to this over approximation vanishes as $K = K_c$ is attained. To achieve this, it is proposed to iteratively solve the problem in [19], replacing the initial controller at each iteration by the optimal controller obtained in the previous one which guarantees monotonic convergence of the objective to a local minimum where $K \approx K_c$ such that the conservatism vanishes.*

B. Error characterization via non-parametric IQCs

In order to characterize the error system, we aim for finding a multiplier $\Pi(e^{j\omega})$ such that the input signal u and the error signal e in Fig. 2 satisfy the IQC defined by $\Pi(e^{j\omega})$ as in (6). Thus, first the frequency spectrum of the signals u and e has to be computed using the available data. To do so, we first simulate H_0 with the same input trajectories used for data collection $\{\{u_k^m\}_{k=0}^{N-1}\}_{m=1}^M$ with initial conditions $\hat{\boldsymbol{\xi}}_0^m = \boldsymbol{\xi}(x_0^m)$ for all $m \in [1, M]$. Next, we obtain the trajectories of e corresponding to the available data as $\{\{e_k^m\}_{k=0}^{N-1}\}_{m=1}^M = \{\{\boldsymbol{\xi}(x_k^m) - \hat{\boldsymbol{\xi}}_k^m\}_{k=0}^{N-1}\}_{m=1}^M$. Then, frequency content of e at each trajectory can be obtained as,

$$E_m(e^{-j\omega}) = \sum_{k=0}^{N-1} e_k^m e^{-j\omega T_s k}, \ \forall \omega \in \Omega, \ \forall m \in [1, M]. \tag{18}$$

Similarly, the frequency spectrum of the plant input u can also be computed $\forall \omega \in \Omega$ and $\forall m \in [1, M]$ following (18).

Additionally, for a known robustly stabilising controller K_c, the IQC stability condition (7) should be satisfied by the resulting $\Pi_{rp}(e^{j\omega}, \gamma)$ as in (8) where γ denotes the robust performance index achieved by K_c. Thus, for a known robustly stabilising initial controller K_c, an IQC multiplier characterizing the error system as well as the achieved robust performance index can be obtained by solving the following FD convex optimization problem,

$$\begin{aligned} \min_{\gamma, \Pi^+, \Pi^-} \quad & \gamma \\ \text{s.t.} \quad & \int_{\omega \in \Omega} \begin{bmatrix} U_m \\ E_m \end{bmatrix}^* \Pi \begin{bmatrix} U_m \\ E_m \end{bmatrix} (e^{j\omega}) d\omega \geq 0, \ \forall m \in [1, M], \\ & \begin{bmatrix} T \\ I \end{bmatrix}^* \Pi_{rp}(\gamma) \begin{bmatrix} T \\ I \end{bmatrix} (e^{j\omega}) \prec 0, \ \forall \omega \in \Omega, \\ & \Pi(e^{j\omega}) = \Pi^+(e^{j\omega}) + \Pi^-(e^{j\omega}), \ \forall \omega \in \Omega, \\ & \Pi_{11}(e^{j\omega}) \succeq 0, \ \Pi_{22}(e^{j\omega}) \preceq 0, \ \forall \omega \in \Omega, \\ & \Pi^+(e^{j\omega}) \succ 0, \ \Pi^-(e^{j\omega}) \preceq 0, \ \forall \omega \in \Omega, \\ & \Pi_{22}^-(e^{j\omega}) \prec 0, \ \forall \omega \in \Omega. \end{aligned} \tag{19}$$

Here, imposing $\Pi_{22}^- \prec 0$ in addition to $\Pi^- \preceq 0$ yields us the desired IQC multiplier such that (17) already guarantees the stability of T, with arbitrarily small conservatism added to the IQC multiplier identification problem.

C. Frequency Sampling

Note that both problems outlined in this paper are framed as FD convex optimization problems with infinitely many constraints known as convex semi-infinite programs (SIPs). A common strategy for solving SIPs is to sample the infinite constraints in the FD at a sufficiently large set of finite frequencies $\Omega_g = \{\omega_1, \ldots, \omega_g\} \subset \Omega$. Since all constraints in (17) and (19) are imposed on Hermitian matrices, it suffices to consider frequencies only in the range $\Omega_g \in [0, \pi/T_s)$, discarding the negative half of the spectrum. While this sampling approach does not provide constraint satisfaction guarantees at all frequencies, probabilistic guarantees dependent on the number of finite frequency points that are considered can be obtained by the scenario approach [11].

As a result of this sampling approach, we obtain the non-parametric IQC multipliers $\Pi(e^{j\omega})$ also at finite number of frequency points Ω_g. While by choosing sufficiently large number of frequency points a robust controller can be synthesized in practice, it should be noted that due the use non-parametric IQCs the number of optimisation variables scale with the number of frequencies in Ω_g.

D. Iterative Approach

While in subsections III-A and III-B we separately addressed the two parts in our problem formulation, for achieving the best possible performance we propose an iterative scheme between the two. Clearly, for the signals u and e there is not a unique IQC multiplier Π characterizing the error model. And since a particular multiplier Π determines a convex set of controllers that we can choose from during the controller synthesis, without iteratively updating the Π and K it is very likely that the achieved performance indexes will be highly conservative. Thus, we propose Algorithm 1.

Algorithm 1: Iterative algorithm over error system characterization and robust controller synthesis

Data: measured trajectories: $\{\{x_k^m, u_k^m\}_{k=0}^{N-1}\}_{m=1}^{M}$,
lifting functions: $\boldsymbol{\xi}(x)$,
initial robustly stabilising controller: K_c.

Preparation:
obtain A and B in (9) by EDMD.
compute $T(e^{j\omega}),\ \forall\omega \in \Omega_g$.
compute $\{(U_m, E_m)(e^{j\omega})\}_{m=1}^{M},\ \forall\omega \in \Omega_g$.
obtain RCF $K_c = X_c Y_c^{-1}$.

Iteration: set $i = 0$.

while γ *converges and* $i \leq i_{max}$ **do**
- update IQC multiplier Π: solve (19) for $\omega \in \Omega_g$, obtain $\Pi^+, \Pi^-\ \forall\omega \in \Omega_g$.
- update controller K: solve (17) for $\omega \in \Omega_g$, (iteratively as in [19]), obtain $K = XY^{-1}$.
- set $i = i + 1$.

end

Result: K, γ.

The presented algorithm yields monotonic decrease of the performance objective γ over each iteration. At the end of the algorithm a controller K that guarantees robust stability against Δ and robust performance on the channel $w \rightarrow z$ with respect to $\Pi_p(\gamma)$ is obtained by only using data trajectories collected from the system and a lifting dictionary. It should be noted that while we synthesize a linear controller in the lifted space, due to the nonlinear state transformation from the state space to the lifted space of observables the resulting controller is nonlinear in the actual state space.

Remark 3. *By properly modifying the optimization problems (19) and (17) in Algorithm 1, it is possible to use the same approach to find an initial robustly stabilising controller. In that scenario, only a nominally stabilising initial controller for H_0 is required by the algorithm, which can be set to 0 for stable systems or easily synthesized using various linear control design methods. Then, instead of optimizing over a performance objective we relax the stability conditions by adding slack variables and optimize over the slack variables until the original stability constraints are met. More precisely, one should set $\Pi_{rp} = \Pi$ and relax the second constraint in (19) and the first constraint in (17) by adding $-\gamma_{s1} I$ and $\gamma_{s2} I$ to the left hand-sides respectively. Then, the iterative approach in Algorithm 1 must be followed while minimizing over γ_{s1} and γ_{s2} when solving (19) and (17), until either $\gamma_{s1} \leq 0$ or $\gamma_{s2} \leq 0$ is achieved. This yields a robustly stabilising controller that can be used as an initial controller for optimizing robust performance afterwards.*

IV. Numerical Example

To demonstrate the proposed method on a simulation example we consider an inverted pendulum which is a commonly used example for validation of nonlinear control methods. The system dynamics are,

$$\dot{x}_1(t) = x_2(t), \tag{20}$$

$$\dot{x}_2(t) = \frac{g}{l}\sin x_1(t) - \frac{b}{ml^2}x_2(t) + \frac{1}{ml^2}u(t), \tag{21}$$

with mass $m = 1\ kg$, length $l = 1\ m$, rotational friction coefficient $b = 0.01$, and gravitational constant $g = 9.81\ m/s^2$. We discretize the dynamics using the 4^{th}-order Runge-Kutta method with sampling time $T_s = 0.01\ s$ and consider the discrete-time model as our true nonlinear system. To collect data, we simulate the discrete-time system for a single trajectory of $N = 5000$ samples with initial condition $x_0 = \begin{bmatrix}0 & 0\end{bmatrix}^T$ and a random input, such that u_k is randomly chosen from $\mathbb{U} = [-10, 10]$ with a uniform distribution for all $k \in [0, N-1]$. By also inferring some knowledge of the dynamics we choose the lifting functions $\boldsymbol{\xi}(x) = \begin{bmatrix}x_1 & x_2 & \sin(x_1)\end{bmatrix}^T$.

After applying the EDMD algorithm the lifted state matrices as in (9) are obtained, yielding a 3-dimensional stable LTI representation of the system. We consider the tracking problem where the pendulum angle x_1 is desired to track the reference w. The performance channel output is defined as $z = \begin{bmatrix}(W_1(w - x_1))^T & (W_2 u)^T\end{bmatrix}^T$ such that the tracking error as well as the control input are penalized during control design. For optimising a desired tracking response we use a low-pass filter W_1 defined by the Matlab command `W1 = 1/makeweight(0.001,1,2,Ts)` and we set $W_2 = 0.1$. We select,

$$\Pi_p = \begin{bmatrix} -\gamma^2 I & 0 \\ 0 & I \end{bmatrix}, \tag{22}$$

such that minimizing $\mathcal{H}_\infty$ norm of T_{zw} is our objective. Next, applying Algorithm 1 with initial controller $K_c = 0$, yields the state feedback controller $K = \begin{bmatrix} 82.98 & 9.076 & -10.64 \end{bmatrix}$ with robust performance index $\gamma^* = 9.6613$.

To observe the benefits of Koopman lifting and error characterization via non-parametric IQCs separately, we consider two other control design methods. For the same robust performance objective, first we consider the case where we did not employ lifting such that $\boldsymbol{\xi}(x) = \begin{bmatrix} x_1 & x_2 \end{bmatrix}$. After identifying the system matrices by solving the EDMD problem, we use Algorithm 1 for robust controller synthesis. This approach yields a robust performance guarantee with index $\gamma_1^* = 31.5295$ achieved by the linear state feedback controller $K_1 = \begin{bmatrix} 278.6 & 22.76 \end{bmatrix}$. Next, to observe the benefit of using non-parametric IQCs, we follow the approach in [10] for the same performance objective. As lifting functions we again use $\boldsymbol{\xi}(x) = \begin{bmatrix} x_1 & x_2 & \sin(x_1) \end{bmatrix}^T$, which yields the same lifted representation obtained earlier. Considering the single measured trajectory, we find a lower bound on the error systems worst case ℓ_2-gain by finding the minimum value of $\gamma_e > 0$ such that

$$\sum_{k=0}^{N-1} \|e_k\|^2 \leq \gamma_e^2 \sum_{k=0}^{N-1} \|u_k\|^2,$$

is satisfied. This yields the lower bound of $\gamma_e^* = 0.0753$ achieved on the worst case ℓ_2-gain of the error system. Next, we apply the linear feedback controller synthesis method from [10, Section 3] which is based on the well known small-gain theorem. This yields a performance index of $\gamma_2^* = 14.5565$. Thus, while all three approaches yield a robust controller that can track a reference in the full operation range $x_1 \in [-\pi, \pi]$, the performance guarantee that is be achieved by the proposed method is significantly better. While we only present state feedback synthesis for simplicity, the proposed method also allows for dynamic output feedback controller synthesis to be used when full state information can not be recovered.

V. Conclusion

The presented method offers a promising approach to robustly control nonlinear systems by leveraging the Koopman operator theory and IQCs. The use of non-parametric IQC multipliers for characterizing the modeling error proves to be a powerful strategy since it yields a tight uncertainty around the lifted LTI model, significantly reducing conservatism for control design. While the iterative algorithm is not guaranteed to converge to the global optimum solution, the monotonic decrease of the performance objective is ensured. Overall, the algorithm enables data-driven control of nonlinear systems with closed-loop guarantees by only using linear control methods and solving convex problems. However, the method relies on the central assumption that the data collected from the system is fully representative of its behaviour in a region of operation. While this assumption can be satisfied by collecting large enough sets of data in practice, quantification of the quality of available data is aimed to be addressed in future research, to further enhance a priori guarantees. Simulation example shows the benefit of the proposed non-parametric IQC based error characterization signifying the main contribution of this work. Future work will also consider the extension of this approach to bilinear/LPV lifted models which promises to yield smaller modeling errors enhancing better closed-loop performance.

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